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Extinction time in growth models subject to binomial catastrophes*

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Abstract. Populations are often subject to catastrophes that lead to significant reductions in the number of individuals. Many stochastic growth models have been considered to explain such dynamics. Among the reported results, it has been considered whether dispersion strategies, at times of catastrophes, increase the survival probability of the population. In this paper, we contrast dispersion strategies by comparing the mean extinction times of a population under conditions of near-certain extinction. Specifically, we consider populations subject to binomial catastrophes, where the population size is reduced according to a binomial law when a catastrophe occurs. Our findings delineate the optimal strategy (dispersion or non-dispersion) based on variations in model parameter values.

Keywords: branching processes, catastrophes, population dynamics

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Contents

1. Introduction

Several stochastic growth models have been considered to represent populations subject to catastrophes. When a catastrophe strikes, a random number of individuals are removed from the population. Survivors may remain together in the same colony (no dispersion) or disperse, forming newly independent colonies. These models are of interest for gaining a deeper understanding of quantities such as population survival probability, extinction time distribution, mean number of individuals removed and the distribution of maximum population size. Previous studies [1, 5–7, 14, 17, 18] pertain to population models where catastrophe survivors remain united in the same colony, whereas the models examined in [11, 12, 19, 20, 22] investigate population dynamics with survivors dispersing to establish new colonies elsewhere. In these papers, different types of catastrophes and different dispersion schemes are considered to analyze whether combining some of these schemes increases population viability. In a biological context, it is known that dispersion plays a central role in both the dynamics and evolution of spatially structured populations. While it could save a small population from local extinction, it also could increase global extinction risk if observed at a very high level; refer to Ronce [21] for additional details.

The models analyzed in [11, 12, 19, 20, 22] aim to establish the best strategy, i.e. dispersion or no dispersion, based on the survival probability of the population. When the survival probability is zero for both strategies, we need to go one step further and consider the expected extinction time. This quantity is of particular importance to estimate the 'minimum viable population size' to guarantee survival for a certain time, as considered in Brockwell [5]. For single-colony models (no dispersion), one can find closed-form formulas for the mean extinction times for different types of catastrophes (see [1, 5, 7]). For models with dispersion, an analogous approach was considered for so-called geometric catastrophes (see [13]). Geometric catastrophes assume that the batch of removed individuals, when a catastrophe strikes, follows a geometric law; that is, the individuals are exposed to the catastrophic effect sequentially and the decline in the population stops at the first individual who survives or when the whole population in the colony becomes extinct.

Here, we consider binomial catastrophes in models with dispersion. In binomial catastrophes, the individuals of a colony are exposed to the catastrophic effect simultaneously and every individual survives a catastrophe with the same probability, independently of anything else. We are able to present closed-form formulas for the mean extinction times and make comparisons with models without dispersion. Our analysis involves comparisons, by numerical and analytical methods, with functions expressed as infinite products, also known as infinite q-products. They are part of the q-series theory (see [4]). Further instances and applications of geometric catastrophes are detailed in [1, 9, 10, 15, 17], while examples and applications of binomial catastrophes can be found in [1, 5, 6, 14, 16, 18].

In conclusion, we propose to find the optimal strategy under conditions of near-certain extinction by considering the mean extinction times for populations subject to binomial catastrophes; that is, when a population is hit by a catastrophe, its size is reduced according to a binomial distribution. In section 2, we present the non-dispersion model proposed in Artalejo *et al* [1] and the models with dispersion proposed in Junior *et al* [11]. We also reach new results for these models. In section 3, we discuss dispersal schemes as strategies for increasing life expectancy. In section 4, we prove the results presented in sections 2 and 3. Finally, in section 5, a numerical algorithm is developed allowing us to make calculations and comparisons with the infinite products that appear in the article.

This type of study provides predictive insights into population dynamics, aiding conservation strategies and risk assessment. By quantifying vulnerability, informing policy decisions and refining models, this line of research contributes to both scientific understanding and practical applications.

2. Models and results

2.1. Binomial catastrophe

Populations are frequently exposed to catastrophic events that result in a significant depletion of their members; for example, habitat destruction, environmental disasters, epidemics, etc. A catastrophe can instantly wipe out the entire population or just a part of it. In order to model such events, it is assumed that when a population is hit by

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a catastrophe, its size is reduced according to some law of probability. For catastrophes that reach individuals simultaneously and independently of everything else, the appropriate model assumes a binomial probability law. That is, if at a catastrophe time the size of the population is i, it is reduced to j with probability

$$\mu_{ij} = \binom{i}{j} p^j (1-p)^{i-j}, \qquad 0 \leqslant j \leqslant i,$$

where $p \in (0,1)$ is the probability that each individual survives the catastrophe. The form of μ_{ij} represents what is called a binomial catastrophe.

