Classical Field Theory

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Introduction

A physical system is usually described in terms of states and observables. (See the discussion in [I-**Faddeev**, §1.1].) In the Hamiltonian framework of classical mechanics, the states form a symplectic manifold $(\mathcal{M}, \Omega)$ and the observables are functions on $\mathcal{M}$. The dynamics of a (time-invariant) system is described by a one-parameter group of symplectic diffeomorphisms; the generating function is the *energy* or *hamiltonian*. The system is said to be *free* if $(\mathcal{M}, \Omega)$ is an affine symplectic space and the motion is by a one-parameter group of affine symplectic transformations. This general description applies to systems which include classical particles, strings, fields, and other types of objects. Often the dynamics of the theory is embedded in a larger symmetry group. For example, in relativistic field theories one assumes that $(\mathcal{M}, \Omega)$ carries a representation of the Poincaré group.

Many classical systems admit a *lagrangian* description in which $(\mathcal{M}, \Omega)$ is derived from a relatively simple expression, called the lagrangian density. One of the main features of a lagrangian description is that the conserved quantity—called the *Noether charge*—corresponding to a symmetry is computed directly from the lagrangian. Furthermore, in field theories there is a local *Noether current* which integrates to the global charge and which gives rise to local conservation laws. This reflects the physical fact that, for example, we can measure energy in any region of space, not just the total energy over all of space. In this text we develop the basic ideas of classical lagrangian field theory. The examples we have in mind are the ones which arise in relativistic quantum field theory as treated in other parts of the book. One should be careful in trying to apply the formalism developed here to constrained systems (e.g., nonholonomic constraints in classical mechanical systems, or constraints imposed on superfields in the superspace descriptions of supersymmetric fields theories). In most of the exposition, we will assume that fields are arbitrary sections of some fixed fiber bundle E over spacetime. Also, we develop the formalism in a purely local way from the lagrangian density, hardly mentioning its integral, the *action*. If one is interested in calculus of variations problems, then the emphasis is different: the action is of primary interest and boundary conditions play a crucial role. A last warning is that not all classical field theories admit a lagrangian description. There is a free example which is important in two-dimensional conformal field theory: a free chiral scalar field in two dimensions.

In Chapter 1 we review some classical mechanics. In a few standard examples we describe the classical equations of motion and the construction of a symplectic structure on the space of classical solutions. Noether's theorem—the construction of a conserved quantity from a one-parameter group of symmetries—is also discussed. We treat both nonrelativistic and relativistic examples, and in §1.4 we show how to obtain nonrelativistic Galilean spacetime as a limit of relativistic Minkowski spacetime.

The general theory of classical fields is laid out in Chapter 2. A classical lagrangian field theory consists of a spacetime M, a space of fields $\mathcal{F}$, and a lagrangian density L. We are mostly interested in Minkowski spacetime, but the theory is quite general and applies to spacetimes which are curved Lorentzian or Riemannian manifolds as well. The fields are some sort of functions on M, more precisely sections of a fiber bundle E over M. The lagrangian density is a density on M for each point of $\mathcal{F}$. It is assumed[1] to be of a local nature on M. More precisely, for some k the value of $L(\phi)$ at a point m of M should depend only on the k-jet of ϕ at m. Usually $k = 1$. The lagrangian density L determines Euler-Lagrange equations $\underline{D}L = 0$, also called equations of motion, which cut out the space of extremals $\mathcal{M} \subset \mathcal{F}$. If ϕ is a field, the tangent space of $\mathcal{F}$ at ϕ is the space of sections of the vector bundle $\phi^*T(E/M)$ over M. The Euler-Lagrange equation $\underline{D}L$ is a morphism of vector bundles from $\phi^*T(E/M)$ to the bundle of densities on M. Its characteristic property is that for a deformation with compact support $\phi[u]$ of ϕ, one has

$$\int \frac{d}{du} L(\phi)[u] \bigg|_{u=0} = \int \underline{D}L \cdot \frac{d}{du}\phi[u] \bigg|_{u=0} .$$

The formalism naturally takes place in the double complex of differential forms on $\mathcal{F} \times M$. This is qualified in two ways: (i) as we want $L(\phi)$ to be a density, rather than a differential form of maximal degree, this double complex should be twisted by the orientation bundle of M; and (ii) we want to consider only (p,q)-forms α which are *local* on M: to a field ϕ and tangent vectors $\xi_1, \dots, \xi_p \in \phi^*T(E/M)$, the form α attaches a q-form $\alpha(\phi; \xi_1, \dots, \xi_p)$ on M, and for some k the value of $\alpha(\phi; \xi_1, \dots \xi_p)$ at a point m of M should depend only on the k-jet at m of ϕ and the ξ_i. The cohomology of the double complex of local forms has been investigated by F. Takens[2]. Write the exterior differential as $D = \delta + d$, with δ of degree $(1,0)$ and d of degree $(0,1)$. Takens' main result is Theorem 2.15: For $p > 0$ the complex $(\Omega^{p;\bullet}_{\text{loc}}, d)$ of local differential forms is exact except in top degree. One can view the $\Omega^{p;q}_{\text{loc}}$ as an inductive limit, in k, of spaces of global sections of soft sheaves on E, and this makes the exactness of $(\Omega^{p;\bullet}_{\text{loc}}, d)$ ($p > 0$, $\bullet \neq$ top) a local question on E. Let $J^k(E)$ be the bundle over M of k-jets of sections of E. Takens also observes that the associated simple complex $\Omega^{\bullet}_{\text{loc}}$ is the inductive limit over k of the de Rham complexes of the $J^k(E)$. As the projections $J^k(E) \to E$ are fibrations, with fibers affine spaces, they induce isomorphisms in cohomology. It follows that the cohomology of E maps isomorphically to that of $\Omega^{\bullet}_{\text{loc}}$. We include a proof of Takens' results in an appendix to Chapter 2.

[1] Although we restrict to lagrangians which depend locally on fields, in quantum field theory one meets *effective lagrangians* which are not local. Some of the formal aspects carry over to nonlocal lagrangians, but we have not taken the trouble to distinguish them.

[2] See References at the end of the manuscript.

If the lagrangian density $L(\phi)$ depends only on the first jet of ϕ, it defines a $(1, n-1)$-form γ (where $n = \dim M$), which we call the *variational 1-form*, characterized by the following properties: (i) the value of the $(n-1)$-form $\gamma(\phi; \xi)$ at $m \in M$ depends only on the 1-jet of ϕ at m, and on the value of ξ at m; and (ii)

$$\underline{D}L = \delta L + d\gamma.$$

The variational $(1, n-1)$-form γ encodes the usual integration by parts argument which occurs in computing Euler-Lagrange equations. For more general lagrangians, the choice of a local $(1, n-1)$-form γ with $\underline{D}L = \delta L + d\gamma$ should be considered part of the definition of the theory. Such a form γ always exists and, by Takens' theorem, it is unique up to the addition of $d\beta$, for β a local $(1, n-2)$-form.

Classical mechanics corresponds to $M = \mathbb{R}$ (time). In classical mechanics, one is used to the following package: (i) the space of extremals $\mathcal{M}$ is a symplectic manifold; (ii) symmetries give symplectic automorphisms of $(\mathcal{M}, \omega)$, and infinitesimal symmetries ξ are given by generating functions Q, with $dQ = -\iota(\xi)\omega$. A particular case is the one-parameter group of time translations, whose generating function is minus the hamiltonian. In (ii), Q is ambiguous up to an additive constant (and its existence can be obstructed by $H^1(\mathcal{M}, \mathbb{R})$). This ambiguity can often be removed by refining (i) to: (i′) the space of extremals carries a canonical principal $\mathbb{R}$-bundle T with connection ∇ whose curvature is ω. Infinitesimal automorphisms of $(\mathcal{M}, T, \nabla)$ can then be identified with pairs (ξ, Q) consisting of a vector field ξ on $\mathcal{M}$ and a function Q such that $dQ = -\iota(\xi)\omega$.

In field theory one has a similar Hamiltonian interpretation if M is given as time $\times$ space, or if one has a suitable notion of space-like hypersurface. A new problem is to obtain counterparts local on M of those constructions. These local counterparts do not require a notion of space-like hypersurface, or a splitting into time $\times$ space, but rather exist for any spacetime M. In particular, they make sense for Euclidean analogs, obtained by a Wick rotation $t \mapsto it$. For the 2-form on $\mathcal{M}$, which we now denote 'Ω', this is done in a paper of G. Zuckerman:[3] if ω is the local $(2, n-1)$-form $\delta\gamma$ on $\mathcal{F} \times M$, then the 2-form Ω on $\mathcal{M}$ is deduced from ω by integration on any space-like hypersurface. Similarly, one can express the generating functions of infinitesimal symmetries as integrals over space-like hypersurfaces of conserved currents, called *Noether currents*. Let ξ be a vector field on $\mathcal{F}$ which is local: ξ_ϕ at $m \in M$ depends only on some jet of ϕ at m. Suppose it is a generalized symmetry of L, in the sense that for some given $(0, n-1)$-local form α_ξ, one has

$$\mathrm{Lie}(\xi)L = d\alpha_\xi.$$

Then the Noether current $j_\xi := \iota(\xi)\gamma - \alpha_\xi$ is *conserved*: $dj_\xi(\phi) = 0$ for all ϕ in $\mathcal{M}$,. The vector field ξ is tangent to $\mathcal{M}$, and the integral Q_ξ of j_ξ on a space-like hypersurface is a corresponding generating function.

It can be more natural to consider symmetries which act on $\mathcal{F}$ and M simultaneously. We will say that a symmetry is *manifest* if it preserves L and γ exactly. For example, time translation is usually a manifest symmetry when it acts on both $\mathcal{F}$ and M. If we let it act only on fields, not on spacetime, it is only a generalized symmetry (with $\alpha = -\iota(\partial_t)L$).

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[3] See References.

Theories whose field content includes a connection or metric possess an infinite dimensional group of local "gauge" symmetries. (For metrics the gauge symmetry group is the group of diffeomorphisms.) We can freeze the metric or connection at fixed values g_0, A_0; then local symmetries which fix g_0 or A_0 act as global symmetries in the theory of the remaining fields. The Noether current for these global symmetries may be computed by differentiating the total lagrangian with respect to the metric or connection, as explained in §2.8 and §2.9. The derivative with respect to the metric is called the *energy-momentum* tensor.

In §2.10 we discuss time-invariant field configurations of finite energy on spacetimes which are time × space. Among these we find *classical vacua* and *solitons*. We also explain briefly perturbations around a classical vacuum and the Higgs mechanism.

Chapter 3 summarizes the basic free lagrangian field theories on Minkowski spacetime. We treat scalar fields, spinor fields, and abelian gauge fields (connections). Our goal is to illustrate the general theory in the simplest case and to record useful formulas.

In classical physics one of the main applications of field theory is to electromagnetism. Chapter 4, which is a discussion of gauge theory in general, begins with a brief treatment of Maxwell's equations in the lagrangian framework. Some familiarity with this material is necessary to understand the intuition behind more complicated models with gauge fields, which are encountered in many lectures in these volumes. In §4.2 we review the basic geometry of connections in principal bundles, paying special attention to the universal connection (in terms of which we write gauge theory lagrangians). Then in §4.3 we write the lagrangian for Yang-Mills theory and describe some additional "θ-terms" which may appear in low dimensions. Finally, we define electric and magnetic charge in §4.4 and discuss the relationship to global gauge transformations.

The general bosonic lagrangians without gravity usually include only scalar fields and gauge fields. We discuss a general lagrangian for these fields in Chapter 5; it includes many important bosonic theories as special cases.

There are special *topological terms* in lagrangians which are invariant under (orientation-preserving) diffeomorphisms. Some, like the θ-terms mentioned above, are related to primary topological invariants. More subtle are the ones associated to secondary invariants, like the Wess-Zumino-Witten term in a σ-model or the Chern-Simons term in three-dimensional gauge theory. In Chapter 6 we briefly introduce the main examples and explain how the action acquires a more subtle geometric meaning. The geometric home for the lagrangian is a "T-calculus" which extends the usual calculus of differential forms, as we indicate in §6.3.

Finally, in Chapter 7 we discuss the "Wick rotation" of a lagrangian from Minkowski spacetime to Euclidean spacetime. For reference we collect the signs and factors of $\sqrt{-1}$ which occur in this analytic continuation.

To a large extent this text presents a preliminary version of this material; we are not satisfied at all with our understanding in many places. Still, we felt it important to include some mathematical framework for the computations in lagrangian field theory. Our treatment is guided by what is needed to follow the lectures and problems recorded in these volumes, and we hope at least to have provided sufficient background for that. Other mathematical accounts appear in the references.

The definition of a classical lagrangian that we use (Definition 2.39) is adopted from a lecture of Joseph Bernstein. The basic formalism appears in the aforementioned paper of Gregg Zuckerman. In preparing this text we also benefited from many discussions with David Kazhdan, John Morgan, Nati Seiberg, and Ed Witten, among others.

CHAPTER 1
Classical Mechanics

The equation of motion of classical mechanics is the Euler-Lagrange equation for extremizing the action integral. From this extremal description we construct a canonical closed 2-form ω on the space $\mathcal{M}$ of classical evolutions. It turns out that ω is nondegenerate, and so $\mathcal{M}$ is a symplectic manifold. More precisely, we construct on $\mathcal{M}$ a canonical $\mathbb{R}$-torsor[4] with connection whose curvature is ω. As usual in symplectic geometry, to an infinitesimal symmetry we associate a function which is called the *Noether charge.* (It is often called the *momentum* or *momentum map.*)

In quantum theory we apply $\exp(\frac{i}{\hbar}\cdot)$ to turn this $\mathbb{R}$-torsor into a unitary line bundle with connection whose curvature is $\frac{i}{\hbar}\omega$.

§1.1. The nonrelativistic particle

We treat three cases: the free particle, a system of particles with potential, and the electromagnetic field.

Free Particle

Let $X = \mathbb{R}^n$ be Euclidean space with its standard inner product $\langle\cdot,\cdot\rangle$. The evolution of a classical *free* particle of mass m is described by a map x from $\mathbb{R}$ (time) to X. The *lagrangian density*[5] is the density on $\mathbb{R}$

$$L = \frac{m}{2}|\dot{x}|^2\,dt, \tag{1.1}$$

where $|\dot{x}|^2$ is the inner product $\langle dx/dt, dx/dt\rangle$. The integral of the lagrangian density

$$S = \int_{t_0}^{t_1} L \tag{1.2}$$

from time t_0 to time t_1 is the *action integral*, or simply *action*. If we deform x we have

$$\begin{aligned} \delta L &= m\langle\dot{x},\delta\dot{x}\rangle\,dt \\ &= -m\langle\ddot{x},\delta x\rangle\,dt - d\Big\{m\langle\dot{x},\delta x\rangle\Big\}. \end{aligned} \tag{1.3}$$

[4]In this context '$\mathbb{R}$-torsor' means 'principal $\mathbb{R}$-bundle'.

[5]We should use the *density* $|dt|$ in place of the *form* dt in (1.1), but in this section we simply orient $\mathbb{R}$ and so identify 1-forms and densities.

Here δ is the differential on the space $\mathcal{F}$ of trajectories x of the particle, d is the differential on $\mathbb{R}$, and the second minus sign[6] in the second line of (1.3) arises since δ and d anticommute on $\mathcal{F} \times \mathbb{R}$. Integrating we find

$$\delta S = \delta \int_{t_0}^{t_1} L = -\int_{t_0}^{t_1} dt\, m\langle \ddot{x}(t), \delta x(t)\rangle + \Big[m\langle \dot{x}(t), \delta x(t)\rangle\Big]_{t_0}^{t_1}. \tag{1.6}$$

The extremality condition is that the 1-form δS on $\mathcal{F}$ vanish on deformations of x with compact support in (t_0, t_1). This leads to the classical equation of motion

$$\ddot{x} = 0 \tag{1.7}$$

whose solutions are uniform motion. The boundary term leads one to consider, for each time t, the 1-form

$$\gamma(t) = m\langle \dot{x}(t), \delta x(t)\rangle \tag{1.8}$$

on the space $\mathcal{F}$ of all paths x. On the subspace $\mathcal{M}$ of extremals (solutions to (1.7)) the action S is a function whose differential is

$$\delta S = \gamma(t_1) - \gamma(t_0) \qquad \text{on } \mathcal{M}. \tag{1.9}$$

It follows that the 2-form on $\mathcal{M}$ defined by

$$\begin{aligned} \omega(t) &:= \delta\gamma(t) \\ &= m\langle \delta\dot{x}(t) \wedge \delta x(t)\rangle \end{aligned} \tag{1.10}$$

is independent of t. It is even independent of t for a specific reason, namely (1.9), with a compatibility among the reasons if three times $t_0 < t_1 < t_2$ are considered:

$$\int_{t_0}^{t_1} L + \int_{t_1}^{t_2} L = \int_{t_0}^{t_2} L. \tag{1.11}$$

This can be rephrased as defining an $\mathbb{R}$-torsor with connection (T, ∇) on $\mathcal{M}$ whose curvature is ω: for each fixed t_0 it is the trivial $\mathbb{R}$-torsor $T(t_0)$ with the connection $\nabla(t_0)$ given by $\gamma(t_0)$. By (1.9) addition of $-\int_{t_0}^{t_1} L$ gives an isomorphism from $\big(T(t_0), \nabla(t_0)\big)$ to $\big(T(t_1), \nabla(t_1)\big)$, and by (1.11) these isomorphisms form a compatible system of isomorphisms. The desired (T, ∇) is the "common value" (projective limit) of the $\big(T(t), \nabla(t)\big)$.

If we fix t_0, the map $x \mapsto \big(x(t_0), \dot{x}(t_0)\big)$ from the space $\mathcal{M}$ of extremals to the tangent bundle TX of X is an isomorphism. If we use (1.8) to map the tangent bundle to the cotangent bundle, then γ (resp. ω) is the pullback of the canonical 1-form (resp. canonical 2-form) on the cotangent bundle.

[6]It is more usual to let δx denote a tangent vector to $\mathcal{F}$, in which case there is no minus sign. We emphasize that our computations take place on $\mathcal{F} \times \mathbb{R}$. For example, '$x$' in (1.1) is the evaluation map $\mathcal{F} \times \mathbb{R} \to \mathbb{R}$ and '$\dot{x}$' is its time derivative. To convert (1.3) into the more usual formula, let $\hat{\xi}$ be a vector (field) on $\mathcal{F}$, and set

$$\hat{\xi} x = \iota(\hat{\xi})\delta x = X. \tag{1.4}$$

Apply the contraction $\iota(\hat{\xi})$ to (1.3). Note that in commuting $\iota(\hat{\xi})$ past d in the last term we pick up a minus sign. Thus we find

$$\iota(\hat{\xi})\delta L = \hat{\xi} L = -m\langle \ddot{x}, X\rangle\, dt + d\Big\{m\langle \dot{x}, X\rangle\Big\}. \tag{1.5}$$

System of Classical Particles with Potential

More generally, if we consider a system of classical particles with rigid constraints, the configuration space is a Riemannian manifold X with Riemannian structure given by twice the kinetic energy. Evolution is described by a map from $\mathbb{R}$ (time) to X. The lagrangian density is

$$L = \frac{1}{2}|\dot{x}|^2\,dt; \tag{1.12}$$

here the masses are included in the metric. If in addition we have an external field of forces depending on a potential, or interaction between the particles described by potentials, then the potentials are encoded by a real-valued function V on X, and the lagrangian density is

$$L = \Big\{\frac{1}{2}|\dot{x}|^2 - V(x)\Big\}\,dt. \tag{1.13}$$

The free story can be repeated with the following changes. The Euler-Lagrange equation (1.7) is now Newton's law

$$\nabla_t \dot{x} + \operatorname{grad} V = 0. \tag{1.14}$$

The 1-form on the space of x is

$$\gamma(t) = \langle \dot{x}(t), \delta x(t)\rangle \tag{1.15}$$

and the symplectic 2-form is

$$\omega(t) = \langle \delta\dot{x}(t) \wedge \delta x(t)\rangle. \tag{1.16}$$

The construction of the $\mathbb{R}$-torsor is as before. The identification of the space of solutions with the initial data at a fixed time depends on suitable completeness assumptions.

Electromagnetic Field

An *electromagnetic field* can be described as an $\mathbb{R}$-torsor with connection (P, ∇) on spacetime $\mathbb{R} \times X$. For now choose a trivialization of P (a "gauge"), and so write ∇ as a 1-form α. We separate the time and space components by

$$\alpha = V\,dt + A. \tag{1.17}$$

Then V is the *scalar potential* and A is the *vector potential*. The evolution of a single charged particle is described as before by a map $x\colon \mathbb{R} \to X$, and for a particle of mass m and charge q the lagrangian density is

$$L = \frac{m}{2}|\dot{x}|^2\,dt - qx^*(\alpha). \tag{1.18}$$

The action $S = \int_{t_0}^{t_1} L$ is the sum of a kinetic energy term and of q times the parallel transport along x from $x(t_0)$ to $x(t_1)$, computed in the chosen gauge. In δL the boundary term is now

$$\gamma(t) = m\langle \dot{x}(t), \delta x(t)\rangle - qA\big(\delta x(t)\big). \tag{1.19}$$

Fix t_0; then under suitable completeness assumptions the map $x \mapsto \big(x(t_0), \dot{x}(t_0)\big)$ identifies the space $\mathcal{M}$ of extremals with the tangent bundle TX. Notice that the map from the tangent bundle to the cotangent bundle given by (1.18) is the previous map shifted by qA.

If we change the gauge, the space of extremals does not change. This becomes clear if we do not interpret $\int_{t_0}^{t_1} L$ as a number, but rather as an isomorphism from $qP_{(t_0,x(t_0))}$ to $qP_{(t_1,x(t_1))}$, the sum of a kinetic energy term and of parallel transport. This isomorphism is manifestly gauge independent. On the space of extremals $\mathcal{M}$ we continue to have an $\mathbb{R}$-torsor with connection whose curvature is ω, independent of the gauge. For t_0 fixed it is naturally $\big(t_0, x(t_0)\big)^*(qP)$ with connection given by $\big(t_0, x(t_0)\big)^*(q\nabla) + m\langle \dot{x}(t_0), \delta x(t_0)\rangle$. Isomorphisms between this $\mathbb{R}$-torsor with connection for different choices of t_0 are given by the action—parallel transport plus a kinetic energy term.

In this example we see that the action need not be a number, but rather can be an element of an $\mathbb{R}$-torsor. We discuss such *topological terms* further in Chapter 6.

§1.2. The relativistic particle

Let X be n-dimensional Minkowski spacetime. This is standard n-dimensional affine space with a Lorentz metric. Fix affine coordinates $t, x^1, x^2, \dots, x^{n-1}$ so that the metric takes the form

$$c^2(dt)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.20}$$

The corresponding basis of the underlying vector space of translations is called an *inertial frame.* Here c is the speed of light. It is often convenient to set $x^0 = ct$.

The worldline of a relativistic particle is represented by a map

$$\begin{aligned} x\colon \mathbb{R} &\longrightarrow X \\ x(\tau) &= \big(t(\tau), x^1(\tau), \dots, x^{n-1}(\tau)\big) \end{aligned} \tag{1.21}$$

with $\langle dx/d\tau, dx/d\tau\rangle \geq 0$ and $dt/d\tau > 0$. The lagrangian density of a free particle of *rest mass* m_0 is

$$L = -m_0 c \left\langle \frac{dx}{d\tau}, \frac{dx}{d\tau} \right\rangle^{1/2} d\tau. \tag{1.22}$$

For a physical particle $dx/d\tau$ lies in the positive light cone; in particular, we have $\langle \frac{dx}{d\tau}, \frac{dx}{d\tau}\rangle > 0$. This lagrangian is invariant by the Poincaré group of symmetries of X, as well as by the group $\mathrm{Diff}^+(\mathbb{R})$ of reparametrizations of x. An action integral is attached to a region R of spacetime bounded by two space-like hypersurfaces H_0 and H_1. One can, for instance, take for R the region $t_0 \leq t \leq t_1$. For a free particle of rest mass m_0, the action integral is $-m_0 c$ times the arc length of the portion of the path x contained in R.

The extremals for this action are straight lines. If S is the action integral for the region R bounded by hypersurfaces H_0, H_1, and if we deform x, then δS is the sum of two terms: (i) an integral $\int_{\tau_0}^{\tau_1}$, where $x(\tau_i) \in H_i$; and (ii) boundary terms. The boundary term for H_1 is

$$\gamma[H_1] = -m_0 c\, \frac{\langle \dot{x}(\tau_1), \delta x(\tau_1)\rangle}{|\dot{x}(\tau_1)|}. \tag{1.23}$$

Note that this expression is Diff($\mathbb{R}$)-invariant. As previously, $\omega := \delta\gamma[H_1]$ is independent of H_1 on the space of extremals and turns it into a symplectic manifold. More precisely, the space of extremals carries an $\mathbb{R}$-torsor T with connection whose curvature is ω. For each choice of space-like hypersurface H it can be identified with the trivial $\mathbb{R}$-torsor, with connection given by $\gamma[H]$. Action integrals give a transitive system of isomorphisms between the descriptions of (T, ∇) given by different H.

To go to the nonrelativistic limit we proceed as follows. We work in our chosen coordinate system and consider velocities which are small relative to the speed of light c. If we take for coordinate τ the time t, then the path is

$$x(t) = \big(t, x^1(t), \dots, x^{n-1}(t)\big). \tag{1.24}$$

The lagrangian is then

$$\begin{aligned} L &= -m_0c^2\sqrt{1 - v^2/c^2}\,dt \\ &= \left\{-m_0c^2 + \frac{1}{2}m_0v^2 + O(\frac{v^4}{c^2})\right\} dt, \end{aligned} \tag{1.25}$$

where

$$v^2 = \sum_i \left(\frac{dx^i}{dt}\right)^2 \tag{1.26}$$

is the velocity squared. The second line of (1.25) shows that the nonrelativistic limit $v/c \to 0$ of L is the kinetic energy $\frac{1}{2}m_0v^2$ minus the potential energy m_0c^2 of the particle at rest. The reader can also check that the nonrelativistic limit of (1.23) for the hypersurface $H_1 : \{t = \text{constant}\}$ is (1.8) (for $m = m_0$).

In the relativistic setting it does not make sense to introduce rigid constraints on a system of particles, nor to introduce potentials. On the other hand, a background electromagnetic field can be introduced on Minkowski spacetime analogously to the nonrelativistic case.

§1.3. Noether's theorem

We have seen a number of examples of how, because equations of motion are Euler-Lagrange equations, the space of classical evolutions $\mathcal{M}$ carries a symplectic structure ω. More precisely, it carries a canonical $\mathbb{R}$-torsor T with connection ∇ whose curvature is ω. Any automorphism of the data used to define the extremality condition induces an automorphism of $(\mathcal{M}, T, \nabla)$. The same holds for infinitesimal automorphisms. Infinitesimal automorphisms of $(\mathcal{M}, T, \nabla)$ can be identified with functions Q on $\mathcal{M}$. Namely, if a vector field $\tilde{\xi}$ on T is such an infinitesimal automorphism, let ξ denote the projection to $\mathcal{M}$ and $\hat{\xi}$ the horizontal lift of ξ to T. Then

$$\tilde{\xi} = \hat{\xi} + Q\zeta, \tag{1.27}$$

where ζ is the infinitesimal action of $\mathbb{R}$ which defines the torsor structure. The infinitesimal automorphism $\tilde{\xi}$ of T respects the connection if and only if

$$dQ = -\iota(\xi)\omega. \tag{1.28}$$

If the curvature ω is nondegenerate, i.e., turns $\mathcal{M}$ into a symplectic manifold, then (1.28) shows that vector field ξ is determined by the generating function Q. For functions Q_1, Q_2 corresponding to vector fields ξ_{Q_1}, ξ_{Q_2}, we compute the vector field which corresponds to the Poisson bracket $\{Q_1, Q_2\}$ to be

$$\xi_{\{Q_1,Q_2\}} = [\xi_{Q_1}, \xi_{Q_2}]. \tag{1.29}$$

(Note: Brackets of infinitesimal automorphisms are given by the *opposite* of Poisson brackets, in the same way that the Lie algebra of diffeomorphisms is the set of vector fields with bracket *opposite* to the usual brackets of vector fields.)

If we have a trivialization of (T, ∇) which is preserved by an infinitesimal automorphism $\tilde{\xi}$, then

$$Q = \iota(\xi)\gamma, \tag{1.30}$$

where γ is the connection form on $\mathcal{M}$ induced from the trivialization. If the trivialization changes, then there is an additional term in (1.30). In §2 we develop a more systematic formalism in which to make computations, so in this section when the trivialization changes we simply report the result.

We apply these ideas to the free particle in both the nonrelativistic and relativistic settings. Consider first a free nonrelativistic particle of mass m_0 moving in $\mathbb{R}^n$. Let $x^1, \dots, x^n$ be standard coordinates on $\mathbb{R}^n$ and δ_{ij} the standard metric. The isometries of $\mathbb{R}^n$ induce automorphisms of $(\mathcal{M}, T, \nabla)$ which fix the trivialization at a fixed time t_0. Thus we can use (1.30) in conjunction with (1.8) to compute the charges. The charge corresponding to the infinitesimal translation $\xi = \partial/\partial x^i$ is the *linear momentum*

$$p_i = m_0 \delta_{ij} \dot{x}^j. \tag{1.31}$$

The charge corresponding to an infinitesimal rotation $\xi = x^i \partial/\partial x^j - x^j \partial/\partial x^i$ is the *angular momentum*

$$M^i_j = m_0 \delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i). \tag{1.32}$$

Now the *hamiltonian* or *energy* E is the charge associated to *minus* time translation in the domain $\mathbb{R}$, and this does not preserve the trivialization of (T, ∇) induced at a fixed time. Rather, the derivative of the trivialization at t_0 is $L_0(t_0)$, where we write the lagrangian density as $L_0\,dt$. (The change in trivialization comes because infinitesimal time translation only preserves the lagrangian up to a total derivative.) Accounting for this we compute the nonrelativistic energy (see Example 2.105).

$$E = \frac{1}{2} m_0 |\dot{x}|^2. \tag{1.33}$$

There is an additional symmetry called a *Galilean boost*. We write the infinitesimal version as a time-varying vector field $\xi = t\partial/\partial x^i$. Again there is an additional term in the formula for the charge N_i since the lagrangian is only preserved up to a total derivative:

$$N_i = m_0 \delta_{ij}(t\dot{x}^j - x^j). \tag{1.34}$$

This is simply minus the initial position times the mass.

Consider now a relativistic particle. The lagrangian (1.22) is invariant under the isometries of X, the Poincaré group, and we would like to compute the corresponding charges. Fix a splitting of Minkowski spacetime into time $\times$ space and use the hypersurface $t = t_0$ to trivialize (T, ∇). We parametrize the worldline as in (1.24). The subgroup of Poincaré which fixes this hypersurface also fixes this trivialization and we use (1.30) and (1.23) to compute the associated charges. Thus the momentum p_i is the charge corresponding to $\xi = \partial/\partial x^i$, which we compute to be

$$p_i = \frac{m_0 \delta_{ij} \dot{x}^j}{\sqrt{1 - v^2/c^2}} = m(v)\delta_{ij}\dot{x}^j. \tag{1.35}$$

Here we have introduced the *relativistic mass*

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \tag{1.36}$$

Note that the nonrelativistic limit $v/c \to 0$ of the relativistic mass $m(v)$ is the rest mass m_0. Similarly, an infinitesimal spatial rotation $\xi = x^i\partial/\partial x^j - x^j\partial/\partial x^i$ has charge the *angular momentum*

$$M^i_j = \frac{m_0 \delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i)}{\sqrt{1 - v^2/c^2}} = m(v)\delta_{jk}(x^i \dot{x}^k - x^k \dot{x}^i). \tag{1.37}$$

Now the charge corresponding to $\xi = -\partial/\partial t$ is the *relativistic energy* E. As in the nonrelativistic case the trivialization changes according to the lagrangian (1.25) (divided by the standard density dt), and so we compute Einstein's famous formula

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = m(v)c^2. \tag{1.38}$$

Note that $(E, -p_i)$ transforms in the dual to the standard representation of the Lorentz group. In $n = 4$ dimensions this quantity is called the *4-momentum*. Finally, the infinitesimal Lorentz boost

$$\xi = \frac{x^i}{c^2}\frac{\partial}{\partial t} + t\frac{\partial}{\partial x^i} \tag{1.39}$$

has corresponding charge

$$N_i = \frac{m_0 \delta_{ij}(t\dot{x}^j - x^j)}{\sqrt{1 - v^2/c^2}} = m(v)\delta_{ij}(t\dot{x}^j - x^j). \tag{1.40}$$

Here again the symmetry does not preserve the trivialization, so there is an extra term to compute. The charge N_i is simply minus the initial position of the particle times the relativistic mass.

