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Wave radiation by a deep submerged cylinder with application to ocean structure

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The radiation problem for a deep submerged cylinder is analysed and an asymptotic expression, uniformly valid in the whole range of frequencies, is derived for the radiation damping and exciting force. The obtained expression is a natural extension of the inertia term in the well known Morison's formula and it may be useful in the analysis of certain ocean structures such as a tension leg platform, for example. In particular it makes it possible to define a geometry tuned to have null excitation at some desired frequency.

1 INTRODUCTION

The hydrodynamic study of an ocean structure, like a tension leg platform (TLP), for example, is an essential step in its design and there are several computer codes, most of them based on Green's function method, dedicated to this issue. These tools, however, are more appropriate for the analysis of a given structure than for its synthesis, and in the initial phase of the design one generally needs a more direct approach that can give some insight about possible geometric forms that would fit better the desired performance of the structure.

In the present paper a simple expression is derived for the radiation damping and wave exciting force on a relatively deep submerged cylinder. This expression can be useful to help define the geometric form of the pontoons of a TLP and it is a natural extension, to the whole range of frequencies, of the inertia term of the well known Morison's formula. The derivation is based on an old study done by Lamb in 1913 about the wave resistance of a submerged circular cylinder;[1] more recent references in related subjects are the works by Leppington and Siew[2] and Grue and Palm.[3]

In Section 2 the motion of a cylinder in an unbounded fluid is briefly reviewed since this result is needed in Section 3, where the free surface effect is incorporated. In Section 4 the expressions for the exciting force and radiation damping are derived and a possible criterion for the optimization of a TPL is discussed. In Section 5 different geometric forms for the cross-sections of the pontoons are analysed, both analytically and numerical, and some simple, perhaps useful, conclusions are derived.

2 MOTION IN UNBOUNDED FLUID

Consider a cross-section as indicated in Fig. 1, symmetric with respect to the y-axis and subjected to a vertical motion with unit velocity. Let b be the half beam of the section and c the radius of the circle that circumscribes it.

If $z = x + i.y$ is the complex variable let $f(z)$ be the complex velocity potential, analytical in the region $|z| > c$. In this region, $f(z)$ can be expanded in a power series in $1/z$ and one has then

$$f(z) = -\sum_{n=1}^{\infty}(-i)^n \frac{B_n}{z^n} \tag{1}$$

For a section symmetric with respect to the y-axis the coefficients B_n are real and if the section is also symmetric with respect to the x-axis then $B_{2n} = 0$. Another important property is the following: if $\{B_n^*,\ n = 1, 2, \ldots\}$ represent the coefficients of a section obtained from the given one by a complete rotation around the x-axis, then

$$\begin{aligned} B_{2n-1}^* &= B_{2n-1} \\ B_{2n}^* &= -B_{2n} \end{aligned} \tag{2}$$

Equation (2) shows that the signs of the even coefficients can be changed by a rotation around the x-axis, a result that will be used later on. The coefficient B_1 can be written in terms of the sectional added mass. In fact, if S is the cross-section area and $C_M = M_a/\rho S$ is the added mass coefficient, then

$$B_1 = -\frac{1}{2\pi}(C_M + 1)S \tag{3}$$

It is usual to analyse the proposed problem in a

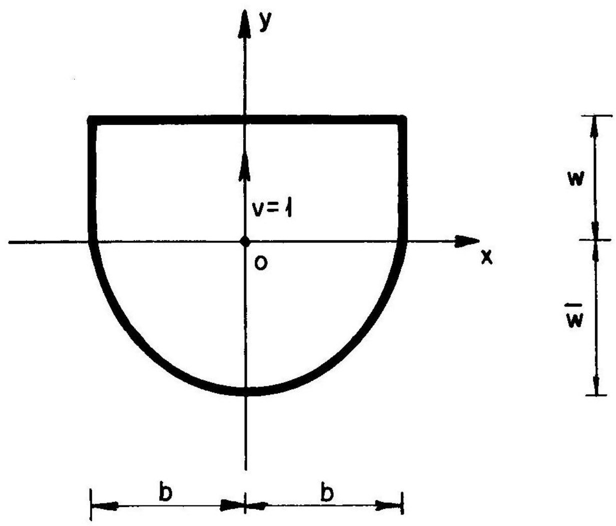

Fig. 1. Geometric definition of a cross-section in vertical motion.

normalized geometry where a certain length a is assumed to be unitary. This geometric scale can be, for example, the half beam of the body ($a = b$) or the equivalent circle's radius, given by $a = [(C_M + 1)S/2\pi]^{1/2}$. If $\{D_n; n = 1, 2, \ldots\}$ are the coefficients of eqn (1) for the normalized geometry the following relations can be derived between D_n and B_n:

$$B_n = a^{n+1} . D_n \tag{4}$$

Observing the identities:

$$\frac{\sin(2n+1)\theta}{r^{2n+1}} = \frac{(-1)^n}{(2n)!} \frac{\partial^{2n}}{\partial y^{2n}} \left(\frac{\sin\theta}{r}\right)$$

$$\frac{\cos 2n\,\theta}{r^{2n}} = \frac{(-1)^{n-1}}{(2n-1)!} \frac{\partial^{2n-1}}{\partial y^{2n-1}} \left(\frac{\sin\theta}{r}\right) \tag{5}$$

the velocity potential can be written as

$$\phi(x,y) = \text{Real}(f(z)) = B_1 . \sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!} \frac{D_n}{D_1} \frac{\partial^{n-1}}{\partial y^{n-1}} \left(\frac{\sin\theta}{r}\right) \tag{6}$$

3 FREE SURFACE EFFECT

Consider now the same section oscillating with frequency ω and suppose that the free surface is at the plane $y = d$. If $K = \omega^2/g$ is the related wavenumber it will be assumed here that the depth of submergence d is so large that the nondimensional parameter $(Ka)^2.e^{-2Kd}$ is small compared with 1 in the whole range of frequencies. This condition is naturally satisfied in a TLP, where usually one has $d \cong 3a$.

