Waves in magnetized quark matter

D.A. Fogaça *, S.M. Sanches Jr., F.S. Navarra

Instituto de Física, Universidade de São Paulo, Rua do Matão Travessa R, 187, 05508-090 São Paulo, SP, Brazil

Abstract

We study wave propagation in a non-relativistic cold quark-gluon plasma immersed in a constant magnetic field. Starting from the Euler equation we derive linear wave equations and investigate their stability and causality. We use a generic form for the equation of state, the EOS derived from the MIT bag model and also a variant of the this model which includes gluon degrees of freedom. The results of this analysis may be relevant for perturbations propagating through the quark matter phase in the core of compact stars and also for perturbations propagating in the low temperature quark-gluon plasma formed in low energy heavy ion collisions, to be carried out at FAIR and NICA.
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1. Introduction

There is a strong belief that quark gluon plasma (QGP) has been formed in heavy ion collisions at RHIC and at LHC [1,2]. Deconfined quark matter may also exist in the core of compact stars [3]. Waves may be formed in the QGP [4–6]. In heavy ion collisions waves may be produced, for example, by fluctuations in baryon number, energy density or temperature caused by inhomogeneous initial conditions [7]. Furthermore, there may be fluctuations induced by energetic partons, which have been scattered in the initial collision of the two nuclei and propagate through the medium, loosing energy and acting as a source term for the hydrodynamical equations.

* Corresponding author.
 E-mail address: david@if.usp.br (D.A. Fogaça).

In [5] we have studied wave propagation in cold and dense matter both in a hadron gas phase and in a quark gluon plasma phase. In deriving wave equations from the equations of hydrodynamics, we have considered both small and large amplitude waves. The former were treated with the linearization approximation while the latter were treated with the reductive perturbation method. Linear waves were obtained by solving an inhomogeneous viscous wave equation and they have the familiar form of sinusoidal traveling waves multiplied by an exponential damping factor, which depends on the viscosity coefficients. Since these coefficients differ by two orders of magnitude, even without any numerical calculation we concluded that, apart from extremely special parameter choices, in contrast to the quark gluon plasma there will be no linear wave propagation in a hadron gas.

In this work we will investigate the effects of a magnetic field on wave propagation in a quark gluon plasma. We shall focus on the stability and causality of these waves. A natural question is "how does the magnetic field affect stability and causality of density waves ?". We will try to answer this question in a, as much as possible, model independent way.

Our conclusions should apply to the deconfined cold quark matter in compact stars and to the cold (or slightly warm) quark gluon plasma formed in heavy ion collisions at intermediate energies, to be performed at FAIR [8] or NICA [9].

In what follows we will carry out a wave analysis which is very frequently used in hydrodynamcis [10]. We will be able to see if the presence of a magnetic field modifies the conclusions reached in [5].

2. Hydrodynamics in an external magnetic field

We shall consider the non-relativistic Euler equation [11] with an external magnetic field $\vec{B}$. The three fermions species (three quarks) have negative or positive charges and due to the external magnetic field they may assume different trajectories [12,13]. As a consequence we must apply the multifluid approach [12,13], which consists in writing one Euler equation for each quark $f = u, d, s$:

$$\rho_{m\,f}\left[\frac{\partial \vec{v}_f}{\partial t} + (\vec{v}_f \cdot \vec{\nabla})\vec{v}_f\right] = -\vec{\nabla}p + \rho_{c\,f}\left(\vec{v}_f \times \vec{B}\right) \tag{1}$$

where $\rho_{m\,f}$ and $\rho_{c\,f}$ are the mass and charge density of the quarks of flavor f respectively. We employ natural units ($\hbar = c = 1$) and the metric used is $g^{\mu\nu} = \text{diag}(+, -, -, -)$.

When we employ the multifluid approach, we are effectively using the approximation of weak interactions between the fluid constituents. In principle in an ideal QGP the interaction between the quark and gluon constituents is weak. In the presence of a strong magnetic field the interaction is even weaker, since the coupling constant decreases with increasing B field [14]. We will work with three equations of state. In the first two of them there is no interaction between the constituents. They are compatible with the multifluid approach. In the third one (called "mean field QCD") we have interactions, but the coupling constant is not large. What justifies the mean field approximation is the high density of sources. So we assume that in all our calculations we are in the weak coupling regime and hence we can borrow all the techniques and approximations (including the multifluid approach) from the plasmas known in electrodynamics.

In what follows we will consider quark matter with three quark flavors: up (u), down (d) and strange (s). As it is usually studied in [15], such quark matter may exist in compact stars. The charges are: $Q_u = 2\,Q_e/3$, $Q_d = -\,Q_e/3$ and $Q_s = -\,Q_e/3$, where $Q_e = 0.08542$ is the abso-

lute value of the electron charge in natural units [15,16]. The masses are [17]: $m_u = 2.2$ MeV, $m_d = 4.7$ MeV and $m_s = 96$ MeV.

