


Equivariant deformations of Hamiltonian stationary Lagrangian submanifolds

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Abstract

We prove an equivariant deformation result for Hamiltonian stationary Lagrangian submanifolds of a Kähler manifold, with respect to deformations of its metric and almost complex structure that are compatible with an isometric Hamiltonian group action. This yields existence of Hamiltonian stationary Lagrangian submanifolds in possibly non-Kähler symplectic manifolds whose metric is arbitrarily close to a Kähler metric.

1 Introduction

Let (M, ω) be a symplectic manifold with a Riemannian metric g . A submanifold Σ of M with $\dim M = 2 \dim \Sigma$ is *Lagrangian* if the restriction $\omega|_{\Sigma}$ of the symplectic form to this submanifold vanishes identically. Multiple constrained variational problems related to minimizing volume of Lagrangian submanifolds have been extensively studied in the literature,

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see for instance [2, 3, 5, 8, 9, 10, 11, 13]. Among these, an important constrained variational problem is related to minimizing volume under *Hamiltonian variations*. The corresponding critical points, called *Hamiltonian stationary Lagrangian submanifolds*, and their possible deformations in an equivariant setup, are the main objects of study in this paper.

Denote by $\mathcal{L}(\Sigma, M)$ the space of Lagrangian submanifolds of M that are diffeomorphic to Σ . A Hamiltonian variation of a Lagrangian submanifold Σ is simply a variation by a Hamiltonian vector field X , i.e., X is a vector field on M along Σ , such that the 1-form $\omega(X, \cdot)|_{\Sigma}$ is exact. In this situation, the submanifolds $\Sigma_t := \{\exp_p(tX_p) : p \in \Sigma\}$, $|t| < \varepsilon$, are also Lagrangian, i.e., $\Sigma_t \in \mathcal{L}(\Sigma, M)$ is a curve through $\Sigma = \Sigma_0$; but generally not all Lagrangians near Σ are obtained this way, see Section 3. More precisely, this curve Σ_t of Lagrangians stays inside the integral leaf through Σ of a certain (integrable) distribution of $\mathcal{L}(\Sigma, M)$, with codimension $b_1(\Sigma)$, that we call the *Hamiltonian distribution*, see Subsection 4.1. The integral leaves of this distribution are locally parametrized by the first de Rham cohomology $H^1(\Sigma, \mathbb{R})$, which is a real vector space of dimension $b_1(\Sigma)$. Given a closed 1-form η on Σ , we denote by $[\eta] \in H^1(\Sigma, \mathbb{R})$ its cohomology class and by $\mathcal{L}(\Sigma, M)_{[\eta]}$ the integral leaf of the Hamiltonian distribution that corresponds to $[\eta]$. When η is exact, i.e., $[\eta] = 0$, then the above means that $\mathcal{L}(\Sigma, M)_{[\eta]}$ is the space of Hamiltonian variations of Σ . A Lagrangian submanifold $\Sigma \subset M$ is a *g -Hamiltonian stationary Lagrangian submanifold* if it has critical volume (with respect to the volume form induced by g) among all its Hamiltonian variations.

Hamiltonian stationary Lagrangian submanifolds have been used, among others, to provide canonical representatives of the Lagrangian homology (the part of the homology generated by Lagrangian cycles), see [13]. Issues related to their existence in various contexts were discussed in [5, 8, 9, 11], and questions regarding their stability were addressed in [2, 3, 10]. In the present paper, we are interested in an *equivariant* rigidity notion, that allows to deform g -Hamiltonian stationary Lagrangian submanifolds according to a suitable deformation of the metric g . This works in a simi-

lar fashion to the implicit function theorem, but taking into account the ambiguity imposed by a group of symmetries of the variational problem. Namely, we assume that there is an isometric Hamiltonian action of a compact Lie group G on M . Such an action carries Lagrangian submanifolds to Lagrangian submanifolds (since the action is by symplectomorphisms), and preserves the volume functional (since the action is by isometries). In this way, a Hamiltonian stationary Lagrangian submanifold Σ is automatically degenerate due to the presence of action fields, and an appropriate *equivariant G -nondegeneracy* condition is introduced, see Definition 4.1. Our main result concerns deformations of such equivariantly nondegenerate submanifolds, corresponding to deformations of the metric g (or of the associated almost complex structure J) that preserve the group of symmetries G . More precisely, we prove the following:

Theorem. Let (M, ω, g_0, J_0) be a Kähler manifold with an isometric Hamiltonian action of a compact Lie group G . Suppose that either:

- (A) There exists a smooth deformation $[-\delta, \delta] \ni t \mapsto g_t \in \text{Met}(M)$ of the metric g_0 , such that G acts by g_t -isometries for all $t \in [-\delta, \delta]$; and let $J_t := J_{g_t}$ be the corresponding family of ω -compatible almost complex structures;¹

or,

- (B) There exists a smooth deformation $[-\delta, \delta] \ni t \mapsto J_t \in \mathcal{J}(M, \omega)$ of J_0 by ω -compatible almost complex structures, such that G acts by J_t -biholomorphisms for all $t \in [-\delta, \delta]$; and let $g_t(\cdot, \cdot) := \omega(\cdot, J_t \cdot)$ be the corresponding family of Riemannian metrics.

Suppose $\Sigma_0 \subset (M, \omega)$ is a G -nondegenerate g_0 -Hamiltonian stationary Lagrangian submanifold. Then there exists $\varepsilon > 0$, a neighborhood \mathcal{V} of $\Sigma_0 \in \mathcal{L}(\Sigma, M)$, a neighborhood \mathcal{E} of $[0] \in H^1(\Sigma, \mathbb{R})$ and a smooth map $\Sigma: (-\varepsilon, \varepsilon) \times \mathcal{E} \rightarrow \mathcal{V}$; such that $\Sigma_{t, [\eta]} := \Sigma(t, [\eta])$ is a g_t -Hamiltonian stationary Lagrangian submanifold in $\mathcal{L}(\Sigma, M)_{[\eta]}$ for all $|t| < \varepsilon$ and all

¹See (5) and Corollary 2.1.

$[\eta] \in \mathcal{E}$, and $\Sigma(0, [0]) = \Sigma_0$. Moreover, if $\Sigma_* \in \mathcal{V}$ is a g_t -Hamiltonian stationary Lagrangian submanifold in $\mathcal{L}(\Sigma, M)_{[\eta]}$ sufficiently close to the G -orbit of Σ_0 , then there exists $\phi \in G$ such that $\phi(\Sigma_*) = \Sigma_{t, [\eta]}$.

Note that the manifolds (M, ω, g_t, J_t) might not be Kähler for $t \neq 0$; the only requirement is that both the metric g_t and the almost complex structure J_t be compatible with the fixed symplectic form ω . In particular, this result abstractly yields existence of g -Hamiltonian stationary Lagrangian submanifolds in certain symplectic manifolds equipped with a metric g that is a small deformation of a Kähler metric, see [5, 8, 9].

The main ingredients in the proof of the above Theorem are the appropriate variational formulation of the problem, which is cast in (quotients of) Hölder spaces due to Fredholmness reasons and the equivariant implicit function theorem with low regularity studied in [4]. The latter is an abstract equivariant formulation of the classic implicit function theorem in a low regularity setup tailored to geometric variational problems. Among the crucial hypotheses are that the linear operator that represents the second variation of the functional in question be a Fredholm operator of index zero. This follows easily in the case of Hamiltonian stationary Lagrangians using standard Schauder estimates, since the corresponding operator is a (fourth order) elliptic operator.

