J. Fluid Mech. (1996), *vol.* 313, *pp.* 39–54

Second-order horizontal steady forces and moment on a floating body with small forward speed

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(Received 13 December 1993 and in revised form 27 October 1995)

In a recent work, a simple formula was derived for the 'wave drift damping' in a two-dimensional floating body and the obtained expression is exact within the context of the related theory, where only leading-order terms in the forward speed are retained. This formula is now generalized for a three-dimensional problem and the coefficients of the 'wave drift damping matrix' are given explicitly in terms of the standard second-order steady forces and moment in the horizontal plane; Munk's yaw moment, related with the steady second-order potential and discussed in Grue & Palm (1993), is not analysed in this paper and the effect of an eventual small angular velocity around the vertical axis is also not considered.

Numerical results agree in general with the proposed formula although in a specific case a consistent disagreement has been observed, as discussed in §5.

1. Introduction

A floating body drifts slowly under the action of the steady second-order wave force but this force is influenced by the small drift velocity caused by it. A possible way to express the interdependent behaviour is to write the excitation as a sum of the standard steady force with a force that depends linearly on the drift velocity, this latter parcel being called 'wave drift damping', after Wichers (1982). In a high sea state it provides a damping mechanism that is usually the dominating one and for this reason this topic has deserved some attention in the literature lately.

The pertinent theory must be linear in the drift velocity U and can be obtained from the general ship motion theory, as described in Newman (1978), by disregarding terms of order U^2 or higher; the basic set of equations can be found in Nossen, Grue & Palm (1991) for a three-dimensional problem.

A simple expression relating the drift force $D_U(\omega)$, influenced by the small forward speed U, with the standard drift force $D_0(\omega)$ has been obtained recently for a two-dimensional problem. In fact, if $c = g/\omega$ is the wave celerity and ω_e the frequency of encounter, it can be shown that (see Aranha 1994)

$$D_U(\omega) = \left[1 - 4\frac{U}{c}\right] D_0(\omega_e), \tag{1}$$

this formula being exact within the context of the related theory, where only terms of leading-order in U/c are considered.

In the reference system moving with the body, one observes a current U from right to left and since terms of order U^2 must be ignored no new waves should be generated by the current and the body; it follows, in this case, that in the far field one should only observe the interaction between the current and the existing far field waves. The known theory on wave propagation in a moving media, discussed in Bretherton & Garret

(1969) and Whitham (1974), can be used then to determine the far field wave amplitudes and so the related steady forces. This is the basic idea behind (1), where one observes the standard result $D_0(\omega)$ being modified kinematically by the frequency of encounter ω_e and dynamically by the factor $1-4U/c$; the kinematics correction alone produces Wichers's approximation $D_U(\omega) \approx D_0(\omega_e)$.

This same sort of reasoning can be used to infer the three-dimensional formula, as briefly explained in this introduction and derived with more details in the rest of the paper.

Consider then a floating body, moving with a forward speed U in the positive x-direction, exposed to an incident wave with amplitude A, frequency ω and propagating in a direction that makes an angle β with the x-axis. Let $D_{x,U}(\omega;\beta)$ be the steady force in surge, $D_{y,U}(\omega;\beta)$ the steady force in sway and $T_{z,U}(\omega;\beta)$ the steady yaw moment; the subscript 'U' refers always to the problem with forward speed. To simplify the notation one may introduce the 'generalized' steady force vector $\boldsymbol{D}_U(\omega;\beta)$ by the expression:

$$\boldsymbol{D}_U(\omega;\beta) = [D_{x,U}\,\boldsymbol{i} + D_{y,U}\,\boldsymbol{j} + T_{z,U}\,\boldsymbol{k}]. \tag{2a}$$

In an analogous way one can write

$$\boldsymbol{D}_0(\omega;\beta) = [D_{x,0}\,\boldsymbol{i} + D_{y,0}\,\boldsymbol{j} + T_{z,0}\,\boldsymbol{k}], \tag{2b}$$

that defines the steady generalized force in the standard problem, where the forward speed is zero; the subscript '0' refers always to this problem.

In the reference system moving with the body one observes a current U in the negative x-direction and a frequency ω_e given by the known expression

$$\omega_e = \left[1 - \frac{U}{c}\cos\beta\right]\omega; \quad c = g/\omega. \tag{3a}$$

Besides this frequency change the incident wave, originally propagating in the β-direction, is also refracted by the current. In fact, the velocity component $U\sin\beta$, perpendicular to the wave direction, changes the angle of the group velocity vector by an amount $(2U/c)\sin\beta$, if only leading-order terms in U/c are retained. As a consequence the actual direction of the incident wave, after refraction, is given by β_1 with

$$\beta_1 = \beta + 2\frac{U}{c}\sin\beta. \tag{3b}$$

If only kinematics corrections were introduced one would obtain, in place of Wichers' approximation $\boldsymbol{D}_0(\omega_e)$, the expression $\boldsymbol{D}_0(\omega_e;\beta_1)$; the dynamic correction, however, implies that this force should be multiplied by the factor $1-4(U\cos\beta/c)$, accordingly to (1) and to the fact that the parcel of the current actually against the incident wave is $U\cos\beta$. In this way one should have

$$\boldsymbol{D}_U(\omega;\beta) = \left[1 - 4\frac{U}{c}\cos\beta\right]\boldsymbol{D}_0(\omega_e;\beta_1). \tag{4}$$

The purpose of the present paper is to show that formula (4) is exact within the context of the related theory, where terms of order U^2 are ignored. Notice that (4) is correct either for a body fixed or free to oscillate, if the corresponding $\boldsymbol{D}_0(\omega_e;\beta_1)$ is used; as shown in Aranha (1994), and corroborated in the present work, all information needed about the oscillatory motion is carried with the standard generalized force $\boldsymbol{D}_0(\omega_e;\beta_1)$.

As pointed out by Grue & Palm (1993), one should add to (4) a moment $\boldsymbol{N} = N\boldsymbol{k}$, related to the coupling between the steady second-order wave potential and the body forward speed $U\boldsymbol{i}$. This moment has a strong formal analogy with the classical Munk's moment and its analysis is omitted in this work.

In §2 the mathematical problem is posed and some basic relations, of conservation of energy and momentum, are stated. In §3, formula (4) is derived and in the following section the wave damping matrix is presented. Numerical results are discussed in §5 and in the Appendix more specific results are elaborated.

2. Mathematical problem and basic results

Let $\boldsymbol{x} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ be the position vector and (r, θ, z) the polar coordinates. Consider a floating body moving in the horizontal plane (x, y) with the speed $U\boldsymbol{i}$ and exposed to an incident wave with amplitude A, frequency ω and being propagated in a direction that makes an angle β with the x-axis. In the reference system moving with the small forward speed $U\boldsymbol{i}$ one observes the frequency of encounter ω_e, defined in (3a), and the following wavenumbers are introduced here for future reference:

$$\left.\begin{aligned} K_e &= \frac{\omega_e^2}{g}, \\ K(\theta) &= K_e\left(1+2\frac{U}{c}\cos\theta\right). \end{aligned}\right\} \tag{5a}$$

From (3a) it follows, in particular, that

$$K(\beta) = K = \frac{\omega^2}{g}, \tag{5b}$$

if terms of order U^2 are neglected, and so the incident wave potential can be written in the form

$$\phi_1(\boldsymbol{x}, t) = \left[-\mathrm{i}\frac{g\mathrm{A}}{\omega}\exp\left(\mathrm{iK}(\beta)\, r\cos(\theta-\beta)\right)\exp\left(\mathrm{K}(\beta)\, z\right)\right]\exp(-\mathrm{i}\omega_e t). \tag{6}$$

As usual, the time factor will be omitted in the sequel except in a few occasions where it may clarify the expression.