In the above equation the pressure is a global feature of the fluid. The velocity, masses and charges are specified for each fermion species. The equation of state contains all fermions of the fluid under the external magnetic field $\vec{B}$. The magnetic field effects are included both in the Euler equation and in the equation of state. We consider an uniform magnetic field of intensity B in the z-direction described by $\vec{B} = B\hat{z}$.

The continuity equation for the baryon density ρ_{Bf} reads [11]:

$$\frac{\partial \rho_{Bf}}{\partial t} + \nabla \cdot (\rho_{Bf}\, \vec{v}_f) = 0 \tag{2}$$

In general, the relationship between the mass density (ρ_m) and the particle density (ρ) is given by $\rho_m = m\rho$, where m is the particle mass. We have then $\rho_{mf} = m_f\, \rho_f$ in (1). Besides, the quark number density can be rewritten in terms of the respective baryon density as $\rho_{mf} = 3m_f\, \rho_{Bf}$, since $\rho_{Bf} = \rho_f/3$. The charge density in (1) of each quark is given by $\rho_{cu} = 2Q_e\, \rho_{Bu}$, $\rho_{cd} = -Q_e\, \rho_{Bd}$ and $\rho_{cs} = -Q_e\, \rho_{Bs}$. In short we have $\rho_{cf} = 3\, Q_f\, \rho_{Bf}$ for each quark f.

3. Non-relativistic equation of state

The equation of state of the quark gluon plasma can be written as:

$$p = c_s{}^2 \epsilon \tag{3}$$

where p, ϵ and c_s are the pressure, energy density and speed of sound respectively. In the presence of an external magnetic field, we may have two different pressures, one parallel ($p_\parallel$) and another perpendicular ($p_\perp$) to the B field direction. Consequently we will also have a parallel ($c_{s\parallel}$) and a perpendicular ($c_{s\perp}$) speed of sound. They are given by [18]:

$$(c_{s\parallel})^2 = \frac{\partial p_\parallel}{\partial \varepsilon} \quad \text{and} \quad (c_{s\perp})^2 = \frac{\partial p_\perp}{\partial \varepsilon} \tag{4}$$

and so $p_\parallel \approx (c_{s\parallel})^2\, \varepsilon$ and also $p_\perp \approx (c_{s\perp})^2\, \varepsilon$. In the non-relativistic limit we have [5]: $\varepsilon \cong \rho_m$, where ρ_m is the volumetric mass density, which can be rewritten as $\rho_m = 3m_f\, \rho_{Bf}$. Considering the pressure anisotropy we have:

$$\vec{\nabla} p \approx 3m_f \left((c_{s\perp})^2\, \frac{\partial \rho_{Bf}}{\partial x}\, ,\ (c_{s\perp})^2\, \frac{\partial \rho_{Bf}}{\partial y}\, ,\ (c_{s\parallel})^2\, \frac{\partial \rho_{Bf}}{\partial z}\hat{z} \right) \tag{5}$$

Inserting (5) into (1), we have for the f-quark:

$$3m_f\, \rho_{Bf} \left[\frac{\partial \vec{v}_f}{\partial t} + (\vec{v}_f \cdot \vec{\nabla})\vec{v}_f \right] =$$

$$-3m_f \left((c_{s\perp})^2\, \frac{\partial \rho_{Bf}}{\partial x}\, ,\ (c_{s\perp})^2\, \frac{\partial \rho_{Bf}}{\partial y}\, ,\ (c_{s\parallel})^2\, \frac{\partial \rho_{Bf}}{\partial z} \right) + 3\, Q_f\, \rho_{Bf} \left(\vec{v}_f \times \vec{B} \right) \tag{6}$$

Linear waves are studied with the dispersion relation obtained through the linearization formalism [5,6]. In this formalism the Euler equation (6) and the continuity equation (2) are rewritten in terms of the perturbed dimensionless variables for the densities, $\hat{\rho}_{Bf}$, and also for the

velocities, $\hat{\vec{v}}_f$, defined from the equilibrium configuration (density ρ_0 and sound speed c_s). The perturbations are described by the corresponding small deviations denoted by δ:

$$\hat{\rho}_{B\,f}(\vec{x}, t) = \frac{\rho_{B\,f}(\vec{x}, t)}{\rho_0} = 1 + \delta\rho_{B\,f}(\vec{x}, t) \tag{7}$$

and

$$\hat{\vec{v}}_f(\vec{x}, t) = \frac{\vec{v}_f(\vec{x}, t)}{c_s} = \delta\vec{v}_f(\vec{x}, t) \tag{8}$$

and only $\mathcal{O}(\delta)$ terms are considered. Inserting (7) and (8) into (6) and into (2), and linearizing both equations, we find:

$$3m_f\,\rho_0\,\frac{\partial}{\partial t}\delta\tilde{\vec{v}}_f + 3m_f\,\rho_0\left((c_{s\perp})^2\frac{\partial}{\partial x}\delta\rho_{B\,f},\ (c_{s\perp})^2\frac{\partial}{\partial y}\delta\rho_{B\,f},\ (c_{s\parallel})^2\frac{\partial}{\partial z}\delta\rho_{B\,f}\right)$$
$$- 3\,Q_f\,\rho_0\left(\delta\tilde{\vec{v}}_f \times \vec{B}\right) = 0 \tag{9}$$

and

$$\frac{\partial}{\partial t}\delta\rho_{B\,f} + \vec{\nabla}\cdot\delta\tilde{\vec{v}}_f = 0 \tag{10}$$

where we have defined $\delta\tilde{\vec{v}}_f = \left(c_{s\perp}\,\delta v_{f_x},\ c_{s\perp}\,\delta v_{f_y},\ c_{s\parallel}\,\delta v_{f_z}\right)$.