The solution of the radiation problem satisfies the imposed kinematic condition on the body's surface, the free surface boundary condition at $y = d$, the radiation condition when $x \to \pm\infty$ and Laplace's equation. Observing that the kinematic condition on the body is automatically satisfied by eqn (6) the radiation potential can be written in the form:

$$\phi_3(x,y,t) = [\phi(x,y) + \psi(x,y)].\cos\omega t + \psi_w(x,y,t) + R(x,y,t) \tag{7}$$

with $\phi(x,y)$ given by eqn (6) and the remaining functions to be defined as explained below.

The function $\psi(x,y)$ is determined by imposing that the term within brackets in eqn (7) satisfies the free surface boundary condition, and $\psi_w(x,y,t)$ is a stationary wave added in order that the radiation condition is fulfilled. The residue $R(x,y,t)$ corrects the body's boundary condition, perturbed by the terms ψ and ψ_w in eqn (7), and must satisfy the remaining conditions. As it will be seen later on this function is of the order $(Ka)^2.e^{-2Kd}$ and will be ignored in the present approximation.

The function $\psi(x,y)$, being a solution of Laplace's equation, can be given by the Fourier integral:

$$\psi(x,y) = \int_0^{\infty} \alpha(k).e^{ky}.\cos kx\, dk \tag{8a}$$

Observing the identity:

$$\frac{\sin\theta}{r} = \frac{y}{x^2+y^2} = \int_0^{\infty} e^{-ky}.\cos kx\, dk \quad (y > 0) \tag{8b}$$

one can write $\phi(x,y)$ in the form:

$$\phi(x,y) = B_1 . \int_0^{\infty} F(ka).e^{-ky}.\cos kx\, dk$$

$$F(ka) = \sum_{n=0}^{\infty} (-1)^n \frac{(ka)^n}{n!} \frac{D_{n+1}}{D_1} \tag{9}$$

The function $F(.)$ is analytic on the complex plane (see Appendix) and depends exclusively on the geometry of the cross-section through the ratios D_n/D_1. In particular, $F(ka) \equiv 1$ for a circle. From the free-surface boundary condition:

$$\frac{\partial}{\partial y}[\phi + \psi] = K[\phi + \psi]; y = d$$

one obtains

$$\alpha(k) = B_1 \frac{k+K}{k-K} F(ka)\,e^{-2kd} \tag{10}$$

The Fourier integral (eqn (8)) can be defined in the complex k-plane, but one must take care of the singularity at $k = K$ introduced by eqn (10). If the contour is indented above (below) the singularity when $x > 0$ $(x < 0)$ the final result can be expressed in the form:

$$\psi(x,y) = \psi_0(x,y) + \psi_e(x,y) \tag{11}$$

with

$$\psi_0(x,y) = -2\pi B_1 K F(Ka) e^{-2Kd} e^{Ky} \sin K|x|$$

$$\psi_e(x,y) = -2\pi B_1$$

$$\times \text{Real}\left[\frac{i}{2\pi}\int_0^{\infty} \frac{im+1}{im-1} F(imKa) e^{-im(2Kd-Ky)} e^{-m|x|} dm\right] \tag{11b}$$

The functions $\psi_0(x,y)$ and $\psi_e(x,y)$ are, respectively, the undulatory and evanescent parcels of $\psi(x,y)$ and one fact should be observed here: the odd derivatives of $\psi_0(x,y)$ with respect to x have discontinuities at $x = 0$ exactly balanced by the discontinuities of the derivatives of $\psi_e(x,y)$. This last result can be easily demonstrated with the help of the Residue Theorem and from the properties of the analytic function $F(z)$ ($F(-imKa) = \bar{F}(imKa)$). In particular, one has

$$\frac{\partial \psi_e}{\partial x}(0,y) - (\text{sign } x).2\pi B_1 K^2 F(Ka) e^{-2Kd} e^{Ky} \tag{12}$$

In the far field only the undulatory parcel $\psi_0(x,y)$ remains, and the radiation condition will be satisfied if the stationary wave:

$$\psi_w(x,y,t) = 2\pi B_1 K F(Ka) e^{-2Kd} e^{Ky}. \cos K|x|. \sin \omega t \tag{13}$$

is added to the solution. The function $R(x,y,t)$ corrects the perturbation introduced in the body's boundary condition by the functions ψ and ψ_w and so

$$\nabla R.\vec{n} = -\nabla(\psi. \cos \omega t + \psi_w).\vec{n} \sim 0(\delta) \tag{14a}$$

where δ is given by (see eqns (11)–(13)):

$$\delta = (C_M + 1)K^2 S.F(Ka).e^{-2K(d-w/2)} \ll 1 \tag{14b}$$

The factor w in eqn (14) comes from the estimation of eqn (14a) at $y = w$. In the far field the radiation potential takes the form:

$$\phi_3(x,y,t) \sim i.2\pi B_1.KF(Ka) e^{-2Kd}.e^{Ky}.e^{i(K|x|-\omega t)}; \; |x| \to \infty \tag{15}$$

an expression that will be used in Section 4.