Consider first the interaction between the incident flow $-U\boldsymbol{i}$ and the body B and let $U\varphi_S(\boldsymbol{x})$ be the related potential. Once terms of order U^2 must be neglected this potential satisfies the 'impermeable' condition $\partial\varphi_S/\partial z = 0$ at the free surface and corresponds to the distortion caused on the incoming flow by the 'double body' immersed in an unbounded fluid. Introducing the operator

$$\boldsymbol{\nabla}_1 = \left[\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}\right]_{z=0}, \tag{7}$$

denoting by ∂B the intersection between the body and the free surface, by $\boldsymbol{n}$ the normal to B pointing out the fluid and $\boldsymbol{n}_1$ the normal to ∂B, the potential φ_S also satisfies:

$$\left.\begin{aligned} \boldsymbol{\nabla}_1\varphi_S\cdot\boldsymbol{n}_1|_{\partial B} &= 0, \\ \frac{\partial\varphi_S}{\partial r} &\to -\cos\theta \quad \text{as } r\to\infty. \end{aligned}\right\} \tag{8}$$

Let $\phi_U(\boldsymbol{x})\exp(-\mathrm{i}\omega_e t)$ be the perturbation caused on $U\varphi_S(\boldsymbol{x})$ by the action of the wave; the subscript U indicates, as before, a variable related to the oscillatory problem when $U \neq 0$. The total potential can be written in the form

$$\Phi(\boldsymbol{x}, t) = U\varphi_S(\boldsymbol{x}) + \phi_U(\boldsymbol{x})\exp(-\mathrm{i}\omega_e t), \tag{9a}$$

and from Bernoulli's equation the oscillatory pressure, linear in ϕ_U, is given by

$$p_U(\boldsymbol{x}) = \rho[\mathrm{i}\omega_e \phi_U - U\boldsymbol{\nabla}\varphi_S \cdot \boldsymbol{\nabla}\phi_U]. \tag{9b}$$

If one designates, for notational convenience, $\boldsymbol{e}_1 = \boldsymbol{i}$; $\boldsymbol{e}_2 = \boldsymbol{j}$; $\boldsymbol{e}_3 = \boldsymbol{k}$, the following coefficients can be introduced (see Newman 1978):

$$\left.\begin{aligned} n_j &= \boldsymbol{n}\cdot\boldsymbol{e}_j; \quad m_j = -[(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla}\varphi_S]\cdot\boldsymbol{e}_j, \\ n_{j+3} &= (\boldsymbol{x}\times\boldsymbol{n})\cdot\boldsymbol{e}_j; \quad m_{j+3} = -[(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{x}\times\boldsymbol{\nabla}\varphi_S)]\cdot\boldsymbol{e}_j. \end{aligned}\right\} \quad (j = 1,2,3) \tag{10}$$

It can be shown, to leading order in U/c, that the potential $\phi_U(\boldsymbol{x})$ must be the solution of the set of equations (see Nossen *et al.* 1991)

$$\left.\begin{aligned} &\nabla^2\phi_U = 0 \quad \text{in } V, \\ &\left.\frac{\partial\phi_U}{\partial z}\right|_{z=0} = K_e\phi_U + \mathrm{i}\frac{U}{c}[2(\boldsymbol{\nabla}_1\varphi_S\cdot\boldsymbol{\nabla}_1\phi_U) + (\nabla_1^2\varphi_S)\,\phi_U], \\ &\boldsymbol{\nabla}\phi_U\cdot\boldsymbol{n}|_B = \sum_j q_{j,U}(\beta)\,[-\mathrm{i}\omega_e n_j + Um_j], \end{aligned}\right\} \tag{11a}$$

where $\{q_{j,U}(\beta); j = 1,2,\ldots,6\}$ are the generalized displacements. If $A_U(\theta;\beta)$ is the non-dimensional amplitude coefficient of the total outgoing wave, the boundary conditions when $z \rightarrow -\infty$ and $r \rightarrow \infty$ are given by

$$\left.\begin{aligned} &\partial\phi_U/\partial z \rightarrow 0, \\ &(\phi_U - \phi_I) \rightarrow -\mathrm{i}\frac{gA}{\omega}\left(\frac{2}{\pi K_e r}\right)^{1/2} A_U(\theta;\beta)\exp\left(\mathrm{i}(K(\theta)\,r - \pi/4)\right)\exp\left(K(\theta)\,z\right). \end{aligned}\right\} \tag{11b}$$

Notice that all variables are functions of the frequency ω and the incidence angle β, although the dependence with respect to ω will be omitted, whenever possible, to simplify the notation.

The oscillatory force coefficients can be defined by

$$F_{j,U}(\beta) = \frac{1}{\rho}\int_B p_U n_j \quad \mathrm{d}B = \int_B [\mathrm{i}\omega_e n_j + Um_j]\,\phi_U\,\mathrm{d}B, \tag{12}$$

with p_U given by (9*b*). The expression on the right-hand side of (12) is a classical result due to Ogilvie & Tuck (1969) and $F_{j,U}(\beta)$ includes both the exciting and reacting hydrodynamic forces, the last ones normally expressed in terms of the added mass and radiation damping matrices.

When the generalized forces $\{F_{j,U}(\beta); j = 1,2,\ldots,6\}$ are known, the generalized displacements $\{q_{j,U}(b); j = 1,2,\ldots,6\}$ can be determined from the equations of motion

$$\left.\begin{aligned} &D_{jl} = -\omega_e^2 M_{jl} + C_{jl}, \\ &\sum_l D_{jl}\,q_{l,U}(\beta) = \rho F_{j,U}(\beta), \end{aligned}\right\} \tag{13}$$

with M_{jl} and C_{jl} being, respectively, the inertia and restoring matrices of the body; notice that the matrix D_{jl} is symmetric.

The energy equation has been deduced by Nossen *et al.* (1991) and can be written, in the present notation, in the form (see (3*b*))

$$\frac{1}{\pi}\int_0^{2\pi}\left[1-2\frac{U}{c}\cos\theta\right]|A_U(\theta;\beta)|^2\,\mathrm{d}\theta+2\left[1-2\frac{U}{c}\cos\beta\right]\mathrm{Re}\,A_U(\beta_1;\beta)=0. \tag{14}$$