To study causality and stability, we follow the procedure adopted in [5,6,10,19–21], where the perturbations are described by plane waves:

$$\delta\varrho = \mathcal{D}\,e^{i\vec{k}\cdot\vec{x}-i\omega t},\quad \delta V_x = \mathcal{V}_x\,e^{i\vec{k}\cdot\vec{x}-i\omega t},\quad \delta V_y = \mathcal{V}_y\,e^{i\vec{k}\cdot\vec{x}-i\omega t}\quad\text{and}\quad \delta V_z = \mathcal{V}_z\,e^{i\vec{k}\cdot\vec{x}-i\omega t} \tag{11}$$

with $\vec{k}\cdot\vec{x} = k_x\,x + k_y\,y + k_z\,z$. The small amplitudes for the dimensionless variables are given by $\mathcal{D}$, $\mathcal{V}_x$, $\mathcal{V}_y$ and $\mathcal{V}_z$. In general, the frequency ω is decomposed as in [6,19–21]: $\omega = \omega_R + i\omega_I$ with $\omega_R \in \mathbb{R}$ and $\omega_I \in \mathbb{R}$. Causality is ensured when the following conditions for ω_R and ω_I are satisfied [22]:

$$\lim_{|\vec{k}|\to\infty}\left|\frac{\omega_R}{|\vec{k}|}\right| < 1 \tag{12}$$

and

$$\lim_{|\vec{k}|\to\infty}\left|\frac{\omega_I}{|\vec{k}|}\right| < \infty \tag{13}$$

The condition (12) is equivalent to stating that the phase velocity $|\vec{v}_p|$ is smaller than unity (the speed of light in natural units), i.e. $|\vec{v}_p| < 1$, where

$$\vec{v}_p = \frac{\omega_R}{|\vec{k}|}\hat{k} = \frac{\omega_R}{|\vec{k}|^2}\vec{k} \tag{14}$$

does not become greater as the wave number increases [19–21]. As a consistency check we evaluate the group velocity, (v_g), which is given by [20–22]:

$$\vec{v}_g = \left(\frac{\partial \omega_R}{\partial k_x}, \frac{\partial \omega_R}{\partial k_y}, \frac{\partial \omega_R}{\partial k_z} \right) \tag{15}$$

and must satisfy $|\vec{v}_g| < \infty$ as the wave number increases. Stability is guaranteed when $\omega_I < 0$, since $e^{i\vec{k}\cdot\vec{x} - i\omega t} = e^{\omega_I t} e^{i\vec{k}\cdot\vec{x} - i\omega_R t}$ and $e^{\omega_I t}$ must be a decreasing function of time.

Inserting (11) into the equations (9) and (10), we are able to rewrite the resulting equations in the following matrix form:

$$A(\omega, \vec{k}) \times \begin{pmatrix} \delta\rho_{Bf} \\ \delta v_{fx} \\ \delta v_{fy} \\ \delta v_{fz} \end{pmatrix} = 0 \tag{16}$$

where $A(\omega, \vec{k})$ is the matrix given by:

$$A(\omega, \vec{k}) =$$
$$\begin{pmatrix} i\,3m_f\,\rho_0\,(c_{s\perp})^2\,k_x & -i\,3m_f\,\rho_0\,\omega\,(c_{s\perp}) & -3\,Q_f\,\rho_0\,B\,(c_{s\perp}) & 0 \\ i\,3m_f\,\rho_0\,(c_{s\perp})^2\,k_y & 3\,Q_f\,\rho_0\,B\,(c_{s\perp}) & -i\,3m_f\,\rho_0\,\omega\,(c_{s\perp}) & 0 \\ i\,3m_f\,\rho_0\,(c_{s\perp})^2\,k_z & 0 & 0 & -i\,3m_f\,\rho_0\,\omega\,(c_{s\parallel}) \\ -i\,\omega & i\,(c_{s\perp})\,k_x & i\,(c_{s\perp})\,k_y & i\,(c_{s\parallel})\,k_z \end{pmatrix} \tag{17}$$

