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Existence and nonexistence of Puiseux inverse integrating factors in analytic monodromic singularities

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Abstract

In this work, we present some criteria about the existence and nonexistence of both Puiseux inverse integrating factors V and Puiseux first integrals H for planar analytic vector fields having a monodromic singularity. These functions are a wide generalization of their formal $\mathbb{R}[[x,y]]$ or algebraic counterpart in Cartesian coordinates (x,y). We prove that none of the functions H and V can be used to characterize degenerate centers although the existence of H is a sufficient center condition.

KEYWORDS

center-focus problem, first integral, inverse integrating factor

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34C05, 34A05, 37G15, 37G10

1 | INTRODUCTION

In this work, we study families of real analytic planar autonomous differential systems

$$\dot{x} = P(x,y;\lambda), \quad \dot{y} = Q(x,y;\lambda), \tag{1}$$

where the dot denotes derivative with respect to an independent time variable. We also will assume that (1) depends analytically on the parameters $\lambda \in \mathbb{R}^p$ of the family. Along the work, we denote by $\mathcal{X} = P(x, y; \lambda)\partial_x + Q(x, y; \lambda)\partial_y$ the vector field associated with (1). We are only interested in the restriction of family (1) to the monodromic parameter space $\Lambda \subset \mathbb{R}^p$ that is defined as the parameter subset for which $(x, y) = (0, 0)$ is a *monodromic singularity*. In other words, the origin is a singular point of (1) where the structure of local flow turns around it for any $\lambda \in \Lambda$. The monodromic set Λ is usually characterized by the blow-up process developed by Dumortier in [1], see also Arnold.[2] In the papers,[3,4] Algaba and coauthors present an algorithmic procedure that allows to specify the parameter conditions that defines Λ in terms of the Newton diagram $\mathbf{N}(\mathcal{X})$ of the vector field $\mathcal{X}$, see also Ref. 5.

Since the origin is monodromic, it possesses a Poincaré return map Π defined in some transversal section with end at the singularity. In [6], Medvedeva shows that the Poincaré map has a Dulac asymptotic expansion $\Pi(x) = \eta x + \sum_j P_j(\log x)x^{\nu_j}$ with $\eta > 0$, $\nu_j > 1$ grow to infinity and P_j have coefficients that are analytic functions of the coefficients of $\mathcal{X}$. We note that the center case corresponds to $\Pi(x) = x$ but it is very difficult to arrive at this identity map since the computation of η and the P_j is a hard problem. So, one needs other tools trying to characterize the subset of Λ defining a center at the origin of (1) and this work explores a possible tool.

Inverse integrating factor $\mathcal{V}(x, y)$ has been widely used in the context of the center-focus problem. For example, in [7], the structure of the Poincaré map of a degenerate monodromic singularity with Puiseux inverse integrating factor was studied. A survey about these functions $\mathcal{V}(x, y)$ was presented in [8], where a thorough overview of its main properties and applications is stated such as: the existence of limit cycles of $\mathcal{X}$ in [9] and the location of polycycles and other limit sets of the flow in [10, 11]. Moreover, in [12], it was proved that any (possibly degenerate) center of an analytic planar system admits a smooth (possibly flat) inverse integrating factor. We emphasize that this kind of inverse integrating factors cannot be used to discern between a center or a focus when it vanishes at the monodromic singularity, even worse we cannot construct a power series method when it is flat at the singularity.

In [13], the authors study the degenerate center-focus problem with characteristic directions assuming that system (1) possesses an inverse integrating factor $\mathcal{V}(x, y)$ well defined in a neighborhood of the origin. All these results exemplify the importance of studying inverse integrating factors.

We are interested in deepening knowledge of methods based on series with exponents in $\mathbb{N}$ (formal series) or in $\mathbb{Z}$ (Laurent series) or in $\mathbb{Q}$ (Puiseux series) applied to the center problem of $\mathcal{X}$. By the classical Poincaré–Lyapunov center theorem, any analytic family $\mathcal{X} = (-y + \cdots)\partial_x + (x + \cdots)\partial_y$ where the dots are higher order terms has a nondegenerate monodromic singular point at the origin and the center case is characterized by the existence of an analytic first integral $\mathcal{H}(x, y) = (x^2 + y^2)/2 + \cdots$, see, for instance, Refs. 14–18, and hence, using polar coordinates by a Puiseux first integral $H(\theta, \rho)$. On the other hand, using (p, q)-weighted polar coordinates, it can be seen that the existence of a Puiseux first integral $H(\theta, \rho)$ also picks up all the centers of the (p, q)-quasi-homogeneous vector fields, see Lemma 4.

Another nice example comes from the analytic family $\mathcal{X} = (y + \cdots)\partial_x + (\cdots)\partial_y$ where the dots are terms of at least second degree having a nilpotent monodromic singular point at the origin where the existence of an analytic first integral $\mathcal{H}(x, y) = y^2 + \cdots$ is not enough to characterize all its centers but when its Andreev number is 2, then they are characterized by the existence of a formal inverse integrating factor $\mathcal{V}(x, y) \in \mathbb{R}[[x, y]]$, hence using (1,2)-weighted polar coordinates by a Puiseux inverse integrating factor $V(\theta, \rho)$. See, for example, the works[19,20] for an account of these phenomena.

Based on the aforementioned center characterizations, our initial objective in this work was to know if we can use Puiseux first integrals or Puiseux inverse integrating factors to solve the center-focus problem. In section 5 of [7] there is an explanation about the origin of Puiseux inverse integrating factors coming from invariant surfaces of the Lagrange characteristic differential equations associated to the linear partial differential equation that defines inverse integrating factors of vector fields in polar coordinates.

More specifically, and focusing only in the use of Puiseux first integrals, we wanted to use convenient weighted polar coordinates (θ,ρ), see its definition in (3), and propose a function $H(\theta,\rho)=\sum_{i\geq 0} h_i(\theta)\rho^{\frac{m+i}{n}}$ with fixed $(m,n)\in\mathbb{Z}\times\mathbb{N}$ and 2π-periodic coefficients h_i as first integral step-by-step in such a way that the obstructions to the existence of the coefficients h_i yield center conditions at the origin of the vector field. This procedure works in the simple case of quasi-homogeneous vector fields, see Lemma 4(ii). Unfortunately, this is not the case for a general vector field because there are centers without Puiseux first integral as we will see in this work. Even worse, we will present some examples of focus without Puiseux inverse integrating factor because of the existence of flat terms in the transformation to Yakovenko's normal form,[21] see Remark 4. It is worth to mention that we have no example of center without a Puiseux inverse integrating factor, so we conjecture that any center must have a Puiseux inverse integrating factor. Indeed, for centers without characteristic directions, the conjecture is true, see Ref. 22. Nevertheless, despite the above conjecture was true, we cannot propose a function $V(\theta,\rho)=\sum_{i\geq 0} v_i(\theta)\rho^{\frac{m+i}{n}}$ with 2π-periodic coefficients v_i as inverse integrating factor step-by-step in order to get obstructions to the existence of the coefficients v_i yielding center conditions because there are focus with Puiseux inverse integrating factor, see Remark 5.

The main result of this work is the following one.

Theorem 1. *Let the origin of family (1) be a monodromic singularity. Then the center cases cannot be characterized either by the existence of a Puiseux first integral H or the existence of a Puiseux inverse integrating factor V. Nevertheless, the existence of H is a sufficient center condition.*

To prove this theorem, first we have obtained some criteria about the existence and nonexistence of $V(\theta,\rho)$ (respectively, $H(\theta,\rho)$), see, for example, Propositions 1 and 2 (respectively, Propositions 9 and 10). Through the work, many examples are introduced showing families (1) where specific members of it can have V or H but these functions are not defined in the whole family.

This work is structured as follows. First, in Section 2, we recall some definitions and preliminary background such as the weighted polar blow-up and its associated Puiseux inverse integrating factors and first integrals. Section 3 is dedicated to statement results about the existence and nonexistence of Puiseux inverse integrating factors V, while Section 4 does the same but with Puiseux first integrals H instead. We relegate to Section 5 the proofs of all the results. At the end, we also add several appendices needed in some parts of the work. We also mention that the computations done were performed utilizing the math software Mathematica 13[23] and Maple 2022.[24]

2 | SOME BACKGROUND AND PRELIMINARY DEFINITIONS

2.1 | The weighted polar blow-up

Let $\mathcal{X}=P(x,y)\partial_x+Q(x,y)\partial_y$ be the analytic vector field

$$P(x,y)=\sum_{(i,j)\in\mathbb{N}^2} a_{ij}x^i y^{j-1},\ Q(x,y)=\sum_{(i,j)\in\mathbb{N}^2} b_{ij}x^{i-1}y^j.$$

To introduce the weighted polar blow-up, we recall briefly the definition of the Newton diagram that will be useful in what follows.

The support of $\mathcal{X}$ is defined by $\operatorname{supp}(\mathcal{X}) = \{(i,j) \in \mathbb{N}^2 : (a_{ij}, b_{ij}) \neq (0,0)\}$. Moreover, the boundary of the convex hull of the set

$$\bigcup_{(i,j)\in \operatorname{supp}(\mathcal{X})} \{(i,j) + \mathbb{R}_+^2\},$$

where the set of positive real numbers is denoted by $\mathbb{R}_+$, consists of two open rays and several segments (or edges). The *Newton diagram* $\mathbf{N}(\mathcal{X})$ of the vector field $\mathcal{X}$ is defined as the union of this polygon and the rays that are not in a coordinate axis, when they exist. Choosing coprimes natural numbers p and q such that q/p determine the tangent of the angle between an edge of $\mathbf{N}(\mathcal{X})$ and the negative direction of the ordinate axis, we obtain the weights $(p,q) \in \mathbb{N}^2$ for which the expansion

$$\mathcal{X} = \sum_{j \geq r} \mathcal{X}_j, \tag{2}$$

where $r \geq 1$ and $\mathcal{X}_j = P_{p+j}(x,y)\partial_x + Q_{q+j}(x,y)\partial_y$ are (p,q)-quasi-homogeneous vector fields of degree j. We say that $\mathcal{X}_r$ is the *leading part* of $\mathcal{X}$. Moreover, the set of all the possible weights of $\mathcal{X}$ in $\mathbf{N}(\mathcal{X})$ is denoted by $W(\mathbf{N}(\mathcal{X})) \subset \mathbb{N}^2$.

Now we define the *weighted polar blow-up* $(x,y) \mapsto (\theta,\rho)$ as

$$x = \rho^p \cos\theta, \quad y = \rho^q \sin\theta, \tag{3}$$

and we consider the Jacobian $J(\theta,\rho) = \rho^{p+q-1}(p\cos^2\theta + q\sin^2\theta)$.

It is clear that when $p = q = 1$, we get the classic polar change of coordinates. In addition, after applying the weighted polar blow-up on the (p,q)-quasi-homogeneous vector field $\mathcal{X}_j = P_{p+j}(x,y)\partial_x + Q_{q+j}(x,y)\partial_y$ of degree j, we obtain

$$\dot{\rho} = \frac{\rho^{j+1}F_j(\theta)}{D(\theta)}, \quad \dot{\theta} = \frac{\rho^j G_j(\theta)}{D(\theta)}, \tag{4}$$

where

$$\begin{aligned} F_j(\theta) &= P_{p+j}(\cos\theta,\sin\theta)\cos\theta + Q_{q+j}(\cos\theta,\sin\theta)\sin\theta, \\ G_j(\theta) &= p\,Q_{p+j}(\cos\theta,\sin\theta)\cos\theta - q\,P_{q+j}(\cos\theta,\sin\theta)\sin\theta, \\ D(\theta) &= p\cos^2\theta + q\sin^2\theta > 0. \end{aligned}$$

Thus, doing a time reparametrization, we can remove the common factor $\rho^r/D(\theta) > 0$ of (4), obtaining

$$\dot{\rho} = \mathcal{R}(\theta,\rho) = \sum_{j\geq r} \rho^{j-r+1}F_j(\theta), \quad \dot{\theta} = \Theta(\theta,\rho) = \sum_{j\geq r} \rho^{j-r}G_j(\theta). \tag{5}$$

Recall that when $\mathcal{X}_r$ has not a monodromic singularity at the origin, then $G_r(\theta)$ has real roots in $[0,2\pi)$. Besides, the set $\{\rho = 0\}$ define a polycycle, because $\dot{\rho} = \mathcal{O}(\rho)$, $\dot{\theta} = G_r(\theta) + \mathcal{O}(\rho)$.