From conservation of linear and angular momentum one obtains expressions for the steady second-order drift forces and yaw moment in terms of the far-field amplitude coefficient $A_U(\theta;\beta)$. The final result is given below and in the Appendix it is shown that it agrees with the one derived by Grue & Palm (1993); recall that Munk's moment N, related to the interaction between the steady second-order potential and the incoming flow, is not included in this analysis:

$$D_{x,U}(\omega;\beta)=-\frac{\rho g A^2}{2K_e}\left\{\frac{1}{\pi}\int_0^{2\pi}\left[1-2\frac{U}{c}(\cos\beta+\cos\theta)\right]\cos\theta|A_U(\theta;\beta)|^2\,\mathrm{d}\theta\right.$$
$$\left.+2\left[1-4\frac{U}{c}\cos\beta\right]\cos\beta_1\,\mathrm{Re}\,A_U(\beta_1;\beta)\right\}, \tag{15a}$$

$$D_{y,U}(\omega;\beta)=-\frac{\rho g A^2}{2K_e}\left\{\frac{1}{\pi}\int_0^{2\pi}\left[1-2\frac{U}{c}(\cos\beta+\cos\theta)\right]\sin\theta|A_U(\theta;\beta)|^2\,\mathrm{d}\theta\right.$$
$$\left.+2\left[1-4\frac{U}{c}\cos\beta\right]\sin\beta_1\,\mathrm{Re}\,A_U(\beta_1;\beta)\right\}, \tag{15b}$$

$$T_{z,U}(\omega;\beta)=\frac{\rho g A^2}{2K_e^2}\,\mathrm{Im}\left\{\frac{1}{\pi}\int_0^{2\pi}\left[1-2\frac{U}{c}(\cos\beta+\cos\theta)\right]A_U^*(\theta;\beta)\frac{\mathrm{d}A_U}{\mathrm{d}\theta}(\theta;\beta)\,\mathrm{d}\theta\right.$$
$$\left.+2\frac{U}{c}\sin\beta_1\,A_U^*(\beta_1;\beta)+2\left[1-4\frac{U}{c}\cos\beta\right]\frac{\mathrm{d}A_U^*}{\mathrm{d}\theta}(\beta_1;\beta)\right\}. \tag{15c}$$

In the above expressions, and in the following, the symbols Re(.) and Im(.) refer to the real and imaginary part of the related variable while * stands for the complex conjugate.

Suppose, in the standard problem (zero forward speed), a wave incident in the direction β_1 with frequency ω_e and let $A_0(\theta;\beta_1)$ be the related outgoing wave amplitude coefficient. If U is taken equal to zero in (14) and $A_0(\theta;\beta_1)$ is used in place of $A_U(\theta;\beta)$, the standard energy relation (optical theorem) is obtained in this case (see Mei 1983); when a similar procedure is followed in (15) the known expressions for $D_{x,0}(\omega_e;\beta_1)$, $D_{y,0}(\omega_e;\beta_1)$ and $T_{z,0}(\omega_e;\beta_1)$ are recovered. If now one can find a relation between $A_U(\theta;\beta)$ and $A_0(\theta;\beta_1)$ then one could hope to express $\boldsymbol{D}_U(\omega;\beta)$ in terms of $\boldsymbol{D}_0(\omega_e;\beta_1)$, as stated in (4); this point will be addressed next.

3. Relation between $A_U(\theta;\beta)$ and $A_0(\theta;\beta_1)$

In this section a relation between $A_U(\theta;\beta)$ and $A_0(\theta;\beta_1)$ will be derived in such a way that it allows one to obtain formula (4). First a more physical argument, based on the theory of wave–current interaction, will be worked out; later on, this same problem will be considered in a more formal way.

3.1. *Wave–current interaction*

As said in the introduction, when terms of order U^2 are ignored no new waves are generated by the body and current and, in the far field, one may consider the interaction of the existing waves with the current. To place this idea in a proper perspective one may consider, initially, the far-field waves in the absence of the current, and then suppose that the current is 'turned on' and increases slowly from its initial zero value to its final value U. From the known kinematics and dynamic theory on the wave–current interaction one can predict then the relation between the final situation, where the amplitude coefficient is $A_U(\theta;\beta)$, and the initial one, when this coefficient is given by $A_0(\theta;\beta_1)$. This plausible argument leads, as shown in Aranha (1994), to a result that can be proved in a strict mathematical sense and motivates its use here in the context of the three-dimensional problem.

To start with one considers first the kinematics interaction between the current and the incident wave, originally propagating in the direction β with frequency ω_e. Let $\boldsymbol{e}_I(\beta) = (\cos\beta;\sin\beta)$ be the direction of the incoming wave and $\boldsymbol{e}_P(\beta) = (-\sin\beta;\cos\beta)$ a direction perpendicular to it. Initially the group velocity is given by the vector $\boldsymbol{c}_{g,0}(\beta) = (g/2\omega_e)\,\boldsymbol{e}_I(\beta)$; after the interaction with the current $-U\boldsymbol{i}$ the group velocity takes the form $\boldsymbol{c}_{g,U}(\beta) = \boldsymbol{c}_{g,0}(\beta) - U\boldsymbol{i}$. If $c_{g,0} = g/2\omega_e$ one can write

$$\boldsymbol{c}_{g,U}(\beta)/c_{g,0} = [1-2(U/c)\cos\beta]\,\boldsymbol{e}_I(\beta) + 2(U/c)\sin\beta\,\boldsymbol{e}_P(\beta),$$

when terms of order $(U/c)^2$ are ignored. Introducing the definition

$$c_{g,U}(\theta) = c_{g,0}\left[1-2\frac{U}{c}\cos\theta\right], \tag{16a}$$

one has then $\boldsymbol{c}_{g,U}(\beta) = c_{g,U}(\beta)\,\boldsymbol{e}_I(\beta_1)$, with β_1 given by (3*b*). This change of wave direction corresponds to the refraction of the wave by the current.

One should next consider the dynamic interaction between the incident wave and the current. Initially the wave has an amplitude A_0 and group velocity $c_{g,0}$; at the end it has the actual amplitude A (see (6)) and the group velocity $c_{g,U}(\beta)$. Assuming that the intrinsic frequency, namely, the frequency measured with respect to the medium, is σ in this last problem, conservation of the wave action implies in the equality $(A_0^2/\omega_e)\,c_{g,0} = (A^2/\sigma)\,c_{g,U}(\beta)$ or

$$\frac{A_0^2}{\omega_e} = \left[1-2\frac{U}{c}\cos\beta\right]\frac{A^2}{\sigma}. \tag{16b}$$

Finally, one considers the interaction between the outgoing wave and current. In the far field this wave is locally plane and it has, initially, an amplitude proportional to $A_0\,A_0(\theta;\beta_1)$, since the incoming wave is in the direction β_1; at the end it has an amplitude proportional to $A\,A_U(\theta;\beta)$ and a group velocity $c_{g,U}(\theta)$. Conservation of wave action implies here in the equality $c_{g,0}(A_0^2/\omega_e)\,|A_0(\theta;\beta_1)|^2 = c_{g,U}(\theta)\,(A^2/\sigma)\,|A_U(\theta;\beta)|^2$ or, with the help of (16*a*) and (16*b*),

$$|A_U(\theta;\beta)|^2 = \left[1-2\frac{U}{c}(\cos\beta-\cos\theta)\right]|A_0(\theta;\beta_1)|^2. \tag{17a}$$

Using this relation in the energy equation (14) and observing also the energy identity for the standard problem one obtains

$$\operatorname{Re} A_U(\beta_1;\beta) = \operatorname{Re} A_0(\beta_1;\beta_1). \tag{17b}$$

From this last result it follows, to leading order in U/c, that $A_U(\beta_1;\beta) = A_0(\beta_1;\beta_1) + \mathrm{i}(U/c)\,\chi$, with χ being a real constant; placing this expression into (17*a*) and ignoring terms of order $(U/c)^2$ one obtains: $2(U/c)\,\chi\,\operatorname{Im}(A_0(\beta_1;\beta_1)) = 0$. Since $\operatorname{Im}(A_0(\beta_1;\beta_1))$ is neither zero nor dependent on U/c, this last equality implies that $\chi = 0$ or

$$A_U(\beta_1;\beta) = A_0(\beta_1;\beta_1). \tag{17c}$$

If (17*a*) and (17*b*) are placed into (15*a*) and (15*b*) and the expressions that relate $D_{x,0}(\omega_e;\beta_1)$ and $D_{y,0}(\omega_e;\beta_1)$ with $A_0(\theta;\beta_1)$ are recalled, the vector equality (4) is obtained in the ***i*** and ***j*** directions; the equality in ***k*** direction will be elaborated in the following.