This paper is organized as follows. In Section 2, we recall basic concepts in symplectic geometry and various Hamiltonian constructions and deformations preserving their symmetry groups. The main aspects of the constrained variational problem of Hamiltonian stationary Lagrangian submanifolds are studied in Section 3, where we also recall the first and second variations in the Kähler case. The rigorous framework for the proof of the above Theorem is discussed in Section 4. Finally, Section 5 contains a few examples of deformations to which this result applies.

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be stranded for several days while no transportation was available, providing a good amount of time to exchange ideas and start a collaboration whose first results led to this note.

2 Preliminaries

In this first section, we recall a few basic facts about the interplay of the isomorphism groups of symplectic, almost complex and Riemannian structures and basic definitions regarding Hamiltonian actions. The reader with a working knowledge of such material may proceed to Section 3.

2.1 Compatible triples

Given a (necessarily even dimensional) real vector space V , consider the following objects on V :

- (i) a nondegenerate skew-symmetric bilinear form $\omega: V \times V \rightarrow \mathbb{R}$;
- (ii) a complex structure $J: V \rightarrow V$;
- (iii) a positive-definite inner product $g: V \times V \rightarrow \mathbb{R}$.

Denote by $\text{Symp}(V, \omega)$ the group of symplectomorphisms of (V, ω) , i.e., automorphisms $T: V \rightarrow V$ such that $\omega(T\cdot, T\cdot) = \omega(\cdot, \cdot)$. Denote by $\text{Aut}(V, J)$ the group of J -biholomorphisms, or automorphisms of (V, J) , i.e., automorphisms $T: V \rightarrow V$ that commute with J . Finally, denote by $\text{O}(V, g)$ the group of g -orthogonal isomorphisms of V , i.e., automorphisms $T: V \rightarrow V$ such that $g(T\cdot, T\cdot) = g(\cdot, \cdot)$.

We say that (ω, J, g) is a *compatible triple* if $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$, or, equivalently, if $\omega(\cdot, J\cdot) = g(\cdot, \cdot)$. If (ω, J, g) is a compatible triple, then:

$$\text{O}(V, g) \cap \text{Aut}(V, J) = \text{O}(V, g) \cap \text{Symp}(V, \omega) = \text{Aut}(V, J) \cap \text{Symp}(V, \omega). \quad (1)$$

In other words, given vector spaces V_i endowed with compatible triples (ω_i, J_i, g_i) , $i = 1, 2$, an isomorphism $T: V_1 \rightarrow V_2$ that preserves any two of the structures in the triple, automatically preserves the third one. In

this way, choosing any two structures among $\{\omega, J, g\}$ on a vector space V determines the third one, so that the triple (ω, J, g) is compatible.

The definition of compatible triples carries over naturally to (even dimensional) smooth manifolds M endowed with a symplectic form ω , an almost complex structure J and a Riemannian metric g . Namely, such a triple (ω, J, g) is *compatible* if at every point $p \in M$, (ω_p, J_p, g_p) is a compatible triple on T_pM . In the special case where J is an integrable almost complex structure (hence a complex structure), the manifold M equipped with a compatible triple (ω, J, g) is called a *Kähler manifold*.

Equation (1) implies that if (ω, J, g) is a compatible triple on M , then

$$\begin{aligned} \text{Iso}(M, g) \cap \text{Symp}(M, \omega) &= \text{Iso}(M, g) \cap \text{Aut}(M, J) \\ &= \text{Aut}(M, J) \cap \text{Symp}(M, \omega). \end{aligned} \quad (2)$$

Note that, $\text{Iso}(M, g)$ is always a (finite-dimensional) Lie group, and it is compact when M is compact. The groups $\text{Aut}(M, J)$ and $\text{Symp}(M, \omega)$ are infinite-dimensional, however both intersections $\text{Iso}(M, g) \cap \text{Symp}(M, \omega)$ and $\text{Iso}(M, g) \cap \text{Aut}(M, J)$ are subgroups of $\text{Iso}(M, g)$ which are closed in the C^1 -topology², and therefore are Lie subgroups of $\text{Iso}(M, g)$.

2.2 Basic Hamiltonian constructions

Suppose (M, ω) is a symplectic manifold and X is a vector field on M . Contracting X with the symplectic form ω , we get a 1-form on M denoted $\iota_X\omega := \omega(X, \cdot)$. For convenience, we also use the special notation

$$\sigma_X := \iota_X\omega = \omega(X, \cdot). \quad (3)$$

When X is a field along a submanifold $\Sigma \subset M$, we also write σ_X for the 1-form on Σ obtained by pulling back the contracted 1-form $\iota_X\omega$, i.e., $\sigma_X = x^*(\iota_X\omega)$. As a word of caution, this 1-form is unrelated to the 2-form given by the pull-back $x^*\omega$.

²Hence, they are also closed in the C^0 -topology, since both topologies coincide on $\text{Iso}(M, g_J)$

A function $h: M \rightarrow \mathbb{R}$ is called a *Hamiltonian function* or *Hamiltonian potential* for the vector field X on M if

$$dh = \sigma_X. \quad (4)$$

In this case, X is called the *symplectic gradient* of h . As a side note, if (M, ω) is Kähler, then (4) is equivalent to $X = -J\nabla h$, where ∇h is the Riemannian gradient of h .

Vector fields on a symplectic manifold that admit a Hamiltonian potential are called *Hamiltonian vector fields*. Equivalently, a vector field is Hamiltonian if $\sigma_X \in B^1(M)$ is an *exact* 1-form. Vector fields such that $\sigma_X \in Z^1(M)$ is a *closed* 1-form are called *symplectic vector fields*. The justification for this name is that the flow of such a vector field preserves the symplectic form ω , i.e., the Lie derivative $\mathcal{L}_X \omega$ vanishes if X is symplectic. Evidently, Hamiltonian fields are always symplectic, and the obstruction for symplectic fields to be Hamiltonian is measured by the first de Rham cohomology $H^1(M, \mathbb{R}) = Z^1(M)/B^1(M)$.

Now, suppose that a Lie group G acts by symplectomorphisms on (M, ω) , i.e., $g: M \rightarrow M$ preserves ω for all $g \in G$. Denote by \mathfrak{g} the Lie algebra of G and by \mathfrak{g}^* its dual. The G -action is said to be *Hamiltonian* if there exists a map $\mu: M \rightarrow \mathfrak{g}^*$, called *moment map*, such that

- (i) $\mu: M \rightarrow \mathfrak{g}^*$ is G -equivariant, where the G -action considered on \mathfrak{g}^* is the coadjoint action;
- (ii) For every $X \in \mathfrak{g}$, denote by $X_p^* := \frac{d}{ds}(\exp(sX) \cdot p)|_{s=0}$ the induced action field on M .³ Then $\langle \mu(\cdot), X^* \rangle: M \rightarrow \mathbb{R}$ is a Hamiltonian potential for the vector field X^* , i.e., $d\langle \mu(\cdot), X^* \rangle = \sigma_{X^*}$, where, as above, $\sigma_{X^*} = \iota_{X^*} \omega = \omega(X^*, \cdot)$.

In other words, every action field is a Hamiltonian field and the moment map μ encodes all the corresponding Hamiltonian potentials.

³i.e., X^* is the vector field on M that is the infinitesimal generator of the 1-parameter group of diffeomorphisms of M generated by the 1-parameter subgroup $\mathbb{R} \ni s \mapsto \exp(sX) \in G$.

