The Standard Model: A Primer

The standard model provides the modern understanding of all of the interactions of subatomic particles, except those due to gravity. The theory has emerged as the best distillation of decades of research.

This book uses the standard model as a vehicle for introducing quantum field theory. In doing this the book also introduces much of the phenomenology on which this model is based. The book uses a modern approach, emphasizing effective field theory techniques, and contains brief discussions of some of the main proposals for going beyond the standard model, such as seesaw neutrino masses, supersymmetry, and grand unification.

Requiring only a minimum of background material, this book is ideal for graduate students in theoretical and experimental particle physics. The book concentrates on getting students to the level of being able to use this theory by doing real calculations with the minimum of formal development. It does so without taking any shortcuts which would leave an incomplete understanding. The book contains several problems, with password-protected solutions available to lecturers at www.cambridge.org/9780521860369

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Preface

The standard model of particle physics, developed in the 1960s and 1970s, has stood for 30 years as “the” theory of particle physics, passing numerous stringent tests. In fact, while many people believe that the standard model is not a complete description of particle physics, it is expected to be, at worst, incomplete rather than wrong; that is, the standard model is at worst a subset of the true theory of particle physics.

For this reason, a good working knowledge of the standard model and its phenomenology is essential for the modern particle physicist. The goal of this book is to provide all the tools for a working, quantitative knowledge of the standard model, with the minimum of formal developments. It presents everything needed to understand the particle spectrum of the standard model, and how to compute decay rates and cross sections at leading order in the weak coupling expansion (tree level). We assume a solid quantum-mechanics background, up to and including canonical quantization and the Dirac equation, but we do not assume familiarity with formal quantum field theory (renormalization, path integrals, generating functionals).

As we see it, this book fills two gaps in the existing literature. The first of these concerns the balance between theoretical sophistication and phenomenological utility. Most treatments of the standard model appear at the end of quantum field theory books. This is rational in the sense that the reader then has the complete set of tools to compute standard-model phenomena at the loop level. This approach has its merits; both authors learned the standard model in this way. Unfortunately, for many, especially experimental practitioners, the quantum field theory preliminaries may be too burdensome. Also, such books frequently do not present the standard model in complete detail, and they generally develop little of its phenomenology. The opposite style of approach is a more “cookbook” book, which introduces quantum field theory at the tree level, typically using electrodynamics as an example, and again presents the standard model at the end. Generally these treatments are incomplete and abbreviated. The intention of this book is to be similar to the latter type of book, except that the presentation of the standard model is complete and contains a discussion of the model's phenomenology and a complete presentation of its Feynman rules.

Our philosophy is that it is important for a particle physicist to have a complete and quantitative knowledge of the standard model; indeed, for many, this is much more important than having a good background in formal quantum field theory. One cannot present the standard model in detail without *some* quantum field theory; but one can get surprisingly far without understanding the details of renormalization and loop effects. Of course, especially for theorists, a good knowledge of quantum field theory is also necessary; indeed, it should be obvious to the reader, at many points in the text, that more formal development is needed to compute to high accuracy. Knowing the material in this book may help the student of more formal quantum field theory by motivating and providing context for that study. Conversely, a student already proficient in quantum field theory can use this book as a succinct presentation of the standard model, and will have the tools to fill in the gaps left in the presentation, where loop corrections are required.

The second gap which we believe this book fills concerns the modern theoretical framework within which the standard model rests: the framework of *effective field theories*. Today we understand the theories we construct to describe nature – including the standard model – to be effective theories which capture the low-energy limit of some more fundamental, microscopic physics. Effective field theories capture a basic experimental fact: although nature comes to us with many scales, it can be understood one scale at a time. For instance, atomic physics can be understood with only limited knowledge of nuclei, and it can because short-distance physics tends to *decouple* from long-distance physics. In the modern understanding it is this observation which ultimately explains the otherwise puzzling requirement of renormalizability which our fundamental theories generally have. This book starts by using the standard model to build up the tools of effective field theory, by showing how and why scattering amplitudes simplify in the low-energy limit. Later chapters then exploit these tools to categorize the kinds of new physics which might ultimately replace the standard model, starting with a discussion of neutrino oscillations and ending with a broad survey of such new physics topics as supersymmetry and grand unified theories.

The first chapter of this book is devoted to introducing the field theory concepts we will need to present the standard model. We present the allowed fields that can make up a quantum field theory (scalars, fermions, and gauge bosons), with particular emphasis on Majorana fermions and on the gauge principle, which appear to play especially important roles in the standard model. We introduce the required rules for formulating the theory's Lagrangian – the "basic principles," such as Lorentz invariance, locality, unitarity, and renormalizability. We see what kinds of interactions are allowed, given the available fields and these basic principles. Then we give a few illustrative examples, including QED and QCD. Supplementary material on group theory, the Lorentz group, and spinors is provided in two appendices.

The second chapter introduces the standard model itself. We present the gauge group and the field content. The Lagrangian then follows as the most general Lagrangian consistent with these fields and with basic principles. This section then explores the consequences, determining the mass eigenstates and their interactions. We present in complete detail what the interaction Hamiltonian of the model is in the mass basis. We also briefly discuss the symmetries of the model, especially the accidental global symmetries of baryon and lepton number, and very briefly discuss anomalies and gauge anomaly cancellation.

The third chapter discusses the S matrix formalism in just enough detail to define and motivate decay rates and cross sections, and to show how they are to be computed in the interaction picture. Together, the first three chapters represent an introduction to the framework of the standard model.

Next, we start using the standard-model interactions to compute processes, introducing the needed technology as we go with the philosophy of "learning by doing" and using specific examples to figure out the patterns. We begin with the simplest processes in the standard model, the decays of heavy bosons, in Chapter 4. The rates of Z^0, $W^{\pm}$, and Higgs-boson decays can be computed using interaction picture perturbation theory and an expansion of the fields in creation and annihilation operators, without much difficulty. In Chapter 5, where we consider the decays of leptons lighter than the W boson mass, we first encounter virtual intermediate particles, requiring the introduction of the propagator. After these examples it is possible to generalize the procedure for computing a decay process. This allows us to introduce the Feynman rules. Chapter 5 ends with a complete presentation of the unitary gauge Feynman rules of the standard model, sufficient for tree level analysis. (The R_ξ gauge Feynman rules appear in Appendix D.)

In Chapter 6 we address scattering processes, concentrating on fermion–fermion scattering. We discuss s-channel scattering in some length, especially near the Z^0 pole, where we first discover the necessity of including loop corrections. We also introduce crossing symmetry and interference between diagrams, external photon states, and initial state radiation.

In Chapter 7 we introduce the notion of effective field theories, using the Fermi theory as the main example. This is especially important as the standard model itself is probably just an effective theory for some more inclusive theory, which is manifested at higher energies. We also present some of the most important results of loop corrections, particularly the running of gauge couplings with scale.

Chapter 8 begins the discussion of hadrons. We motivate why the running of couplings causes the confinement of quarks and gluons within hadrons, and we describe and motivate the spectrum of heavy-light and light-light mesons and of baryons, emphasizing the use of approximate symmetries.

Chapter 9 discusses hadronic interactions. It explains why both the low- and high-energy regimes are somewhat tractable, but the intermediate energy regime is not. We discuss deep inelastic scattering and the partonic structure of hadrons, up to and including the Altarelli–Parisi (DGLAP) equations. Then we discuss chiral perturbation theory, leptonic meson decays, and oscillation phenomena in the K and B meson systems.

The last part of the book gives a brief survey of what may lie beyond the standard model. We begin in Chapter 10 with a discussion of neutrino masses. Technically, these cannot lie beyond the standard model, because they have been observed, and the meaning of the standard model must be enlarged to accommodate them. However, as we discuss, there are two viable ways to do so, Majorana neutrino masses and Dirac neutrino masses, and we do not (yet) know which is correct. We discuss the Majorana possibility at some length in the context of non-renormalizable field theories. We discuss oscillation phenomena in some length, including the MSW effect, and briefly cover neutrinoless double beta decay. We also give examples of high-energy physics that could lead to the non-renormalizable operator responsible for Majorana neutrino masses.

Finally, Chapter 11 discusses what *may* lie beyond the standard model. We organize this material in terms of problems with the standard model, which can in turn be organized in terms of the dimensionality of the operator presenting the problem. The hierarchy problem appears because of the dimension-2 Higgs mass term, and may be solved by supersymmetry. The strong CP problem appears because of the dimension-4 Θ term in QCD, and may be solved by the axion mechanism. The baryon-number conservation "problem" (opportunity) arises because of the possibility of dimension-6 operators in the standard model; these might arise at an interesting level within grand unified theories.

Our approach is modern and synthetic; we present the model first and then explore its phenomenology, without first presenting the experimental evidence which has led us to the field content of the model. We also do not cite previous literature in the text, leaving references to our (hopefully sufficient) bibliography. We also adopt what we hope is a modern and streamlined notation. In most respects our nomenclature is that in conventional use, even where this does not correctly reflect the historical development of ideas. For instance, we refer to the Higgs mechanism and the Higgs boson, rather than the (more correct but cumbersome) Anderson-Brout-Englert-Higgs-Guralnik-Hagen-Kibble mechanism.

There is one respect in which we do not follow the most conventional set of conventions. Namely, we have used the metric convention, $\eta_{\mu\nu} = \mathrm{Diag}[-1 + 1 + 1 + 1]$, which is the less common convention within the phenomenology community. However, to ease the text's use, we present in Appendix E a clear discussion of how to convert between conventions, culminating in a metric convention conversion table.

In our experience it is possible to cover most of this book in a high-paced, one-semester first-year graduate level course. To do so, it is necessary to shave some corners. Most of Chapter 1, and Chapter 2 through Section 2.4, are essential, but Section 2.5 can be skipped without too much loss to the continuity. Similarly Section 4.2 and Section 4.3 can be given as problems instead of covered as sections. Chapter 5 and Chapter 6 should be covered in full, but then material from the remaining chapters can be picked and chosen as time and interest allow. The material in Chapter 10 does not rely on Chapter 8 or Chapter 9. A full year course should quite easily be able to cover all of the material in this book.

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Part I

Theoretical framework

1
Field theory review

Quantum field theory is the language in terms of which the laws of physics are cast, and so we start with a whirlwind summary of some of its main features. Interspersed amongst the introductory topics in this chapter we also discuss some of the more general features that are usually demanded of any reasonable field-theoretic description of nature.

1.1 Hilbert space, creation and annihilation operators

Quantum field theories are special kinds of quantum mechanical theories which describe the behavior of particles. As quantum mechanical theories, their most basic objects are the Hilbert space of possible states $\mathcal{H}$, and the Hamiltonian H which describes time evolution in that Hilbert space.

The possible kinds of states are zero-particle states, one-particle states, two-particle states, and so on. Therefore, the Hilbert space in which all operators live is the sum of the zero-particle space with the one-particle space with the two-particle space, and so on:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \tag{1.1}$$

Here

$$\mathcal{H}_0 = \{|0\rangle\} \tag{1.2}$$

denotes the one-dimensional space spanned by the zero-particle state or vacuum: $|0\rangle$.

$$\mathcal{H}_1 = \{|\mathbf{p}, k\rangle\} \tag{1.3}$$

is similarly the span of all one-particle states with the basis states chosen to be eigenstates of linear momentum. Here $\mathbf{p}$ represents the momentum of a state, and k denotes all of the other particle labels.