In fact, (17*a*) and (17*c*) suggest a little stronger relation between $A_U(\theta;\beta)$ and $A_0(\theta;\beta_1)$, namely

$$A_U(\theta;\beta) = \left[1 - \frac{U}{c}(\cos\beta - \cos\theta)\right] A_0(\theta;\beta_1). \tag{18a}$$

Expression (18*a*) implies that the phase of $A_0(\theta;\beta_1)$ is not changed by the current, only its magnitude is affected; this result is consistent with (17*c*) and it is further elaborated in the next item.

Taking now the derivative of (18*a*) with respect to θ one obtains

$$\frac{\mathrm{d}A_U}{\mathrm{d}\theta}(\theta;\beta) = \left[1 - \frac{U}{c}(\cos\beta - \cos\theta)\right]\frac{\mathrm{d}A_0}{\mathrm{d}\theta}(\theta;\beta_1) - \frac{U}{c}\sin\theta A_0(\theta;\beta_1), \tag{18b}$$

and placing (18*a*) and (18*b*) into (15*c*) the equality (4) in the ***k*** direction can be derived; one should just observe that $\mathrm{Im}[\sin\theta|A_0(\theta;\beta_1)|^2] = 0$ and recalls the definition of $T_{z,0}(\omega_c;\beta_1)$ in terms of $A_0(\theta;\beta_1)$.

3.2. *Mathematical proof of* (18*a*)

In this item, expression (18*a*), inferred from a general argument and used to derive formula (4), is proved in a strict mathematical sense. In order to do so one takes here, as before, an incident wave with frequency ω in the earth-fixed system and wavenumber $K = \omega^2/g$, but considers the body advancing with velocities $\pm U\boldsymbol{i}$; the reversed flow problem $-U\boldsymbol{i}$ is used in ship hydrodynamics to derive some reciprocity relations and it is also needed in the present context.

For this reason the sign (+) is reserved, in the following, to define the problem when the velocity is $+U\boldsymbol{i}$ while the sign (−) will be used when the velocity is $-U\boldsymbol{i}$; in particular, the incident wave directions will be assumed as $\beta^{\pm}$, where the angle β^{+} is not supposed to be related, in any way, to the angle β^{-}. The following parameters are introduced accordingly:

$$\left.\begin{aligned}
\omega_e^{\pm} &= \omega\left[1 \mp \frac{U}{c}\cos\beta^{\pm}\right],\\
K_e^{\pm} &= \frac{(\omega_e^{\pm})^2}{g} = K\left[1 \mp 2\frac{U}{c}\cos\beta^{\pm}\right],\\
K^{\pm}(\theta) &= K_e^{\pm}\left[1 \pm 2\frac{U}{c}\cos\theta\right] = K\left[1 \pm 2\frac{U}{c}(\cos\theta - \cos\beta^{\pm})\right],\\
K &= K^{\pm}(\beta^{\pm}).
\end{aligned}\right\} \tag{19a}$$

Let $\phi^{\pm}_{I,U}(\beta^{\pm})$ denote the incident wave in the directions $\beta^{\pm}, \phi^{\pm}_{D,U}(\beta^{\pm})$ the related scattered wave and $\phi^{\pm}_{U}(\beta^{\pm})$ be the total potential, where

$$\left.\begin{aligned}
\phi^{\pm}_{U}(\beta^{\pm}) &= \phi^{\pm}_{I,U}(\beta^{\pm}) + \phi^{\pm}_{D,U}(\beta^{\pm}),\\
\phi^{\pm}_{I,U}(\beta^{\pm}) &= \exp(\mathrm{i}Kr\cos(\theta - \beta^{\pm}))\exp(Kz).
\end{aligned}\right\} \tag{19b}$$

The potentials $\phi^{\pm}_{U}(\beta^{\pm})$ are solutions of (11), (12) and (13) if one uses $\pm U$ in place of U, $K_e^{\pm}$ in place of K_e, $q^{\pm}_{j,U}(\beta^{\pm})$ in place of $q_{j,U}(\beta)$, $A^{\pm}_{U}(\theta;\beta^{\pm})$ in place of $A_U(\theta;\beta)$, $K^{\pm}(\theta)$

in place of $K(\theta)$ and $F^{\pm}_{j,U}(\beta^{\pm})$ in place of $F_{j,U}(\beta)$; the factor $-\mathrm{i}gA/\omega$ of the incident wave is assumed unitary.

One considers also the standard problems (no current) at the frequencies $\omega^{\pm}_e$, assuming incident waves in the 'refracted directions' $\beta^{\pm}_1$ where

$$\beta^{\pm}_1 = \beta^{\pm} \pm 2\frac{U}{c}\sin\beta^{\pm}. \tag{20a}$$

Notice, for future reference, that

$$\frac{K^+_e}{K}\cos(\theta-\beta^+_1)+\frac{K^-_e}{K}\cos(\theta-\beta^-_1) = \cos(\theta-\beta^+)+\cos(\theta-\beta^-). \tag{20b}$$

Again, if $\phi^{\pm}_{I,0}(\beta^{\pm}_1)$ and $\phi^{\pm}_{D,0}(\beta^{\pm}_1)$ are, respectively, the incident and scattered waves in the standard problem, then the total potential $\phi^{\pm}_0(\beta^{\pm}_1)$ can be written as

$$\left.\begin{aligned} \phi^{\pm}_0(\beta^{\pm}_1) &= \phi^{\pm}_{I,0}(\beta^{\pm}_1)+\phi^{\pm}_{D,0}(\beta^{\pm}_1),\\ \phi^{\pm}_{I,0}(\beta^{\pm}_1) &= \exp(\mathrm{i}K^{\pm}_e\, r\cos(\theta-\beta^{\pm}_1))\exp(K^{\pm}_e\, z), \end{aligned}\right\} \tag{21a}$$

and satisfy the set of equations

$$\left.\begin{aligned} &\nabla^2\phi^{\pm}_0 = 0 \quad \text{in} \quad V;\\ &\left.\frac{\partial\phi^{\pm}_0}{\partial z}\right|_{z=0} = K^{\pm}_e\,\phi^{\pm}_0;\\ &\nabla\phi^{\pm}_0\cdot\boldsymbol{n}|_B = \sum_j q^{\pm}_{j,0}(\beta^{\pm}_1)(-\mathrm{i}\omega^{\pm}_e\, n_j);\\ &\partial\phi^{\pm}_0/\partial z \to 0 \quad \text{for} \quad z\to-\infty;\\ &\phi^{\pm}_{D,0}\to\left(\frac{2}{\pi K^{\pm}_e\, r}\right)^{1/2} A^{\pm}_0(\theta;\beta^{\pm}_1)\exp(\mathrm{i}(K^{\pm}_e\, r-\pi/4))\exp(K^{\pm}_e\, z) \quad \text{for} \quad r\to\infty, \end{aligned}\right\} \tag{21b}$$

together with

$$\left.\begin{aligned} F^{\pm}_{j,0}(\beta^{\pm}_1) &= \int_B [\mathrm{i}\omega^{\pm}_e\, n_j]\,\phi^{\pm}_0(\beta^{\pm}_1)\,\mathrm{d}B;\\ \sum_1 D_{jl}\, q^{\pm}_{l,0}(\beta^{\pm}_1) &= \rho F^{\pm}_{j,0}(\beta^{\pm}_1). \end{aligned}\right\} \tag{21c}$$

