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Reliability analysis of a two-unit general parallel system with $(n-2)$ warm standbys

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ABSTRACT

A parallel $(2, n-2)$-system is investigated here where two units start their operation simultaneously and any one of them is replaced instantaneously upon its failure by one of the $(n-2)$ warm standbys. We assume availability of n non-identical, non-repairable units. The unit-lifetimes in full operational mode and in partial operational mode have general distribution functions G_i and H_i $(i = 1, \ldots, n)$ respectively. The system reliability is evaluated by recursive relations.

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1. Introduction

Redundant systems have attracted the attention of several researchers working in the field of reliability theory. There are two main categories of redundant standby systems, namely cold and warm. In the first category many workers, including Murari and Goel [13], Gupta and Goel [5], Goel et al. [2] and Gupta and Chaudhary [7], have investigated the two-unit standby system models assuming that, on the failure of one unit, it is replaced by the standby unit instantaneously. Gupta and Kishan [8] consider situations of two-unit system where the standby unit does not operate instantaneously but a fixed preparation time is required to put standby and repaired units into operation. Several workers including Gopalan et al. [4], Gupta and Goel [6], Murari and Maruthachalam [12], have analyzed two-unit system models under a variety of assumptions using the regenerative point technique. Recently, Lam [11] and Zhang and Wang [18], consider cold standby repairable systems applying the geometric process repair model. In all these system models it is assumed that the lifetimes are uncorrelated random variables. The structure of these system models allows the operation of only one unit during the repair time of the other.

Papageorgiou and Kokolakis [14] have investigated a two-unit parallel system supported by $(n-2)$ cold standbys. The unit-lifetimes are considered random, not necessarily independently distributed, with a general joint distribution F. The main result there is the analytic evaluation of the system reliability, unlike most earlier results which provide bounds under partial information about the joint pdf, or refer to specific independent unit-lifetime distributions. Papageorgiou and Kokolakis [15] extend the above main result by deriving a reliability formula which is efficient and easy to use for simulation techniques.

Within the second category of warm standby systems several workers including Wang et al. [17] and Ke and Wang [10] have analyzed a K out of $M+W$ warm standby system. In the former, the system constitutes of M operating machines and W standbys together with a repairable service station. Failure and service times are exponentially distributed. In the later, there are R repairmen and the balking and reneging of units permitted. Failure and repair times are considered exponentially distributed. Other workers including Kalpakam and Hameed [9] studied the asymptotic behavior of the residual lifetime distribution of a two-unit warm standby redundancy system supported by a single repair facility. They conclude in this case that the asymptotic distribution is always exponential no matter what the individual distributions of the units lifetimes and repair times are.

The availability and reliability of a n-unit system with $(n-1)$ warm standbys and a single repair facility are considered by Gopalan [3]. The failure times are assumed exponentially distributed while the pdf of the repair time is arbitrary. Srinivasan and Subramanian [16] analyzed a three-unit system consisting of a single unit working online and two warm standbys together with a single repair facility. The failure time of the online unit and the repair time have general distributions while in standby situation the failure rate is constant.

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Nomenclature

U_i	the ith unit introduced into the system
T	system lifetime
c	fixed required period of system operation
T_i	lifetime of ith unit in full operation mode
T_{ip}	lifetime of ith unit in partial operation mode
$g_i(\cdot), G_i(\cdot)$	PDF and CDF of the ith unit-lifetime in full operation mode
$h_i(\cdot), H_i(\cdot)$	PDF and CDF of the ith unit-lifetime in partial operation mode
$F_i(\cdot)$	CDF of the $(2, i-2)$ system lifetime
$\overline{F}_i(\cdot)$	reliability function of the $(2, i-2)$ system
S_i	starting time of full operation mode of the ith unit
R_i	remaining lifetime of the ith unit
n	number of available non-repairable units
$\mathscr{S}$	the event $\{T \geqslant c\}$

Here, we examine a related problem. It refers to a parallel $(2, n-2)$-system, where two units start their operation simultaneously and the unit that fails first is replaced instantaneously upon its failure by one of the $(n-2)$ warm standbys. The main difference of the model we examine in this paper is that we have available a fixed number of n non-repairable units, the lifetimes of which have general distribution functions G_i and H_i $(i = 1, \ldots, n)$, when in full operational mode and in partial operational mode respectively. Like in Papageorgiou and Kokolakis [14], our system has in parallel operation two units until, at least, the entrance of the last available standby unit. The difference from the above is that here we have warm standbys in the place of cold standbys there. Our main result is the evaluation of the system reliability by using recursive analysis.