2.2. Growth model without dispersion

Artalejo *et al* [1] present a model for a population that sticks together in one colony, without dispersion. That colony gives birth to new individuals at rate $\lambda > 0$, while binomial catastrophes happen at rate μ.

The population size (number of individuals in the colony) at time t is a continuous-time Markov process $\{X(t) : t \geqslant 0\}$ that we denote by $C(\lambda, p)$. With the intention of making the formulas more straightforward and simplifying the analysis, we take $\mu = 1$ and set $X(0) = 1$.

Artalejo *et al* [1] uses the word 'extinction' to describe the event that $X(t) = 0$, for some $t > 0$, for a process where state 0 is not an absorbing state. In fact, the extinction time here is the first hitting time to the state 0,

$$\tau_A := \inf\{t > 0 : X(t) = 0\}.$$

The probability of extinction of $C(\lambda, p)$ is denoted by $\psi_A = \mathbb{P}[\tau_A < \infty]$. Its complement, $1 - \psi_A$, is called survival probability. Artalejo *et al* [1] proved that $\psi_A = 1$ (extinction occurs almost surely) for all $\lambda > 0$ and $0 < p < 1$. The next result establishes the mean time of extinction for $C(\lambda, p)$.

Theorem 2.1 (Artalejo *et al* [1]). *For the process $C(\lambda, p)$,*

$$\mathbb{E}[\tau_A] = \frac{1}{\lambda}\left(\prod_{k=0}^{\infty}\left(1 + \lambda p^k\right) - 1\right).$$

Remark 2.2. The infinite product $\prod_{k=0}^{\infty}(1 + \lambda p^k)$ is convergent for all $|p| < 1$ and $\lambda \in \mathbb{R}$. For series representations and other properties of this infinite product, see [4, corollary 2.3] and [8, theorem 10.10].

2.3. Growth models with dispersion and spatial restriction

Let $\mathbb{T}_d^+$ be an infinite rooted tree whose vertices have degree $d+1$, except the root that has degree d. Let us define a process with dispersion on $\mathbb{T}_d^+$, starting from a single colony placed at the root of $\mathbb{T}_d^+$, with just one individual. The number of individuals in a colony grows following a Poisson process of rate $\lambda > 0$. We associate an exponential time of mean 1 to each colony, which indicates when the binomial catastrophe strikes the colony. Each individual that survives the catastrophe randomly picks a neighbor vertex between the d neighboring vertices furthest from the root to create new colonies. Among the survivors that go to the same vertex to create a new colony there, only one succeeds; the others die. Therefore, in this case, when a catastrophe occurs in a colony, that colony is replaced by 0,1, ... or d colonies. Let us denote this process with by $C_d(\lambda, p)$.

$C_d(\lambda, p)$ is a continuous-time Markov process with state space $\{0,1,2,3,\dots\}^{\mathbb{T}^d}$. For each particular realization of this process, we say that it *survives* if for any instant of time there is at least one colony somewhere. Otherwise, we say that it *dies out*. We denote by ψ_d, the probability of extinction of $C_d(\lambda, p)$. Junior *et al* [11, theorem 2.8] showed that $\psi_d < 1$ if and only if $p > \frac{d}{d+(d-1)\lambda}$, showing that there is a phase transition with respect to the parameter p.

It is clear that when $\psi_d < 1$, the extinction mean time for the process $C_d(\lambda, p)$ is infinite. In the next results, we derive the extinction mean time when extinction almost surely occurs, when $d = 2$ and $d = 3$.

Theorem 2.3. *Let τ_d be the extinction time of the process $C_d(\lambda, p)$.*

(i) If $p < \frac{2}{\lambda+2}$, then

$$\mathbb{E}[\tau_2] = \frac{(\lambda p + 1)(\lambda p + 2)}{\lambda p^2 (\lambda + 1)} \ln \left[\frac{(1-p)(\lambda p + 2)}{(1-p)(\lambda p + 2) - \lambda p^2 (\lambda + 1)} \right].$$

If $p = \frac{2}{\lambda+2}$, then $\mathbb{E}[\tau_2] = \infty$.