Now we recall the definition of the *cylinder*

$$C = \left\{(\theta,\rho) \in \mathbb{S}^1 \times \mathbb{R} \,:\, 0 \le \rho \ll 1\right\} \text{ with } \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}),$$

the *local curves of zero angular speed* $\Theta^{-1}(0) = \{(\theta,\rho) \in C \,:\, \Theta(\theta,\rho) = 0\}$, and of the set of *characteristics directions* of (5)

$$\Omega_{pq} = \{\theta^* \in \mathbb{S}^1 \,:\, G_r(\theta^*) = 0\}.$$

Note that if $\Omega_{pq} \neq \emptyset$, then $\Theta^{-1}(0) \neq \emptyset$. Besides, we can have $\Theta^{-1}(0)\backslash\{\rho = 0\} = \emptyset$. For more details, see Ref. 7. It is worth to emphasize that we can always take $0 \notin \Omega_{pq}$ performing a linear change of coordinates, if necessary.

Now we consider the ordinary differential equation associated to (5)

$$\frac{d\rho}{d\theta} = \frac{\mathcal{R}(\theta,\rho)}{\Theta(\theta,\rho)} = \mathcal{F}(\theta,\rho) = \sum_{i\ge 1} \mathcal{F}_i(\theta)\rho^i, \tag{6}$$

where we denote

$$\mathcal{F}_1(\theta) = F_r(\theta)/G_r(\theta). \tag{7}$$

Thus, (6) is a function well defined in $C\backslash\Theta^{-1}(0)$. We also consider, for every weight $(p,q) \in W(\mathbf{N}(\mathcal{X}))$, the following subset of Λ:

$$\Lambda_{pq} = \{\lambda \in \Lambda \,:\, \Theta^{-1}(0)\backslash\{\rho = 0\} \neq \emptyset\},$$

and the *critical parameters subset* $\Lambda^* \subset \Lambda$ as the intersection

$$\Lambda^* = \bigcap_{(p,q)\in W(\mathbf{N}(\mathcal{X}))} \Lambda_{pq}.$$

Definition 1. The *class* $\mathrm{Mo}^{(p,q)}$ *is formed by the vector field (2) whose leading part $\mathcal{X}_r$ with respect to some weights $(p,q) \in W(\mathbf{N}(\mathcal{X}))$ has a monodromic singularity at the origin.*

Note that $\mathcal{X} \in \mathrm{Mo}^{(p,q)}$ if, and only if, $G_r(\theta)$ is nonvanishing in $[0, 2\pi)$. Thus, if $\mathcal{X} \in \mathrm{Mo}^{(p,q)}$, then $\Theta^{-1}(0) = \emptyset$.

2.2 | Puiseux inverse integrating factors and Puiseux first integrals

In this section, we recall some important definitions and results useful in the sequence.

A not locally null real $C^1(C\backslash\Theta^{-1}(0))$ function $V(\theta,\rho)$ is an *inverse integrating factor* of (6) if the following linear partial differential equation is satisfied

$$\frac{\partial V}{\partial\theta}(\theta,\rho) + \frac{\partial V}{\partial\rho}(\theta,\rho)\,\mathcal{F}(\theta,\rho) = \frac{\partial \mathcal{F}}{\partial\rho}(\theta,\rho)\,V(\theta,\rho). \tag{8}$$

Moreover, if $V(\theta,\rho)$ can be expanded in a convergent Puiseux series about $\rho = 0$ (except may be at points $(\theta,\rho) = (\theta^*, 0)$ with $\theta^* \in \Omega_{pq}$) of the form

$$V(\theta,\rho) = \sum_{i\geq 0} v_i(\theta)\rho^{\frac{m+i}{n}}, \tag{9}$$

it is said to be a *Puiseux inverse integrating factor*. Here, the coefficients $v_i : \mathbb{S}^1\backslash\Omega_{pq} \to \mathbb{R}$ are C^1 functions. The leading coefficient $v_m(\theta) \not\equiv 0$ and $(m,n) \in \mathbb{Z}\times\mathbb{N}^*$ with $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$ are known as *multiplicity* and *index*, respectively. Moreover, when $n = 1$, we deal with a Laurent inverse integrating factor, and when $n = 1$ and $m \geq 0$, we obtain an analytic inverse integrating factor if v_i are analytic, for all $i \in \mathbb{N}$. If we do not care about convergence issues in the series (9) and the formal series still satisfies the partial differential equation term-by-term then we say that (9) is a *formal Puiseux inverse integrating factor*.

Remark 1. Recall that the inverse integrating factors of $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ are C^1 functions $\mathcal{V}(x,y)$ such that $(P(x,y)\,dy - Q(x,y)\,dx)/\mathcal{V}(x,y)$ is a closed differential 1-form off the zero set $v^{-1}(0)$. Moreover, given $\mathcal{V}(x,y)$ we can construct an inverse integrating factor of (6) in $C\backslash\{\Theta^{-1}(0)\cup\{\rho=0\}\}$ explicitly given by

$$V(\theta,\rho) = \frac{\mathcal{V}(\rho^p\cos\theta, \rho^q\sin\theta)}{J(\theta,\rho)\,\Theta(\theta,\rho)\,\rho^r}. \tag{10}$$

We recall the next example provided in [7].

Example 1. The equation $d\rho/d\theta = \rho^3$ possesses the inverse integrating factor $V(\theta,\rho) = \rho^3\,\sin(2\theta+\rho^{-2})$ that is a $C^1(C\backslash\{\rho=0\})$ function. Still, it is not a Puiseux inverse integrating factor since it cannot be expanded in the Puiseux series about $\rho = 0$.

We say that a nonconstant real $C^1(C\backslash\Theta^{-1}(0))$ function $H(\theta,\rho)$ is a *first integral* of (6) if

$$\frac{\partial H}{\partial\theta}(\theta,\rho) + \frac{\partial H}{\partial\rho}(\theta,\rho)\,\mathcal{F}(\theta,\rho) \equiv 0. \tag{11}$$

Analogous to what we have defined for inverse integrating factors, when a first integral of (6) could be expanded in a convergent Puiseux series about $\rho = 0$ (except may be at points $(\theta,\rho) = (\theta^*, 0)$ with $\theta^* \in \Omega_{pq}$), we say that it is a *Puiseux first integral* of (6).

We recall two well-known relations between first integrals and inverse integrating factors that will be useful in the sequel. Let $V_i(\theta,\rho)$, $i = 1,2$, and $H(\theta,\rho)$ be two Puiseux inverse integrating factors and a Puisex first integral of (6), respectively. Then $\widetilde{H}(\theta,\rho) = V_1(\theta,\rho)/V_2(\theta,\rho)$ is a first integral of (6), that is well defined for all (θ,ρ) for which $V_2(\theta,\rho) \neq 0$. Moreover, $\widetilde{V}(\theta,\rho) = H(\theta,\rho)V_1(\theta,\rho)$ is an inverse integrating factor of (6). For more details, see the survey[8] and references therein.

In what follows we present the definition of an important quantity. To do that we consider the expansion of $\mathcal{F}(\theta)$ given by (6), in $C\backslash\Theta^{-1}(0)$, where $\mathcal{F}_1(\theta)$ is given by (7). So, we define

$$\xi_{pq} = PV\int_0^{2\pi} \mathcal{F}_1(\theta)\,d\theta,$$

where PV denotes the Cauchy principal value

$$PV\int_I f(\theta)d\theta = \lim_{\epsilon\to 0^+}\int_{I_\epsilon} f(\theta)d\theta,$$

with $I_\epsilon = I \setminus J_\epsilon$ and $J_\epsilon = \cup_{i=1}^{l}(\theta_i^* - \epsilon, \theta_i^* + \epsilon)$. It is clear that it cannot exist.

As we saw before, if $\mathcal{X} \in \mathrm{Mo}^{(p,q)}$, then $\Theta^{-1}(0) = \emptyset$. Besides, when the origin is a center of $\mathcal{X} \in \mathrm{Mo}^{(p,q)}$, then it is also a center of its leading part $\mathcal{X}_r$ (see the reciprocal of Theorem 5, in [25], for example). Moreover, a necessary and sufficient condition for the origin to be a center of $\mathcal{X}_r$ is that $\int_0^{2\pi} \mathcal{F}_1(\theta)\,d\theta = 0$, for more details see Ref. 26, for instance. These information imply that $\int_0^{2\pi} \mathcal{F}_1(\theta)\,d\theta = 0$ is a necessary center condition at the origin of $\mathcal{X} \in \mathrm{Mo}^{(p,q)}$.

3 | EXISTENCE AND NONEXISTENCE OF PUISEUX INVERSE INTEGRATING FACTORS

In this section, we present some conditions related to the existence of a formal Puiseux inverse integrating factor.

Inverse integrating factors $\mathcal{V}(x,y)$ in Cartesian coordinates are defined as C^1 functions such that the rescaled vector field $\mathcal{X}/\mathcal{V}$ is Hamiltonian, see also Remark 1. We say that $\mathcal{V}$ is a formal inverse integrating factor when $\mathcal{V} \in \mathbb{R}[[x,y]]$ and that $\mathcal{V}$ is an algebraic inverse integrating factor in case that $\mathcal{V}(x,y) = f^\lambda(x,y)$ where $f \in \mathbb{R}[[x,y]]$ and $\lambda \in \mathbb{Q}$.

Due to formula (10), it is clear that the existence of a formal or algebraic inverse integrating factor $\mathcal{V}(x,y)$ implies the existence of a Puiseux inverse integrating factor $V(\theta,\rho)$, which justifies why we analyze the functions $V(\theta,\rho)$ in this work. The following example explicitly shows that $V(\theta,\rho)$ is a more general object than just formal $\mathcal{V}(x,y)$.

Example 2. System

$$\dot{x} = y + x^4, \quad \dot{y} = -x^5 + 3x^3y, \tag{12}$$

has a nilpotent time-reversible center at the origin. Moreover, in [27], it was proved that (12) does not admit any formal inverse integrating factor in the Cartesian coordinates. However, taking the unique weights $(p,q) = (1,3) \in W(\mathbf{N}(\mathcal{X}))$ and doing the weighted polar blow-up (3), we obtain an equation (5) of the form

$$\frac{d\rho}{d\theta} = \frac{\rho\cos^2(\theta)\Big(\tan(\theta) - \sin(\theta)\cos^3(\theta) + \rho\cos^3(\theta) + 3\rho\sin^2(\theta)\cos(\theta)\Big)}{-3\sin^2(\theta) - \cos^6(\theta)}$$

having the Puiseux inverse integrating factor $V(\theta,\rho) = v_2(\theta)\rho^2$ with $v_2(\theta) = (-33\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta) + 58)^{1/6}$.

When ξ_{pq} does not exist, we can establish the following result.

Proposition 1. *We take $0 \notin \Omega_{pq}$. If ξ_{pq} does not exist, then any Puiseux inverse integrating factor must have $m = n$.*

Remark 2. There are monodromic singularities where ξ_{pq} does not exist for some $(p,q) \in W(\mathbf{N}(\mathcal{X}))$. An example with $(p,q) = (1,1)$ is given in [28] where it is shown that the origin of the following family

$$\dot{x} = y(\alpha x^2 + bxy + cy^2), \;\; \dot{y} = y^2(\alpha x + by) + x^5, \tag{13}$$

with $\Lambda = \{(\alpha, b, c) \in \mathbb{R}^4 \,:\, \alpha < 0, c < 0\}$ is monodromic and

$$\xi_{11} = -\frac{4b}{c} \lim_{\varepsilon \to 0^+} \cot(\varepsilon).$$

Hence, ξ_{11} does not exist when $b \neq 0$. Thus, an eventual inverse Puiseux inverse integrating factor of (13) would have $m = n$ according to Proposition 1.

The following lemma is a direct consequence of Lemma 2 of [7] and its proof. We state it here because it will be useful in some examples.

Lemma 1. *Assume that there exists a Puiseux inverse integrating factor $V(\theta,\rho) = v_m(\theta)\rho^{m/n} + \cdots$, with $m \neq n$. Also assume that the restriction $v_m|_{\Omega_{pq}}$ takes finite values and that the possible zero values have finite multiplicity. Then, ξ_{pq} exists and its value is $\xi_{pq} = 0$.*

Now we give some necessary conditions for Equation (6) to have a formal Puiseux inverse integrating factor.