The dispersion relation is found by solving the equation $det\,A(\omega, \vec{k}) = 0$. It may be written as:

$$\omega^4 - \left[(c_{s\perp})^2\,k_x{}^2 + (c_{s\perp})^2\,k_y{}^2 + (c_{s\parallel})^2\,k_z{}^2 + \left(\frac{B^2\,Q_f{}^2}{m_f^2} \right) \right] \omega^2$$
$$+ \left(\frac{B^2\,Q_f{}^2}{m_f^2} \right) (c_{s\parallel})^2\,k_z{}^2 = 0 \tag{18}$$

which implies that

$$\omega^2{}_\pm = \frac{(c_{s\perp})^2\,k_x{}^2}{2} + \frac{(c_{s\perp})^2\,k_y{}^2}{2} + \frac{(c_{s\parallel})^2\,k_z{}^2}{2} + \left(\frac{B^2\,Q_f{}^2}{2m_f^2} \right)$$
$$\pm \sqrt{ \frac{1}{4} \left[(c_{s\perp})^2\,k_x{}^2 + (c_{s\perp})^2\,k_y{}^2 + (c_{s\parallel})^2\,k_z{}^2 + \left(\frac{B^2\,Q_f{}^2}{m_f^2} \right) \right]^2 - \left(\frac{B^2\,Q_f{}^2}{m_f^2} \right) (c_{s\parallel})^2\,k_z{}^2 } \tag{19}$$

The four solutions of (18) are then $\omega(\vec{k}) = \pm\sqrt{\omega^2{}_\pm}$. In this case we notice that $\omega(\vec{k}) \in \mathbb{R}$ and $\omega_I = 0$ ensures stability. The phase velocity is calculated from (14):

$$\vec{v}_p = \frac{\omega}{|\vec{k}|}\hat{k} = \pm \left\{ \frac{(c_{s\perp})^2\,k_x{}^2}{2|\vec{k}|^2} + \frac{(c_{s\perp})^2\,k_y{}^2}{2|\vec{k}|^2} + \frac{(c_{s\parallel})^2\,k_z{}^2}{2|\vec{k}|^2} + \left(\frac{B^2\,Q_f{}^2}{2m_f^2\,|\vec{k}|^2} \right) \right.$$
$$\left. \pm \sqrt{ \left[\frac{(c_{s\perp})^2\,k_x{}^2}{2|\vec{k}|^2} + \frac{(c_{s\perp})^2\,k_y{}^2}{2|\vec{k}|^2} + \frac{(c_{s\parallel})^2\,k_z{}^2}{2|\vec{k}|^2} + \left(\frac{B^2\,Q_f{}^2}{2m_f^2\,|\vec{k}|^2} \right) \right]^2 - \left(\frac{B^2\,Q_f{}^2}{m_f^2} \right) (c_{s\parallel})^2\,\frac{k_z{}^2}{|\vec{k}|^4} } \right\}^{1/2} \hat{k} \tag{20}$$

With the above expression we can take the limit (12):

$$\lim_{|\vec{k}|\to\infty}\left|\frac{\omega}{\vec{k}}\right| \cong \lim_{|\vec{k}|\to\infty}\sqrt{(c_{s\perp})^2 + [(c_{s\parallel})^2 - (c_{s\perp})^2]\frac{k_z^2}{|\vec{k}|^2}}$$
$$= \sqrt{(c_{s\perp})^2 + [(c_{s\parallel})^2 - (c_{s\perp})^2]\cos^2(\theta)} \tag{21}$$

where θ is the angle between the direction of the magnetic field and the direction of the wave propagation. We can see that the above limit takes values between $c_{s\parallel}$ and $c_{s\perp}$. Causality is always satisfied.

The components of the group velocity (15) are given by:

$$\frac{\partial\omega}{\partial k_x} = \pm\frac{1}{2\omega}\left\{(c_{s\perp})^2 k_x\right.$$
$$\left.\pm\frac{\left[(c_{s\perp})^2|\vec{k}|^2 - \left[(c_{s\perp})^2 - (c_{s\parallel})^2\right]k_z^2 + \left(\frac{B^2 Q_f{}^2}{m_f^2}\right)\right](c_{s\perp})^2 k_x}{2\sqrt{\left[(c_{s\perp})^2\frac{|\vec{k}|^2}{2} - \left[(c_{s\perp})^2 - (c_{s\parallel})^2\right]\frac{k_z^2}{2} + \left(\frac{B^2 Q_f{}^2}{2m_f^2}\right)\right] - \left(\frac{B^2 Q_f{}^2}{m_f^2}\right)(c_{s\parallel})^2 k_z^2}}\right\} \tag{22}$$

$$\frac{\partial\omega}{\partial k_y} = \pm\frac{1}{2\omega}\left\{(c_{s\perp})^2 k_y\right.$$
$$\left.\pm\frac{\left[(c_{s\perp})^2|\vec{k}|^2 - \left[(c_{s\perp})^2 - (c_{s\parallel})^2\right]k_z^2 + \left(\frac{B^2 Q_f{}^2}{m_f^2}\right)\right](c_{s\perp})^2 k_y}{2\sqrt{\left[(c_{s\perp})^2\frac{|\vec{k}|^2}{2} - \left[(c_{s\perp})^2 - (c_{s\parallel})^2\right]\frac{k_z^2}{2} + \left(\frac{B^2 Q_f{}^2}{2m_f^2}\right)\right] - \left(\frac{B^2 Q_f{}^2}{m_f^2}\right)(c_{s\parallel})^2 k_z^2}}\right\} \tag{23}$$