2. System description

The problem of the successful control of a process by a two-unit parallel system supported by warm standbys is considered in this paper. Here, we suppose that there is available a fixed number of n non-repairable and non-identical units. Let T_i and T_{ip} $(i = 1, \ldots, n)$ denote the time to failure for the ith unit when fully energized and partially energized, respectively. The unit-lifetimes T_i $(i = 1, \ldots, n)$ are random independently distributed with general distributions G_i. Similarly, the unit-lifetimes T_{ip} $(i = 1, \ldots, n)$ are random independently distributed with general distributions H_i. We expect that $T_{ip} > T_i$ since the rate of the deterioration is less when the unit is partially energized than when fully energized, specifically we assume $G_i(t) > H_i(t)$ for all $t > 0$. The process has a fixed duration c. The control of the process is considered successful when the system is up, i.e. at least one of its two units is in operation state, during the required time interval c. The process is initially controlled by two units and the remaining $(n-2)$ units are warm standbys. The two initial units start their fully energized operation simultaneously, i.e. at time $t = S_1 = S_2 = 0$ and the $(n-2)$ standby units start their partially energized operation at the same time. The fully energized unit that fails first is replaced upon its failure instantaneously by a warm standby, i.e. the third unit in the system starts its fully energized operation at time $S_3 = \min\{T_1, T_2\}$, provided that it has not failed before that time as a partially energized unit. Similarly the forth unit in the system starts its fully energized operation upon the failure time of the first failed unit among the two working ones, i.e. at time $S_4 = \min\{\max\{T_1, T_2\}, S_3 + R_3\}$, provided it has not failed before, where R_3 denotes the third unit's remaining lifetime. For a better understanding in Fig. 1 we present a few cases of a (2,2)-parallel system failure.

In general, the ith unit in the system starts its full operation at time $S_i = \min\{\max\{T_1, T_2, S_3 + R_3, \ldots, S_{i-2} + R_{i-2}\}, S_{i-1} + R_{i-1}\}$, provided that has not failed before and it works in parallel with the fully energized working one. Thus, the times S_i $(i = 3, 4, \ldots, n)$ are random, and the process is simultaneously controlled by two working units until, at least, the entrance of the last available unit. The system fails when all units fail and the system reliability depends entirely on the distributions of unit-lifetimes.

The model analyzed here is general and could be applicable in a number of real life situations such as emergency power generators and navigator components. It is particulary important in cases where low power consumption is mandatory, such as in space craft systems.

3. Model analysis

Let T be the system lifetime and $\mathscr{S}$ the event of the successful control of the process during the required period of time c, i.e. $\mathscr{S} = \{T \geqslant c\}$. Let also T_i $(i = 1, \ldots, n)$ be the unit-lifetimes in fully operational mode. These are considered independently distributed with distribution functions G_i $(i = 1, \ldots, n)$.

The remaining time R_i of the unit U_i that has survived up to time X, in partially energized mode, is given by $R_i = T_i - G_i^{-1}(H_i(X))$ in fully operational mode. This is based on the following argument cf. Blischke and Murthy [1]. The unit U_i that has survived for a period X in partially energized mode is equivalent to it surviving for a period Z in fully energized mode, with the quantities X and Z being related by $G_i(Z) = H_i(X)$. Thus the effective age is $Z = G_i^{-1}(H_i(X))$ or, equivalently, $Z = \overline{G}_i^{-1}(\overline{H}_i(X))$.

From the above description it follows that the system is non-stop functioning with two units in full operation mode until, at least, the entrance of the last available warm standby unit that has survived in partial operation mode up to time $S_n < c$. We are interested in evaluating the system reliability $P[\mathscr{S}] = P[T \geqslant c]$.

We will derive a recursive relation to evaluate $P[\mathscr{S}]$ for any number, n, of independent G_i distributed lifetimes $(i = 1, \ldots, n)$.

Here, we have:

$$S_1 = S_2 = 0,$$
$$S_3 = \min\{T_1, T_2\},$$

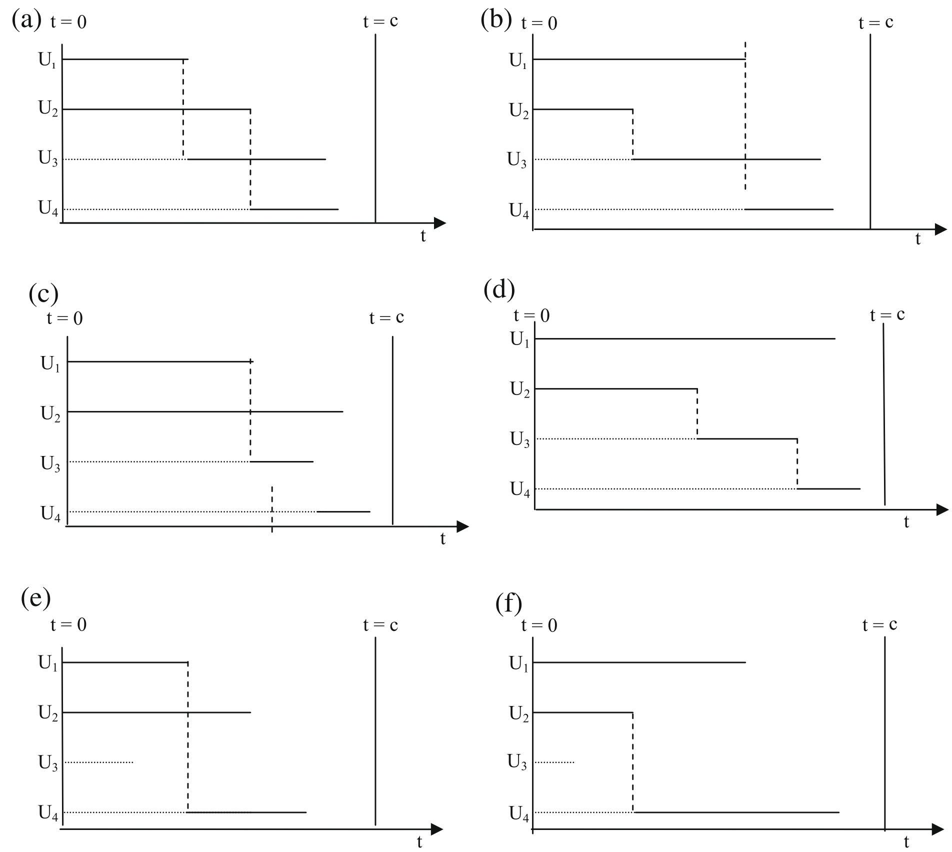

Fig. 1. Units U_3 and U_4 start their partial energized operation at time $t = 0$ and their fully energized operation upon the failure time of: (a) U_1 and U_2, (b) U_2 and U_1, (c) U_1 and U_3, and (d) U_2 and U_3, respectively. Unit U_4 starts its fully energized operation upon the failure time of the (e) U_1 and (f) U_2 while unit U_3 has failed before as a partially energized unit. The process is not successfully controlled for the required time $t = c$. The solid line represents full unit operation and the dotted line partial operation.

and in general for $i \geqslant 4$,

$$S_i = \min\{\max\{T_1, T_2, R_3 + S_3, \ldots, R_{i-2} + S_{i-2}\}, R_{i-1} + S_{i-1}\} \quad (i = 4, \ldots, n). \tag{1}$$

Let us define now the following random variables:

$$M_i = \max\{\max\{T_1, T_2, R_3 + S_3, \ldots, R_{i-2} + S_{i-2}\}, R_{i-1} + S_{i-1}\} \quad (i = 4, \ldots, n), \tag{2}$$

and

$$M_2 = T_1 \quad \text{and} \quad M_3 = \max\{T_1, T_2\}.$$

Then, for the system lifetime, we have:

$$T = \max\{T_1, T_2, R_3 + S_3, \ldots, R_n + S_n\} = M_{n+1}.$$

If we put

$$V_i = R_i + S_i \quad (i = 2, \ldots, n), \tag{3}$$

with $R_2 = T_2$ and $S_2 = 0$, we have

$$S_i = \min\{M_{i-1}, V_{i-1}\} \quad (i = 3, \ldots, n),$$

and

$$M_i = \max\{M_{i-1}, V_{i-1}\} \quad (i = 3, \ldots, n).$$

To clarify the problem we evaluate the above probability with $n = 3$ and 4.

3.1. (2, 1)-System

We first deal with the case $n = 3$. Here, the units U_1 and U_2 are fully energized and the unit U_3 is partial energized when the system is first put in use. Let T_i be the lifetimes of the units U_i $(i = 1, 2)$, respectively. Let S_3 denote the time to failure of the first failed unit, so $S_3 = \min\{T_1, T_2\}$. At this time the third unit U_3 starts its fully energized operation in the system, provided that it has not failed before. The (2, 1)-system lifetime is:

$$T = \max\{T_1, T_2, V_3\} = \max\{\max\{T_1, T_2\}, R_3 + S_3\} = \max\{M_3, R_3 + S_3\} = \max\{M_3, V_3\},$$

with cdf

$$F_3(t) = P[\max\{M_3, R_3 + S_3\} \leqslant t] = P[\{M_3 \leqslant t\}\{R_3 + S_3 \leqslant t\}] = \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} P[0 \leqslant R_3 \leqslant t - s_3 | S_3 = s_3, M_3 = m_3] dK_2(s_3, m_3),$$

where $K_2(s_3, m_3)$ the joint cdf of the quantities $S_3 = \min\{T_1, T_2\}$ and $M_3 = \max\{T_1, T_2\}$. Specifically we have $K_2(s_3, m_3) = P[S_3 \leqslant s_3, M_3 \leqslant m_3] = G(s_3)(2G(m_3) - G(s_3))$.