Example 2.1. If G is a closed connected Lie subgroup of $U(n+1)$, then the restriction of the transitive $U(n+1)$ -action on CP^n to G is a Hamiltonian action, with moment map $\mu([z]) = \pi_{\mathfrak{g}}(-i z z^* / 2 \|z\|^2)$, where $\pi_{\mathfrak{g}}$ is the orthogonal projection onto \mathfrak{g} , with respect to an Ad-invariant inner product in $U(n+1)$.

2.3 Deformations preserving symmetries

Constructing compatible triples on a manifold is quite elementary. We are interested in a slightly more elaborate problem, namely that of constructing 1-parameter families (ω, J_t, g_t) of compatible triples for a fixed symplectic form ω , that preserve a nontrivial subgroup G of (2) that acts in a Hamiltonian way on M , and for $t = 0$ turn M into a Kähler manifold. In general, most deformations of this type do not produce other Kähler structures, i.e., J_t is non-integrable for $t > 0$, but this is not an issue for our applications. For an example in which integrability is preserved, see Subsection 5.4.

We now observe that, due to (2), such deformations (ω, J_t, g_t) can be obtained by first considering a deformation g_t of the metric g_0 that preserves the isometric G -action; and then considering the corresponding deformation J_t of the almost complex structure J_0 with respect to the fixed symplectic form ω , see Corollary 2.1. A few concrete constructions of deformations of compatible triples are described in Section 5. Let us give more details on how the above works.

An almost complex structure J is called ω -compatible if the triple (ω, J, g_J) , where $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$, is a compatible triple. In other words, J is ω -compatible if g_J is a Riemannian metric. Define the spaces:

$$\mathcal{J}(M, \omega) := \{J : TM \rightarrow TM : J \text{ is an } \omega\text{-compatible almost complex structure}\};$$

$$\text{Met}(M, \omega) := \{g \in \text{Met}(M) : g = g_J \text{ for some } J \in \mathcal{J}(M, \omega)\}.$$

The map $\mathcal{J}(M, \omega) \ni J \mapsto g_J \in \text{Met}(M, \omega)$ is clearly a bijection, whose inverse will be denoted by $\text{Met}(M, \omega) \ni g \mapsto J_g \in \mathcal{J}(M, \omega)$. Let us

recall the following standard result that, in particular, implies that \mathbf{M} is homotopically equivalent to $\text{Met}(M)$ hence contractible, see [6, Prop 2.50, 2.51].

Proposition 2.1. Let (M, ω) be a symplectic manifold. There exists a smooth retraction $r: \text{Met}(M) \rightarrow (M, \omega)$.

Proof. Given $g \in \text{Met}(M)$, there exists a unique skew-symmetric $(1, 1)$ -tensor A_g on M that satisfies $\omega(\cdot, \cdot) = g(A_g \cdot, \cdot)$. Since ω is everywhere nondegenerate, A_g is everywhere nonsingular. The pointwise polar decomposition of A_g provides two unique $(1, 1)$ -tensors⁴ on M ,

$$P_g = (A_g A_g^*)^{\frac{1}{2}} \quad \text{and} \quad J_g = P_g^{-1} A_g, \quad (5)$$

such that:

- (i) $P_g J_g = J_g P_g = A_g$;
- (ii) P_g is positive, i.e., $g(P_g \cdot, \cdot)$ is symmetric and positive-definite;
- (iii) J_g is g -orthogonal, i.e., $J_g^* = J_g^{-1}$.

Since A_g is skew-symmetric and P_g is symmetric, then

$$J_g^{-1} = J_g^* = A_g^* P_g^{-1} = -A_g P_g^{-1} = -J_g,$$

i.e., J_g is an almost complex structure. The desired map r is given by $r(g) := g(P_g \cdot, \cdot)$. Since $P_g = J_g^{-1} A_g = J_g^* A_g$, then

$$r(g) = g(P_g \cdot, \cdot) = g(J_g^* A_g \cdot, \cdot) = g(A_g \cdot, J_g \cdot) = \omega(\cdot, J_g \cdot) \in \text{Met}(M, \omega). \quad (6)$$

By the uniqueness of the polar decomposition, it is also immediate to see that $r(g) = g$ if $g \in \text{Met}(M, \omega)$, i.e., r is a retraction. Smoothness follows immediately from the smoothness of the polar decomposition in the open set of invertible operators. ■

⁴Here, T^* denotes the g -adjoint of a $(1, 1)$ -tensor T on M , defined by $g(T^* \cdot, \cdot) = g(\cdot, T \cdot)$.

As an immediate consequence of the above result and (2), we get a way of deforming compatible triples preserving their symmetry, whose sole input is a metric deformation preserving an isometric action. This process fits the set of hypotheses (A) in the Theorem in the Introduction. This choice of deformation intuitively allows more examples, since it is generally easier to deform a metric preserving a group action than deforming an almost complex structure preserving its automorphism group.

Corollary 2.1. *Let (ω, J_0, g_0) be a compatible triple on M , and suppose that G is a Lie group that acts on M by symplectomorphisms and g_0 -isometries (hence by J_0 -biholomorphisms). Assume g_t , $t \in [-\delta, \delta]$, is a deformation of g_0 such that the G -action is by g_t -isometries for $t \in [-\delta, \delta]$, and let $J_t := J_{g_t}$ be the almost complex structure obtained from g_t as in (5). Then the triple (ω, J_t, g_t) , $t \in [-\delta, \delta]$, is compatible and a deformation of (ω, J_0, g_0) , so that G acts by g_t -isometries and J_t -biholomorphisms, $t \in [-\delta, \delta]$.*

3 Hamiltonian stationary Lagrangian submanifolds

3.1 Lagrangian submanifolds

Let (M, ω) be a symplectic manifold and Σ be a compact manifold with $\dim \Sigma = \frac{1}{2} \dim M$. An embedding $x: \Sigma \hookrightarrow (M, \omega)$ is called *Lagrangian* if $x^*\omega = 0$. In this case, we say $x(\Sigma) \subset M$ is a Lagrangian submanifold. Lagrangian submanifolds play a central role in symplectic geometry, see [6] and references therein. A smooth family $x_s: \Sigma \rightarrow M$, $s \in (-\varepsilon, \varepsilon)$, of embeddings is called a *Lagrangian* (respectively, *Hamiltonian*) *deformation* of $x_0 = x$ if its derivative $X = \frac{d}{ds}x_s|_{s=0}$ is a symplectic (respectively, Hamiltonian) vector field along x , i.e., if the 1-form $\sigma_X := x^*(\omega(X, \cdot))$ on Σ is *closed* (respectively, *exact*), see Subsection 2.2.

Example 3.1. Consider the sphere $S^2 = \mathbb{C}P^1$ with its standard Kähler structure. Any great circle Σ on S^2 is Lagrangian (and minimal), and

divides S^2 into two domains of same area. Any deformation of Σ through other smooth curves is a Lagrangian deformation; however only those deformations through curves that still bisect the area of S^2 are Hamiltonian, by Stokes' Theorem.

A particularly interesting feature of a Lagrangian submanifold Σ of a Kähler manifold (M, ω) is the following. Denote by H the mean curvature vector of Σ in M and consider the contracted 1-form σ_H . Then $d\sigma_H = \text{Ric}|_{\Sigma}$, where Ric is the Ricci 2-form on M . In particular, if (M, ω) is Kähler-Einstein, i.e., $\text{Ric} = \kappa\omega$, then $d\sigma_H = 0$ and thus H is a symplectic vector field, i.e., an infinitesimal Lagrangian deformation. As observed by [13], this suggests that it is natural to consider variational problems for volume of submanifolds with a Lagrangian constraint, as follows.