The space of N-particle states is constructed as the tensor product of N copies of the one-particle space. For instance, $\mathcal{H}_2$ is the set of all two-particle states,

$$\mathcal{H}_2 = \{|\mathbf{p}_1, k_1; \mathbf{p}_2, k_2\rangle = \pm|\mathbf{p}_2, k_2; \mathbf{p}_1, k_1\rangle\} \tag{1.4}$$

etc.The sign, $\pm$, is $+$ for bosons and $-$ for fermions. A Hilbert space constructed in this way is conventionally referred to as a Fock space.

It is convenient to express the operators that act within this space in terms of a basic set of *creation* and *annihilation* operators in the following way. The *annihilation operator*, $a_{\mathbf{p}k}$, is the operator that removes the particle with quantum numbers $\mathbf{p}$ and k from a given state. If the state on which $a_{\mathbf{p}k}$ acts does not contain the particle in question then the operator is defined to give zero. That is,

$$\begin{aligned} a_{\mathbf{p}k}|0\rangle &= 0 \\ a_{\mathbf{p}k}|\mathbf{q}, l\rangle &= 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{kl}|0\rangle \\ a_{\mathbf{p}k}|\mathbf{q}, l; \mathbf{k}, m\rangle &= 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{kl}|\mathbf{k}, m\rangle \\ &\quad \pm 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta_{km}|\mathbf{q}, l\rangle \end{aligned} \tag{1.5}$$

and so on. Here, $E_{\mathbf{p}}$ is the energy of a particle of spatial momentum $\mathbf{p}$, namely, $\sqrt{\mathbf{p}^2 + m^2}$, with m the mass of a particle with labels k. The sign in this last result is $\pm$ according to the statistics of particles $|\mathbf{p}, k\rangle$ and $|\mathbf{q}, l\rangle$. Here and throughout, we use units for which $\hbar = c = 1$. The normalization is chosen to make Lorentz invariance more manifest, as discussed below.

This definition implies that the Hermitian conjugate, $a^*_{\mathbf{p}k}$, of $a_{\mathbf{p}k}$ is a *creation operator* for the same particle type; i.e.

$$a^*_{\mathbf{p}i}|0\rangle = |\mathbf{p}, i\rangle \tag{1.6}$$

$$a^*_{\mathbf{p}i}|\mathbf{q}, j\rangle = |\mathbf{p}, i; \mathbf{q}, j\rangle \tag{1.7}$$

etc.(Our notation is to use an asterisk for complex conjugation of c-numbers and Hermitian conjugation of operators, and to reserve a dagger, †, for Hermitian conjugation of matrices.)

These definitions, together with the normalization convention

$$\langle \mathbf{p}, i|\mathbf{q}, j\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{ij} \tag{1.8}$$

imply the following properties. For bosons,

$$|\mathbf{p}, i; \mathbf{q}, j\rangle = |\mathbf{q}, j; \mathbf{p}, i\rangle \tag{1.9}$$

$$[a_{\mathbf{p}i}, a_{\mathbf{q}j}] = \left[a^*_{\mathbf{p}i}, a^*_{\mathbf{q}j}\right] = 0 \tag{1.10}$$

$$\left[a_{\mathbf{p}i}, a^*_{\mathbf{q}j}\right] = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{ij} \tag{1.11}$$

and for fermions,

$$|\mathbf{p}, i; \mathbf{q}, j\rangle = -|\mathbf{q}, j; \mathbf{p}, i\rangle \tag{1.12}$$

$$\{a_{\mathbf{p}i}, a_{\mathbf{q}j}\} = \{a^*_{\mathbf{p}i}, a^*_{\mathbf{q}j}\} = 0 \tag{1.13}$$

$$\{a_{\mathbf{p}i}, a^*_{\mathbf{q}j}\} = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{ij} \tag{1.14}$$

in which $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

A few comments are in order about the field normalizations above. First, note that momentum integrations $\mathrm{d}p/2\pi$ always have factors of 2π in the denominator, and momentum delta functions $2\pi\delta(p-q)$ always have factors of 2π multiplying them. Following these rules,

- momentum space and energy integrations always involve $\int \mathrm{d}^3\mathbf{p}/(2\pi)^3$, $\int \mathrm{d}E/2\pi$;
- delta functions are always of form $(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$ or $(2\pi)\delta(E_1 - E_2)$,

accounts for all 2π factors we will ever encounter.

Second, the momentum delta functions we have written are accompanied by factors of $2E_{\mathbf{p}}$, and the same $2E_{\mathbf{p}}$ appears in the denominator in momentum integrations. This normalization, called relativistic normalization, is convenient in a Lorentz invariant theory, because it makes it easier to make Lorentz invariance manifest. Note in particular, that

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}2\pi\delta(p^2+m^2)\theta(p^0) \tag{1.15}$$

which is manifestly Lorentz invariant. [Note that our metric convention is that $\eta_{\mu\nu} = \mathrm{Diag}[-1, 1, 1, 1]$, so $p^2 \equiv \eta_{\mu\nu}p^\mu p^\nu = -(p^0)^2 + \mathbf{p}^2$.] This expression can be verified by performing the p^0 integration, using the δ function. Its Lorentz invariance is not quite manifest, since the step function $\theta(p^0)$ does not look invariant, as it refers to the time component; but the $2\pi\delta(p^2+m^2)$ forces p^μ to be timelike for $m^2 > 0$ and lightlike for $m^2 = 0$, which ensures that the sign of p^0 does not change under (orthochronous) Lorentz transformations. Throughout this book, whenever there is an integral $\int \mathrm{d}^3p/(2\pi)^3 2E_{\mathbf{p}}$, we will always implicitly define $p^0 = E_{\mathbf{p}}$ inside the integral.

The fundamental claim now to be made is that *any* operator acting on our Hilbert space, $\mathcal{H}$, can be written as a linear combination of monomials of the as and a^*s; i.e.,

$$\mathcal{O} = A_{0,0} + \sum_i \int \frac{\mathrm{d}^3p}{2E_{\mathbf{p}}(2\pi)^3}\Big[A_{0,1}(\mathbf{p}, i)a_{\mathbf{p}i} + A_{1,0}(\mathbf{p}, i)a^*_{\mathbf{p}i}\Big] \tag{1.16}$$

$$+\sum_{ij}\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{4E_{\mathbf{p}}E_{\mathbf{q}}(2\pi)^6}\Big[A_{0,2}(\mathbf{p},i;\mathbf{q},j)a_{\mathbf{p}i}a_{\mathbf{q}j}+A_{1,1}(\mathbf{p},i;\mathbf{q},j)a^*_{\mathbf{p}i}a_{\mathbf{q}j}$$
$$+A_{2,0}(\mathbf{p},i;\mathbf{q},j)a^*_{\mathbf{p}i}a^*_{\mathbf{q}j}\Big]+\cdots \tag{1.17}$$

The operators, $\mathcal{O}$, are in one-to-one correspondence with the coefficient functions $\{A_{0,0}, A_{1,0}(\mathbf{p},i), A_{0,1}(\mathbf{p},i),\ldots\}$. This can be shown inductively by explicitly solving for these coefficients in terms of the matrix elements of $\mathcal{O}$: $\langle\psi|\mathcal{O}|\phi\rangle$. For example $\langle 0|\mathcal{O}|0\rangle = A_{0,0}$, $\langle 0|\mathcal{O}|\mathbf{p},i\rangle = A_{0,1}(\mathbf{p},i)$, and so on.

In particular, the Hamiltonian for a system of free particles has a simple expression in terms of the as and a^*s:

$$H_0 = E_0 + \sum_i\int\frac{\mathrm{d}^3p}{2E_{\mathbf{p}}(2\pi)^3}\,\varepsilon(\mathbf{p},i)a^*_{\mathbf{p}i}a_{\mathbf{p}i}. \tag{1.18}$$

To learn the interpretation of the coefficients E_0 and $\varepsilon(\mathbf{p},i)$, calculate the action of H_0 on various states. On the vacuum H_0 gives

$$H_0|0\rangle = E_0|0\rangle \tag{1.19}$$

since $a_{\mathbf{p}i}|0\rangle = 0$. E_0 is clearly the energy of the no-particle state $|0\rangle$, i.e., the vacuum energy. Similarly,

$$H_0|\mathbf{q},j\rangle = \left[E_0+\varepsilon(\mathbf{q},j)\right]|\mathbf{q},j\rangle \tag{1.20}$$

and

$$H_0|\mathbf{q}_1,j_1;\ldots;\mathbf{q}_N,j_N\rangle = \left[E_0+\sum_{k=1}^{N}\varepsilon(\mathbf{q}_k,j_k)\right]|\mathbf{q}_1,j_1;\ldots;\mathbf{q}_N,j_N\rangle \tag{1.21}$$

etc.The many-particle momentum eigenstates, $|\mathbf{q}_1,j_1;\ldots;\mathbf{q}_N,j_N\rangle$ are also eigenstates of the energy, H_0, with eigenvalue

$$E = E_0+\sum_{k=1}^{N}\varepsilon(\mathbf{q}_k,j_k). \tag{1.22}$$

This implies that the energy of a single-particle state $|\mathbf{p},i\rangle$ relative to the vacuum is $\varepsilon(\mathbf{p},i)$. Relativistic kinematics then determines the momentum-dependence of ε on $\mathbf{p}$ as

$$\varepsilon(\mathbf{p},i)=\sqrt{\mathbf{p}^2+m_i^2}=E_{\mathbf{p}} \tag{1.23}$$

where m_i is the mass of particle type i. Notice that the energy of a many-particle state relative to the vacuum is just the sum of the single-particle energies, showing that the particles described by H_0 do not interact.

We emphasize that this is a special property of free field theories; in general, even if single-particle states are eigenstates of the Hamiltonian, many-particle states are in general not eigenstates of the Hamiltonian. This means that they can undergo non-trivial time evolution. Indeed, almost all interesting phenomena in particle physics arise from the fact that many-particle states are not eigenstates of the Hamiltonian.

1.2 General properties of interactions

We are interested in writing down a Hamiltonian

$$H = H_0 + H_{\text{int}} \tag{1.24}$$

that describes the interactions of the particles we know. The present section is devoted to summarizing the minimal requirements for a physically reasonable theory. These properties translate into a set of restrictions on what form will be allowed for H. The purpose of this process is to arrive at the general class of theories from which the standard model is to be chosen. Being aware of the alternatives available gives some feeling for which features may be changed and which are inviolable.

We now return to a statement of these requirements. A sketch of their justification is given in the next subsection, but for a complete discussion the reader should consult a field theory text.

1.2.1 Physical constraints on H

The basic principles we demand of any candidate physical theory are:

(i) Unitarity: (i.e. conservation of probability)
The requirement here is to ensure that time evolution preserve the property that the sum of probabilities over all mutually exclusive events gives one. This requires that the time-evolution operator

$$U = \mathrm{e}^{-\mathrm{i}Ht} \tag{1.25}$$

be unitary. Equivalently the Hamiltonian must be Hermitian:

$$H = H^*. \tag{1.26}$$

(ii) Cluster decomposition: (i.e. locality)
This requirement is that physics be independent at different points in space at a given time. Specifically we require that amplitudes (and hence probabilities) for events that are well separated from one another factorize into a product of independent amplitudes. Such a

factorization is what would be expected for statistically independent events.