Proposition 2. *Consider system (1) and the associated ordinary differential equation (6) with $0 \notin \Omega_{pq}$ without lost of generality. If there exists a formal Puiseux inverse integrating factor $V(\theta,\rho) = \sum_{k\geq 0} v_{m+k}(\theta)\rho^{(m+k)/n}$ of (6), then its coefficients satisfy the linear differential equation*

$$v'_{m+k}(\theta) = \alpha_{m+k}\mathcal{F}_1(\theta)v_{m+k}(\theta) + b_{m+k}(\theta), \tag{14}$$

in $\mathbb{S}^1 \backslash \Omega_{pq}$, where $\alpha_{m+k} \in \mathbb{R}$ and $b_{m+k} : \mathbb{S}^1 \backslash \Omega_{pq} \to \mathbb{R}$, for any $k \geq 0$. Moreover, assuming that both ξ_{pq} and $\Lambda_{m+k} := PV \int_0^{2\pi} \Phi^{-1}_{m+k}(\theta) b_{m+k}(\theta) d\theta$, with $\Phi_{m+k}(\theta) = \exp(\alpha_{m+k}\, PV \int_0^{\theta} \mathcal{F}_1(\sigma) d\sigma)$, exist for all $k \geq 0$, then $(m-n)\xi_{pq} = 0$ and the following statements hold:

(i) If $\xi_{pq} \neq 0$, then $m = n$ and for each $n \in \mathbb{N}$, the formal Puiseux inverse integrating factor is unique up to multiplicative nonzero constants.
(ii) If $\xi_{pq} = \Lambda_{m+k} = 0$ for any $k \geq 0$, then there can be infinitely many formal Puiseux inverse integrating factors.
(iii) If $\xi_{pq} = 0$ and $\Lambda_{m+k} \neq 0$ for some $k \geq 0$, then there is no Puiseux inverse integrating factor.

With the following example, we emphasize that the condition $m \neq n$ in Lemma 1 is necessary.

Example 3. Consider the family of vector fields

$$\begin{aligned} \dot{x} &= \lambda_1(x^6 + 3y^2)(-y + \mu x) + \lambda_2(x^2 + y^2)(y + Ax^3), \\ \dot{y} &= \lambda_1(x^6 + 3y^2)(x + \mu y) + \lambda_2(x^2 + y^2)(-x^5 + 3Ax^2 y). \end{aligned} \tag{15}$$

In [7], it is proved that the origin is a monodromic singularity if, and only if, the parameters belong to the set

$$\Lambda = \{(\lambda_1, \lambda_2, \mu, A) \in \mathbb{R}^4 \,:\, 3\lambda_1 - \lambda_2 > 0,\ \ \lambda_1 - \lambda_2 > 0\}.$$

Moreover, restricted to $\Lambda \setminus \Lambda^*$, where

$$\Lambda^* = \{\alpha_{11} \geq 0,\ \alpha_{13} \geq 0,\ A\lambda_2 + \sqrt{\alpha_{11}} \leq 0,\ -\mu\lambda_1 + \sqrt{\alpha_{13}} \geq 0\},$$

with $\alpha_{11} = -3\lambda_1^2 + 4\lambda_1\lambda_2 + (-1 + A^2)\lambda_2^2$ and $\alpha_{13} = (-3 + \mu^2)\lambda_1^2 + 4\lambda_1\lambda_2 - \lambda_2^2$, the center condition is

$$3\lambda_1\mu + \sqrt{3}A\lambda_2 = 0.$$

Since the weights $(p, q) = (1, 1)$ are on the Newton diagram of (15), we take polar coordinates and we see that $0 \in \Omega_{11}$. Therefore, we perform the linear change of coordinates $(x, y) \mapsto (y, x)$ in order that $0 \notin \Omega_{11}$. The outcome is that, in polar coordinates, $d\rho/d\theta = \mathcal{F}_1(\theta)\rho + \mathcal{O}(\rho^2)$ with $\mathcal{F}_1(\theta) = -(3\mu\lambda_1 + \lambda_2 \tan\theta)/(3\lambda_1 - \lambda_2)$. Then,

$$\xi_{11} = PV \int_0^{2\pi} \mathcal{F}_1(\theta) d\theta = -\frac{6\pi\mu\lambda_1}{3\lambda_1 - \lambda_2}.$$

On the other hand, family (15) has the inverse integrating factor $\hat{\mathcal{V}}(x, y) = (x^2 + y^2)(x^6 + 3y^2)$ that is modified as $\mathcal{V}(x, y) = \hat{\mathcal{V}}(y, x)$ yielding the transformed Puiseux inverse integrating factor

$$V(\theta, \rho) = \frac{3}{(\lambda_2 - 3\lambda_1)}\rho + \mathcal{O}(\rho^3).$$

This is an example of a family having a Puiseux inverse integrating factor V with $m = n = 1$ and $v_m(\theta)$ constant. Hence, it satisfies the conditions on $v_m|_{\Omega_{11}}$ imposed in Lemma 1, although ξ_{11} can be different from zero when $\mu\lambda_1 \neq 0$. Observe that this example does not contradict Lemma 1, because $m = n$.

We also want to note that, according to Proposition 2(i), family (15) with $\xi_{11} \neq 0$ cannot possess another formal Puiseux inverse integrating factor with index $n = 1$.

3.1 | Monodromic families without Puiseux inverse integrating factor

In this section, we show two examples of monodromic families without Puiseux inverse integrating factor. In Proposition 4, we give a family where ξ_{pq} does not exist for $(p, q) = (1, 1) \in W(\mathbf{N}(\mathcal{X}))$ and that $V(\theta, \rho)$ also does not exist for these weights. Moreover, in Proposition 5, we show a family of monodromic nilpotent vector fields where the focus does not have Puiseux inverse integrating factors. First, we established a preliminary result.

We define the quantity $\chi_{pq}(\theta)$ as the following Cauchy principal value:

$$\chi_{pq}(\theta) = PV \int_0^{\theta} \mathcal{F}_1(\sigma) d\sigma, \tag{16}$$

if it exists, that is, if its value is finite. Notice that $\chi_{pq}(2\pi) = \xi_{pq}$.

Proposition 3. *Assume that the origin is a monodromic singular point of (1) with $0 \notin \Omega_{pq}$ for some $(p,q) \in W(\mathbf{N}(\mathcal{X}))$. If there is $\hat{\theta} \in \mathbb{S}^1 \backslash \Omega_{pq}$ such that $(n-m)\chi_{pq}(\hat{\theta}) > 0$ and $\chi_{pq}(\hat{\theta}) = \pm\infty$, then there is no Puiseux inverse integrating factor with $m \neq n$.*

We are going to see how Proposition 3 can be used to show a family of analytic monodromic singularities without Puiseux inverse integrating factor using the weighted polar coordinates associated to one edge of $\mathbf{N}(\mathcal{X})$.

Proposition 4. *The Newton diagram of family (13) has associated weights $W(\mathbf{N}(\mathcal{X})) = \{(1,1),(2,1)\}$. Using polar coordinates, there is no Puiseux inverse integrating factor with arbitrary fixed (m,n) in the hole family (13) restricted to the monodromic parameter space $\Lambda = \{(\alpha, b, c) \in \mathbb{R}^4 : \alpha < 0, c < 0\}$.*

Remark 3. Recall that if $\mathcal{V}(x,y) \in \mathbb{R}[x,y]$ is an inverse integrating factor of $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$, then $V(\theta,\rho)$, given by (10), exists for all $(p,q) \in W(N(\mathcal{X}))$. See Remark 1. As in Proposition 4 $V(\theta,\rho)$ does not exist for $(p,q) = (1,1) \in W(N(\mathcal{X}))$, it follows that (13) does not admit a formal Puiseux inverse integrating factor in the Cartesian coordinates.

Before the next proposition, we recall some results about nilpotent planar vector fields. First, any analytic planar vector field with a monodromic nilpotent singular point is orbitally analytically conjugate to the Liénard differential system

$$\dot{x} = -y, \ \dot{y} = x^{2N-1} + yB(x), \tag{17}$$

where $N \geq 2$ is called the Andreev number, $B(x) = \sum_{j \geq \beta} B_j x^j$ with $B_\beta \neq 0$, and one of the following conditions hold: (i) $\beta > N-1$; (ii) $\beta = N-1$ and $B_\beta^2 - 4N < 0$; (iii) $B(x) \equiv 0$. These conditions are easily deduced from the Andreev characterization of monodromic nilpotent singular points, see Ref. 29. Moreover, the origin of system (17) is a center if, and only if, $B(x)$ is an odd function, see Refs. 30, 31.

Proposition 5. *When $a \neq 0$, the family*

$$\dot{x} = y, \ \dot{y} = -x^5 + ax^4 y, \tag{18}$$

has a focus at the origin without Puiseux inverse integrating factor.

Remark 4. By a result in [21], in a neighborhood of $\rho = 0$ in the cylinder C, there is a C^∞ diffeomorphism $(\theta,\rho) \mapsto \phi(\theta,\rho) = (\varphi, r)$ mapping $\hat{\mathcal{X}} = \partial_\theta + \mathcal{F}(\theta,\rho)\partial_\rho$ with $\mathcal{F} \in C^\omega(C)$ into $\phi_* \hat{\mathcal{X}}$ with associated equation

$$\begin{aligned} \frac{dr}{d\varphi} &= \lambda r \quad \text{with } \lambda \neq 0, & &\text{if} \quad m = 1, \\ \frac{dr}{d\varphi} &= r^m + a\, r^{2m-1} \quad \text{with } a \in \mathbb{R}, & &\text{if} \quad m > 1, \end{aligned}$$

where m is the multiplicity of the limit cycle $\rho = 0$ of $\hat{\mathcal{X}}$. In particular,

$$\bar{V}(\varphi, r) = \begin{cases} r & \text{if} \quad m = 1, \\ r^m + a\, r^{2m-1} & \text{if} \quad m > 1, \end{cases}$$

is an inverse integrating factor of this equation and, undoing the change of coordinates, we have a smooth inverse integrating factor $V(\theta, \rho)$ of the differential equation (6) associated to $\hat{\mathcal{X}}$. Clearly, if ϕ is nonflat in ρ at $\rho = 0$, then $V(\theta, \rho)$ would be smooth and nonflat. Consequently, $V(\theta, \rho)$ would be a Puiseux inverse integrating factor with index 1 and positive multiplicity. This consequence is in contradiction with the example provided in Proposition 5 that has a nilpotent focus at the origin when $a \neq 0$ with Andreev number $N = 3$ so that using $(1, N)$-quasi-homogeneous weighted polar coordinates, one gets $\Omega_{1N} = \emptyset$ and consequently the associated vector field $\hat{\mathcal{X}}$ is analytic in C. In summary, a straight consequence of Proposition 5 is that Lemma 21 of [32] and its by-product written in statement (a) of Theorem 1 in [22] are wrong and the main reason is that the C^∞ diffeomorphism ϕ mapping $\hat{\mathcal{X}}$ into its normal form is given by a formal power series with possibly flat terms not taken into account in [32].

4 | EXISTENCE AND NONEXISTENCE OF PUISEUX FIRST INTEGRALS

The next lemma is used in the proof of the forthcoming Proposition 6. It was stated for the first time in Lemma 18(iii) of [7] with a proof in the particular case that the Puiseux inverse integrating factor $V(\theta, \rho)$ is well defined in a neighborhood of $\rho = 0$, that is, when the coefficients $v_i(\theta)$ in the expansion (9) are functions well defined in $\mathbb{S}^1$. The proof of this lemma in the general case when $v_i(\theta)$ are functions only defined in $\mathbb{S}^1 \backslash \Omega_{pq}$ is given in [33].

Lemma 2. *Let the origin be a monodromic singularity of system (1). If $V(\theta, \rho)$ is a Puiseux inverse integrating factor of its associated differential equation (6), then $V^{-1}(0) \backslash \{\rho = 0\} = \emptyset$.*

When (6) has a Puiseux first integral $H(\theta, \rho)$, then it also possesses a Puiseux inverse integrating factor $V(\theta, \rho)$ as it can be easily checked from the relations $V = 1/(\partial_\rho H) = -\mathcal{F}/(\partial_\theta H)$. In general, the converse is not true as the example (19) in Section 4.1 shows. Anyway now we present a specific situation where that converse holds.

Proposition 6. *Let the origin be a center of system (1) restricted to $\Lambda \backslash \Lambda_{pq}$. Assume that $V(\theta, \rho)$ is a Puiseux inverse integrating factor of its associated differential equation (6) well defined in $C \backslash \{\rho = 0\}$. Then, (6) admits a Puiseux first integral $H(\theta, \rho)$.*

There are monodromic families where the centers are characterized by the existence of a Puiseux first integral, see, for example, Proposition 8, although this is not always the case. The next lemma gives a sufficient center condition.