(ii) If $p < \frac{3}{2\lambda+3}$, then

$$\mathbb{E}[\tau_3] = \frac{2\lambda p + 3}{2g(\lambda, p)} \ln \left[\frac{3 - 3p - \lambda p + g(\lambda, p)}{3 - 3p - \lambda p - g(\lambda, p)} \right],$$

where

$$g(\lambda, p) = \sqrt{\frac{\lambda^2 p^3 (\lambda + 1)(6 + \lambda p - 3p)}{(\lambda p + 3)(\lambda p + 1)}}. \tag{2.1}$$

If $p = \frac{3}{2\lambda+3}$, then $\mathbb{E}[\tau_3] = \infty$.

2.4. Growth model with dispersion but no spatial restrictions

Consider a population of individuals divided into separate colonies. Each colony begins with an individual. The number of individuals in each colony increases independently according to a Poisson process of rate $\lambda > 0$. We associate an exponential time of mean 1 to each colony, which indicates when the binomial catastrophe strikes the colony. Each individual that survived the catastrophe begins a new colony independently of everything else. We denote this process by $C_*(\lambda, p)$ and consider it starting from a single colony with just one individual.

For each particular realization of $C_*(\lambda,p)$, we say that it *survives* if for any instant of time there is at least one colony somewhere. Otherwise, we say that it *dies out.* We denote by ψ_* the probability of extinction of $C_*(\lambda,p)$. Junior *et al* [11, theorem 2.3] showed that $\psi_* < 1$ if and only if $p > \dfrac{1}{\lambda+1}$.

It is clear that when $\psi_* < 1$, the extinction mean time of $C_*(\lambda,p)$ is infinite. The following theorem establishes the mean time of extinction for $C_*(\lambda,p)$ when $\psi_* = 1$.

Theorem 2.4. *Let τ_* the extinction time of the process $C_*(\lambda,p)$. Then*

$$\mathbb{E}[\tau_*] = \begin{cases} 1 - \dfrac{\lambda+1}{\lambda}\ln\left[1-\dfrac{\lambda p}{1-p}\right] & , \quad \textit{if } p < \dfrac{1}{\lambda+1}; \\ \infty & , \quad \textit{if } p = \dfrac{1}{\lambda+1}. \end{cases}$$

2.5. Connections with branching processes

Models $C_d(\lambda,p)$ and $C_*(\lambda,p)$ are special versions of branching processes. Next, we present an alternative description of these models.

Let $N_\cdot$ be a Poisson process with rate λ and $N_0 = 1$. Let J be an exponential random variable with rate 1, independent of $N_\cdot$. Consider a population of size N_J that undergoes a catastrophe; each of its elements survives with probability p, independently of the rest. After the catastrophe, the surviving population size is then $Z = \mathrm{Bin}(N_J, p)$ (a binomial random variable). If this number is zero, then set $B = 0$. Otherwise, label each element of the surviving population with a type $1,\ldots,d$, independently of the others, and let B be the number of distinct resulting types.

Next, consider a continuous-time branching process with rate 1 and offspring distribution B. The resulting model is $C_d(\lambda,p)$. The limit corresponding to $d\to\infty$ (number of types is the same as the size of the surviving population) is $C_*(\lambda,p)$ and we refer to it informally as the $d=\infty$ case. In particular, for $C_*(\lambda,p)$, the offspring distribution conditioned on Z is δ_Z. As for $C_d(\lambda,p)$, $d<\infty$, the offspring distribution conditioned on $Z=n$ is equal to $k=0,1,\ldots,d$ with probability $p_{n,k}$. In light of the above construction, $p_{0,\cdot} = \delta_0$, and for $n\geqslant 1$, and $k = 1,\ldots,d\wedge n$, a combinatorial calculation gives

$$p_{n,k} = \frac{1}{d^n}\binom{d}{k}\sum_{r_1,\ldots,r_k\geqslant 1, r_1+\cdots+r_k=n}\frac{n!}{r_1!\cdots r_k!}$$

(for all other values $p_{n,k}=0$). The formula above represents the proportion of ways to label n items resulting in exactly k distinct labels from the set $\{1,\ldots,d\}$. The expression for $p_{n,k}$ is manageable when $d = 2,3$ and gets more complicated as d gets larger. Note, however, that as $d\to\infty$, $p_{n,\cdot}\to\delta_n$, which is exactly observed for $d=\infty$.