The condition that physics at spatially separated positions be independent comes in two parts. The first is that physical observables must commute at spatially separated points and the second is that time evolution must preserve this property. We consider each of these in turn:

(a) Microcausality
The first condition is to require that physical observables may be separately measurable at different positions and equal times. In a quantum theory we must therefore demand that all physical observables commute at space-like separations. That is:

$$[A(x), B(y)] = 0 \quad \text{for} \quad (x-y)^2 > 0. \tag{1.27}$$

Condition (1.27) is sometimes referred to as the requirement of microcausality.

(b) Locality
We next require that this property, that spatially separated physical amplitudes must factorize, be preserved by time evolution, provided, of course, that no physical signals propagate from one point to the other. Since the time-evolution operator, Eq. (1.25), is the exponential of the Hamiltonian, the property that it factorizes turns out to require that the Hamiltonian should be the sum of those for each of the spatially separated regions. The Hamiltonian must therefore have the form

$$H = \int \mathrm{d}^3x \; \mathcal{H}(\mathbf{x}, t) \tag{1.28}$$

which boils down to requiring that the total energy be a sum of the energy of the degrees of freedom at each point. This is consistent with the intuition that the degrees of freedom at each point of space at a given time are independent, since the total energy for a set of independent systems is the sum of the energies of the independent constituents.

(iii) Invariance under Lorentz transformations and translations (Poincaré invariance)
Here we build in the requirements of special relativity and translation invariance in space and time. In quantum mechanics this implies the existence of corresponding conserved charges, P^μ and $J^{\mu\nu} = -J^{\nu\mu}$

(with $\mu, \nu = 0, 1, 2, 3$), representing four-momentum and angular momentum respectively. In particular, the total energy is given by

$$H = P^0$$

The particle states transform under unitary representations of the Poincaré group given by the operators:

$$U(a,\omega) = \exp\left[-\mathrm{i}a_\mu P^\mu + \frac{\mathrm{i}}{2}\omega_{\mu\nu}J^{\mu\nu}\right] \tag{1.29}$$

generated by these conserved charges. The states, $|\mathbf{p}, \sigma, j\rangle$, may then be labelled by their three-momenta, $\mathbf{p}$, mass, m, total spin, s, and spin-projection, σ, together with any other internal labels, j. The labels m and s are generally not explicitly indicated.

The Minkowski-space conventions used in what follows are:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.30}$$

$$P^\mu = (E, \mathbf{p}), \quad \text{and} \quad P_\mu = (-E, \mathbf{p}) \tag{1.31}$$

$$x^\mu = (t, \mathbf{x}), \quad \text{and} \quad x_\mu = (-t, \mathbf{x}) \tag{1.32}$$

$$\epsilon^{0123} = +1 \tag{1.33}$$

implying that the invariant product $x^2 = -(x^0)^2 + \mathbf{x}^2$ is negative for timelike vectors and positive for space-like vectors. We provide a review of Lorentz symmetry in Appendix C.

(iv) Stability:
The final condition to be imposed is that the spectrum of H be bounded from below. This is necessary if the vacuum state, defined as the state of lowest energy, is to exist.

1.2.2 Renormalizability

A further condition to be imposed on the standard model that is not as fundamental as those just described is the requirement of renormalizability. In fact, perfectly good theories, such as general relativity, are not renormalizable and yet are still very successful at accounting for experiments. Some explanation is therefore required to justify this demand.

The physical motivation comes from the idea that physical theories generically come with an implicit minimum distance, d, (or maximum energy, Λ)

beyond which they are not meant to apply. For example, the quantum electrodynamics of electrons and photons is only physically correct up to an energy of twice the mass of the lightest particle that is heavier than the electron: $\Lambda = 2m_\mu$, twice the muon mass. At energies higher than this, muons can no longer be neglected, since they can be pair-produced in the final state even if they are not present initially. The correct theory for physics at energies above Λ becomes the quantum electrodynamics of photons, electrons and muons. This theory is in turn only valid up to the next threshold, the pion mass, and so on.

Classically, it is not important to specify this "cut-off" carefully. In a quantum theory, however, since all states can contribute to any given process as intermediate (or "virtual") particles, any quantum calculation will depend explicitly on the cut-off scale, Λ. This may be seen, for example, by considering the expression, in time-independent perturbation theory, for the quadratic energy shift due to a perturbing Hamiltonian,

$$\delta E_\psi = \sum_n{}' \frac{|\langle\psi|H|n\rangle|^2}{E_\psi - E_n} \qquad |n\rangle \neq |\psi\rangle \tag{1.34}$$

Clearly any state, $|n\rangle$, contributes to Eq. (1.34) regardless of its energy. Given our ignorance of the spectrum above the energy Λ, it only makes sense to include those states with energy less than Λ in this sum. The result therefore depends explicitly on Λ in a potentially complicated way.

If detailed knowledge of physics at the Λ scale is necessary in order to calculate probability amplitudes for processes at energies lower than Λ, then the theory is called *non-renormalizable*. These theories have less predictive power, since predictions depend on physics at the scale Λ, about which we are by assumption quite ignorant.

In *renormalizable* theories, on the other hand, Λ only appears in physical predictions (for large Λ) through a small number of parameters, such as the masses and charges of some or all of the particles involved. All other processes may then be computed in terms of these parameters. Once the few incalculable parameters are taken from experiment, definite predictions may be made.

Whether or not a renormalizable theory should be expected to describe a given system depends therefore on the properties of the system. Physically, successful description in terms of a renormalizable theory is equivalent to the statement that the physics of interest, at energies $E \ll \Lambda$, is largely insensitive to the higher-energy physics appropriate to the scale Λ. In general, a renormalizable description of the physics at an energy E is justified to the extent that contributions of order E/Λ are not important. Otherwise non-renormalizable interactions must be included.

As an example, consider the theory describing the energy levels of the hydrogen atom. Neglecting the structure of the nucleus, this theory is given by the quantum electrodynamics of pointlike electrons, protons, and photons. Ignoring nuclear structure (such as the proton magnetic moment) means neglecting powers of $E_{\rm atom}/M_{\rm proton}$, and the resulting theory is renormalizable. Within this theory atomic physics depends only on the electron and proton mass and charge. If we demand accuracy higher than $E_{\rm atom}/M_{\rm proton}$, the proton structure cannot be ignored, leading to a non-renormalizable description.

An example of a situation for which no renormalizable theory should be expected is provided by the theory describing the nuclear scattering of the deuteron. Suppose that in this theory we wish to ignore the fact that the deuteron consists of a proton and neutron bound by these same nuclear interactions, instead taking the deuteron as a point particle. The corresponding theory that describes the scattering data cannot be renormalizable. This reflects the fact that in this case the scale, Λ, of the physics being neglected (the nuclear binding) and the scale, E, of the physics being studied (the nuclear scattering) are essentially the same. Non-renormalizability is the theory's way of telling us that effects of order E/Λ cannot be neglected.

Turning this argument around, we can use the renormalizability of a theory to tell us what the next scale, Λ, of new physics is. If we succeed in describing all data at presently accessible energies, E, in terms of a renormalizable theory then we learn that the scale of any new physics can be large: $\Lambda \gg E$. If a non-renormalizable theory is required, we learn that we are still missing some fundamental ingredients.

This physical picture implies that renormalizability is the minimal criterion for a theory which purports to describe *all* of the physics appropriate to any given scale. Demanding renormalizability for the standard model then amounts to the assumption that no hitherto unknown particles or interactions are required to understand present experiments. As judged by the splendid success of the standard model, this turns out to be a fairly good assumption. The sole exception (at the time of this writing) is the physics of neutrino oscillations, which appears to demand new physics; this can be understood within the standard model as the existence of non-renormalizable interactions. We return to this point at some length in Chapter 10 (and more generally to the issue of renormalizability and high-dimension operators in Chapter 7). Note, for the current purposes, that the scale required to explain neutrino masses is $\Lambda \sim 10^{14}$ GeV. This is so much higher than the intrinsic scales in the electroweak theory that, if the standard model is correct up to this scale, there are virtually no other consequences of the high-energy physics expected, and therefore we are (otherwise) very well justified in treating the standard model as a renormalizable theory (with one possible exception, see Section 11.5).

1.2.3 Canonical quantization

We now turn to the problem of how to ensure that a given set of interactions incorporates the properties listed above. The most efficient way to do so is to set up the formalism in terms of the action

$$S = \int L(t)\, \mathrm{d}t \tag{1.35}$$

rather than the Hamiltonian. The conditions listed above for H then become relatively simple conditions for S.

H is related to S by the usual canonical methods. That is, given a set of physical variables q^i and a Lagrangian, $L(q, \dot{q})$, define the canonical momenta by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{1.36}$$

The Hamiltonian, H, is then given by

$$H = \sum_i p_i \dot{q}^i - L \tag{1.37}$$

In this last expression, Eq. (1.36) is supposed to be inverted to allow the elimination of $\dot{q}^i$ in favor of p_i. The formalism may be generalized in the case when this cannot be done, or when L depends on higher time-derivatives of q such as $\ddot{q}^i$ etc.

We consider the implications for S of each of the properties of the previous sections in turn.

(i) Unitarity:
H is real provided that the action, S, is real.

(ii) Locality:
In order for H to be a local function,

$$H = \int \mathrm{d}^3x\, \mathcal{H}(\mathbf{x}, t) \tag{1.38}$$

we require that L must also be expressed as an integral over a *Lagrangian density*:

$$L = \int \mathrm{d}^3x \, \mathcal{L}(\mathbf{x}, t) \tag{1.39}$$

$$\text{so} \qquad S = \int \mathrm{d}^4x \, \mathcal{L}(\mathbf{x}, t) \tag{1.40}$$

It is a customary abuse of language in quantum field theory to refer to the Lagrangian density as the Lagrangian.

Recall that $\mathcal{H}$ and $\mathcal{L}$, like any operators, are to be expressed in terms of the creation and annihilation operators, $a_{\mathbf{p}i}$ and $a^*_{\mathbf{p}i}$. But $\mathcal{H}$ and $\mathcal{L}$ are built of operators at a single spacetime point, which means that they must be built from the Fourier transforms of $a_{\mathbf{p}i}$ and $a^*_{\mathbf{p}i}$:

$$A_\alpha(\mathbf{x}, t) = \sum_k \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} u_\alpha(\mathbf{p}, k) a_{\mathbf{p}k} \, \mathrm{e}^{\mathrm{i}px} \tag{1.41}$$

In this equation $px = p^\mu x_\mu = -p^0 x^0 + \mathbf{p} \cdot \mathbf{x}$, with $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. α denotes any labels that distinguish the fields due to one particle type from another. The coefficients $u_\alpha(\mathbf{p}, k)$ ensure that both sides of the equation transform the same way under Lorentz transformations. The Lagrange density then becomes

$$\mathcal{L} = \mathcal{L}(x, A_\alpha(x), \partial_\mu A_\alpha(x), \ldots) \tag{1.42}$$

We return to the related consequences of causality after first considering Poincaré invariance.