The nonrelativistic limits of (1.35), (1.37), (1.38), (1.40) give (1.31), (1.32), (1.33), (1.34). (From E in (1.38) we must subtract the rest energy m_0c^2 to obtain the nonrelativistic energy (1.33).)

§1.4. Synthesis

Let M_c denote n-dimensional Minkowski spacetime with metric g_c given in (1.20). Let M_∞ denote the limiting space as $c \to \infty$. Of course, the affine spaces which underlie M_c and M_∞ are identical. Let W be the vector space of translations of M_∞. The limit $\lim_{c\to\infty} g_c^{-1} = g_\infty^{-1}$ of the inverse metric is a degenerate quadratic form on W^*. It has a one-dimensional kernel $S^\circ \subset W^*$ whose annihilator $S \subset W$ determines a codimension one foliation of M_∞ by affine subspaces. Then g_∞^{-1} on W^*/S° is inverse to a negative definite metric on S. We think of this foliation (with the metric on the leaves) as defining the simultaneous spatial events in M_∞. We also fix a scale $dt \in S^\circ$ for time; then there is a one-dimensional space of distinguished affine time functions which differ by a constant. The group G_∞ of affine transformations of M_∞ which preserve g_∞^{-1} and dt is the *Galilean group*. Let G_c denote the group of affine transformations of M_c which preserve g_c; the double cover of its identity component is the Poincaré group. Note that G_c, G_∞ act on the same affine space, so there is a well-defined sense in which $G_c \to G_\infty$ as $c \to \infty$.

The translations sit inside G_c and G_∞. In G_∞ the spatial translations lie in a subgroup H of dimension $2(n-1)$ generated by vector fields $t\partial + \partial'$, where t is any affine time function and ∂, ∂' are spatial translations. Vector fields $t\partial$ generate affine transformations called *Galilean boosts*; they are the $c \to \infty$ limit of Lorentz boosts (1.39).

There is a nontrivial central extension of G_∞ which restricts nontrivially on H. For a geometric picture we begin with a fixed $(n+1)$-dimensional Minkowski spacetime N with vector space of translations V. For each spacelike vector $v \in V$ we consider the subgroup G_v of the Poincaré group which fixes v. It acts on the quotient affine space $M_v = N/\mathbb{R}\cdot v$. This quotient space M_v inherits a Minkowski metric we describe in two equivalent ways: it is the metric induced from the orthogonal space $(\mathbb{R}\cdot v)^\perp \subset V$, or equivalently its inverse is the subspace metric on the annihilator $(V/\mathbb{R}\cdot v)^* \cong (\mathbb{R}\cdot v)^\circ \subset V^*$. The subgroup G_v is the trivial central extension of the Poincaré group of M_v by translations along v. Now consider $\ell \in V$ a lightlike vector. Set $M_\ell = N/\mathbb{R}\cdot\ell$, and let G_ℓ be the subgroup of the Poincaré group of N which fixed ℓ. Now the inverse metric on V^* restricts to a degenerate form on $(V/\mathbb{R}\cdot\ell)^* \cong (\mathbb{R}\cdot\ell)^\circ \subset V^*$. Also, we take the functional $\langle \ell, \cdot\rangle$ to be the dt above. The group of affine transformations of M_ℓ which preserves the degenerate form and ℓ is the Galilean group. Thus G_ℓ is the central extension of this group by translations along v. Note that the subspace of $(V/\mathbb{R}\cdot\ell)^* \cong (\mathbb{R}\cdot\ell)^\circ$ previously called S° is here $(\ell^\perp)^\circ$, the annihilator of the orthogonal subspace to ℓ. (Since ℓ is lightlike, we have $\mathbb{R}\cdot\ell \subset \ell^\perp$, which is why it makes sense to define $dt = \langle \ell, \cdot\rangle$.) Also, S is simply $\ell^\perp/\mathbb{R}\cdot\ell$.

For an explicit description introduce coordinates $t^+, t^-, x^1, \dots x^{n-1}$ on N so that the metric is[7]

$$dt^+ dt^- - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.41}$$

[7] Here the speed of light is inessential, so we set it to one.

Take $\ell = \partial_+ = \partial/\partial t^+$. Then $(\ell^\perp)^\circ$ is the span of dt^-, and the possible time functions are $\{at^- + b : a, b \in \mathbb{R}\}$. The Lie algebra of G_ℓ is spanned by

$$(1.42)\qquad \begin{aligned} &\partial_+ && \text{(central element)} \\ \partial_- = \frac{\partial}{\partial t^-},\quad &\partial_i = \frac{\partial}{\partial x^i} && \text{(translations)} \\ &x^i\partial_j - x^j\partial_i && \text{(spatial rotations)} \\ B_i = &\, t^-\partial_i + x^i\partial_+ && \text{(Galilean boosts)} \end{aligned}$$

The nontrivial bracket

$$(1.43)\qquad [\partial_i, B_j] = \delta_{ij}\partial_+$$

reflects the nontrivial central extension. Note from (1.31) and (1.34) that for the free nonrelativistic particle the Poisson bracket of the Noether charges corresponding to ∂_i and B_j is

$$(1.44)\qquad \{p_i, N_j\} = -\delta_{ij}m_0.$$

Comparing (1.43) and (1.44) we see that the conserved quantity corresponding to ∂_+ in a theory should be identified with the mass. Thus in the Galilean theory there is an additional conserved quantity over the relativistic theory—the mass. (The bracket of elements in the centrally extended Galilean algebra and the Poisson bracket of the corresponding Noether charges are opposite. This is the usual situation for *left* group actions on symplectic manifolds; see §5 of [I-**Signs**].)

Quite generally, suppose we have a classical theory whose state space $(\mathcal{M}, \omega)$ carries a symplectic action of either the Galilean group or the Poincaré group. Infinitesimally we have an antihomomorphism from the Lie algebra $\mathfrak{g}$ of that group into the Lie algebra of vector fields on $\mathcal{M}$. We assume that each vector field ξ in the image satisfies $-\iota(\xi)\omega$ is exact, i.e., is the symplectic gradient of some function. For a general group G, the existence and uniqueness of a lift $\mathfrak{g} \to C^\infty(\mathcal{M})$ which is an antihomomorphism of Lie algebras is measured by $H^1(\mathfrak{g})$ and $H^2(\mathfrak{g})$. For the Poincaré group there is a unique lift; for the Galilean group there is a lift of the central extension we constructed above, unique up to a shift of the total energy. (If spacetime has dimension 3, then we can also shift angular momentum.) The central element then maps to a locally constant function which is the total mass of the system. In the Poincaré case the vector space V of translations maps to a vector space of functions which—after choosing an inertial frame—is $(-E, p_1, \dots, p_{n-1})$, where E is energy and p_i are linear momenta. Under Lorentz transformations this transforms as the coefficients of an element in[8] V^*, and out of it we construct an invariant using the metric:

$$(1.45)\qquad E^2/c^2 - |p|^2 = m_0^2c^2.$$

This defines the "rest mass" m_0 of the system. It is a Poincaré invariant, though in general not locally constant, function on $\mathcal{M}$. In the Galilean case there is a Galilean-invariant codimension one subspace $S \subset V$ of translations in spatial directions, and

[8] Note that the vector dual to $(E, -p)$ is $(E/c^2, p)$.

S carries a metric. If the Galilean algebra, and not a central extension, were to lift to functions, then we would conclude that the norm square $|p|^2$ of the total momentum is Galilean-invariant. But clearly $|p|^2$ typically changes under boosts, and this shows why boosts and spatial translations have a nonzero commutator in the central extension.

We can formulate the theory of the free nonrelativistic particle by considering its worldline in the space M_∞. The difference from the relativistic situation is that the time functions provide distinguished parametrizations of the worldline up to translation. This allows us to "couple" the free nonrelativistic particle to an arbitrary potential function V, as in (1.13); the integrated lagrangian (action) does not change if we translate time and so is Galilean invariant. By contrast, as stated earlier we cannot introduce a Poincaré invariant coupling of the relativistic particle to a potential function. Rather, we can couple it to *fields*, specifically to an electromagnetic field (abelian connection) and to a gravitational field (variable Lorentz metric). In the next chapter we take up the general theory of fields.

CHAPTER 2
Lagrangian Theory of Classical Fields

§2.1. Dimensional analysis

It is often useful to follow the advice we give to beginning students: Check units!

Every physical quantity has units attached to it. The basic units are units of mass, length, and time. The number measuring a physical quantity Q depends on the choice of units, and Q is said to be of dimension $M^a L^b T^c$ if the number q measuring Q is multiplied by $\lambda^{-a}\mu^{-b}\nu^{-c}$ when the units are multiplied by λ, μ, ν. Notation: $[Q] = M^a L^b T^c$. For example, the action integral S of (1.2), where the integral is taken between prescribed instants, has the dimension of an action:

$$[S] = ML^2T^{-1}. \tag{2.1}$$

The dimension of a p-form is defined to be the dimension of its value on a fixed p-vector. The lagrangian density (1.1) (a 1-form on the time line), the variational 1-form γ of (1.8) (a 1-form on the space of trajectories), and the closed 2-form ω all have the dimension of an action.

The dimension of the conserved quantity Q corresponding to an infinitesimal Galilean transformation η is given by

$$[Q] = [\text{action}][\eta]. \tag{2.2}$$

In the relativistic setting we will usually impose $c = 1$, so that a unit of length gives one of time. Then [action] becomes ML, and the conserved quantities corresponding to infinitesimal generators of the Poincaré group once again have

$$[Q] = [\text{action}][\eta]. \tag{2.3}$$

We can consider electric charge to have units C and take the constant q in (1.18) to have $[q] = C$; then the 1-form α in (1.17) which represents the electromagnetic field has units of action divided by charge:

$$[\alpha] = ML^2T^{-1}C^{-1}. \tag{2.4}$$

Alternatively, we can work in a system of units in which charge is expressed in terms of mass, length, and time by

$$C^2 = ML^3T^{-2}. \tag{2.5}$$

This comes from declaring the constant k in Coulomb's law "$F = kq_1q_2/r^2$" to be dimensionless.

Universal physical constants allow us to convert units. In relativistic theories the speed of light

$$[c] = LT^{-1} \tag{2.6}$$

identifies time with length. In quantum theories Planck's constant

$$[\hbar] = ML^2T^{-1} \tag{2.7}$$

has units of action, which eliminates one of M, L, T. In theories of gravity the Newton constant G in the Newton force law "$F = Gm_1m_2/r^2$" has units

$$[G] = L^3M^{-1}T^{-2}, \tag{2.8}$$

which again allows us to eliminate one of M, L, T. In a relativistic quantum field theory we can use $c, \hbar$ to express all units in terms of mass, and so each physical quantity has a *mass dimension*.[9] If the theory includes gravity (e.g. string theory), then we can use $c, \hbar, G$ to express everything in terms of dimensionless quantities.

§2.2. Densities and twisted differential forms

In this section M is an ordinary (not super) manifold.

If M is oriented of dimension n, then n-forms with compact support can be integrated on M. Changing the orientation multiplies the integral by -1. This leads to the consideration of *densities*, defined to be sections of

$$\operatorname{Dens} M := \textstyle\bigwedge^n T^*M \otimes o_M, \tag{2.9}$$

where o_M is the orientation line bundle. On any M, oriented (and orientable) or not, the integral of a density ω with compact support is unambiguously defined: one writes ω as a sum of densities ω_i with support in orientable local charts U_i, one orients each U_i, and doing so identifies ω_i with an n-form on U_i, and one defines $\int_M \omega := \sum \int_{U_i} \omega_i$.

More generally, we will have to consider the complex of *twisted forms*, the tensor product of $\Omega^\bullet_M$ with the local system o_M. We treat o_M as being in cohomological degree $-n$, and define

$$\Omega_M^{|-p|} := \Omega_M^{n-p} \otimes o_M. \tag{2.10}$$

A compactly supported element of $\Omega_M^{|-p|}$ can be integrated on a normally oriented submanifold of M of codimension p. Elements of $\Omega_M^{|-p|}$ can be viewed as sections of

$$\textstyle\bigwedge^p TM \otimes \operatorname{Dens} M; \tag{2.11}$$

the isomorphism with the description (2.10) comes by contracting a p-vector field with a density.

In the sequel we often say simply 'form' for 'twisted form'.

In the case of supermanifolds, densities are related not to an exterior power of the cotangent bundle, but to its Berezinian, and (2.11) defines the components of the complex of integral densities. See [**I-Supersymmetry**, §§3.9–12] for details.

On a Riemannian manifold M the Hodge $*$-operator is an isomorphism

$$*\colon \Omega^q_M \longrightarrow \Omega_M^{|-q|}. \tag{2.12}$$

---

[9]Then for a physical quantity Q instead of writing $[Q] = M^n$ we usually write $[Q] = n$.

§2.3. Fields and lagrangians

Let M be a smooth manifold of dimension n. We formulate field theory on M. In the standard physical setup $M = M^n$ is affine Minkowski spacetime. For $n = 1$ this is M^1, which is (affine) time; it is the appropriate "spacetime" M for classical mechanics. In usual examples of field theory we can analytically continue to imaginary time and so obtain a field theory on Euclidean space. Often field theories can also be formulated for curved metrics—of Lorentz or Euclidean signature. We also allow the spacetime M to be a supermanifold. As we cautioned in the introduction, our framework is not adequate for many types of constrained mechanical systems.

Fields on M are (smooth) sections of a given fiber bundle $E \to M$. Let $\mathcal{F}$ denote the space of all sections. For example, the basic field in a *σ-model* is a map $\phi\colon M \to X$ for some auxiliary manifold X. In this case we simply have $E = M \times X$. We can also study a twisted version in which E is not a product. An important case is when E is a vector bundle. The basic field in a *gauge theory* is a connection A with gauge group some specified Lie group G. If we fix a principal G-bundle $P \to M$, then A is a section of a certain associated bundle of affine spaces. For many purposes it is best not to fix P and rather to view the collection of all connections as a category. However, in this section we simply view P as fixed. There is an evaluation map

$$e\colon \mathcal{F} \times M \longrightarrow E. \tag{2.13}$$

Again we allow E—and so also $\mathcal{F}$—to be a supermanifold, even if M is an ordinary manifold.

Of course, a field theory typically contains several fields ϕ_i and correspondingly $E = \times E_i$ is a fiber product. Roughly speaking, each E_i decomposes into an intrinsic part times an extrinsic part. The intrinsic part is associated to the principal frame bundle of M via a representation of GL (or Spin), and the representation determines the type of field. Thus a *scalar field* is associated to the trivial representation. The basic field $\phi\colon M \to X$ in a σ-model is a typical example. A scalar field may take values in a nonlinear space, but the extrinsic values of other types of fields are linear. A *p-form field* is a p-form on M; it may take values in some vector bundle over M. Physicists often use the word 'vector field' to refer to a 1-form field. They also use 'vector field' or 'gauge field' to refer to a *connection*, which is a type of field which was discussed above. There are also "connectionlike" versions of p-forms (see §6.3), but only for abelian structure groups. If M is a spin manifold then we can also consider *spinor fields*, which are sections of a spin bundle possibly tensored with another extrinsic vector bundle. The precise choice of spinor bundle varies with the example. Theories of gravity also include a *metric* on X, sometimes called a 'gravitational field'. In theories of supergravity there is also a *Rarita-Schwinger field*, which is a section of an irreducible subbundle of $S \otimes T^*M$, where S is a spin bundle.

The *spin* of a field depends on the representation of Spin (or GL) which defines it. Namely, we decompose the complexified representation under the Spin(2) subgroup which double covers the group of rotations in some 2-plane. We obtain a sum of one-dimensional representations which we label by half-integers $0, \pm 1/2, \pm 1, \dots$. Replace these numbers by their absolute values; then the largest number which occurs is the spin of the field. Thus a scalar field has spin 0, a spinor field spin 1/2, a p-form field or connection spin 1, a Rarita-Schwinger field spin 3/2, and a metric spin 2.

There are physical reasons to restrict to these values of the spin in quantum field theories (see [**II-Dynamics of QFT**, §2.4]).

In unitary quantum field theories there is a connection between spin and *statistics*, the statistics being whether the field is even or odd (in the sense of supergeometry): fields of integral spin are even and fields of half-integral spin are odd. The usual physical terminology is that even fields are *bosons* and odd fields are *fermions*. This spin-statistics connection is violated in some nonunitary topological field theories.

In some theories there are local gauge symmetries (see Definition 2.93) which act on the fields. This occurs for fields which are p-forms ($p \geq 1$) and for connections in gauge theory. (A p-form α is gauge equivalent to $\alpha + d\beta$ for β any $(p-1)$-form.) The gauge symmetries of a connection are as usual. Local symmetries are sections of a bundle of groups over M. Let $\overline{\mathcal{F}}$ denote the quotient of the space of fields $\mathcal{F}$ by the action of local symmetries. In our notation we treat $\overline{\mathcal{F}}$ as a manifold, though this may not be true and we may need to work equivariantly on some space which projects onto $\overline{\mathcal{F}}$. (For example, in gauge theory we often fix a basepoint on each component of M. Then the group of gauge transformations which equal the identity at the basepoints acts freely on the space of connections, and the finite dimensional group of automorphisms at the basepoints acts on the quotient.)

Our formulation is not meant to include theories of gravity, where one of the fields is a metric and the group of diffeomorphisms of M acts as a local symmetry. We will, however, use the metric as a *background field* when we discuss the energy-momentum tensor. Then in some cases (e.g., with spinor fields) the fiber bundle $E \to M$ varies with the metric. The appropriate modification of this setup is described later when we discuss the energy-momentum tensor.

We will use a complex which is, basically, the de Rham complex of $\mathcal{F} \times M$. Reflecting the product structure of $\mathcal{F} \times M$, it is a double complex. Suppose first that M is oriented, of dimension n. The complex $\Omega^{\bullet,|\bullet|}$ we will use is then the de Rham complex shifted by n: in other words, $\Omega^{p,|-q|}(\mathcal{F} \times M)$ is the space of p-forms on $\mathcal{F}$ with values in the space of $(n-q)$-forms on M. We let δ be the exterior derivative of $\mathcal{F}$, d the exterior derivative[10] of forms on M, and $D = \delta + d$ the total exterior derivative. For general M we use twisted $(n-q)$-forms: instead of the de Rham complex of M, we use $\Omega_M^{|\bullet|} = \Omega_M^{\bullet} \otimes o_M$.

More relevant is the subcomplex $\Omega_{\text{loc}}^{\bullet,|\bullet|}$ of *local* elements of $\Omega^{\bullet,|\bullet|}$, where $\alpha \in \Omega^{p,|-q|}$ is said to be local if for some k, at any $\phi \in \mathcal{F}$ the value of the twisted $(n-q)$-form $\alpha(\phi; \hat{\xi}_1, \dots, \hat{\xi}_p)$ at $m \in M$ depends only on the k-jet at m of $\phi, \hat{\xi}_1, \dots, \hat{\xi}_p$. Here $\hat{\xi}_i$ are tangents to $\mathcal{F}$ at ϕ. If $\mathcal{F}$ is the space of sections of a bundle E over M, and if $\pi^{(k)}\colon J^kE \to M$ is the bundle of k-jets of sections of $\pi\colon E \to M$, such an α is a global section over J^kE of $\Omega^p_{J^kE/M} \otimes \pi^{(k)^*}\Omega_M^{|-q|}$. Note that k cannot be kept fixed, as d maps $\Omega^p_{J^kE/M} \otimes \pi^{(k)^*}\Omega_M^{|-q|}$ to $\Omega^p_{J^{k+1}E/M} \otimes \pi^{(k+1)^*}\Omega_M^{|-q+1|}$.

As mentioned in the Introduction, the double complex $\Omega_{\text{loc}}^{\bullet,|\bullet|}$ has been investigated by Takens, who proved the following.

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[10]Our sign convention is that for $\alpha \in \Omega^p(\mathcal{F})$ and $\beta \in \Omega^{|-q|}(M)$,

$$d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d\beta. \tag{2.14}$$

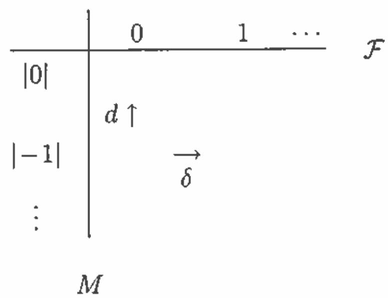
Theorem 2.15 (Takens). *For $p > 0$ the complex $\left(\Omega_{\mathrm{loc}}^{p,|\bullet|}(\mathcal{F} \times M), d\right)$ of local differential forms is exact except in the top degree $|\bullet| = 0$.*

This theorem plays a crucial role in the discussion of generalized symmetries (§2.6), and also plays a part in the general discussion of this section. We also make use of the following generalization, proved by the same method.

Theorem 2.16 (Takens). *Let $V_i \to E$, $i = 1, \ldots, p$, be vector bundles, where we require $p \geq 1$. Let $V = \times_E V_i$ be the fiber product. For $\phi \in \mathcal{F}$ a section of E, let $\mathcal{V}_\phi$ be the space of sections of $\phi^* V \to M$. Consider in $\Omega_{\mathrm{loc}}^{0,|\bullet|}(\mathcal{V}_\phi \times M)$ the subcomplex of forms $\alpha(\phi; \zeta_1, \ldots, \zeta_p)$ which are $\mathbb{R}$-multilinear in ζ_i. Then this subcomplex is exact except in the top degree $|\bullet| = 0$.*

In the appendix to Chapter 2 we give a proof of these theorems.

We will use the following picture to depict elements in the double complex:



We remark that certain topological (terms in) lagrangians do not fit into this formalism. We discuss the necessary modifications in Chapter 6.

The basic ingredient in a classical field theory is a *lagrangian (density)*

$$L \in \Omega_{\mathrm{loc}}^{0,|0|}(\mathcal{F} \times M); \tag{2.17}$$

it has units of action:

$$[L] = ML^2T^{-1}. \tag{2.18}$$

The notation in (2.15) indicates that for each field ϕ in $\mathcal{F}$ we have a density $L(\phi)$ on M, while (2.16) indicates that the integral of $L(\phi)$ on a fixed compact region of M has the dimension of an action. Typically, $L(\phi)$ is a local functional of ϕ and, for most fundamental lagrangians, the value of $L(\phi)$ at a point m depends only on the 1-jet of ϕ at m (gravity is an exception).

Typically, the integral $\int_M L(\phi)$ is divergent. Even if, as often, one can restrict one's attention to fields decaying rapidly at spatial infinity, so that the integral between two space-like hypersurfaces converges, the integration over time will diverge. Computations given below show that otherwise the expected symplectic structure on the space of extremals would vanish. However, for a deformation with compact support of ϕ, and if we assume L local, then $\delta L(\phi)$ is a density with compact support, which can be integrated. One says that ϕ_0 is *extremal* if $\int \delta L = 0$ for any such deformation of ϕ_0. We now make this more precise. One considers families of fields $\phi[u]$, with $\phi[0] = \phi_0$, and with $\phi[u]$ independent of u in the complement of a compact region of M. The density $\frac{d}{du}L(\phi[u])\big|_{u=0}$ then has compact support. The extremality condition is that its integral should vanish, for any $\phi[u]$. It is suggestive, if abusive, to write the condition $\frac{d}{du}\int_M L(\phi[u]) = 0$.

Let us suppose that the fields ϕ are sections of a bundle E over M. For a deformation $\phi[u]$ of ϕ_0, the density $\frac{d}{du}L(\phi[u])$ at $u = 0$ will depend only on the field of vertical vectors $\frac{d}{du}\phi$ on E, along ϕ_0, which is a section $\hat{\xi}$ of $\phi_0^*T(E/M)$. The $\mathbb{R}$-linear form

$$\hat{\xi} \longmapsto \int_M \frac{d}{du}L(\phi) \tag{2.19}$$

on the sections with compact support of $\phi_0^*T(E/M)$ can be uniquely written as

$$\int_M \underline{D}L(\hat{\xi}) \tag{2.20}$$

for $\underline{D}L$ a $(\phi_0^*T(E/M))^*$-valued density. The unicity of $\underline{D}L$ is clear, and its existence is a local question. It is proved by the usual integration by parts argument leading to Euler-Lagrange equations. Extremality of ϕ_0 means that $\underline{D}L = 0$ at ϕ_0 (Euler-Lagrange equations). We note that $\underline{D}L$ is in $\Omega^{1,|0|}$, with a value at $m \in M$ depending only on some jet of ϕ at m, and on the value at m of $\hat{\xi}$.

This formalism does not hold if, as is typical of superspace formulations of supersymmetric theories, the fields ϕ are sections of a bundle E subjected to constraints. The problem is that $\hat{\xi}$ is no longer arbitrary in $\phi_0^*T(E/M)$. Let $\mathcal{M}$ be the space of extremals. We now explain that if spacetime M is time $\times$ space, or at least if there is a suitable notion of space-like hypersurface, then the construction on $\mathcal{M}$ of a closed 2-form—on the model of Chapter 1—uses only conditions on L which hold for such constrained superspace formulations. The crucial condition is that the constraints allow ϕ to be deformed independently in disjoint regions of M. (Formally: the sheaf of fields, a subsheaf of the sheaf of C^∞-sections of E/M, is soft). In practice, for superspace formulations, this means that superfields can be described in components.

Let H_1 and H_2 be space-like hypersurfaces with, to simplify the picture, H_1 before H_2. In superspace formulations, H_1 and H_2 should be codimension $1|0$ submanifolds of M. We consider

$$\int_{H_1}^{H_2} L(\phi) \ , \tag{2.21}$$

which we at first assume to converge. If ϕ_0 is extremal, and if $\phi[u]$ deforms ϕ_0 ($\phi[0] = \phi_0$), then

$$\frac{d}{du}\int_{H_1}^{H_2} L(\phi[u]) \tag{2.22}$$

will in general not vanish at $u = 0$, except if the support of the deformation is between H_1 and H_2. For deformations with compact support, it will be the difference of boundary terms attached to H_1 and H_2:

$$\frac{d}{du}\int_{H_1}^{H_2} L(\phi[u]) = \Gamma_2(\hat{\xi}) - \Gamma_1(\hat{\xi}) \ . \tag{2.23}$$

In this formula, Γ_i is a 1-form on $\mathcal{F}$, defined only on $\mathcal{M}$, whose value at a tangent vector $\hat{\xi}$ at an extremal ϕ_0 depends only on some jet of $\hat{\xi}$ along H_i. We now restrict those 1-forms to (the tangent bundle of) $\mathcal{M}$. Equation (2.23) means that, with δ the de Rham differential on $\mathcal{M}$,

$$\delta \int_{H_1}^{H_2} L = \Gamma_2 - \Gamma_1 \ , \tag{2.24}$$

so that the exterior derivative $\delta\Gamma_i$ is independent of i. One defines the canonical closed 2-form on $\mathcal{M}$ by

$$\Omega := \delta\Gamma_i \ . \tag{2.25}$$

Remark 2.26. If the integral of $L(\phi)$ on the whole of M did make sense, this construction would collapse: $\Gamma_2(\phi)$ would be δ of the integration of $L(\phi)$ from the infinite past to H_2, giving $\Omega = \delta\Gamma_2 = 0$.

Remark 2.27. Instead of assuming $\int_{H_1}^{H_2}$ to exist, one may consider only fields which coincide with a fixed extremal ϕ_* at spatial infinity. The given construction then defines Γ_i for tangent vectors to $\mathcal{M}$ whose support intersects the region of spacetime in between any two space-like hypersurfaces in a compact set.

Remark 2.28. As in Chapter 1, the construction gives an $\mathbb{R}$-torsor T with connection ∇ on $\mathcal{M}$, whose curvature is Ω. The choice of a space-like hypersurface H trivializes T, and the trivializations given by H_1 and H_2 differ by $\int_{H_1}^{H_2} L(\phi)$.

Remark 2.29. Adding to the lagrangian density an exact term $d\alpha$, for α in $\Omega_{\text{loc}}^{0,|-1|}$, does not change the space of extremals. For H a space-like hypersurface, it changes the corresponding 1-form Γ on $\mathcal{M}$ by

$$\hat{\xi} \longmapsto \int_H \frac{d}{du}\alpha(\phi[u]) \quad \text{at} \quad u = 0 \ . \tag{2.30}$$

The torsor with connection (T, ∇) for L, and the one (T', ∇') for $L + d\alpha$, can be identified, by sending the trivialization 0_H of T corresponding to a space-like hypersurface H to the trivialization $0'_H - \int_H \alpha$ of T'. In Chapter 1 (particle in an electromagnetic field), we saw one example where a change of gauge changes $L(\phi)$ by an exact term, and where the torsor T of Remark 2.28 is defined, but where a trivialization of T is given by the choice of H and of a gauge along H.

Remark 2.31. We now give a variant of the definition of Ω, to make clearer on what it depends. Let $\phi[u, v]$ be a 2-parameter family of extremals. At $\phi_0 = \phi[0, 0]$ in $\mathcal{M}$, we have the two tangent vectors $\hat{\xi}_1 = \frac{\partial}{\partial u}\phi[u, v]$ at $(0, 0)$ and $\hat{\xi}_2 = \frac{\partial}{\partial u}\phi[u, v]$ at $(0, 0)$. What is $\Omega(\hat{\xi}_1, \hat{\xi}_2)$? Let H be a space-like hypersurface. It has a past and a future side, and we identify a neighborhood of H with $H \times (-1, 1)$, with $h \in H$ corresponding to $(h, 0)$ and the future side of H corresponding to $H \times [0, 1)$. Let $\phi_1[u, v]$ be a family of fields, deforming $\phi[u, 0]$, agreeing with $\phi[u, v]$ in the future, and with $\phi[u, 0]$ in the past: for some a, b with $-1 < a < b < 1$,

$$\begin{aligned} \phi_1[u, 0] &= \phi[u, 0]; \\ \phi_1[u, v] &= \phi[u, v], \qquad \text{at } (h, t) \text{ for } t > b; \\ \phi_1[u, v] &= \phi[u, 0], \qquad \text{at } (h, t) \text{ for } t < a. \end{aligned} \tag{2.32}$$

Let $\phi_2[u,v]$ be a similar family, with the roles of u and v interchanged. Then

$$\Omega(\hat{\xi}_1, \hat{\xi}_2) = \int \partial_u \partial_v L(\phi_1) - \partial_u \partial_v L(\phi_2) \quad \text{at} \quad (u,v) = (0,0) \,. \tag{2.33}$$

The integrand vanishes for $t > b$ (where $L(\phi_1) = L(\phi_2)$) and for $t < a$ (where $\partial_v L(\phi_1) = \partial_u L(\phi_2) = 0$). If $\phi[u,v] = \phi_0$ at (h,t) for h outside a compact of H, one can take $\phi = \phi_1 = \phi_2$ outside of such a compact, and the integrand has then compact support.

We now relate the definition of Ω to the previous one. We first consider a one-parameter family $\phi[u]$ of extremals, with $\phi[0] = \phi_0$. Let $\phi_1[u]$ agree with $\phi[u]$ for $t > b$, and with ϕ_0 for $t < a$ $(-1 < a < b < 0)$. We have

$$\Gamma_H(\hat{\xi}) = \int_{t=-1}^{t=0} \frac{d}{du} L(\phi_1[u]) \quad \text{at} \quad u = 0, \tag{2.34}$$

with Γ_H the boundary term attached to H. For commuting vector fields, we have $d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X)$. Applying this to the (u,v) plane, and to ∂_u, ∂_v to compute $d\Gamma_H$, we get the formula given for $\Omega(\hat{\xi}_1, \hat{\xi}_2)$.

The construction of the closed 2-form Ω on the space $\mathcal{M}$ of extremals is reassuring, but not particularly useful. What is more interesting are analogs local on M of Ω and for Ω nondegenerate of functions on $\mathcal{M}$ corresponding to infinitesimal symmetries. Such local analogs will continue to make sense after a Wick rotation, in Euclidean field theory.