3. Discussion

In the presence of binomial catastrophes, dispersion is a good strategy to increase the probability of survival of the population. When there is no dispersion, the probability of survival is always zero (see Artalejo *et al* [1, theorem 3.1]). However, when there is dispersion, the probability of survival can be positive depending on the parameters λ and p (see Junior *et al* [11, theorems 2.3 and 2.8] for details). An interesting question is to determine whether, when the processes $C(\lambda,p)$, $C_d(\lambda,p)$ and $C_*(\lambda,p)$ almost surely dies out, dispersion is an advantage or not for extending the population's life span. The answer is not trivial. Note that the growth and catastrophe rates are $n\lambda$ and n, respectively, whenever there are n colonies in the whole population. Moreover, a catastrophe is more likely to wipe out a smaller colony than a larger one. On the other hand, multiple colonies provide multiple chances for survival (because the catastrophe only affects the colony where it occurs) and this may be a critical advantage of the processes $C_d(\lambda,p)$ and $C_*(\lambda,p)$ over the process $C(\lambda,p)$. Also note that in the $C_d(\lambda,p)$ process, due to space constraints, during dispersion, some individuals may end up at the same spatial location. In this case, all but one individual dies. As a consequence, there is a conundrum: on the one hand, dispersion creates independent populations and thus contributes to survival. On the other hand, dispersion leads to death due to competition for space.

The next result provides a comparison of the average times until extinction between processes $C(\lambda,p)$ and $C_2(\lambda,p)$, under the condition that extinction almost surely happens in both processes.

Proposition 3.1. *Assume $p < \frac{2}{2+\lambda}$. Then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_2]$ if and only if*

$$\prod_{k=0}^{\infty}\left(1+\lambda p^k\right) < 1 + \frac{(\lambda p+1)(\lambda p+2)}{p^2(\lambda+1)}\ln\left[\frac{(1-p)(\lambda p+2)}{(1-p)(\lambda p+2)-\lambda p^2(\lambda+1)}\right]. \quad (3.1)$$

Moreover, $\mathbb{E}[\tau_A] = \mathbb{E}[\tau_2]$ if and only if we have an equality in (3.1).

Proposition 3.1 is a consequence of theorems 2.1 and 2.3(i). In section 5, we develop a numerical algorithm that allows the computation and comparison of the function $f(p,\lambda) = \prod_{k=0}^{\infty}(1+\lambda p^k)$. In particular, we can verify whether and where, in terms of the parametric space, inequality (3.1) holds. From proposition 3.1 we can conclude that dispersion is a better strategy compared to non-dispersion, when the parameters (λ,p) fall in the gray region of figure 1. The opposite (non-dispersion is a better strategy than dispersion) holds in the yellow region. Furthermore, Junior *et al* [11, theorem 2.8] show that the extinction probabilities in the white region of figure 1 satisfies $\psi_2 < 1 = \psi_A$. In conclusion, still in the white region, dispersion is a better strategy than non-dispersion.

Example 3.2. Both processes, $C(1/2,p)$ and $C_2(1/2,p)$, die out if and only if $p \leqslant 4/5$. In this case, considering (3.1), we obtain $p_l \approx 0.38$ and $p_u \approx 0.75$ such that:

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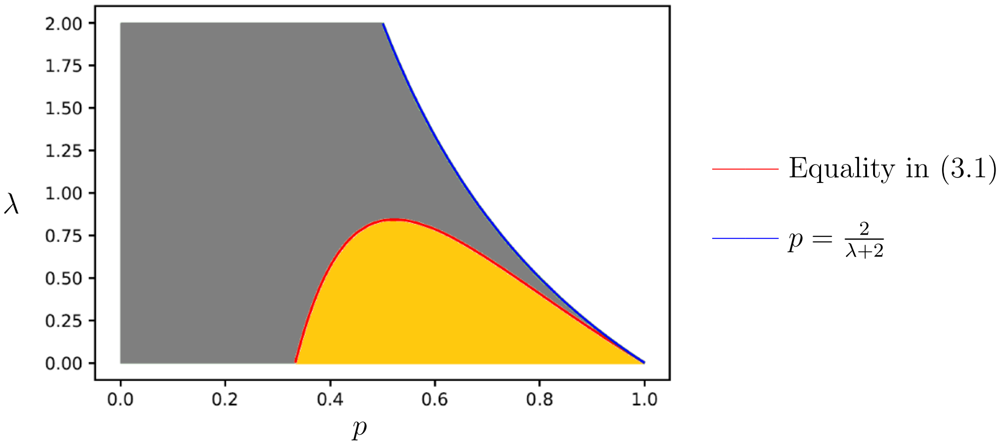

Figure 1. In the gray region, $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_2]$. In the yellow region, $\mathbb{E}[\tau_A] > \mathbb{E}[\tau_2]$.