4 EXCITING FORCE AND RADIATION DAMPING

The exciting force due to an incident wave with potential ϕ_I can be computed from the radiation solution with the help of Haskind's relation. In the present case, one obtains, with eqn (15), the simple expression:

$$f_3(t) = \rho S(C_M + 1).F(Ka).\frac{dw_0}{dt}$$

$$w_0(t) = \frac{\partial \phi_I}{\partial y} = -iwA e^{Kd} e^{-i\omega t} \tag{16}$$

where A is the wave amplitude, and $f_3(t)$ the heave sectional force.

Observing that $F(Ka) \to 1$ when $Ka \to 0$, eqn (16) reduces to the inertia part of Morison's formula in this limit. So eqn (16) is the natural extension of this formula to the whole range of frequencies with an error of the form $(1 + 0(\delta))$, δ being given by eqn (14b). For a circle, in particular, $F(Ka) \equiv 1$ and eqn (16) reduces to Morison's formula independent of the value of Ka. This simplified expression is compared in Ogilvie[4] with the exact solution (see Fig. 2 of Ogilvie's work) and one can check directly that, for a given Ka, the error tends to zero as Kd increases.

If A_0 is the amplitude of the imposed motion on the cylinder and $A(\omega)$ is the far field wave amplitude generated by this motion, from eqn (15) one obtains

$$\left|\frac{A(\omega)}{A_0}\right| = K^2 S(C_M + 1)F(Ka)\, e^{-Kd} \tag{17a}$$

and from energy conservation it follows that the sectional radiation damping can be written as

$$c_r = \frac{\rho g^2}{\omega^3}\left|\frac{A(\omega)}{A_0}\right|^2 \tag{17b}$$

The pontoons of a TLP, for example, are relatively slender and once can use eqn (17), together with strip theory, to estimate the radiation damping in the three vertical modes heave, pitch and roll. In particular the radiation damping in the heave motion is given in this approximation by $c_r.l$, where l is the total perimeter of the pontoons. Supposing that M is the total mass of the structure, and $M_a = C_M \rho S l$ is the added mass, then the percentage of the critical damping ζ_r is given by

$$c_r.l = 2(M + M_a)\zeta_r \omega_n$$
$$= \rho\omega l(KS)^2(C_M + 1)^2 F^2(Ka)\, e^{-2Kd}$$

where ω_n is the heave natural frequency. In the resonant condition one has

$$\zeta_r = 0.5\frac{(C_M + 1)}{C_M + (M/\rho S l)}.e^{-2Kd}.K^2(C_M + 1)S.F^2(Ka) \tag{18}$$

Equation (18) splits the radiation damping into three distinct contributions: the first depends on the depth of submergence through the exponential term in $-2Kd$; the second on the 'hydrodynamic size' of the pontoons through the term proportional to $(C_M + 1)S$; the third depends on the geometric form of the cross-section through the function of form $F(Ka)$. Observing that the natural period of a TLP is short (of the order of 4 s) one has $F(Ka)$ relatively different from $1(Ka \geqslant 0(1))$ and so the form of the cross-section can affect strongly the value of ζ_r for these structures.

One of the main concerns in the design of a TLP is the fatigue life of the tendons. Within the context of the linear analysis it can be shown that the r.m.s. value of the vertical displacement is proportional to q, where

$$q = \left(\frac{c_r}{c_i + c_v + c_r}\right)^{1/2} \tag{19}$$

In eqn (19), c_i, c_v, c_r are, respectively, the damping coefficients due to the internal dissipation in the structure, to viscous dissipation in the fluid and to wave radiation to infinity. This result, derived by Vandiver,[5] has a simple explanation: since the resonant peak is very large the random response is dominated by the behaviour of the system in the vicinity of the resonance and so the average value of the heave amplitude squared is proportional to the force spectrum, at the natural frequency, times the transfer function. The force spectrum is related, through Haskind's relations, to the radiation damping, and the transfer function at the resonance is inversely proportional to the total damping.

The parameter q is monotonically increasing with c_r, an observation that leads to the following result: the r.m.s. value of the vertical displacement (and so the dynamic tension in the tendons) increases with the function of form $F(Ka)$. The fatigue life of the tendons is very sensitive to the dynamic tension, and a simple design criteria can be derived in this case: the geometric form of the pontoons' cross-sections must be chosen in order to minimize the function of form $F(Ka)$.

This criteria may be criticized since important second-order effects have been ignored, but, it is believed, it represents a first step attempting to define a suitable form of the pontoons; anyway, in Section 6, a digression about second-order forces is made in order to place the discussion in a proper perspective.

5 THE FUNCTION OF FORM $F(Ka)$

In this section a more systematic study of the function $F(Ka)$ will be made, aiming to shed some light on the geometric features that lead to the minimization of this function. First, an analysis is made for a family of simple geometries and in section (5.2) some ideas related to the geometric synthesis of a desirable function of form are discussed. The last item introduces a numerical method based on Hamilton's principle and presents numerical results for different geometries.