Lemma 3. *If system (1) has a monodromic singularity at the origin and $H(\theta, \rho)$ is a Puiseux first integral of its associated differential equation (6), then the origin is a center.*

In general, the converse of Lemma 3 is not true. Indeed, in the next proposition, we show that family (15) contains centers without Puiseux first integrals.

Proposition 7. *There are centers in the family (15) without a Puiseux first integral.*

Remark 5. There are families of vector fields $\mathcal{X} = \mathcal{X}_r + \cdots$ with a monodromic singularity at the origin whose members have both center and foci and the origin is a center of $\mathcal{X}_r$ for which there is a Puiseux inverse integrating factor in the complete family. An example of it is the family (19) in the work,[13] where $\mathcal{X}_r$ is Hamiltonian.

Remark 6. Recall that the monodromic class $\mathrm{Mo}^{(p,q)}$ is defined as the set of analytic vector fields $\mathcal{X}$ with a monodromic singularity at the origin such that its associated Newton diagram has only one edge with weights $(p,q) \in \mathbb{N}^2$ and $\Omega_{pq} = \emptyset$, see Definition 1. This is the simplest monodromic class and it contains all the analytic vector fields with monodromic singularities that are either nondegenerate or nilpotent. Clearly, the Poincaré map Π of a vector field in the monodromic class $\mathrm{Mo}^{(p,q)}$ is an analytic map at the origin that could be write as $\Pi(\rho_0) = \eta\rho_0 + \mathcal{O}(\rho_0)$. It is well known that $\eta = \exp(\pm\xi_{pq})$, where the positive sign is taken when the flow rotates counterclockwise and the negative sign otherwise. It is important to mention that when we restrict to vector fields lying in $\mathrm{Mo}^{(p,q)}$, Proposition 2 can be viewed as a generalization of statements (i) and (ii), of Corollary 17, in [32], taking into account that analytic inverse integrating factors in Cartesian coordinates become Puiseux inverse integrating factors in (p,q)–weighted polar coordinates.

4.1 | Monodromic family having focus without Puiseux first integral

In this section, we show an example of a monodromic family that admits a Puiseux first integral if, and only if, the center condition is satisfied.

Consider the system

$$\dot{x} = y^3 + 2ax^3y + 2x(\alpha x^4 + \beta xy^2),\ \dot{y} = -x^5 - 3ax^2y^2 + 3y(\alpha x^4 + \beta xy^2), \tag{19}$$

previously studied in [13, 34]. Note that this system is in the class $\mathrm{Mo}^{(2,3)}$ when the monodromic condition $|a| < 1/\sqrt{6}$ is satisfied. Moreover, the origin is a center when the parameters are in the center parameter space $C = \{(a,\alpha,\beta) \in \mathbb{R}^3 : |a| < 1/\sqrt{6},\ \alpha f(a) + \beta g(a) = 0\}$, where

$$f(a) = -\int_0^{2\pi} \frac{\cos^4\theta(2\cos^2\theta + 3\sin^2\theta)}{(\hat{D}(\theta;a))^{13/12}} d\theta,$$

$$g(a) = -\int_0^{2\pi} \frac{\cos\theta\sin^2\theta(2\cos^2\theta + 3\sin^2\theta)}{(\hat{D}(\theta;a))^{13/12}} d\theta,$$

$$\hat{D}(\theta;a) = 2\cos^6\theta + 12a\cos^3\theta\sin^2\theta + 3\sin^4\theta.$$

Family (19) possesses an algebraic inverse integrating factor given by $\mathcal{V}(x,y) = [(x^3 + 3ay^2)^2 + \frac{3}{2}(1-6a^2)y^4]^{13/12}$ independently of the monodromic nature of the origin. Using (10) with $(p,q) = (2,3)$ and $J(\theta,\rho) = -\rho^4(-5 + \cos 2\theta)/2$ yields $V(\theta,\rho) = \rho^2(\hat{D}(\theta;a))^{1/12}$. Nevertheless, family (19) only has a Puiseux first integral in the center case as the following proposition shows.

Proposition 8. *A system in the family (19) admits a Puiseux first integral if, and only if, the origin is a center.*

5 | PROOFS

5.1 | Proof of Theorem 1

Proof. The fact that centers are not characterized by the existence of H is a consequence of Proposition 7, whereas the noncharacterization of centers by the existence of V follows from the coexistence of foci and V, see, for example, Lemma 4(i). The fact that the existence of H is a sufficient center condition is proved in Lemma 3. □

5.2 | Proof of Proposition 1

Proof. Inserting the expansion $V(\theta,\rho) = v_m(\theta)\rho^{m/n} + \mathcal{O}(\rho^{m/n})$ and $\mathcal{F}(\theta,\rho) = \mathcal{F}_1(\theta)\rho + \mathcal{O}(\rho)$ into the partial differential equation (8) gives the linear homogeneous differential equation

$$n\, v_m'(\theta) + (m-n)\mathcal{F}_1(\theta)v_m(\theta) = 0, \tag{20}$$

whose general solution is

$$v_m(\theta) = v_m(0)\exp\left(\frac{n-m}{n} PV \int_0^\theta \mathcal{F}_1(s)\,ds\right). \tag{21}$$

Evaluating at 2π gives

$$v_m(2\pi) = v_m(0)\exp\left(\frac{n-m}{n}\xi_{pq}\right),$$

if ξ_{pq} would exists. Using that v_m is a 2π-periodic function together with the nonexistence of ξ_{pq}, we must have $m = n$. □

5.3 | Proof of Proposition 2

Proof. We follow the same first step of the proof of Proposition 1, introducing the expansion

$$V(\theta,\rho) = \sum_{i=m}^{m+4n} v_i(\theta)\rho^{i/n} + \mathcal{O}(\rho^{(m+4n)/n}), \tag{22}$$

together with $\mathcal{F}(\theta,\rho) = \sum_{i\geq 1}^{4} \mathcal{F}_i(\theta)\rho^i + \mathcal{O}(\rho^5)$, into the partial differential equation (8), which defines V, we obtain

$$\begin{aligned}&\sum_{i=m}^{m+4n} v_i'(\theta)\rho^{\frac{i}{n}} + \mathcal{O}(\rho^{\frac{m+4n}{n}}) - \left(\sum_{i=1}^{4} i\mathcal{F}_i(\theta)\rho^{i-1} + \mathcal{O}(\rho^4)\right)\left(\sum_{i=m}^{m+4n} v_i(\theta)\rho^{\frac{i}{n}} + \mathcal{O}\left(\rho^{\frac{m+4n}{n}}\right)\right)\\ &+ \frac{\sum_{i=1}^{4}\mathcal{F}_i(\theta)\rho^i + \mathcal{O}(\rho^5)}{n}\left(\sum_{i=m}^{m+4n} i v_i(\theta)\rho^{\frac{i-n}{n}} + \mathcal{O}\left(\rho^{\frac{m+3n}{n}}\right)\right) = 0.\end{aligned} \tag{23}$$

Equating the coefficients of the powers $\rho^{(m+j)/n}$, with $0 \leq j \leq n-1$, we see that $v_{m+j}(\theta)$ satisfies the linear ordinary differential equation

$$n\, v'_{m+j}(\theta) + (m-n+j)\mathcal{F}_1(\theta) v_{m+j}(\theta) = 0. \tag{24}$$

We point out that $v_m \not\equiv 0$, because it is the leading term of (22). Besides, when $j = 0$, in (24), we obtain (20).

Moreover, taking into account the next powers $\rho^{(m+n+j)/n}$, with $0 \leq j \leq n-1$, in (23), we obtain

$$n v'_{m+n+j}(\theta) + (m+j)\mathcal{F}_1(\theta) v_{m+n+j}(\theta) = (2n-m-j)\mathcal{F}_2(\theta) v_{m+j}(\theta). \tag{25}$$

When we consider the coefficients of $\rho^{(m+2n+j)/n}$, $0 \leq j \leq n-1$, we see that $v_{m+2n+j}(\theta)$ satisfies the linear ordinary differential equation

$$\begin{aligned} n\, v'_{m+2n+j}(\theta) + (m+n+j)\mathcal{F}_1(\theta) v_{m+2n+j}(\theta) = {} & (3n-m-j)\mathcal{F}_3(\theta) v_{m+j}(\theta) \\ & + (n-m-j)\mathcal{F}_2(\theta) v_{m+n+j}(\theta). \end{aligned} \tag{26}$$

In Appendix B, we show the ordinary differential equations that appear in the next four steps, because it will be useful in the examples.

Using (23), we also obtain

$$\alpha_{m+k} = \frac{n-m-k}{n}, \tag{27}$$

for all $k \in \mathbb{N} \cup \{0\}$. In particular, $\alpha_{m+\ell n} = \frac{n(1-\ell)-m}{n}$, for all $\ell \in \mathbb{N} \cup \{0\}$. It is worth to mention that $\alpha_{m+k} = 0$ if, and only if, $m + k = n \in \mathbb{N}$. Moreover, $m = n$ occurs only when $k = 0$.

Suppose that $\xi_{pq} \neq 0$ and $m \neq n$ (hence $\alpha_m \neq 0$). Note that the linear ordinary differential equation (20) has the unique solution (21). So, the unique 2π-periodic solution of (20) must be $v_m(\theta) \equiv 0$, which is a contradiction since v_m is the leading coefficient of V. Therefore,

$$(m-n)\xi_{pq} = 0$$

is a necessary condition for the existence of a Puiseux inverse integrating factor. In particular, when $m = n$, $v_m(\theta) = v_m(0)$ is constant.

Now, assuming the existence of both $\Phi_{m+k}(\theta)$ and Λ_{m+k}, from the variation of constants formula applied to the linear differential equation (14), we obtain

$$v_{m+k}(2\pi) = \Phi_{m+k}(2\pi) v_{m+k}(0) + \Phi_{m+k}(2\pi)\, \Lambda_{m+k}.$$

Recall that v_{m+k} are functions well defined in $\mathbb{S}^1 \backslash \Omega_{pq}$, so $v_{m+k}(2\pi)$ and $v_{m+k}(0)$ are real numbers. The periodicity condition $v_{m+k}(2\pi) = v_{m+k}(0)$ holds if, and only if, the initial condition $v_{m+k}(0)$ is a solution of the algebraic linear system

$$(1 - \Phi_{m+k}(2\pi)) v_{m+k}(0) = \Phi_{m+k}(2\pi)\, \Lambda_{m+k}. \tag{28}$$

We start with the simplest analysis. If $\xi_{pq} = 0$, then $\Phi_{m+k}(2\pi) = 1$. So, using (28), either all the solutions of equation (14) or none of them are 2π–periodic according to whether Λ_{m+k} is zero or not, respectively. This proves statements (ii) and (iii).

In the case $\xi_{pq} \neq 0$, we have $m = n$. Then, from (27), $\alpha_{m+k} = -k/n$. So, $\alpha_m = 0$ and $\alpha_{m+k} \neq 0$ for all $k > 0$. Using (24), for $0 < k < n-1$, and that v_{m+k} is 2π-periodic, we conclude that $v_{m+k} \equiv 0$ when $\xi_{pq} \neq 0$. Repeating the process, using (25) and that $v_{m+k} \equiv 0$, for $1 \leq k \leq n-1$, yields that all the coefficient functions $v_{m+n+k}(\theta) \equiv 0$. Going further in this recursively process, we conclude that $v_{m+k} \equiv 0$, for all $k \neq \ell n$ with $\ell \in \mathbb{N}$. To finish, by (28), $v_{m+n\ell}(\theta)$ are also 2π-periodic functions if, and only if, $v_{m+\ell n}(0) = \Phi_{m+\ell n}(2\pi)\Lambda_{m+\ell n}/(1 - \Phi_{m+\ell n}(2\pi))$. So, $v_{m+\ell n}(\theta)$ is the unique solution of (14) with the former initial condition. Thus, the statement (i) follows.