For the third unit's remaining lifetime R_3 we have either $R_3 = T_3 - G_3^{-1}(H_3(s_3))$, provided that has not failed before s_3, or $R_3 = 0$, when $T_{3p} \leqslant s_3$. Thus we have:

$$\begin{aligned} F_3(t) &= \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} \{P[R_3 = 0 | S_3 = s_3, M_3 = m_3] + P[0 < R_3 \leqslant t - s_3 | S_3 = s_3, M_3 = m_3]\} dK_2(s_3, m_3) \\ &= \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} \{P[T_{3p} \leqslant s_3] + P[0 < R_3 \leqslant t - s_3 | S_3 = s_3]\} dK_2(s_3, m_3), \end{aligned}$$

since the third unit's remaining lifetime depends only on S_3. We have therefore,

$$\begin{aligned} F_3(t) &= \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} \{H_3(s_3) + P[0 < T_3 - G_3^{-1}(H_3(s_3)) \leqslant t - s_3 | S_3 = s_3]\} dK_2(s_3, m_3) \\ &= \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} \{H_3(s_3) + P[G_3^{-1}(H_3(s_3)) < T_3 \leqslant t - s_3 + G_3^{-1}(H_3(s_3)) | S_3 = s_3]\} dK_2(s_3, m_3) \\ &= \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} \{H_3(s_3) + G_3(t - s_3 + G_3^{-1}(H_3(s_3))) - G_3(G_3^{-1}(H_3(s_3)))\} dK_2(s_3, m_3) \\ &= \int_{\{0 \leqslant s_3 \leqslant m_3 \leqslant t\}} G_3(t - s_3 + G_3^{-1}(H_3(s_3))) dK_2(s_3, m_3). \end{aligned} \tag{4}$$

3.2. (2,2)-System

For $n = 4$ we have:

$$T = \max\{T_1, T_2, R_3 + S_3, R_4 + S_4\} = \max\{\max\{\max\{T_1, T_2\}, R_3 + S_3\}, R_4 + S_4\} = \max\{M_4, V_4\},$$

and similarly its cdf

$$F_4(t) = P[\max\{M_4, V_4\} \leqslant t] = P[\{M_4 \leqslant t\}\{R_4 + S_4 \leqslant t\}] = \int_{\{0 \leqslant s_4 \leqslant m_4 \leqslant t\}} G_4(t - s_4 + G_4^{-1}(H_4(s_4))) dK_3(s_4, m_4), \tag{5}$$

where K_3 the joint cdf of the quantities S_4 and M_4. This will be evaluated by a recursive relation according to the Theorem presented in the next section.

3.3. General case

We will evaluate the system lifetime cdf in the general case of n-units. This is provided by a recursive relation as the following theorem states.

Theorem 1. *The system reliability is*

$$\overline{F}_i(t) = 1 - F_i(t) = 1 - \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} G_i(t - s_i + G_i^{-1}(H_i(s_i))) dK_{i-1}(s_i, m_i),$$

where $K_{i-1}(s_i, m_i)$ is the joint cdf of the random variables S_i and M_i $(i = 3, \ldots, n)$, and it is given by:

$$K_{i-1}(s_i, m_i) = L_{i-2}(s_i, m_i) + L_{i-2}(m_i, s_i) - F_{i-1}(s_i),$$

where $L_{i-1}(m_i, v_i)$ is the joint cdf of the random variables M_i and V_i, $(i = 3, \ldots, n)$, given by:

$$\begin{aligned} L_{i-1}(m_i, v_i) = &\int_{\{m_{i-1} \leqslant v_i, m_{i-1} \leqslant v_{i-1} \leqslant m_i\}} G_i(v_i - m_{i-1} + G_i^{-1}(H_i(m_{i-1}))) dL_{i-2}(m_{i-1}, v_{i-1}) \\ &+ \int_{\{v_{i-1} \leqslant v_i, v_{i-1} \leqslant m_{i-1} \leqslant m_i\}} G_i(v_i - v_{i-1} + G_i^{-1}(H_i(v_{i-1}))) dL_{i-2}(m_{i-1}, v_{i-1}). \end{aligned}$$