and

$$\frac{\partial\omega}{\partial k_z} = \pm\frac{1}{2\omega}\left\{(c_{s\parallel})^2 k_z\right.$$
$$\left.\pm\frac{\left[(c_{s\perp})^2|\vec{k}|^2 - \left[(c_{s\perp})^2 - (c_{s\parallel})^2\right]k_z^2 + \left(\frac{B^2 Q_f{}^2}{m_f^2}\right)\right](c_{s\parallel})^2 k_z - \left(\frac{2B^2 Q_f{}^2}{m_f^2}\right)(c_{s\parallel})^2 k_z}{2\sqrt{\left[(c_{s\perp})^2\frac{|\vec{k}|^2}{2} - \left[(c_{s\perp})^2 - (c_{s\parallel})^2\right]\frac{k_z^2}{2} + \left(\frac{B^2 Q_f{}^2}{2m_f^2}\right)\right] - \left(\frac{B^2 Q_f{}^2}{m_f^2}\right)(c_{s\parallel})^2 k_z^2}}\right\} \tag{24}$$

where we verify that $|\vec{v}_g| < \infty$ as the wave number increases. From the results given by (21) and from the limit $\lim_{|\vec{k}|\to\infty}|\vec{v}_g| < \infty$ we conclude that causality is satisfied. Two particular cases have special interest:

(i) *No B field* ($c_{s\parallel} = c_{s\perp} = c_s$): $\omega(\vec{k}) = \pm(c_s)|\vec{k}|$ and $|\vec{v}_p| = c_s$

(ii) *Very strong B field* ($B^2 \to \infty$): $\omega(\vec{k}) \cong \pm\left(\dfrac{B\,Q_f}{m_f}\right)$ and $|\vec{v}_p| = \left(\dfrac{B\,Q_f}{m_f|\vec{k}|}\right)$

In order to have an idea of the numbers involved, we remember that the relevant strong magnetic fields are of the order of (or smaller than) 10^{19} G. These values correspond to $BQ_e \simeq m_\pi^2 \simeq 0.02$ GeV^2, with $m_\pi \simeq 140$ MeV, 1 GeV$^2 = 1.44 \times 10^{19}$ G [23] and to a phase velocity of

$$|\vec{v}_p| = v_p \simeq \frac{B\,Q_e}{m_f\,|\vec{k}|} \simeq \frac{m_\pi^2}{m_f\,|\vec{k}|} \tag{25}$$

and hence $|\vec{v}_p| < 1$ when $|\vec{k}| > 1000$ MeV, for example.

The above results for the non-relativistic equation of state are model independent and allow for quantitative estimates of some quantities, as long as we stay far from the very high velocity regime.

4. The MIT bag model equation of state

The thermodynamical properties of the hot QGP can be calculated from first principles in lattice QCD. On the other hand, the equation of state of the cold quark gluon plasma is not yet known with the same level of precision and we need to use models. For simplicity we often use the equation of state derived from the MIT bag model, which describes a gas of noninteracting quarks and gluons and takes into account non-perturbative effects through the bag constant $\mathcal{B}$. This constant is interpreted as the energy needed to create a bubble (or bag) in the QCD physical vacuum. In our case the quarks move under the action of an external magnetic field.

The energy density (ε_{MIT}), the parallel pressure ($p_{f\parallel MIT}$) and the perpendicular pressure ($p_{f\perp MIT}$), are given respectively by [24]:

$$\varepsilon_{MIT} = \mathcal{B} + \frac{B^2}{8\pi} + \sum_{f=u}^{d,s} \frac{|Q_f|B}{2\pi^2} \sum_{n=0}^{n_{max}^f} 3(2 - \delta_{n0}) \int_0^{k_{z,F}^f} dk_z \sqrt{m_f^2 + k_z^2 + 2n|Q_f|B} \tag{26}$$

$$p_{\parallel MIT} = -\mathcal{B} - \frac{B^2}{8\pi} + \sum_{f=u}^{d,s} \frac{|Q_f|B}{2\pi^2} \sum_{n=0}^{n_{max}^f} 3(2 - \delta_{n0}) \int_0^{k_{z,F}^f} dk_z \frac{k_z^2}{\sqrt{m_f^2 + k_z^2 + 2n|Q_f|B}} \tag{27}$$

$$p_{\perp MIT} = -\mathcal{B} + \frac{B^2}{8\pi} + \sum_{f=u}^{d,s} \frac{|Q_f|^2 B^2}{2\pi^2} \sum_{n=0}^{n_{max}^f} 3(2 - \delta_{n0})n \int_0^{k_{z,F}^f} \frac{dk_z}{\sqrt{m_f^2 + k_z^2 + 2n|Q_f|B}} \tag{28}$$