(iii) Translation invariance:

Translation invariance implies that $\mathcal{L}$ depends on the spacetime coordinates $\mathbf{x}$ and t only implicitly through its dependence on $A_\alpha(x)$ and its derivatives:

$$\mathcal{L}(x, A_\alpha(x), \partial_\mu A_\alpha(x), \ldots) = \mathcal{L}(A_\alpha(x), \partial_\mu A_\alpha(x), \ldots) \tag{1.43}$$

(iv) Lorentz invariance:

Noether's theorem (see Subsection 1.4.2) allows the construction of the conserved charges P^μ and $J^{\mu\nu}$ provided that the action, S, is invariant under Poincaré transformations (unless there is an anomaly, see Subsection 2.5.3). From Eq. (1.40) this implies that $\mathcal{L}$ must be constructed out of the $A_\alpha(x)$ in such a way as to be a Lorentz scalar. In order to do so it is convenient to choose the fields, $A_\alpha(x)$, to transform in (finite-dimensional) representations of the Lorentz group

$$U(\omega)A_\alpha(x)U(\omega)^* = D_{\alpha\beta}A_\beta(\exp[\omega]\cdot x) \tag{1.44}$$

This, together with the transformation law for the single-particle states, determines the coefficients, $u_\alpha(\mathbf{p},k)$ appearing in Eq. (1.41). This is the main topic of Section 1.3. $\mathcal{L}$ must then be constructed from various combinations of the fields, their derivatives and the invariant tensors $\eta_{\mu\nu}$ and $\epsilon_{\mu\nu\lambda\rho}$.

(v) Causality:
Causality implies that bilinears of fields, such as the Hamiltonian density, must commute at spacelike separations. This is a strong condition, since the fields defined by Eq. (1.41) satisfy

$$[A_\alpha(\mathbf{x},t), A^*_\alpha(\mathbf{y},t)] \neq 0 \tag{1.45}$$

Causality is ensured provided that, for each particle, there exists another particle (its antiparticle) of equal mass and spin, described by the field

$$B_\alpha(x) = \sum_k \int \frac{\mathrm{d}^3p}{2E_\mathbf{p}(2\pi)^3} v_\alpha(\mathbf{p},k) b_{\mathbf{p}k}\, \mathrm{e}^{\mathrm{i}px} \tag{1.46}$$

$\mathcal{L}$ must depend on the fields $A(x)$ and $B(x)$ only through the combination

$$\phi_\alpha(x) = A_\alpha(x) + \xi B^*_\alpha(x) \tag{1.47}$$

in which ξ is a phase, since in this case

$$[\phi_\alpha(\mathbf{x},t), \phi^*_\alpha(\mathbf{y},t)] = 0 \tag{1.48}$$

In general the antiparticle need not be distinct from the particle. If the particle and antiparticle are identical, $a_{\mathbf{p}k} = b_{\mathbf{p}k}$, then ξ can be chosen such that $\phi = \phi^*$.

This observation has three physical consequences.

(a) Antiparticles exist and couple with a strength identical to particles. This is called *crossing symmetry*. Since H_{int} involves $a_{\mathbf{p}k}$ and $b_{\mathbf{p}k}$ only in the schematic combination $a_{\mathbf{p}k} + b^*_{\mathbf{p}k}$ there are *no* interactions that can conserve the total number of particles.

(b) For fermions the fields must anticommute at spacelike separations. For general spins the condition that bilinears, such as H_0, commute for space-like separations implies that integer-spin particles must be bosons and half-integer-spin particles must be fermions – the spin-statistics theorem.

(c) The behavior of particles and antiparticles under symmetries such as parity or gauge transformations are related. In particular the electric charge of a particle is the opposite of that of the antiparticle.

(vi) Stability:
The generalization of the canonical method to theories with higher time derivatives shows that the Hamiltonian is in this case generically linear in one of its variables. Such a Hamiltonian cannot be bounded from below. Stability then implies that the Lagrangian must be a function of at most one time derivative of the fields. In practice, this forbids the appearance of more than quadratic powers of derivatives of fields.

(vii) Renormalizability:
Renormalizability may be summarized as the requirement that all parameters that appear in the Lagrangian must have positive dimension in powers of mass. That is to say, if the operator $\mathcal{O}$ appears in $\mathcal{L}$ with a coefficient c:

$$\mathcal{L} = c\,\mathcal{O} \tag{1.49}$$

then c must have dimension M^d for $d \geq 0$. Since all of the constituents, $A_\alpha(x)$ and ∂_μ, of $\mathcal{O}$ each have dimension M^p for $p > 0$ and $\mathcal{L}$ has dimension M^4, this severely limits the allowed interactions to only include operators for which $d \leq 4$. Generally, all such interactions which are consistent with the assumed symmetries must be included.

1.3 Free field theory

In this book we will generally be interested in theories which, at least on some energy scale, can be described in terms of *weakly coupled* particles; that is, by a Hamiltonian which is dominated by a "free theory" piece H_0, with interactions H_I which can be treated by perturbation theory. The standard model turns out to be such a theory, and most of the tools we have available to study quantum field theories are based on this assumption. In most of this book we will only treat corrections to the free-theory approximation at the leading order, that is, at the lowest power in the interaction Hamiltonian H_I at which the phenomena of interest happen.

To proceed with this project we first need to see what the most general free field theories can look like. We focus on particles with spins zero through one since all known non-gravitational experiments appear to be describable in terms of these, and since renormalizability seems to require an interacting field theory to be composed of such particles.

Recall that the Hamiltonian for a system of free particles is given by

$$H_0 = E_0 + \sum_i \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} E_{\mathbf{p}} a^*_{\mathbf{p}i} a_{\mathbf{p}i} \tag{1.50}$$

which is quadratic in the operators $a_{\mathbf{p}i}$. We wish to construct the corresponding Lagrangian in terms of the fields, $A_\alpha(x)$. Since the fields are linear in the creation and annihilation operators the desired Lagrangian density, $\mathcal{L}_0$, must also be at most quadratic in the A_αs and their derivatives.

The discussion will use properties of the Lorentz group, which are reviewed in Appendix C.

1.3.1 Spin-zero particles

Spin-zero particles are described by fields that transform as scalars under Lorentz transformations. That is,

$$U(\omega)\phi(x)U(\omega)^* = \phi(\Lambda \cdot x) \tag{1.51}$$

where $\Lambda^\mu{}_\nu = (\exp\omega)^\mu_\nu$ is a Lorentz-transformation matrix. In terms of creation and annihilation operators

$$\phi(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} \left[a_{\mathbf{p}}\, \mathrm{e}^{\mathrm{i}px} + \overline{a}^*_{\mathbf{p}}\, \mathrm{e}^{-\mathrm{i}px}\right] \tag{1.52}$$

in which

$$E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2} \tag{1.53}$$

and $px = -E_{\mathbf{p}}x^0 + \mathbf{p}\cdot\mathbf{x}$. The field $\phi(x)$ has been chosen real, as may be done without loss of generality because any complex field can always be decomposed into its real and imaginary parts. The energy relation, Eq. (1.53), implies that the four-momentum p^μ satisfies

$$p^\mu p_\mu = -E^2_{\mathbf{p}} + \mathbf{p}^2 = -m^2 \tag{1.54}$$

which becomes the Klein–Gordon equation

$$(-\partial_\mu\partial^\mu\phi + m^2\phi) = 0 \tag{1.55}$$

in position space. In the canonical approach these conditions are derived as equations of motion from the action rather than the representation theory of the Poincaré group.

We now consider the most general possible theory of several scalars, and show that it always reduces to a set of independent scalars, with potentially different masses. Consider then, a system of N types of spinless particles. Such a system may be described in terms of N real fields, $\phi^i(x)$, with $i = 1, \ldots, N$. The most general Lagrangian that is Poincaré invariant, involves only two time derivatives (stability), and is quadratic in these N fields, is

$$\mathcal{L}_0 = -\frac{1}{2}A_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{2}B_{ij}\phi^i\phi^j - C \tag{1.56}$$

A sum from 1 to N is implied over repeated indices.

A term such as

$$D_{ij}\phi^i\partial^\mu\partial_\mu\phi^j$$

is not included since it is equivalent to

$$-D_{ij}\partial^\mu\phi^i\partial_\mu\phi^j$$

after an integration by parts. This Lagrangian is real (unitarity) provided that the (symmetric) coefficients A_{ij}, B_{ij} and C all are.

The corresponding conjugate momentum and Hamiltonian are:

$$\pi_i(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{\phi}^i} = A_{ij}\dot{\phi}^j(x) \tag{1.57}$$

$$\text{so} \quad \mathcal{H}_0 = \pi_i\dot{\phi}^i - \mathcal{L}_0$$

$$= +\frac{1}{2}\Big[A_{ij}\dot{\phi}^i\dot{\phi}^j + A_{ij}\nabla\phi^i\cdot\nabla\phi^j + B_{ij}\phi^i\phi^j\Big] + C \tag{1.58}$$

This Hamiltonian is bounded below provided that the matrices A_{ij} and B_{ij} are non-negative definite. We assume in what follows that A_{ij} is strictly positive definite, since there would otherwise be a particle without any kinetic energy.

There are considerably more parameters appearing in the Lagrangian, Eq. (1.56), than appeared in the Hamiltonian, Eq. (1.50). This is because many of the constants in Eq. (1.56) may be absorbed into redefinitions of the field variables by putting $\mathcal{L}_0$ into *canonical form*. Only linear transformations

$$\phi^i = M^i_j\phi'^j \equiv (M\phi')^i \tag{1.59}$$

need be considered since these are the only ones that ensure that $\mathcal{L}_0$ remains quadratic when expressed in terms of the new variable, ϕ'^j. We use this freedom to put A_{ij} and B_{ij} into standard form.