- If $p \in (p_l, p_u)$, then $\mathbb{E}[\tau_2] < \mathbb{E}[\tau_A] < \infty$.
- If $p = p_l$ or $p = p_u$, then $\mathbb{E}[\tau_A] = \mathbb{E}[\tau_2] < \infty$.
- If $p \in (0, p_l) \cup (p_u, 4/5)$, then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_2] < \infty$.
- If $p \geqslant 4/5$, then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_2] = \infty$.

The following result establishes a comparison between the mean extinction times for the processes $C(\lambda, p)$ and $C_3(\lambda, p)$, when extinction almost surely occurs in both processes.

Proposition 3.3. *Assume $p < \frac{3}{2\lambda+3}$. Then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_3]$ in and only if*

$$\prod_{k=0}^{\infty} \left(1 + \lambda p^k\right) < 1 + \frac{\lambda(2\lambda p + 3)}{2g(\lambda, p)} \ln\left[\frac{3 - 3p - \lambda p + g(\lambda, p)}{3 - 3p - \lambda p - g(\lambda, p)}\right], \tag{3.2}$$

where $g(\lambda, p)$ is given by (2.1). Moreover, $\mathbb{E}[\tau_A] = \mathbb{E}[\tau_3]$ if and only if we have an equality in (3.2).

Proposition 3.3 is a consequence of theorems 2.1 and 2.3(ii). From proposition 3.3, we can conclude that dispersion is a better strategy compared to non-dispersion, when the parameters (λ, p) fall in the gray region of figure 2. The opposite (non-dispersion is a better strategy than independent dispersion) holds in the yellow region. Furthermore, Junior *et al* [11, theorem 2.8] show that the extinction probabilities in the white region of figure 2 satisfies $\psi_3 < 1 = \psi_A$. Thus, in the white region, dispersion is a better strategy than non-dispersion.

Example 3.4. Both processes, $C(1/5, p)$ and $C_3(1/5, p)$, die out if and only if $p \leqslant 15/17$. In this case, considering (3.2), we obtain $p_l \approx 0.58$ and $p_u \approx 0.80$ such that:

- If $p \in (p_l, p_u)$, then $\mathbb{E}[\tau_3] < \mathbb{E}[\tau_A] < \infty$.
- If $p = p_l$ or $p = p_u$, then $\mathbb{E}[\tau_A] = \mathbb{E}[\tau_3] < \infty$.

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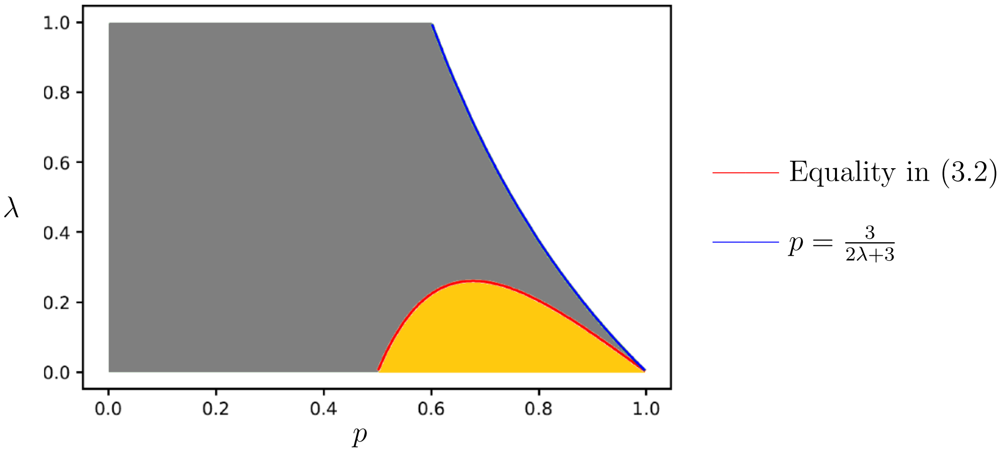

Figure 2. In the gray region, $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_3]$. In the yellow region, $\mathbb{E}[\tau_A] > \mathbb{E}[\tau_3]$.