5.1 Family of ellipses

The conformal mapping:

$$z = 0.5(Z + 1/Z) \tag{20a}$$

transforms the unit circle in the Z-plane into a flat plate and a circle with radius $R > 1$ into an ellipse. Denoting members of this family by the ratio w/b, see Fig. 1, the following simple relation can be derived between the coefficients in the series in eqn (1) ($a = b = 1$):

$$\frac{D_{2n+1}}{D_1}(w/b) = [1 - (w/b)^2]^n \frac{D_{2n+1}}{D_1}(0) \tag{20b}$$

For the flat plate the complex potential has the form:

$$f(z) = -i\frac{1}{z + (z^2 - 1)^{1/2}} \tag{21a}$$

this expanded in a powers series in $1/z$ gives

$$D_3/D_1 = -1/4; \; D_7/D_1 = -5/64 \quad (21b)$$

$$D_5/D_1 = 1/8; \; D_9/D_1 = 7/128 \quad (21b)$$

Using eqns (21b) and (20b) in eqn (9) the following approximation is obtained for the function of the form ($a = b$ in this case):

$$F(Ka; w/b) = 1 - \frac{\lambda^2}{8} + \frac{\lambda^4}{192} - \frac{\lambda^6}{9216} + \frac{\lambda^8}{737\,280}$$

$$\lambda = [1 - (w/b)^2]^{1/2}.Ka \tag{22}$$

The function of form decreases then with w/b, and Fig. 2 shows the graph of $F(Ka; w/b)$ as a function of Ka for different values of w/b. Notice that the behaviour shown in eqn (22) is typical for sections that are symmetrical with respect to the x-axis: the first correction to the circle result $F(Ka) \equiv 1$ appears at the power $(Ka)^2$ since these sections have zero even coefficients in eqn (1). For values of Ka that are not too large one should prefer then a section nonsymmetric with respect to the x-axis since; in this case, it is always possible to obtain a first negative term in eqn (22) that is linear in Ka (see eqn (2)). This point will be elaborated further on.

5.2 Geometric synthesis of a function of form

The hydrodynamic problem is, in general, posed in a direct way (namely, to determine the function of form for a given geometry) although the inverse problem may be more interesting from a design point of view. In this case a function of form, with some desirable properties, is defined and one must then find the geometry that fits the given function $F(Ka)$.

The intention here is not to exhaust this problem and only some simple examples will be discussed. Observing the exponential nature of the series in eqn (9) the following functions of form are proposed:

$$F_1(Ka) = \mathrm{e}^{-\alpha Ka}$$

$$F_2(Ka) = \cos(\alpha Ka) \tag{23}$$

$$F_3(Ka) = 1 - \sin(\alpha Ka)$$

The first function of form $F_1(Ka)$ leads to a trivial result that could have been anticipated, as explained below. The two other functions, however, give nontrivial results and one can choose α to tune the structure to give zero response at some desired frequency. For example, if $a = 8$ m, and T_n, the natural period, is 4 s, then $Ka = 2$ at resonance and

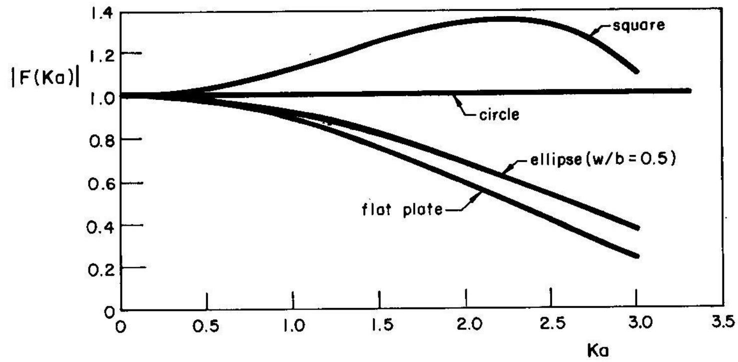

Fig. 2. Function of the form $F(Ka)$ for the family of ellipses. Also shown is the function of form for a square ($a = b = 1$ in all cases).

$F_2(Ka) = F_3(Ka) = 0$ when $Ka = 2$ and $\alpha = \pi/4$. In this situation the resonant response is eliminated.

Furthermore, since $F(Ka)$ depends only on the ratios D_n/D_1, a freedom exists to choose the value of D_1 and this is an important point. In fact, if one takes $D_1 = -1$, then all geometries will have the same 'hydrodynamic size' $(C_M + 1)S$ of the circle and will differ, one from the other, only in the form of the cross-section (see eqn (18)). In this circumstance, changes in the geometry will affect weakly other parameters such as the natural frequency, for example, and this makes the comparison between geometries more realistic. The normalization of the coefficient D_1 implies, however, that one is using the length $a = [(C_M + 1)S/2\pi]^{1/2}$, instead of the half beam, as the geometric scale of the problem. This fact will be assumed in the following.

Expanding the function $F_1(Ka)$ in a power series in Ka and comparing with eqn (9) one obtains $D_n/D_1 = \alpha^{n-1}$. Using these relations in eqn (1) the complex potential takes the form:

$$f_1(z) = (iD_1/z + D_2/z^2).[1 - (\alpha/z)^2 + (\alpha/z)^4 - \cdots]$$
$$= (iD_1 z + D_2)/(z^2 + \alpha^2)$$

and since $D_2/D_1 = \alpha$ the following result can be obtained when $D_1 = -1$:

$$f_1(z) = -i\frac{1}{z + i\alpha} \tag{24}$$

Since eqn (24) is the result for the normalized geometry the conclusion is simple: the function of form $F_1(Ka)$ is related to a circle of radius a and centred in a point with coordinate $y = -\alpha a$. This result could have been anticipated: if one takes $F(Ka) = e^{-\alpha Ka}$ in eqn (16) the heave force on a circle, $d + \alpha a$ distant from the free surface, is obtained. In the Appendix the question of the invariance of eqn (16) under coordinate translation is further discussed.