Since A_{ij} is assumed positive definite, its eigenvalues $a_1, \ldots, a_N$ are all positive and its square root and inverse exist. If we define the new fields ϕ'^i as

$$\phi^i = (A^{-1/2}\phi')^i \tag{1.60}$$

then $\mathcal{L}_0$ becomes

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi'^i\partial^\mu\phi'^i - \frac{1}{2}B'_{ij}\phi'^i\phi'^j - C \tag{1.61}$$

where

$$B'_{ij} \equiv (A^{-1/2}BA^{-1/2})_{ij} \tag{1.62}$$

This does not exhaust the freedom (1.59) to redefine fields. Indeed, the transformation $\phi' = \mathcal{O}\varphi$ in which $\mathcal{O}^T\mathcal{O} = I$ preserves the form (1.61). Recall now that any real symmetric matrix can be diagonalized by an orthogonal transformation

$$\mathcal{O}^T B'\mathcal{O} = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & & \\ & & & b_N \end{pmatrix} \tag{1.63}$$

with $b_k \geq 0$ from stability. The redefinition $\phi' = \mathcal{O}\varphi$ with this $\mathcal{O}$ then diagonalizes the *mass matrix*, B'_{ij}, giving:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\varphi^i\partial^\mu\varphi^i - \frac{1}{2}b_i\varphi^i\varphi^i - C \tag{1.64}$$

$$\text{and } \mathcal{H}_0 = \frac{1}{2}\dot{\varphi}^i\dot{\varphi}^i + \frac{1}{2}(\nabla\varphi^i)\cdot(\nabla\varphi^i) + \frac{1}{2}b_i\varphi^i\varphi^i + C \tag{1.65}$$

Unless some of the eigenvalues of the matrix B_{ij} are degenerate, this exhausts our freedom to linearly redefine fields. Equation (1.64) is then the standard form for $\mathcal{L}_0$. The equations of motion are

$$(-\partial^\mu\partial_\mu + b_i)\varphi^i = 0 \tag{1.66}$$

The parameters appearing in $\mathcal{L}_0$ may be related to the physical vacuum energy, E_0, and masses, m_i, by expressing the total Hamiltonian, Eq. (1.65), in terms of $a_{\mathbf{p}i}$ and comparing to Eq. (1.50):

$$\begin{aligned} H_0 &= \int d^3x\, \mathcal{H}(x) \\ &= E_0 + \sum_{i=1}^{N}\int \frac{d^3\mathbf{p}}{2E_\mathbf{p}(2\pi)^3}E_\mathbf{p}a^*_{\mathbf{p}i}a_{\mathbf{p}i} \end{aligned} \tag{1.67}$$

$$\text{with } E_\mathbf{p} = \sqrt{\mathbf{p}^2 + b_i} \tag{1.68}$$

$$\text{and } E_0 = C\int d^3x + \sum_i \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} (2\pi)^3 \delta^3(0) \tag{1.69}$$

Clearly the eigenvalues $b_i = m_i^2$ give the square of the particle masses. The vacuum energy is more delicate since it diverges at both long and short distances. The long-distance divergence may be regularized by putting the system within a space of finite, but large, volume Ω. The divergence of E_0 as $\Omega \to \infty$ merely indicates that the total energy is not the quantity of physical interest, since the total energy is by construction an extensive variable that grows with the size of the system. The well behaved quantity in this limit is the *energy density*, $\rho = E_0/\Omega$. Using

$$(2\pi)^3\delta^3(0) = \int_\Omega d^3x \, e^{i(\mathbf{p}=0)\cdot\mathbf{x}} = \Omega \tag{1.70}$$

the energy density is

$$\begin{aligned} \frac{E_0}{\Omega} &= C + \sum_{i=1}^{N} \int_0^\Lambda \frac{1}{2} \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \\ &= C + \frac{1}{16\pi^2} \sum_{i=1}^{N} \left[\Lambda^4 + m_i^2 \Lambda^2 - \frac{1}{4} m_i^4 \log\left(\frac{\Lambda^2}{m_i^2}\right) + o\left(\frac{m_i^2}{\Lambda^2}\right) \right] \end{aligned} \tag{1.71}$$

The short distance divergence has been regulated by cutting off the integration at a maximum momentum, $|p| < \Lambda$. The Λ-dependence can then be renormalized by canceling it with a Λ-dependent constant C.

1.3.2 Spin-half particles

We assume familiarity with the Dirac equation and the Lorentz group in the following; readers unfamiliar with one or both may consult Appendix C.

Spin-half particle states are labeled by $|\mathbf{p}, \sigma\rangle$, in which the label $\sigma = \pm\frac{1}{2}$ represents the projection of intrinsic angular momentum along some axis. Representation theory of the Poincaré group implies that spin-$\frac{1}{2}$ particles are most easily represented by *spinor* fields. Four-component spinor fields transform as follows under Lorentz transformations:

$$U(\omega)\psi(x)U(\omega)^* = D(-\omega)\psi(\Lambda \cdot x) \tag{1.72}$$

in which $D(\omega)$ is the four-by-four matrix given explicitly by

$$D(\omega) = \exp\left[\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right] \tag{1.73}$$

with the matrices $\mathcal{J}^{\mu\nu}$ given, in the chiral basis which will be used throughout this book, by

$$\mathcal{J}_k = \frac{1}{2}\epsilon_{klm}\mathcal{J}^{lm} = \begin{pmatrix} \frac{1}{2}\sigma_k & 0 \\ 0 & \frac{1}{2}\sigma_k \end{pmatrix} \tag{1.74}$$

$$\mathcal{K}_k = \mathcal{J}_{k0} = \begin{pmatrix} -\frac{\mathrm{i}}{2}\sigma_k & 0 \\ 0 & \frac{\mathrm{i}}{2}\sigma_k \end{pmatrix} \tag{1.75}$$

Here the two-by-two matrices, σ_k with $k = 1, 2, 3$, denote the usual Pauli spin matrices.

It is clear that this representation is block-diagonal and so is *reducible*. That is, the upper two components of a spinor field never "mix" with the lower two components under any Poincaré transformation. Therefore, it is consistent to consider quantum field theories in which only the upper or lower components of a spinor exist as fields of the theory. Though this does not happen for quantum electrodynamics–the electron can be represented by a 4-component Dirac spinor – it turns out that it *does* happen for *every* spinor field in the standard model.

There are two equivalent choices of notation to handle such fields, which we will now list.

(i) *Weyl* spinors:

A Weyl spinor is one for which the upper two or lower two components are zero. That is, define *left-handed* and *right-handed* spinors by

$$\psi_{\mathrm{L}} = \frac{1}{2}(1+\gamma_5)\psi = P_{\mathrm{L}}\,\psi = \begin{pmatrix} \xi \\ 0 \end{pmatrix} \tag{1.76}$$

$$\psi_{\mathrm{R}} = \frac{1}{2}(1-\gamma_5)\psi = P_{\mathrm{R}}\,\psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \tag{1.77}$$

in which ξ and χ are two-component objects and γ_5 is the following four-by-four matrix:

$$\gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (= -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3) \tag{1.78}$$

I here denotes the two-by-two unit matrix, and γ^μ are defined below.

(ii) *Majorana* spinors:

Alternately, we may work in terms of 4-component spinors where the bottom two components are not independent but are determined by the upper two components. Specifically, first define a two-by-two, real antisymmetric matrix ε,

$$\varepsilon \equiv \mathrm{i}\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{1.79}$$

Now note that if ξ is left-handed under Lorentz transformations, then $\chi = \varepsilon\xi^*$ is right-handed. This follows from the property

$$\varepsilon\sigma_i^* = -\sigma_i\varepsilon \tag{1.80}$$

With this in mind, a Majorana spinor is then defined by

$$\psi_{\mathrm{M}} \quad = \quad \begin{pmatrix} \xi \\ \varepsilon\xi^* \end{pmatrix} \tag{1.81}$$

These two formulations of fermions with two independent components are equivalent, and the choice of which one to use to formulate a theory is a matter of taste. It is our preference in this book to work with the Majorana notation, mostly because it is simple to make contact with the 4 component γ-matrix algebra in which calculations are generally performed.

The relation between a Majorana spinor field and the creation and annihilation operators is

$$\psi(x) = \sum_{\sigma=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} \left[u(\mathbf{p},\sigma)a_{\mathbf{p}\sigma}\mathrm{e}^{\mathrm{i}px} + v(\mathbf{p},\sigma)a^*_{\mathbf{p}\sigma}\mathrm{e}^{-\mathrm{i}px}\right] \tag{1.82}$$

In this expression $\psi(x)$, $u(\mathbf{p},\sigma)$, and $v(\mathbf{p},\sigma)$ are all 4-component objects with $v(\mathbf{p},\sigma)$ defined in terms of $u(\mathbf{p},\sigma)$ by

$$v\left(\mathbf{p},\sigma=\pm\frac{1}{2}\right) = \pm\gamma_5 u\left(\mathbf{p},\sigma=\mp\frac{1}{2}\right) \tag{1.83}$$

It turns out that for this decomposition to be consistent with Lorentz invariance, the spinor u in the rest frame, $\mathbf{p}=0$, must satisfy

$$m\beta u\left(\mathbf{p}=0,\sigma=\pm\frac{1}{2}\right) = mu\left(\mathbf{p}=0,\sigma=\pm\frac{1}{2}\right) \tag{1.84}$$

where β denotes the following matrix:

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \left(=\mathrm{i}\gamma^0\right) \tag{1.85}$$

The non-zero $\mathbf{p}$ generalization of Eq. (1.84) can be found by applying a boost, using Eq. (1.73). The mass m on the right-hand side becomes the four-vector p_μ, which in the rest frame has a single component, $E=m$. The matrix $-i\beta$ is really the time component of a four-vector of matrices, the *Dirac* matrices γ^μ, so Eq. (1.84) in a general frame becomes

$$(\mathrm{i}\not{p} + m)u(\mathbf{p}, \sigma) = 0 \tag{1.86}$$

with $\not{p}$ defined by $\not{p} = \gamma_\mu p^\mu$ (and in general $\not{a} \equiv \gamma^\mu a_\mu$). Equation (1.73) uniquely determines the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \gamma_k = \begin{pmatrix} 0 & -\mathrm{i}\sigma_k \\ \mathrm{i}\sigma_k & 0 \end{pmatrix} \tag{1.87}$$

In position space, Eq. (1.86) is the *Dirac equation*:

$$(\not{\partial} + m)\psi = 0 \tag{1.88}$$

The Dirac or gamma matrices γ^μ used here differ by a factor of i from the form they would take if we adopted a $\eta_{\mu\nu} = \mathrm{diag}[+---]$ Lorentz metric notation. The reader should be aware of this notation choice. This is discussed in some detail in Appendix E.

The matrix $\gamma^0 = -\mathrm{i}\beta$ is anti-Hermitian, while the spatial γ matrices are Hermitian. Therefore the Dirac matrices transform differently under Hermitian conjugation. Similarly, the matrices $\mathcal{J}_k$ which perform rotations are Hermitian, while the matrices $\mathcal{K}_k$ which perform boosts are anti-Hermitian; so $D^\dagger(\omega)$ does not equal $D^{-1}(\omega)$ in general. However, the matrix β satisfies

$$\beta = \beta^\dagger = \beta^T = \beta^{-1}, \quad \beta\gamma_\mu^\dagger = -\gamma_\mu\beta, \quad \beta\gamma_5 = -\gamma_5\beta \tag{1.89}$$

Also, since $\mathcal{J}^{\mu\nu} = -\mathrm{i}[\gamma^\mu, \gamma^\nu]/4$, these imply that

$$\mathcal{K}_k^\dagger\beta = \beta\mathcal{K}_k, \quad \mathcal{J}_k^\dagger\beta = \beta\mathcal{J}_k \tag{1.90}$$

Because of these properties of β, it is convenient to define the *Dirac conjugate* of a spinor, $\overline{\psi}$, as

$$\overline{\psi} \equiv \psi^\dagger\beta \tag{1.91}$$

which transforms under Lorentz transformations as

$$U(\omega)\overline{\psi}(x)U(\omega)^* = \overline{\psi}(\Lambda \cdot x)D^{-1}(-\omega) \tag{1.92}$$

Therefore $\overline{\psi}\psi$ transforms as a Lorentz scalar. As can be readily checked, $D^{-1}(\omega)\gamma^\mu D(\omega) = \Lambda^\mu{}_\nu\gamma^\nu$, so $\overline{\psi}\gamma^\mu\psi$ transforms as a vector.