Proof. Firstly, we will derive a formula to determine the joint cdf of the random variables M_i and V_i, namely L_{i-1}, $(i = 3, \ldots, n)$.

$$\begin{aligned} L_{i-1}(m_i, v_i) = &P[M_i \leqslant m_i, V_i \leqslant v_i] = P[\{\max\{M_{i-1}, V_{i-1}\} \leqslant m_i\} \cdot \{R_i + S_i \leqslant v_i\}] = P[\{\max\{M_{i-1}, V_{i-1}\} \leqslant m_i\} \\ &\cdot \{R_i + \min\{M_{i-1}, V_{i-1}\} \leqslant v_i\}] = P[\{\max\{M_{i-1}, V_{i-1}\} \leqslant m_i\} \cdot \{R_i + \min\{M_{i-1}, V_{i-1}\} \leqslant v_i\} \cdot \{M_{i-1} \leqslant V_{i-1}\}] \\ &+ P[\{\max\{M_{i-1}, V_{i-1}\} \leqslant m_i\} \cdot \{R_i + \min\{M_{i-1}, V_{i-1}\} \leqslant v_i\} \cdot \{M_{i-1} > V_{i-1}\}]. \end{aligned}$$

If now $M_{i-1} \leqslant V_{i-1}$ then $M_i = V_{i-1}$, $S_i = M_{i-1}$ and if $M_{i-1} > V_{i-1}$ then $M_i = M_{i-1}$, $S_i = V_{i-1}$. Thus, the cdf $L_{i-1}(m_i, v_i)$ can be written:

$$L_{i-1}(m_i, v_i) = P[\{V_{i-1} \leqslant m_i\} \cdot \{R_i + M_{i-1} \leqslant v_i\} \cdot \{M_{i-1} \leqslant V_{i-1}\}] + P[\{M_{i-1} \leqslant m_i\} \cdot \{R_i + V_{i-1} \leqslant v_i\} \cdot \{M_{i-1} > V_{i-1}\}] = P[\{V_{i-1} \leqslant m_i\} \cdot \{T_i - G_i^{-1}(H_i(m_{i-1})) \leqslant v_i - M_{i-1}\} \cdot \{M_{i-1} \leqslant V_{i-1}\}] + P[\{M_{i-1} \leqslant m_i\} \cdot \{T_i - G_i^{-1}(H_i(v_{i-1})) \leqslant v_i - V_{i-1}\} \cdot \{M_{i-1} > V_{i-1}\}] = P[\{V_{i-1} \leqslant m_i\} \cdot \{T_i \leqslant v_i - M_{i-1} + G_i^{-1}(H_i(m_{i-1}))\} \cdot \{M_{i-1} \leqslant V_{i-1}\}] + P[\{M_{i-1} \leqslant m_i\} \cdot \{T_i \leqslant v_i - V_{i-1} + G_i^{-1}(H_i(v_{i-1}))\} \cdot \{M_{i-1} > V_{i-1}\}]$$

and therefore

$$L_{i-1}(m_i, v_i) = \int_{\{m_{i-1} \leqslant v_i, m_{i-1} \leqslant v_{i-1} \leqslant m_i\}} G_i(v_i - m_{i-1} + G_i^{-1}(H_i(m_{i-1})))dL_{i-2}(m_{i-1}, v_{i-1}) + \int_{\{v_{i-1} \leqslant v_i, v_{i-1} \leqslant m_{i-1} \leqslant m_i\}} G_i(v_i - v_{i-1} + G_i^{-1}(H_i(v_{i-1})))dL_{i-2}(m_{i-1}, v_{i-1}). \tag{6}$$

Since $M_2 = T_1, V_2 = T_2$ and the quantities $T_1,\ T_2$ are independent we have:

$$L_1(m_2, v_2) = P[M_2 \leqslant m_2, V_2 \leqslant v_2] = P[T_1 \leqslant m_2, T_2 \leqslant v_2] = \int_0^{m_2}\int_0^{v_2} g(t_1, t_2)dt_2dt_1 = \int_0^{m_2}\int_0^{v_2} g_1(t_1)g_2(t_2)dt_2dt_1. \tag{7}$$

We can now use the joint cdf L_{i-1} to find the joint cdf K_{i-1} of the random variables S_i and M_i $(i = 3, \ldots, n)$. Specifically, we have:

$$K_{i-1}(s_i, m_i) = P[S_i \leqslant s_i, M_i \leqslant m_i] = P[M_i \leqslant m_i] - P[S_i > s_i, M_i \leqslant m_i] = P[\max\{M_{i-1}, V_{i-1}\} \leqslant m_i] - P[s_i < M_{i-1} \leqslant m_i, s_i < V_{i-1} \leqslant m_i] = F_{i-1}(m_i) - L_{i-2}(m_i, m_i) + L_{i-2}(s_i, m_i) + L_{i-2}(m_i, s_i) - L_{i-2}(s_i, s_i).$$