We will see in §2.4 that if the space of fields is the space of sections of a bundle E over M, and if the value of $L(\phi)$ at $m \in M$ depends only on the first jet of ϕ at m, there is a unique

$$\gamma \in \Omega_{\mathrm{loc}}^{1,|-1|}(\mathcal{F} \times M) \tag{2.35}$$

such that

$$\underline{D}L = \delta L + d\gamma \tag{2.36}$$

and that γ is "linear over functions": at ϕ, for $\hat{\xi}$ a tangent vector to $\mathcal{F}$ at ϕ, identified with a section of $\phi^* T(E/M)$, the value of the form $\gamma(\hat{\xi})$ at $m \in M$ should depend only on $\hat{\xi}$ at m. More generally, we have the following

Definition 2.37. A form $\beta \in \Omega_{\mathrm{loc}}^{1,|\bullet|}(\mathcal{F} \times M)$ is *linear over functions* at (ϕ, m) if for every $\hat{\xi} \in T_\phi \mathcal{F}$ and every function f on M,

$$\beta_{(\phi,m)}(f\hat{\xi}) = f(m)\beta_{(\phi,m)}(\hat{\xi}). \tag{2.38}$$

This γ is a local counterpart to the 1-form Γ_H on $\mathcal{M}$ attached to a space-like hypersurface H. Indeed, Γ_H is the integral of γ on H. Taking an exterior derivative in the $\mathcal{F}$ direction, we obtain a local analog of the 2-form Ω: the 2-form Ω is the integral on H of $\omega := \delta\gamma$ in $\Omega_{\mathrm{loc}}^{2,|-1|}(\mathcal{F} \times M)$. This integral is a 2-form on the whole of $\mathcal{F}$, but it is independent of H only as a 2-form on $\mathcal{M}$.

In the case of gauge theories, even if L and γ are invariant by gauge transformations, one should not expect γ to vanish in the direction of the gauge orbits.

For more general Lagrangians, to localize the construction of Ω, one should *choose* γ in $\Omega_{\mathrm{loc}}^{1,|-1|}(\mathcal{F} \times M)$ such that $\underline{D}L = \delta L + d\gamma$. We call γ the *variational 1-form*. The pair (L, γ) defines a field theory.

Definition 2.39. $\mathcal{L} = L + \gamma$ defines a *classical (lagrangian) field theory* if

$$(DL)^{1,|0|} = \delta L + d\gamma \tag{2.40}$$

is linear over functions.

Equation (2.40) is a simple rewriting of (2.36).

Since $\underline{D}L$ is uniquely determined by L, and is local, it follows from Theorem 2.15 that the difference between any two choices for the variational 1-form γ is d-exact. A d-exact change in γ leads to a d-exact change in the local symplectic form ω, defined below in (2.44). In the Hamiltonian situation, where spacetime is time $\times$ space, the symplectic form Ω on the space of classical solutions and the $\mathbb{R}$-torsor with connection whose curvature is Ω do not depend on the choice of γ.

For $K \in \Omega_{\text{loc}}^{0,|-1|}(\mathcal{F} \times M)$ we can form a new lagrangian $L + dK$. Then $\gamma + \delta K$ is a valid choice of variational 1-form, and the new total lagrangian is $\mathcal{L} + DK$. In this case the local symplectic form ω is unchanged, as are the equations of motion $(D\mathcal{L})^{1,|0|}$. In the Hamiltonian situation the global symplectic form Ω is unchanged, and there is an isomorphism of the $\mathbb{R}$-torsors for L and $L + dK$ constructed from K.

If γ is linear over functions, in the sense of Definition 2.37, we write

$$\gamma = \delta\phi \wedge \pi. \tag{2.41}$$

In this formula ϕ is a section of E over $\mathcal{F} \times M$, $\delta\phi$ is the corresponding local $(1,0)$-form with values in $\phi^*T(E/M)$, and π is a local $(0,|-1|)$-form with values in $\phi^*T^*(E/M)$. If ϕ is a real scalar field $\phi\colon M \to \mathbb{R}$, then $\pi \in \Omega_{\text{loc}}^{0,|-1|}(\mathcal{F} \times M)$ is called the *conjugate momentum* (density) to ϕ. Roughly speaking, at least locally we can choose a coordinate system (on the fibers of E) to write any set of fields as a collection of real scalar fields and so obtain conjugate momenta.

As explained above, we have the

Definition 2.42. The space $\mathcal{M} \subset \mathcal{F}$ of *classical solutions* is the space of ϕ such that the restriction of $(D\mathcal{L})^{1,|0|}$ to $\{\phi\} \times M$ vanishes:

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma = \underline{D}L = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.43}$$

Physicists refer to the submanifold $\mathcal{M}$ (or $\mathcal{M} \times M$) as *on-shell*; its complement in $\mathcal{F}$ (or $\mathcal{F} \times M$) is *off-shell*, though that term is usually used to describe all of $\mathcal{F}$ (or $\mathcal{F} \times M$). We let $\overline{\mathcal{M}}$ denote the quotient of $\mathcal{M}$ by gauge symmetries; as with $\overline{\mathcal{F}}$ we treat $\overline{\mathcal{M}}$ as a manifold. In some contexts $\overline{\mathcal{M}}$ is called the *moduli space*.

Define

$$\omega := \delta\gamma, \qquad \text{in } \Omega_{\text{loc}}^{2,|-1|}(\mathcal{F} \times M). \tag{2.44}$$

Restricted to the space of classical solutions we have

$$\omega = D\mathcal{L} \qquad \text{on } \mathcal{M} \times M, \tag{2.45}$$

and so

$$D\omega = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.46}$$

We call ω the *local symplectic form*, since in the Hamiltonian situation its integral over a spacelike hypersurface is the global symplectic form on the space of classical solutions. ω also has units of action:

$$[\omega] = ML^2T^{-1}. \tag{2.47}$$

The picture of our on-shell data is:

$$\begin{array}{c|ccccc} & 0 & & 1 & & 2 \\ \hline |0| & L & \rightarrow & 0 & & \\ & & & \uparrow & & \\ |-1| & & & \gamma & \rightarrow & \omega \end{array} \quad \mathcal{M}$$

$$M$$

Elements of $\Omega_{\mathrm{loc}}^{0,|-1|}(\mathcal{F} \times M)$ are called *currents*. A current j is *conserved* if

$$dj = 0 \qquad \text{on } \mathcal{M} \times M. \tag{2.48}$$

Below we discuss symmetries and show how they lead to conserved (Noether) currents. Such currents are local counterparts of generating functions, on the symplectic manifold $\mathcal{M}$ of extremal fields, of infinitesimal symplectic transformations. More precisely, their integral on a spacelike hypersurface, to the extent that it makes sense, is such a generating function. The current being conserved, its integral is invariant under suitable deformations of the spacelike hypersurface. Another source of conserved currents is topology. Namely, in some theories the space of fields has nontrivial topology and topological invariants are constructed by integrating topological currents. Note that the units of a current are not fixed; see (2.3).

§2.4. First order lagrangians

Let $\pi\colon E \to M$ be a fiber bundle. For s_1 and s_2 two local sections of E/M defined in a neighborhood of $m \in M$, the relation "in local coordinates, the derivatives of order $\le k$ at m of s_1 and s_2 coincide" is an equivalence relation. The equivalence classes are called the k-jets of sections of E/M at m. They form the fiber at m of a new fiber bundle $J^k(E) \to M$. Local coordinate systems on $J^k(E)$ can be obtained as follows. One fixes near $m \in M$ a local coordinate system $\{x^\mu\}$ on M, a trivialization $E = M \times F$ of E, and local coordinates $\{y^a\}$ on F. Sections of $E \to M$ near m are given by functions $\phi^a(x^\mu)$ from M to F. The x^μ and $\partial^n \phi^a$ with $|n| \le k$ form a local coordinate system.

A local section ϕ of $E \to M$ defines a section $j^k(\phi)$ of $J^k(E) \to M$, with value at $m \in M$ the k-jet of ϕ at m.

Definition 2.49. A lagrangian $L \in \Omega_{\mathrm{loc}}^{0,|0|}(\mathcal{F} \times M)$ *depends only on the k-jet of the fields* if there exists a morphism $\ell\colon J^k(E) \to \mathrm{Dens}(M)$ of bundles over M such that

$$L(\phi) = \ell\big(j^k(\phi)\big). \tag{2.50}$$

Example 2.51. Most "fundamental" lagrangians L depend only on the 1-jet of the fields. (A notable exception occurs in theories of gravity, since the curvature of a metric depends on the 2-jet of the metric.) A simple example to keep in mind is a theory of a scalar field $\phi\colon M \to X$ with values in a Riemannian manifold X. Then $E = M \times X$ and the fiber of $J^1E \to E$ at (m, x) is $\mathrm{Hom}(T_mM, T_xX)$. Let $V\colon X \to \mathbb{R}$ be a (potential energy) function. The theory of the scalar field in the potential V has

$$\ell(m, x, T) = \frac{1}{2}|T|^2 - V(x), \tag{2.52}$$

where $T \in \mathrm{Hom}(T_mM, T_xX)$.

We now assume that L is a lagrangian density depending only on the first jet of the field ϕ. The condition $\underline{D}L = \delta L + d\gamma$ means, in integrated form, that for U a compact integration domain with smooth boundary, one has, for $\xi \in \Gamma\big(M, \phi^*T(E/M)\big)$ a tangent vector of $\mathcal{F}$ at ϕ,

$$\xi \int_U L(\phi) = \int_U \iota(\xi)\underline{D}L + \int_{\partial U} \iota(\xi)\gamma. \tag{2.53}$$

The last sign comes from the fact that $\iota(\xi)d\gamma = -d\iota(\xi)\gamma$. If γ is assumed to satisfy $\iota(f\xi)\gamma = f\iota(\xi)\gamma$ for f a function on M, i.e., if $\gamma(\phi, \xi)$ at m depends only on the value of ξ at m, this formula makes it clear that γ is unique. Indeed, the formula determines $\int_{\partial U} f\iota(\xi)\gamma$ for any U and f, hence $\iota(\xi)\gamma$ for any ξ. As any ξ can be decomposed into sections of $\Gamma\big(M, \phi^*T(E/M)\big)$ with small support, existence is a local question on E and M. Locally on M, one can trivialize E as $M \times F$. Let us choose local coordinate systems $\{x^\mu\}$ on M and $\{y^a\}$ on F. This gives a local coordinate system $\{x^\mu, y^a, y^a_\mu\}$ on $J^1(E)$. By assumption, $L = \ell(j^1(\phi))$ with

$$\ell = \ell(x^\mu, y^a, y^a_\mu)|d^nx|. \tag{2.54}$$

The $|-1|$-form $\iota(\xi)\gamma$ on M is then given by the standard integration by parts:

$$\iota(\xi)\gamma = \sum \frac{\partial \ell}{\partial y^a_\mu}\xi^a\iota(\partial_\mu)|d^nx|. \tag{2.55}$$

§2.5. Hamiltonian theory

Suppose that spacetime is $M = M^1 \times N$ for some manifold N, where M^1 is affine one-dimensional Minkowski space, i.e., the affine real line with its standard metric. We view M^1 as time and N as space. We also assume that the fiber bundle $E \to M^1 \times N$ is pulled back from a fixed bundle on N, or equivalently that time translation has been lifted to E. In this case we integrate the local symplectic form ω to obtain a closed 2-form Ω on the space of classical solutions $\mathcal{M}$:

$$\Omega = \int_{\{t\}\times N} \omega \in \Omega^2(\mathcal{M}). \tag{2.56}$$

Typically N is noncompact and so to ensure convergence we only evaluate Ω on tangent vectors to $\mathcal{M}$ with compact support in spatial directions, or at least with sufficient decay at spatial infinity. The hyperbolicity of the classical equations of motion implies finite propagation speed of the classical solutions, and so the decay conditions are uniform in time. By (2.46) the right hand side of (2.56) is independent of $t \in M^1$, and also Ω is a closed 2-form on $\mathcal{M}$.

A generalized infinitesimal local symmetry is a construction as follows (see Definition 2.93). One gives: (i) a vector bundle V over M; (ii) for each section ζ of V, a vector field X_ζ on the space $\mathcal{F}$ of fields and a $(0,|-1|)$-form α_ζ such that $\mathrm{Lie}(X_\zeta)L = d\alpha_\zeta$. One requires that X and α be local: the value at $m \in M$ of $X_\zeta(\phi) \in \Gamma(M, \phi^*T(E/M))$ should depend only on the k-jet of ϕ and ζ at m, and similarly for α. One also requires X and α to be additive in ζ. The basic example is the algebra of infinitesimal gauge symmetries, for which V is the adjoint bundle of a principal bundle.

In the presence of such symmetries, one cannot hope for the closed 2-form Ω on $\mathcal{M}$ to be nondegenerate. Indeed, the X_ζ are tangent to $\mathcal{M}$, because X_ζ is a generalized symmetry (see Definition 2.71). We claim that the X_ζ preserve Ω and are in the kernel of Ω, at least for ζ with compact support so that everything is well-defined. Indeed, decomposing ζ one can assume it has a small support. Let us choose t such that $\{t\} \times N$ does not meet the support of ζ. If we compute Ω using $\{t\} \times N$, then Ω does not see ζ and the claim follows. Therefore, Ω is the pullback of a 2-form $\overline{\Omega}$ on the quotient $\overline{\mathcal{M}}$ of $\mathcal{M}$ by the local symmetries (or on the quotient by any subgroup). The best one can hope for, then, is that $\overline{\Omega}$ is a symplectic structure.

If $j \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a conserved current then the associated *charge* Q_j is

$$Q_j = \int_{\{t\}\times N} j. \tag{2.57}$$

If ϕ is in $\mathcal{M}$, then since $dj = 0$ the right hand side is independent of t. This is a global conservation law. Local conservation laws are obtained by considering a domain $U \subset N$. For simplicity assume the closure of U is compact with smooth boundary ∂U. Let

$$q_t = \int_{\{t\}\times U} j \tag{2.58}$$

be the total charge contained in U at time t. Write

$$j = dt \wedge j_1 + j_2, \tag{2.59}$$

where j_1, j_2 do not involve dt. Stokes' theorem applied to integration over the fibers of the projection $M^1 \times U \to M^1$ implies

$$\frac{dq_t}{dt} + \int_{\{t\}\times \partial U} j_1 = 0. \tag{2.60}$$

This says that the rate of change of the total charge in U is minus the flux through the boundary. The units of a charge are the same as that of the current.

In theories with local symmetry the global charge is gauge invariant since we require the local current to be gauge invariant up to an exact form.

A field ϕ on $M = M^1 \times N$ is *static* if its time derivative $\mathrm{Lie}(\partial/\partial t)\phi$ vanishes. (Recall that ϕ is a section of $E \to M^1 \times N$ and we have fixed a lift of $\partial/\partial t$ to E.) When there are gauge symmetries we say that ϕ is static if locally on N there is a gauge in which $\mathrm{Lie}(\partial/\partial t)\phi = 0$. For a p-form field α such a local gauge exists if and only if the gauge-invariant field $d\alpha$ is invariant under t. The same assertion holds for abelian connections α, where $d\alpha$ is replaced by the curvature. The proof of this assertion is straightforward.[11] The local gauge in which α is static is unique up to static local gauge transformations. There does not seem to be an analogous criterion for static nonabelian connections in terms of the curvature.

Let $\mathcal{F}_N$ denote the space of static fields and $\overline{\mathcal{F}}_N$ the quotient by static gauge symmetries. In §2.10 we define the energy of a static field and so a subspace $\mathcal{FE}_N$ of static fields of finite energy.

§2.6. Symmetries and Noether's theorem

We assume that the space of fields $\mathcal{F}$ is the space of sections of a fiber bundle E over M, and that L in $\Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a lagrangian density. An automorphism $g\colon \mathcal{F} \to \mathcal{F}$ of $\mathcal{F}$ is *local* if for some k the value of $g(\phi)$ at $m \in M$ depends only on the k-jet of ϕ at m, and if the same condition holds for the inverse g^{-1} of g. A *generalized symmetry* of L is a local automorphism g of $\mathcal{F}$, given with α in $\Omega^{0,|-1|}_{\mathrm{loc}}$ such that

$$L(g(\phi)) - L(\phi) = d\alpha(\phi) \; . \tag{2.66}$$

By the locality assumption, deformations with compact support of ϕ and $g\phi$ correspond to each other, and by (2.66), the integral of the variation of $L(\phi)$ is preserved. It follows that g preserves the space $\mathcal{M}$ of extremals. As in §2.3, the lagrangian density L gives rise to an $\mathbb{R}$-torsor with connection (T, ∇) on $\mathcal{M}$. A generalized symmetry acts on (T, ∇) as follows. Let H be a space-like hypersurface. It defines a trivialization 0_H of T. We map 0_H at ϕ to

$$g(0_H \text{ at } \phi) = \left(0_H - \int_H \alpha(\phi)\right) \text{ at } g(\phi) \; . \tag{2.67}$$

---

[11] Write

$$\begin{aligned} \alpha &= dt \wedge A(t) + B(t), \qquad A(t) \in \Omega^{p-1}_N, \quad B(t) \in \Omega^p_N, \\ \beta &= dt \wedge P(t) + Q(t), \qquad P(t) \in \Omega^{p-2}_U, \quad Q(t) \in \Omega^{p-1}_U \end{aligned} \tag{2.61}$$

on some small open set $U \subset N$. Then $\alpha + d\beta$ is time independent if and only if

$$\dot{A} - d\dot{P} + \ddot{Q} = 0 \tag{2.62}$$

$$\dot{B} + d\dot{Q} = 0, \tag{2.63}$$

whereas $d\alpha$ is time independent if and only if

$$\ddot{B} - d\dot{A} = 0 \tag{2.64}$$

$$d\dot{B} = 0. \tag{2.65}$$

The integrability condition for (2.63) is (2.65), so we can solve for $\dot{Q}$. Then we solve (2.62) for $\dot{P}$; the integrability condition is (2.64). (A slightly different argument is required for $p = 1$ since in that case $P \equiv 0$.)

The resulting map $g\colon T \to T$ is independent of the choice of H. Indeed, if H_1 and H_2 are two space-like hypersurfaces, with H_1 before H_2, we have at ϕ

$$0_{H_2} - 0_{H_1} = \int_{H_1}^{H_2} L(\phi) \; . \tag{2.68}$$

As $\int_{H_1}^{H_2} L(g\phi) = \int_{H_1}^{H_2} L(\phi) + [\int \alpha(\phi)]_{H_1}^{H_2}$, this is compatible with 0_{H_1} and 0_{H_2} at ϕ to be mapped, respectively, to

$$0_{H_1} - \int_{H_1} \alpha(\phi) \quad \text{and} \quad 0_{H_2} - \int_{H_2} \alpha(\phi) \tag{2.69}$$

at $g(\phi)$. Indeed, at $g(\phi)$,

(2.70)
$$\begin{aligned}\Big(0_{H_2} - \int_{H_2} \alpha(\phi)\Big) - \Big(0_{H_1} - \int_{H_1} \alpha(\phi)\Big) &= \int_{H_1}^{H_2} L(g(\phi)) - \Big[\int \alpha(\phi)\Big]_{H_1}^{H_2} \\ &= 0_{H_2} - 0_{H_1} \text{ at } \phi.\end{aligned}$$

We now consider the infinitesimal analog of this construction. A vector field $\hat{\xi}$ on $\mathcal{F}$ is *local* if for some k the value of $\hat{\xi}_\phi \in \Gamma\big(M, \phi^*T(E/M)\big)$ at $m \in M$ depends only on the k-jet of ϕ at m. In other words, if q is the projection from $J^k(E)$ to E, then $\hat{\xi}$ is given by a section of $q^*T(E/M)$ on $J^k(E)$.

Definition 2.71. A *generalized infinitesimal symmetry* of L is a local vector field $\hat{\xi}$ on $\mathcal{F}$, given with $\alpha_{\hat{\xi}}$ in $\Omega_{\text{loc}}^{0,|-1|}$ such that

$$\operatorname{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M. \tag{2.72}$$

Here, $\operatorname{Lie}(\hat{\xi})L$ is simply $\iota(\hat{\xi})\delta L$. Such a symmetry should not be expected to integrate to a generalized (local) symmetry, as defined above. For instance, the vector field on $\mathcal{F}$ corresponding to an infinitesimal time translation is local. A finite time translation on $\mathcal{F}$ is not. To make it local, one would need to act both on $\mathcal{F}$ and M, as we will do later.

One can, however, repeat the previous construction, if one views a generalized infinitesimal symmetry as a generalized symmetry (g, α) depending on an infinitesimal parameter ε, with $(g(0), \alpha(0)) = (\text{Identity}, 0)$. Formally, this means working over $\operatorname{Spec}(\mathbb{R}[\varepsilon]/(\varepsilon^2))$. Doing so, we get a lifting of $g(\varepsilon)$ to T, i.e., a lifting of the vector field $\hat{\xi}$ on $\mathcal{M}$ to T, respecting the connection. Such a lifting corresponds to a function $Q_{\hat{\xi}}$ on $\mathcal{M}$, such that

$$dQ_{\hat{\xi}} = -\iota(\hat{\xi})\Omega \; , \tag{2.73}$$

for Ω the curvature form of T. The function $Q_{\hat{\xi}}$ is the difference between the lifting of $\hat{\xi}$ and its horizontal lifting. For any space-like hypersurface H

$$Q_{\hat{\xi}} = \Gamma_H(\hat{\xi}) - \int_H \alpha_{\hat{\xi}} = \int_H \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} \; . \tag{2.74}$$

The *Noether current* of $(\hat{\xi}, \alpha_{\hat{\xi}})$ is the local $(0, |-1|)$-form

$$j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} \; . \tag{2.75}$$

By construction, its integral on a space-like hypersurface H is the charge $Q_{\hat{\xi}}$, giving the lifting of the vector field $\hat{\xi}$ on $\mathcal{M}$ to (T, ∇). For ϕ in $\mathcal{M}$, $\int_H j_{\hat{\xi}}$ is hence independent of H. It follows that $j_{\hat{\xi}}$ is a conserved current.

To prove that the Noether current $j_{\hat{\xi}}$ is conserved, i.e. that $j_{\hat{\xi}}(\phi)$ is closed when ϕ is extremal, we have used a global argument, relying on a notion of "space-like hypersurface" H. This leads to extraneous convergence problems. They could be avoided by using instead hypersurfaces homologous to zero, in a small neighborhood of a point. The corresponding line bundle on $\mathcal{M}$ has trivial curvature, but this does not spoil the argument. Here is another way out, which gives as well a local counterpart of (2.73).

Proposition 2.76. *Suppose $(\hat{\xi}, \alpha_{\hat{\xi}})$ is a generalized infinitesimal symmetry of $\mathcal{L}$. Then, for some $\beta_{\hat{\xi}}$ in $\Omega^{1,|-2|}_{\mathrm{loc}}(\mathcal{F} \times M)$, the identity*

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M \tag{2.77}$$

holds on shell. The Noether current $j_{\hat{\xi}} := \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}}$ is conserved,

$$dj_{\hat{\xi}} = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.78}$$

and

$$\delta j_{\hat{\xi}} = -\iota(\hat{\xi})\omega + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M. \tag{2.79}$$

Remark 2.80. Both (2.77) and (2.79) hold as equalities, on $\mathcal{M} \times M$, of $(1, |-1|)$-forms on $\mathcal{F} \times M$: at ϕ in $\mathcal{M}$, they give rise to an equality of $|-1|$-forms on M when evaluated against any tangent vector η of $\mathcal{F}$ at ϕ, whether or not it is tangent to $\mathcal{M}$.

The formula (2.79) is the promised local counterpart of (2.73). We summarize (2.72) and (2.77) in the diagrams:

M \ $\mathcal{F}$	0
$\|0\|$	$\mathrm{Lie}(\hat{\xi})L$
	$\uparrow$
$\|-1\|$	$\alpha_{\hat{\xi}}$

M \ $\mathcal{M}$	0		1
$\|0\|$	$\mathrm{Lie}(\hat{\xi})L$		
	$\uparrow$		
$\|-1\|$	$\alpha_{\hat{\xi}}$	$\rightarrow$	$\mathrm{Lie}(\hat{\xi})\gamma$
			$\uparrow$
$\|-2\|$			$\beta_{\hat{\xi}}$

Proof. The vector field $\hat{\xi}$ is tangent to $\mathcal{M}$. Since on $\mathcal{M} \times M$ we have $\underline{D}\,L = \delta L + d\gamma = 0$, we also have on $\mathcal{M} \times M$

$$\mathrm{Lie}(\hat{\xi})\underline{D}\,L = \delta\,\mathrm{Lie}(\hat{\xi})L + d\,\mathrm{Lie}(\hat{\xi})\gamma = 0, \tag{2.81}$$

in other words,

$$d(-\delta\alpha_{\hat{\xi}} + \mathrm{Lie}(\hat{\xi})\gamma) = 0. \tag{2.82}$$

If ϕ is in $\mathcal{M}$, then this identity holds when evaluated on any tangent vector $\eta \in \Gamma(M, \phi^*T(E/M))$ of $\mathcal{F}$ at ϕ. As $\iota(\eta)(-\alpha_{\hat{\xi}} + \mathrm{Lie}(\hat{\xi})\gamma)$ is local in η, it follows from Takens that for some $\beta_{\hat{\xi}}(\phi;\eta)$, local in η, one has

$$-\delta\alpha_{\hat{\xi}}(\phi) + \mathrm{Lie}(\hat{\xi})\gamma(\phi) = d\beta_{\hat{\xi}}\ . \tag{2.83}$$

Suppose we have chosen locally on M coordinate systems $U_i \hookrightarrow \mathbb{R}^n$, trivializations of E as $F \times U_i$ and local coordinate systems on F. In terms of those, and of a partition of unity attached to the resulting covering of E, $\beta_{\hat{\xi}}$ is given by Takens by explicit local formulas. Those formulas define $\beta_{\hat{\xi}}$ on $\mathcal{F} \times M$, with (2.77) holding on $\mathcal{M} \times M$. *A fortiori*, (2.77) holds as an identity of $(1,|-1|)$-forms on $\mathcal{M} \times M$, i.e., when applied to vectors tangent to $\mathcal{M} \times M$ only. Applying δ to it, we obtain that $\hat{\xi}$ preserves the $(2,|-1|)$-form ω on $\mathcal{M} \times M$ up to an exact derivative:

$$\mathrm{Lie}(\hat{\xi})\omega = d(-\delta\beta_{\hat{\xi}})\ . \tag{2.84}$$

As $\hat{\xi}$ is a vector field on $\mathcal{F}$, Cartan's formula takes the form

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\iota(\hat{\xi})\gamma + \iota(\hat{\xi})\delta\gamma = \delta\iota(\hat{\xi})\gamma + \iota(\hat{\xi})\omega\ . \tag{2.85}$$

Plugging this in (2.77), we obtain (2.79) on $\mathcal{M} \times M$.

A vector field ξ on $\mathcal{F} \times M$ is said to be *decomposable and local* if it is the sum of a local vector field $\hat{\xi}$ on $\mathcal{F}$ and of a vector field η on M. For such a vector field, the Lie derivative $\mathrm{Lie}(\xi)$ preserves the bigrading of $\Omega^{\bullet,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M)$. We say that (ξ, α_ξ) is a *generalized infinitesimal symmetry* if α_ξ in $\Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is such that

$$\mathrm{Lie}(\xi)L = d\alpha_\xi\ . \tag{2.86}$$

The Lie derivative $\mathrm{Lie}(\xi)$ is the sum of $\mathrm{Lie}(\hat{\xi})$ and of $\mathrm{Lie}(\eta)$. As $\mathrm{Lie}(\eta)L = d\iota(\eta)L$, it hence follows from (2.86) that

$$\mathrm{Lie}(\hat{\xi})L = d(\alpha_\xi - \iota(\eta)L)\ \ : \tag{2.87}$$

the vector field $\hat{\xi}$ on $\mathcal{F}$, and $\alpha_{\hat{\xi}} := \alpha_\xi - \iota(\eta)L$, form a generalized infinitesimal symmetry. The corresponding Noether current is

$$j_\xi = \iota(\hat{\xi})\gamma + \iota(\eta)L - \alpha_\xi = \left[\iota(\xi)\mathcal{L}\right]^{0,|-1|} - \alpha_\xi\ . \tag{2.88}$$

In this formula, as before, $\mathcal{L} = L + \gamma$. We say that ξ is a *manifest* symmetry of L if $\mathrm{Lie}(\xi)L = 0$. For a manifest symmetry, one can take $\alpha_\xi = 0$, hence $\alpha_{\hat{\xi}} = -\iota(\eta)L$, and the formula (2.88) for the Noether current then simplifies to

$$j_\xi = \left[\iota(\xi)\mathcal{L}\right]^{0,|-1|} \qquad \text{(for a manifest symmetry of } L\text{)}. \tag{2.89}$$

We say that ξ is a *manifest* symmetry of $\mathcal{L} = L + \gamma$ if $\mathrm{Lie}(\xi)\mathcal{L} = 0$, i.e. if ξ preserves L and γ. As $\mathrm{Lie}(\xi) = \mathrm{Lie}(\hat{\xi}) + \mathrm{Lie}(\eta)$, this gives

$$\mathrm{Lie}(\hat{\xi})\gamma = -(d\iota(\eta)\gamma + \iota(\eta)d\gamma) \ . \tag{2.90}$$

On shell, $\iota(\eta)d\gamma = -\iota(\eta)\delta L = \delta\iota(\eta)L = -\delta\alpha_{\hat{\xi}}$, hence $\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d(-\iota(\eta)\gamma)$: the formula (2.77) holds for the generalized symmetry $(\hat{\xi}, \alpha_{\hat{\xi}})$ and $\beta_{\hat{\xi}} = -\iota(\eta)\gamma$. For this choice of $\beta_{\hat{\xi}}$, (2.79) becomes

$$\delta j_\xi = -\iota(\hat{\xi})\omega - d\iota(\eta)\gamma \ . \tag{2.91}$$

Proposition 2.92. *Suppose that L depends only on the first jet of the fields, that γ is the canonical variational form, that $\xi = \hat{\xi} + \eta$ is a manifest symmetry of L, and that it is induced by an infinitesimal automorphism of the bundle $E \to M$, that is, by a vector field X on E projecting to η on M. Then, ξ is a manifest symmetry of $\mathcal{L}$.*

Integrating ξ, this reduces to the statement that if an automorphism of the bundle $E \to M$ preserves L, it also preserves γ. As γ is canonically deduced from L, this is an application of transport of structures.

This argument seems to assume that X can be integrated, i.e. that $\exp(tX)e$ does not go to infinity in finite time. However, as the question is local, integrability is immaterial. One only needs to know that for $t \leq t_0$ and for suitable $E' \to M'$ with E' open in E and M' in M, $\exp(tX)$ and $\exp(t\eta)$ map E' and M' isomorphically to $E'_t \subset E$ and to $M'_t \subset M$.

Following Zuckerman, we give the formal definition of a local, or gauge, symmetry.

Definition 2.93. A *generalized infinitesimal local symmetry* of a theory $\mathcal{L} = L + \gamma$ is specified by a vector bundle $V \to M$ and linear maps

$$\begin{aligned} \zeta &\longmapsto X_\zeta \\ \zeta &\longmapsto \alpha_\zeta , \end{aligned} \tag{2.94}$$

where ζ is a section of V and X_ζ is a generalized infinitesimal symmetry of $\mathcal{L}$:

$$\mathrm{Lie}(X_\zeta)L = d\alpha_\zeta \qquad \text{on } \mathcal{F} \times M. \tag{2.95}$$

We require that the dependence of X_ζ and α_ζ on ζ be local.

Two examples: In gauge theory V is the adjoint bundle of a principal bundle and X_ζ acts by infinitesimal gauge transformations. In theories of gravity $V = TM$ and X_ζ acts by the Lie derivative of ζ.

Proposition 2.96. *The Noether current attached to a generalized infinitesimal local symmetry is d-exact on $\mathcal{M} \times M$.*

Proof. This follows directly from Theorem 2.16.

Assuming suitable decay at infinity, we see that the Noether charge associated to a local infinitesimal symmetry vanishes.

Finally, we introduce a bracket operation on Noether currents. There is a standard Poisson bracket of functions on $\overline{\mathcal{M}}$ in the Hamiltonian situation. If $j = j^\mu(x)\,\partial_\mu \otimes |d^n x|$ is a current, then we can view $j^\mu(x)$ for x fixed as defining a (singular) function on $\overline{\mathcal{M}}$ and so compute Poisson brackets of components of currents. This is a standard procedure in physics. Such computations go under the name *current algebra*. The bracket we introduce on *Noether* currents is defined in arbitrary field theories: we need not be in a Hamiltonian situation and there is no nondegeneracy assumption. The bracket (2.101) below is often simpler to compute in practice than the standard Poisson bracket of distributions on $\overline{\mathcal{M}}$.