- If $p \in (0, p_l) \cup (p_u, 15/17)$, then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_3] < \infty$.
- If $p \geqslant 15/17$, then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_3] = \infty$.

The following result establishes that the mean extinction time for the process without dispersion, $C(\lambda, p)$, is less than for the process with dispersion and no spatial restriction, $C_*(\lambda, p)$, when extinction almost surely occurs in both processes.

Proposition 3.5. *If $p < \frac{1}{\lambda+1}$, then $\mathbb{E}[\tau_A] < \mathbb{E}[\tau_*]$.*

Proposition 3.5 leads us to the conclusion that, in the absence of spatial constraints and under binomial catastrophes, dispersion is a more effective strategy than non-dispersion in extending the population's lifespan.

4. Proofs

Lemma 4.1 (Lemma 4.1 in Junior *et al* [12]). *Let $(Y_t)_{t\geqslant 0}$ be a continuous time branching process, where each particle survives an exponential time of rate 1 and right before death produces a random number of particles with probability generating function*

$$f(s) = \sum_{k=0}^{\infty} p_k s^k.$$

Suppose that $Y_0 = 1$ and $f'(1) \leqslant 1$. Let $\tau = \inf\{t > 0 : Y_t = 0\}$, the extinction time of the process $(Y_t)_{t\geqslant 0}$.

(i) If $p_2 \neq 0$ and $p_k = 0$ for $k \geqslant 3$, then

$$\mathbb{E}[\tau] = \begin{cases} \dfrac{1}{p_2} \ln\left(\dfrac{p_0}{p_0 - p_2}\right) & , \text{ if } f'(1) < 1, \\ \infty & , \text{ if } f'(1) = 1. \end{cases}$$

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(ii) If $p_3 \neq 0$ and $p_k = 0$ for $k \geqslant 4$, then

$$\mathbb{E}[\tau] = \begin{cases} \dfrac{1}{\sqrt{4p_0p_3 + (p_2+p_3)^2}} \ln\left[\dfrac{2p_0 - p_2 - p_3 + \sqrt{4p_0p_3 + (p_2+p_3)^2}}{2p_0 - p_2 - p_3 - \sqrt{4p_0p_3 + (p_2+p_3)^2}}\right] & , \text{if } f'(1) < 1, \\ \infty & , \text{if } f'(1) = 1. \end{cases}$$

(iii) If $p_0 = \beta$ and $p_n = \alpha c^n$ for $n \geqslant 1$, where α, β and c are positive constants, then

$$\mathbb{E}[\tau] = \begin{cases} 1 - \dfrac{1-\beta}{c} \ln\left[1 - \dfrac{c}{\beta}\right] & , \text{if } f'(1) < 1, \\ \infty & , \text{if } f'(1) = 1. \end{cases}$$

In order to prove theorems 2.3 and 2.4, observe that the probability distribution of the number of survivals right after the catastrophe (but before the dispersion) is given by

$$\mathbb{P}(N=0) = \beta, \mathbb{P}(N=n) = \alpha c^n, n = 1, 2, \ldots,$$

where

$$\beta = \frac{1-p}{\lambda p + 1}, \ \alpha = \frac{\lambda + 1}{\lambda(\lambda p + 1)} \text{and} \quad c = \frac{\lambda p}{\lambda p + 1}. \tag{4.1}$$

For details see Machado *et al* [11, equation (4.1)].

Proof of theorem 2.3. Let Z_t be the number of colonies at time t in the model $C_d(\lambda, p)$. Observe that Z_t is a continuous-time branching process with $Z_0 = 1$. Each particle (colony) in Z_t survives an exponential time of rate 1 and right before death produces $k \leqslant d$ particles (colonies are created right after a catastrophe) with probability p_k given by

$$p_k = \begin{cases} \beta & , \text{if } k = 0; \\ \alpha \dbinom{d}{k} \displaystyle\sum_{n=k}^{\infty} T(n,k) \left(\frac{-d}{c}\right)^n & , \text{if } 1 \leqslant k < d; \\ 1 - \displaystyle\sum_{j=0}^{d-1} p_j & , \text{if } k = d; \end{cases}$$

where $T(n,k)$ denotes the number of surjective functions $f: A \to B$, with $|A| = n$ and $|B| = k$.