Since we have $F_i(t) = P[\max\{M_i, V_i\} \leqslant t] = P[M_i \leqslant t, V_i \leqslant t] = L_{i-1}(t, t)$, we get:

$$K_{i-1}(s_i, m_i) = L_{i-2}(s_i, m_i) + L_{i-2}(m_i, s_i) - F_{i-1}(s_i) \quad (i = 3, \ldots, n). \tag{8}$$

With the cdf's L_{i-2} and K_{i-1} known, we may now proceed to derive a recursive formula for the $(2, i-2)$- system cdf $F_i,\ (i = 3, \ldots, n)$.

$$\begin{aligned} F_i(t) &= P[\max\{M_i, R_i + S_i\} \leqslant t] = P[\{M_i \leqslant t\}\{R_i + S_i \leqslant t\}] \\ &= \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} \{P[R_i = 0|S_i = s_i, M_i = m_i] + P[0 < R_i \leqslant t - s_i|S_i = s_i, M_i = m_i]\}dK_{i-1}(s_i, m_i) \\ &= \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} \{P[T_{ip} \leqslant s_i] + P[0 < R_i \leqslant t - s_i|S_i = s_i]\}dK_{i-1}(s_i, m_i) \\ &= \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} \{H_i(s_i) + P[0 < T_i - G_i^{-1}(H_i(s_i)) \leqslant t - s_i|S_i = s_i]\}dK_{i-1}(s_i, m_i) \\ &= \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} \{H_i(s_i) + P[G_i^{-1}(H_i(s_i)) < T_i \leqslant t - s_i + G_i^{-1}(H_i(s_i))|S_i = s_i]\}dK_{i-1}(s_i, m_i) \\ &= \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} \{H_i(s_i) + G_i(t - s_i + G_i^{-1}(H_i(s_i))) - G_i(G_i^{-1}(H_i(s_i)))\}dK_{i-1}(s_i, m_i) \end{aligned}$$

and thus

$$F_i(t) = \int_{\{0 \leqslant s_i \leqslant m_i \leqslant t\}} G_i(t - s_i + G_i^{-1}(H_i(s_i)))dK_{i-1}(s_i, m_i). \qquad \square \tag{9}$$

Remark 1. An other way to evaluate F_i could be to use the relation $F_i(t) = L_{i-1}(t, t)$ but, the evaluation of $L_{i-1}(t_1, t_2)$, instead of $K_{i-1}(t_1, t_2)$, is more complicated especially for large values of i.

4. Applications

In this section, applying Theorem 1, we give the final expression for the $(2, n-2)$ system reliability in the case of $n = 3$ and $n = 4$, i.e. with one and two warm standbys respectively and with particular lifetime distributions.

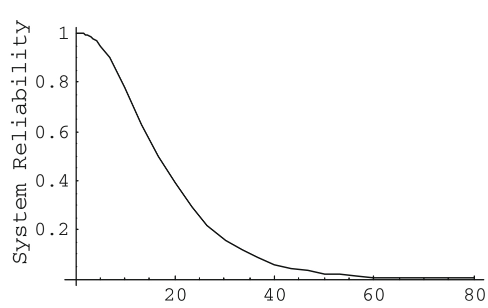

Fig. 2. System reliability $\overline{F}_3(t)$ for $b_f = \frac{1}{10}$ and $b_p = \frac{1}{30}$.

4.1. (2, 1)-System with exponential lifetimes

We first study the case of $n = 3$ units with lifetimes exponentially distributed with parameter b_f, when fully energized, and b_p, when partially energized. In this case the joint cdf $K_2(s_3, m_3)$, of the random quantities $S_3 = \min\{T_1, T_2\}$ and $M_3 = \max\{T_1, T_2\}$, is given by:

$$K_2(s_3, m_3) = 1 - 2e^{-b_f m_3} - e^{-2b_f s_3} + 2e^{-b_f(m_3+s_3)}. \tag{10}$$

Applying (4), or (9) for $i = 3$, the lifetime distribution $F_3(t)$ of the considered system is as follows:

$$F_3(t) = 1 + e^{-2b_f t} - 2e^{-b_f t} - \frac{2b_f e^{-2b_f t}(e^{b_f t} - 1)}{b_p} + \frac{2b_f^2 e^{-t(2b_f+b_p)}(e^{t(b_f+b_p)} - 1)}{b_p(b_f + b_p)}. \tag{11}$$

In Fig. 2 we present the system reliability $\overline{F_3}(t)$ for $b_f = \frac{1}{10}$ and $b_p = \frac{1}{30}$, i.e. for $E[T_i] = 10$ $(i = 1, 2, 3)$ and $E[T_{3p}] = 30$.