By definition one has

$$\left.\begin{aligned} \phi^{\pm}_U(\beta^{\pm}) &= \phi^{\pm}_0(\beta^{\pm}_1)+\frac{U}{c}\,\delta\phi^{\pm}_0,\\ q^{\pm}_{j,U}(\beta^{\pm}) &= q^{\pm}_{j,0}(\beta^{\pm}_1)+\frac{U}{c}\,\delta q^{\pm}_j,\\ F^{\pm}_{j,U}(\beta^{\pm}) &= F^{\pm}_{j,0}(\beta^{\pm}_1)+\frac{U}{c}\,\delta F^{\pm}_j, \end{aligned}\right\} \tag{22a}$$

with $\{\delta\phi^{\pm};\delta q^{\pm}_j;\delta F^{\pm}_j\}\cong O(1)$, and from the symmetry of D_{jl} one obtains

$$\left.\begin{aligned} \sum_j F^{+}_{j,U}(\beta^{+})\, q^{-}_{j,U}(\beta^{-}) &= \sum_j F^{-}_{j,U}(\beta^{-})\, q^{+}_{j,U}(\beta^{+}),\\ \sum_j F^{+}_{j,0}(\beta^{+}_1)\, q^{-}_{j,0}(\beta^{-}_1) &= \sum_j F^{-}_{j,0}(\beta^{-}_1)\, q^{+}_{j,0}(\beta^{+}_1). \end{aligned}\right\} \tag{22b}$$

The fluid volume V is bounded by the free surface F by the body B and by a cylindrical surface C with radius $r \to \infty$. Suppose the equality $\nabla^2 \phi_U^+(\beta^+)\,\phi_U^-(\beta^-) = \nabla^2 \phi_U^-(\beta^-)\,\phi_U^+(\beta^+)$ integrated in V. If Green's identity is used, together with the first expression in (22*b*), and the parameter

$$\Lambda = (K_e^+ - K_e^-) \int_F \phi_0^+(\beta_1^+)\,\phi_0^-(\beta_1^-)\,\mathrm{d}F - \mathrm{i}\left[\frac{\omega_e^+ - \omega_e^-}{\omega_e^-} \sum_j F_{j,0}^-(\beta_1^-)\,q_{j,0}^+(\beta_1^+) + \frac{\omega_e^+ - \omega_e^-}{\omega_e^+} \sum_j F_{j,0}^+(\beta_1^+)\,q_{j,0}^-(\beta_1^-)\right]$$

is introduced, one gets

$$\Lambda + \int_C \left[\frac{\partial \phi_U^+}{\partial r}\phi_U^- - \frac{\partial \phi_U^-}{\partial r}\phi_U^+\right]\mathrm{d}C - 2\mathrm{i}\frac{U}{c}\int_0^{2\pi} \cos\theta[\phi_0^+\,\phi_0^-]_{z=0}\, r\,\mathrm{d}\theta = 0.$$

In deriving this equality Green's identity was used also at the free surface together with (8), resulting in the term proportional to $-2\mathrm{i}U/c$; in this parcel, and also in the one that defines Λ, (22*a*) has been used to replace $\phi_U^\pm$ by $\phi_0^\pm$, since terms of order $(U/c)^2$ are ignored.

Considering now the identity $\nabla^2 \phi_0^+(\beta_1^+)\,\phi_0^-(\beta_1^-) = \nabla^2 \phi_0^-(\beta_1^-)\,\phi_0^+(\beta_1^+)$ and repeating the procedure stated above one obtains:

$$\Lambda + \int_C \left[\frac{\partial \phi_0^+}{\partial r}\phi_0^- - \frac{\partial \phi_0^-}{\partial r}\phi_0^+\right]\mathrm{d}C = 0.$$

From both expressions it follows then, in the limit $r \to \infty$:

$$\int_C \left[\frac{\partial \phi_U^+}{\partial r}\phi_U^- - \frac{\partial \phi_U^-}{\partial r}\phi_U^+\right]\mathrm{d}C - 2\mathrm{i}\frac{U}{c}\int_0^{2\pi} \cos\theta[\phi_0^+(\beta_1^+)\,\phi_0^-(\beta_1^-)]_{z=0}\, r\,\mathrm{d}\theta = \int_C \left[\frac{\partial \phi_0^+}{\partial r}\phi_0^- - \frac{\partial \phi_0^-}{\partial r}\phi_0^+\right]\mathrm{d}C. \quad (23a)$$

Notice that the above equality holds either for an oscillating body ($q_{j,U}^\pm(\beta^\pm) \neq 0$) or else for a body fixed in waves ($q_{j,U}^\pm(\beta^\pm) = 0$). In the surface C one can use the far-field conditions for the scattered waves and so

$$\left.\begin{aligned} \frac{\partial \phi_U^\pm}{\partial r} &= -\mathrm{i}K\left[\frac{K^\pm(\theta)}{K} - \cos(\theta - \beta^\pm)\right]\phi_{I,U}^\pm(\beta^\pm) + \mathrm{i}K^\pm(\theta)\,\phi_U^\pm(\beta^\pm), \\ \frac{\partial \phi_0^\pm}{\partial r} &= -\mathrm{i}K_e^\pm[1 - \cos(\theta - \beta_1^\pm)]\,\phi_{I,0}^\pm(\beta_1^\pm) + \mathrm{i}K_e^\pm\,\phi_0^\pm(\beta_1^\pm). \end{aligned}\right\}$$

Placing these expressions into (23*a*) and observing that

$$\mathrm{i}(K^+(\theta) - K^-(\theta)) \int_\infty^0 \phi_U^+(\beta^+)\,\phi_U^-(\beta^-)\,\mathrm{d}z - 2\mathrm{i}\frac{U}{c}\cos\theta[\phi_0^+(\beta_1^+)\,\phi_0^-(\beta_1^-)]_{z=0} - \mathrm{i}(K_e^+ - K_e^-)\int_\infty^0 \phi_0^+(\beta_1^+)\,\phi_0^-(\beta_1^-)\,\mathrm{d}z = 0,$$

if terms of order $(U/c)^2$ are ignored, one obtains, after multiplying both sides of (23*a*) by $\frac{1}{2}\mathrm{i}\pi K$,

$$I_U^-(Kr) - I_U^+(Kr) + J(Kr) = I_0^-(Kr) - I_0^+(Kr), \quad (23b)$$

with

$$\left.\begin{aligned} I_U^{\pm}(Kr) &= \tfrac{1}{2}\pi K\int_C\left[\frac{K^{\mp}(\theta)}{K}-\cos(\theta-\beta^{\mp})\right]\phi^{\mp}_{I,U}(\beta^{\mp})\,\phi^{\pm}_{D,U}(\beta^{\pm})\,K\,\mathrm{d}C,\\ I_0^{\pm}(Kr) &= \tfrac{1}{2}\pi K\int_C\frac{K_e^{\mp}}{K}[1-\cos(\theta-\beta_1^{\mp})]\,\phi^{\mp}_{I,0}(\beta_1^{\mp})\,\phi^{\pm}_{D,0}(\beta_1^{\pm})\,K\,\mathrm{d}C, \end{aligned}\right\}\quad(23c)$$

and

$$\begin{aligned} J(Kr) = \tfrac{1}{2}\pi K\int_C\Bigg[&\frac{K^{+}(\theta)-K^{-}(\theta)}{K}\phi^{+}_{I,U}(\beta^{+})\,\phi^{-}_{I,U}(\beta^{-})-\frac{K_e^{+}-K_e^{-}}{K}\phi^{+}_{I,0}(\beta_1^{+})\,\phi^{-}_{I,0}(\beta_1^{-})\\ &+(\cos(\theta-\beta^{-})-\cos(\theta-\beta^{+}))\,\phi^{+}_{I,U}\,\phi^{-}_{I,U}\\ &-\left(\frac{K_e^{-}}{K}\cos(\theta-\beta_1^{-})-\frac{K_e^{+}}{K}\cos(\theta-\beta_1^{+})\right)\phi^{+}_{I,0}\,\phi^{-}_{I,0}\Bigg]K\,\mathrm{d}C. \end{aligned}$$