We point out that the statements are still valid when $\mathcal{F}_1 \equiv 0$. In this case, $\xi_{pq} = 0$, which provides directly that $(m - n)\xi_{pq} = 0$. Moreover, for all $k \geq 0$, we have $\Phi_{m+k}(\theta) \equiv 1$ and $\Lambda_{m+k} := PV \int_0^{2\pi} b_{m+k}(\theta) d\theta$. Thus, (28) is simply $\Lambda_{m+k} = 0$. □

5.4 | Proof of Proposition 3

Proof. If $v_m(\theta)$ would exists and $0 \notin \Omega_{pq}$, we can put $v_m(0) = 1$, without loss of generality. Indeed, $v_m(\theta)$ must satisfy the linear Cauchy problem

$$nG_r(\theta)v_m'(\theta) = (n - m)v_m(\theta)F_r(\theta), \quad v_m(0) = 1, \tag{29}$$

where $\mathcal{F}_1(\theta) = F_r(\theta)/G_r(\theta)$. See Equation (20). It may occur that the solution $v_m(\theta)$ of the Cauchy problem (29) be divergent at $\theta \in \Omega_{pq}$.

For any $\theta \in \mathbb{S}^1$ and $\varepsilon > 0$ sufficiently small, we consider the interval $I_\varepsilon(\theta) = [0, \theta] \backslash \cup_{i=1}^{\ell} (\theta_i^* - \varepsilon, \theta_i^* + \varepsilon)$ where the set of characteristic directions is assumed to be $\Omega_{pq} = \{\theta_1^*, \dots, \theta_\ell^*\}$. Integrating (29) over $I_\varepsilon(\theta)$ and taking the limit when $\varepsilon \to 0^+$, we obtain

$$\log |v_m(\theta)| = PV \int_0^\theta \frac{v_m'(\theta)}{v_m(\theta)} d\theta = \frac{n-m}{n} PV \int_0^\theta \frac{F_r(\theta)}{G_r(\theta)} d\theta = \frac{n-m}{n} \chi_{pq}(\theta),$$

for those $\theta \notin \Omega_{pq}$ where $\chi_{pq}(\theta)$ exists. In particular, defining $\alpha_m = (n - m)/n$ and assuming $m \neq n$, we get that $|v_m(\theta)| = \exp\left(\alpha_m \chi_{pq}(\theta)\right)$ when $\theta \notin \Omega_{pq}$ and $\chi_{pq}(\theta)$ exists. As a by-product, if there is $\hat{\theta} \in \mathbb{S}^1 \backslash \Omega_{pq}$ such that $\alpha_m \chi_{pq}(\hat{\theta}) > 0$ and $\chi_{pq}(\hat{\theta}) = \pm\infty$, then $v_m(\theta)$ is not defined in $\mathbb{S}^1 \backslash \Omega_{pq}$. Consequently, $V(\theta, \rho)$ does not exist, proving the proposition. □

5.5 | Proof of Proposition 4

Proof. Family (13) restricted to $\Lambda = \{(\alpha, b, c) \in \mathbb{R}^4 : \alpha < 0, c < 0\}$ has $W(\mathbf{N}(\mathcal{X})) = \{(1,1), (2,1)\}$. We only perform an analysis taking the weights $(p, q) = (1, 1)$ for which $0 \in \Omega_{11}$. Therefore, interchanging the variables $(x, y) \mapsto (y, x)$ the system becomes

$$\dot{x} = x^2(\alpha y + bx) + y^5, \quad \dot{y} = x(\alpha y^2 + bxy + cx^2). \tag{30}$$

After the polar changing of coordinates, we can see that $0 \notin \Omega_{11} = \{\pi/2, 3\pi/2\}$. Some computations give that

$$\mathcal{F}_1(\theta) = \tan\theta + \frac{1}{c}\sec^2\theta(b + \alpha\tan\theta)$$

has the continuous primitive

$$P(\theta) = -\log(|\cos\theta|) + \frac{1}{c}\left(\frac{\alpha}{2}\sec^2\theta + b\tan\theta\right)$$

in $\mathbb{S}^1\backslash\Omega_{11}$ that can be used, according to (16), to compute

$$\chi_{11}(\theta) = \begin{cases} -\frac{\alpha}{2c} - \log(|\cos\theta|) + \frac{\alpha}{2c}\sec^2\theta + \frac{b}{c}\tan\theta & \text{if} \quad 0 \le \theta \le \pi/2, \\ \frac{b}{c}\,\infty & \text{if} \quad \pi/2 < \theta \le 2\pi. \end{cases}$$

In particular,

$$\chi_{11}(2\pi) = \frac{4b}{c}\lim_{\varepsilon\to 0^+}\cot(\varepsilon).$$

As $c < 0$, $\chi_{11}(2\pi) = -\infty$ if $b > 0$ and $\chi_{11}(2\pi) = \infty$ if $b < 0$. So, $(n-m)\chi_{11}(2\pi) = \infty$ when $b(n-m) < 0$. In consequence, the hole family in Λ cannot possess a Puiseux inverse integrating factor with fixed (m, n) when $m \ne n$, according to Proposition 3 with $\hat{\theta} = 2\pi$.

We continue the analysis when $m = n$ in which case $v_n(\theta) = 1$ and, since $\mathcal{F}_2(\theta) \equiv 0$, one has

$$v_{2n}(\theta) = \Phi_{2n}(\theta)v_{2n}(0),$$

with $\Phi_{2n}(\theta) = \exp(-\chi_{11}(\theta))$ provided the principal value exists. We continue assuming that $b > 0$ so that $\Phi_{2n}(2\pi) = \infty$. Hence, one needs to take $v_{2n}(0) = 0$ and therefore $v_{2n}(\theta) \equiv 0$. Then, $v_{3n}(\theta)$ adopts the form

$$v_{3n}(\theta) = \Phi_{3n}(\theta)(v_{3n}(0) + I(\theta)), \quad \text{where } I(\theta) = PV\int_0^\theta 2\Phi_{3n}^{-1}(\sigma)\mathcal{F}_3(\sigma)\,d\sigma,$$

with $\Phi_{3n}(\theta) = \exp(-2\chi_{11}(\theta))$.

As we search an example without a Puiseux inverse integrating factor, we fix the parameter values $\{\alpha, b, c\} = \{-1, 1, -2\} \in \Lambda$ from now on. In this case, $\Phi_{3n}(0) = 1$, $\lim_{\theta\to\pi/2^-}\Phi_{3n}(\theta) = 0$ and $\Phi_{3n}(\theta) = \infty$, for any $\pi/2 < \theta < 2\pi$. See Figure 1(A). We also see that the function $\Gamma(\theta) = 2\Phi_{3n}^{-1}(\theta)\mathcal{F}_3(\theta)$ has a vertical asymptote in $\pi/2$ from the left, that is, $\lim_{\theta\to\pi/2^-}\Gamma(\theta) = -\infty$ and $\Gamma(\theta) = 0$, for any $\pi/2 < \theta < 2\pi$. See Figure 1(B). Moreover, $\lim_{\theta\to\pi/2^-} I(\theta) = \int_0^\theta \Gamma(\sigma)\,d\sigma = -\infty$. So, $I(\theta) = -\infty$ for $\theta \ge \pi/2$. As $v_{3n}(0)$ is a finite real number, $v_{3n}(\theta) = \Phi_{3n}(\theta)(v_{3n}(0) + I(\theta))$ does not exist, for any $\theta > \pi/2$, which concludes the proof. □

5.6 | Proof of Proposition 5

Proof. Clearly, (18) is a family of monodromic nilpotent vector fields with Andreev number $N = 3$. Moreover, the unique weight on the Newton diagram is $(p, q) = (1, 3)$. Besides, when $a = 0$, the

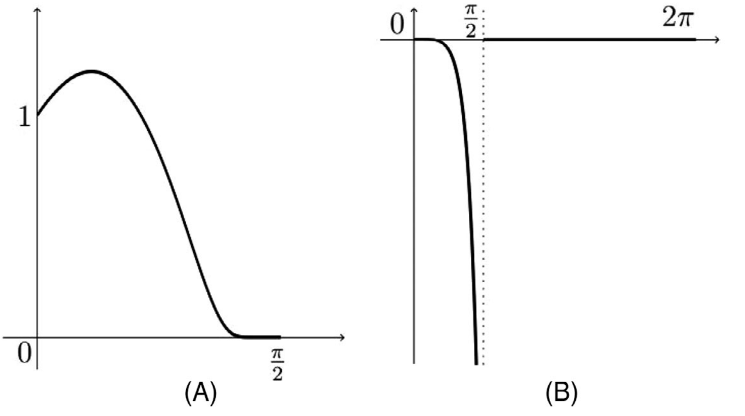

FIGURE 1 (A) Behavior of $\Phi_{3n}(\theta)$, $0 < \theta < \pi/2$; (B) behavior of $\Gamma(\theta)$, $0 < \theta < 2\pi$.

origin is an integrable center with first integral $H(x, y) = y^2/2 + x^6/6$. So, in the following, we focus on the case $a \neq 0$.

Doing the weighted polar blow-up (3) and the time reparametrization explained in Section 2.1, its associated differential equation (6) is given by

$$\frac{d\rho}{d\theta} = \sum_{i\geq 1} \mathcal{F}_i(\theta)\rho^i = \frac{\rho\cos(\theta)\sin(\theta)(1-\cos(\theta)^4 + a\rho^2\cos(\theta)^3\sin(\theta))}{-\cos(\theta)^6 + a\rho^2\cos(\theta)^5\sin(\theta) - 3\sin(\theta)^2},$$

where the first functions $\mathcal{F}_i(\theta)$ are $\mathcal{F}_2 = \mathcal{F}_4 = \mathcal{F}_6 = \mathcal{F}_8 \equiv 0$,

$$\mathcal{F}_1 = \frac{\cos(\theta)(-1+\cos(\theta)^4)\sin(\theta)}{\cos(\theta)^6 + 3\sin(\theta)^2}, \qquad \mathcal{F}_3 = \frac{a\cos(\theta)^4(-2+\cos(2\theta))\sin(\theta)^2}{(\cos(\theta)^6 + 3\sin(\theta)^2)^2},$$

$$\mathcal{F}_5 = \frac{a^2\cos(\theta)^9(-2+\cos(2(\theta))\sin(\theta)^3}{(\cos(\theta)^6 + 3\sin(\theta)^2)^3}, \quad \mathcal{F}_7 = \frac{a^3\cos(\theta)^14(-2+\cos(2(\theta))\sin(\theta)^4}{(\cos(\theta)^6 + 3\sin(\theta)^2)^4}.$$

Moreover, $P(\theta) = -\log\left(3\sin^2(\theta) + \cos^6(\theta)\right)/6$ is a primitive of $\mathcal{F}_1(\theta)$. Hence, $\xi_{13} = \text{PV}\int_0^{2\pi} \mathcal{F}_1(\sigma)\mathrm{d}\sigma = 0$.

We follow the step-by-step and the notation of the proof of Proposition 2. First, using (20), (25), (26), (B1), (B2), (B3), and (B4), we calculate b_k, with $k = m, m+n, \dots, m+6n$.

In the following, we would like to determine if there exist $k \in \mathbb{N}$ such that $\Lambda_{m+k} \neq 0$, where

$$\Phi_{m+k}(\theta) = \exp\left(\alpha_{m+k}\,\text{PV}\int_0^{\theta} \mathcal{F}_1(\sigma)\mathrm{d}\sigma\right), \quad \Lambda_{m+k} = \text{PV}\int_0^{2\pi} \Phi_{m+k}^{-1}(\theta)b_{m+k}(\theta)d\theta,$$

with $\alpha_{m+k} = (n-m-k)/n$. Using (20), we obtain

$$v_m(\theta) = c_0(-33\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta) + 58)^{\frac{1}{6}\left(\frac{m}{n}-1\right)}.$$

As the leading term $v_m(\theta) \not\equiv 0$, we can choose, without loss of generality, that the integrating constant is $c_0 = 1$. As $b_m = 0$ (see (20)), it follows that $\Lambda_m = 0$.

Now, using (25), we obtain $b_{m+n} = 0$, $\Lambda_{m+n} = 0$ and

$$v_{m+n}(\theta) = c_1(-33\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta) + 58)^{\frac{m}{6n}},$$

where c_1 is the integrating constant. Moreover, using (26), we can calculate $v_{m+2n}, \Phi_{m+2n}, b_{m+2n}$, and then, we obtain

$$\Lambda_{m+2n} = -\frac{(m-3n)}{n} A(\theta),$$

with $A(\theta) = (a\sin^2(\theta)\cos^4(\theta)(\cos(2\theta) - 2)(-33\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta) + 58)^{\frac{1}{6}(\frac{m}{n}-1)}$ $(3\sin^2(\theta) + \cos^6(\theta))^{-\frac{m+13n}{6n}}$. Note that $A(\theta) > 0$, for all θ. Then $\Lambda_{m+2n} = 0$ if, and only if, $m = 3n$. So, by Proposition 2(iii), it follows that $V(\theta, \rho)$ does not exist when $a \neq 0$ and $m \neq 3n$.