It is also convenient to introduce the *charge conjugation matrix* C, as the matrix which relates a Majorana spinor to its Dirac conjugate:

$$C = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} (= \gamma^2\beta), \quad \text{so} \quad \psi_{\mathrm{M}} = C\overline{\psi}_{\mathrm{M}}^T, \quad \text{and} \quad \psi_{\mathrm{M}}^T = -\overline{\psi}_{\mathrm{M}}C \tag{1.93}$$

Its properties are

$$-C = C^\dagger = C^{-1} = C^T\,, \quad \gamma_\mu^T C = -C\gamma_\mu\,, \quad C\beta = -\beta C\,, \quad C\gamma_5 = \gamma_5 C \tag{1.94}$$

Returning to Eq. (1.86), we can solve explicitly for the spinor $u(\mathbf{p}, \sigma)$, giving

$$u(\mathbf{p}, \sigma) = \frac{1}{\sqrt{2}} \begin{pmatrix} A_+ - A_- \sigma \cdot \hat{\mathbf{p}} & 0 \\ 0 & A_+ + A_- \sigma \cdot \hat{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \chi(\sigma) \\ \chi(\sigma) \end{pmatrix} \tag{1.95}$$

where

$$\chi\left(\sigma = +\frac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi\left(\sigma = -\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.96}$$

$\hat{\mathbf{p}}$ is the unit vector $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, and the coefficients $A_\pm$ are the following functions of the particle energy $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$:

$$A_\pm(\mathbf{p}) = \sqrt{E_{\mathbf{p}} \pm m} \tag{1.97}$$

As defined by Eq. (1.95), $u(\mathbf{p}, \sigma)$ satisfies the normalization condition:

$$\overline{u}(\mathbf{p}, \sigma')u(\mathbf{p}, \sigma) = 2m\delta_{\sigma\sigma'} \tag{1.98}$$

The dyadics $u\overline{u}$ and $v\overline{v}$ are often encountered in calculations. They can be thought of as matrices, with values

$$u(\mathbf{p}, \sigma)\overline{u}(\mathbf{p}, \sigma) = \frac{1}{2}(m - \mathrm{i}\not{p})(1 + \mathrm{i}\gamma_5 \not{s}) \tag{1.99}$$

and

$$v(\mathbf{p}, \sigma)\overline{v}(\mathbf{p}, \sigma) = -\frac{1}{2}(m + \mathrm{i}\not{p})(1 + \mathrm{i}\gamma_5 \not{s}) \tag{1.100}$$

In these expressions $s^\mu(\sigma)$ is the *spin axial four-vector*. It is defined in the following way. Suppose the spin projection, $\sigma = \pm\frac{1}{2}$, is measured along the direction defined by the unit vector $\mathbf{e}$ in the particle rest frame. Define s^μ in this frame by $s^0 = 0$ and $\mathbf{s} = \pm\mathbf{e}$ in which the sign $\pm$ denotes the sign of σ. The result in any other frame is found by performing the appropriate Lorentz boost. Notice that this definition implies the following invariant properties:

$$s^2 = s^\mu s_\mu = +1 \qquad \text{and} \qquad s \cdot p = s^\mu p_\mu = 0 \tag{1.101}$$

Now we repeat the exercise of showing that it is always possible to write a free theory of spin-half particles in a canonical form. Consider the Lagrangian description of a system of N non-interacting spin-half particles. Just as there is no loss in choosing our scalar fields to be real, we may always take our spinor fields to be Majorana. The Lagrangian must then be a Lorentz-invariant function of N Majorana spinors, ψ^m, that is at most quadratic in the fields and involves the fewest (nonzero) number of derivatives. The most general such Lagrangian is

$$\mathcal{L}_0 = -\frac{1}{2}A_{mn}\overline{\psi}^m \partial\!\!\!/ \psi^n - \frac{\mathrm{i}}{2}B_{mn}\overline{\psi}^m\gamma_5\partial\!\!\!/\psi^n - \frac{1}{2}C_{mn}\overline{\psi}^m\psi^n - \frac{\mathrm{i}}{2}D_{mn}\overline{\psi}^m\gamma_5\psi^n - E \tag{1.102}$$

The Lagrangian must be Hermitian; together with the results of problem 1.1, this implies that A_{mn}, B_{mn}, C_{mn}, D_{mn}, and E must all be real. We may also take the matrices A, C, and D symmetric and B antisymmetric, since the operators multiplying them have the same property.

As usual, most of the parameters in this Lagrangian may be eliminated by performing field redefinitions. The purpose of the remainder of this section is to use this freedom to put the Lagrangian (Eq. (1.102)) into a standard form in which all parameters have an obvious physical significance. Consider then the following field redefinition:

$$\psi^m = V^m_n\psi'^n + \mathrm{i}U^m{}_n\gamma_5\psi'^n \tag{1.103}$$

with real matrices V and U. This is the most general transformation that preserves the Majorana character of the spinors and the quadratic form of the Lagrangian. It is convenient in what follows to handle the left- and right-handed parts of the fields separately. We therefore rewrite Eq. (1.102) and Eq. (1.103) as

$$P_{\mathrm{L}}\,\psi^m = (V + \mathrm{i}U)^m_n P_{\mathrm{L}}\,\psi'^n \tag{1.104}$$

$$P_{\mathrm{R}}\,\psi^m = (V - \mathrm{i}U)^m_n P_{\mathrm{R}}\,\psi'^n \tag{1.105}$$

$$\mathcal{L}_0 = -\frac{1}{2}\left[(A+\mathrm{i}B)_{mn}\overline{\psi}^m P_{\mathrm{L}}\,\partial\!\!\!/\psi^n + (C+\mathrm{i}D)_{mn}\overline{\psi}^m P_{\mathrm{L}}\,\psi^n\right] + \mathrm{h.c.} - E \tag{1.106}$$

Define the complex matrices $\mathcal{A} = (A + \mathrm{i}B)$, $\mathcal{C} = (C + \mathrm{i}D)$, and $\mathcal{V} = (V + \mathrm{i}U)$. The properties of A, B, C, and D then imply that $\mathcal{A}$ is Hermitian and $\mathcal{C}$ is symmetric. For stability we require that $\mathcal{A}$ be positive definite. In terms of the new variables the Lagrangian is then:

$$\mathcal{L}_0 = -\frac{1}{2}\left[(\mathcal{V}^T\mathcal{A}\mathcal{V}^*)_{mn}\overline{\psi}'^m P_{\mathrm{L}}\,\partial\!\!\!/\psi'^n - (\mathcal{V}^T\mathcal{C}\mathcal{V})_{mn}\overline{\psi}'^m P_{\mathrm{L}}\,\psi'^n\right] + \mathrm{h.c.} - E \tag{1.107}$$

In order to simplify $\mathcal{L}_0$ choose $\mathcal{V}$ as follows:

$$\mathcal{V} = (\mathcal{A}^*)^{-\frac{1}{2}}\mathcal{M} \tag{1.108}$$

in which $\mathcal{M}$ is the unitary matrix that satisfies the following property:

$$\mathcal{M}^T \mathcal{C}' \mathcal{M} = \begin{pmatrix} c_1 & & \\ & c_2 & \\ & & c_N \end{pmatrix} \tag{1.109}$$

$\mathcal{C}'$ is the complex symmetric matrix $\mathcal{C}' = [\mathcal{A}^{-\frac{1}{2}}\mathcal{C}(\mathcal{A}^*)^{-\frac{1}{2}}]$. For any such matrix, a unitary matrix, $\mathcal{M}$, satisfying (1.109) always exists (see Problem 1.6). In fact, $\mathcal{M}$ may always be chosen such that the numbers $c_k, k = 1, \ldots, N$ are all real and non-negative. It must be emphasized that since Eq. (1.109) is *not* a similarity transformation, the c_k are *not* the eigenvalues of the matrix $\mathcal{C}$ or $\mathcal{C}'$. Instead, c_k^2 turn out to be the eigenvalues of the Hermitian matrix $\mathcal{C}'^\dagger \mathcal{C}'$.

Having made this redefinition, the Lagrangian is in canonical form:

$$\mathcal{L}_0 = -\frac{1}{2}\overline{\psi}^m \partial\!\!\!/ \psi^m - \frac{1}{2} c_m \overline{\psi}^m \psi^m - E \tag{1.110}$$

The equation of motion for this action is

$$(\partial\!\!\!/ + c_m)\psi^m = 0 \tag{1.111}$$

which is recognized as the Dirac equation with mass c_m. To confirm this connection we compare the resulting free Hamiltonian with the general form (1.50):

$$\begin{aligned} H_0 &= \int \mathrm{d}^3x \, \overline{\psi}^m (\gamma \cdot \nabla + c_m)\psi^m + E \\ &= E_0 + \sum_{m=1}^{N} \sum_{\sigma=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3\mathbf{p}}{2E_\mathbf{p}(2\pi)^3} E_\mathbf{p} a^*_{\mathbf{p}\sigma m} a_{\mathbf{p}\sigma m} \qquad (1.112) \\ \text{with } E_0 &= E \int \mathrm{d}^3x - \sum_m \sum_\sigma \frac{1}{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_\mathbf{p} (2\pi)^3 \delta^3(0) \qquad (1.113) \end{aligned}$$

The corresponding vacuum energy density is

$$\frac{E_0}{\Omega} = E - \frac{1}{8\pi^2} \sum_{i=1}^{N} \left[\Lambda^4 + m_i^2 \Lambda^2 - \frac{1}{4} m_i^4 \log\left(\frac{\Lambda^2}{m_i^2}\right) + o\left(\frac{m_i^2}{\Lambda^2}\right) \right] \tag{1.114}$$

Notice the relative factor of -2 between the zero-point energy, Eq. (1.114), of free spin-half Majorana fermions and that, Eq. (1.71), of free real scalars.

1.3.3 Spin-one particles

The fields that are most convenient for representing spin-one particles differ for massive and massless particles. This is as might have been expected given that massive and massless spin-one particles have differing numbers of spin states. The particle states are labeled by $|\mathbf{p}, \lambda\rangle$ in which $\lambda = \pm 1$ for massless particles and $\lambda = 0, \pm 1$ for massive ones.

1.3.3.1 Massive spin-one particles

Massive particles are most conveniently represented in terms of a four-vector field, V^μ. This transforms under a Lorentz transformation according to

$$U(\omega) V^\mu(x) U(\omega)^* = (\Lambda^{-1})^\mu{}_\nu V^\nu(\Lambda \cdot x) \tag{1.115}$$

The relation between such a field and the creation and annihilation operators for a massive spin-one particle is,

$$V^\mu(x) = \sum_{\lambda=-1}^{1} \int \frac{\mathrm{d}^3\mathbf{p}}{2E_\mathbf{p}(2\pi)^3} \Big[\epsilon^\mu(\mathbf{p},\lambda) a_{\mathbf{p}\lambda}\, \mathrm{e}^{\mathrm{i}px} + \epsilon^{\mu *}(\mathbf{p},\lambda) a^*_{\mathbf{p}\lambda}\, \mathrm{e}^{-\mathrm{i}px}\Big] \tag{1.116}$$

Here the three four-vectors $\epsilon^\mu(\mathbf{p}, \lambda)$ denote the three linearly independent directions that correspond to each polarization λ. For example, for linearly polarized particles these would correspond to the three unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ in the particle rest frame. For circularly polarized particles choose instead the combinations $\mathbf{e}_z$ and $\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y)$. These polarization vectors are all characterized by the covariant constraint that is the analogue of Eq. (1.101):

$$p_\mu \epsilon^\mu(\mathbf{p}, \lambda) = 0 \tag{1.117}$$

They satisfy the normalization condition

$$\epsilon^{\mu *}(\mathbf{p}, \lambda) \epsilon_\mu(\mathbf{p}, \lambda') = \delta_{\lambda\lambda'} \tag{1.118}$$

and completeness relation

$$\sum_{\lambda=-1}^{1} \epsilon_\mu(\mathbf{p}, \lambda) \epsilon_\nu{}^*(\mathbf{p}, \lambda) = \eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \tag{1.119}$$

Together with the condition $p^2 + m^2 = 0$, Eq. (1.115) implies that in position space $V^\mu(x)$ must satisfy

$$(-\partial^2 + m^2) V^\mu = 0 \qquad \text{and} \qquad \partial^\mu V_\mu = 0 \tag{1.120}$$

These are the conditions that V^μ must satisfy in order to represent massive spin-one particles.