Observing that $\mathrm{d}C = r\,\mathrm{d}\theta\,\mathrm{d}z$ and integrating in z the expression that defines $J(Kr)$ one obtains, after using (20*b*) and the expressions for $\{\phi^{\pm}_{I,U}(\beta^{\pm});\phi^{\pm}_{I,0}(\beta_1^{\pm})\}$, given in (19*b*) and (21*a*), the result

$$\begin{aligned} J(Kr) = \tfrac{1}{2}\pi Kr&\left(1+\frac{2K}{K_e^{+}-K_e^{-}}\right)\\ &\times\int_0^{2\pi}[\cos(\theta-\beta^{+})-\cos(\theta-\beta^{-})]\exp(\mathrm{i}Kr[\cos(\theta-\beta^{+})+\cos(\theta-\beta^{-})])\,\mathrm{d}\theta. \end{aligned}$$

The phase $\cos(\theta-\beta^{+})+\cos(\theta-\beta^{-})$ is stationary when $\sin(\theta-\beta^{+})+\sin(\theta-\beta^{-}) = 0$; assuming $\beta^{-}\neq\beta^{+}+\pi$ this condition implies in $\cos(\theta-\beta^{+}) = \cos(\theta-\beta^{-})$ and so $J(Kr)\to 0$ when $Kr\to\infty$. Notice that the next term in the asymptotic expansion comes from the end values of the integral and it is also zero owing to the periodicity of the integrand; so $J(Kr) = O(1/Kr^{1/2})$ as $Kr\to\infty$. The case $\beta^{-} = \beta^{+}+\pi$ will be addressed at the end of this item. It follows then that

$$I_U^{+}(Kr)-I_0^{+}(Kr) = I_U^{-}(Kr)-I_0^{-}(Kr);\quad Kr\to\infty. \quad(24a)$$

Using the asymptotic expressions for the scattered waves in (23*c*) one has

$$\left.\begin{aligned} I_U^{\pm}(Kr) &= \frac{1}{2}\left(\frac{\pi Kr}{2}\right)^{1/2}\exp(-\mathrm{i}\pi/4)\left(\frac{K}{K_e^{\pm}}\right)^{1/2}\\ &\quad\times\int_0^{2\pi}\frac{2K}{K+K^{\pm}(\theta)}\left[\frac{K^{\mp}(\theta)}{K}-\cos(\theta-\beta^{\mp})\right]A_U^{\pm}(\theta;\beta^{\pm})\\ &\quad\times\exp(\mathrm{i}(U/c)\,Kr\varphi_U(\theta))\,\mathrm{d}\theta;\\ \varphi_U(\theta) &= \cos(\theta-\beta^{\mp})+\frac{K^{\pm}(\theta)}{K}, \end{aligned}\right\}\quad(24b)$$

and

$$\left.\begin{aligned} I_0^{\pm}(Kr) &= \frac{1}{2}\left(\frac{\pi Kr}{2}\right)^{1/2}\exp(-\mathrm{i}\pi/4)\left(\frac{K}{K_e^{\pm}}\right)^{1/2}\int_0^{2\pi}\frac{2K}{K_e^{+}+K_e^{-}}\frac{K_e^{\mp}}{K}\\ &\quad\times[1-\cos(\theta-\beta_1^{\mp})]\,A_0^{\pm}(\theta;\beta_1^{\pm})\\ &\quad\times\exp(\mathrm{i}(U/c)\,Kr\varphi_0(\theta))\,\mathrm{d}\theta;\\ \varphi_0(\theta) &= \frac{K_e^{\mp}}{K}\cos(\theta-\beta_1^{\mp})+\frac{K_e^{\pm}}{K}. \end{aligned}\right\}\quad(24c)$$

In (24*b*) the phase $\varphi_U(\theta)$ is stationary when

$$\varphi_U'(\theta) = -\sin(\theta-\beta^{\mp}) \pm 2(U/c)\sin\theta = 0$$

or $\{\theta = \beta_1^{\mp}; \theta = \beta_1^{\mp}+\pi\}$. In the first case the integrand is zero while in the second it is equal to $2[1 \pm U/c(\cos\beta_1^{\mp}+\cos\beta^{\pm})]$; placing this result in the known expression obtained from the method of the stationary phase one gets, in the limit $Kr \to \infty$:

$$I_U^{\pm}(Kr) = \left(\frac{K_e^{\mp}}{K_e^{\pm}}\right)^{1/2} \frac{2K}{K_e^{+}+K_e^{-}}\left[1 \pm \frac{U}{c}(\cos\beta_1^{\mp}+\cos\beta^{\pm})\right]$$
$$\times A_U^{\pm}(\beta_1^{\mp}+\pi;\beta^{\pm})\exp(\mp 2\mathrm{i}(U/c)\,Kr[\cos\beta^{+}+\cos\beta^{-}]). \quad (25a)$$

From (24*c*), $\varphi_0'(\theta) = -(K_e^{\mp}/K)\sin(\theta-\beta_1^{\mp})$ and so $\varphi_0'(\theta)=0$ when $\{\theta=\beta_1^{\mp}; \theta = \beta_1^{\mp}+\pi\}$; again the integrand is zero in the first case and it is equal to 2 in the second one. The final result is

$$I_0^{\pm}(Kr) = \left(\frac{K_e^{\mp}}{K_e^{\pm}}\right)^{1/2} \frac{2K}{K_e^{+}+K_e^{-}} A_0^{\pm}(\beta_1^{\mp}+\pi;\beta_1^{\pm})\exp(\mp 2\mathrm{i}(U/c)\,Kr[\cos\beta^{+}+\cos\beta^{-}]). \quad (25b)$$

Placing (25*a*, *b*) into (24*a*) the following identity is obtained:

$$\left(\frac{K_e^{-}}{K_e^{+}}\right)^{1/2}\left[\left(1+\frac{U}{c}(\cos\beta_1^{-}+\cos\beta^{+})\right)A_U^{+}(\beta_1^{-}+\pi;\beta^{+}) - A_0^{+}(\beta_1^{-}+\pi;\beta_1^{+})\right]$$
$$\times\exp(-2\mathrm{i}(U/c)\,Kr[\cos\beta^{+}+\cos\beta^{-}])$$
$$= \left(\frac{K_e^{+}}{K_e^{-}}\right)^{1/2}\left[\left(1-\frac{U}{c}(\cos\beta_1^{+}+\cos\beta^{-})\right)A_U^{-}(\beta_1^{+}+\pi;\beta^{-}) - A_0^{-}(\beta_1^{+}+\pi;\beta_1^{-})\right]$$
$$\times\exp(+2\mathrm{i}(U/c)\,Kr[\cos\beta^{+}+\cos\beta^{-}]). \quad (25c)$$

The above equality holds for every r, preserving the condition $r\to\infty$; since $\beta^{-} \neq \beta^{+}+\pi$ then $[\cos\beta^{+}+\cos\beta^{-}] \neq 0$ and the phase $\pm\mathrm{i}\alpha = \pm 2\mathrm{i}(U/c)\,Kr[\cos\beta^{+}+\cos\beta^{-}]$ is arbitrary as $Kr\to\infty$. It follows that the expressions within brackets on both sides of (25*c*) must be zero. Returning to the notation of the last item ($\beta=\beta^{+}$) and recalling that β_1^{-} is not related with $\beta=\beta^{+}$, one may write θ instead of $\beta_1^{-}+\pi$ in the expression on the left-hand side of (25*c*). Relation (18*a*) is then obtained if terms of order $(U/c)^2$ are ignored again.