From now on we suppose that $m = 3n$. With this assumption and repeating the process explained below, we can choose integration constants such that $\Lambda_{(3+k)n} = 0$, for $k = 0, 1 \dots, 5$. Further, using (B4), we get $\Lambda_{9n} = \int_0^{2\pi} \Gamma(\theta)$, where

$$\Gamma(\theta) = -\frac{a^3 10485762^{2/3}\sin^2(\theta)\cos^{14}(\theta)(\cos(2\theta) - 2)(527\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta) - 502)}{5(-33\cos(2\theta) + 6\cos(4\theta) + \cos(6\theta) + 58)^5}.$$

Moreover, we can see that Λ_{9n} is nonvanishing when $a \neq 0$. So, by Proposition 2(iii), it follows that $V(\rho, \theta)$ does not exist when $a \neq 0$ and $m = 3n$.

Therefore, $V(\rho, \theta)$ could exists only when $a = 0$ and, as we saw before, in such case the origin is a center of (18). □

5.7 | Proof of Proposition 6

Proof. In $\Lambda \backslash \Lambda_{pq}$, $\mathcal{F} \in C^\omega(C \backslash \{\rho = 0\})$. Let $\omega = d\rho - \mathcal{F}(\theta, \rho)d\theta$. By Lemma 2, the differential 1-form ω/V is closed in $C \backslash \{\rho = 0\}$. Indeed, we are going to see that it is exact. To do it, we consider any noncontractible cycle $\gamma \subset C \backslash \{\rho = 0\}$. It means that γ is not homotopic to a point. By De Rham's theorem (see, e.g., Ref. 35) and due to the cylinder topology, it follows that ω/V is exact in $C \backslash \{\rho = 0\}$ if, and only if, the value of the line integral $\oint_\gamma \omega/V$ is zero. Let γ be any periodic orbit of (6), so $\omega|_\gamma \equiv 0$ and then $\oint_\gamma \omega/V = 0$. For more details, see the proof of Theorem 4 in [7]. Thus, $\omega/V = dH$ for a certain C^2 first integral $H(\theta, \rho)$ of (6) defined in $C \backslash \{\rho = 0\}$. It means that H is a 2π-periodic function in θ. We can check that, actually, this H has a Puiseux development at $\rho = 0$ just by looking at the relations $\partial_\rho H = 1/V$ and $\partial_\theta H = -\mathcal{F}/V$. □

5.8 | Proof of Lemma 3

Proof. There exists a Puiseux inverse integrating factor $V(\theta, \rho)$ of (6) since it has a Puiseux first integral $H(\theta, \rho)$. It follows that H^s is also a Puiseux first integral of (6), for any $s \in \mathbb{Q}$. So, we can find $s^* \in \mathbb{Q}$ such that VH^{s^*} is a Puiseux inverse integrating factor of (6) with negative multiplicity m. Then, by Theorem 4 of [7], it follows that the origin is a center. □

5.9 | Proof of Proposition 7

Proof. First, we note that after the linear change of coordinates $(x, y) \mapsto (y, x)$, the only weights on the Newton diagram are $(p, q) = (1, 1)$ and $(p, q) = (3, 1)$.

We start analyzing the case $(p, q) = (1, 1)$ with $\xi_{11} \neq 0$, that is, $\lambda_1 \mu \neq 0$. In this case, if (15) admits a Puiseux first integral, by Proposition 10 in the Appendix, we obtain $m = 0$. Moreover, in the next step, when $m = 0$, (C7) is simply

$$nh_n'(\theta) + n\mathcal{F}_1(\theta)h_n(\theta) = 0, \tag{31}$$

and has the solution $h_n(\theta) = c_1 \exp(\int_0^\theta \mathcal{F}_1(s)ds)$, where c_1 is the integration constant. Then, $h_n(\theta)$ is 2π-periodic if, and only if, $c_1 = 0$ so that $h_n(\theta) \equiv 0$. In the next step, using (C8), we obtain $nh_{2n}'(\theta) + 2n\mathcal{F}_1(\theta)h_{2n}(\theta) = 0$, whose solution is $h_{2n}(\theta) = c_2 \exp(\int_0^\theta \mathcal{F}_1(s)ds)$. To ensure that h_{2n} is 2π periodic, we get $h_{2n} \equiv 0$ since $\xi_{11} \neq 0$. With a recursive process, we can see that the ordinary differential equations obtained from (11) are always homogeneous. More specifically, we obtain

$$nh_{\ell n}'(\theta) + \ell n\mathcal{F}_1(\theta)h_{\ell n}(\theta) = 0.$$

So, the integration constant $c_\ell = 0$, for any $\ell > 1$. Thus, $H(\theta, \rho) \equiv 1$ and (15) does not admit a Puiseux first integral in this case.

Finally, we analyze the case $(p, q) = (3, 1)$ with $\xi_{31} \neq 0$. Using the weighted polar blow-up (3), we obtain the associated polar equation $d\rho/d\theta = \mathcal{F}_1(\theta)\rho + \mathcal{O}(\rho^2)$ with

$$\mathcal{F}_1(\theta) = -\frac{32\sin^5(\theta)\big(\csc^3(\theta)\big(A\lambda_2(\cos(2\theta) + 2) + 3\lambda_1\cot^3(\theta) + \lambda_2\cot(\theta)\big) + \cos(\theta)(\lambda_1 - \lambda_2)\big)}{(33\cos(2\theta) + 6\cos(4\theta) - \cos(6\theta) + 58)(\lambda_1 - \lambda_2)}.$$

Then,

$$\xi_{31} = PV\int_0^{2\pi} \mathcal{F}_1(\theta)\,d\theta = \frac{2\pi A\lambda_2}{\sqrt{3}(\lambda_2 - \lambda_1)}.$$

Hence, if (15) admits a Puiseux first integral, it must be with $m = 0$. In the next step, we see that (C7) is homogeneous and has the form (31). Therefore, the integration constant c_1 must vanish by the 2π-periodicity of $h_n(\theta)$. Doing a similar analysis as in the previous case, we conclude that $H(\theta, \rho) \equiv 1$ and (15) does not admit a Puiseux first integral.

With this computations, we conclude that the centers in the family (15) for which $3\lambda_1\mu + \sqrt{3}A\lambda_2 = 0$, but with both $\lambda_1\mu \neq 0$ and $A\lambda_2 \neq 0$ do not admit a Puiseux first integral. □

5.10 | Proof of Proposition 8

Proof. Doing the weight polar blow-up (3) and the time reparametrization explained in Section 2.1, we obtain

$$\begin{aligned}\dot{\rho} &= \rho\cos(\theta)(3\sin^3(\theta) - 3\sin(\theta)\cos^4(\theta) + 2\sin(\theta)\cos^3(\theta) - 3\sin^3(\theta)\cos(\theta) \\ &\quad + 6\alpha\rho\cos^5(\theta) + 9\alpha\rho\sin^2(\theta)\cos^3(\theta) + 9\beta\rho\sin^4(\theta) + 6\beta\rho\sin^2(\theta)\cos^2(\theta))/3, \\ \dot{\theta} &= -3\sin^4(\theta) - 2\cos^6(\theta) - 4\sin^2(\theta)\cos^3(\theta).\end{aligned}$$

Moreover,

$$\mathcal{F}_1(\theta) = \frac{\sin(\theta)\cos(\theta)(-6a\cos(\theta) - 10a\cos(3\theta) + 8\cos(2\theta) + \cos(4\theta) - 1)}{8\left(12a\sin^2(\theta)\cos^3(\theta) + 3\sin^4(\theta) + 2\cos^6(\theta)\right)},$$

$$\mathcal{F}_2(\theta) = \frac{\cos(\theta)(\cos(2\theta) - 5)\left(\alpha\cos^3(\theta) + \beta\sin^2(\theta)\right)}{3a\sin^3(2\theta)\csc(\theta) + 6\sin^4(\theta) + 4\cos^6(\theta)},$$

$$\mathcal{F}_i(\theta) \equiv 0, \text{ for all } i \geq 3.$$

We also observe that $\xi_{23} = \int_0^{2\pi} \mathcal{F}_1(\sigma)d\sigma = 0$, and that

$$\int_0^{\theta} \mathcal{F}_1(\sigma)d\sigma = \log(2^5/f_1(\theta))^{1/12},$$

where

$$f_1(\theta) = f_1(\theta; a) = 24a\cos(\theta) - 12a\cos(3\theta) - 12a\cos(5\theta) - 9\cos(2\theta) + 12\cos(4\theta) + \cos(6\theta) + 28.$$

Furthermore, for all $k \geq 0$,

$$\widetilde{\Phi}_{m+k}(\theta) = \exp\left(\widetilde{\alpha}_{m+k}\int_0^{\theta} \mathcal{F}_1(\sigma)d\sigma\right) = \left(\frac{2^{5/12}}{f_1(\theta)^{1/12}}\right)^{\widetilde{\alpha}_{m+k}},$$

$$\widetilde{\Lambda}_{m+k} = \int_0^{2\pi} \widetilde{\Phi}^{-1}_{m+k}(\theta)\widetilde{b}_{m+k}(\theta)d\theta = \int_0^{2\pi} \widetilde{b}_{m+k}(\theta)\left(\frac{2^{5/12}}{f_1(\theta)^{1/12}}\right)^{-\widetilde{\alpha}_{m+k}} d\theta.$$

To simplify the notation, let

$$f_2(\theta) = f_2(\theta; (\alpha, \beta)) = \cos(\theta)(\cos(2\theta) - 5)\left(\alpha\cos^3(\theta) + \beta\sin^2(\theta)\right),$$

and observe that $\mathcal{F}_2(\theta) = 2^3 f_2(\theta)/f_1(\theta)$.

We follow the steps and the notation of the proof of Proposition 10. As (C1) is a homogeneous linear ordinary differential equation, the leading term $h_m(\theta)$ of a Puiseux first integral $H(\theta, \rho)$ of (19) must be

$$h_m(\theta) = c_0 f_1(\theta)^{m/12n},$$

where c_0 is an integration constant, and $c_0 \neq 0$, because $h_m(\theta) \not\equiv 0$. So, without loss of generality, we take $c_0 = 1$. Moreover, as $\widetilde{b}_m = 0$, we also obtain $\widetilde{\Lambda}_m = 0$. In the next step, using (C7), we obtain $\widetilde{b}_{m+n} = -m\mathcal{F}_2(\theta)h_m(\theta)/n = -2^3 m f_2(\theta) f_1(\theta)^{(m/12n)-1}/n$, and

$$\widetilde{\Lambda}_{m+n} = \int_0^{2\pi} \widetilde{b}_{m+n}(\theta)\left(\frac{2^{5/12}}{f_1(\theta)^{1/12}}\right)^{-\widetilde{\alpha}_{m+n}} d\theta = -\frac{m 2^3 2^{5(m+n)/12n}}{n}\int_0^{2\pi} \frac{f_2(\theta)}{f_1(\theta)^{13/12}} d\theta.$$

With straightforward computations, we can see that

$$\int_0^{2\pi} \frac{f_2(\theta)}{f_1(\theta)^{13/12}} d\theta = 0 \tag{32}$$

if, and only if, $(\alpha, \beta) \in C$. So, by Proposition 10(iii), if $m \neq 0$ and $(\alpha, \beta) \notin C$, (19) does not admit a Puiseux first integral.