Consider a theory $\mathcal{L} = L + \gamma$ on $\mathcal{F} \times M$ for some manifold M.

Definition 2.97. We call (j, ξ) a *Noether pair* if $j \in \Omega^{0,|-1|}_{\text{loc}}(\mathcal{F} \times M)$ is a current, ξ is a generalized infinitesimal symmetry of $\mathcal{L}$, and if

$$\text{Lie}(\xi)L = d\alpha \qquad \text{on } \mathcal{F} \times M, \tag{2.98}$$

where

$$\alpha = \big[\iota(\xi)\mathcal{L}\big]^{0,|-1|} - j. \tag{2.99}$$

Let (j_i, ξ_i), $i = 1, 2$ be Noether pairs. We define the bracket

$$\big\{(j_1, \xi_1)\,,\,(j_2, \xi_2)\big\} = (\text{Lie}(\xi_1)j_2 + C\,,\,[\xi_1, \xi_2]), \tag{2.100}$$

where

$$C = C(\xi_1, j_1; \xi_2, j_2) = \Big[-\iota(\xi_2)\,\text{Lie}(\xi_1)\mathcal{L} + \text{Lie}(\xi_2)\big(\iota(\xi_1)\mathcal{L} - j_1\big)\Big]^{0,|-1|}. \tag{2.101}$$

A straightforward computation shows that the right hand side of (2.100) is a Noether pair.

Remark 2.102. The formula for C may be changed by an exact term, though this particular choice is nice: for this choice the quantity α defined in (2.99) for the bracket is

$$\text{Lie}(\xi_1)\alpha_2 - \text{Lie}(\xi_2)\alpha_1, \tag{2.103}$$

where α_i are the corresponding quantities for ξ_i. The precise formula for C is not valuable; the important term in the bracket of currents is $\text{Lie}(\xi_1)j_2$.

For manifest symmetries $C = 0$, and in all cases C is exact on-shell (after imposing the equations of motion). The bracket makes the space of Noether pairs into a Lie algebra (off-shell). For nonmanifest symmetries, this Lie algebra is typically infinite dimensional and becomes finite dimensional only after we impose the equations of motion. Then it is nice to work modulo exact forms. Hence define an e-Noether pair $[j,\xi]$ to be an equivalence class of Noether pairs (j,ξ) where $(j_1,\xi) \sim (j_2,\xi)$ if $j_2 - j_1$ is d-exact. Thus j in $[j,\xi]$ is determined on-shell by ξ up to an element of the local cohomology group $H^{0,|-1|}_{\text{loc}}(\mathcal{M} \times M)$. Under the bracket (2.100) the set of e-Noether pairs is a Lie algebra on-shell. If we restrict the symmetries ξ to lie in some Lie algebra $\mathfrak{h}$, then the corresponding set of e-Noether pairs is a central extension of $\mathfrak{h}$ by $H^{0,|-1|}_{\text{loc}}(\mathcal{M} \times M)$.

Example 2.110 in §2.7 illustrates the difference between on-shell and off-shell symmetries. Note that the algebra of symmetries generated off-shell is infinite dimensional.

Suppose now that we are in the Hamiltonian situation $M = M^1 \times N$ and that $(\overline{\mathcal{M}}, \overline{\Omega})$ is symplectic. We work with Noether pairs (j,ξ) such that ξ projects to a vector field $\bar{\xi}$ on $\overline{\mathcal{M}}$ and j is gauge invariant. Then by (2.73) $\bar{\xi}$ is the symplectic gradient of the global charge $\overline{Q} \in \Omega^0(\overline{\mathcal{M}})$. The charge $\overline{Q}$ only depends on the equivalence class of (j,ξ) in the set of e-Noether pairs. Let $[j_i,\xi_i]$ be e-Noether pairs with global charges $\overline{Q}_i$. Set $[j,\xi] = \{[j_1,\xi_1], [j_2,\xi_2]\}$. Then the global charge $\overline{Q}$ associated to j is the Poisson bracket

$$\overline{Q} = \{\overline{Q}_1, \overline{Q}_2\}_{\overline{\mathcal{M}}}. \tag{2.104}$$

Equation (2.104) relates the bracket on Noether pairs to the Poisson bracket on $\overline{\mathcal{M}}$; it follows directly from (2.100), the definition of Poisson bracket, and Stokes' theorem.

Recall that if $\mathfrak{h}$ is a Lie algebra of Hamiltonian vector fields on a symplectic manifold, then the sub-Poisson algebra of functions whose symplectic gradients lie in $\mathfrak{h}$ is a central extension of $\mathfrak{h}$ by the space of locally constant functions. For a Lie algebra $\mathfrak{h}$ of symmetries of $\mathcal{L}$ as above, the set of e-Noether pairs $[j,\xi]$ with $\xi \in \mathfrak{h}$ is a local version of the global central extension defined by the charges.

§2.7. More on symmetries

This section is not strictly speaking needed for the theoretical development. Rather, we present examples of symmetries and Noether currents. These examples also illustrate the calculus we use to compute in function spaces.

Example 2.105. For a system of nonrelativistic point particles (1.13) time translation is a symmetry. As in the discussion of Example 2.110 it is manifest[12] for the vector field ξ in (2.120). Explicitly,[13]

$$\begin{aligned} \iota(\xi)\delta x &= -\dot{x}, \\ \iota(\xi)dt &= 1. \end{aligned} \tag{2.107}$$

[12]In Example 2.131 below we discuss time translation as a nonmanifest symmetry.

[13]Beware that usually the vector field $\hat{\xi}$ is omitted and one simply writes

$$\delta x = -\dot{x} \tag{2.106}$$

instead of the first equation in (2.107). In (2.106) 'δ' does not denote the differential on $\mathcal{F}$, but rather the vector field $\hat{\xi}$. Then signs are different when commuting d and δ.

Then we compute directly from (1.13) and (1.15) that $\mathrm{Lie}(\xi)\mathcal{L} = 0$. The associated Noether current

$$\begin{aligned} j_\xi &= \iota(\xi)\gamma + \iota(\xi)L \\ &= -|\dot{x}|^2 + \bigl(\frac{1}{2}|\dot{x}|^2 - V(x)\bigr) \\ &= -\bigl(\frac{1}{2}|\dot{x}|^2 + V(x)\bigr) \end{aligned} \tag{2.108}$$

is *minus* the *hamiltonian*. The Hamiltonian formulation of classical mechanics involves the hamiltonian function (2.108) and the symplectic form (1.16) on the symplectic manifold $\mathcal{M}$. Noether currents coincide with charges.

In this example the field is a map $M^1 \to X$, where M^1 is affine time and X is a Riemannian manifold. One should view 'x' in these formulas as the evaluation map (2.13)

$$x\colon \mathcal{F} \times M^1 \longrightarrow X, \tag{2.109}$$

though we often also denote a specific field $x\colon M^1 \to X$ with the same letter. The coordinate t on M^1 is a real-valued function on $\mathcal{F} \times M^1$ which is constant in the $\mathcal{F}$ direction.

Example 2.110. Let X be a real inner product space of dimension r with inner product $\langle\cdot,\cdot\rangle$, and for $x\colon M^1 \to X$ let

$$L = \Bigl\{\frac{1}{2}|\dot{x}|^2 - \frac{1}{2}|x|^2\Bigr\}\,|dt|. \tag{2.111}$$

This lagrangian describes a system of r identical harmonic oscillators. The equation of motion (1.14) is

$$\ddot{x} = -x. \tag{2.112}$$

(We omit the argument 't' from equations for readability.) Consider the vector field $\hat{\xi}$ on $\mathcal{F} = \mathrm{Map}(M^1, X)$ given by

$$\iota(\hat{\xi})\delta x = Ax + B\dot{x} \tag{2.113}$$

for some $A, B \in \mathrm{End}(X)$. Then

$$\iota(\hat{\xi})\delta\dot{x} = A\dot{x} + B\ddot{x}. \tag{2.114}$$

Assume that A is skew-symmetric and B is symmetric. We compute

$$\mathrm{Lie}(\hat{\xi})L = \iota(\hat{\xi})\delta L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M^1, \tag{2.115}$$

for

$$\alpha_{\hat{\xi}} = \frac{1}{2}\langle B\dot{x}, \dot{x}\rangle - \frac{1}{2}\langle Bx, x\rangle. \tag{2.116}$$

Thus $\hat{\xi}$ is a generalized infinitesimal symmetry for A skew-symmetric and B symmetric. Also,

$$\text{Lie}(\hat{\xi})\gamma = \iota(\hat{\xi})\delta\gamma + \delta\iota(\hat{\xi})\gamma = \langle B\dot{x}, \delta\dot{x}\rangle + \langle B\ddot{x}, \delta x\rangle. \tag{2.117}$$

From (2.116) we find

$$\delta\alpha_{\hat{\xi}} = \langle B\dot{x}, \delta\dot{x}\rangle - \langle Bx, \delta x\rangle, \tag{2.118}$$

so that on-shell—that is, after imposing the equation of motion (2.112)—we have

$$\text{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M^1,. \tag{2.119}$$

as implied by Proposition 2.76.

Only when $B = 0$ is the symmetry manifest. In that case it is the internal symmetry generated by an infinitesimal rotation in X. For $A = 0$ and $B = \text{id}$ we see that $\hat{\xi}$ is the vector field on $\mathcal{F}$ induced from the vector field $\partial/\partial t$ on M^1. We make this symmetry manifest by forming the vector field

$$\xi = -\hat{\xi} + \partial/\partial t; \tag{2.120}$$

then $\text{Lie}(\xi)\mathcal{L} = 0$. In other cases the symmetry is nonmanifest and is not induced by any symmetry of M^1. Also, note that on-shell the group of symmetries generated by (2.113), (2.114) is isomorphic to the unitary group $U(r)$, whereas off-shell it is infinite dimensional. We meet this same phenomenon in the more complicated situation of supersymmetric theories with no off-shell formulation, i.e., no auxiliary fields. Then the supersymmetry algebra only closes on-shell and the supersymmetry is not manifest.

The conserved quantity associated to a diagonal element of the Lie algebra $\mathfrak{u}(r)$ with one nonzero entry is the energy of one of the r oscillators.

Next, we give an example in field theory where β_ξ in Proposition 2.76 is nonzero.

Example 2.121. Let $M = M^2$ denote two-dimensional Minkowski spacetime with coordinates x^0, x^1. Here x^0 is the speed of light times a standard time coordinate. For convenience fix the orientation $\{x^0, x^1\}$ and use it to identify twisted forms with forms. Note that $*dx^0 = dx^1$ and $*dx^1 = dx^0$. Let $\mathcal{F} = \{\phi\colon M^{1,1} \to \mathbb{R}\}$ be the set of real scalar fields. The free (massless) lagrangian is

$$\begin{aligned} L &= \frac{1}{2} d\phi \wedge *d\phi \\ &= \frac{1}{2}|d\phi|^2\, dx^0 \wedge dx^1 \\ &= \frac{1}{2}\left\{(\partial_0\phi)^2 - (\partial_1\phi)^2\right\} dx^0 \wedge dx^1. \end{aligned} \tag{2.122}$$

From this we derive

$$\gamma = \partial_0\phi\ \delta\phi \wedge dx^1 + \partial_1\phi\ \delta\phi \wedge dx^0 \tag{2.123}$$

$$\omega = \partial_0\delta\phi \wedge \delta\phi \wedge dx^1 + \partial_1\delta\phi \wedge \delta\phi \wedge dx^0 \tag{2.124}$$

and the equation of motion

$$\partial_0^2\phi - \partial_1^2\phi = 0 \qquad \text{on } \mathcal{M}\times M. \tag{2.125}$$

Consider infinitesimal translation in the x^0 (time) direction. It defines a vector field ξ on $\mathcal{F}\times M$ by

$$\begin{aligned} \iota(\xi)\delta\phi &= -\partial_0\phi \\ \iota(\xi)dx^0 &= 1 \\ \iota(\xi)dx^1 &= 0. \end{aligned} \tag{2.126}$$

Then a routine computation shows that ξ is a *manifest* symmetry: $\mathrm{Lie}(\xi)L = \mathrm{Lie}(\xi)\gamma = 0$ on $\mathcal{F}\times M$. (By Proposition 2.92 we need only check $\mathrm{Lie}(\xi)L = 0$.) But we can also regard infinitesimal time translation as a *nonmanifest* symmetry $\hat{\xi}$ by letting it operate only along $\mathcal{F}$:

$$\begin{aligned} \iota(\hat{\xi})\delta\phi &= -\partial_0\phi \\ \iota(\hat{\xi})dx^0 &= 0 \\ \iota(\hat{\xi})dx^1 &= 0. \end{aligned} \tag{2.127}$$

Then we compute

$$\mathrm{Lie}(\hat{\xi})L = d\alpha \qquad \text{on } \mathcal{F}\times M, \tag{2.128}$$

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha + d\beta \qquad \text{on } \mathcal{M}\times M, \tag{2.129}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2}\left[(\partial_0\phi)^2 - (\partial_1\phi)^2\right]dx^1 \in \Omega_{\mathrm{loc}}^{0,|-1|}, \\ \beta &= \partial_1\phi\,\delta\phi \qquad\qquad\qquad \in \Omega_{\mathrm{loc}}^{1,|-2|}. \end{aligned} \tag{2.130}$$

A simple example from mechanics illustrates the importance of locality in Definition 2.71.

Example 2.131. Consider a free particle $x\colon M^1 \to \mathbb{R}$ with lagrangian $L = \frac{m}{2}\dot{x}^2\,dt$ and the canonical $\gamma = m\dot{x}\,\delta x$. Let $\hat{\xi}$ be infinitesimal time translation, considered[14] as a motion only along $\mathcal{F}$:

$$\begin{aligned} \iota(\hat{\xi})\delta x &= -\dot{x}, \\ \iota(\hat{\xi})dt &= 0. \end{aligned} \tag{2.132}$$

Then

$$\mathrm{Lie}(\hat{\xi})L = -m\dot{x}\ddot{x}\,dt = da_{\hat{\xi}} \tag{2.133}$$

[14] In Example 2.105 we discuss time translation as a manifest symmetry.

for

$$a_{\hat{\xi}} = -\frac{1}{2}m\dot{x}^2, \tag{2.134}$$

so that $\hat{\xi}$ is a generalized infinitesimal symmetry. Imposing the equation of motion $\ddot{x} = 0$ we compute

$$\mathrm{Lie}(\hat{\xi})\gamma - \delta a_{\hat{\xi}} = 0 \qquad \text{on } \mathcal{M} \times M^1, \tag{2.135}$$

as it must by Proposition 2.76. Now fix a time t_0 and a function $F\colon M^1 \to \mathbb{R}$ and consider instead

$$\tilde{a}_{\hat{\xi}} = -\frac{1}{2}m\dot{x}^2 + F\bigl(x(t_0)\bigr). \tag{2.136}$$

(We could put any function on $\mathcal{F}$ here; we take one of this explicit form for ease of writing.) Since $F\bigl(x(t_0)\bigr)$ is independent of t we have $\mathrm{Lie}(\hat{\xi})L = d\tilde{a}_{\hat{\xi}}$, but now

$$\mathrm{Lie}(\hat{\xi})\gamma - \delta\tilde{a}_{\hat{\xi}} = -F'\bigl(x(t_0)\bigr)\,\delta x(t_0) \qquad \text{on } \mathcal{M} \times M^1 \tag{2.137}$$

is nonzero unless F is constant.

Note that $F\bigl(x(t_0)\bigr)$ is a *nonlocal* function of x—its value at $t \in M^1$ does not just depend on a finite jet of x at t. This example shows that without locality in Definition 2.71, the energy would be completely ill-defined.

§2.8. Computing Noether's current by gauging symmetries

Let us make the following assumptions: (a) the space $\mathcal{F}$ of fields is the space of maps from M to a manifold X, that is, E is a product $M \times X$; (b) the lagrangian density L depends only on the first order jet of fields; (c) γ is the canonical variational form; (d) a Lie group G acts on X; it preserves L, hence γ.

Let P be a principal G-bundle on M, and let E^P be the twisted form of E defined by P: a section p of P over an open subset U of M defines an isomorphism $\sigma(p)\colon E_U \to E^P_U$, and for g a map from U to G, $\sigma(pg) = \sigma(p)g$. Let $\mathcal{F}_1$ be the space of pairs (A, ϕ), where A is a connection on P and where ϕ is a section of E^P. On $\mathcal{F}_1 \times M$, let $L_1(A, \phi)$ be the lagrangian density whose value at m in M is computed as follows. Choose a local trivialization p of P which is horizontal (for A) at m. The section $\sigma(p)^{-1}\phi$ of E can be identified with a map from M to X, and

$$L_1(A, \phi) \text{ at } m := L(\sigma(p)^{-1}\phi) \text{ at } m\ . \tag{2.138}$$

The assumptions (b) and (d) ensure that the second member does not depend on the choice of p. The value of $L_1(A, \phi)$ at m depends only on the value of A at m and on the first jet of ϕ at m. We let γ_1 be the canonical variational 1-form.

In local coordinates, L is computed as follows. Over a local coordinate system $x\colon U \hookrightarrow \mathbb{R}^n$ of M, one chooses a trivialization of P. This identifies E^P with $M \times X$. The connection A is a 1-form on M with values in Lie(G), which the action of G on X turns into a 1-form on M with values in vector fields on X. The original lagrangian L is of the form

$$L(\phi) = \ell(\phi; \phi_i)dx \tag{2.139}$$

with $\phi_i = \partial\phi/\partial x_i$. The new one is

$$L_1(A,\phi) = \ell(\phi;\phi_i + A_i(\phi))dx\ . \tag{2.140}$$

It follows that if the variational 1-form $\gamma(\phi)$ is $\sum c_I(\phi;\phi_i)dx^I$, then

$$\gamma_1(A,\phi) = \sum c_I(\phi,\phi_i + A_i(\phi))dx^I\ . \tag{2.141}$$

The gauge group $\mathcal{G}_P$ of automorphisms of P acts on $\mathcal{F}_1$ by transport of structures. As L_1 and γ_1 have been defined (in terms of structures preserved by $\mathcal{G}_P$), they are preserved. If P is the trivial principal bundle G, so that $E^P = E$, and if A is the trivial connection A_0, the group G acting on P (on the left) by constant gauge transformations respects A_0; its action on ϕ is the action of G on $\mathcal{F}$, and

$$L(A_0,\phi) = L(\phi)\ . \tag{2.142}$$

The construction of $\mathcal{F}_1$, L_1 from $\mathcal{F}$, L and the action of G is called "gauging the symmetry". Our aim in this section is to explain how the Noether current attached to the manifest infinitesimal symmetry of L given by the action of Lie(G) can be computed by differentiating $L_1(A,\phi)$ in A. We will work in a more general framework than the one above, which covers as well the case where the fields are a map $\phi\colon M \to X$ and a ϕ^*TX-valued spinor.

Let P be a fixed principal G-bundle on M, and let con(P) be the bundle over M whose sections are the connections on P. Let us first assume only that E is a fiber space over con(P), so that the space of fields $\mathcal{F}$ fibers over the space $\mathcal{A}$ of connections, that the action of the gauge group $\mathcal{G}_P$ of automorphisms of P on $\mathcal{A}$ is lifted to an action of $\mathcal{F}$, that this lifted action is local—$g\phi$ at $m \in M$ depends only on some jet of g and ϕ at m—and that it preserves a lagrangian density L. For A in $\mathcal{A}$, let $\mathcal{F}_A$ be the fiber of $\mathcal{F}$ over $\mathcal{A}$ at A. It is the space of sections of E_A, the inverse image in E of the section A of con(P). The lagrangian density L induces a lagrangian density L_A on $\mathcal{F}_A \times M$. Let ϕ in $\mathcal{F}_A$ be an extremal for L_A. This means that $\underline{D}L$ at ϕ, a morphism of vector bundles from $\phi^*T(E/M)$ to densities on M, factors through $A^*T(\mathrm{con}(M)/M)$. As $\mathrm{con}(M)/M$ is an affine space bundle over M, the latter is independent of A. It is the vector bundle of 1-forms with values in the adjoint bundle $\mathfrak{g}^P$.

The bundle $\underline{\mathrm{Hom}}(\Omega^1(\mathfrak{g}^P),\Omega^{|0|})$ is identified with $\Omega^{|-1|}(\mathfrak{g}^{P\vee})$: to an element $\beta\otimes\xi'$ in $\Omega^{|-1|}(\mathfrak{g}^{P\vee})$ corresponds the morphism $\alpha\otimes\xi \mapsto \alpha\wedge\beta.\,\langle\xi,\xi'\rangle$. To ϕ in $\mathcal{F}_A$, extremal for L_A, we[15] have attached a section J_1 of $\Omega^{|-1|}\otimes\mathfrak{g}^{P\vee}$. The gauge invariance of L gives:

Proposition 2.144. *One has*

$$d_A J_1 = 0.$$

In particular, if the infinitesimal gauge transformation ξ is horizontal for A, then $J_1(\xi) = \langle\xi, J_1\rangle$ is a conserved current.

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[15]Physicists typically summarize the definition of J_1 by the equation

$$\delta S(\phi) = \int_M J_1^\mu(\phi)\,\delta A_\mu, \tag{2.143}$$

where S is the action.

Proof. Let ξ be an infinitesimal gauge transformation, a section of g^P. Then, ξ induces a vector field on $\mathcal{A}$ whose value at A is $-d_A\xi$. If ξ has compact support, one has

$$\int \langle -d_A\xi \wedge J_1 \rangle = 0 \,. \tag{2.145}$$

Suppose indeed that $g(u)$ is a family of sections with compact support of the gauge group, with $\frac{d}{du}g(u) = \xi$ at $u = 0$. By gauge invariance of L and extremality of ϕ, we have

$$\int \langle -d_A\xi \wedge J_1 \rangle = \int \left\langle \underline{D}L, \frac{d}{du}g(u)\phi \right\rangle = \int \frac{d}{du}L(g(u)\phi) = 0 \,. \tag{2.146}$$

By integration by parts, i.e., using that

$$d\,\langle \xi, J_1 \rangle = \langle d_A\xi \wedge J_1 \rangle + \langle \xi \wedge d_A J_1 \rangle \,, \tag{2.147}$$

equation (2.145) gives that for any ξ with compact support,

$$\langle \int \xi \wedge d_A J_1 \rangle = 0 \,, \tag{2.148}$$

so that $d_A J_1 = 0$. The second statement follows from (2.147).

We now give conditions under which the current $J_1(\xi)$ can be interpreted as a Noether current. We suppose that:

(a) the fiber bundle E over M is a fiber product $\mathrm{con}(P) \times E'$, so that the space $\mathcal{F}$ of fields is a product $\mathcal{A} \times \mathcal{F}'$;

(b) the action of $\mathcal{G}_P$ on E is deduced from its action on $\mathrm{con}(P)$, and from an action of G^P on E': the value at m of $g\phi'$, for g in $\mathcal{G}_P$ and ϕ' a section of E', depends only on the values of g and ϕ' at m;

(c) the value of $L(A, \phi')$ at m depends only on the value of A at m, and of the 1-jet of ϕ at m.

The assumptions (a) (b) (c) hold for a lagrangian obtained by gauging a symmetry. The assumption (c) ensures that the canonical variational 1-form γ depends only on $\delta\phi'$, not on δA. It agrees with the γ_A for the L_A on $\mathcal{F}' = \mathcal{F}_A$.

As $\mathcal{G}_P$ acts on $\mathcal{F}'$, an infinitesimal gauge transformation ξ induces a vector field $\xi^\wedge$ on $\mathcal{F}'$. If ξ is horizontal for A, i.e. fixes A, then the vector field $\xi^\wedge$ on $\mathcal{F}'$ is a manifest symmetry of L_A.

Proposition 2.149. *If the lagrangian density L is invariant by gauge transformations, if the assumptions (a) (b) (c) above hold, and if ξ is an infinitesimal gauge transformation which fixes the connection A on P, i.e. if $d_A\xi = 0$, then the Noether current for the manifest symmetry $\xi^\wedge$ of the lagrangian L_A is the conserved current $J_1(\xi)$ of Proposition 2.144.*

Proof. Let ξ be an infinitesimal gauge transformation. By gauge invariance of L,

$$\mathrm{Lie}(\xi)L = \iota(\xi)\delta L = \iota(-d_A\xi)\delta L + \iota(\xi^\wedge)\delta L_A = 0\ . \tag{2.150}$$

As L depends only on the 0^{th}-jet of A, we have

$$\iota(-d_A\xi)\delta L = \iota(-d_A\xi)\underline{D}L = -d_A\xi \wedge J_1\ . \tag{2.151}$$

We also have

$$\iota(\xi^\wedge)\delta L_A = \iota(\xi^\wedge)\underline{D}L_A - \iota(\xi^\wedge)d\gamma_A = \underline{D}L_A(\xi^\wedge) + d\iota(\xi^\wedge)\gamma_A\ , \tag{2.152}$$

and so

$$d\iota(\xi^\wedge)\gamma_A = d_A\xi \wedge J_1 - \underline{D}L_A(\xi^\wedge)\ . \tag{2.153}$$

For any function f on M, we have $(f\xi)^\wedge = f\xi^\wedge$. Taking (2.153) for $f\xi$, and subtracting f times (2.153) for ξ, we get for any f

$$df \wedge \iota(\xi^\wedge)\gamma_A = df \wedge J_1(\xi)\ , \tag{2.154}$$

hence

$$\iota(\xi^\wedge)\gamma_A = J_1(\xi)\ . \tag{2.155}$$

For $d_A\xi = 0$, the left side is the Noether current corresponding to the manifest symmetry $\xi^\wedge$.

Elements in the center of $\mathfrak{g}$ determine infinitesimal "global" gauge transformations which act trivially on all of $\mathcal{A}$, and so by Proposition 2.144 lead to currents defined on the entire space of classical solutions of a theory $\mathcal{L}$. In §4.4 we relate these currents to *electric charge*.

§2.9. The energy-momentum tensor

First approach

Let M denote (affine) n-dimensional Minkowski spacetime. Fix a coordinate system $x^0, x^1, \dots, x^{n-1}$ with respect to which the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2, \tag{2.156}$$

and the natural density is

$$|d^nx| = |\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1}\,|. \tag{2.157}$$

Here $x^0 = ct$ is the speed of light times the standard time coordinate. Of course, the metric and density have their geometric units, a power of length:

$$[g] = L^2, \qquad [\,|d^nx|\,] = L^n. \tag{2.158}$$

Suppose $\mathcal{L} = L + \gamma$ describes a field theory on M. We assume that the Poincaré group is a manifest symmetry group of the theory. Now the constant vector field

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{2.159}$$

on M induces a vector field on $\mathcal{F} \times M$, which we are assuming is a symmetry, and we denote *minus* the $(0, |-1|)$-component of the associated Noether current by

$$\Theta_{\mu\nu} * dx^\nu = \Theta_{\mu\nu} g^{\nu\nu'}\ \iota(\partial_{\nu'})|d^n x| \tag{2.160}$$

for some functions

$$\Theta_{\mu\nu} \colon \mathcal{F} \times M \longrightarrow \mathbb{R}. \tag{2.161}$$

The tensor $\Theta = (\Theta_{\mu\nu})$ is called the energy-momentum tensor. The conservation law (2.78) is

$$\sum_\nu \partial_\nu \Theta_{\mu\nu} = 0. \tag{2.162}$$

The units of the functions $\Theta_{\mu\nu}$ are those of a spatial momentum density:

$$[\Theta_{\mu\nu}] = \frac{M}{L^{n-2}T} = \frac{ML}{T}\frac{1}{L^{n-1}}. \tag{2.163}$$

Physically correct units for the components involving $x^0 = ct$ are obtained by re-expressing Θ in the coordinate system $\{t, x^1, \dots, x^n\}$.

A vector field $\eta = \eta^\mu \partial_\mu$ on M defines a current

$$\eta \cdot \Theta = \Theta_{\mu\nu}\eta^\mu * dx^\nu. \tag{2.164}$$

Lemma 2.165. *Θ is symmetric if and only if $\eta \cdot \Theta$ is conserved for every infinitesimal Lorentz transformation η.*

Proof. Let $g_{\mu\nu}$ denote the Minkowski metric. Then a skew form $B = (B^{\nu\lambda})$ corresponds to an infinitesimal Lorentz transformation

$$\eta = B^{\mu\lambda} g_{\lambda\sigma} x^\sigma \partial_\mu. \tag{2.166}$$

Using (2.162) we find

$$\begin{aligned} d(\eta \cdot \Theta) &= d(\Theta_{\mu\nu} B^{\mu\lambda} g_{\lambda\sigma} x^\sigma * dx^\nu) \\ &= \Theta_{\mu\nu} B^{\mu\lambda} g_{\lambda\sigma} g^{\sigma\nu}\, |d^n x| \\ &= \Theta_{\mu\nu} B^{\mu\nu}\, |d^n x|. \end{aligned} \tag{2.167}$$

The conclusion follows.

In practice Θ is symmetric only for theories of scalar fields. For other types of fields there is an "improved" symmetric energy-momentum tensor, as we discuss below.

The various components of the energy-momentum tensor have a physical interpretation, as the name suggests. (See the general discussion at the end of §1.4.) In this paragraph we work on-shell. We also insert factors of c to obtain correct physical units. Let

$$|d^{n-1}x| = |\,dx^1 \wedge \cdots \wedge dx^{n-1}\,| \tag{2.168}$$

be the canonical density on a time slice $\{x^0 = \text{constant}\}$. Then $c\Theta_{00}\,|d^{n-1}x|$ is the *energy density* and $-\Theta_{i0}\,|d^{n-1}x|$ $(i \geq 1)$ the *momentum density*. Assuming suitable decay at infinity, we integrate these densities over space to obtain minus the charges associated to the infinitesimal translations ∂_μ:

$$\begin{aligned} E &= \int_{\{x^0=\text{constant}\}} c\Theta_{00}\,|d^{n-1}x|, \\ P_i &= \int_{\{x^0=\text{constant}\}} -\Theta_{i0}\,|d^{n-1}x| \end{aligned} \tag{2.169}$$

(The minus sign comes since $\Theta_{i\nu}$ is *minus* the Noether current associated to ∂_i, whereas the momentum P_i is the Noether charge associated to ∂_i. Recall that $-E$ is the Noether charge associated to $c\partial_0$.) The conservation law asserts that on-shell E and P_i are independent of time. E is the *total energy* or *hamiltonian* and P_i the *momentum* in the i^{th} spatial direction. The square of the *(rest) mass* of a field configuration is

$$M^2 = E^2/c^4 - \sum_i P_i^2/c^2, \tag{2.170}$$

as in (1.45). The remaining components of Θ comprise the *stress tensor* $(\Theta_{ij})_{i,j\geq 1}$.

Second approach

For field theories which can be formulated in an arbitrary background metric, another approach to energy-momentum tensors is available. The relation with Noether currents is to be obtained by arguments parallel to those of §2.8, but with the group of diffeomorphisms replacing the group of gauge transformations.

Let $\text{met}(M) \to M$ denote the fiber bundle of Lorentzian metrics on the tangent bundle. Taking the inverse metric, we will view it as an open subbundle of the bundle of contravariant symmetric 2-tensors $g^{\mu\nu}$, and identify its relative tangent bundle $T\big(\text{met}(M)/M\big)$ with the pull-back from M of the bundle $\text{Sym}^2(T)$ of contravariant symmetric 2-tensors. In a Euclidean context, one would rather take for $\text{met}(M)$ the fiber bundle whose sections are the Riemannian structures on M.

In §2.8, we assumed the fiber bundle E over M to be a product $\text{con}(P) \times_M E'$. To similarly assume here that $E = \text{met}(M) \times_M E'$ is possible only at the cost of excluding spinor (or Rarita-Schwinger) fields. Indeed, for V a vector space given with a symmetric bilinear form g, the corresponding space of spinors depends on g (and on a lifting of the structural group from an orthogonal to a spin group). When g varies, the spaces of spinors form a vector bundle on the space of g's. Because of this, we will only assume that E is a fiber bundle $E \to \text{met}(M)$ over $\text{met}(M)$.