The special case $\beta^{-}=\beta^{+}+\pi$ corresponds to $\theta=\beta_1$ and in this circumstance the desired expression can be obtained either by invoking the continuity of $\{A_U(\theta;\beta); A_0(\theta;\beta_1)\}$ in θ or else with the help of the energy relation (14).

4. Wave drift damping matrix

Expression (4) can be rewritten in the form below when terms of order $(U/c)^2$ are disregarded and (3) is used:

$$\boldsymbol{D}_U(\omega;\beta) = \boldsymbol{D}_0(\omega;\beta) - \left[\left(\omega\frac{\partial \boldsymbol{D}_0}{\partial\omega}(\omega;\beta)+4\boldsymbol{D}_0(\omega;\beta)\right)\cos\beta - 2\sin\beta\frac{\partial \boldsymbol{D}_0}{\partial\beta}(\omega;\beta)\right]\frac{U}{c}. \quad (26)$$

The parcel within brackets in (26) is just the 'wave drift damping' and the ***i*** component of this expression has been heuristically obtained by Clark, Malenica & Molin (1992).

So far only the case with a forward speed U in the x-direction was considered; one will now consider the reference system fixed in the body and suppose a forward speed $\boldsymbol{U} = U_x \boldsymbol{i} + U_y \boldsymbol{j}$ with components U_x and U_y in the x- and y-direction, respectively.

For notational convenience the following vectors are introduced:

$$\left.\begin{aligned} \boldsymbol{b}_W(\omega;\beta) &= \left[\omega \frac{\partial \boldsymbol{D}_0}{\partial \omega}(\omega;\beta) + 4\boldsymbol{D}_0(\omega;\beta)\right]\frac{1}{c}, \\ \boldsymbol{b}_R(\omega;\beta) &= -\left[2\frac{\partial \boldsymbol{D}_0}{\partial \beta}(\omega;\beta)\right]\frac{1}{c}, \end{aligned}\right\} \quad (27a)$$

where $\boldsymbol{b}_W$ is related to the action of the current in the wave direction while $\boldsymbol{b}_R$ is associated with the wave refraction.

If $\boldsymbol{I}$ is the (3×3) identity matrix one can define the (3×2) matrix $\boldsymbol{B}(\omega;\beta)$ by the expression

$$\boldsymbol{B}(\omega;\beta) = [\boldsymbol{I}\,|\,\boldsymbol{I}]_{(3\times 6)} \begin{bmatrix} \cos\beta\boldsymbol{b}_W(\omega;\beta) & \sin\beta\boldsymbol{b}_W(\omega;\beta) \\ \sin\beta\boldsymbol{b}_R(\omega;\beta) & -\cos\beta\boldsymbol{b}_R(\omega;\beta) \end{bmatrix}_{(6\times 2)} \quad (27b)$$

and write the exciting steady force in the matrix form

$$\boldsymbol{D}_U(\omega;\beta) = \boldsymbol{D}_0(\omega;\beta) - \boldsymbol{B}(\omega;\beta)\begin{Bmatrix} U_x \\ U_y \end{Bmatrix}. \quad (27c)$$

The matrix $\boldsymbol{B}(\omega;\beta)$ may be called the 'wave drift damping matrix' but it should be observed that it does not contain all information needed for the analysis of the motion in the horizontal plane; it lacks a third column related to the slow angular velocity around the vertical axis. This point has been not subjected to analysis in the present work. The matrix (27c) also does not include the effect of the Munk's moment $\boldsymbol{N}$ introduced by Grue & Palm (1993).

All results derived here are valid for infinite water depth. The extension for finite water depth can be obtained following the same arguments as presented in this work but using the proper expression for $\boldsymbol{D}_U(\omega;\beta)$ instead of (15) and the pertinent relation between the phase and group velocities.

5. Numerical results

In this section one intends to compare the obtained expression for the 'wave damping, either in the form (4) or else in the form (26), with numerical results known in the literature and to comment on observed discrepancies between them.

Clark *et al.* (1992) compared the $\boldsymbol{i}$ component of (26) with numerical results obtained from the solution of (11) for a circular cylinder (or an array) orthogonal to the free surface. The agreement was exact within six significant figures but these authors reported, also, a 'less good' agreement when the body is free to oscillate; this latter observation is not supported by the theory presented here, once formula (4) is correct whether or not the body is free to oscillate, and it is further commented on at the end of this section. Figure 1, for instance, compares directly expression (4), when $\beta = 0$, with numerical results obtained by Faltinsen (1994) for a hemisphere free to oscillate in heave and surge; the maximum value of the drift force corresponds to heave resonance and the information about the body oscillation is carried with the force for the standard problem $U = 0$.

Just to check that the agreement is not restricted to special geometries, expression (26), again in the $\boldsymbol{i}$ direction, has been confronted with numerical results obtained by

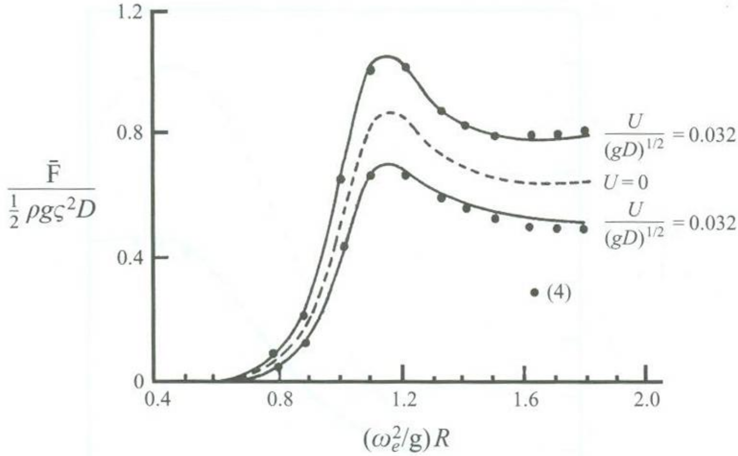

FIGURE 1. Drift force influenced by the forward speed U; hemisphere free to oscillate in surge and heave. D = diameter; $R = \frac{1}{2}D$; ω_e = frequency of encounter (Faltinsen 1994).

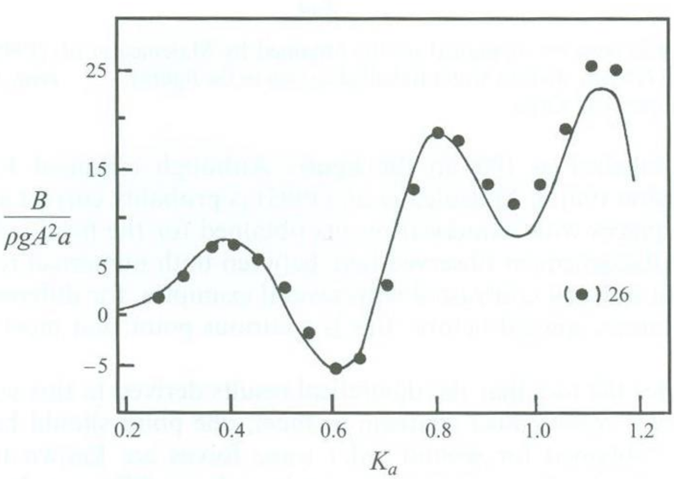

FIGURE 2. Comparison between (26) and numerical results obtained by Nossen *et al.* (1991) for a semi-submersible with ring-like pontoon and four columns.