Turn now to the Lagrangian formulation of a system of free massive spin-one particles. We must construct the most general quadratic, Lorentz-invariant etc. Lagrangian whose equations of motion imply Eq. (1.120). The new feature here is that the condition that the equations of motion be equivalent to Eq. (1.120) will be found to impose conditions on what form we may entertain for the Lagrangian. This is unlike what we encountered for spin-zero and spin-half particles, where the most general Lagrangian automatically implied the analogues of Eq. (1.120), i.e. the Klein–Gordon or Dirac equations. This new feature arises because, unlike for scalar or spinor fields, a four-vector may a priori represent particles of more than one spin. It may correspond to either spin zero or spin one. (Schematically, a vector represents a spin-zero particle when it is the gradient of a scalar.)

To see how this works consider the most general quadratic Lagrangian for a single vector field, given by

$$\mathcal{L}_0 = -\frac{1}{2}A\partial_\mu V_\nu \partial^\mu V^\nu - \frac{1}{2}B\partial_\mu V_\nu \partial^\nu V^\mu - \frac{1}{2}CV^\mu V_\mu - D \tag{1.121}$$

The constants A, B, C, and D must all be real. The equations of motion for such a Lagrangian are

$$A\Box V^\mu + B\partial^\mu \partial_\nu V^\nu - CV^\mu = 0 \tag{1.122}$$

Taking the divergence of Eq. (1.122) gives the further equation,

$$[(A+B)\Box - C]\partial^\mu V_\mu = 0 \tag{1.123}$$

These equations only imply that $\partial \cdot V = 0$ when $A + B = 0$ and $C \neq 0$. In this case they are equivalent to Eq. (1.120). We may also always rescale V^μ to ensure that $A = 1$. The Lagrangian must therefore be

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{2}(\partial_\mu V_\nu \partial^\mu V^\nu - \partial_\mu V_\nu \partial^\nu V^\mu) - \frac{1}{2}C' V^\mu V_\mu - D \\ &= -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{2}C' V^\mu V_\mu - D\end{aligned} \tag{1.124}$$

in which $f_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, which is called the *field strength*.

Comparison with Eq. (1.120) or the expression for the corresponding free Hamiltonian implies that $C' = C/A = m^2$ has the interpretation of the squared mass of the particle being described. The vacuum energy is similarly

$$\frac{E_0}{\Omega} = D + \frac{3}{16\pi^2}\sum_{i=1}^{N}\left[\Lambda^4 + m_i^2\Lambda^2 - \frac{1}{4}m_i^4 \log\left(\frac{\Lambda^2}{m_i^2}\right) + O\left(\frac{m_i^2}{\Lambda^2}\right)\right] \tag{1.125}$$

For N massive spin-one particles the argument above, together with one that exactly parallels that given for scalar fields, implies that the most general Lagrangian,

$$\mathcal{L}_0 = -\frac{1}{2}A_{ab}\partial_\mu V^a_\nu \partial^\mu V^{b\nu} - \frac{1}{2}B_{ab}\partial_\mu V^a_\nu \partial^\nu V^{b\mu} - \frac{1}{2}C_{ab}V^{a\mu}V^b_\mu - D \quad (1.126)$$

may be rewritten as

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{2}(\partial_\mu V^a_\nu \partial^\mu V^{a\nu} - \partial_\mu V^a_\nu \partial^\nu V^{a\mu}) - \frac{1}{2}C'_a V^{a\mu}V^a_\mu - D \\ &= -\frac{1}{4}f^a_{\mu\nu}f^{a\mu\nu} - \frac{1}{2}C'_a V^{a\mu}V^a_\mu - D \end{aligned} \quad (1.127)$$

1.3.3.2 Massless spin-one particles

Massless spin-one particles are, on the other hand, most conveniently represented in terms of an antisymmetric tensor field, $f_{\mu\nu}$. The relation between such a field and the creation and annihilation operators for a massless spin-one particle are:

$$f_{\mu\nu}(x) = \sum_{\lambda=\pm1}\int \frac{d^3\mathbf{p}}{2E_\mathbf{p}(2\pi)^3}\Big[(\mathrm{i}p_\mu \epsilon_\nu(\mathbf{p},\lambda) - \mathrm{i}p_\nu \epsilon_\mu(\mathbf{p},\lambda))\, a_{\mathbf{p}\lambda}\, \mathrm{e}^{\mathrm{i}px} + \mathrm{h.c.}\Big] \quad (1.128)$$

Here the two quantities $\epsilon^\mu(\mathbf{p},\lambda)$ denote the linearly independent directions that correspond to each polarization λ. For particles moving along the Z axis, linearly polarized particles correspond to the choice of the unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ perpendicular to the particle motion. The alternative combinations $\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y)$ correspond instead to circularly polarized particles.

Notice that Eq. (1.128) only determines the polarization vector, ϵ^μ, up to the *gauge* freedom

$$\epsilon^\mu(\mathbf{p},\lambda) \to \epsilon^\mu(\mathbf{p},\lambda) + p^\mu \quad (1.129)$$

This freedom may be used to ensure that ϵ^μ satisfies the following Lorentz-covariant properties:

$$\overline{p}_\mu \epsilon^\mu(\mathbf{p},\lambda) = p_\mu \epsilon^\mu(\mathbf{p},\lambda) = 0 \quad (1.130)$$

in which $\overline{p}_\mu$ is a null vector $\overline{p}_\mu \overline{p}^\mu = p_\mu p^\mu = 0$ satisfying $p_\mu \overline{p}^\mu = -1$. The normalization and completeness relations satisfied by such polarization vectors are

$$\epsilon^{\mu *}(\mathbf{p},\lambda)\epsilon_\mu(\mathbf{p},\lambda') = \delta_{\lambda\lambda'} \quad (1.131)$$

and

$$\sum_{\lambda=\pm1}\epsilon_\mu(\mathbf{p},\lambda)\epsilon^*_\nu(\mathbf{p},\lambda) = \eta_{\mu\nu} + p_\mu \overline{p}_\nu + p_\nu \overline{p}_\mu \quad (1.132)$$

Note that the null vector $\overline{p}$ is not unique; indeed, the substitution

$$\overline{p}^{\mu} \to \overline{p}^{\mu} + a\epsilon^{\mu} + \frac{a^2}{2}p^{\mu} \tag{1.133}$$

for any spacelike ϵ^{μ} satisfying $\epsilon \cdot \overline{p} = 0 = \epsilon \cdot p$ and $\epsilon^*_{\mu}\epsilon^{\mu} = 1$, yields a new vector satisfying the required properties for $\overline{p}$. However, if we choose particular polarization vectors $\epsilon_{\mu}(\lambda)$ and require $\overline{p} \cdot \epsilon(\lambda) = 0$ for each λ, then the choice is made unique.

In position space, the conditions, Eq. (1.128) through Eq. (1.130), imply that

$$\begin{aligned} f_{\mu\nu} &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} & (1.134)\\ A_{\mu}(x) &= \sum_{\lambda=\pm1} \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3}\Big[\epsilon_{\mu}(\mathbf{p},\lambda)a_{\mathbf{p}\lambda}\mathrm{e}^{\mathrm{i}px} + \cdots\Big] & (1.135) \end{aligned}$$

in which the *gauge potential*, $A^{\mu}(x)$, is only defined up to the freedom, Eq. (1.129)

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\omega(x) \tag{1.136}$$

where $\omega(x)$ is an arbitrary function. The mass-shell condition $p^2 = 0$ then becomes

$$\partial_{\mu}f^{\mu\nu} = 0 \tag{1.137}$$

or, using Eq. (1.136) to impose the *gauge condition* $\partial^{\mu}A_{\mu} = 0$, equivalently:

$$\Box A^{\mu} = 0 \tag{1.138}$$

The corresponding free Lagrangian then is

$$\mathcal{L}_0 = -\frac{1}{4}\sum_{a=1}^{N} f^a_{\mu\nu}f^{a\mu\nu} \tag{1.139}$$

It is crucial to realize that, whereas the field-strength $f_{\mu\nu}$ defined in this way is a tensor under Lorentz transformations, the gauge potential, A_{μ}, is *not* a four-vector. Rather, it transforms as:

$$U(\omega)A^{\mu}(x)U(\omega)^* = \Lambda^{\mu}_{\nu}A^{\nu}(\lambda \cdot x) + \partial^{\mu}\omega(x) \tag{1.140}$$

for some scalar field $\omega(x)$. That is, A^{μ} transforms as a four-vector only up to a gauge transformation. This is a crucial observation because if we wish to write down interactions that do not vanish in the zero-momentum limit between massless spin-one particles and other particles (such as, for example, the Coulomb interaction in electromagnetism) then we must build our Lagrangian from the field A^{μ} rather than $f^{\mu\nu}$. Since A^{μ} is only a Lorentz four-vector up to gauge transformations, we see that Lorentz invariance of the Lagrangian requires that the interactions be invariant under the gauge transformations of Eq. (1.136). In this way we see gauge invariance emerge as a consequence of Lorentz invariance for massless particles of high spin. (A similar argument may be made for massless particles with spin-3/2 or -2, leading to supersymmetry or general covariance.)

1.4 Implications of symmetries

We pause here for a short aside on the general symmetry features that may arise in a Lagrangian. There are two motivations for this aside, corresponding to the two roles played by symmetries in what follows. First, symmetries are useful because they often allow us to make exact statements, even without a detailed understanding of a theory's dynamics. Namely, they can provide general conservation laws and spectral degeneracies familiar from quantum mechanics. Second, symmetries play a crucial role in the couplings of massless (or light) spin-one particles, by virtue of the requirement of gauge invariance that must be imposed. In this section we address the first of these roles in the first two subsections and return to the issue of gauge invariance in the last subsection.

1.4.1 Symmetries and conservation laws

Perhaps the simplest example of the connection between symmetry and a conservation law is given by the example of a discrete symmetry. For example, suppose the Hamiltonian of a system has a symmetry, in the sense that it remains unchanged after the replacement $\phi(x) \to -\phi(x)$; i.e.

$$H(-\phi, -\partial_\mu \phi) = H(\phi, \partial_\mu \phi) \tag{1.141}$$

identically for any field configuration $\phi(x)$. This ensures that there is a conservation law, inasmuch as it is possible to define a unitary operator, $\mathcal{X}$, which represents this replacement in the following sense:

$$\mathcal{X}\phi(x)\mathcal{X}^* = -\phi(x) \tag{1.142}$$

and so

$$\mathcal{X}a_p\mathcal{X}^* = -a_p \, , \tag{1.143}$$

Such an operator necessarily satisfies the symmetry property of a quantum symmetry: $\mathcal{X}H = H\mathcal{X}$.