3.2 Minimizing volume

Let (ω, J, g) be a compatible triple on M and consider the corresponding volume form vol_g on M . This provides a way to measure the volume of an embedding $x: \Sigma \rightarrow M$, by setting

$$\text{Vol}_g(x(\Sigma)) = \int_{\Sigma} x^*(\text{vol}_g). \quad (7)$$

A Lagrangian embedding $x_0: \Sigma \rightarrow M$ is called *g-Lagrangian* (respectively, *g-Hamiltonian stationary*) if it has critical volume with respect to any Lagrangian (respectively, Hamiltonian) deformations $x_s: \Sigma \rightarrow M$, $s \in (-\varepsilon, \varepsilon)$, of x_0 , i.e.,

$$\left. \frac{d}{ds} \text{Vol}_g(x_s(\Sigma)) \right|_{s=0} = \int_{\Sigma} g(H, X) = 0, \quad (8)$$

where H is the mean curvature vector of Σ in (M, g) , $X_p = \left. \frac{d}{ds} x_s(p) \right|_{s=0}$ is the variation field and the integration is with respect to the pulled-back volume form $x^*(\text{vol}_g)$. In some references, Hamiltonian stationary Lagrangian submanifolds are also called *H-minimal submanifolds*. This reflects the fact that they minimize the volume functional in some directions, the *Hamiltonian* directions; while minimal submanifolds are critical

points of the volume functional with respect to all directions (hence, in particular, are Hamiltonian stationary).

Notice that the above functional remains invariant if we replace $x(\Sigma)$ with $\phi(x(\Sigma))$, where $\phi: M \rightarrow M$ is an isometric symplectomorphism. Namely, $\phi(x(\Sigma))$ is still Lagrangian because ϕ is a symplectomorphism, and it has the same volume as $x(\Sigma)$ because ϕ is an isometry.

Let us derive the Euler-Lagrange equations for this functional. If X is a Hamiltonian variation of x_0 , then $\sigma_X = dh$, for some $h: \Sigma \rightarrow \mathbb{R}$. Thus, if x_0 is g -Hamiltonian stationary Lagrangian, then (8) reads

$$0 = \int_{\Sigma} g(H, X) = \int_{\Sigma} g(\sigma_H, \sigma_X) = \int_{\Sigma} g(\sigma_H, dh) = \int_{\Sigma} (\delta\sigma_H)h,$$

where δ is the codifferential, i.e., the formal adjoint of the exterior derivative operator d . Since the above vanishes for all h , we get that the Euler-Lagrange equations for a Hamiltonian stationary Lagrangian embedding x_0 are

$$\delta\sigma_H = 0. \tag{9}$$

As pointed out before, if (M, ω, J) is Kähler-Einstein, then $d\sigma_H = 0$. Consequently, in the Kähler-Einstein case, (9) is equivalent to $\Delta\sigma_H = 0$, where $\Delta = d\delta + \delta d$ is the (nonnegative) Laplace-de Rham operator; i.e., σ_H is a harmonic 1-form. In particular, it follows by Hodge theory that if $H^1(\Sigma, \mathbb{R}) = 0$, then the Hamiltonian stationary Lagrangian embedding x_0 is actually minimal.

Example 3.2. An example of a Hamiltonian stationary Lagrangian submanifold that is not minimal is given by the standard tori $T = S^1(r_1) \times \cdots \times S^1(r_n) \subset \mathbb{C}^n$. Note that \mathbb{C}^n is Kähler-Einstein however the first de Rham cohomology of T is not trivial. Another interesting family of examples of Hamiltonian stationary Lagrangians is given by curves with constant geodesic curvature on a Riemann surface.

3.3 Second variation

Following [8, Def 2.6] and [11, Thm 3.4], we now describe the second variation formula of the functional (7), assuming that the compatible triple (ω, J, g) turns M into a Kähler manifold. Given a g -Hamiltonian stationary Lagrangian embedding $x_0: \Sigma \rightarrow M$ and any Hamiltonian deformation x_s of x_0 ,

$$\begin{aligned} \frac{d^2}{ds^2} \text{Vol}_g(x_s(\Sigma)) \Big|_{s=0} &= \int_{\Sigma} [g(\Delta dh, dh) - \text{Ric}(Jdh, Jdh) \\ &\quad - 2g(dh \otimes dh \otimes \sigma_H, S) + g(dh, \sigma_H)^2], \end{aligned}$$

where h is a Hamiltonian potential for the variation $X_p = \frac{d}{ds}x_s(p)|_{s=0}$, i.e., $\sigma_X = dh$, and S is a $(0, 3)$ -tensor on Σ defined by $S(X, Y, Z) = g(J(B(X, Y)), Z)$, where B is the second fundamental form of $x_0(\Sigma) \subset M$. In particular, we get an expression for the corresponding Jacobi operator Jac_{x_0} , which represents the above quadratic form and is the linearized operator of the Euler-Lagrange equations (9),

$$\text{Jac}_{x_0}(h) = \Delta^2 h + \delta\sigma_{\text{Ric}^\perp(J\nabla h)} - 2\delta\sigma_{B(JH, \nabla h)} - JH(JH(h)), \quad (10)$$

where $\text{Ric}^\perp(X)$ for a normal vector X to Σ is defined by $g(\text{Ric}^\perp(X), Y) = \text{Ric}(X, Y)$ for all Y normal to Σ .

4 The variational framework

Let us describe the appropriate variational framework that yields the proof of the Theorem in the Introduction, as an application of the equivariant implicit function theorem, as formulated, e.g., in [4, Thm 3.2].

4.1 Unparametrized embeddings

We denote by $\text{Emb}^{\mathcal{L}}(\Sigma, M)$ the space of Lagrangian embeddings $x: \Sigma \hookrightarrow (M, \omega)$ of class $C^{k, \alpha}$, $k \geq 4$, where the regularity choice is due to Fredholmness reasons. The space $\text{Emb}^{\mathcal{L}}(\Sigma, M)$ endowed with the corresponding $C^{4, \alpha}$ topology is a smooth Banach manifold, and its tangent space at

$x \in \text{Emb}^{\mathcal{L}}(\Sigma, M)$ can be identified with the space of $C^{4,\alpha}$ vector fields along x that are Lagrangian variations of x , see Subsection 3.1.

There is a natural action of the diffeomorphism group $\text{Diff}(\Sigma)$ on $\text{Emb}^{\mathcal{L}}(\Sigma, M)$. Two Lagrangian embeddings $x_i \in \text{Emb}^{\mathcal{L}}(\Sigma, M)$, $i = 1, 2$, are *congruent* if there exists a diffeomorphism $\psi: \Sigma \rightarrow \Sigma$ such that $x_1 = x_2 \circ \psi$, i.e., if they belong to the same orbit of this action. Given a Lagrangian embedding $x \in \text{Emb}^{\mathcal{L}}(\Sigma, M)$, we denote by $[x]$ its congruence class, i.e., the orbit of x . Denote by $\mathcal{L}(\Sigma, M)$ the orbit space of *unparametrized* Lagrangian embeddings of Σ in (M, ω) :

$$\mathcal{L}(\Sigma, M) := \text{Emb}^{\mathcal{L}}(\Sigma, M) / \text{Diff}(\Sigma), \quad (11)$$

i.e., the set of congruence classes of Lagrangian embeddings of Σ into M . In other words, an element $[x] \in \mathcal{L}(\Sigma, M)$ is a class of embeddings of Σ in M whose elements can be obtained from one another by reparametrizations. The set $\mathcal{L}(\Sigma, M)$ can be thus identified with the set of Lagrangian submanifolds of M (of class $C^{4,\alpha}$) that are diffeomorphic to Σ . We consider $\mathcal{L}(\Sigma, M)$ endowed with the induced quotient topology. The action of $\text{Diff}(\Sigma)$ is neither free nor proper, and the orbit space $\mathcal{L}(\Sigma, M)$ fails to be a smooth Banach manifold.