In §2.8, we assumed that the gauge group acts on $\mathcal{F}$. Here, if among the fields there is a connection over some principal G-bundle P, it is not convenient to assume an action on $\mathcal{F}$ of the group of diffeomorphisms. What will act is an extension of the group of diffeomorphisms by the gauge group: the group of pairs (f, φ) of a diffeomorphism f and of a lifting φ: $P \xrightarrow{\sim} f^*P$ of f to P. If P is trivial, a trivialization of P defines a splitting of this extension. Often, it is more natural to use that a connection on P provides a lifting to P of infinitesimal diffeomorphisms (= vector fields on M), the horizontal lifting. One should however keep in mind that this horizontal lifting is not compatible with brackets of vector fields.

Because of this, we will at first only assume given a lagrangian density L on $\mathcal{F} \times M$ which obeys the weak version of diffeomorphism invariance explained below. A diffeomorphism of M acts on $\mathrm{Met}(M)$. An infinitesimal diffeomorphism of M, that is, a vector field ξ, induces on $\mathrm{Met}(M)$ the vector field whose value at a metric g is $-\mathrm{Lie}(\xi)(g^\vee)$, for g the inverse metric. One has

$$-\mathrm{Lie}(\xi)(g^\vee)^{\mu\nu} = (\nabla^\mu \xi)^\nu + (\nabla^\nu \xi)^\mu, \tag{2.171}$$

where the tensor indices are moved up and down by the metric tensor and its inverse. At m, this is easily checked in a coordinate system in which the metric g is constant $+O(x^2)$, so that the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ vanish at m. The diffeomorphism invariance assumption is the following: for any vector field ξ on M, there is a local vector field $\tilde{\xi}$ on $\mathcal{F}$, projecting to $-\mathrm{Lie}(\xi)(g^\vee)$ on $\mathrm{Met}(M)$, which is a generalized symmetry of the lagrangian density L.

For g in $\mathrm{Met}(M)$, let $\mathcal{F}_g$ be the fiber of $\mathcal{F} \to \mathrm{Met}(M)$ at g. If E_g is the inverse image of g in E, then $\mathcal{F}_g$ is the space of sections of the fiber bundle E_g over M. The lagrangian density L induces a lagrangian density L_g on $\mathcal{F}_g \times M$.

Suppose that ϕ in $\mathcal{F}_g$ is an extremal for L_g. The Euler-Lagrange equation

$$\underline{D}L\colon \phi^*T(E/M) \to \text{ densities on } M \tag{2.172}$$

then vanishes on $\phi^*T\big(E/\operatorname{met}(M)\big)$, hence factors through

$$\underline{D}_g L\colon g^*T\big(\operatorname{met}(M)/M\big) \to \text{ densities on } M\ . \tag{2.173}$$

This expresses that if $\phi[u]$ is a deformation with compact supports of ϕ, inducing a deformation $g[u]$ of the metric, then

$$\int \frac{d}{du} L(\phi[u]) \qquad \text{at } u = 0 \tag{2.174}$$

depends only on $\frac{d}{du}(g[u])$ at $u = 0$, and not on the concomitant variation of the other fields. This allows to unambiguously attach to ϕ (assumed to be extremal for L_g) a symmetric 2-tensor $T_{\mu\nu}$ with values in densities on M such that (2.173) is[16]

$$a^{\mu\nu} \longmapsto \frac{1}{2}\, a^{\mu\nu}\, T_{\mu\nu} \tag{2.176}$$

for $a^{\mu\nu}$ in $g^*T(E/M)$ a symmetric contravariant 2-tensor. In local coordinates $\{x^\mu\}$ on M we write the energy-momentum tensor as

$$T_g = T_{\mu\nu}(g, \phi)\, dx^\mu \otimes dx^\nu\ \mu_g(x), \tag{2.177}$$

where μ_g is the canonical density given by the Riemannian metric g.

[16]Physicists write (2.176) as

$$\delta S(\phi) = \frac{1}{2} \int_M T_{\mu\nu}(\phi)\, \delta g^{\mu\nu}, \tag{2.175}$$

where S is the action. In our definition we restrict to ϕ which are extremal for L_g.

Remark 2.178. We have defined $T_{\mu\nu}$ only on-shell, that is, for ϕ an extremal of L_g. This does not provide a unique differential expression in ϕ giving $T_{\mu\nu}$. Suppose, however that we obtained a differential expression θ in ϕ which (even locally on M) gives $T_{\mu\nu}$ for any extremal. Suppose also that θ depends only on the k-jet of the section ϕ of E_g: θ is a section on $J^k(E_g/M)$ of the pull-back of $\mathrm{Sym}^2(T)\otimes$ densities on M. Then, if any k-jet is the k-jet of an extremal (the validity of this depends on which Cauchy data the Euler Lagrange equations for L_g require), then θ is unique and defines $T_{\mu\nu}$ off-shell.

Proposition 2.179. *Under the assumptions above, if ϕ in $\mathcal{F}_g$ is extremal for L_g, the corresponding $T_{\mu\nu}$ obeys the following conservation law: its covariant derivative $T_{\mu\nu;\rho}$ is such that its contraction ${T_{\mu\nu}}^{\nu}$ vanishes.*

Proof. Let ξ be a vector field with compact support on M. It induces the vector field $-\mathrm{Lie}(\xi)(g^\vee)$ on $\mathrm{Met}(M)$ and, by assumption, this vector field can be lifted to $\hat{\xi}$ on $\mathcal{F}$ in such a way that $\mathrm{Lie}(\hat{\xi})L$ is an exact differential. It follows that

$$\int_M \langle T, -\mathrm{Lie}(\xi)(g^\vee)\rangle = 0\ . \tag{2.180}$$

Our computations will be local. Let us choose an orientation of M to identify densities with forms of maximal degree n. Instead of viewing T as a section of $\Omega^1\otimes\Omega^1\otimes\Omega^n$, one can use g to identify Ω^1 with T^1, and view T as a section of $\Omega^1\otimes T^1\otimes\Omega^n$, mapped isomorphically by contraction to $\Omega^1\otimes\Omega^{n-1}$. Raising and lowering indices by g, $-\mathrm{Lie}(\xi)(g^\vee)$ similarly corresponds to a symmetrization of $\nabla\xi$ in $\Omega^1\otimes T^1$, and in (2.180), the symbol $\langle\quad\rangle$ denotes contraction of Ω^1 with T^1 while simultaneously wedging Ω^{n-1} and Ω^1. One has, for $\langle\quad\rangle$ denoting only contraction of Ω^1 and T^1,

$$\nabla\langle T,\xi\rangle = \langle\nabla T,\xi\rangle + \langle T,\nabla\xi\rangle\ . \tag{2.181}$$

After wedging, $\nabla\langle T,\xi\rangle$ becomes $d\langle T,\xi\rangle$ and (2.181) gives

$$\int_M \langle\nabla T \text{ wedged },\xi\rangle = 0\ . \tag{2.182}$$

As this holds for all ξ, it follows that ∇T in $\Omega^1\otimes\Omega^1\otimes\Omega^{n-1}$, projected to $\Omega^1\otimes\Omega^n$ by $\alpha\otimes\beta\otimes\gamma\mapsto\beta\otimes(\alpha\wedge\gamma)$, vanishes. This is equivalent to ${T_{\mu\nu}}^{\nu}=0$.

Corollary 2.183. *Let us view T as a 1-form with values in $(n-1)$-forms (or better, with values in $\Omega^{|-1|}$). If ϕ in $\mathcal{F}_g$ is extremal for L_g, and if ζ is a Killing vector field, i.e., an infinitesimal isometry, then $T(\zeta)$ is closed.*

Proof. As $\nabla\zeta=0$, we have

$$(\nabla T)(\zeta) = \nabla(T(\zeta))\ , \tag{2.184}$$

which, applying $\wedge$: $\Omega^1\otimes\Omega^{n-1}\to\Omega^n$, gives

$$0 = dT(\zeta)\ . \tag{2.185}$$

We managed to relate the conserved currents in Corollary 2.183 to Noether's currents only under the following restrictive assumptions: (a) E is a fiber product $\mathrm{met}(M) \times_M E'$, making the space $\mathcal{F}$ of fields a product $\mathrm{Met}(M) \times \mathcal{F}'$; (b) the Lagrangian density $L(g, \phi')$ at $m \in M$ depends only on the value of g at m and on the 1-jet of the section ϕ' of E' at m; (c) we take for γ the canonical choice; (d) to each vector field ζ on M is attached, by a local rule, a vector field ξ_ζ on $\mathcal{F}'$, so that $(\zeta, -\mathrm{Lie}(\zeta)(g^\vee), \xi_\zeta)$ is a manifest symmetry of L; (e) the map $\zeta \mapsto \xi_\zeta$ is linear over functions.

Remarks 2.186. (i) (a) is needed to make sense of (b); it fails for spinor fields.
(ii) Condition (d) fails for tensor fields, if $\zeta \mapsto \xi_\zeta$ is given by a Lie derivative. However, it holds for connections on a fixed principal G-bundle P, if ξ_ζ is defined as follows. Given a connection ∇_A, use it to lift ζ to P. The flow $\exp(t\zeta)$ generated by ζ is then lifted to P, and one takes

$$\xi_\zeta = \frac{d}{dt}(\exp(-t\zeta)^*(\nabla_A)) \quad \text{at} \quad t = 0 . \tag{2.187}$$

The vector field ξ_ζ is obtained by contracting the curvature 2-form F_A with ζ, and is hence linear over the functions.
(iii) Condition (a) allows us to define T off-shell as the restriction to the tangent space to $\mathrm{Met}(M)$ of $\underline{D}L$ (or δL; this amounts to the same by (b)).

Assuming (a) to (e), we now repeat the arguments of §2.8. The manifest symmetry of L means the vanishing of a Lie derivative of L: at (g, ϕ')

$$d\iota(\zeta)L + \underline{D}L(-\mathrm{Lie}(\zeta)g^\vee) + \iota(\xi_\zeta)\delta L = 0 . \tag{2.188}$$

Writing $I(\zeta)$ for this identity, we now express that $I(f\zeta) - fI(\zeta)$ vanishes. The first term contributes $df \wedge \iota(\zeta)L$. As $-\mathrm{Lie}(\zeta)(g^\vee)$ is $\nabla\zeta$, changed to a twice contravariant tensor and symmetrized and that $\nabla(f\zeta) - f\nabla\zeta = df \otimes \zeta$, the second term contributes $\langle T, \mathrm{grad}\, f \otimes \zeta\rangle$. As $\delta L = \underline{D}L - d\gamma$, with $\underline{D}L$ linear over functions, and that $\iota(\xi)d\gamma - d\iota(\xi)\gamma$, the third term contributes $df \wedge \iota(\xi_\zeta)\gamma$. With the notations of Corollary 2.183, the contribution of the second term can be written $df \wedge T(\zeta)$, giving

$$df \wedge (\iota(\zeta)L + T(\zeta) + \iota(\xi_\zeta)\gamma) = 0 . \tag{2.189}$$

As this holds for all f, we can suppress the $df\wedge$. If ζ is an infinitesimal isometry of g, then (ζ, ξ_ζ) is a manifest infinitesimal symmetry of L_g, and we get

Proposition 2.190. *Under the assumptions made, if ζ is a Killing vector field, the conserved current $T(\zeta)$ of* (3.5) *is the opposite of the Noether current of the corresponding symmetry of L_g.*

§2.10. Finite energy configurations, classical vacua, and solitons

In this section we work in the Hamiltonian framework $M = M^1 \times N$. Let t be an affine coordinate on M^1 and ∂_t the vector field which generates unit time translation. Recall that a field ϕ is *static* if

$$\frac{\partial\phi}{\partial t} = \iota(\hat{\xi}_t)\delta\phi = 0, \tag{2.191}$$

where $\hat{\xi}_t$ is the vector field on $\mathcal{F}$ induced by the action of ∂_t. Let $\mathcal{F}_N$ denote the space of static fields and $\overline{\mathcal{F}}_N$ the quotient by gauge symmetries. Equation (2.191) asserts that the vector field $\hat{\xi}_t$ vanishes on $\mathcal{F}_N$.

Consider a theory $\mathcal{L} = L + \gamma$ which we assume is manifestly invariant under time translation.[17] The *energy density* Θ is *minus* the canonical Noether current for ∂_t:

$$\Theta = -\big(\iota(\xi_t)\mathcal{L}\big)^{0,|-1|}, \tag{2.192}$$

where $\xi_t = \partial_t - \hat{\xi}_t$. The energy at time t of a field ϕ is (see (2.169))

$$E_\phi(t) = \int_{\{t\}\times N} \Theta(\phi). \tag{2.193}$$

For a static field the energy is constant in time. Let $\mathcal{FE}_N$ denote the space of finite energy static fields and $\overline{\mathcal{FE}}_N$ the quotient by gauge symmetries. These are subspaces of $\mathcal{F}_N$ and $\overline{\mathcal{F}}_N$, respectively. Define $\mathcal{M}_N \subset \mathcal{FE}_N$ (and $\overline{\mathcal{M}}_N \subset \overline{\mathcal{FE}}_N$) to be the space of static classical solutions of finite energy.

The energy density of a static field is simply related to the lagrangian density.

Proposition 2.194. *(i) We have*

$$\Theta = -\iota(\partial_t)L \qquad on\ \mathcal{F}_N; \tag{2.195}$$

(ii) If ϕ is a critical point of energy on $\mathcal{FE}_N$, then ϕ is a solution to the classical equations, i.e., $\phi \in \mathcal{M}_N$. Conversely, every element of $\mathcal{M}_N$ is a critical point of energy.

Proof. By (2.191) we have $\iota(\hat{\xi}_t)\gamma = 0$ on $\mathcal{F}_N$. Hence on $\mathcal{F}_N$

$$\Theta = -\iota(\xi_t)\mathcal{L} = -\iota(\partial_t)L + \iota(\hat{\xi}_t)\gamma = -\iota(\partial_t)L, \tag{2.196}$$

which is (i). For (ii) we first note that on $\mathcal{F}_N$ we have by (2.191) that

$$\begin{aligned} 0 = \operatorname{Lie}(\xi_t)\gamma &= d\iota(\partial_t)\gamma + \iota(\partial_t)d\gamma - \delta\iota(\hat{\xi}_t)\gamma - \iota(\hat{\xi}_t)\delta\gamma \\ &= d\iota(\partial_t)\gamma + \iota(\partial_t)d\gamma. \end{aligned} \tag{2.197}$$

Using (2.195) we compute that on $\mathcal{F}_N$

$$\begin{aligned} \delta\Theta &= \iota(\partial_t)\delta L \\ &= \iota(\partial_t)(D\mathcal{L})^{1,|0|} - \iota(\partial_t)d\gamma \\ &= \iota(\partial_t)(D\mathcal{L})^{1,|0|} + d\iota(\partial_t)\gamma. \end{aligned} \tag{2.198}$$

This is the integration by parts equation for the functional E on $\mathcal{F}_N$—analogous to (2.40) for the functional L—and so the Euler-Lagrange equation for E is

$$\iota(\partial_t)(D\mathcal{L})^{1,|0|} = 0, \tag{2.199}$$

[17]This excludes, for example, the theory of a nonrelativistic particle moving in a time-dependent potential.

which is equivalent to the equation of motion $(D\mathcal{L})^{1,|0|} = 0$.

The global minima of energy are called *vacuum solutions*. For field theories on Minkowski spacetime the set of vacuum solutions is referred to as the *classical moduli space of vacua*. We denote it $\mathcal{M}_{\text{vac}}$. A vacuum solution in Minkowski spacetime is usually assumed to be Poincaré invariant (if the theory is Poincaré invariant). This means that scalar fields are constant, gauge fields are gauge equivalent to a trivial connection, spinor fields vanish, and p-form fields ($p \geq 1$) are exact. The classical moduli space is then the space of constant values of the scalar fields. Now the energy density $\Theta(\phi_0)$ of a constant scalar field $\phi_0 \colon M \to X$ satisfies

$$dt \wedge \Theta(\phi_0) = V(\phi_0)\,|dt|\,|d^{n-1}x| \tag{2.200}$$

for a *potential energy* function $V \colon X \to \mathbb{R}$. Assume[18] that $V \geq 0$. The only finite energy constant scalar fields have $V(\phi_0) = 0$, and so

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \tag{2.202}$$

is the classical moduli space. If there is also a connection field in the theory with gauge group G, then G acts on X and V is an invariant function. In that case the effect of dividing by global gauge transformations is that the classical moduli space is the quotient

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{2.203}$$

Returning for a moment to a general field theory $\mathcal{L} = L+\gamma$ on a manifold M, fix a field configuration $\phi_0 \in \mathcal{F}$. Then there is a *perturbation theory* for the fluctuations around ϕ_0 in which the space of fields is $T_{\phi_0}\mathcal{F}$ and the N^{th} order perturbative lagrangian is the N^{th} order jet of L at ϕ_0.

On Minkowski spacetime M we often perturb around a vacuum solution.[19] Recall that at a vacuum all gauge fields A_0 are trivial, and we can use them to trivialize all bundles. Then any scalar ϕ_0 is a constant in a manifold X. In the perturbation theory the fluctuations of a trivial connection A_0 on a principal G-bundle P lie in $\Omega^1_M(\operatorname{ad} P) \cong \Omega^1_M(\mathfrak{g})$, where we use the trivialization. The fluctuations $\tilde{\phi}$ of the constant scalar ϕ_0 lie in $T_{\phi_0}X$. Spinor fields, Rarita-Schwinger fields, and p-form fields all vanish at a vacuum configuration and we consider the fields in the original lagrangian to be fluctuations about zero. Now since we are at a vacuum solution the perturbative lagrangian starts out with quadratic terms. In the quadratic approximation $\tilde{\phi}$ is typically a free scalar field in the Euclidean space $T_{\phi_0}X$ with mass[20] $\operatorname{Hess}_{\phi_0} V$; there may be higher derivative terms as well. There is a mass matrix for the spinor fields which also depends on ϕ_0. The *massless* fluctuations of the gauge field lie in $\Omega^1_M(\mathfrak{g}_{\phi_0})$, where $\mathfrak{g}_{\phi_0}$ is the Lie algebra of the stabilizer subgroup G_{ϕ_0} at ϕ_0 of the G action on X. Often G_{ϕ_0} is called the *unbroken* gauge group, though this terminology is confusing. Other components of the gauge fluctuations are massive due to the *Higgs mechanism*, which we illustrate in the following example.

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[18]In many problems the energy density is nonnegative for all fields: $\Theta_t(\phi) \geq 0$ for all ϕ, where

$$dt \wedge \Theta(\phi) = \Theta_t(\phi)\,|dt|\,|d^{n-1}x|. \tag{2.201}$$

[19]This is the first step in a perturbative construction of a quantum theory around the chosen vacuum.

[20]Free fields and this use of 'mass' are discussed in §3.

Example 2.204. We work on n-dimensional Minkowski spacetime M^n. The fields are a complex scalar field ϕ, a real spinor field[21] ψ, and a $\mathbb{T} \times \mathbb{T}$ connection A. As in any gauge theory, we work on the quotient of the space of fields by gauge transformations. Let $\mathbb{T}\times\mathbb{T}$ act on $\mathbb{C}$ by $(\lambda_1, \lambda_2)\cdot z = \lambda_1 z \lambda_2^{-1}$, and suppose that ϕ is a section of the associated hermitian line bundle. The spinor field ψ is not coupled to A. Let

$$L = \Big(\| d_A\phi \|^2 - \frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \psi \not{D} \psi\rangle - \|\phi\|^2 (1 - \|\phi\|^2)^2 - \phi\psi\psi\Big) |d^n x|. \tag{2.205}$$

The potential energy is

$$V(\phi_0) = \|\phi_0\|^2 (1 - \|\phi_0\|^2)^2 \tag{2.206}$$

and $V^{-1}(0)$ consists of the origin and the unit circle. The classical moduli space is the quotient of $V^{-1}(0)$ by $\mathbb{T} \times \mathbb{T}$ and so consists of two isolated points. They are the two classical vacua. At the origin the perturbative lagrangian is the same as (2.205), except that A should now be viewed as a perturbation $A = A_0 + \alpha$ of a trivial connection; α is an $(i\mathbb{R} \oplus i\mathbb{R})$-valued 1-form on M. We use A_0 to construct a trivializing section of the hermitian line bundle, and so view ϕ as a map $M \to \mathbb{C}$. Then the perturbative lagrangian is

$$L_0 = \Big(\| d\phi + \alpha\cdot\phi \|^2 - \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\langle \psi \not{D} \psi\rangle - \|\phi\|^2 (1 - \|\phi\|^2)^2 - \phi\psi\psi\Big) |d^n x|. \tag{2.207}$$

From the quadratic part of L_0 we read off that ϕ is massive (with mass 1), ψ is massless, and α is massless. The entire gauge group $\mathbb{T} \times \mathbb{T}$ is unbroken.

Now we expand around $\phi_0 = 1$, which is most easily accomplished by substituting $\phi = 1 + \tilde{\phi}$ into L_0:

$$\begin{aligned} L_1 &= \Big(\| d\tilde\phi + \alpha\cdot(1+\tilde\phi) \|^2 - \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\langle \psi \not{D} \psi\rangle \\ &\qquad - \|1+\tilde\phi\|^2 (1 - \|1+\tilde\phi\|^2)^2 - (1+\tilde\phi)\psi\psi\Big) |d^n x| \\ &= \Big(\| d\tilde\phi \|^2 - \frac{1}{2}|d\alpha|^2 + \frac{1}{2}\langle \psi \not{D} \psi\rangle - 4|(\operatorname{Re}\tilde\phi)|^2 \\ &\qquad - \psi\psi + \|\alpha_1 - \alpha_2\|^2\Big) |d^n x| + \text{higher order terms.} \end{aligned} \tag{2.208}$$

In the last line we wrote only the quadratic part; it contains the information about masses. Also, we wrote $\alpha = (\alpha_1, \alpha_2)$; the last term is the norm square of $\alpha \cdot 1 = \alpha_1 - \alpha_2$. The imaginary part of the complex scalar field is gauged away by constant gauge transformations (which preserve the trivial connection A_0). Thus the only scalar field is the *real* scalar field $\operatorname{Re}\tilde\phi$ with mass square 4. The spinor ψ is now massive with mass 2 (and so mass square 4). Set[22] $\sqrt{2}\beta = \alpha_1 - \alpha_2$ and $\sqrt{2}\gamma = \alpha_1 + \alpha_2$. Then the quadratic part of the lagrangian involving α is

$$-\frac{1}{2}|d\beta|^2 - \frac{1}{2}|d\gamma|^2 + 2\|\beta\|^2. \tag{2.209}$$

[21] The precise spinor representation S of $\mathrm{Spin}(1, n-1)$ is not crucial, but we do assume that there is an invariant skew form $\bigwedge^2 S \to \mathbb{R}$ so that we can write a mass term.

[22] Here the massless field γ is canonical, but the massive field β is somewhat arbitrary.

So we see that β has mass square 4 and γ is massless. Note that γ lies in the Lie algebra of the stabilizer of $1 \in \mathbb{C}$, which is the diagonal $\mathbb{T} \subset \mathbb{T} \times \mathbb{T}$. This is the unbroken gauge group. The appearance of a mass for β is an example of the Higgs mechanism.

In general, if H is a group of global symmetries of $\mathcal{L}$, then H acts on the space of static classical solutions $\mathcal{M}_N$. For $\phi \in \mathcal{M}_N$ we have the subgroup $\operatorname{Stab}\phi \subset H$ which fixes ϕ. We say that ϕ *spontaneously breaks* H down to $\operatorname{Stab}\phi$, and that $\operatorname{Stab}\phi$ is the group of *unbroken* symmetries. (For example, we asserted above that in Minkowski spacetime Poincaré symmetry is unbroken at a vacuum solution.) If ϕ_0 is a vacuum solution, the homogeneous space $H/\operatorname{Stab}\phi_0$ is embedded in $V^{-1}(0)$ by the H action. Then the perturbative scalar fields with values in $\mathfrak{h}/\mathfrak{stab}\,\phi_0 \subset T_{\phi_0}X$ are massless. These fields are called (classical) *Goldstone bosons*. They are massless scalar fields guaranteed by the symmetry. Of course, there may be other massless scalar fields which are not related to symmetry.

Example 2.210. Consider a theory of a complex scalar $\phi\colon M \to \mathbb{C}$ with potential (2.206). Now there is a global $\mathbb{T}$ symmetry and the moduli space consists of the origin and the unit circle. We do *not* divide by the global symmetry; rather, at a classical vacuum on the unit circle there is a single (real) Goldstone boson field due to the symmetry. At the origin $\mathbb{T}$ acts trivially and the entire complex scalar is massive.

A *soliton* is a static classical solution whose energy is not a global minimum. In many examples the space $\overline{\mathcal{FE}}_N$ of finite energy static fields is not connected, and a soliton is a minimum energy configuration in a component where the global minimum is not achieved. For example, consider a scalar field $\phi\colon M^2 \to \mathbb{R}$ on two-dimensional Minkowski spacetime with lagrangian density

$$L = \Big(\frac{1}{2}|d\phi|^2 - V(\phi)\Big)\,|d^2x|. \tag{2.211}$$

Suppose $V \geq 0$ and $V^{-1}(0) = \{a, b\}$. Then the space of finite energy static fields has 4 components: A static field $\phi(t,x) = \phi(x)$ depends only on the spatial variable, and the finite energy condition means $\lim_{x\to\infty}\phi(x)$ and $\lim_{x\to-\infty}\phi(x)$ lie in $V^{-1}(0) = \{a, b\}$. There are 2 vacuum solutions $\phi(x) \equiv a$ and $\phi(x) \equiv b$. There are solitons with $\lim_{x\to-\infty}\phi(x) = a$, $\lim_{x\to\infty}\phi(x) = b$ and also (anti-)solitons with $\lim_{x\to-\infty}\phi(x) = b$, $\lim_{x\to\infty}\phi(x) = a$.

§2.11. Dimensional reduction

Suppose $\mathcal{L} = L + \gamma$ is a Poincaré invariant field theory on n-dimensional Minkowski spacetime M^n. We obtain a theory on M^{n-1} as follows. As usual let $x^0, \dots, x^{n-1}$ denote coordinates on M^n, and $x^0, \dots, x^{n-2}$ coordinates on M^{n-1}, viewed as the quotient of M^n by translations in the x^{n-1} direction. Let $\hat{\xi}_{n-1}$ denote the vector field on $\mathcal{F}$ induced by the action of ∂_{n-1}, and define

$$\mathcal{F}_{n-1} = \{\phi \in \mathcal{F} : \operatorname{Lie}(\hat{\xi}_{n-1})\phi = 0\}. \tag{2.212}$$

These are the fields which are constant in the x^{n-1} direction. We identify $\mathcal{F}_{n-1}$ with a space of fields on M^{n-1}. Then the dimensionally reduced theory is

$$\mathcal{L}_{n-1} = \mathcal{L} \otimes \partial_{n-1} \ \in \Omega^0_{\mathrm{loc}}(\mathcal{F}_{n-1} \times M^{n-1}). \tag{2.213}$$

A scalar field on M^n reduces to a scalar field on M^{n-1}, but for fields of higher spin the identification of $\mathcal{F}_{n-1}$ with fields on M^{n-1} is more complicated. For example, a 1-form on M^n reduces to a 1-form plus a scalar field on M^{n-1}.

More generally, if $M \to M'$ is a fiber bundle with a section of $\operatorname{Det} T(M/M')$, then we can reduce a field theory on M to a field theory on M'. Instead of restricting to fields constant along the fibers (dimensional reduction) we can also include fields which fluctuate along the fibers (compactification); from the point of view of M' there is an infinite number of such fields.

Appendix: Takens' acyclicity theorem

This appendix gives the proof of Theorem 2.16. Our setting is slightly different from that of Takens, who works on the infinite jet bundle $J^\infty_{E/M}$ of E/M. To go from local to global results, he uses that if $0 \to \mathcal{F}_0 \to \cdots \to \mathcal{F}_n \to \mathcal{F}_{n+1}$ is an exact sequence of sheaves, with $\mathcal{F}_i$ soft for $i \le n$, the sequence of groups of global sections is again exact. In order that the space of "local" lagrangian densities, or forms, be a space of global sections of some sheaf, he is led to define "local" to mean "depending only on k-jets of fields and their variations, where k is bounded *locally* on $J^\infty_{E/M}$". For us, k is globally bounded, and the sheaf theoretic argument has to be unraveled, replaced by a direct use of partitions of unity.

We fix a submersion $E \to M$. For fixed $p > 0$, we are to prove that the corresponding complex $(\Omega^{p,*}_{\text{loc}}, d)$ is acyclic, except in top degree. It will be more convenient to prove a more general statement. Fix vector bundles $V_1, \dots, V_p$ on E. Let V be their product. The complex $(\Omega^{0,*}_{\text{loc}}, d)$ for $V \to M$ is the complex of forms α on M depending (locally) on a section ϕ of E/M and on sections $\xi_1, \dots, \xi_p$ of $\phi^* V_i$. For some k, the value of $\alpha(\phi, \xi_1, \dots, \xi_p)$ at $m \in M$ should depend only on the k-jet of ϕ and the k-jet of $\xi_1, \dots, \xi_p$ at m. We denote $\Omega_{\text{loc,multi}}$ the subcomplex consisting of the α depending $\mathbb{R}$-linearly on each of $\xi_1, \dots, \xi_p$. The following is a restatement of Theorem 2.16.

Theorem 2.214. *If $p > 0$, the complex $\Omega_{\text{loc,multi}}$ is acyclic except in top degree.*

If we take all V_i to be the relative tangent bundle $T_{E/M}$, then $(\Omega^{p,*}_{\text{loc}}, d)$ is a direct factor of $\Omega_{\text{loc,multi}}$: it is the antisymmetric part for the action of the symmetric group S_p. The acyclicity of $(\Omega^{p,*}_{\text{loc}}, d)$ for $p > 0$ (except in top degree) hence follows from the theorem.

Proof. For e_0 in E, with image m_0 in M, one can choose (a) a neighborhood U of m_0, and a local coordinate system $x\colon U \hookrightarrow \mathbb{R}^d$, (b) a neighborhood W of e_0, a decomposition $W = F \times U$ and a local coordinate system $F \hookrightarrow \mathbb{R}^f$, and (c) trivializations of the V_i on W. In such a local coordinate system, α in $\Omega_{\text{loc,multi}}$ can uniquely be written as a finite sum

$$\alpha(\phi; \xi_1, \dots, \xi_p) = \sum \alpha_{n_1,\dots,n_p}(\phi) \partial^{n_1}\xi_1 \dots \partial^{n_p}\xi_p \,. \tag{2.215}$$

In the formula, each n_i is a multi-index $(n_{i1}, \dots, n_{id})$ and $\partial^{n_i} := \partial_1^{n_{i1}} \dots \partial_d^{n_{id}}$ is the corresponding iterated derivative in the coordinate system $(x_1, \dots, x_d)$. We define as usual $|n_i| = \sum n_{ij}$. For some k each $\alpha_{n_1,\dots,n_p}$ depends only on the k-jet of ϕ and takes values in $\otimes V_i^\vee \otimes \Omega^*_M$: it is a section on $J^k(E/M)$ of the tensor product of the inverse images of the $V_i^\vee$ and of Ω^*_M. We write F_* for the increasing filtration by

$\sum |n_i|$. It does not depend on the local coordinate systems used. The differential d maps F_N to F_{N+1}.

Replacing in (2.215) each ∂^{n_i} by its symbol in $\mathrm{Sym}^{|n_i|}(T_M)$, we obtain the following description of $\mathrm{Gr}^F(\Omega^q_{\mathrm{loc,multi}})$. It is the inductive limit in k of the space of sections over $J^k_{E/M}$ of the vector bundle

$$\otimes_{*\leq k} \mathrm{Sym}^*(T_M) \otimes V^{i\vee} \otimes \Omega^q_M \tag{2.216}$$

with each factor to be replaced by its pull-back to $J^k_{E/M}$. The degree N is the sum of the symmetric algebra degrees.