The baryon density (ρ_B) is written as:

$$\rho_B = \sum_{f=u}^{d,s} \frac{|Q_f|B}{2\pi^2} \sum_{n=0}^{n_{max}^f} (2 - \delta_{n0})\, k_{z,F}^f(n) \quad \text{with } n \leq n_{max}^f = int\left[\frac{\mu_f^2 - m_f^2}{2|Q_f|B}\right] \tag{29}$$

where the Fermi momentum is given by:

$$k_{z,F}^f(n) = \sqrt{\mu_f{}^2 - m_f^2 - 2n|Q_f|B}, \tag{30}$$

where μ_f is the chemical potential of the quark f and $int[a]$ denotes the integer part of a. The parallel and perpendicular speed of sound in this case are given by (4): $(c_{s\parallel})^2 = \partial p_{\parallel MIT}/\varepsilon_{MIT}$ and $(c_{s\perp})^2 = \partial p_{\perp MIT}/\varepsilon_{MIT}$.

In order to appreciate more easily the effect of the magnetic field, we will consider the particular case of a very strong field, i.e., we consider $|Q_f|B > \mu_f{}^2$ such that $n_{max}^f = 0$ in (29). We choose a common chemical potential μ which satisfies $|Q_f|B > \mu^2 > m_f^2$ for all quark flavors and defines the following Fermi momentum: $k_{z,F}^f(n) \to k_F = \mu$. The baryon density (29) is then given by:

$$\rho_B = \sum_{f=u}^{d,s} \frac{|Q_f|B}{2\pi^2}\mu \tag{31}$$

In this limit the energy density and the pressures are given by (26), (27) and (28):

$$\varepsilon_{MIT} = \mathcal{B} + \frac{B^2}{8\pi} + \sum_{f=u}^{d,s} \frac{3|Q_f|B}{2\pi^2}\left[-\frac{m_f^2}{4}\ln\left(m_f^2\right) + \frac{m_f^2}{2}\ln\left(2k_F\right) + \frac{k_F^2}{2}\right] \tag{32}$$

$$p_{\parallel MIT} = -\mathcal{B} - \frac{B^2}{8\pi} + \sum_{f=u}^{d,s} \frac{3|Q_f|B}{2\pi^2}\left[\frac{m_f^2}{4}\ln\left(m_f^2\right) - \frac{m_f^2}{2}\ln\left(2k_F\right) + \frac{k_F^2}{2}\right] \tag{33}$$

$$p_{\perp MIT} = -\mathcal{B} + \frac{B^2}{8\pi} \tag{34}$$

Using the above expressions, the pressure gradient is given by:

$$\vec{\nabla}p = \left(\frac{\partial}{\partial x}p_\perp, \frac{\partial}{\partial y}p_\perp, \frac{\partial}{\partial z}p_\parallel\right)$$
$$= \left(0, 0, -\frac{3|Q_f|B\,m_f^2}{4\pi^2\,\rho_{Bf}}\frac{\partial\rho_{Bf}}{\partial z} + \frac{6\pi^2}{|Q_f|B}\rho_{Bf}\frac{\partial\rho_{Bf}}{\partial z}\right) \tag{35}$$

Repeating the same calculations of the last sections, the matrix $A(\omega, \vec{k})$ in this case is:

$$A(\omega,\vec{k}) = \begin{pmatrix} 0 & -i\,3m_f\,\rho_0\,\omega\,(c_{s\perp}) & -3\,Q_f\,\rho_0\,B\,(c_{s\perp}) & 0 \\ 0 & 3\,Q_f\,\rho_0\,B\,(c_{s\perp}) & -i\,3m_f\,\rho_0\,\omega\,(c_{s\perp}) & 0 \\ i\,\Omega_s\,k_z & 0 & 0 & -i\,3m_f\,\rho_0\,\omega\,(c_{s\parallel}) \\ -i\,\omega & i\,(c_{s\perp})\,k_x & i\,(c_{s\perp})\,k_y & i\,(c_{s\parallel})\,k_z \end{pmatrix} \tag{36}$$

where

$$\Omega_s \equiv \left(\frac{6\pi^2\,\rho_0{}^2}{|Q_f|B} - \frac{3|Q_f|B\,m_f^2}{4\pi^2}\right) \tag{37}$$

and the dispersion relation is:

$$\omega^4 - \left[(\mathcal{V}_s)^2 k_z^2 + \left(\frac{B^2\,Q_f^2}{m_f^2}\right)\right]\omega^2 + \left(\frac{B^2\,Q_f^2}{m_f^2}\right)(\mathcal{V}_s)^2 k_z^2 = 0 \tag{38}$$

with the parameter $\mathcal{V}_s$ identified as:

$$(\mathcal{V}_s)^2 \equiv \frac{2\pi^2 \rho_0}{|Q_f| B\, m_f} - \frac{|Q_f| B\, m_f}{4\pi^2 \rho_0} \tag{39}$$