Let us briefly describe the structure of $\mathcal{L}(\Sigma, M)$. A classic result due to Weinstein [14] states that given a smooth (i.e., C^∞) Lagrangian embedding $x_0: \Sigma \rightarrow M$, there exists a smooth symplectomorphism Ψ from a neighborhood U of $x_0(\Sigma)$ in (M, ω) to a neighborhood V of the zero section of the cotangent bundle $T\Sigma^*$ endowed with its canonical symplectic structure, such that $\Psi \circ x_0$ is the inclusion of the zero section into $T\Sigma^*$. It is an easy observation that the image of a 1-form is a Lagrangian submanifold of $T\Sigma^*$ if and only if this 1-form is closed. Consequently, small Lagrangian deformations of $x_0: \Sigma \rightarrow M$ are parametrized by closed 1-forms on Σ . More precisely, given any Lagrangian embedding $x: \Sigma \rightarrow U$ of class $C^{4,\alpha}$ sufficiently close to x_0 , there exists a (unique) closed 1-form η_x on Σ such that $\Psi(x(\Sigma))$ is the image of $\eta_x: \Sigma \rightarrow T\Sigma^*$. In other words, the Lagrangian embeddings $\Psi \circ x$ and η_x are congruent. The map $[x] \mapsto \eta_x$

gives a continuous bijection from a neighborhood of $[x_0]$ in $\mathcal{L}(\Sigma, M)$ to a neighborhood of the origin in the Banach space of closed 1-forms on Σ of class $C^{4,\alpha}$. As x varies in the set of smooth Lagrangian embeddings of Σ into M , such bijections form an atlas of charts for the topological manifold $\mathcal{L}(\Sigma, M)$.

We observe, however, that the transition maps between two of these charts are in general only continuous, and not differentiable. This is due to a subtle technicality, which in particular implies that right-composition with a diffeomorphism of class $C^{4,\alpha}$ is not a differentiable map in the set of maps of class $C^{4,\alpha}$ between two smooth manifolds. This and other relevant issues concerning the lack of regularity of the space of unparametrized embeddings are discussed thoroughly in [1]. As explained in this reference, since we are only interested in local questions around a smooth embedding, we can use the above chart as an identification and formally treat $\mathcal{L}(\Sigma, M)$ as a smooth manifold. In this way, for convenience of notation we henceforth refer to, e.g., tangent spaces, distributions and smooth functions on $\mathcal{L}(\Sigma, M)$, and the implicit rigorous version of these objects are the corresponding objects defined in a small neighborhood of the origin of the Banach space of closed 1-forms on Σ of class $C^{4,\alpha}$.

Under the above convention, if $x_0 \in \text{Emb}^{\mathcal{L}}(\Sigma, M)$ is smooth, the tangent space at $[x_0]$ to $\mathcal{L}(\Sigma, M)$ can be identified as

$$T_{[x_0]}\mathcal{L}(\Sigma, M) = Z^1(\Sigma), \quad (12)$$

i.e., with the Banach space $Z^1(\Sigma)$ of closed 1-forms on Σ of class $C^{4,\alpha}$. Note this is the image of the surjective linear map $T_{x_0}\text{Emb}^{\mathcal{L}}(\Sigma, M) \ni X \mapsto \sigma_X \in Z^1(\Sigma)$, whose kernel corresponds to variations tangent to x_0 , i.e., reparametrizations, which form the tangent space to the orbit of $\text{Diff}(\Sigma)$ that passes through x_0 . This linear map is precisely the linearization of the orbit space projection $\text{Emb}^{\mathcal{L}}(\Sigma, M) \rightarrow \mathcal{L}(\Sigma, M)$ at the smooth embedding x_0 .

The tangent space (12) has a distinguished subspace, namely $B^1(\Sigma)$, formed by exact 1-forms on Σ of class $C^{4,\alpha}$. This subspace corresponds

to Hamiltonian variations of x_0 , and gives rise to an integrable distribution⁵ of $\mathcal{L}(\Sigma, M)$ with codimension $b_1(\Sigma) = \dim H^1(\Sigma, \mathbb{R})$, see [15]. We call this distribution the *Hamiltonian distribution* in $\mathcal{L}(\Sigma, M)$. Given a Lagrangian embedding $x_0: \Sigma \rightarrow M$, the integral leaves of the Hamiltonian distribution near $[x_0]$ are parametrized by elements of the first de Rham cohomology $H^1(\Sigma, \mathbb{R})$. Given a closed 1-form η on Σ , we denote by $[\eta]$ its cohomology class and by $\mathcal{L}(\Sigma, M)_{[\eta]}$ the integral leaf of the Hamiltonian distribution corresponding to $[\eta]$. In particular, when η is exact, i.e., $[\eta] = 0$, then $\mathcal{L}(\Sigma, M)_{[\eta]}$ is the integral leaf through $[x_0]$, i.e., consists of all the Hamiltonian deformations of x_0 .

4.2 Volume functional

We now describe how to encode the variational problem described in Subsection 3.2 in the above setup, for a varying family of metrics. Namely, we start from a family of volume functionals parametrized by a family $g_t \in \text{Met}(M)$, $t \in [-\delta, \delta]$, of metrics on M ,

$$\text{Vol}: \text{Emb}^{\mathcal{L}}(\Sigma, M) \times [-\delta, \delta] \rightarrow \mathbb{R}, \quad \text{Vol}(x, t) = \text{Vol}_{g_t}(x(\Sigma)).$$

This functional is clearly invariant under reparametrizations, i.e., under the action of $\text{Diff}(\Sigma)$. Hence, it passes to the quotient, defining a continuous map

$$\text{Vol}: \mathcal{L}(\Sigma, M) \times [-\delta, \delta] \rightarrow \mathbb{R}, \tag{13}$$

that is smooth in every local chart around $[x_0] \in \mathcal{L}(\Sigma, M)$, where $x_0 \in \text{Emb}^{\mathcal{L}}(\Sigma, M)$ is smooth, see [1, Cor 4.4].

For any fixed $t \in [-\delta, \delta]$, the critical points of (13) are exactly the g_t -Lagrangian stationary Lagrangian embeddings of Σ in M . Consider now a Lagrangian embedding $x_0: \Sigma \rightarrow M$, and let η be a closed 1-form on Σ whose cohomology class $[\eta]$ belongs to a sufficiently small

⁵Since we are working locally around smooth points, consider this distribution defined in an open subset that is the domain of a chart; where it is integrable in the usual sense that it is tangent to a foliation of this subset.

neighborhood of 0 in the vector space $H^1(\Sigma, \mathbb{R})$. For any fixed $t \in [-\delta, \delta]$, the critical points of the restriction of (13) to $\mathcal{L}(\Sigma, M)_{[\eta]}$ are precisely the g_t -Hamiltonian stationary Lagrangian embeddings of Σ in M that belong to $\mathcal{L}(\Sigma, M)_{[\eta]}$. This is the constrained variational problem that we use to prove our main result. Under the appropriate identifications, the Euler-Lagrange equation and Jacobi operator of this variational problem coincide with the ones discussed in the previous section.