The differential d induces $\mathrm{Gr}^F_N(d)$: $\mathrm{Gr}^F_N(\Omega^q_{\mathrm{loc,multi}}) \to \mathrm{Gr}^F_{N+1}(\Omega^{q+1}_{\mathrm{loc,multi}})$, which is linear over the functions. It is deduced from a morphism of vector bundles on M

$$\left(\bigotimes_1^p \mathrm{Sym}^*(T_M)\right) \otimes \Omega^q_M \to \left(\bigotimes_1^p \mathrm{Sym}^*(T_M)\right) \otimes \Omega^{q+1}_M \tag{2.217}$$

by pull-back and tensorization by $\otimes V^{i\vee}$. Locally on M, for $\{e_j\}$ a basis of T_M, (2.217) is

$$s \otimes \alpha \longmapsto \sum_j \left(\left(\sum_{i=1}^p 1 \otimes \cdots \otimes (e_j \text{ at } i^{\text{th}} \text{ place}) \otimes \cdots \otimes 1 \right) \cdot s \right) \otimes e^j \wedge \alpha \,. \tag{2.218}$$

The crux of the matter is now the following

Lemma 2.219. *On M, for each integer N, the complex of vector bundles with components*

$$(\textit{part of degree } (N+q) \textit{ of } \bigotimes_1^p \mathrm{Sym}^*(T_M)) \otimes \Omega^q_M$$

and differential the morphisms (2.218) *of vector bundles is acyclic, except in top degree.*

Proof. This is to be checked point by point. One then recognizes in the complex a variant of the Koszul complex, and one can write an explicit homotopy. In more detail: for T the tangent space at a point of M, one has to consider a complex with components

$$\left(\bigotimes_1^p \mathrm{Sym}^*(T)\right) \otimes \wedge^q(T^\vee) \,. \tag{2.220}$$

One has $\bigotimes_1^p \mathrm{Sym}^*(T) = \mathrm{Sym}^*\left(\bigoplus_1^p T\right)$. If we identify T with its image $\Delta T \subset \bigoplus_1^p T$ by the diagonal embedding, and if S is any supplement, we have

$$\mathrm{Sym}^*\left(\bigoplus_1^p T\right) = \mathrm{Sym}^*(S) \otimes \mathrm{Sym}^*(T) \,, \tag{2.221}$$

and the complex (2.220) becomes the tensor product of $\mathrm{Sym}^*(S)$ (of cohomological degree zero) by the complex whose component of degree q is

$$\mathrm{Sym}^*(T) \otimes \wedge^q T^\vee \,, \tag{2.222}$$

and whose differential is

$$d(s \otimes \alpha) = \sum_j (e_j s) \otimes (e^j \wedge \alpha) \tag{2.223}$$

for e_j a basis of T and e^j the dual basis of $T^\vee$.

The complex (2.222) is a version of the Koszul complex. It is multiplicative in T: for $T = T' \oplus T''$, it is the tensor product of the similar complexes of T' and T''. To check its acyclicity except in top degree, it hence suffices to check it for $\dim(T) = 1$, in which case the complex reduces to

$$\mathbb{R}[x] \xrightarrow{x} \mathbb{R}[x] \tag{2.224}$$

in degree 0 and 1. One can also write an explicit homotopy H: for P and α homogeneous, define $H(P\alpha)$ to be 0 if P is of degree 0 and α of maximal degree $\dim(T)$. Otherwise

$$H(P\alpha) := \left(\sum \partial_{e^j}(P) \otimes \iota(e_j)\alpha\right) / (\deg P + \dim T - \deg \alpha) \tag{2.225}$$

One checks that $dH + Hd$ is the identity minus the projection to the part of bidegree 0 in P and $\dim(T)$ in α.

Lemma 2.219 implies Theorem 2.214, as the acyclicity (except at the top) of an associated graded to $\Omega_{\mathrm{loc,multi}}$ implies the same acyclicity for $\Omega_{\mathrm{loc,multi}}$ itself. From the proof of Corollary 2.183 one can in addition deduce a local formula to attach to α such that $d\alpha = 0$ a β for which $\alpha = d\beta$. Indeed, let us cover E by open sets W_ℓ for which one has local coordinates as in (a), (b), (c) before (2.215). On each W_ℓ, a choice of local coordinates splits the filtration F, and the proof of Lemma 2.219 gives a homotopy operator H_ℓ of filtration -1 such that outside of top degree $\mathrm{Id} - (dH_\ell + H_\ell d)$ is of filtration -1, i.e., maps F_N to F_{N-1}. Let χ_ℓ be a partition of unity subordinate to the covering W_ℓ and define $H(\alpha) = \sum H_\ell(\chi_\ell \alpha)$. We again have $\mathrm{Id} - (dH + Hd)$ of filtration -1. If α is in F_N, and closed,

$$\alpha' := \alpha - (dH + Hd)\alpha = \alpha - d(H\alpha) \tag{2.226}$$

is in F_{N-1}. One has $\alpha = dH\alpha + \alpha'$, α' is closed, and repeating the argument for α', one eventually obtains β with $\alpha = d\beta$.

CHAPTER 3
Free Field Theories

§3.1. Coordinates on Minkowski spacetime

Let M denote affine n-dimensional Minkowski spacetime. Fix a coordinate system $x^0, x^1, \ldots, x^{n-1}$ with respect to which the metric is

$$g = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{3.1}$$

Note that $x^0 = ct$ for t the standard time coordinate. The wave operator on functions is

$$\begin{aligned} \Box &= (-1)^{n-1} * d * d \\ &= -d^* d \\ &= \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 \\ &= g^{\mu\nu} \partial_\mu \partial_\nu, \end{aligned} \tag{3.2}$$

where we use the notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{3.3}$$

The canonical density is

$$|d^n x| = |\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \,|. \tag{3.4}$$

For a function f we have

$$d * df = \Box f \; |d^n x|. \tag{3.5}$$

The symbol 'g' usually denotes the Minkowski metric, though occasionally it denotes a general metric as is clear from the context.

Lagrangians in Minkowski spacetime are real (see Chapter 7). Thus equations involving complex quantities have complex conjugate equations which also hold, and which we usually omit.

We define global charges by integrating over the spacelike submanifold $\{x^0 = 0\}$ with canonical density

$$|d^{n-1} x| = |\, dx^1 \wedge \cdots \wedge dx^{n-1} \,|. \tag{3.6}$$

Indices $\mu, \nu, \ldots$ run from 0 to $n-1$; indices $i, j, \ldots$ run from 1 to $n-1$.

§3.2. Real scalar fields

A real scalar field is a map $\phi\colon M \to \mathbb{R}$. The free lagrangian for a field of mass $m \geq 0$ is

$$(3.7)\qquad \begin{aligned} L &= \left\{\frac{1}{2}\,|d\phi|^2 - \frac{m^2}{2}\,\phi^2\right\}|d^nx| \\ &= \frac{1}{2}\,d\phi \wedge *d\phi - \frac{m^2}{2}\,\phi^2\,|d^nx| \\ &= \left\{\frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \frac{m^2}{2}\phi^2\right\}|d^nx|. \end{aligned}$$

It is instructive to check the units in the lagrangian. Comparing the two terms we see that m must have units L^{-1} (to match the units of d). In a relativistic quantum theory we can replace m by $mc/\hbar$; then m has units of mass. Of course, in that context we usually work in a natural system of units with $c = \hbar = 1$. In that case ϕ has mass dimension $(n-2)/2$. The differential of L along the space of fields ϕ is

$$(3.8)\qquad \begin{aligned} \delta L &= -d\delta\phi \wedge *d\phi - m^2\phi\,\delta\phi\,|d^nx| \\ &= -\delta\phi \wedge \left\{d * d\phi + m^2\phi\,|d^nx|\right\} - d\{\delta\phi \wedge *d\phi\}. \end{aligned}$$

So from (3.5) the classical field equation is

$$(3.9)\qquad (\Box + m^2)\phi = 0$$

and the variational 1-form is

$$(3.10)\qquad \gamma = \delta\phi \wedge *d\phi.$$

The local symplectic form $\omega = \delta\gamma$ is

$$(3.11)\qquad \omega = *d\delta\phi \wedge \delta\phi.$$

Equation (3.9) is most easily analyzed through the Fourier transform

$$(3.12)\qquad \begin{aligned} \hat\phi(k) &= \frac{1}{(2\pi)^{n/2}}\int_V e^{-\sqrt{-1}\langle k,x\rangle}\,\phi(x)\,|d^nx|, \\ \phi(x) &= \frac{1}{(2\pi)^{n/2}}\int_{V^*} e^{+\sqrt{-1}\langle k,x\rangle}\,\hat\phi(k)\,|d^nk|. \end{aligned}$$

Then equation (3.9) transforms to

$$(3.13)\qquad (-|k|^2 + m^2)\hat\phi(k) = 0.$$

Thus the Fourier transform $\hat\phi$ is supported on the *mass shell*

$$(3.14)\qquad \mathcal{O}_m = \{k \in M : |k|^2 = m^2\}.$$

Note that since ϕ is real we have

$$(3.15)\qquad \hat\phi(-k) = \overline{\hat\phi(k)}.$$

Write $k = (k_0, \dots, k_{n-1})$; then k_i has units L^{-1} and is called the *wave number*. The *frequency* $\omega = k_0 c$ has units T^{-1}.

We can formulate the theory for an arbitrary metric g on M as

$$L_g = \bigl(\tfrac{1}{2} g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi - \frac{m^2}{2}\,\phi^2\bigr)\, \mu_g(x), \tag{3.16}$$

where μ_g is the canonical density associated to the metric g. Now a computation shows

$$\delta\mu_g = \frac{1}{2} g^{-1} \cdot \delta g\ \mu_g = -\frac{1}{2}\delta g^{-1} \cdot g\ \mu_g, \tag{3.17}$$

where

$$\delta g^{-1} \cdot g = \delta g^{\mu\nu} g_{\mu\nu}. \tag{3.18}$$

So the energy-momentum tensor, as defined in (2.176), is

$$T_g = d\phi \cdot d\phi\ \mu_g - g\, L_g. \tag{3.19}$$

In local coordinates we write (see (2.177))

$$T_{\mu\nu} = \partial_\mu\phi\, \partial_\nu\phi + \bigl(-\frac{1}{2}|d\phi|^2 + \frac{m^2}{2}\phi^2\bigr)\, g_{\mu\nu}. \tag{3.20}$$

Specialize to Minkowski spacetime. The energy density $T_{00}\, |d^{n-1}x|$ is given by

$$T_{00} = \bigl(\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}\sum_{i=1}^{n-1}(\partial_i\phi)^2 + \frac{m^2}{2}\phi^2\bigr). \tag{3.21}$$

Note that this is nonnegative and only vanishes for $\phi \equiv 0$. Minus the momentum density $T_{i0}\, |d^{n-1}x|$ is given by

$$T_{i0} = \partial_i\phi\partial_0\phi, \qquad i = 1, 2, \dots, n-1. \tag{3.22}$$

By Proposition 2.190 this agrees with the energy-momentum tensor (2.160) given by the canonical Noether currents of translations, as we now verify directly. The vector field ξ_μ induced by ∂_μ is

$$\begin{aligned} \iota(\xi_\mu) dx^\nu &= \delta^\nu_\mu \\ \iota(\xi_\mu)\delta\phi &= -\partial_\mu\phi. \end{aligned} \tag{3.23}$$

Then *minus* the associated Noether current is

$$\begin{aligned} -\iota(\xi_\mu)(L+\gamma) &= -L\, * dx^\mu + \partial_\mu\phi\, * d\phi \\ &= (-g_{\mu\nu} L + \partial_\mu\phi\partial_\nu\phi)\, * dx^\nu, \end{aligned} \tag{3.24}$$

where $g_{\mu\nu}$ is the Lorentz metric.

The "trace" of T is

$$g^{\mu\nu} T_{\mu\nu} = (1 - \frac{n}{2}) g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi + \frac{m^2 n}{2}\phi^2, \tag{3.25}$$

which vanishes if $n = 2$ and $m = 0$. Thus the theory of a massless scalar field is conformally invariant in 2 dimensions.

§3.3. Complex scalar fields

Our convention is that if '$\langle\cdot,\cdot\rangle$' denotes a real bilinear form, then it also denotes the extension to a *bilinear* form over the complex numbers. If $\langle\cdot,\cdot\rangle$ is a real inner product on a real vector space W, then the associated hermitian norm on the complexification $W_{\mathbb{C}}$ is

$$w \longmapsto \langle \bar{w}, w\rangle. \tag{3.26}$$

Over the reals or complexes we always use the notation

$$|w|^2 = \langle w, w\rangle. \tag{3.27}$$

A complex scalar field is a map $\Phi\colon M \to \mathbb{C}$. The free lagrangian for a field of mass $m \ge 0$ is

$$L = \Big\{ \langle d\overline{\Phi}, d\Phi\rangle - m^2\langle \overline{\Phi}, \Phi\rangle \Big\}\, |d^nx|. \tag{3.28}$$

This theory is equivalent to a theory of two uncoupled free real scalars ϕ_1, ϕ_2; simply set

$$\Phi = \frac{\phi_1 + \sqrt{-1}\,\phi_2}{\sqrt{2}}. \tag{3.29}$$

Then (3.28) reduces to the sum of two copies of (3.7). The variation of (3.28) is

(3.30)
$$\begin{aligned} \delta L &= \big\{-\langle d\delta\overline{\Phi} \wedge d\Phi\rangle + \langle d\overline{\Phi} \wedge d\delta\Phi\rangle - m^2\delta\overline{\Phi}\,\Phi - m^2\overline{\Phi}\,\delta\Phi\big\}\,|d^nx| \\ &= -\delta\overline{\Phi}\big\{d*d\Phi + m^2\Phi\,|d^nx|\big\} - \delta\Phi\big\{d*d\overline{\Phi} + m^2\overline{\Phi}\,|d^nx|\big\} - d\gamma \end{aligned}$$

for the variational 1-form

$$\gamma = \delta\overline{\Phi} \wedge *d\Phi + \delta\Phi \wedge *d\overline{\Phi}. \tag{3.31}$$

So the equation of motion is

$$(\Box + m^2)\Phi = 0. \tag{3.32}$$

The analysis of (3.32) proceeds as in the real case. Note the significant difference that there is no reality condition (3.15) in the Fourier transform.

The circle group $\mathbb{T} \subset \mathbb{C}$ operates on Φ by scalar multiplication, and this symmetry manifestly preserves the lagrangian. (In the language of §2 it is an "internal" symmetry.) The corresponding infinitesimal symmetry ξ is

$$\iota(\xi)\delta\Phi = \sqrt{-1}\Phi. \tag{3.33}$$

(Of course, we also have the conjugate equation $\iota(\xi)\delta\overline{\Phi} = -\sqrt{-1}\;\overline{\Phi}$.) So the associated Noether current (see (2.88)) is

$$\begin{aligned} j_\xi &= \iota(\xi)\big(L + \gamma\big) \\ &= \sqrt{-1}\big(\Phi * d\overline{\Phi} - \overline{\Phi} * d\Phi\big). \end{aligned} \tag{3.34}$$

The corresponding global charge is

$$Q_\xi = \sqrt{-1}\int_{\{x^0=0\}} (\Phi\partial_0\overline{\Phi} - \overline{\Phi}\partial_0\Phi)\,|d^{n-1}x|. \tag{3.35}$$

The energy-momentum tensor is

$$T_g = d\overline{\Phi}\cdot d\Phi\;\mu_g - g\,L_g. \tag{3.36}$$

or (see (2.177))

$$T_{\mu\nu} = \partial_\mu\overline{\Phi}\,\partial_\nu\Phi + \Big(-\frac{1}{2}\langle d\overline{\Phi}, d\Phi\rangle + \frac{m^2}{2}\langle\overline{\Phi},\Phi\rangle\Big)\,g_{\mu\nu}. \tag{3.37}$$

We evaluate the energy-momentum tensor for a plane wave

$$\Phi(x) = ae^{ik(x)}, \tag{3.38}$$

where $k = (k_0,\dots,k_{n-1})$ is in the dual of V, the vector space underlying M. We restore the constants necessary to treat m as a mass, so replace m by $mc/\hbar$. Components of the *wave number* $(k_1,\dots,k_{n-1})$ have units L^{-1} and the *frequency* $\omega = k_0c$ has units T^{-1}. The complex constant a has units $\sqrt{M/TL^{n-4}}$. By the equation of motion (3.32), we have

$$|k|^2 = m^2c^2/\hbar^2. \tag{3.39}$$

Then (3.37) reduces to

$$T_{\mu\nu} = |a|^2 k_\mu k_\nu. \tag{3.40}$$

§3.4. Spinor fields

Let V be the vector space underlying Minkowski spacetime M, and let $\mathrm{Spin}(V)$ denote the Lorentz group. Suppose S is a *real* spin representation of $\mathrm{Spin}(V)$. This means that there are symmetric pairings

$$\begin{aligned}\Gamma\colon S^*\otimes S^* &\longrightarrow V\\ \tilde{\Gamma}\colon S\otimes S &\longrightarrow V\end{aligned} \tag{3.41}$$

which satisfy a Clifford relation. Let $\{e_\mu\}$ be a basis of V and $\{f^a\}$ a basis of S. We use the dual bases $\{e^\mu\},\{f_a\}$ for V^*, S^*. Write

$$\begin{aligned}\Gamma(f_a, f_b) &= \Gamma^\mu_{ab}e_\mu,\\ \tilde{\Gamma}(f^a, f^b) &= \tilde{\Gamma}^{\mu ab}e_\mu;\end{aligned} \tag{3.42}$$

then the Clifford relation is

$$\tilde{\Gamma}^{\mu ab}\Gamma^\nu_{bc} + \tilde{\Gamma}^{\nu ab}\Gamma^\mu_{bc} = 2g^{\mu\nu}\delta^a_c. \tag{3.43}$$

Let ΠS denote the odd vector space which is S with odd parity. A *spinor field* (spin 1/2 fermion) is a map $\psi\colon M \to \Pi S$. With respect to the basis we write

$$\psi(x) = \psi_a(x) f^a. \tag{3.44}$$

A *dual spinor field* is a map $\lambda\colon M \to \Pi S^*$. The analysis of spinor fields below adapts easily to dual spinor fields.

A mass pairing for spinor fields is a skew-symmetric pairing

$$M\colon \bigwedge^2 S \longrightarrow \mathbb{R}. \tag{3.45}$$

Nonzero pairings need not exist.[23] If there is a nonzero mass pairing M, then there exists a *normalized* skew-symmetric pairing

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R}; \tag{3.46}$$

the normalization condition is

$$\tilde{\Gamma}^{\mu ab} = \Gamma^{\mu}_{a'b'} \epsilon^{aa'} \epsilon^{bb'}. \tag{3.47}$$

For simplicity assume that ϵ is irreducible. Then we can write

$$M = m\epsilon, \qquad m \in \mathbb{R}. \tag{3.48}$$

We write

$$\psi M \psi = M^{ab} \psi_a \psi_b, \tag{3.49}$$

where the name of a bilinear form is written between the arguments. The kinetic term in the free lagrangian is the *Dirac form*

$$\psi \not{D} \psi = \tilde{\Gamma}^{\mu}(\psi, \partial_\mu \psi) = \tilde{\Gamma}^{\mu ab} \psi_a\, \partial_\mu \psi_b. \tag{3.50}$$

Equation (3.50) defines a symmetric (in the graded sense) bilinear form up to an exact term, so it is exactly symmetric after integrating over M, assuming no contribution at infinity. The free lagrangian is

$$L = \left\{ \frac{1}{2}\, \psi \not{D} \psi - \frac{1}{2}\, \psi M \psi \right\} |d^n x|. \tag{3.51}$$

The variation of L is

$$\delta L = \left\{ \frac{1}{2} \delta\psi\, \not{D} \psi + \frac{1}{2} \psi\, \not{D} \delta\psi - \delta\psi\, M\, \psi \right\} |d^n x|. \tag{3.52}$$

Now

$$\begin{aligned} \psi\, \not{D} \delta\psi &= \tilde{\Gamma}^{\mu ab} \psi_a\, \partial_\mu \delta\psi_b \\ &= -\tilde{\Gamma}^{\mu ab} \partial_\mu \psi_a\, \delta\psi_b + \tilde{\Gamma}^{\mu ab} \partial_\mu (\psi_a\, \delta\psi_b), \end{aligned} \tag{3.53}$$

[23] For example, for $n = 2$ take S to be a half-spinor representation. It is one-dimensional, so $\bigwedge^2 S = 0$.

and so

$$\delta L = \delta\psi\Big(\not{D}\psi - M\,\psi\Big)\,|d^n x| - d\gamma \tag{3.54}$$

for the variational 1-form

$$\begin{aligned} \gamma &= \frac{1}{2}\tilde{\Gamma}^{\mu}(\psi,\delta\psi)\,\iota(\partial_\mu)|d^n x| \\ &= \frac{1}{2}(\tilde{\Gamma}^{\mu ab}\psi_a\,\delta\psi_b)\,\iota(\partial_\mu)|d^n x|. \end{aligned} \tag{3.55}$$

Now M determines a map $S \to S^*$ and $\tilde{\Gamma}$ a map $S \to V \otimes S^*$, so that both M and $\not{D}$ define operators from spinor fields to dual spinor fields. We use the same letters $M, \not{D}$ to denote these operators. Then the equation of motion is

$$\not{D}\psi = M\psi. \tag{3.56}$$

We analyze (3.56) via the Fourier transform (3.12):

$$\sqrt{-1}\,\tilde{\Gamma}^{\mu ab}k_\mu\hat{\psi}_b(k) = M^{ab}\hat{\psi}_b(k) \qquad \text{for all } a. \tag{3.57}$$

Multiply both sides of (3.57) by $\sqrt{-1}\,\tilde{\Gamma}^{\nu}_{ca}k_\nu$. After some simplification using (3.43), (3.47), and (3.57) we find

$$|k|^2\hat{\psi}_c(k) = m^2\hat{\psi}_c(k) \qquad \text{for all } c. \tag{3.58}$$

Thus the Fourier transform of a solution is supported on the mass shell $\mathcal{O}_m$. Furthermore, equation (3.57) defines a subbundle $\Pi S'$ of the trivial bundle $\mathcal{O}_m \times \Pi S$, and $\hat{\psi}$ is a section of $\Pi S'$. The rank of S' is[24] $\dim S/2$. Again we have a reality condition (3.15). For $m > 0$ the subgroup of $\mathrm{Spin}(V)$ which stabilizes $k \in \mathcal{O}_m$ is isomorphic to $\mathrm{Spin}(n-1)$; the fiber of S' at k is a spin representation of $\mathrm{Spin}(n-1)$. For $m = 0$ the stabilizer has reductive part isomorphic to $\mathrm{Spin}(n-2)$, and the fiber of S' at k is a spin representation of $\mathrm{Spin}(n-2)$.

In some cases the minimal real spin representation S admits an action of $\mathbb{C}$ or $\mathbb{H}$ commuting with $\mathrm{Spin}(V)$. Then if $\tilde{\Gamma}$ is sesquilinear we have an action of $\mathbb{T}$ or Sp_1 which preserves the kinetic term $\psi\not{D}\psi$. (The mass pairing (3.45) may break this symmetry.) If there are multiple copies of the spin representation, then there is a larger compact group of manifest symmetries which rotate the various copies. Each such symmetry has a Noether current which may be computed from (3.55).

As for the energy-momentum tensor we first compute minus the canonical Noether current associated to translations, as in (2.160). Translation ∂_μ induces a vector field ξ_μ with

$$\begin{aligned} \iota(\xi_\mu)dx^\nu &= \delta^\nu_\mu \\ \iota(\xi_\mu)\delta\psi &= -\partial_\mu\psi. \end{aligned} \tag{3.59}$$

Using (3.51) and (3.55) we compute $-\iota(\xi_\mu)(L+\gamma) = \Theta_{\mu\nu} * dx^\nu$, where

$$\Theta_{\mu\nu} = \frac{1}{2}\tilde{\Gamma}^{\sigma ab}\psi_a\partial_\mu\psi_b g_{\nu\sigma} - (\frac{1}{2}\psi\not{D}\psi - \frac{1}{2}\psi M\psi)g_{\mu\nu}. \tag{3.60}$$

This is *not* symmetric in μ, ν (as expected).

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[24]There is an exceptional case: For a chiral spinor field in 2 dimensions equation (3.57) says that the Fourier transform is supported on half of the forward lightcone. In other words, a solution to the Dirac equation is either left- or right-moving, depending on the chirality. See [**I-Supersolutions**, §2.6] for the precise formulas.

§3.5. Abelian gauge fields

We restrict ourselves to the massless case. In 3 dimensions the Chern-Simons functional serves as a mass term (see Problem FP4 of [**I-Homework**]). In any dimension the Higgs mechanism can be used to introduce a mass. For another discussion of this material, see the solution to Problem FP3 of [**I-Homework**].

The standard physical interpretation (Maxwell's equations) is discussed in §4.1. The basics of principal bundles and connections are reviewed in §4.2.

We choose as gauge group the circle group $\mathbb{T}$ of unit norm complex numbers. On the Lie algebra $\mathrm{Lie}(\mathbb{T}) \cong \sqrt{-1}\,\mathbb{R}$ we have the positive definite inner product

$$\langle a, b\rangle = -ab, \qquad a, b \in \sqrt{-1}\,\mathbb{R}. \tag{3.61}$$

The field in a gauge theory is a connection A on M with structure group $\mathbb{T}$, and we work with fields up to gauge equivalence. The space of equivalence classes is a real affine space. The lagrangian is

$$\begin{aligned} L &= -\frac{1}{2}\langle F_A \wedge *F_A\rangle \\ &= -\frac{1}{2}|F_A|^2\,|d^nx| \\ &= -\frac{1}{4}\langle F_{\mu\nu}, F_{\mu'\nu'}\rangle\, g^{\mu\mu'} g^{\nu\nu'}\,|d^nx|, \end{aligned} \tag{3.62}$$

where the curvature is

$$F_A = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu. \tag{3.63}$$

Note that since

$$*(dx^\mu \wedge dx^\nu) = g^{\mu\mu'} g^{\nu\nu'}\,\iota(\partial_{\nu'})\iota(\partial_{\mu'})|d^nx|, \tag{3.64}$$

we have

$$*F_A = \frac{1}{2}F_{\mu\nu}\, g^{\mu\mu'} g^{\nu\nu'}\,\iota(\partial_{\nu'})\iota(\partial_{\mu'})|d^nx|. \tag{3.65}$$

Now δA is an imaginary 1-form on $\mathcal{F}_A \times M$, and

$$\delta F_A = -d\delta A; \tag{3.66}$$

the sign comes from commuting δ past d. Thus the variation of the lagrangian is

$$\begin{aligned} \delta L &= \langle d\delta A \wedge *F_A\rangle \\ &= d\langle \delta A \wedge *F_A\rangle - \langle \delta A \wedge d * F_A\rangle. \end{aligned} \tag{3.67}$$

Hence the canonical 1-form is

$$\begin{aligned} \gamma &= -\langle \delta A \wedge *F_A\rangle \\ &= -\langle \delta A_\mu, F_{\nu\sigma}\rangle g^{\mu\sigma} g^{\nu\rho}\,\iota(\partial_\rho)|d^nx| \end{aligned} \tag{3.68}$$

and the classical equation of motion is

$$d * F_A = 0, \tag{3.69}$$

or

$$g^{\rho\mu}\,\partial_\rho F_{\mu\nu} = 0 \qquad \text{for all } \nu. \tag{3.70}$$

This is an affine equation for A, and since gauge transformations act by an affine action, the moduli space $\overline{\mathcal{M}}$ of classical solutions modulo gauge equivalence is an affine subspace of the space of fields modulo equivalence. The local symplectic form is

$$\begin{aligned}\omega &= \langle \delta A \wedge *d\delta A\rangle \\ &= \langle(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \wedge \delta A_\sigma\rangle\, g^{\nu\sigma} g^{\mu\rho}\, \iota(\partial_\rho)|d^n x|.\end{aligned} \tag{3.71}$$

Of course, the detailed analysis involves choosing an origin, so we may as well from the beginning work with the vector space of translations associated to connections. Hence in this paragraph only we take the field to be a *real* 1-form α up to addition of an exact real function. The lagrangian is

$$L = -\frac{1}{2} d\alpha \wedge *d\alpha = -\frac{1}{2}|d\alpha|^2\,|d^n x|, \tag{3.72}$$

and the equation of motion is

$$d * d\alpha = 0. \tag{3.73}$$

As usual, we consider the Fourier transform $\hat{\alpha} = \hat{\alpha}_\mu dx^\mu$ on V^*. The vector space of classical solutions is the first cohomology space of the complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{*d*d} \Omega^1(M). \tag{3.74}$$

On the Fourier transforms we have

$$\begin{aligned} d\hat{f}(k) &= \sqrt{-1}\,\hat{f}(k)k \\ *d*d\hat{\alpha}(k) &= |k|^2\hat{\alpha}(k) - \langle\hat{\alpha}(k), k\rangle k. \end{aligned} \tag{3.75}$$

An easy argument shows that the first cohomology is isomorphic to the collection of Real functions $\{\hat{\alpha}\colon \text{lightcone} \to V^* \otimes \mathbb{C}\}$ on the lightcone which satisfy $\langle\hat{\alpha}(k), k\rangle = 0$ modulo the set of functions $\{\hat{\alpha}(k) = \hat{f}(k)k\}$, where $\hat{f}$ ranges over the complex functions on the lightcone. (The Reality condition is that of the Fourier transform.) The subgroup of $\mathrm{Spin}(V)$ which stabilizes k has reductive part isomorphic to $\mathrm{Spin}(n-2)$, and $\hat{\alpha}$ is a section of a bundle whose fiber at k is the vector representation of $\mathrm{Spin}(n-2)$.

Return now to the abelian gauge field and lagrangian (3.62) To compute the energy-momentum tensor, we first compute the Noether current j_μ of *minus* translation by ∂_μ (see (2.160)). Note that ∂_μ induces the vector field $\hat{\xi}_\mu$ on the space $\mathcal{F}_A$ of fields defined by (see (4.40))

$$\hat{\xi}_\mu A = -\iota(\partial_\mu)F_A. \tag{3.76}$$

Let $\xi_\mu = \partial_\mu + \hat{\xi}_\mu$. Then

$$j_\mu = -\iota(\partial_\mu + \hat{\xi}_\mu)(L+\gamma) \tag{3.77}$$
$$= -\frac{1}{2}\big[\iota(\partial_\mu)F_A \wedge *F_A - F_A \wedge \iota(\partial_\mu) * F_A\big].$$

We find after some computation that

$$\Theta_{\mu\nu} = -\langle F_{\mu\rho}, F_{\nu\sigma}\rangle\, g^{\rho\sigma} + \frac{1}{2}|F_A|^2 g_{\mu\nu}, \tag{3.78}$$

where $\Theta_{\mu\nu}$ is defined in (2.160). Notice that it is symmetric.

We can compute the energy-momentum tensor more directly using the alternative definition (2.176). Couple the lagrangian (3.62) to an arbitrary metric g:

$$L_g = -\frac{1}{2}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle\, g^{\mu\rho} g^{\nu\sigma} \mu_g(x). \tag{3.79}$$

Using (3.17) and differentiating with respect to g^{-1} we find the energy-momentum tensor to be (see (2.177))

$$T_{\mu\nu} = -\langle F_{\mu\rho}, F_{\nu\sigma}\rangle g^{\rho\sigma} + \frac{1}{2}|F_A|^2 g_{\mu\nu}, \tag{3.80}$$

which agrees with (3.78), as it must by Proposition 2.190. In (4.8) we rewrite (3.80) in terms of the electric and magnetic components.

For a 1-form field α the energy-momentum tensor computed by differentiating the metric is again (3.80) (where we interpret $F_{\mu\nu} = \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu$). However, the canonical Noether current j_μ differs from (3.77) since the induced vector field $\hat{\xi}_\mu$ acts with an additional term over (3.76):

$$\hat{\xi}_\mu \alpha = -\operatorname{Lie}(\partial_\mu)\alpha = -\iota(\partial_\mu)d\alpha - d\iota(\partial_\mu)\alpha. \tag{3.81}$$

Then j_μ picks up an extra term

$$-d\iota(\partial_\mu)\alpha \wedge *d\alpha = -d\big(\iota(\partial_\mu)\alpha \cdot *d\alpha\big) + \iota(\partial_\mu)\alpha \cdot d * d\alpha. \tag{3.82}$$

This is written as the sum of an exact term and a term which vanishes on-shell.

The "trace" of T,

$$g^{\mu\nu}T_{\mu\nu} = (\frac{n}{2} - 2)|F_A|^2, \tag{3.83}$$

vanishes when $n = 4$. This says that the lagrangian (3.62) is conformally invariant in $n = 4$ dimensions.