Considering (31) as the background density, we can rewrite (39) as:

$$(\mathcal{V}_s)^2 = \frac{\tilde{Q}\, \mu}{|Q_f|\, m_f} - \frac{|Q_f|\, m_f}{2\, \tilde{Q}\, \mu} \tag{40}$$

where $\tilde{Q} \equiv \sum_{j=u}^{d,s} |Q_f|$. We clearly notice in (40) that $(\mathcal{V}_s)^2 > 0$ because $\tilde{Q} > |Q_f|$ and $\mu > m_f$. Inserting the above expression into (38) we can solve it, finding ω and then the phase and group velocities. The resulting expressions coincide with equations (19) to (24), once we set in these latter $c_{s\perp} = 0$ and $c_{s\parallel} \to \mathcal{V}_s$. The dispersion relation (38) has only real roots ($\omega_I = 0$) and always satisfies the stability condition (13). In particular, the new version of eq. (21) is:

$$\lim_{|\vec{k}|\to\infty} \left| \frac{\omega}{|\vec{k}|} \right| = \lim_{|\vec{k}|\to\infty} |\vec{v}_p| \cong \lim_{|\vec{k}|\to\infty} \sqrt{\frac{(\mathcal{V}_s)^2\, k_z^2}{|\vec{k}|^2}} = \mathcal{V}_s \cos(\theta) \tag{41}$$

where θ is, as before, the angle between the vector $\vec{k}$ and the z direction. Since $\mathcal{V}_s$ is always larger than one, causality is guaranteed only for certain directions of propagation. Perturbations propagating along the direction of the magnetic field (for which $\theta = 0$ and $k_z = |\vec{k}|$), will have $|\vec{v}_p| > 1$. This is unphysical and is an indication of the inadequacy of the formalism for these extreme conditions.

5. Improved MIT bag model

In this section we shall use the equation of state which we call mQCD and which was derived in [15,25]. With mQCD we improve the MIT bag model including explicitly the gluonic degrees of freedom and also new non-perturbative effects. We assume that the quarks and gluons in the cold QGP are deconfined but can interact, forming the QGP. This means that the coupling is nonzero and also that there are remaining non-perturbative interactions and gluon condensates. We split the gluon field into two components $G^{a\mu} = A^{a\mu} + \alpha^{a\mu}$, where $A^{a\mu}$ ("soft" gluons) and $\alpha^{a\mu}$ ("hard" gluons) are the components of the field associated with low and high momentum modes respectively. The expectation values of $A^{a\mu} A_\mu^a$ and $A^{a\mu} A_\mu^a A^{b\nu} A_\nu^b$ are non-vanishing in a non-trivial vacuum and from them we obtain an effective gluon mass (m_G) and also a contribution ($\mathcal{B}_{QCD}$) to the energy and to the pressure of the system similar to the one of the MIT bag model. Since the number of quarks is very large and their coupling to the gluons is not small, the high momentum levels of the gluon field will have large occupation numbers and hence the $\alpha^{a\mu}$ component of the field can be approximated by a classical field. This is the same mean field approximation very often applied to models of nuclear matter, such as the Walecka model [5,26, 27].

The energy density (ε), the parallel pressure ($p_{f\parallel}$) and the perpendicular pressure ($p_{f\perp}$), are given respectively by [15]:

$$\varepsilon = \frac{27g_h^2}{16m_G^2} (\rho_B)^2 + \mathcal{B}_{QCD} + \frac{B^2}{8\pi}$$

$$+ \sum_{f=u}^{d,s} \frac{|Q_f|B}{2\pi^2} \sum_{n=0}^{n_{max}^f} 3(2-\delta_{n0}) \int_0^{k_{z,F}^f} dk_z \sqrt{m_f^2 + k_z^2 + 2n|Q_f|B} \tag{42}$$

$$p_\parallel = \frac{27 g_h^2}{16 m_G^2}(\rho_B)^2 - \mathcal{B}_{QCD} - \frac{B^2}{8\pi}$$

$$+ \sum_{f=u}^{d,s} \frac{|Q_f|B}{2\pi^2} \sum_{n=0}^{n_{max}^f} 3(2-\delta_{n0}) \int_0^{k_{z,F}^f} dk_z \frac{k_z^2}{\sqrt{m_f^2 + k_z^2 + 2n|Q_f|B}} \tag{43}$$

$$p_\perp = \frac{27 g_h^2}{16 m_G^2}(\rho_B)^2 - \mathcal{B}_{QCD} + \frac{B^2}{8\pi}$$

$$+ \sum_{f=u}^{d,s} \frac{|Q_f|^2 B^2}{2\pi^2} \sum_{n=0}^{n_{max}^f} 3(2-\delta_{n0})n \int_0^{k_{z,F}^f} \frac{dk_z}{\sqrt{m_f^2 + k_z^2 + 2n|Q_f|B}} \tag{44}$$

The baryon density (ρ_B) is given by (29) [15].