Note that if there is an isometric Hamiltonian action of a Lie group G on M , the induced G -action by left-composition on the space of embeddings $\text{Emb}^{\mathcal{L}}(\Sigma, M)$ commutes with the $\text{Diff}(\Sigma)$ -action and hence induces a G -action on $\mathcal{L}(\Sigma, M)$. It is a straightforward observation that the Hamiltonian distribution in $\mathcal{L}(\Sigma, M)$ is preserved by this G -action, and that the above constrained variational problem is also invariant under such action.

4.3 G -nondegenerate embeddings

Since the variational problem in question is G -invariant, the linearization of deformations that correspond to the G -action will automatically produce elements in the kernel of the Jacobi operator (10). More precisely, for each x_0 consider the linear map

$$\mathfrak{g} \ni X \longmapsto \sigma_{X^*} \in B^1(\Sigma), \quad (14)$$

which associates to $X \in \mathfrak{g}$ the exact 1-form on Σ given by $\sigma_{X^*} = x_0^*(\omega(X^*, \cdot))$, where X^* is the action field corresponding to X . Such 1-form is exact, because the G -action is Hamiltonian. Since the action preserves ω , g_0 and the Hamiltonian distribution, the image \mathcal{N}_{x_0} of the linear map (14) is contained in the kernel of Jac_{x_0} , since this is the second variation of the volume functional (restricted to the integral leaf $\mathcal{L}(\Sigma, M)_{[0]}$ through $[x_0] \in \mathcal{L}(\Sigma, M)$ of the Hamiltonian distribution). Here we are identifying the space of exact 1-forms $B^1(\Sigma)$ of class $C^{k,\alpha}$ on Σ with the space of real-valued functions modulo constants $C^{k,\alpha}(\Sigma)/\mathbb{R}$; and \mathcal{N}_{x_0} can be identified with the tangent space at $[x_0]$ to the G -orbit of $[x_0] \in \mathcal{L}(\Sigma, M)_{[0]}$.

Definition 4.1. The g_0 -Hamiltonian stationary embedding x_0 is said to be G -nondegenerate if \mathcal{N}_{x_0} coincides with the kernel of Jac_{x_0} in $B^1(\Sigma)$.

In other words, x_0 is G -nondegenerate if the kernel of Jac_{x_0} is as small as it can be, since fields originating from (14) are necessarily in it. Examples of G -nondegenerate embeddings will be given in Section 5.

4.4 Proof of main result

We are now ready for the proof of the Theorem in the Introduction. For convenience, we restate it below in the language of compatible triples and unparametrized embeddings.

Theorem. Let (M, ω, g_0, J_0) be a Kähler manifold with an isometric Hamiltonian action of a compact Lie group G . Let (ω, g_t, J_t) be a deformation of the compatible triple (ω, g_0, J_0) , such that G acts by g_t -isometries and J_t -biholomorphisms, for all $t \in [-\delta, \delta]$. Suppose $x_0: \Sigma \rightarrow (M, \omega)$ is a G -nondegenerate g_0 -Hamiltonian stationary Lagrangian embedding. Then, there exists $\varepsilon > 0$, a neighborhood \mathcal{V} of $[x_0] \in \mathcal{L}(\Sigma, M)$, a neighborhood \mathcal{E} of $[0] \in H^1(\Sigma, \mathbb{R})$ and a map $x: (-\varepsilon, \varepsilon) \times \mathcal{E} \rightarrow \mathcal{V}$; such that $x(0, 0) = [x_0]$ and $x(t, [\eta])$ is a g_t -Hamiltonian stationary Lagrangian unparametrized embedding in $\mathcal{L}(\Sigma, M)_{[\eta]}$, for all $|t| < \varepsilon$ and all $[\eta] \in \mathcal{E}$. Moreover, given $(t, [\eta]) \in (-\varepsilon, \varepsilon) \times \mathcal{E}$, if $[x_*] \in \mathcal{V}$ is a g_t -Hamiltonian stationary Lagrangian unparametrized embedding in $\mathcal{L}(\Sigma, M)_{[\eta]}$ sufficiently close to the G -orbit of $[x_0]$, then there exists $\phi \in G$ such that $\phi([x_*]) = x(t, [\eta])$.

Proof. By assumption, the G -action preserves ω and g_t , for all $t \in [-\delta, \delta]$. Thus, the induced G -action by left-composition on the space of embeddings $x: \Sigma \rightarrow M$ preserves Lagrangian embeddings, as well as their g_t -volume. Moreover, since the action is assumed Hamiltonian, it also preserves the leaves of the Hamiltonian distribution. This means that by choosing a G -nondegenerate g_0 -Hamiltonian stationary Lagrangian embedding x_0 we are in the setup of the G -equivariant implicit function theorem studied in [4].

Using identification (12) and the canonical splitting $Z^1(\Sigma) = B^1(\Sigma) \oplus H^1(\Sigma, \mathbb{R})$, we can write a sufficiently small neighborhood \mathcal{U} of $[x_0] \in \mathcal{L}(\Sigma, M)$ as a product $\mathcal{U} = \mathcal{U}_B \times \mathcal{U}_H$, where \mathcal{U}_B is a neighborhood of $0 \in B^1(\Sigma)$ and \mathcal{U}_H is a neighborhood of $[0] \in H^1(\Sigma)$. In this way, any $[x] \in \mathcal{U}$ corresponds to a unique pair $(\beta, [\eta]) \in \mathcal{U}_B \times \mathcal{U}_H$, and $[x_0]$ corresponds to $(0, [0])$. Moreover, each slice $\mathcal{L}_{[\eta]} := \{(\beta, [\eta]) : \beta \in \mathcal{U}_B\}$ of $\mathcal{U}_B \times \mathcal{U}_H$ corresponds to the intersection of the leaf $\mathcal{L}(\Sigma, M)_{[\eta]}$ with \mathcal{U} . The abstract implicit function theorem is applied to the volume functional Vol in (13), considered as a function of three variables in the neighborhood \mathcal{U} of $[x_0]$; $\text{Vol}(t, \beta, [\eta])$, where t varies in $[-\delta, \delta]$, β varies in \mathcal{U}_B and $[\eta]$ varies in \mathcal{U}_H . Thus, g_t -Hamiltonian stationary Lagrangian embeddings correspond to points where the derivative of $\text{Vol}(t, \beta, [\eta])$ with respect to β vanishes.