Moreover, $\tau_d^i = \inf\{t > 0 : Z_t = 0\}$.

- For $d = 2$, we have that

$$p_0 = \beta, \ p_1 = \frac{2\alpha c}{2-c} \text{ and } p_2 = 1 - \beta - \frac{2\alpha c}{2-c}.$$

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Furthermore, the condition $p<\frac{2}{2+\lambda}$ is equivalent to $p_1+2p_2<1$. Thus, from lemma 4.1(i), we have that

$$\begin{aligned}\mathbb{E}\left[\tau_2\right] &= \frac{1}{p_2}\ln\left(\frac{p_0}{p_0-p_2}\right)\\ &= \frac{(\lambda p+1)(\lambda p+2)}{\lambda p^2(\lambda+1)}\ln\left[\frac{(1-p)(\lambda p+2)}{(1-p)(\lambda p+2)-\lambda p^2(\lambda+1)}\right],\end{aligned}$$

where the last line has been obtained using (4.1).

When $p=\frac{2}{2+\lambda}$, we have that $p_1+2p_2=1$. Thus, from lemma 4.1(i), it follows that $\mathbb{E}[\tau_2]=\infty$.

- For $d=3$, we have that

$$p_0=\beta, p_1=\frac{3\alpha c}{3-c}, p_2=\frac{6\alpha c^2}{(3-2c)(3-c)} \text{ and } p_3=1-\beta-\frac{3\alpha c}{3-c}-\frac{6\alpha c^2}{(3-2c)(3-c)}.$$

Furthermore, the condition $p<\frac{3}{2\lambda+3}$ is equivalent to $p_1+2p_2+3p_3<1$. Thus, from lemma 4.1(ii), we have that

$$\begin{aligned}\mathbb{E}\left[\tau_3\right] &= \frac{1}{\sqrt{4p_0p_3+(p_2+p_3)^2}}\ln\left[\frac{2p_0-p_2-p_3+\sqrt{4p_0p_3+(p_2+p_3)^2}}{2p_0-p_2-p_3-\sqrt{4p_0p_3+(p_2+p_3)^2}}\right]\\ &= \frac{2\lambda p+3}{2g(\lambda,p)}\ln\left[\frac{3-3p-\lambda p+g(\lambda,p)}{3-3p-\lambda p-g(\lambda,p)}\right],\end{aligned}$$

where the last line has been obtained using (4.1) and $g(\lambda,p)$ is given by (2.1).

When $p=\frac{3}{2\lambda+3}$, we have that $p_1+2p_2+3p_3=1$. Thus, from lemma 4.1(ii), it follows that $\mathbb{E}[\tau_3]=\infty$.

□

Proof of theorem 2.4. Analogously to the proof of theorem 2.3, in this case, $p_0=\beta$, and $p_n=\alpha c^n, n=1,2,\ldots$. □

Proof of proposition 3.5. Assume that $p<\frac{1}{\lambda+1}$ (or equivalently $\lambda p<1-p$). From theorems 2.1 and 2.4 we have that $\mathbb{E}[\tau_A]<\mathbb{E}[\tau_*]$ if and only if

$$\prod_{k=1}^{\infty}\left(1+\lambda p^k\right)<1-\ln\left(1-\frac{\lambda p}{1-p}\right). \tag{4.2}$$

To show that inequality (4.2) holds, note that the series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{n}\lambda^n p^{kn}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{\lambda^n p^n}{1-p^n},$$

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converges absolutely if $\lambda p < 1 - p$ (use the root test). Thus, by using the Taylor expansion (see [3, chapter 9]) of the function $\ln(1+x)$ and Fubini's theorem (see [2, chapter 10]), we have

$$\begin{aligned}
\ln\left[\prod_{k=1}^{\infty}\left(1+\lambda p^{k}\right)\right] &= \sum_{k=1}^{\infty}\ln\left(1+\lambda p^{k}\right)\\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\lambda^{n}p^{kn}\\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{n}\lambda^{n}p^{kn}\\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{\lambda^{n}p^{n}}{1-p^{n}}.
\end{aligned}$$