As in [15,28] we define $\xi \equiv g_h/m_G$. Choosing $\xi = 0$ we recover the MIT EOS (26), (27) and (28). For a given magnetic field intensity, we choose the values for the chemical potentials ν_f which determine the density ρ_B. We also choose the other parameters: ξ and $\mathcal{B}_{QCD}$. The background density (upon which small perturbation occur) is given by ρ_0, and it is usually given as multiples of the ordinary nuclear matter density $\rho_N = 0.17\ fm^{-3}$ [15].

Performing the same calculations shown in the previous sections, we obtain the following matrix:

$$A(\omega, \vec{k}) =$$

$$\begin{pmatrix} i\left(\frac{27 g_h^2 \rho_0^2}{8 m_G^2}\right)k_x & -i\,3m_f\,\rho_0\,\omega\,(c_{s\perp}) & -3\,Q_f\,\rho_0\,B\,(c_{s\perp}) & 0 \\ i\left(\frac{27 g_h^2 \rho_0^2}{8 m_G^2}\right)k_y & 3\,Q_f\,\rho_0\,B\,(c_{s\perp}) & -i\,3m_f\,\rho_0\,\omega\,(c_{s\perp}) & 0 \\ i\left(\frac{27 g_h^2 \rho_0^2}{8 m_G^2}\right)k_z & 0 & 0 & -i\,3m_f\,\rho_0\,\omega\,(c_{s\parallel}) \\ -i\,\omega & i\,(c_{s\perp})\,k_x & i\,(c_{s\perp})\,k_y & i\,(c_{s\parallel})\,k_z \end{pmatrix} \tag{45}$$

which yields the following dispersion relation:

$$\omega^4 - \left[(\tilde{c}_s)^2(k_x^2 + k_y^2 + k_z^2) + \left(\frac{B^2 Q_f^2}{m_f^2}\right)\right]\omega^2 + \left(\frac{B^2 Q_f^2}{m_f^2}\right)(\tilde{c}_s)^2 k_z^2 = 0 \tag{46}$$

where we identify the "effective sound speed" $\tilde{c}_s$,

$$(\tilde{c}_s)^2 \equiv \frac{9 g_h^2 \rho_0}{8 m_f m_G^2} \tag{47}$$

which depends on the features of the EOS. We can solve eq. (46) obtaining ω and the phase and group velocities, which become identical with equations (19) to (24) when we set $c_{s\perp} = c_{s\parallel} = \tilde{c}_s$ in the latter. We can then conclude that stability and causality are satisfied in the present case.

Let us look at the following particular cases:

(*i*) *No B field ($B = 0$):* $\omega(\vec{k}) = \pm(\tilde{c}_s)|\vec{k}|$ and $|\vec{v}_p| = \tilde{c}_s$
    In this case we recover the results found in [5].

(*ii*) *Very strong B field ($|Q_f|B > \mu^2 > m_f^2$):* The dispersion relation is:

$$\omega^4 - \left[(\tilde{c}_s)^2 |\vec{k}|^2 + (\mathcal{V}_s)^2 k_z^2 + \left(\frac{B^2 Q_f^2}{m_f^2} \right) \right] \omega^2 + \left(\frac{B^2 Q_f^2}{m_f^2} \right) (\mathcal{V}_s)^2 k_z^2$$

$$+ \left(\frac{B^2 Q_f^2}{m_f^2} \right) (\tilde{c}_s)^2 k_z^2 = 0 \tag{48}$$

where $(\mathcal{V}_s)^2$ is given by (40). The condition (21) is written as:

$$\lim_{|\vec{k}| \to \infty} \left| \frac{\omega}{|\vec{k}|} \right| = \lim_{|\vec{k}| \to \infty} |\vec{v}_p| \cong \lim_{|\vec{k}| \to \infty} \sqrt{(\tilde{c}_s)^2 + \frac{(\mathcal{V}_s)^2 k_z^2}{|\vec{k}|^2}}$$

$$\cong \sqrt{(\tilde{c}_s)^2 + (\mathcal{V}_s)^2 \cos^2(\theta)} \tag{49}$$

The same discussion made below Eq. (41) applies here. Causality may be satisfied for appropriate choices of g_h/m_G and θ.

6. Conclusions

We have studied the effects of a constant magnetic field on the propagation of waves in non-relativistic cold and ideal quark matter. Using the equations of non-relativistic ideal hydrodynamics in an external magnetic field, we have derived the dispersion relation for density and velocity perturbations. The magnetic field was included both in the equation of state and in the equations of motion, where the term of the Lorentz force was considered. We have used three equations of state: a generic non-relativistic one, the MIT bag model EOS and the mQCD EOS. The anisotropy effects caused by the B field were also manifest in the parallel and perpendicular sound speeds. We proved that the introduction of the magnetic field does not lead to instabilities in the velocity and density waves. In the case of the non-relativistic EOS the propagation of these waves was found to respect causality. As for the MIT and mQCD equations of state, we found situations where the phase velocity might be larger than one. In particular, this might happen for waves moving along the direction of the (very strong) magnetic field. In spite of its limitations, our study could determine the situations in where we are "safe" and where we might expect problems with instabilities and causality.

Acknowledgements

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