In this setup, the only hypothesis in the equivariant implicit function theorem that requires additional explanation is the Fredholmness of the Jacobi operator Jac_{x_0} , which corresponds to the second variation of Vol with respect to the variable β at the point $(0, 0, [0])$. This operator is defined on the tangent space to \mathcal{U}_B at the origin, which is $B^1(\Sigma)$, that we now write as $B_{k,\alpha}^1(\Sigma)$ to emphasize that its elements are exact 1-forms on Σ of class $C^{k,\alpha}$. Such linear space is canonically identified with $C^{k,\alpha}(\Sigma)/\mathbb{R}$, the real-valued functions Hölder space $C^{k,\alpha}$ modulo constants. Although so far we were implicitly using this identification for convenience, it is now important to write it explicitly. The Jacobi operator of the variational problem above mentioned is given by the composition of the linear maps in the bottom line of the following commutative diagram:

$$\begin{array}{ccccc}
 C^{k,\alpha}(\Sigma) & \xrightarrow{\text{Jac}_{x_0}} & C^{k-4,\alpha}(\Sigma) & & \\
 \downarrow & \nearrow & \downarrow & & \\
 B_{k,\alpha}^1(\Sigma) & \xrightarrow{\cong} & C^{k,\alpha}(\Sigma)/\mathbb{R} & \xrightarrow{[\text{Jac}_{x_0}]} & C^{k-4,\alpha}(\Sigma)/\mathbb{R} \xrightarrow{\cong} B_{k-4,\alpha}^1(\Sigma),
 \end{array}$$

where the vertical arrows are the natural projections and the top line operator Jac_{x_0} is given by formula (10). This formula shows that Jac_{x_0} is

a fourth-order (formally self-adjoint) linear elliptic operator. Thus, from standard Schauder estimates, such an operator is Fredholm of index 0. Since constant functions are in the kernel of Jac_{x_0} , it induces an operator from the quotient $C^{k,\alpha}(\Sigma)/\mathbb{R}$, denoted by the dotted arrow. Such operator has the same image as Jac_{x_0} and its kernel is $(\ker \text{Jac}_{x_0})/\mathbb{R}$. Hence, it is a Fredholm operator of index -1 . The Jacobi operator $[\text{Jac}_{x_0}]: C^{k,\alpha}(\Sigma)/\mathbb{R} \rightarrow C^{k-4,\alpha}(\Sigma)/\mathbb{R}$ is given by the composition of the latter with the projection $C^{k-4,\alpha}(\Sigma) \rightarrow C^{k-4,\alpha}(\Sigma)/\mathbb{R}$, which is a Fredholm operator of index $+1$. Thus, $[\text{Jac}_{x_0}]$ is also Fredholm and its index is the sum of the indices of its factors, which is 0. This proves the desired Fredholmness condition. From⁶ [4, Thm 3.2], we now get that there exists a map $\beta(t, [\eta])$ (defined locally) such that the map $x(t, [\eta]) := (\beta(t, [\eta]), [\eta]) \in \mathcal{U}_B \times \mathcal{U}_H = \mathcal{U} \subset \mathcal{L}(\Sigma, M)$ satisfies the desired conditions. ■

5 Examples of deformations

In this section, we describe some deformations of compatible triples, and a few examples to which the result proved above applies.

5.1 Cheeger deformations

We now briefly outline an important example of metric deformation preserving symmetries, the so-called *Cheeger deformation*, and the corresponding deformation of almost complex structures via Corollary 2.1. Cheeger deformations are very important tools in Riemannian geometry (see [16, 17]), and its counterpart in almost complex manifolds given by the above correspondence apparently has not yet been thoroughly explored.

Let $J_0 \in \mathcal{J}(M, \omega)$ be an ω -compatible almost complex structure on (M, ω) , and $g_0(\cdot, \cdot) = \omega(\cdot, J_0 \cdot)$. Suppose G is a compact Lie group of

⁶For further details, the reader may follow the proof of the constant mean curvature hypersurfaces version of the equivariant implicit function theorem discussed in [4], which has many analogies with the application discussed here.

symplectomorphisms of (M, ω) , that acts on (M, g_0) by isometries. This gives a partition of M into G -orbits, and the deformation g_t of g_0 we now describe essentially works by rescaling g_0 in the direction of these orbits, leaving it unchanged in the complementary directions. This will then automatically cause g_t to be G -invariant, i.e., the deformation will be through metrics that still have an isometric G -action. Such deformations have been extensively used in many situations, see [16, 17], and we outline its construction following the notation of the above references.

Fix a bi-invariant metric Q on G , and consider the product manifold $M \times G$ endowed with the product metric $g + \frac{1}{t}Q$. Denote by $g \cdot p$ the action of $g \in G$ on $p \in M$ and define a submersion

$$\rho: M \times G \rightarrow M, \quad \rho(p, g) = (g^{-1}) \cdot p.$$

Let $g_t \in \text{Met}(M)$ be the unique metric that makes ρ a Riemannian submersion. It is immediate from its definition that g_t is G -invariant, i.e., G acts isometrically on (M, g_t) , $t > 0$. The curve of metrics g_t , $t > 0$, extends smoothly across $t = 0$, and coincides with the original metric g_0 at this point. In this sense, g_t is a deformation of g_0 .

In order to analyze how g_t , $t > 0$, differs from g_0 , we have to introduce some more notation. Let G_p be the isotropy group at $p \in M$ and \mathfrak{g}_p its Lie algebra. Fix the Q -orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{m}_p$, and identify \mathfrak{m}_p with the tangent space $T_p G(p)$ to the G -orbit through p , via action fields; i.e., $X \in \mathfrak{m}_p$ is identified with $X_p^* = \frac{d}{ds} \exp(sX)|_{s=0} \in T_p G(p)$. This induces a g_t -orthogonal decomposition $T_p M = \mathcal{V}_p \oplus \mathcal{H}_p$ in vertical space $\mathcal{V}_p = \{X_p^* \in T_p G(p) : X \in \mathfrak{m}_p\}$ and horizontal space $\mathcal{H}_p = \mathcal{V}_p^\perp$, where $^\perp$ is the g_t -orthogonal complement. Let

$$P_t: \mathfrak{m}_p \rightarrow \mathfrak{m}_p, \quad Q(P_t(X), Y) = g_t(X_p^*, Y_p^*).$$

Then P_t is a Q -symmetric automorphism that represents g_t in terms of Q , and it can be easily computed (see [16, Prop 1.1]) that $P_t = P_0 (\text{id} + tP_0)^{-1}$, $t \geq 0$. Thus, defining

$$C_t: T_p M \rightarrow T_p M, \quad g(C_t(X), Y) = g_t(X, Y),$$

we get

$$C_t(X) = P_0^{-1}P_t(X^\mathcal{V}) + X^\mathcal{H},$$

where $X^\mathcal{V}$ and $X^\mathcal{H}$ are the vertical and horizontal components of X respectively. If P_0 has eigenvalues λ_i , then by the above formula, C_t has eigenvalues $\frac{1}{1+t\lambda_i}$ in the vertical directions and 1 in the horizontal directions. This means that as t increases, the metric g_t shrinks in the direction of the G -orbits and stays unchanged in the remaining directions.

By Corollary 2.1, given the above deformation g_t of g_0 , there is a corresponding deformation $J_t = J_{g_t}$ of J_0 , such that the G -action by symplectomorphisms on M is also by g_t -isometries and J_t -biholomorphisms, for $t > 0$.