Nossen *et al.* (1991) for a semi-submersible platform with a ring like pontoon and four columns, free to oscillate in surge. The comparison is shown in figure 2 and the agreement is again good, the observed discrepancy in high frequency probably being due to unavoidable discretization errors in this range.

In one class of problems, specifically, for a circular cylinder (or an array) perpendicular to the free surface and free to oscillate, a consistent discrepancy between numerical results and (26) has been observed as shown, for instance, in figure 3.

In this example the water depth is equal to the radius of the cylinder (shallow water) and the cylinder is free to respond to waves in surge and sway. The problem has been analysed by means of two different numerical methods. One, said to be 'semi-analytical', is an extension of the method proposed by Emmerhoff & Sclavounos (1992) for the fixed cylinder, and it has been used by Malenica, Clark & Molin (1995); the other is a pure numerical method worked by Grue. Both numerical results agree between themselves but do not agree with the extension of (26) for shallow water, this

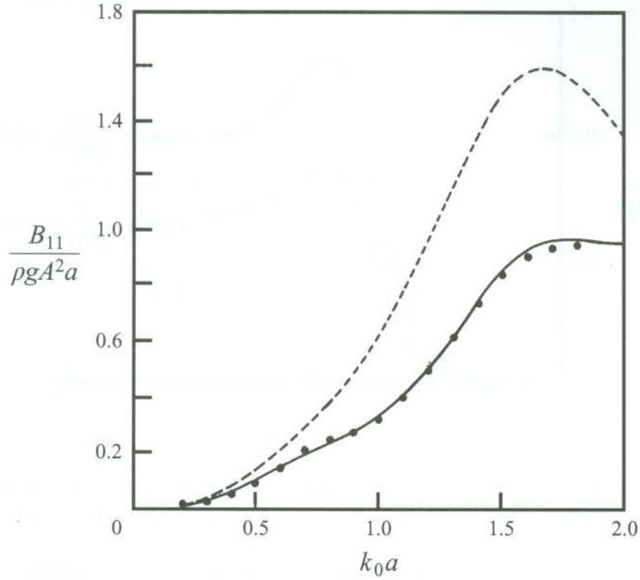

FIGURE 3. Comparison between numerical results obtained by Malenica *et al.* (1995) and by Grue with the extension of (26) for shallow water (labelled as (90) in the figure). ——, semi-analytical; ---, formula (90); ..., numerical, Grue.

extension being labelled as (90) in the figure. Although obtained by a heuristic approach, expression (90) in Malenica *et al.* (1995) is probably correct since it agrees to four decimals places with numerical results obtained for the fixed cylinder.

The consistent disagreement observed here between both numerical results and the analytical formula must be contrasted with several examples, for different geometries and modes of motions, quoted before; this is a curious point that must be looked at with attention.

However, besides the fact that the theoretical results derived in this work are exact and so all numerical results must conform to them, one point should be noted here: numerical results obtained for second-order wave forces are known to be not yet outstanding. Specifically, for a truncated cylinder a huge difference between steady forces in harmonic waves, computed either from direct pressure integration or else by momentum theory, has been obtained, as discussed in Faltinsen (1994). This latter result, together with the former one for the 'wave drift damping', seems to indicate that these geometries have, when free to oscillate, a sort of numerical ill-behaviour not yet circumvented by the existing numerical methods.

This work has been partially supported by DEPRO, the Production Department of PETROBRAS; a research grant from CNPq and FAPESP must also be acknowledged. The author is specially indebted to Professor Paul Sclavounos for his careful reading, useful suggestions and public support of this work.

Appendix

This Appendix aims to compare expressions (15) for the steady forces and moment with similar expressions derived by Grue & Palm (1993). These authors used the symbol ν instead of K_e and the far field coefficient $H(\theta)$ in place of the factor $(2/\pi K_e)^{1/2} A_U(\theta;\beta)\, e^{-i\pi/4}$; see expressions (36) and (39) of their work.

Observing that $K_e/K = 1-2U/c\cos\beta$ and using the present notation in the Grue & Palm result, one can check that the expression for the yaw moment coincides with (15c) when the Munk's parcel is ignored; the related expressions for the forces are given below:

$$\left.\begin{aligned} \frac{D_{x,U}}{\frac{1}{2}\rho g A^2/K_c} &= -\frac{1}{\pi}\int_0^{2\pi}\left[\left(1-2\frac{U}{c}\cos\beta\right)\cos\theta+2\frac{U}{c}\sin^2\theta\right]|A_U(\theta;\beta)|^2\,\mathrm{d}\theta \\ &\qquad -2\cos\beta(1-2(U/c)\cos\beta)\,\mathrm{Re}\,A_U(\beta_1;\beta), \\ \frac{D_{y,U}}{\frac{1}{2}\rho g A^2/K_c} &= -\frac{1}{\pi}\int_0^{2\pi}\left[\left(1-2\frac{U}{c}\cos\beta\right)\sin\theta-2\frac{U}{c}\sin 2\theta\right]|A_U(\theta;\beta)|^2\,\mathrm{d}\theta \\ &\qquad -2\sin\beta(1-2(U/c)\cos\beta)\,\mathrm{Re}\,A_U(\beta_1;\beta). \end{aligned}\right\} \quad (\mathrm{A}\ 1)$$

It is not difficult to show now that (A 1) coincides with the expressions (15) of the present work. In fact, if the term $2U/c\cos^2\theta$ is added and subtracted within the brackets of the polar integral for $D_{x,U}$ and the energy identity is used, one obtains, when terms of order $(U/c)^2$ are ignored, the same polar integral as in (15a) plus a term of the form

$$\chi_1 = -2\cos\beta\left(1-2\frac{U}{c}\cos\beta\right)\mathrm{Re}\,A_U(\beta_1;\beta)+4\frac{U}{c}\mathrm{Re}\,A_U(\beta_1;\beta).$$

One can now write

$$\chi_1 = -\left(1-4\frac{U}{c}\cos\beta\right)\left[\cos\beta\left(1+2\frac{U}{c}\cos\beta\right)-2\frac{U}{c}\right]2\,\mathrm{Re}\,A_U(\beta_1;\beta),$$

and so the above expression for $D_{x,U}$ coincides with (15a) since the term within brackets is just $\cos\beta_1$.

The polar integral for $D_{y,U}$ is the same in (A 1) and (15b); the coefficient

$$\chi_2 = -2\sin\beta\left(1-2\frac{U}{c}\cos\beta\right)\mathrm{Re}\,A_U(\beta_1;\beta)$$

can be written as

$$\chi_2 = -\left(1-4\frac{U}{c}\cos\beta\right)\left[\sin\beta+2\frac{U}{c}\sin\beta\cos\beta\right]2\,\mathrm{Re}\,A_U(\beta_1;\beta),$$

where the term within brackets is $\sin\beta_1$.

It turns out then that expressions (15) coincide, within the accepted error of order $(U/c)^2$, with the expressions obtained by Grue & Palm (1993).

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