The same procedure applied to the function of form $F_2(Ka)$ leads to the result:

$$f_2(z) = -\frac{i}{2}\frac{1}{z + \alpha} - \frac{i}{2}\frac{1}{z - \alpha} \tag{25}$$

which has a simple meaning: it is the body formed by two y-dipoles placed at the points $x = \pm\alpha$. For $\alpha \ll 1$ this body tends to a circle with radius $a = 1$, but for $\alpha \gg 1$ one has two circles centred at $x = \pm\alpha$ and with radius $1/2^{1/2}$. For $\alpha = \pi/4$ one obtains the geometry shown in Fig. 3, where each pontoon is split into two parallel cylinders with indicated cross-sections.

The function of form $F_3(Ka)$ is associated with the complex potential:

$$f_3(z) = -\frac{i}{z} + 0{\cdot}5\frac{1}{z - \alpha} - 0{\cdot}5\frac{1}{z + \alpha} \tag{26}$$

that corresponds to a y-dipole at the origin and two x-dipoles with opposite signs at $x = \mp\alpha$. The geometry is shown in Fig. 3 for $\alpha = \pi/4$ and one confirms a result already pointed out in eqn (20): sections nonsymmetric with respect to the x-axis have a function of form that decays faster with Ka. In the present case, in particular, the value of $F_3(Ka)$ is very small in a large vicinity of $Ka = 2$, a peculiarity that it is indeed desirable.

Others functions of form can be analysed, but it is not the purpose of this work to explore further this issue.

5.3 Numerical determination of $F(Ka)$ for a given geometry

In this section attention is turned to the direct problem, where a geometry is given and the related function of form $F(Ka)$ must be determined. In this case, one must resort to a numerical method generally based, in hydrodynamics, on an integral equation defined with the help of Green's function. A different approach, perhaps more direct and expedite, can be developed from Hamilton's principle of mechanics, as elaborated below.

Let y be the position of the mechanical system (fluid) and $Y(t)$ the external force applied by the body. The potential of this force is $-y.Y(t)$ although, in a fluid system, it is more natural to use the form $(dy/dt).\int_c^t Y(\xi)d\xi$, where dy/dt is the body's velocity. Notice that one form differs from the other by an exact time differential that is immaterial in the Principle of the Least Action (Hamilton's principle). If $\rho U(\phi)$ is the

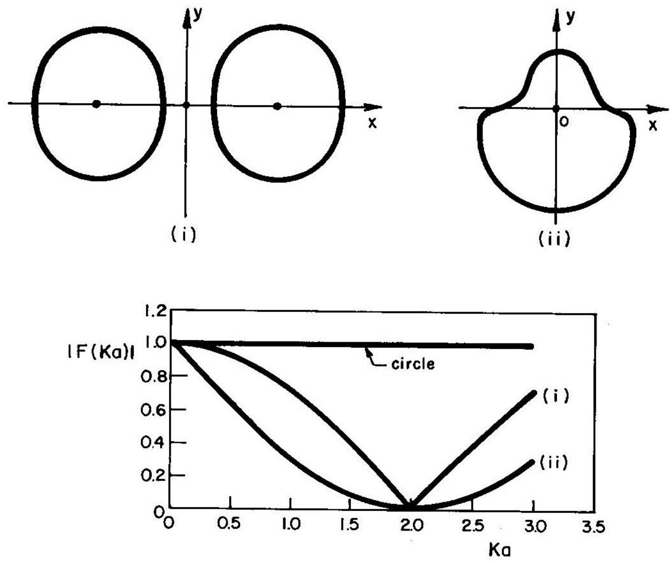

Fig. 3. Functions of form: (i) cos (αKa) and (ii) $1 - \sin(\alpha Ka)$ with related geometries $(\alpha = \pi/4)$.

'potential energy' and $dy/dt = 1$, then

$$\rho.U(\phi) = \int_c^t Y(\xi)\,d\xi = \rho.\int_S \phi.n_y.dS \tag{27a}$$

where $\phi(x,y)$ is the velocity potential, S is the body's surface, and n_y is the y component of the normal $\vec{n}$ pointing towards the body. If V is the region occupied by the fluid the kinetic energy is given by

$$T(\phi) = \tfrac{1}{2}\rho G(\phi;\phi) = \tfrac{1}{2}\rho\int_V (\nabla\phi)^2 dV \tag{27b}$$

The Lagrangean is defined by the expression:

$$L(\phi) = \tfrac{1}{2}\rho G(\phi;\phi) - \rho U(\phi) \tag{27}$$

and, in a steady problem, Hamilton's principle reduces to the stationary condition of $L(\phi)$, in other words, the actual velocity potential is the one that makes $\delta L = 0$.

The discretization of the problem is immediate in this formulation: if $\{T_n(x,y); n = 1,2,\ldots,N\}$ is a set of linearly independent functions with finite energy one can write

$$\phi_N(x,y) = \sum_{n=1}^{N} q_n.T_n(x,y) \tag{28a}$$

as an approximation for $\phi(x,y)$ in the space spanned by the functions $\{T_n(x,y)\}$. In this case the Langrangean takes the form:

$$L(q_n) = \tfrac{1}{2}\rho\sum\sum G(T_n;T_m)q_nq_m - \rho\sum U(T_n)q_n \tag{28b}$$

and the stationary condition $\delta L(q_n) = 0$ reduces to the linear system:

$$[G(T_n;T_m)].\{q_n\} = \{U(T_m)\}. \tag{28c}$$

Notice that if all $T_n(x,y)$ satisfy Laplace's equation and decay at least as fast as a dipole, one can show that

$$G(T_n;T_m) = \int_S (\nabla T_n.\vec{n}).T_m.dS \tag{29}$$

and so all the matrices in eqn (28c) can be obtained by integration over the body's surface.