4.2. (2, 1)-System with exponential and Weibull lifetimes

In a (2, 1)-system with standby unit-lifetime following a Weibull distribution with parameters $b = 2$ and $\lambda = 2$, i.e. $H_3(t) = 1 - e^{\frac{-t^2}{4}}$ and $E[T_{3p}] = \sqrt{\pi}$, and with initial units U_1 and U_2 lifetimes exponentially distributed with parameter $b_f = 2$, when fully energized, i.e. $G_i(t) = 1 - e^{-2t}$ and $E[T_i] = 0.5$ for $i = 1, 2$, we get from the Theorem 1:

$$F_3(t) = e^{-4t}\left(\left(e^{2t} - 1\right)^2 + 4\sqrt{\pi}\left(e^{2t+4}\left(Erf[2] - Erf\left[2 + \frac{t}{2}\right]\right) + Erf\left[\frac{t}{2}\right]\right)\right),$$

where

$$Erf[z] = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

4.3. (2, 2)-System with exponential lifetimes

Now we study the case of $n = 4$ units with lifetimes exponentially distributed with parameter b_f, when fully energized, and b_p, when partially energized. We first evaluate the joint cdf of the random variables M_3 and V_3, namely L_2 by applying (6) and (7). Then we evaluate the joint cdf K_3 of the random variables S_4 and M_4 from (8) for $i = 4$ and we introduce it into (5), or (9) for $i = 4$, and we get the lifetime distribution F_4 of the considered system. Specifically we have:

For $i = 3$, we have from (6)

$$L_2(m_3, v_3) = 1 - 2e^{-b_f m_3} + e^{-2b_f m_3} - \frac{2b_f}{b_f + b_p} e^{-b_f v_3} + \frac{2b_f}{b_p} e^{-b_f(m_3+v_3)} - \frac{2b_f^2}{b_p(b_f + b_p)} e^{-(b_f+b_p)m_3 - b_f v_3}, \quad m_3 \leqslant v_3,$$

and

$$L_2(m_3, v_3) = 1 - 2e^{-b_f m_3} - e^{-2b_f v_3} + \frac{2(b_f + b_p)}{b_p} e^{-b_f(m_3+v_3)} - \frac{2b_f}{b_f + b_p} e^{-b_f v_3} + \frac{2b_f}{(b_f + b_p)} e^{-(2b_f+b_p)v_3} - \frac{2b_f}{b_p} e^{-b_f m_3 - (b_f+b_p)v_3}, \quad m_3 > v_3. \tag{12}$$

Introducing (11) and (12) into (8) for $i = 4$ we get

$$K_3(s_4, m_4) = 1 + \left(2 + \frac{4b_f}{b_p}\right) e^{-b_f(m_4+s_4)} + \frac{2b_f}{b_p} e^{-s_4(2b_f+b_p)} - \frac{2b_f + b_p}{b_p} e^{-2b_f s_4} - \frac{2(2b_f + b_p)}{b_p + b_f} e^{-b_f m_4} - \frac{2b_f(2b_f + b_p)}{b_p(b_f + b_p)} e^{-b_f(m_4+s_4) - b_p s_4}. \tag{13}$$

Finally, applying (5), or (9) for $i = 4$, the lifetime distribution $F_4(t)$ of the considered system is as follows:

$$F_4(t) = \frac{1}{b_p^2(b_f + 2b_p)} [e^{-2t(b_f+b_p)} \times (2(-1 + e^{tb_p})^2 b_f^3 + (1 - 8e^{tb_p} + 7e^{2tb_p}) b_f^2 b_p + e^{2tb_p}(7 - 8e^{tb_f} + e^{2tb_f}) b_f b_p^2 + 2e^{2tb_p}(-1 + e^{tb_f})^2 b_p^3)],$$

or after a rearrangement of the terms we get:

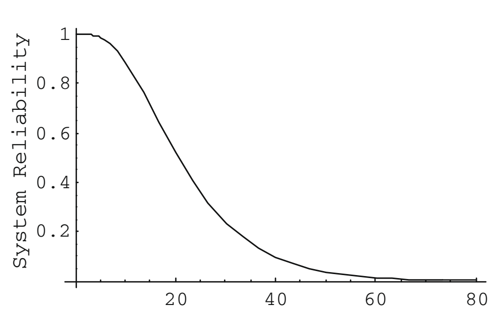

Fig. 3. System reliability $\overline{F}_4(t)$ for $b_f = \frac{1}{10}$ and $b_p = \frac{1}{30}$.