As (19) is in the class $\mathrm{Mo}^{(2,3)}$, when $(\alpha, \beta) \in C$, it admits an analytic first integral (see Theorem 1 of [22]). So, we fix our attention on the case $(\alpha, \beta) \notin C$ and $m = 0$. In such case $h_0(\theta) \equiv 1$, $\widetilde{b}_n = 0$, $\widetilde{\Lambda}_n = 0$, and $h_n(\theta) = c_1 f_1(\theta)^{1/12}$, where $c_1 \in \mathbb{R}$ is an integration constant. We also observe that, $\widetilde{\alpha}_{\ell n} = -\ell$, for all $\ell \in \mathbb{N}$. Moreover, as $\mathcal{F}_i \equiv 0$, for all $i \geq 3$, (C8) becomes

$$n\, h'_{2n}(\theta) + 2n\mathcal{F}_1(\theta) h_{2n}(\theta) = -\, n\mathcal{F}_2(\theta) h_n(\theta). \tag{33}$$

Thus, $\widetilde{b}_{2n} = -2^3 c_1 f_2(\theta)/f_1(\theta)^{11/12}$, and

$$\widetilde{\Lambda}_{2n} = c_1 2^3 2^{5/6} \int_0^{2\pi} \frac{f_2(\theta)}{f_1(\theta)^{13/12}} d\theta.$$

Using again that (32) is valid if, and only if, $(\alpha, \beta) \in C$, we need to impose $c_1 = 0$ to continue. It implies that $h_n(\theta) \equiv 0$ and that (33) is homogeneous. Therefore, its solution is $h_{2n}(\theta) = c_2 f_1(\theta)^{2/12}$. More generally, as $\mathcal{F}_i \equiv 0$, for all $i \geq 3$, the ordinary differential equations obtained from (C5) are simply

$$n\, h'_{\ell n}(\theta) + \ell n\mathcal{F}_1(\theta) h_{\ell n}(\theta) = -\,(\ell - 1) n\mathcal{F}_2(\theta) h_{\ell n - 1}(\theta),\ \ell \geq 1.$$

So, $\widetilde{b}_{\ell n} = -2^3(-1 + \ell) c_{\ell - 1} f_2(\theta)/f_1(\theta)^{(-13+\ell)/12}$, where $c_{\ell n - 1}$ is an integration constant, and

$$\widetilde{\Lambda}_{\ell n} = (-1 + \ell) c_{\ell - 1} 2^3 2^{5\ell/12} \int_0^{2\pi} \frac{f_2(\theta)}{f_1(\theta)^{13/12}} d\theta.$$

Thus, with a recursive process, we obtain $c_{\ell n} = 0$, for all $\ell \geq 1$, which implies that $h_{\ell n} = 0$, for all $\ell \geq 1$. Therefore, $H(\theta, \rho) \equiv 1$ and (19) does not admit a Puiseux first integral when $(\alpha, \beta) \notin C$. □

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The authors state that there are no known conflicts of interests related with the development of this paper.

DATA AVAILABILITY STATEMENT

The study was conducted without any datasets being generated or analyzed.

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REFERENCES

1. Dumortier F. Singularities of vector fields on the plane. *J Differential Equations*. 1977;23:53-106.
2. Arnold VI. *Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires*. Editions Mir; 1980.
3. Algaba A, García C, Reyes M. Characterization of a monodromic singular point of a planar vector field. *Nonlinear Anal*. 2011;74:5402-5414.
4. Algaba A, García C, Reyes M. A new algorithm for determining the monodromy of a planar differential system. *Appl Math Comput*. 2014;237:419-429.
5. García IA, Giné J., Grau M. A necessary condition in the monodromy problem for analytic differential equations on the plane. *J Symbolic Comput*. 2006;41:943-958.
6. Medvedeva NB. On the analytic solvability of the problem of distinguishing between center and focus. *Proc Steklov Inst Math*. 2006;254:7-93.
7. García IA, Giné J. The Poincaré map of degenerate monodromic singularities with Puiseux inverse integrating factor. *Adv Nonlinear Anal*. 2023;12:20220314.
8. García IA, Grau M. A survey on the inverse integrating factor. *Qual Theory Dyn Syst*. 2010;9(1-2):115-166.
9. Giacomini H, Llibre J, Viano M. On the nonexistence, existence and uniqueness of limit cycles. *Nonlinearity*. 1996;9:501-516.
10. Enciso A, Peralta-Salas D. Existence and vanishing set of inverse integrating factors for analytic vector fields. *Bull Lond Math Soc*. 2009;41:1112-1124.
11. García IA, Shafer DS. Integral invariants and limit sets of planar vector fields. *J Differential Equations*. 2005;217:363-376.
12. Giné J, Peralta-Salas D. Existence of inverse integrating factors and Lie symmetries for degenerate planar centers. *J Differential Equations*. 2012;252(1):344-357.
13. García IA, Giné J. Center problem with characteristic directions and inverse integrating factors. *Commun Nonlinear Sci Numer Simul*. 2022;108:106276.
14. Algaba A, García C, Giné J. Geometric criterium in the center problem. *Mediterr J Math*. 2016;13:2593-2611.
15. Giné J. On the centers of planar analytic differential systems. *Internat J Bifur Chaos Appl Sci Engrg*. 2007;17(9):3061-3070.
16. Giné J. On the degenerate center problem. *Internat J Bifur Chaos Appl Sci Engrg*. 2011;21(5):1383-1392.
17. Giné J, Maza S. The reversibility and the center problem. *Nonlinear Anal*. 2011;74(2):695-704.
18. Romanovski VG, Shafer DS. *The center and cyclicity problems: a computational algebra approach*. Birkhäuser Boston, Inc.; 2009.
19. Algaba A, García C, Giné J. Nilpotent centres via inverse integrating factors. *Eur J Appl Math*. 2016;27(5):781-795.
20. García IA. Formal inverse integrating factors and the nilpotent center problem. *Internat J Bifur Chaos Appl Sci Engrg*. 2016;26:1650015.
21. Yakovenko SY. Smooth normalization of a vector field near a semistable limit cycle. *Ann Inst Fourier (Grenoble)*. 1993;43:893-903.
22. García IA, Maza S. A new approach to center conditions for simple analytic monodromic singularities. *J Differential Equations*. 2010;248:363-380.
23. Wolfram Research, Inc. *Mathematica, Version 13.0*. Champaign, IL; 2021.
24. Maple 2022. Maplesoft, a division of Waterloo Maple Inc. Waterloo, Ontario.
25. Algaba A, Fuentes N, García C. Centers of quasi-homogeneous polynomial planar systems. *Nonlinear Anal Real World Appl*. 2012;13:419-431.
26. Giacomini H, Giné J, Llibre J. The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems. *J Differential Equations*. 2006;227(2):406-426; *Corrigendum, ibid*. 2007;232:702.
27. Algaba A, García C, Reyes M. Existence of an inverse integrating factor, center problem and integrability of a class of nilpotent systems. *Chaos Solitons Fractals*. 2012;45(6):869-878.

28. Gasull A, Llibre J, Mañosa V, Mañosas F. The focus-centre problem for a type of degenerate system. *Nonlinearity*. 2000;13:699-729.
29. Andreev A. Investigation on the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point. *Translations Amer Math Soc*. 1958;8:187-207.
30. Berthier M, Moussu R. Réversibilité et classification des centres nilpotents. *Ann Inst Fourier (Grenoble)*. 1994;44:465-494.
31. Moussu R. Symétrie et forme normale des centres et foyers dégénérés. *Ergodic Theory Dynam Systems*. 1982;2:241-251.
32. García IA, Giacomini H, Grau M. Generalized Hopf bifurcation for planar vector fields via the inverse integrating factor. *J Dyn Differ Equ*. 2011;23:251–281.
33. García IA, Giné J. *Principal Bautin ideal of monodromic singularities with inverse integrating factors*. Preprint, Universitat de Lleida; 2024.
34. Algaba A, García C, Giné J. Analytic integrability for some degenerate planar vector fields. *J Differential Equations*. 2014;257:549-565.
35. Fulton W. *Algebraic Topology. A First Course*. Graduate Texts in Mathematics, Vol 153. Springer-Verlag; 1995.

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APPENDIX A: MONODROMIC QUASI-HOMOGENEOUS SYSTEMS

Recall that the weighted polar blow-up $(x, y) \mapsto (\theta, \rho)$, given in (3), brings the (p, q)-quasi-homogeneous vector field $\mathcal{X}_r = P_{p+r}(x, y)\partial_x + Q_{q+r}(x, y)\partial_y$ into a linear differential equation (6) of the form

$$\frac{d\rho}{d\theta} = \mathcal{F}_1(\theta)\,\rho. \tag{A1}$$

The following lemma is a generalization of the results presented in Appendix A of [26]. For more details, see Ref. 26 and references therein.

Lemma 4. *Let the origin be a monodromic singularity of a quasi-homogeneous vector field $\mathcal{X}_r$ and consider the associated differential equation (A1). Then the following holds:*

(i) $V(\theta, \rho) = \rho$ is a Puiseux inverse integrating factor of (A1);
(ii) the origin is a center if, and only if, (A1) admits a Puiseux first integral;
(iii) if the origin is a focus, then $V(\theta, \rho) = \rho$ is the unique Puiseux inverse integrating factor of (A1), up to a nonzero multiplicative constant.

Proof. The proof of (i) is direct, checking that $V(\theta, \rho) = \rho$ satisfies Equation (8) with $\mathcal{F}(\theta, \rho) = \mathcal{F}_1(\theta)\,\rho$.

We are going to prove the sufficient part of item (ii). Assume that there is a Puiseux first integral $H = H(\theta, \rho) = \sum_{i \geq m} h_i(\theta)\rho^{i/n}$, where the coefficients functions $h_i(\theta)$, defined in $\mathbb{S}^1 \backslash \Omega_{pq}$, are 2π-periodic. It is straightforward to see that the leading term $H_m(\theta, \rho) = h_m(\theta)\rho^{m/n}$ of H is also a Puiseux first integral of (A1).

On the other hand, since (A1) is linear, we also obtain that

$$\hat{H}(\theta,\rho) = \rho \exp\left(-\int_0^{\theta} \mathcal{F}_1(s)\,ds \right) \tag{A2}$$

is a first integral. Clearly, we have the relation $\hat{H}^{m/n} = H_m$, that is,

$$h_m(\theta) = \exp\left(-\frac{m}{n} \int_0^{\theta} \mathcal{F}_1(s)\,ds \right).$$

Note that m cannot be zero because; otherwise, $H_m \equiv 1$ becomes a constant. Since $h_m(\theta)$ is 2π-periodic, we get $\xi_{pq} = \int_0^{2\pi} \mathcal{F}_1(\theta)\,d\theta = 0$. The proof finishes recalling that $\mathcal{X}_r$ has a center at the origin if, and only if, $\xi_{pq} = 0$. See the Appendix Section of [26], for more details.

To prove the necessary part of (ii), we assume that the origin is a center, that is, $\xi_{pq} = 0$. As we saw before, since (A1) is linear, (A2) satisfies (11). These conditions assure the existence of a solution $H(\theta,\rho) = h_m(\theta)\rho^{m/n}$, with $m \neq 0$, and where $h_m(\theta)$ are 2π-periodic functions of the homogeneous partial differential equation $\mathcal{X}_r(H) \equiv 0$. It implies that $nh_m'(\theta) + m\mathcal{F}_1(\theta)\,h_m(\theta) = 0$. Clearly, this H is a Puiseux first integral of $\mathcal{X}_r$.

Statement (iii) is an immediate consequence of statement (ii) and the fact that the quotient of two Puiseux inverse integrating factors is a Puiseux first integral. □

APPENDIX B: HIGHER ORDER ODES FOR THE COEFFICIENTS OF *V*

In this appendix, we show four more linear ordinary differential equations that we obtained using (23) and that were used in Example 5. So, we enlarge the ones obtained in the proof of Proposition 2.