It is natural to use typical flow singularities (poles, dipoles, vortex line, etc.) to define the basis $\{T_n(x,y), n = 1,2,\ldots,N\}$ of the approximation (eqn (28a)), where the singular point must be inside the body. In particular a 'vortex line' is defined here as two counter rotating vortices with a branch cut coincident with the segment that joins the vortices. This singularity is then defined by the position of the two vortices and the remaining ones (poles, dipoles, etc.) by the position of the singular point.

As discussed in Aranha and Pesce,[6] a physical visualization of the flow field can help define a minimal set of singularities for a given problem. For example, an ellipse with $w/b = 0.1$ can be represented by 'vortices lines' that imitate the rotation of the velocity field around the ends of the ellipse. Using four vortices lines $(N = 4)$, placed at $(\pm 0{\cdot}98;0)$, $(\pm 0{\cdot}8;0)$, $(\pm 0{\cdot}7;0)$, $(\pm 0{\cdot}6;0)$, one obtains the results shown in Table 1, where the coefficients D_n have been obtained by the Fourier decomposition of eqn (28a) in a circle with radius $c = 2$. Comparison with the exact solution, derived in eqn (20), shows a very good agreement. A square with $b = 1$ was also analysed with three trial functions: a y-dipole at the origin (circle's solution) and two vortices lines, at $(\pm 0{\cdot}9;0{\cdot}9)$ and $(\pm 0{\cdot}9;-0{\cdot}9)$, to imitate the flow rotation around the corners. In this case the only known coefficient is D_1, related to the added mass, and the agreement is good again.

The function $F(Ka)$ is defined by the series in eqn (9) whose convergence is relatively fast. For example, if the

Table 1. Coefficients D_n and $F(3)$ for an ellipse with $w/b = 0.1$ and a square

$D_n; F(3)$	Ellipse			Square		
	$N=4$	Exact	Error (%)	$N=3$	Exact	Error (%)
D_1	−0·541	−0·5500	1·6	−1·353	−1·393	2·9
D_2	0·135	0·1361	0·8	−0·357	*	*
D_5	−0·066	−0·0674	2·0	0·347	*	*
D_7	0·041	0·0417	1·7	0·402	*	*
D_9	−0·029	−0·0289	0·4	−0·506	*	*
$F(3)$	0·2238	0·2318	3·4	1·084	*	*

series is oscillatory, a common circumstance, and the sum is truncated at m the error is smaller than $((Ka)^m/m!).D_{m+1}/D_1$. Since $D_n/D_1 \to 0$ as $n \to \infty$ an upper bound for the error is given by $(Ka)^m/m!$, a factor smaller than 1·63% when $Ka \leqslant 3$ and $m = 10$. In the following the analysis will be restricted to the range of frequencies $0 \leqslant Ka \leqslant 3$ and the series in eqn (9) will be truncated at $m = 10$. The last line of Table 1 compares the values of $F(3)$ for the numerical and exact solutions in the case of the ellipse.

As has been observed before, sections nonsymmetric with respect to the x-axis have a nonzero value for the coefficient D_2/D_1, the one that multiplies Ka in eqn (9). Furthermore, by a rotation around the x-axis, one can change the sign of this ratio (see eqn (2)). So, for a given nonsymmetric section (or for the section obtained from this one by a rotation) the function of form decays linearly with Ka for small values of the frequency, certainly a desirable feature. This simple result can be helpful to select the pontoon's geometry, and Fig. 4 presents the function of form for some nonsymmetric sections. These results, again, are intended only to exemplify the theory, not to exhaust it, but they show the advantages of this class of geometries

6 A DIGRESSION ON SECOND-ORDER FORCES

As pointed out in Section 1 the former motivation of this work aimed to analyse the resonant response of a TLP, where the natural period is short and the linear exciting force on the pontoons, proportional to e^{-Kd}, is small. In this context the second-order 'sum frequency' force may become important, since it decreases linearly with depth and not exponentially; as a matter of fact this force appears to be the dominating parcel for the usual geometric configuration of a TLP, where the columns are large and the pontoons relatively slim, and a critical analysis of this trend may be of importance in order to place the discussion of this work in a proper perspective.

Observing that a second-order effect is larger the larger is the diffraction and that the interaction between the pontoons and the incident wave can be neglected in the short wave regime, one can restrict the attention to the columns that intercept the free surface. In this way one considers first a circular column with radius a_0 and draft D excited by a harmonic wave with frequency ω and amplitude A; if Newman's approximation[7] for the second-order sum frequency is further approximated for relatively small Ka_0 one obtains that the vertical second-order force on the column is given by

$$Z_{2,C}^{(i)}(t) \cong \frac{5\pi}{4}\rho g a_0 A^2 (a_0/D)(Ka_0)^2\, e^{i\pi/4}\, e^{-2i\omega t}$$

Suppose now a TLP with four pontoons disposed in a squared configuration with side $2l$ and four columns placed at the corners of the square. Suppose also that the pontoons have a *circular* cross-section and let V be the displaced volume of the structure, V_C the volume of the columns, and V_P the pontoon's displacement; obviously $V = V_C + V_P$. Assuming a force like the above one in each column one obtains, as resultant, the expression:

$$Z_{2,C}(t) \cong \mathscr{F}_{2,C}\,\rho V g G_2(Kl;\beta)\, e^{i\pi/4}\, e^{-2i\omega t}$$
$$\mathscr{F}_{2,C} = \tfrac{5}{16}(KA)^2 \tfrac{V}{\pi D^3}(V_C/V)^2 \tag{30a}$$

where $G_2(Kl;\beta) = \cos(Kl\sin\beta).\cos(Kl\cos\beta)$ takes care of the phase difference between the forces in each column, and β is the angle of the incident wave.