Let $a_n = \frac{(-1)^{n+1}}{n}\frac{\lambda^n p^n}{1-p^n}$. Observe that for $\lambda p < 1 - p$,

$$\begin{aligned}
a_{2n}+a_{2n+1} &= -\frac{\lambda^{2n}p^{2n}}{2n\left(1-p^{2n}\right)}+\frac{\lambda^{2n+1}p^{2n+1}}{(2n+1)\left(1-p^{2n+1}\right)}\\
&= -\frac{\lambda^{2n}p^{2n}}{2n\left(2n+1\right)}\left[\frac{2n+1}{1-p^{2n}}-\frac{2n\lambda p}{1-p^{2n+1}}\right]\\
&< -\frac{\lambda^{2n}p^{2n}}{2n\left(2n+1\right)}\left[\frac{2n+1}{1-p^{2n}}-\frac{2n\left(1-p\right)}{1-p^{2n+1}}\right]\\
&= -\frac{\lambda^{2n}p^{2n}}{2n\left(2n+1\right)}\left[\frac{2n\left[p^{2n}\left(1-p\right)+p\left(1-p^{2n}\right)\right]}{\left(1-p^{2n}\right)\left(1-p^{2n+1}\right)}+\frac{1}{1-p^{2n}}\right]\\
&< 0.
\end{aligned}$$

Thus,

$$\ln\left[\prod_{k=1}^{\infty}\left(1+\lambda p^{k}\right)\right] = a_1+\sum_{n=1}^{\infty}\left(a_{2n}+a_{2n+1}\right) < a_1 = \frac{\lambda p}{1-p}.$$

Therefore, using the Taylor expansions of the functions e^x and $\ln(1-x)$, we have that

$$\begin{aligned}
\prod_{k=1}^{\infty}\left(1+\lambda p^{k}\right) &\leqslant \exp\left(\frac{\lambda p}{1-p}\right)\\
&= 1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\lambda p}{1-p}\right)^{n}\\
&< 1+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\lambda p}{1-p}\right)^{n}\\
&= 1-\ln\left(1-\frac{\lambda p}{1-p}\right).
\end{aligned}$$

□

5. Numerical analysis

This section presents the development of a numerical method to identify the regions within the parametric space of $p \times \lambda$ where inequalities (3.1) and (3.2) hold, as well as the regions where they do not.

Let

$$f(p,\lambda) = \prod_{k=0}^{\infty} \left(1+\lambda p^k\right).$$

Let g_1, g_2, h_1 and h_2, functions of p and λ such that inequalities (3.1) and (3.2) correspond to $f < g_1$ and $f < g_2$, restricted to $h_1 > 0$ and $h_2 > 0$, respectively. Note that

$$h_1(p,\lambda) = 2 - p(\lambda+2)$$

$$h_2(p,\lambda) = 3 - p(2\lambda+3).$$

In order to calculate and compare the function f, we use the lower and upper bounds given by the following lemma.

Lemma 5.1. *If $p < \frac{a}{b\lambda+a}$. Then, for all $M \in \mathbb{N}$,*

$$\prod_{k=0}^{M} \left(1+\lambda p^k\right) \leqslant f(p,\lambda) \leqslant \exp\left(\frac{a}{b}p^M\right) \prod_{k=0}^{M} \left(1+\lambda p^k\right).$$

Proof. The first inequality holds since $1+\lambda p^k \geqslant 1$ for all $k \geqslant 1$. In order to prove the second inequality, we observe that as $p < \frac{a}{a+b\lambda}$, then

$$\lambda < \frac{a}{b}\frac{(1-p)}{p}.$$

Thus, using $(1+x) \leqslant e^x$ for all $x \in \mathbb{R}$,
we have that

$$\begin{aligned}
\prod_{k=0}^{\infty}\left(1+\lambda p^k\right) &\leqslant \left[\prod_{k=0}^{M}\left(1+\lambda p^k\right)\right] \exp\left(\sum_{k=M+1}^{\infty} \lambda p^k\right) \\
&= \left[\prod_{k=0}^{M}\left(1+\lambda p^k\right)\right] \exp\left(\frac{\lambda p^{M+1}}{1-p}\right) \\
&< \left[\prod_{k=0}^{M}\left(1+\lambda p^k\right)\right] \exp\left(\frac{a}{b}p^M\right).
\end{aligned}$$

□

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We now consider the following task. Given particular values of p and λ, determine if f is lower than or greater than g. We do this by recursion on M: if the upper bound of f is below g then f is lower than g, if the lower bound of f is above g then f is greater than g, in other case we try again with a bigger value of M. Notice that the upper and lower bounds of f tend to f when M tends to infinity.

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