CHAPTER 4
Gauge Theory

§4.1. Classical electromagnetism

As in §3.5 we consider an abelian gauge field on Minkowski space M. To make contact with the usual formulas of electromagnetism we take the gauge group to be the multiplicative group $\mathbb{R}^{>0}$ rather than the circle group $\mathbb{T}$. (In classical physics the electric charge is not quantized, so this makes more physical sense.) Thus the gauge field A is a connection on a principal $\mathbb{R}^{>0}$-bundle. The curvature F_A is a *real* 2-form. The formulas of §3.5 hold, only now (3.61) is replaced by

$$\langle a,b\rangle = ab, \qquad a,b\in\mathbb{R}. \tag{4.1}$$

We work in any dimension n, though of course the electromagnetism of our world is $n=4$. Choose a splitting $M=M^1\times N$ of Minkowski spacetime into time × space. For now N can be any Riemannian manifold. Let $x^0=ct$ be the standard coordinate on M^1. Define the *electric* and *magnetic* fields

$$\begin{aligned} E_A &\in \Omega^1(N)\\ B_A &\in \Omega^2(N)\end{aligned} \tag{4.2}$$

by

$$F_A = B_A - dt\wedge E_A. \tag{4.3}$$

E_A and B_A depend on time t, but neither contains dt:

$$\iota(\partial_0)E_A = \iota(\partial_0)B_A = 0. \tag{4.4}$$

Let $*_N$ be the Hodge star operator for the *positive* definite metric on N. Then if $*$ is the Hodge $*$ operator on $M^1\times N$ (with its usual metric of signature $+--\cdots$), we have

$$*F_A = \frac{1}{c} *_N E_A + c\,dt\wedge *_N B_A. \tag{4.5}$$

So the lagrangian (3.62) is

$$L = \frac{1}{2}\Big(\frac{|E_A|^2}{c^2} - |B_A|^2\Big)\ |d^nx|, \tag{4.6}$$

where the norms of the forms E_A, B_A are computed in the positive definite metric on N. A multiplicative constant is usually inserted in L, depending on the units used; we omit it.[25] The equation of motion (3.69) together with the Bianchi identity $dF_A = 0$ comprise Maxwell's law in empty space:

$$\begin{aligned} dB_A &= 0 & dE_A &= -\frac{\partial B_A}{\partial t} \\ d{*_N}E_A &= 0 & c^2\, d{*_N}B_A &= {*_N}\frac{\partial E_A}{\partial t} \end{aligned} \tag{4.7}$$

The initial components of the energy-momentum tensor (3.80) reduce to

$$\begin{aligned} T_{00} &= \frac{1}{2}\Big(\frac{|E_A|^2}{c^2} + |B_A|^2\Big) \\ -T_{i0} &= \sum_{j=1}^{n-1}(B_A)_{ij}(E_A)_j, \end{aligned} \tag{4.8}$$

the classical expressions for the energy density and the Poynting vector, while the space components T_{ij} comprise the Maxwell stress tensor. (Recall that $-T_{i0}$ integrates to the field momentum.)

We can couple the electromagnetic field to a current $J \in \Omega^{|-1|}(M)$. The lagrangian density is

$$L = -\frac{1}{2}F_A \wedge {*F_A} + J \wedge A. \tag{4.9}$$

The second term is not well-defined in general, since A is a connection, not a 1-form. In fact, if J is a *conserved* current

$$dJ = 0, \tag{4.10}$$

then we can write $J = dK$ and so by Stokes' theorem

$$\int_M J \wedge A = -\int_M K \wedge F_A \tag{4.11}$$

is well-defined, assuming suitable decay at infinity. (We discuss such "topological terms" in Chapter 6.) A typical conserved current is the Poincaré dual of a *closed* curve in M, in which case (4.11) is the holonomy of A around the curve. In coordinates we write

$$J = J^\mu\, \iota(\partial_\mu)|d^n x|, \tag{4.12}$$

where $J^0 = c\rho$ is c times the *charge density* ρ and minus the spatial components $-J^i$ comprise the *current density* j:

$$J = c\rho\, |d^{n-1}x| - c\, dt \wedge j. \tag{4.13}$$

[25]In the "mks" system of units, the constant is written $\epsilon_0 c^2 = 10^7/4\pi$.

We can treat J as an external field, which is fixed, or as a dynamical field which varies. For example, we can take J to be Poincaré dual to the path of a dynamical moving point charge of rest mass m_0 and charge q. In that case we add the kinetic term (1.22) for the point particle to the lagrangian. Then the fields are the particle $x\colon \mathbb{R}^1 \to M$ and the connection A on M. The classical action is

$$S = \int_{T_1}^{T_2} -m_0 c \left|\frac{dx}{d\tau}\right| |d\tau| - q\int_{T_1}^{T_2} x^*A - \int_M \frac{1}{2} F_A \wedge *F_A. \tag{4.14}$$

In these formulas we compute the action for the proper time interval $[T_1, T_2] \subset \mathbb{R}$. The second term should be interpreted as a multiple of the logarithm of the parallel transport of A along a piece of the path x; see §6.1. It is not a real number, but rather lives in a real torsor (depending on the endpoints). Note that the path is oriented.

The equation of motion for the lagrangian (4.9) is

$$d * F_A = J. \tag{4.15}$$

Rewriting in terms of E_A, B_A, ρ, j we have

$$\begin{aligned} d{*_N} E_A &= c^2\rho\, |d^{n-1}x| \\ c^2 d{*_N} B_A &= {*_N}\frac{\partial E_A}{\partial t} + c^2 j, \end{aligned} \tag{4.16}$$

which are two of Maxwell's equations. (The top two equations in (4.7) remain unchanged.)

The action (4.14) is invariant under reparametrizations of the path x. We compute the equation of motion ignoring boundary terms. The equation for A is a special case of (4.16) where the current J is Poincaré dual to q times the path x. The variation of the first term with respect to the path x—after integration by parts—leads to a contribution to the equation of motion of (see (1.25))

$$-\frac{\partial}{\partial t}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right), \tag{4.17}$$

where v is the velocity measured in our fixed system of coordinates. To vary the second term with respect to x, we use the formula that the variation of the holonomy is the integral of the curvature evaluated on the variation of the path. Thus the contribution of this term to the equation of motion is the vector associated via the metric to the 1-form

$$q\iota(\partial/\partial t + v)F_A = qE_A - q\iota(v)B_A. \tag{4.18}$$

(We omit the dt component of the 1-form.) So the equation for x is

$$\frac{\partial}{\partial t}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = q(E_A - \iota(v)B_A)^*, \tag{4.19}$$

'$*$' denoting the dual vector. In four dimensions this is the Lorentz force law.

§4.2. Principal bundles and connections

In this section we establish some notation and review standard notions. Let M be an arbitrary (super)manifold. All vector fields are even.

Let $P \to M$ be a principal bundle with structure group a Lie group G, often called the *gauge group*. If M is a supermanifold, then so is P. We take G to act on the *right*. Elements $\zeta \in \mathfrak{g}$ in the Lie algebra of G induce vertical vector fields $\hat{\zeta}$ on P, and $\widehat{[\zeta_1, \zeta_2]} = [\hat{\zeta}_1, \hat{\zeta}_2]$.

Suppose V is a space with a *left* G action. In many interesting cases V is a linear space and the action is linear, but it need not be. There is an associated bundle $V^P \to M$ whose sections are equivariant maps $f\colon P \to V$. So f satisfies

$$f(pg) = g^{-1} \cdot f(p), \qquad p \in P, \quad g \in G, \tag{4.20}$$

where '$\cdot$' denotes the action of G on V. Write $g = \exp(t\zeta)$ and differentiate with respect to t to find

$$\hat{\zeta} f = -\zeta \cdot f, \qquad \zeta \in \mathfrak{g}, \tag{4.21}$$

where now '$\cdot$' denotes the infinitesimal action of $\mathfrak{g}$ on V.

The bundle associated to the adjoint action of G on $\mathfrak{g}$ is the adjoint bundle $\mathfrak{g}^P = \operatorname{ad} P = P \times_G \mathfrak{g}$. A section of the adjoint bundle is then a map $\epsilon\colon P \to \mathfrak{g}$ with $\epsilon(pg) = \operatorname{Ad}_{g^{-1}} \epsilon(p)$. Using the infinitesimal $\mathfrak{g}$ action on P we can identify ϵ with a G-invariant vertical vector field $\hat{\epsilon}$ on P. Namely, define $\hat{\epsilon}_p$ to be the vertical vector corresponding to the infinitesimal action by $\epsilon(p)$. Then by (4.21)

$$\hat{\epsilon}_1(\epsilon_2) = -[\epsilon_1, \epsilon_2] \tag{4.22}$$

and the corresponding vertical vector field is $[\hat{\epsilon}_1, \hat{\epsilon}_2]$. The fiber of $\operatorname{ad} P$ at $m \in M$ is the Lie algebra of infinitesimal automorphisms of the fiber of P at m. It acts on the *left*, which explains the minus in (4.22). Equation (4.22) is a special case of the following: If $f\colon P \to V$ is a section of an associated bundle, then

$$\hat{\epsilon} f = -\epsilon \cdot f. \tag{4.23}$$

(Compare (4.21).)

There is also an adjoint bundle of groups $P \times_G G \to M$. Sections of this bundle act as automorphisms of P, often called *gauge transformations*.

A *connection* on P is a G-invariant distribution on P which projects isomorphically onto TM. Thus a vector field η on M has a horizontal lift $\tilde{\eta}$ which is a G-invariant vector field on P. Equivalently, a connection is encoded in a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$ which satisfies

$$\begin{aligned} \iota(\hat{\zeta})A &= \zeta, & \zeta &\in \mathfrak{g}; \\ R_g^* A &= \operatorname{Ad}_{g^{-1}}(A), & g &\in G. \end{aligned} \tag{4.24}$$

Here $R_g\colon P \to P$ is the (right) action of g on P. The *curvature* $F_A \in \Omega^2_P(\mathfrak{g})$ is

$$F_A = dA + \frac{1}{2}[A \wedge A]. \tag{4.25}$$

It follows from (4.24) that $F_A \in \Omega^2_M(\operatorname{ad} P)$ is a 2-form on the base M with values in the adjoint bundle. If η_1, η_2 are vector fields on M, then

$$-\Big([\tilde\eta_1, \tilde\eta_2] - \widetilde{[\eta_1, \eta_2]}\Big) \tag{4.26}$$

is a G-invariant vertical vector field on P, and by (4.25) it corresponds to the section

$$\iota(\eta_2)\iota(\eta_1)F_A = -\iota([\tilde\eta_1, \tilde\eta_2])A \tag{4.27}$$

of the adjoint bundle.

Let $\epsilon\colon P \to \mathfrak{g}$ be a section of the adjoint bundle and $\hat\epsilon$ the associated G-invariant vertical vector field. Then for any vector field η on M,

$$[\tilde\eta, \hat\epsilon] = \widehat{\tilde\eta\epsilon}. \tag{4.28}$$

A connection A on P induces a covariant derivative ∇ on any associated bundle $V^P = P \times_G V$. For a vector field η on M, and a section $f\colon P \to V$ of V^P,

$$\nabla_\eta f = \tilde\eta f. \tag{4.29}$$

The covariant derivative may be viewed as an operator $d_A\colon \Omega^0(V^P) \to \Omega^1(V^P)$, defined by

$$d_A f = (d + A)f. \tag{4.30}$$

If G is a linear algebraic group, then we have the Tannakian statement: A connection on P is equivalent to a system of connections on all associated vector bundles which is compatible with the tensor product of representations.

Automorphisms of P—gauge transformations—act on connections by pullback. An automorphism $\varphi\colon P \to P$ may be represented by an equivariant map $g\colon P \to G$, defined by

$$\varphi(p) = p \cdot g(p). \tag{4.31}$$

Then if $A \in \Omega^1_P(\mathfrak{g})$ is a connection, we have

$$\varphi^* A = \operatorname{Ad}_{g^{-1}} A + g^{-1} dg, \tag{4.32}$$

where $g^{-1}dg$ is the pullback by g of the Maurer-Cartan form on G.

The *Bianchi identity* is simply the Jacobi identity for horizontal vector fields on P:

$$[[\nabla_{\eta_1}, \nabla_{\eta_2}], \nabla_{\eta_3}] + [[\nabla_{\eta_2}, \nabla_{\eta_3}], \nabla_{\eta_1}] + [[\nabla_{\eta_3}, \nabla_{\eta_1}], \nabla_{\eta_2}] = 0. \tag{4.33}$$

Equivalently,

$$d_A F_A = 0, \tag{4.34}$$

where $d_A\colon \Omega^2_M(\operatorname{ad} P) \to \Omega^3_M(\operatorname{ad} P)$ is the extension of the differential d using the connection A.

The set of all connections on P is an affine space $\mathcal{A}_P$. One sees this easily from (4.24), which are affine equations. The associated vector space of translations is $\Omega^1_M(\operatorname{ad} P)$. The group $\mathcal{G}_P$ of gauge transformations acts on $\mathcal{A}_P$ on the left using pullback by the inverse (or pushforward of the associated horizontal distribution).

Let $\operatorname{Aut} P$ denote the group of all diffeomorphisms $\varphi\colon P \to P$ which commute with the right G action. Such a φ induces a diffeomorphism of M, and there is an exact sequence

$$1 \longrightarrow \mathcal{G}_P \longrightarrow \operatorname{Aut} P \longrightarrow \operatorname{Diff} M. \tag{4.35}$$

The last map may not be onto; the image is the subgroup of Diff M which preserves the topological type of $P \to M$. Also, the sequence (4.35) is not usually split. Infinitesimally—at the level of Lie algebras—we do have a short exact sequence

$$0 \longrightarrow \operatorname{Lie}(\mathcal{G}_P) \longrightarrow \operatorname{Lie}(\operatorname{Aut} P) \longrightarrow \mathcal{X}(M) \longrightarrow 0. \tag{4.36}$$

Furthermore, a connection A gives a splitting as vector spaces (but not as Lie algebras): to a vector field $\eta \in \mathcal{X}(M)$ we attach its horizontal lift $\tilde{\eta}$.

It is often more geometric (and more physical) not to fix a particular bundle P. Then instead of a single affine space of connections, we study the *category* $\mathcal{C}_M(G)$ of all connections on all principal G-bundles over M. A morphism in $\mathcal{C}_M(G)$ is an isomorphism of principal bundles which preserves the given connections. The set of equivalence classes $\overline{\mathcal{C}_M(G)}$ may be identified as a disjoint union of spaces $\mathcal{A}_P/\mathcal{G}_P$, where P runs over a set of representatives of topological types of G-bundles on M. In a lagrangian field theory including connections, the category $\mathcal{C}_M(G)$ is part of the "space" of fields.

Fix a bundle P. On the product bundle

$$\mathbf{P} = \mathcal{A}_P \times P \longrightarrow \mathcal{A}_P \times M \tag{4.37}$$

there is a universal connection $\mathbf{A}$. Its restriction to $\{A\} \times P$ is A, and its restriction to $\mathcal{A}_P \times \{p\}$ is zero. It is straightforward to compute the curvature of $\mathbf{A}$ at (A, m): evaluated on $\eta_1, \eta_2 \in T_mM$ and $\alpha_1, \alpha_2 \in \Omega^1_M(\operatorname{ad} P)$ we obtain

$$\begin{aligned} \iota(\eta_2)\iota(\eta_1)F_{\mathbf{A}} &= \iota(\eta_2)\iota(\eta_1)F_A \\ \iota(\eta)\iota(\alpha)F_{\mathbf{A}} &= \iota(\eta)\alpha \\ \iota(\alpha_2)\iota(\alpha_1)F_{\mathbf{A}} &= 0. \end{aligned} \tag{4.38}$$

These are equations for elements of the fiber $(\operatorname{ad} P)_m$.

The group $\operatorname{Aut} P$ acts on $\mathbf{P}$ (on the left) by the product of its actions on $\mathcal{A}_P$ and P. The universal connection $\mathbf{A}$ is invariant under this action. Lagrangians in gauge theory are gauge invariant functions (partial densities) on $\mathcal{A}_P \times M$ which are computed from $\mathbf{A}$, and the $(\operatorname{Aut} P)$-invariance of $\mathbf{A}$ implies that such lagrangians are invariant under subgroups of Diff M.

From (4.35) we see that Diff M acts on the space $\overline{\mathcal{C}_M(G)}$ of equivalence classes of connections. Also, the discussion after (4.36) shows that there is a canonical lift of $\mathcal{X}(M)$ to vector fields on the universal bundle $\mathbf{P}$. First, if $A \in \mathcal{A}_P$ is a connection and η a vector field on M, then η acts on A using the horizontal lift $\tilde{\eta}$:

$$\operatorname{Lie}(\tilde{\eta})A = \iota(\tilde{\eta})dA = \iota(\eta)F_A. \tag{4.39}$$

Since diffeomorphisms of M act on fields by pullback, the induced vector field $\hat{\xi}_\eta$ on $\mathcal{A}_P$ is

$$\hat{\xi}_\eta = -\iota(\eta)F_A \qquad \text{at } A \in \mathcal{A}_P. \tag{4.40}$$

Thus

$$\xi_\eta = -\iota(\bar{\eta})F_A + \eta \tag{4.41}$$

is the vector field on $\mathcal{A}_P \times M$ induced by η. Its horizontal lift $\tilde{\xi}_\eta$ to $\mathbf{P}$ (using the universal connection $\mathbf{A}$) is the desired induced vector field on $\mathbf{P}$. For vector fields η_1, η_2 on M, we compute

$$[\hat{\xi}_{\eta_1}, \hat{\xi}_{\eta_2}] = \hat{\xi}_{[\eta_1,\eta_2]} - d_A\big(F_A(\eta_1,\eta_2)\big) \tag{4.42}$$

$$[\tilde{\eta}_1, \tilde{\xi}_{\eta_2}] = \widetilde{[\eta_1,\eta_2]} - F_A(\eta_1,\eta_2) \tag{4.43}$$

Equation (4.42) asserts that the map $\eta \mapsto \hat{\xi}_\eta$ is a homomorphism[26] up to infinitesimal gauge transformations, as is evident from the sequence (4.36). Equation (4.43) implies that if η_1 and η_2 commute, then the symmetry ξ_{η_2} commutes with η_1 up to a gauge transformation. This is used in superspace formulations of supersymmetric gauge theories, where the symmetry generated by the vector fields Q_a on superspace commute up to a gauge transformation with the action of vector fields D_a used to build lagrangians.

§4.3. Pure Yang-Mills theory

The data which define a pure gauge theory are:

$$\begin{array}{ll} G & \text{Lie group with Lie algebra } \mathfrak{g} \\ \langle \cdot, \cdot \rangle & \text{bi-invariant inner product on } \mathfrak{g} \end{array} \tag{4.44}$$

We work on Minkowski n-space M. The field in the theory is a connection A on a principal G-bundle over M. As explained in §4.2, the collection of fields is best regarded as a category. We write lagrangians which are invariant under gauge transformations.

Recall (4.25) that for a nonabelian group the curvature F_A is a nonlinear function of the connection A. So nonabelian gauge theories are not free—the lagrangian is not quadratic in the field A. For perturbation theory one introduces a coupling constant (or several if G is a product of simpler groups). We simply absorb these constants in the inner product $\langle \cdot, \cdot \rangle$.

The pure Yang-Mills lagrangian is given by the same formulas as (3.62):

$$L = -\frac{1}{2}\langle F_A \wedge *F_A \rangle. \tag{4.45}$$

[26] When a Lie algebra acts on a manifold on the left we expect an *anti*homomorphism. But if we view the space of vector fields as the Lie algebra of the group of diffeomorphisms, then the induced bracket is *minus* the usual Lie bracket of vector fields.

The manipulations following (3.62) are valid for arbitrary gauge groups, except that we must replace d by d_A in the nonabelian case. Thus the equation of motion is

$$d_A * F_A = 0, \tag{4.46}$$

and the variational 1-form and local symplectic form are

$$\begin{aligned} \gamma &= -\langle \delta A \wedge *F_A \rangle, \\ \omega &= \langle \delta A \wedge *d_A \delta A \rangle. \end{aligned} \tag{4.47}$$

Equation (4.46) is called the *Yang-Mills equation*. Of course, the connection A always satisfies the Bianchi identity (4.34).

The discussion of the energy-momentum tensor carries over without change, and formula (3.80) holds:

$$T_{\mu\nu} = -\langle F_{\mu\rho}, F_{\nu\sigma} \rangle g^{\rho\sigma} + \frac{1}{2}|F|^2 g_{\mu\nu}. \tag{4.48}$$

Pure Yang-Mills theory is conformally invariant in 4 dimensions (see (3.83)).

The energy density is

$$T_{00} = \frac{1}{2}\sum_i |F_{0i}|^2 + \frac{1}{2}\sum_{i<j} |F_{ij}|^2, \tag{4.49}$$

where $i, j = 1, \dots, n-1$ run over spatial indices. Therefore, the field configurations of minimal energy are flat, and the moduli space of vacua on Minkowski space is a point—the equivalence class of the trivial connection.

Fundamental lagrangians in physical theories are constrained by renormalizability. With that criterion there are a few terms one can add to the Yang-Mills lagrangian (4.45) in dimensions 2, 3, and 4. These terms are topological in nature. We often refer to them as "θ-terms".

In $n = 2$ dimensions suppose

$$\langle\!\langle \cdot \rangle\!\rangle : \mathfrak{g} \longrightarrow \mathbb{R} \tag{4.50}$$

is a trace on $\mathfrak{g}$, i.e., a linear map for which $\langle\!\langle [a, b] \rangle\!\rangle = 0$ for all $a, b \in \mathfrak{g}$. Then an additional possible term in the lagrangian is

$$L_2 = \frac{\theta}{2\pi}\langle\!\langle F_A \rangle\!\rangle. \tag{4.51}$$

We include the constant $\theta \in \mathbb{R}$ since this is the form in which this θ-term is usually written. There is a constant θ for each independent trace on $\mathfrak{g}$. On a compact 2-manifold M (or on the space of fields with finite action on Minkowski space) the integral of (4.51) is locally constant on the space of fields. Thus L_2 does not affect the equations of motion. It does, however, enter into the formula for the variational 1-form. Namely, the variational 1-form for pure Yang-Mills plus L_2 is

$$\gamma = -\langle \delta A \wedge *F_A \rangle + \frac{\theta}{2\pi}\langle\!\langle \delta A \rangle\!\rangle. \tag{4.52}$$

The local symplectic form and energy-momentum tensor are unchanged.

In $n = 3$ dimensions there is a Chern-Simons term; we discuss it in §6.2. It is different than the θ-terms (4.51) and (4.54): the Chern-Simons term is not locally constant on the space of fields, and so does have a dynamical effect.

In $n = 4$ dimensions there is a θ-term associated to an invariant bilinear form

$$\langle\langle\cdot\rangle\rangle\colon \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathbb{R}. \tag{4.53}$$

(The form need not be an inner product; in particular, it need not be identical to (4.44).) The θ-term is

$$L_4 = \frac{\theta}{16\pi^2}\langle\langle F_A \wedge F_A\rangle\rangle. \tag{4.54}$$

As with L_2, the integral of L_4 is locally constant on the space of fields. The modified variational 1-form is

$$\gamma = -\langle \delta A \wedge *F_A\rangle + \frac{\theta}{8\pi^2}\langle\langle \delta A \wedge F_A\rangle\rangle. \tag{4.55}$$

§4.4. Electric and magnetic charge

To begin, let spacetime be a product $M = M^1 \times N$. For a classical electromagnetic field A, as in §4.1, the electric charge "enclosed" by a closed codimension 1 submanifold $\Sigma \subset N$ is

$$\mathcal{E}_A(\Sigma) = \int_\Sigma *F_A = \frac{1}{c}\int_\Sigma *_N E_A, \tag{4.56}$$

and the magnetic charge enclosed by a closed 2-dimensional submanifold $\Sigma \subset N$ is

$$\mathcal{B}_A(\Sigma) = \int_\Sigma F_A = \int_\Sigma B_A. \tag{4.57}$$

In each case Σ need not be a boundary, hence the quotes around 'enclosed'. We use the same terminology for a general spacetime M, which may also be Euclidean—then only the first equations in (4.56) and (4.57) make sense. For a gauge field with general gauge group G, electric and magnetic charge are associated to a trace

$$\tau\colon \mathfrak{g} \longrightarrow \mathbb{R} \tag{4.58}$$

on the Lie algebra $\mathfrak{g}$:

$$\begin{aligned} \mathcal{E}^\tau_A(\Sigma) &= \int_\Sigma *\tau(F_A),\\ \mathcal{B}^\tau_A(\Sigma) &= \int_\Sigma \tau(F_A). \end{aligned} \tag{4.59}$$

Magnetic charge is an example of a *topological* charge; $\mathcal{B}^\tau_A(\Sigma)$ is a topological invariant which depends only on the topological type of the principal bundle underlying A. Hence it is a locally constant function on the space of fields—in particular, the space of classical solutions. It is *central* in the sense that it Poisson commutes with any other charge.

Electric charge is not generally conserved in a finite region. For example, in classical electromagnetism it follows from (4.16) that the time derivative of the electric charge enclosed by Σ is proportional to the integral of the spatial current j through Σ. Also, if $\Sigma = \partial\Omega$ then the electric charge enclosed by Σ is proportional to the integral of the charge density ρ over Ω. This is the usual statement of Gauss' law. We generalize to field theory, where the current J is computed from matter fields, as follows. We work with a gauge field A for a general gauge group G. Elements ϵ in the center of $\mathfrak{g}$ give infinitesimal global gauge transformations ϵ which act trivially on the space of connections: $d_A\epsilon = 0$ for all A. From §2.8 it follows that if L'_A is a gauge invariant "matter lagrangian" in which A is a fixed background field and which only depends on the 1-jet of the matter fields ϕ, then the Noether current j_ϵ for ϵ in the theory $\mathcal{L}'_A$ is computed by differentiating L'_A with respect to A (see Proposition 2.149). We now consider A as a dynamical variable by adding the Yang-Mills lagrangian to L'_A. We write the signs for a Lorentz manifold.

Proposition 4.60. *Let $L'_A(\phi)$ be the matter lagrangian described above, and set*

$$L(A,\phi) = -\frac{1}{2}\langle F_A \wedge *F_A\rangle + L'_A(\phi). \tag{4.61}$$

Suppose Ω is a spacelike $(n-1)$-dimensional region with smooth boundary Σ. Then if ϵ is the global gauge transformation corresponding to a central element $\epsilon \in \mathfrak{g}$, on-shell we have

$$\mathcal{E}_A^{\tau(\epsilon)}(\Sigma) = \int_\Omega j_\epsilon, \tag{4.62}$$

where $\tau(\epsilon) \in \mathfrak{g}^*$ is the trace $\zeta \mapsto \langle \epsilon, \zeta\rangle$ on $\mathfrak{g}$ and j_ϵ is the canonical Noether current for ϵ in the theory $\mathcal{L}'_A$.

Remark 4.63. In Minkowski space we often take Ω to be a spacelike hyperplane—the limit of large balls—in which case Σ is the sphere at "spatial infinity"—the limit of large spheres. Then the limit of $\mathcal{E}_A^{\tau(\epsilon)}(\Sigma)$ is the total electric charge in the system.

Proof. There is a canonical variational 1-form γ since the lagrangian only depends on the 1-jet of the fields. Let $\mathcal{L} = L + \gamma = \mathcal{L}_{YM} + \mathcal{L}'_A$ be the total lagrangian. The equation of motion is

$$(D\mathcal{L}_{YM})^{1,|0|} + (D\mathcal{L}'_A)^{1,|0|} = 0. \tag{4.64}$$

Evaluate this on $\alpha\epsilon$, where $\alpha \in \Omega^1_M$ is an arbitrary 1-form and $\alpha\epsilon$ is viewed as a tangent to the space of connections. By manipulations as in §3.5 we have

$$\iota(\alpha\epsilon)(D\mathcal{L}_{YM})^{1,|0|} = -\alpha \wedge \langle \epsilon, d_A * F_A\rangle. \tag{4.65}$$

By Proposition 2.149 the second term is $j_\epsilon(\alpha)$. Since α is arbitrary,

$$j_\epsilon = \langle \epsilon, d_A * F_A\rangle = d\langle \epsilon, *F_A\rangle. \tag{4.66}$$

Equation (4.62) is the integral version of (4.66).

Formula (4.62) is not modified if we add a θ-term (4.51) or (4.54) to (4.61). (This has the effect of adding a multiple of F_A to $*F_A$ in (4.65), but that extra piece vanishes by Bianchi.) However, formula (4.62) is modified in the theory of monopoles; see [II-**Dynamics of QFT**, §9.6].

CHAPTER 5
σ-Models and Coupled Gauge Theories

§5.1. Nonlinear σ-models

We begin with some preliminary remarks. First, if M is a manifold and $E \to M$ a real or complex vector bundle with connection ∇, then we form the twisted de Rham complex

$$\Omega^0_M(E) \xrightarrow{d_\nabla = \nabla} \Omega^1_M(E) \xrightarrow{d_\nabla} \Omega^2_M(E) \xrightarrow{d_\nabla} \cdots . \tag{5.1}$$

It is not a complex in general, but rather

$$d^2_\nabla = R_\nabla, \tag{5.2}$$

where R_∇ is the curvature of ∇. Next, if $\phi\colon M \to X$ is a map between manifolds, then $d\phi \in \Omega^1_M(\phi^* TX)$. If now TX has a connection ∇, then

$$d_\nabla d\phi = \phi^* T_\nabla, \tag{5.3}$$

where T_∇ is the torsion of the connection ∇. We will apply this to the Levi-Civita connection of a Riemannian manifold; its torsion vanishes.

The data which define a nonlinear σ-model are:

$$\begin{array}{ll} X & \text{Riemannian manifold} \\ V\colon X \longrightarrow \mathbb{R} & \text{potential energy function} \end{array} \tag{5.4}$$

The field in the theory is a map $\phi\colon M \to X$; the space of fields is $\mathcal{F} = \mathrm{Map}(M, X)$. Diffeomorphisms of X act on $\mathcal{F}$ by composition, and diffeomorphisms of M act on $\mathcal{F}$ using composition by the *inverse*. In each case the subgroup of isometries preserves the lagrangian (5.6) below. There is a canonical evaluation map

$$e\colon \mathcal{F} \times M \longrightarrow X \tag{5.5}$$

which is invariant under the product action of $\mathrm{Diff}\, M$ on $\mathcal{F} \times M$. Infinitesimal symmetries act on derivatives of the field ϕ via the Levi-Civita covariant derivative ∇. This introduces curvature terms into the bracket of infinitesimal symmetries in general. Since e is invariant under $\mathrm{Diff}\, M$, and since the lagrangian (5.6) below can be written in terms of e, it is easy to check that isometries of M are manifest symmetries (as are isometries of X).

The σ-model lagrangian with potential is

$$\begin{aligned} L &= \left\{\frac{1}{2}|d\phi|^2 - \phi^*V\right\}|d^nx| \\ &= \frac{1}{2}\langle d\phi \wedge *d\phi\rangle - \phi^*V\,|d^nx|. \end{aligned} \tag{5.6}$$

Note that $|d\phi|$ is computed using both the metric on M and the metric on X. We carry out the familiar analysis:

$$\begin{aligned} \delta L &= \langle \delta_\nabla d\phi \wedge *d\phi\rangle - \langle \delta\phi, \phi^*\operatorname{grad} V\rangle\,|d^nx| \\ &= -d\{\,\langle \delta\phi \wedge *d\phi\rangle\,\} - \langle \delta\phi \wedge d_\nabla * d\phi\rangle - \langle \delta\phi, \phi^*\operatorname{grad} V\rangle\,|d^nx|. \end{aligned} \tag{5.7}$$

Hence the equation of motion is

$$\Box_\nabla \phi = -\phi^*\operatorname{grad} V, \tag{5.8}$$

where $\Box_\nabla$ is the covariant extension of the wave operator (3.2). Despite this notation one shouldn't lose sight of Newton's law, which is the special case $n = 1$ of equation (5.8). The variational 1-form and local symplectic form are

$$\begin{aligned} \gamma &= \langle \delta\phi \wedge *d\phi\rangle, \\ \omega &= \langle *d_\nabla \delta\phi \wedge \delta\phi\rangle. \end{aligned} \tag{5.9}$$

It is easy to couple this theory to an arbitrary metric g on M:

$$L_g = \left(\frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \phi^*V\right)\mu_g(x). \tag{5.10}$$

The computation leading to (3.19) is essentially unchanged, and so the energy-momentum tensor is

$$T_g = \langle d\phi \cdot d\phi\rangle \mu_g - g\,L, \tag{5.11}$$

or more explicitly (see (2.177))

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi + \left(-\frac{1}{2}|d\phi|^2 + \phi^*V\right) g_{\mu\nu}. \tag{5.12}$$

On Minkowski space the moduli space of vacua—field configurations which minimize the energy density T_{00}—is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0) \tag{5.13}$$

assuming that 0 is the minimum value of the potential energy V.