Let $\omega_1 = 2\omega$ be the natural frequency of a TLP and consider the linear exciting force on the pontoons at this frequency. Assuming a circular cross-section ($F(Ka) = 1$) and a phase function $G_1(Kl;\beta)$ analogous to $G_2(Kl;\beta)$, then the vertical linear exciting force can be written as

$$Z_{1,P}(t) = \mathscr{F}_{1,P}\,\rho V g G_1(Kl;\beta)\, e^{-i\omega_1 t}$$
$$\mathscr{F}_{1,P} = 2(K_1 A_1)\, e^{-K_1 d}(V_P/V) \tag{30b}$$

with $K_1 = \omega_1^2/g$ and A_1 being the amplitude of the wave with frequency ω_1. If one defines now the coefficient:

$$\mathscr{F} = \mathscr{F}_{1,P} + \mathscr{F}_{2,C} \tag{30c}$$

the total exciting force can be expressed as

$$Z(t) = Z_{1,P}(t) + Z_{2,C}(t) = \mathscr{F}\,\rho V g\, G(Kl;\beta)\, e^{-i\omega_1 t} \tag{30d}$$

where $G(Kl;\beta)$ takes care of the phase difference

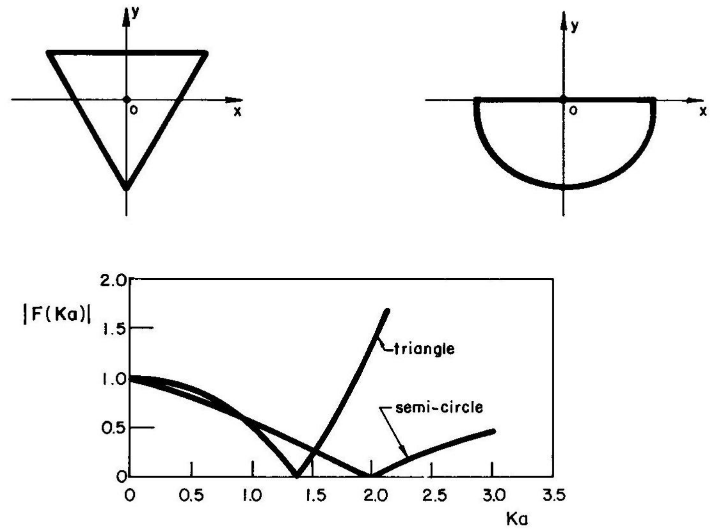

Fig. 4. Functions of form $F(Ka)$ for nonsymmetrical geometries.

between the distinct force components and of the angle of incidence.

Ignoring, for the sake of brevity, the influence of this phase function, eqns (30c) and (30d) show that the hydrodynamic design of a TLP should search for a minimum of the force coefficient $\mathcal{F}$; assuming, for the sake of evaluation, a medium size steel platform with displacement $V = 32\,000\,\mathrm{m}^3$, draft $D = 20\,\mathrm{m}$, with the pontoons centre line distant $d = 15\,\mathrm{m}$ from the free surface and a natural period $T_1 = 2\pi/\omega_1 = 4\,\mathrm{s}$, one obtains $\mathcal{F}_{1,\mathrm{P}} \cong 3{\cdot}84 \times 10^{-3}(V_\mathrm{P}/V)$ and $\mathcal{F}_{2,\mathrm{C}} \cong 2{\cdot}56 \times 10^{-3}(V_\mathrm{C}/V)^2$ if $KA = K_1A_1 = 0{\cdot}08$, a usual value for a significant wave. In this case the minimum of $\mathcal{F}$ occurs when $V_\mathrm{C}/V = 0{\cdot}75$ where then

$$\mathcal{F}_{1,\mathrm{P}} = 0{\cdot}96 \times 10^{-3};\ \mathcal{F}_{2,\mathrm{C}} = 1{\cdot}44 \times 10^{-3} \quad (V_\mathrm{C}/V = 0{\cdot}75) \qquad (31\mathrm{a})$$
$$\mathcal{F} = 2{\cdot}40 \times 10^{-3}$$

The above result justify, in part at least, the actual trend in the design of a TLP, where large columns and relatively slim pontoons are used, and one observes also that the second-order effect is the dominating one for this configuration.

If instead of this choice one uses an opposite one, with large pontoons and slim columns, where $V_\mathrm{P}/V = 0{\cdot}75$ and $V_\mathrm{C}/V = 0{\cdot}25$, for example, one obtains the values:

$$\mathcal{F}_{1,\mathrm{P}} = 2{\cdot}88 \times 10^{-3};\ \mathcal{F}_{2,\mathrm{C}} = 0{\cdot}16 \times 10^{-3} \quad (V_\mathrm{C}/V = 0{\cdot}25) \qquad (31\mathrm{b})$$
$$\mathcal{F} = 3{\cdot}04 \times 10^{-3}$$

the total force being roughly 30% larger than in the 'minimum' configuration case (eqn (31a)).