The equation associated to the power $\rho^{(m+3n+j)/n}$, with $0 \le j \le n-1$, gives that $v_{m+3n+j}(\theta)$ satisfies the linear ordinary differential equation

$$\begin{aligned} n\,v'_{m+3n+j}(\theta) + (m+2n+j)\mathcal{F}_1(\theta)v_{m+3n+j}(\theta) &= (4n-m-j)\mathcal{F}_4(\theta)v_{m+j}(\theta) \\ &+ (2n-m-j)\mathcal{F}_3(\theta)v_{m+n+j}(\theta) - (m+j)\mathcal{F}_2(\theta)v_{m+2n+j}(\theta). \end{aligned} \tag{B1}$$

In the sequence, analyzing the equation associated to the power $\rho^{(m+4n+j)/n}$, with $0 \le j \le n-1$, we obtain that $v_{m+4n+j}(\theta)$ satisfies

$$\begin{aligned} n\,v'_{m+4n+j}(\theta) + (m+3n+j)\mathcal{F}_1(\theta)v_{m+4n+j}(\theta) &= (5n-m-j)\mathcal{F}_5(\theta)v_{m+j}(\theta) \\ &+ (3n-m-j)\mathcal{F}_4(\theta)v_{m+n+j}(\theta) + (n-m-j)\mathcal{F}_3(\theta)v_{m+2n+j}(\theta) \\ &- (m+n+j)\mathcal{F}_2(\theta)v_{m+3n+j}(\theta). \end{aligned} \tag{B2}$$

On the next step, we see that, for $0 \le j \le n-1$, $v_{m+5n+j}(\theta)$ satisfies the linear ordinary differential equation

$$\begin{aligned} n\,v'_{m+5n+j}(\theta) + (m+4n+j)\mathcal{F}_1(\theta)v_{m+5n+j}(\theta) &= (6n-m-j)\mathcal{F}_6(\theta)v_{m+j}(\theta) \\ &+ (4n-m-j)\mathcal{F}_5(\theta)v_{m+n+j}(\theta) + (2n-m-j)\mathcal{F}_4(\theta)v_{m+2n+j}(\theta) \\ &- (m+j)\mathcal{F}_3(\theta)v_{m+3n+j}(\theta) - (m+2n+j)\mathcal{F}_2(\theta)v_{m+4n+j}(\theta). \end{aligned} \tag{B3}$$

Finally, the equation associated to the power $\rho^{(m+6n+j)/n}$, with $0 \le j \le n-1$ gives that $v_{m+6n+j}(\theta)$ satisfies the linear ordinary differential equation

$$\begin{aligned} n\,v'_{m+6n+j}(\theta) &+ (m+5n+j)\mathcal{F}_1(\theta)v_{m+6n+j}(\theta) = (7n-m-j)\mathcal{F}_7(\theta)v_{m+j}(\theta) \\ &+ (5n-m-j)\mathcal{F}_6(\theta)v_{m+n+j}(\theta) + (3n-m-j)\mathcal{F}_5(\theta)v_{m+2n+j}(\theta) \\ &+ (n-m-j)\mathcal{F}_4(\theta)v_{m+3n+j}(\theta) - (n+m+j)\mathcal{F}_3(\theta)v_{m+4n+j}(\theta) \\ &- (m+3n+j)\mathcal{F}_2(\theta)v_{m+5n+j}(\theta). \end{aligned} \tag{B4}$$

APPENDIX C: ODES FOR THE COEFFICIENTS OF PUISEUX FIRST INTEGRALS

In this section, we present some conditions related to the existence of a Puiseux first integral. The results are analogous to the ones obtained for Puiseux inverse integrating factors in Section 3. More specifically, the next Propositions 9 and 10 are the analogous to Propositions 1 and 2, respectively.

Proposition 9. *We take $0 \notin \Omega_{pq}$ performing a linear change of coordinates, if necessary. If ξ_{pq} does not exist, then any Puiseux first integral must have $m = 0$.*

Proof. Inserting the expansion $H(\theta,\rho) = h_m(\theta)\rho^{m/n} + \mathcal{O}(\rho^{m/n})$ and $\mathcal{F}(\theta,\rho) = \mathcal{F}_1(\theta)\rho + \mathcal{O}(\rho)$ into the partial differential equation (11) gives the linear homogeneous differential equation

$$n\,h'_m(\theta) + m\mathcal{F}_1(\theta)h_m(\theta) = 0, \tag{C1}$$

whose general solution is

$$h_m(\theta) = h_m(0)\exp\left(-\frac{m}{n}PV\int_0^\theta \mathcal{F}_1(s)\,ds\right). \tag{C2}$$

Therefore, when ξ_{pq} exists, it follows that

$$h_m(2\pi) = h_m(0)\exp\left(-\frac{m}{n}\xi_{pq}\right),$$

whereas if ξ_{pq} does not exist, since $h_m(2\pi) = h_m(0) \in \mathbb{R}$ because $0 \notin \Omega_{pq}$, then $m = 0$. □

Proposition 10. *Consider system (5) and the associated ordinary differential equation (6) with $0 \notin \Omega_{pq}$, without loss of generality. If there exists a formal Puiseux first integral $H(\theta,\rho) = \sum_{k\ge 0} h_{m+k}(\theta)\rho^{(m+k)/n}$ of (6), then its coefficients satisfy the linear differential equation*

$$h'_{m+k}(\theta) = \widetilde{\alpha}_{m+k}\mathcal{F}_1(\theta)h_{m+k}(\theta) + \widetilde{b}_{m+k}(\theta), \tag{C3}$$

in $\mathbb{S}^1\backslash\Omega_{pq}$, where $\widetilde{\alpha}_{m+k} \in \mathbb{R}$ and $\widetilde{b}_{m+k} : \mathbb{S}^1\backslash\Omega_{pq} \to \mathbb{R}$ for any $k \ge 0$. Moreover, assuming that both ξ_{pq} and $\widetilde{\Lambda}_{m+k} := PV\int_0^{2\pi} \widetilde{\Phi}^{-1}_{m+k}(\theta)\widetilde{b}_{m+k}(\theta)d\theta$, with $\widetilde{\Phi}_{m+k}(\theta) = \exp(\widetilde{\alpha}_{m+k}\,PV\int_0^\theta \mathcal{F}_1(\sigma)d\sigma)$, exist for all $k \ge 0$, then $m\xi_{pq} = 0$ and the following statements hold:

(i) If $\xi_{pq} \neq 0$, then $m = 0$ and for each $n \in \mathbb{N}$, the formal Puiseux first integral is unique up to multiplicative nonzero constants.
(ii) If $\xi_{pq} = \widetilde{\Lambda}_{m+k} = 0$ for any $k \geq 0$, then there can be infinitely many formal Puiseux first integrals.
(iii) If $\xi_{pq} = 0$ and $\widetilde{\Lambda}_{m+k} \neq 0$ for some $k \geq 0$, then there is no Puiseux first integral.

Proof. We follow the same steps of the proof of Proposition 2. Introducing the expansion

$$H(\theta,\rho) = \sum_{i\geq m} h_i(\theta)\rho^{i/n} = \sum_{i=m}^{m+4n} h_i(\theta)\rho^{i/n} + \mathcal{O}(\rho^{(m+4n)/n}), \tag{C4}$$

together with $\mathcal{F}(\theta,\rho) = \sum_{i=1}^{4} \mathcal{F}_i(\theta)\rho^i + \mathcal{O}(\rho^5)$, where $\mathcal{F}_1(\theta) = F_k(\theta)/G_k(\theta)$, into the partial differential equation (11), that defines H, we obtain

$$\begin{aligned} &\sum_{i=m}^{m+4n}\left(\frac{\mathrm{d}}{\mathrm{d}\theta}h_i(\theta)\right)\rho^{\frac{i}{n}} + \mathcal{O}(\rho^{\frac{m+4n}{n}}) \\ &+ \frac{\sum_{i=1}^{4}\mathcal{F}_i(\theta)\rho^{i-1} + \mathcal{O}(\rho^4)}{n}\left(\sum_{i=m}^{m+4n} i h_i(\theta)\rho^{\frac{i}{n}} + \mathcal{O}\left(\rho^{\frac{m+4n}{n}}\right)\right) = 0 \end{aligned} \tag{C5}$$

Equating the coefficients of the powers $\rho^{(m+j)/n}$, with $0 \leq j \leq n-1$, we see that the $h_{m+j}(\theta)$ satisfies the linear ordinary differential equation

$$n\,h'_{m+j}(\theta) + (m+j)\mathcal{F}_1(\theta)h_{m+j}(\theta) = 0. \tag{C6}$$

We point out that $h_m \not\equiv 0$, because it is the leading term of (C4). Besides, when $j = 0$, in (C6), we obtain (C1).

Moreover, taking into account the next powers $\rho^{(m+n+j)/n}$, with $0 \leq j \leq n-1$, in (C5), we obtain

$$nh'_{m+n+j}(\theta) + (m+n+j)\mathcal{F}_1(\theta)h_{m+n+j}(\theta) = -(m+j)\mathcal{F}_2(\theta)h_{m+j}(\theta). \tag{C7}$$

When we consider the coefficients of $\rho^{(m+2n+j)/n}$, $0 \leq j \leq n-1$, we see that $h_{m+2n+j}(\theta)$ satisfies the linear ordinary differential equation

$$\begin{aligned} n\,h'_{m+2n+j}(\theta) + (m+2n+j)\mathcal{F}_1(\theta)h_{m+2n+j}(\theta) = &-(m+j)\mathcal{F}_3(\theta)h_{m+j}(\theta) \\ &-(m+n+j)\mathcal{F}_2(\theta)h_{m+n+j}(\theta). \end{aligned} \tag{C8}$$

Note that, using (C5), we also obtain

$$\widetilde{\alpha}_{m+k} = -\frac{m+k}{n}, \tag{C9}$$

for all $k \in \mathbb{N} \cup \{0\}$. It is worth to mention that $\widetilde{\alpha}_{m+k} = 0$ if, and only if, $m + k = 0 \in \mathbb{N}$. In particular, if $m = 0$, it only occurs when $k = 0$.

Suppose that $\xi_{pq} \neq 0$ and $m \neq 0$ (hence $\widetilde{\alpha}_m \neq 0$). Note that the linear ordinary differential equation (C1) has the unique solution (C2). So, the unique 2π-periodic solution of (C1) must be $h_m(\theta) \equiv 0$, which is a contradiction, because the leading term $h_m(\theta)$ of H is not identically null. Therefore,

$$m\xi_{pq} = 0$$

is a necessary condition for the existence of a Puiseux first integral. Note that when $m = 0$, $h_m(\theta) = h_m(0)$ is constant.

Now, assuming the existence of both $\widetilde{\Phi}_{m+k}(\theta)$ and $\widetilde{\Lambda}_{m+k}$, from the variation of constants formula applied to the linear differential equation (C3), we obtain

$$h_{m+k}(2\pi) = \widetilde{\Phi}_{m+k}(2\pi)h_{m+k}(0) + \widetilde{\Phi}_{m+k}(2\pi)\,\widetilde{\Lambda}_{m+k}.$$

Recall that h_{m+k} are functions well defined in $\mathbb{S}^1\backslash\Omega_{pq}$, so $h_{m+k}(2\pi)$ and $h_{m+k}(0)$ are real numbers. The periodicity condition $h_{m+k}(2\pi) = h_{m+k}(0)$ holds if, and only if, the initial condition $h_{m+k}(0)$ is a solution of the algebraic linear system

$$(1 - \widetilde{\Phi}_{m+k}(2\pi))h_{m+k}(0) = \widetilde{\Phi}_{m+k}(2\pi)\,\widetilde{\Lambda}_{m+k}. \tag{C10}$$

We start with the simplest analysis. If $\xi_{pq} = 0$, then $\widetilde{\Phi}_{m+k}(2\pi) = 1$. So, using (C10), either all the solutions of Equation (C3) or none of them are 2π–periodic according to whether $\widetilde{\Lambda}_{m+k}$ is zero or not, respectively. This proves statements (ii) and (iii).

In the case $\xi_{pq} \neq 0$, we have $m = 0$. Then, from (C9), $\widetilde{\alpha}_k = -k/n$. So, $\widetilde{\alpha}_0 = 0$ and $\widetilde{\alpha}_k \neq 0$ for all $k > 0$. Using (C1), for $0 < k < n-1$, and that h_k is 2π-periodic, we conclude that $h_k \equiv 0$, when $\xi_{pq} \neq 0$. Repeating the process, using (C7) and that $h_k \equiv 0$, for $1 \leq k \leq n-1$, yields that all the coefficient functions $h_{n+k}(\theta) \equiv 0$. Going further in this recursively process, we conclude that $h_k \equiv 0$, for all $k \neq \ell n$ with $\ell \in \mathbb{N}$. To finish, by (C10), $h_{n\ell}(\theta)$ are also 2π-periodic functions if, and only if, $h_{\ell n}(0) = \widetilde{\Phi}_{m+\ell n}(2\pi)\widetilde{\Lambda}_{\ell n}/(1 - \widetilde{\Phi}_{\ell n}(2\pi))$. So, $h_{\ell n}$ is the unique solution of (C3), with $\widetilde{\alpha}_k \in \mathbb{R}$ and $\widetilde{b}_k : \mathbb{S}^1\backslash\Omega_{pq} \to \mathbb{R}$ with the former initial condition $h_{\ell n}(0)$. Thus, the statement (i) follows.

We point out that the statements are still valid when $\mathcal{F}_1 \equiv 0$. In this case $\xi_{pq} = 0$, which provides directly that $m\xi_{pq} = 0$. Moreover, for all $k \geq 0$, we have $\widetilde{\Phi}_{m+k}(\theta) \equiv 1$ and $\widetilde{\Lambda}_{m+k} := PV \int_0^{2\pi} \widetilde{b}_{m+k}(\theta)d\theta$. So, (C10) is simply $\widetilde{\Lambda}_{m+k} = 0$. □