If ξ is a Killing vector field (infinitesimal isometry) on X, then the associated Noether current is

$$j = \langle \xi, *d\phi\rangle. \tag{5.14}$$

There are also topological terms one can add to a pure σ-model, usually called *Wess-Zumino terms*. We describe them in Chapter 6.

§5.2. Gauge theory with bosonic matter

The theory in this section is the most general bosonic theory without gravity[27] (though we do not include all possible terms—e.g. topological terms—in the lagrangian). We can also describe it as a gauged nonlinear σ-model, or as a gauge theory with bosonic matter. The data which defines the theory are:

(5.15)

G	Lie group with Lie algebra $\mathfrak{g}$
$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on $\mathfrak{g}$
X	Riemannian manifold on which G acts by isometries
$V: X \longrightarrow \mathbb{R}$	potential function invariant under G

The fields are

(5.16)

A	connection on some principal G-bundle $P \longrightarrow M$
ϕ	section of the associated bundle $P \times_G X \longrightarrow M$

It is often convenient to view ϕ as an equivariant map $\phi\colon P \to X$. The lagrangian combines (4.45) and (5.6):

$$L = \left(-\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^*V\right)|d^nx|. \tag{5.17}$$

There is a new term

$$\langle \delta A \cdot \phi, *d_A\phi\rangle \tag{5.18}$$

in δL from the coupling of A and ϕ, so a new term in the equations of motion. The variational 1-form is

$$\gamma = \langle \delta\phi \wedge *d_A\phi\rangle - \langle \delta A \wedge *F_A\rangle. \tag{5.19}$$

The energy-momentum tensor is

$$T_{\mu\nu} = (\partial_A)_\mu\phi\,(\partial_A)_\nu\phi - \langle F_{\mu\rho}, F_{\nu\sigma}\rangle\, g^{\rho\sigma} + \left(\frac{1}{2}|F_A|^2 - \frac{1}{2}|d_A\phi|^2 + \phi^*V\right) g_{\mu\nu} \tag{5.20}$$

The moduli space of vacua is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)\,/\,G \tag{5.21}$$

assuming that 0 is the minimum value of the potential energy V.

In §2.8 we described a simple example of "gauging a symmetry".

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[27] By this we mean a theory of scalar and gauge fields only. There are also models with p-form fields for $p \geq 2$, for example. The lagrangian here covers most *fundamental* (vs. *effective*) lagrangians without gravity.

CHAPTER 6
Topological Terms

We have already introduced "θ-terms" in gauge theory in §4.3. These are a sort of topological term related to *primary* topological invariants, in this case characteristic classes of principal bundles. Corresponding terms occur in a σ-model as well. Namely, if $\phi\colon M \to X$ is a field in a σ-model defined on an oriented spacetime M^n, and $\omega \in \Omega^n(X)$ is a closed differential form, then we can insert a term

$$-c\,\phi^*\omega \tag{6.1}$$

into a lagrangian. (The minus sign indicates that (6.1) is a contribution to the potential energy.) Here $c \in \mathbb{R}$ is a constant; if ω has periods which are $2\pi\hbar$ times integers, then only $c \pmod{\mathbb{Z}}$ enters into the quantum theory. In this chapter we consider a different type of topological term which is related to *secondary* topological invariants. For example, in a gauge theory the holonomy of a connection is a secondary invariant associated to a first Chern class; in three dimensions the Chern-Simons invariant is associated to a four-dimensional characteristic class. In σ-models we meet *Wess-Zumino-Witten terms* which are secondary invariants associated to cohomology classes in the target space X.

To fit such terms into the general theory of Chapter 2, we need local differential-geometric objects which integrate to these secondary invariants. We indicate an extension of ordinary calculus to include these objects briefly, though a systematic development of foundations for this extension is lacking. In the first few sections we focus instead on examples.

For both θ-terms and topological terms we need to work over an oriented spacetime. The orientation allows us to pass from differential forms to densities. Note that neither θ-terms nor topological terms depend on a metric on spacetime.

§6.1. Gauge theory

The simplest example of a topological term was already discussed at the end of §1.1 and in §4.1. Namely, consider a theory on a spacetime M^n which includes a connection A for the group $\mathbb{R}^{>0}$. Suppose $x\colon \mathbb{R} \to M$ is a parametrized path which may be fixed or variable. Then we can introduce a term

$$-q\,x^*A \tag{6.2}$$

into the lagrangian of the theory, where $q \in \mathbb{R}$ is a constant. Note that (6.2) is meant to be integrated over $\mathbb{R}$, not over spacetime M. It is unchanged by orientation-preserving diffeomorphisms of $\mathbb{R}$; in other words, it only depends on the image of x viewed as an oriented submanifold of M.

Now $-qx^*A$ is different from other terms in lagrangians we have seen so far. A connection A on an $\mathbb{R}^{>0}$-bundle $P \to M$ is an element of $\Omega^1(P)$, and x^*A a 1-form on the pullback x^*P. It is *not* a 1-form on the base $\mathbb{R}$. To compute the action we can fix a trivialization of x^*P. Taking into account the dependence on this trivialization, the action over an interval $[T_1, T_2] \subset \mathbb{R}$ is a homomorphism of $\mathbb{R}$-torsors, the $\mathbb{R}$-torsors being q times the fibers of x^*P at T_1 and T_2. In the lagrangian approach to the quantum theory it is not the action S, but rather the exponential $\exp(\sqrt{-1}\,S/\hbar)$ which enters. Then this term in the action is parallel transport in the circle bundle associated to P via the homomorphism

$$\begin{aligned} \mathbb{R}^{>0} &\longrightarrow \mathbb{T} \\ x &\longmapsto (\sqrt{-1})^{qx/\hbar} \end{aligned} \tag{6.3}$$

For usual kinetic and potential terms, the exponentiated action is an element of $\mathbb{T}$. This parallel transport is instead a homomorphism of circle torsors.

The equation of motion (Lorentz force law) arising from (6.2) in classical electromagnetism is discussed at the end of §4.1.

In the quantum theory of electromagnetism the electric charge q is quantized in suitable units. It makes sense from the beginning to regard the gauge field A as a connection in a $\mathbb{T}$-bundle (as opposed to an $\mathbb{R}^{>0}$-bundle). More generally, in quantum Yang-Mills theories the gauge field is a connection for a *compact* gauge group G. Then it makes sense to write a term $x^*\tau(A)$ in the lagrangian, where the trace $\tau\colon \mathfrak{g} \to \mathbb{R}$ is $1/\sqrt{-1}$ times the differential of an abelian character $G \to \mathbb{T}$. This term in the exponentiated action is again interpreted as a homomorphism of circle torsors.

In three-dimensional gauge theory there is a topological term due to Chern and Simons. Let M^3 be a 3-dimensional oriented spacetime and consider a theory with gauge field A. If G is the gauge group, and $\langle\cdot,\cdot\rangle$ an invariant inner product on the Lie algebra $\mathfrak{g}$, then the *Chern-Simons term* is

$$L_{CS} = \langle A \wedge F_A\rangle - \frac{1}{6}\langle A \wedge [A \wedge A]\rangle. \tag{6.4}$$

If A is a connection on P, then this term is a 3-form on the total space of P. Assume for convenience that $M^3 = M^1 \times \Sigma^2$ is the product of time M^1 and a space Σ^2 which is a closed oriented surface. Then once again the exponentiated action on $[T_1, T_2] \times \Sigma$ is best interpreted as a homomorphism between $\mathbb{T}$-torsors $\mathcal{T}_1$ and $\mathcal{T}_2$ defined from (6.4) at $\{T_1\} \times \Sigma$ and $\{T_2\} \times \Sigma$. (For this we need $\langle\cdot,\cdot\rangle$ to lie in a distinguished lattice of inner products.) For trivializable bundles P the torsors are trivialized by a choice of trivialization of P, and may be constructed by such trivializations. We omit the details. (For nontrivializable bundles we need a refinement of the inner product $\langle\cdot,\cdot\rangle$ to an element of $H^4(BG;\mathbb{Z})$ in order to define the torsors and the exponentiated action.)

Locally we can break gauge invariance and fix a trivialization of the bundle which carries the connection A. Then (6.4) pulls down to a local 3-form on M. We compute

$$\delta L_{CS} + d\gamma = \langle \delta A \wedge 2F_A\rangle, \tag{6.5}$$

where

$$\gamma = \langle A \wedge \delta A \rangle. \tag{6.6}$$

So for pure Chern-Simons theory the equation of motion is

$$F_A = 0. \tag{6.7}$$

For the particular spacetime $M^3 = M^1 \times \Sigma^2$, the space of solutions $\overline{\mathcal{M}}$ up to gauge equivalence is the moduli space of flat connections on Σ. The local symplectic form

$$\omega = \langle \delta A \wedge \delta A \rangle \tag{6.8}$$

integrates on Σ to (twice) the usual symplectic form on $\overline{\mathcal{M}}$.

The equations of motion (6.7) and the local symplectic form (6.8) are gauge invariant, whereas the variational 1-form (6.6) is not. This suggests that a Noether charge is no longer a well-defined real number. Indeed, if ϵ is a central element of $\mathfrak{g}$, interpreted as a constant gauge transformation, then the associated Noether current from (6.6) is formally

$$d\langle \epsilon \wedge A \rangle. \tag{6.9}$$

The integral over a region $\Omega \in \Sigma$ is minus the logarithm of the holonomy around $\partial\Sigma$ of the bundle associated to the exponential of the trace $\langle \epsilon, \cdot \rangle$ on $\mathfrak{g}$. This logarithm is well-defined only up to integer shifts. (If the Chern-Simons term is added to the lagrangian of a standard 3-dimensional theory, as in (4.61), then this term contributes to the electric charge (4.62).)

§6.2. Wess-Zumino-Witten terms

Consider a σ-model with field $\phi\colon M \to X$, where the spacetime M is an oriented n-manifold. Under the simplifying hypothesis $H_{n-1}(X) = H_n(X) = 0$, we construct a term in an action associated to a closed form $\Omega \in \Omega^{n+1}(X)$ with periods in $2\pi\hbar\mathbb{Z}$. To remove the simplifying topological assumption we need a more precise form of this data, as we explain in §6.3. For simplicity, consider a product spacetime $M = M^1 \times N^{n-1}$, where we assume space N to be a compact oriented n-manifold.

We first construct a $\mathbb{T}$-torsor $\mathcal{T}_f = \mathcal{T}_f(\Omega)$ for each map $f\colon N \to X$ as follows. Since $H_{n-1}(X) = 0$ there exist extensions $F\colon Y^n \to X$ to oriented n-chains Y^n with $\partial F = f$. Each such F trivializes $\mathcal{T}_f$. If F_1, F_2 are two such extensions, then since $H_n(X) = 0$ we can find an oriented $(n+1)$-chain $H\colon Z^{n+1} \to X$ with $\partial H = F_2 - F_1$. Then the isomorphism from the F_1-trivialization to the F_2-trivialization is

$$\exp\Bigl[\frac{\sqrt{-1}}{\hbar}\int_Z H^*\Omega\Bigr]. \tag{6.10}$$

Because of the assumption on the periods of Ω, this is independent of the choice of H.

We can allow $H_{n-1}(X)$ to be torsion in this construction if we specify a cohomology class $c \in H^n(X; 2\pi\hbar\mathbb{Z})$ such that the de Rham class of Ω is the image $c_{\mathbb{R}}$

of c in real cohomology. Then a multiple of f bounds, and we essentially use the choice of c to divide by that multiple in the abelian group $\mathbb{T}$.

For another perspective, consider the usual diagram

$$\begin{array}{ccc} \mathrm{Map}(N,X)\times N & \xrightarrow{e} & X \\ \downarrow{\scriptstyle \pi} & & \\ \mathrm{Map}(N,X) & & \end{array} \tag{6.11}$$

Then $\pi_* e^*\Omega$ is a 2-form on $\mathrm{Map}(N,X)$ with periods in $2\pi\hbar\mathbb{Z}$, and $\pi_* e^* c$ is a refinement in $H^2(\mathrm{Map}(N,X); 2\pi\hbar\mathbb{Z})$. So there exist principal $\mathbb{T}$-bundles with connection over $\mathrm{Map}(N,X)$ whose curvature is $-(\sqrt{-1}/\hbar)\pi_* e^*\Omega$ and whose Chern class is $\pi_* e^* c/2\pi\hbar$. The construction above gives a particular such $\mathbb{T}$-bundle; it can be extended to produce a connection as well.

Now if $\phi\colon [T_1,T_2]\times N\to X$ is a field, the exponentiated action $\exp(\sqrt{-1}\,S/\hbar)$ of ϕ is naturally an element of the circle torsor

$$\mathcal{T}_{\partial\phi} = \mathcal{T}_{\phi|\{T_2\}\times N}\cdot \mathcal{T}^{-1}_{\phi|\{T_1\}\times N}. \tag{6.12}$$

Namely, if F_i is a trivialization of $\mathcal{T}_{\phi|\{T_i\}\times N}$, then $F_2+\phi-F_1$ bounds an $(n+1)$-chain $H\colon Z^{n+1}\to X$, and we use formula (6.10) to compute the action in this trivialization.

We remark that the gauge theory constructions of §6.1 may be understood—at least heuristically—as special cases in which $X=BG$ is the classifying space of the gauge group G.

§6.3. Smooth Deligne cohomology

Fix a smooth manifold X and an integer[28] n. Let us consider on X the constant sheaf $\mathbb{Z}$, the constant sheaf $\mathbb{R}$, and the following complex of sheaves, denoted $F^n(\Omega)$ (or simply $F(\Omega)$): the subcomplex of the de Rham complex given by

$$F(\Omega)^p = \begin{cases} 0, & \text{if } p<n; \\ \Omega^p, & \text{if } p\ge n. \end{cases} \tag{6.13}$$

Both the cohomology of X with coefficients in $\mathbb{Z}$ and the hypercohomology of X with coefficients in $F(\Omega)$ map to the cohomology of X with coefficients in $\mathbb{R}$. The first because $\mathbb{Z}$ maps to $\mathbb{R}$. The second is just the cohomology of the complex $\Omega^n(X)\to\Omega^{n+1}(X)\to\cdots$ starting in degree n, which is a subcomplex of the de Rham complex computing the real cohomology. Suppose we have somehow lifted those maps at the cochain level. In other words, suppose we have found natural complexes $(C^*_{\mathbb{Z}},d_{\mathbb{Z}})$, $(C^*_{F^n},d_{F^n})$ (or simply (C^*_F,d_F)), and $(C^*_{\mathbb{R}},d_{\mathbb{R}})$ with $H^*(C^*_{\mathbb{Z}})=H^*(X,\mathbb{Z})$, $H^*(C^*_F)=\mathbb{H}^*(X,F(\Omega))$, and $H^*(C^*_{\mathbb{R}})=H^*(X,\mathbb{R})$, and found morphisms of complexes

$$\varphi_{\mathbb{Z}},\varphi_F\colon C^*_{\mathbb{Z}},C^*_{F^n}\longrightarrow C^*_{\mathbb{R}} \tag{6.14}$$

[28] Our choice of indexing makes the description of products below more natural. However, there is a shift in the application to lagrangian field theory: it is cocycles for the cohomology group D^{n+1} which enter lagrangians for n-dimensional spacetimes.

inducing the maps we described in cohomology. This can be done in many ways—one way will be described later—but the philosophy of cohomological algebra tells that they are essentially equivalent.

One can then form a mapping cone (K, d) with

$$K^p := C^p_{\mathbb{Z}} \oplus C^p_F \oplus C^{p-1}_{\mathbb{R}}$$
$$d := d_{\mathbb{Z}} + d_F - d_{\mathbb{R}} + \varphi_{\mathbb{Z}} - \varphi_F. \tag{6.15}$$

A p-cycle $c = (c_{\mathbb{Z}}, c_F, c_{\mathbb{R}})$ of K is the data of p-cycles $c_{\mathbb{Z}}$ and c_F of $C_{\mathbb{Z}}$ and C_F, and of a homology between their images in $C_{\mathbb{R}}$. The mapping cone K behaves as if one had a short exact sequence of complexes $0 \to K \to C_{\mathbb{Z}} \oplus C_F \to C_{\mathbb{R}} \to 0$. For instance, one has a long exact sequence of cohomology groups. The *smooth Deligne cohomology* group $D^{p,n}$ is $H^p(K)$. For $p < n$ it is $H^{p-1}(X, \mathbb{R}/\mathbb{Z})$. For $p > n$ it is $H^p(X, \mathbb{Z})$. We will be mainly interested in $D^n := D^{n,n}$. It sits in an exact sequence

$$H^{n-1}(X,\mathbb{Z}) \to H^{n-1}(X,\mathbb{R}) \to D^n \to H^n(X,\mathbb{Z}) \oplus (\text{closed } n\text{-forms}) \to H^n(X,\mathbb{R}) \tag{6.16}$$

and it is an extension of the group of closed n-forms with integral periods by $H^{n-1}(X, \mathbb{R}/\mathbb{Z})$.

This description of D^n is not the most economical, but it suggests the functorial properties to be expected. *Products*: One has product maps on $\mathbb{Z}$ and $\mathbb{R}$, as well as product maps $F^n \otimes F^m \to F^{n+m}$. If the corresponding cup-product in cohomology is expressed at the cochain level, giving products in $C_{\mathbb{Z}}$, $C_{\mathbb{R}}$, and C_F compatible with $\varphi_{\mathbb{Z}}$ and $\varphi_{\mathbb{R}}$, one obtains products $D^{p,n} \otimes D^{q,m} \to D^{p+q,n+m}$. Indeed, homologies c and c' between the images of $c_{\mathbb{Z}}$ and c_F (resp. $c'_{\mathbb{Z}}$ and c'_F) give a homology between the images of $c_{\mathbb{Z}}c'_{\mathbb{Z}}$ and $c_F c'_F$. In fact, there are two naturally cohomologous homologies: $\varphi(c_{\mathbb{Z}})c' + c\varphi(c'_F)$ and $c\varphi(c'_{\mathbb{Z}}) + \varphi(c_F)c'$. *Integration*: If a proper submersion $f\colon X \to Y$ of relative dimension d has oriented fibers, integration along the fibers f has meaning in integral and real cohomology, as well as from F^n on X to F^{n-d} on Y. Expressed compatibly at the cochain level, it should provide $\int_{X/Y}\colon D^{p,n}(X) \to D^{p-d,n-d}(Y)$, and in particular $D^n(X) \to D^{n-d}(Y)$.

One way to find complexes $C^*_{\mathbb{Z}}$, C^*_F, and $C^*_{\mathbb{R}}$ is to use the Čech method for computing cohomology. If $\{U_i\}_{i \in I}$ is an open covering of X such that the nonempty intersections $U_{i_0 \dots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$ are contractible, one can use for $C^*_{\mathbb{Z}}$ the Čech complex

$$C^p_{\mathbb{Z}} = \prod \Gamma(U_{i_0 \dots i_p}, \mathbb{Z}), \tag{6.17}$$

and for C^*_F and $C^*_{\mathbb{R}}$ the simple complex associated to the double complex

$$C^{p,q}_F = \prod \Gamma(U_{i_0 \dots i_p}, F(\Omega)^q)$$
$$C^{p,q}_{\mathbb{R}} = \prod \Gamma(U_{i_0 \dots i_p}, \Omega^q). \tag{6.18}$$

For $n > 0$ the map $\mathbb{Z} \oplus F \to \Omega^*$ of complexes of sheaves on X is injective. If so is $\varphi_{\mathbb{Z}} - \varphi_F\colon C^*_{\mathbb{Z}} \oplus C^*_F \to C^*_{\mathbb{R}}$, the mapping cone K has the same cohomology, up to a shift of index by 1, as the cokernel of $\varphi_{\mathbb{Z}} - \varphi_F$, a complex whose cohomology is the hypercohomology of the complex of sheaves

$$C^\infty(\text{values in } \mathbb{R}/\mathbb{Z}) \longrightarrow \Omega^1 \longrightarrow \dots \longrightarrow \Omega^{n-1} \longrightarrow 0. \tag{6.19}$$

If we denote by $\tilde{\Omega}^*$ this complex, we have in the Čechist computation

$$\operatorname{coker}(\varphi_{\mathbb{Z}} - \varphi_F) = \prod \Gamma(U_{i_0 \dots i_p}, \tilde{\Omega}^p), \tag{6.20}$$

and $D^{p,n}$ is the cohomology group H^{p-1} of the complex.

For $n = 1$ we find that D^1 is the group of C^∞ maps from X to the circle $\mathbb{R}/\mathbb{Z}$. For $n = 2$ elements of D^2 are represented by cocycles

$$\begin{array}{ccccc} \alpha_i & \longrightarrow & 0 & & \\ & & \uparrow & & \\ & & g_{ij} & \longrightarrow & 0 \end{array} \tag{6.21}$$

where the diagram indicates the equations

$$\begin{aligned} g_{ij} + g_{jk} &= g_{ik} \\ \alpha_i - \alpha_j &= dg_{ij}. \end{aligned} \tag{6.22}$$

The $\exp(2\pi\sqrt{-1}\, g_{ij})$ are the transition functions for a $\mathbb{T}$-bundle, trivialized on the U_i, and the $2\pi\sqrt{-1}\,\alpha_i$, viewed as local connection 1-forms, provide this $\mathbb{T}$-bundle with a connection. The group D^2 is the group of isomorphism classes of $\mathbb{T}$-bundles with connection. More precisely, a cocycle defines a bundle with connection and a homology defines an isomorphism between bundles.

In §6.2, what we needed to avoid assumptions on the homology of X is not just a closed $(n+1)$-form with integral periods, but rather a class in D^{n+1}, or rather a cocycle giving such a class. If $H^n(X, \mathbb{R}/\mathbb{Z}) = 0$ (equivalently: $H_n(X, \mathbb{Z}) = 0$), one has

$$D^{n+1} \xrightarrow{\sim} (\text{closed } (n+1)\text{-forms with integral periods}). \tag{6.23}$$

If $H^{n-1}(X, \mathbb{R}/\mathbb{Z}) = 0$ (equivalently: $H_{n-1}(X, \mathbb{Z}) = 0$), the ambiguity in the choice of a cocycle becomes irrelevant: if c_1 and c_2 are two cocycles representing the same class in D^{n+1}, not only are they homologous, $c_1 - c_2 = dc$, but any two choices c' and c'' for c are homologous: $c' - c'' = d\tilde{c}$.

If a cocycle c is chosen and if ϕ is a field, i.e., a map from M to X, then ϕ^*c is a cocycle giving a class in $D^{n+1}(M)$. For N in M a compact subvariety of dimension $n-1$, integration of ϕ^*c on N should produce a $\mathbb{T}$-principal homogeneous space—a one-dimensional complex vector space $\mathcal{L}(N, \phi^*c)$ with metric attached to N and ϕ^*c. Indeed, integration on N maps $D^{n+1}(M)$ to D^2 of a point. If N is the boundary of a singular chain S, integration of ϕ^*c on S should produce a unit vector in $\mathcal{L}(N, \phi^*c)$. For instance, in a Hamiltonian picture, a space-like hypersurface N, supposed here compact, would give $\mathcal{L}(N, \phi^*c)$ and the slice S between two such hypersurfaces N_1 and N_2 would provide an isomorphism (the action integral) from $\mathcal{L}(N_1, \phi^*c)$ to $\mathcal{L}(N_2, \phi^*c)$.

A systematic treatment of those expectations has yet to be given.

CHAPTER 7

Wick Rotation: From Minkowski Space to Euclidean Space

A basic constraint on a Minkowski space action is that it be real. An action S_M is the integral of a lagrangian density L_M over Minkowski space M:

$$S_M = \int_M L_M. \tag{7.1}$$

Choose a time t on M. Then we (Wick) rotate to Euclidean space E by introducing imaginary time

$$\tau = \sqrt{-1}\,t. \tag{7.2}$$

By convention the Euclidean action is $1/\sqrt{-1}$ times the rotated Minkowski action:

$$\frac{1}{\sqrt{-1}} S_M = S_E = \int_E L_E. \tag{7.3}$$

Note that $e^{\sqrt{-1}\,S_M} = e^{-S_E}$. Also, S_E is not real in general.

We describe the continuation to Euclidean space more precisely for a σ-model. The field is a map $\phi : M \longrightarrow X$ into some Riemannian manifold. The complexification of the space of maps $M \longrightarrow X$ is the space of *holomorphic* maps $M_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ between the complexified spaces. The lagrangian extends to a holomorphic function on this space, and the Euclidean action is the restriction of this continuation to maps $E \longrightarrow X$. (Note that $E_{\mathbb{C}} = M_{\mathbb{C}}$ so $E \subset M_{\mathbb{C}}$.) There is a similar picture for other types of fields.

We consider four types of terms which typically occur in an action: kinetic terms for bosons, potential terms, topological terms (also θ-terms), and kinetic terms for fermions.

In this chapter set the speed of light $c = 1$. We use the conventions in §3.1 for Minkowski space M. So the metric is

$$g_M = dt^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{7.4}$$

On Euclidean space E we use the positive definite metric

$$g_E = d\tau^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2. \tag{7.5}$$

So as to avoid confusion, we fix the standard orientations $t, x^1, \dots, x^{n-1}$ on M and $\tau, x^1, \dots, x^{n-1}$ on E, and we write lagrangians as forms rather than densities, though we often omit the '$\wedge$' sign. Let

$$d^{n-1}x = dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{7.6}$$

§7.1. Kinetic terms for bosons

Consider a particle of mass m moving in some Riemannian manifold X. It is described by a map $x : M^1 \to X$. Then the kinetic energy density is (see (1.13))

$$L_M = \frac{m}{2}\left|\frac{dx}{dt}\right|^2 dt. \tag{7.7}$$

The continuation to imaginary time—after dividing by $\sqrt{-1}$—is

$$L_E = \frac{m}{2}\left|\frac{dx}{d\tau}\right|^2 d\tau. \tag{7.8}$$

In higher dimensions, we consider a real scalar field on Minkowski space, which is a real function $\phi : M \to \mathbb{R}$. The kinetic lagrangian is

$$L_M = \frac{1}{2}\;|d\phi|^2_M\; dt\, d^{n-1}x, \tag{7.9}$$

where $|\cdot|_M$ is the norm (7.4) on M. The continuation to E is

$$L_E = \frac{1}{2}\,|d\phi|^2_E\; d\tau\, d^{n-1}x, \tag{7.10}$$

where $|\cdot|_E$ is the Euclidean norm (7.5).

For a gauge field[29] A the kinetic term is the Yang-Mills lagrangian

$$L_M = -\frac{1}{2}|F_A|^2_M\, dt\, d^{n-1}x. \tag{7.11}$$

The continuation to Euclidean space is

$$L_E = \frac{1}{2}|F_A|^2_E\, d\tau\, d^{n-1}x. \tag{7.12}$$

§7.2. Potential terms

For the particle $x : \mathbb{R} \to X$, the potential energy is described by a function $V : X \to \mathbb{R}$. The corresponding term in the lagrangian is

$$L_M = -\,V(x(t))\,dt. \tag{7.13}$$

The continuation to imaginary time is

$$L_E = V(x(\tau))\,d\tau. \tag{7.14}$$

The extension to higher dimensions is the same: *Potential terms appear with a $-$ sign in Minkowski actions and with a $+$ sign in Euclidean actions.*

[29] For a concrete example of rotating a gauge field to Euclidean space, see the solution to Problem 11 of [II-**Dynamics of QFT, Exercises**].

§7.3. Topological terms and θ-terms

Let α be a *real* 1-form on X and consider the lagrangian (for $x\colon \mathbb{R} \to X$)

$$L_M = -x^*\alpha. \tag{7.15}$$

The continuation to imaginary time is innocuous except for the conventional division by $\sqrt{-1}$:

$$L_E = \sqrt{-1}\, x^*\alpha. \tag{7.16}$$

Hence in the Euclidean (imaginary time) lagrangian the topological term is imaginary. A similar factor is introduced into the continuation of any topological term (in the sense of Chapter 6).

The θ-terms in gauge theory (§4.3) behave similarly. For example, the continuation of

$$L_M = \frac{\theta}{2\pi}\langle\!\langle F_A\rangle\!\rangle \tag{7.17}$$

to Euclidean space is

$$L_E = -\sqrt{-1}\,\frac{\theta}{2\pi}\langle\!\langle F_A\rangle\!\rangle. \tag{7.18}$$

§7.4. Kinetic terms for fermions

We begin with the particle $x : \mathbb{R} \to X$ and add an odd field ψ which is a section of $x^*\Pi TX$, the parity-reversed pullback of the tangent bundle. Then ψ is real and in real time its kinetic term in the lagrangian is

$$L_M = \frac{m}{2}\,\langle\psi, \frac{d\psi}{dt}\rangle\, dt. \tag{7.19}$$

Rotating to imaginary time and dividing by $\sqrt{-1}$, we obtain

$$L_E = -\sqrt{-1}\,\frac{m}{2}\,\langle\psi, \frac{d\psi}{d\tau}\rangle\, d\tau. \tag{7.20}$$

Consider now a real spinor field ψ in n-dimensional Minkowski spacetime. We retain the notation of §3.4, but we set the mass $M = 0$. (The mass term is a potential energy term, so is covered by the discussion of §7.2.) The lagrangian in Minkowski space is

$$\begin{aligned} L_M &= \frac{1}{2}\psi \not{D}_M \psi\, dt\, d^{n-1}x \\ &= \frac{1}{2}(\tilde{\Gamma}_M)^{\mu ab}\psi_a\partial_\mu\psi_b\, dt\, d^{n-1}x. \end{aligned} \tag{7.21}$$

The first observation when rotating to Euclidean space is that a real spin representation S of $\mathrm{Spin}(1, n-1)$ extends to a *complex* representation $S_{\mathbb{C}}$ of the complex spin group $\mathrm{Spin}_{\mathbb{C}}(n)$, but the restriction to $\mathrm{Spin}(n) \subset \mathrm{Spin}_{\mathbb{C}}(n)$ is not necessarily real. Thus in the Euclidean theory we take the field

$$\psi\colon E \longrightarrow \Pi S_{\mathbb{C}} \tag{7.22}$$

to be a *complex* spinor field. Again: *There are no reality conditions on spinor fields in Euclidean space.* Define pairings

$$\begin{aligned} \Gamma_E &: S^*_{\mathbb{C}} \otimes S^*_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \\ \tilde{\Gamma}_E &: S_{\mathbb{C}} \otimes S_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} \end{aligned} \tag{7.23}$$

by

$$\begin{aligned} (\Gamma_E)^0_{ab} &= -\sqrt{-1}\,(\Gamma_M)^0_{ab}, & (\tilde{\Gamma}_E)^{0ab} &= -\sqrt{-1}\,(\tilde{\Gamma}_M)^{0ab}, \\ (\Gamma_E)^i_{ab} &= -(\Gamma_M)^i_{ab}, & (\tilde{\Gamma}_E)^{iab} &= -(\tilde{\Gamma}_M)^{iab}, \end{aligned} \tag{7.24}$$

where $i = 1, \dots, n-1$ runs over the spatial indices. Then $\Gamma_E, \tilde{\Gamma}_E$ satisfy a Clifford relation

$$(\tilde{\Gamma}_E)^{\mu ab}(\Gamma_E)^{\nu}_{bc} + (\tilde{\Gamma}_E)^{\nu ab}(\Gamma_E)^{\mu}_{bc} = -2g^{\mu\nu}_E\,\delta^a_c. \tag{7.25}$$

The Euclidean Dirac form is

$$\psi \not{D}_E \psi = (\tilde{\Gamma}_E)^{\mu ab}\psi_a \partial_\mu \psi_b, \tag{7.26}$$

and the factors in (7.24) are chosen so that the lagrangian (7.21) rotates to

$$L_E = \frac{1}{2}\psi \not{D}_E \psi \, d\tau\, d^{n-1}x \tag{7.27}$$

in Euclidean space.

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There are many mathematical accounts of classical mechanics, for example

V. I. Arnold, *Mathematical methods of classical mechanics*, Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Springer, New York, 1989.

Many elements of the formalism we adopt here for classical lagrangian field theory may be found in

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