These results, however, were derived under the assumption that the pontoon has a circular cross-section. If one supposes now that the cross-section is 'tuned' to the natural period in such a way that $\mathcal{F}_{1,\mathrm{P}} = 0$, as discussed in the preceding section, the optimum choice for V_C/V is the one that minimizes $\mathcal{F}_{2,\mathrm{C}}$; in this case one should prefer slim columns and large pontoons, a choice that is opposite to the actual trend in the TLP design. In particular, if $V_\mathrm{C}/V = 0{\cdot}25$ one would obtain for the 'tuned' geometry an exciting force that is 7% of the 'optimum' value indicated in eqn (31a), since then $\mathcal{F} = 0{\cdot}16 \times 10^{-3}$ (see eqn (31b) with $\mathcal{F}_{1,\mathrm{P}} = 0$).

It is not necessary to emphasize the pedagogic nature of this digression; several important points in the actual design were ignored and the real problem is much more complex than the overview given here. The intention was just to stress the possible flexibility introduced in the design when one can change the geometric configuration of the pontoon's cross-section.

7 CONCLUSION

In this work an extension of the Morison formula, valid over the whole range of frequencies, was derived for a relatively deep submerged cylinder. As should be expected the high frequency response is strongly affected by the cross-section geometry, in contrast to the low frequency response where only the size of the section has importance. A TLP has a relatively short natural period in the vertical modes and so its resonance response should be influenced by the form of the pontoons. This was the practical motivation of the work.

The obtained expression may be useful in an initial phase of the design and a possible criterion for optimization was proposed in the present paper: the function of form $|F(Ka)|$ should be minimum. One could then derive some simple conclusions (for example, nonsymmetric sections are preferable) or even to address the crucial design problem: the geometric synthesis of a desirable response.

It should be emphasized, however, that some aspects of the problem have been intentionally ignored. On the one hand the analysis assumed a two-dimensional flow and, in this way, the hydrodynamic interaction between the pontoons (and columns also) was not considered here; on the other hand, important second-order effects, as the one discussed in Section 6, were not studied in depth. It does not seem difficult to extend the analysis to a full three-dimensional problem or to use the derived solution to estimate the second-order effect. The final result, however, will likely be much more complex than the one obtained here and of questionable importance for the geometric synthesis of the structure.

In spite of the above mentioned limitations it is felt that the proposed expression can be useful for the purpose it aims at: to help define the geometry of an ocean structure. For a strict analysis of a specific geometry one must resort to more sophisticated mathematical models or even to an experimental investigation.

APPENDIX

From the convergence of the power series (eqn (1)) when $|z| \geq \mu c$, $\mu > 1$, one obtains that there exists an M, independent of n, for which $|B_n| < M(\mu c)^n$.[8] From eqn (4) it follows then that $|D_n| < M/a(\mu c/a)^n$ and from eqn (9) one has

$$|F(z)| = \left|\sum_{n=0}^{\infty}(-1)^n \frac{D_{n+1}}{D_1}\frac{z^n}{n!}\right| \leq \sum_{n=0}^{\infty}\left|\frac{D_{n+1}}{D_1}\right|\frac{|z^n|}{n!}$$

$$< \frac{\mu Mc}{D_1 a^2}\sum_{n=0}^{\infty}\frac{(\mu c|z|/a)^n}{n!} = \frac{\mu Mc}{D_1 a^2}\exp\left[\mu c|z|/a\right]$$

The above inequality shows that $F(z)$ is analytic in the whole complex plane.

The function of form $F(Ka)$ depends, to some extent, on the position of the origin of the coordinated system used. In fact, since the eqns (16) and (18) must be invariant under a translation of the coordinated axis, then one must have

$$F_1(Ka) = e^{-Ka\Delta y}.F(Ka) \qquad \text{(A.1)}$$

where $F_1(Ka)$ is the function of form when the origin is at $0_1 = (0; \Delta y)$.

To prove eqn (A.1) one should start with the series expansion in eqn (1). If $D_n(\Delta y)$ are the coefficients of $f(z)$ in relation to the origin 0_1 and $z_1 = z - i\Delta y$ is the related complex variable, then one can place $z = z_1 + i\Delta y$ in eqn (1) and expand the parcels $(1 + i\Delta y/z_1)^{-n}$ in a power series of $i\Delta y/z_1$. From the equalities of the terms that multiply z_1^{-n} the following identity can be derived:

$$D_{n+1}(\Delta y) = \sum_{m=0}^{n} D_{m+1}\frac{n!}{m!(n-m)!}(\Delta y)^{n-m}$$

Since $D_1(\Delta y) = D_1$ from the definition (eqn (9)) one has

$$F_1(Ka) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^n\frac{(Ka)^n(\Delta y)^{n-m}}{m!(n-m)!}\frac{D_{m+1}}{D_1}$$

or

$$F_1(Ka) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(-1)^n\frac{(Ka)^n(\Delta y)^{n-m}}{m!(n-m)!}\frac{D_{m+1}}{D_1}$$

If $p = n - m$, one obtains

$$F_1(Ka) = \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{p+m}\frac{(Ka)^{p+m}(\Delta y)^p}{m!p!}\frac{D_{m+1}}{D_1}$$

which proves eqn (A.1) with the help of eqn (9).

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