5.2 Minimal Lagrangians in Kähler-Einstein manifolds

Hamiltonian stationary Lagrangians in Kähler-Einstein manifolds have been extensively studied in the literature, see e.g. [2, 3, 5, 9, 10, 11, 12]. In this subsection, we are interested in the particular case of *minimal* Lagrangians Σ of these manifolds, which are automatically Hamiltonian stationary, see Subsection 3.2. In this case, the Jacobi operator (10) assumes a very simple form. Namely, if κ is the Einstein constant of (M, g_0) , i.e., $\text{Ric} = \kappa g_0$ and $x_0: \Sigma \rightarrow M$ is a minimal Lagrangian submanifold, then

$$\text{Jac}_{x_0}(h) = \Delta^2 h + \kappa \delta \sigma_{J\nabla h} = \Delta(\Delta h - \kappa h), \quad (15)$$

since $\sigma_{J\nabla h} = -dh$. Let us analyze the kernel of this operator. Since Σ is compact, the function $\Delta h - \kappa h$ is harmonic if and only if it is constant. We are working on $C^{k,\alpha}(\Sigma)/\mathbb{R}$, i.e., modulo constants, so it follows that elements in the kernel of the Jacobi operator are precisely the exact 1-forms dh such that

$$\Delta h - \kappa h = 0, \quad (16)$$

i.e., h is an eigenfunction of the Laplacian on Σ , corresponding to the Einstein constant κ of M .

The *Hamiltonian stability* of such minimal Lagrangian submanifolds, i.e., whether their Jacobi operator is nonnegative, plays an important role in the theory. An immediate conclusion from the above is that Σ is Hamiltonian stable if and only if $\lambda_1(\Sigma) \geq \kappa$, where $\lambda_1(\Sigma)$ is the first eigenvalue of the Laplacian on (functions on) Σ , cf. [10, Thm 4.4].

For our equivariant setup, the Kähler-Einstein manifold M has an isometric Hamiltonian G -action, hence every action field not tangent to Σ induces an element of its Jacobi operator through the map (14). In particular, (16) always has nontrivial solutions, i.e., the Einstein constant κ is an eigenvalue of the Laplacian on Σ . Such a minimal Lagrangian is G -nondegenerate if and only if all solutions h of (16) are of this form, i.e., if there exists an action field X^* such that $h = \langle \mu(\cdot), X^* \rangle|_{\Sigma}$, where μ is the moment map of the action. For example, one way to verify this condition is by dimensional reasons. Namely, if the dimension of the span of action fields normal to $x_0(\Sigma) \subset M$ is larger than or equal to the multiplicity of κ as an eigenvalue of the Laplacian on Σ , then Σ is G -nondegenerate. This provides a setup to which our main result applies, considering the 1-parameter family of compatible triples (ω, J_t, g_t) obtained by a Cheeger deformation of g_0 with respect to the G -action.

Let us give a few concrete examples in the case (M, g_0) is $\mathbb{C}P^n$ with its standard Kähler structure, for which $\kappa = 2n + 2$. More precisely, we will consider totally real minimal submanifolds $x_0: \Sigma \rightarrow \mathbb{C}P^n$ with parallel second fundamental form. These were classified by Naitoh and Takeuchi, see [2, Sec 2]. Following the classification, Amarzaya and Ohnita [2] determined which of those submanifolds are Hamiltonian stable. Namely, they obtained the following, see [2, Thm 4.1].

Theorem 5.1 (Amarzaya-Ohnita). Let Σ be a n -dimensional totally real minimal submanifold embedded in $\mathbb{C}P^n$ with parallel second fundamental form in the following table:

Σ	n
$SU(p)/\mathbb{Z}_p$	$p^2 - 1$
$SU(p)/SO(p)\mathbb{Z}_p$	$(p - 1)(p + 2)/2$
$SU(2p)/Sp(p)\mathbb{Z}_{2p}$	$(p - 1)(2p + 1)$
$E_6/F_4\mathbb{Z}_3$	26

Then Σ is a Hamiltonian stable minimal Lagrangian submanifold in $\mathbb{C}P^n$. Moreover, the kernel of the Jacobi operator of Σ is exactly the span of the normal projections of Killing vector fields on $\mathbb{C}P^n$.

With the above result at hand, one can apply our deformation methods to a such Σ using any Cheeger deformation of $\mathbb{C}P^n$ with respect to a G -action that has the following extra property: the normal space at each $p \in x_0(\Sigma)$ must be contained in the tangent space $T_pG(p)$ to the G -orbit through p . It then automatically follows that the image of the map (14) is precisely the kernel of the Jacobi operator Jac_{x_0} , i.e., x_0 is G -nondegenerate.

5.3 Sasaki metrics on tangent bundles

Let (N, g) be a Riemannian manifold, and consider its tangent bundle $M = TN$. We recall a standard construction of a Riemannian metric on M , starting from a metric on N , see also [7]. Let $\pi: TN \rightarrow N$ denote the canonical projection; for $v \in TN$, write $T_v(TN) = \text{Ver}_v \oplus \text{Hor}_v$, where Ver_v is the *vertical subspace*, i.e., the tangent space to the fiber T_pN , where $p = \pi(v)$, and Hor_v is the *horizontal subspace* determined by the Levi-Civita connection of g . Given $\xi \in T_v(TN)$, we denote by ξ^{ver} and ξ^{hor} its vertical and horizontal component, respectively.

The spaces Ver_v and Hor_v are g_S -orthogonal. There is a canonical isomorphism $\text{Ver}_v = T_v(T_pN) \rightarrow T_pN$; the restriction of g_S to Ver_v is defined to be equal to the pull-back of g_p through such isomorphism. Moreover, the restriction of the differential $d\pi(p)|_{\text{Hor}_v}: \text{Hor}_v \rightarrow T_pN$ is an isomorphism, the restriction of g_S to Hor_v is defined to be equal to the pull-back

of g_p through such isomorphism. This defines a smooth Riemannian tensor g_S on $M = TN$, called the *Sasaki metric associated to g* .

In addition, a symplectic form ω_g can be defined on M , as follows. Given $v \in TN$, and $\xi, \eta \in T_v(TN)$, let

$$\omega_g(\xi, \eta) := g(\xi^{\text{ver}}, \eta^{\text{hor}}) - g(\xi^{\text{hor}}, \eta^{\text{ver}}).$$

This symplectic structure interacts well with the Sasaki metric, due to the following observation.

Lemma 5.1. *Let $f: N \rightarrow N$ be a g -isometry. Then, $df: TN \rightarrow TN$ preserves both the Sasaki metric g_S and the symplectic form ω_g . Moreover, the map*

$$\text{Iso}(N, g) \ni f \mapsto df \in \text{Iso}(TN, g_S) \cap \text{Symp}(TN, \omega_g)$$

is an injective Lie group homomorphism with closed image. ■

In particular, if G is a Lie group acting by isometries on a Riemannian manifold (N, g) , we also have a G -action on M by g_S -isometries that preserve ω_g . If the G -action on N preserves a 1-parameter family of Riemannian metrics g_t (e.g., a Cheeger deformation), then the corresponding action on M provides an example of the situation considered in the Theorem in the Introduction, with the choice of a G -nondegenerate Hamiltonian stationary Lagrangian of M .

5.4 Kähler-Ricci flow

One situation in which the deformation (ω, J_t, g_t) preserves the integrability of J_t , i.e., the fact that (M, ω, J_t, g_t) is Kähler, is when g_t evolves by the Kähler-Ricci flow. This means g_t is a solution of the evolution equation $\frac{\partial}{\partial t} g_t = -2 \text{Ric}(g_t)$, which also clearly preserves the isometries of the initial metric g_0 . Thus, if there is a Hamiltonian isometric action of G on (M, g_0) , we automatically get that G acts on M by g_t -isometries and J_t -biholomorphisms. In this way, G -nondegenerate g_0 -Hamiltonian stationary Lagrangians of (M, ω, J_0, g_0) may be deformed to g_t -Hamiltonian

stationary Lagrangians up to the G -action, where g_t is the Kähler-Ricci flow of g_0 .

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