$$F_4(t) = 1 - \frac{4(2b_f + b_p)}{b_f + 2b_p} e^{-tb_f} + \frac{(2b_f + b_p)(b_f + b_p)}{b_p^2} e^{-2tb_f} - \frac{4b_f^2}{b_p^2} e^{-(2b_f+b_p)t} + \frac{b_f^2(2b_f + b_p)}{b_p^2(b_f + 2b_p)} e^{-(2b_f+2b_p)t}. \quad (14)$$

In Fig. 3 we present the system reliability $\overline{F}_4(t)$ for $b_f = \frac{1}{10}$ and $b_p = \frac{1}{30}$, i.e. for $E[T_i] = 10$ $(i = 1, \dots, 4)$ and $E[T_{ip}] = 30$ $(i = 1, 2)$.

Remark 2. In the above three cases the application of Theorem 1 was rather straightforward. In systems with a rather large number of warm standby units and/or less tractable lifetime distributions, the implementation of Theorem 1 would require the application of computationally intensive methods such as modern MCMC algorithms. This will be the subject of a future work.

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References

[1] R.W. Blischke, D.N.P. Murthy, Reliability: Modeling, Prediction, and Optimization, John Wiley & Sons, Inc., 2000.
[2] L.R. Goel, P. Srivastava, R. Gupta, Two unit cold standby system with correlated failures and repairs, International Journal of Systems Science 23 (1992) 379–391.
[3] M.N. Gopalan, Probabilistic analysis of a single server n-unit system with $(n-1)$ warm standbys, Operations Research 23 (1975) 591–595.
[4] M.N. Gopalan, T.K. Ramesh, K.V. Krishna Kumar, Analysis of one server two-unit parallel system subject to degradation, Microelectron Reliability 26 (1986) 657–664.
[5] R. Gupta, L.R. Goel, Profit analysis of two unit priority standby system with administrative delay in repair, International Journal of Systems Science 20 (1989) 1703–1712.
[6] R. Gupta, L.R. Goel, Cost benefit analysis of a two-unit parallel standby system with administrative delay in repair, International Journal of Systems Science 21 (1990) 1369–1379.
[7] R. Gupta, A. Chaudhary, A two-unit priority system subject to random shocks and Rayleigh failure time distributions, Microelectronics and Reliability 32 (1992) 1713–1723.
[8] R. Gupta, R. Kishan, On the profit comparison of two stochastic models each pertaining to a two-unit standby system with fixed preparation time and hyperexponential repair time distributions, International Journal of Systems Science 30 (12) (1999) 1309–1317.
[9] S. Kalpakam, S.M.A. Hameed, Quasi-stationary distribution of a two-unit warm-standby redundant system, Journal of Applied Probability 20 (2) (1983) 429–435.
[10] J.-C. Ke, K.-H. Wang, The reliability analysis of balking and reneging in a repairable system with warm standbys, Quality and Reliability Engineering International 18 (2002) 467–478.
[11] Y. Lam, A geometric process maintenance model with preventive repair, European Journal of Operational Research 182 (2007) 806–819.
[12] K. Murari, C. Maruthachalam, Two-units parallel system with periods of working and rest, IEEE Transactions on Reliability 30 (1981) 91.
[13] K. Murari, V. Goel, Comparison of two unit cold standby reliability models with three types of repair facilities, Microelectronics and Reliability 24 (1984) 35–39.
[14] E. Papageorgiou, G. Kokolakis, A two unit parallel system supported by $(n-2)$ standbys with general and non-identical lifetimes, International Journal of Systems Science 35 (1) (2004) 1–12.
[15] E. Papageorgiou, G. Kokolakis, A two unit general parallel system with $(n-2)$ cold standbys – Analytic and simulation approach, European Journal of Operational Research 176 (2007) 1016–1032.
[16] S.K. Srinivasan, R. Subramanian, Reliability analysis of a three unit warm standby redundant system with repair, Annals of Operations Research 143 (2006) 227–235.
[17] K.-H. Wang, Y.-J. Lai, J.-B. Ke, Reliability and sensitivity analysis of a system with warm standbys and a repairable service station, International Journal of Operations Research 1 (1) (2004) 61–70.
[18] Y.-L. Zhang, G.-J. Wang, A deteriorating cold standby repairable system with priority in use, European Journal of Operational Research 183 (2007) 278–295.