If any *Hermitian* operator, $\mathcal{X}$, satisfies the condition $[\mathcal{X}, H] = 0$, it defines a conservation law. (For instance, in the example being discussed the condition $\mathcal{X}^2 = I$ together with the unitarity of $\mathcal{X}$ automatically ensures $\mathcal{X}$ is Hermitian.) It defines a conservation law because the fact that $\mathcal{X}$ commutes with H ensures that energy eigenstates may be labeled consistently by the eigenvalues of $\mathcal{X}$: $\mathcal{X}|E, x\rangle = x|E, x\rangle$. Furthermore, this label is conserved because it cannot change under time evolution:

$$\mathcal{X}|E, x; t\rangle = \mathcal{X}\, \mathrm{e}^{-\mathrm{i}Ht}|E, x\rangle = \mathrm{e}^{-\mathrm{i}Ht}\mathcal{X}|E, x\rangle = x|E, x; t\rangle \tag{1.144}$$

If it is true that $\mathcal{X}^2 = I$, then the eigenvalues satisfy $x = \pm 1$.

It bears emphasis that this conservation is an exact statement, provided only that $\mathcal{X}$ commutes with the exact Hamiltonian of the system, and so can have very powerful consequences. It implies, for example, that the lowest-energy state having eigenvalue $x = -1$ must be absolutely stable. It must be stable since it cannot decay into lower energy states, since energy conservation requires that any decay products have lower energy and yet they must also share the eigenvalue $x = -1$. Since no states satisfy both requirements, the decay cannot occur.

1.4.2 Local conservation laws: continuous symmetries

A particularly important class of conservation laws arises in the case when the theory has a continuous symmetry: $U(g)H = HU(g)$, where $U(g)$ is a unitary operator and g is any element $g \in G$ of a continuous group (whose properties are reviewed in appendix B). Since any element of the group g can be written as $g = \exp(\mathrm{i}\epsilon_a t_a)$ with t_a the Lie algebra elements of the group, the unitary operator can be written $U(g) = \exp(\mathrm{i}\epsilon_a Q_a)$. The operators Q defined in this way satisfy both $[Q, H] = 0$ and $Q^* = Q$, with the latter condition following as a consequence of the unitarity of $U(g)$.

This connection between a conserved charge, Q, and a symmetry holds equally well regardless of whether one is interested in classical mechanics, "ordinary" quantum mechanics of a few degrees of freedom, or field theory. For example, the symmetries of time translation, spatial translation or spatial rotations imply the conservation of energy, linear, and angular momentum respectively.

A new feature which appears in field theories having continuous symmetries is that the resulting conservation law holds *locally* through the existence of a spacetime-dependent conserved current, according to *Noether's theorem*. This local conservation may be seen as follows.

Suppose the Lagrangian density, $\mathcal{L}[\phi, \partial_\mu \phi]$, is invariant with respect to a local transformation of the field variables, $\phi^i(x)$

$$\delta\phi^i(x) = \epsilon^a F^i_a[\phi, \partial_\mu\phi; x] \tag{1.145}$$

in which ϵ^a represent a set, $a = 1, \ldots, N$ of *spacetime-independent* infinitesimal parameters and F^i_a indicates a local functional of the fields. The invariance of the action may be expressed as

$$\begin{aligned} 0 &\equiv \delta\mathcal{L} \\ &= \frac{\partial\mathcal{L}}{\partial\phi^i(x)}\epsilon^a F^i_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\epsilon^a\partial_\mu F^i_a \\ &= \left[\frac{\partial\mathcal{L}}{\partial\phi^i(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}\right)\right]\epsilon^a F^i_a + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}F^i_a\right]\epsilon^a \end{aligned} \tag{1.146}$$

The first term in the final line of Eq. (1.146) vanishes once the equations of motion for ϕ^i are used. The final line then shows that the equations of motion imply that the four-vector *Noether current*,

$$j^\mu_a(x) \equiv -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))}F^i_a \tag{1.147}$$

is conserved; $\partial_\mu j^\mu_a(x) = 0$ for each a. (The overall minus sign is conventional.) This last equation expresses conservation because it implies that the *charge*, Q, defined by

$$Q_a(t) \equiv \int \mathrm{d}^3x\; j^0_a(\mathbf{x}, t) \tag{1.148}$$

is time-independent:

$$\frac{dQ_a}{dt} = \int d^3x \frac{\partial j^0}{\partial t} = -\int d^3x\; \nabla\cdot\vec{j} = \oint d^2x\; \vec{n}\cdot\vec{j}\; = 0.$$

We assume here that there is no net flux going out of the boundary at infinity.

A symmetry for which the Lagrangian density is invariant as in Eq. (1.146) is known as an *internal symmetry*. This is to distinguish it from *spacetime symmetries* such as Poincaré transformations. In general, symmetries that act on spacetime coordinates as well as the fields cannot leave the Lagrangian density invariant because the Lagrangian density is not constant throughout spacetime. In this case a slightly more general form for Noether's theorem is necessary.

Suppose, then, that under the transformations

$$\begin{aligned} \delta\phi^i(x) &= \epsilon^a F^i_a[\phi, \partial_\mu\phi; x] \\ \delta x^\mu &= \epsilon^a \xi^\mu_a(x) \end{aligned} \tag{1.149}$$

the Lagrangian density transforms into a total derivative (so the action $\int \mathcal{L}\, \mathrm{d}^4x$ is invariant)

$$\delta\mathcal{L} \equiv \epsilon^a \partial_\mu V^\mu_a \tag{1.150}$$

for some Lorentz-vector fields, $V^\mu_a[\phi^i, \partial\psi]$, that are local functionals of $\phi^i(x)$. Repeating the arguments leading to Eq. (1.146) again implies conserved currents, $\partial_\mu j^\mu_a(x) = 0$, with $j^\mu_a(x)$ given by

$$j^\mu_a(x) = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))} F^i_a + V^\mu_a(x) \tag{1.151}$$

Conservation laws such as these are significant because they are exact results, and so allow conclusions even in the absence of a detailed understanding of the dynamics of a particular system. In a quantum theory the conserved charges, Q_a, are of particular interest since they are Hermitian and commute with the system Hamiltonian (since they are conserved!). They are therefore ideal operators for labeling the individual particle states. With particle states labeled in this way, conservation laws imply general selection rules concerning how quantum numbers must be related before and after collision processes.

It is often true that the symmetry transformation law given in the first line of Eq. (1.149) is more general than is necessary for a particular physical situation. It is often sufficient to consider symmetry transformations that are linear in the field variables

$$\delta\phi^i(x) = i\epsilon^a (T_a)^i_j \phi^j(x) \tag{1.152}$$

1.4.3 Spectral relations

The second major conclusion that may be drawn from symmetry properties of the Lagrangian of a system concerns the system's energy spectrum. The general statement is that states that are related by a symmetry transformation must have the same energy. This is a simple consequence of the fact that the conserved charge, Q_a, commutes with the system Hamiltonian. If, for instance, two energy eigenstates are related by $|\psi\rangle = Q_a|\chi\rangle$, then

$$\begin{aligned} H|\psi\rangle &= HQ_a|\chi\rangle \\ &= Q_a H|\chi\rangle \\ &= E_\chi Q_a|\chi\rangle \\ &= E_\chi|\psi\rangle \end{aligned} \tag{1.153}$$

It follows that $|\psi\rangle$ and $|\chi\rangle$ have the same energy eigenvalue, $E_\psi = E_\chi$, or are *degenerate*.

In general, states in the Hilbert space fall into unitary representations of the symmetry and all of the elements of a given representation must have the same energy.

Now, in a field theory we would like to apply this reasoning to the single-particle states in order to derive relations among the particle masses. This can be done subject to a single *caveat*: the ground state of the theory must be invariant under the symmetry transformations. That is to say, if the symmetry transformations are represented in the Hilbert space by the unitary transformations $U(\epsilon) = \exp(\mathrm{i}\epsilon^a Q_a)$, then the invariance of the ground state, $|0\rangle$, is expressed by: $U(\epsilon)|0\rangle = |0\rangle$ or, equivalently, $Q_a|0\rangle = 0$.

The connection between the invariance of the vacuum and symmetry relations among particle masses arises because symmetry transformations in field theory are usually defined as acting on the fields representing the various particles. If the fields representing a particular two particles are related by a symmetry transformation, it does not necessarily follow that the corresponding particle states are related by this same symmetry. It is this link between the fields and the particles that relies on the invariance of the ground state.

To see this in some detail, suppose that the fields, $\phi_1(x)$ and $\phi_2(x)$, corresponding to particle types "1" and "2", are related by the action of some symmetry:

$$\phi_1(x) = \mathrm{i}[Q, \phi_2(x)] \tag{1.154}$$

where $Q^* = Q$ is Hermitian. Then the same is true for the corresponding creation and annihilation operators:

$$a_1 = \mathrm{i}[Q, a_2] \tag{1.155}$$

The particle states are therefore related as follows:

$$\begin{aligned} |1\rangle &= a_1^*|0\rangle \\ &= \mathrm{i}[Q, a_2^*]|0\rangle \\ &= \mathrm{i}Qa_2^*|0\rangle - \mathrm{i}a_2^*Q|0\rangle \\ &= \mathrm{i}Q|2\rangle - \mathrm{i}a_2^*Q|0\rangle \end{aligned} \tag{1.156}$$

The particle states therefore satisfy $|1\rangle = iQ|2\rangle$ *if* the no-particle state is invariant: $Q|0\rangle = 0$. Once it is known that the particle states are related in this way, the arguments leading to Eq. (1.153) may be used to infer that they have equal masses.

To summarize, the general quantum-mechanical result, which implies that states that are related by symmetry transformations must be degenerate, applies equally well within the field-theoretical context. It does *not* follow, however, that particles whose representative fields are related by symmetry transformations must be degenerate (i.e. have equal masses). This last implication does hold, though, if the ground state of the system is invariant under the action of the symmetry. It is a general feature of field theories that the ground state need not be invariant with respect to symmetry transformations. If the ground state is not invariant, the symmetry is said to be *spontaneously broken*. For spontaneously broken symmetries it is generic that naive symmetry relationships among masses fail.

The conserved currents discussed in the previous section, however, exist regardless of whether a symmetry is spontaneously broken or not, because Noether's theorem only uses the invariance of the action. It is true, however, that spontaneous breaking of a symmetry makes it impossible to use the corresponding charge to define conserved quantum numbers for particle states.

1.5 Renormalizable interactions

We now turn to the construction of general interactions involving particles with spin-zero, -half, or -one. The goal is to construct the most general form for these interactions that is consistent with the five principles outlined in Section 1.2. In this section the general form for renormalizable interactions involving particles of spins zero through one is summarized, largely without proof. The purpose is to outline the general features of these interactions.

1.5.1 Spin-zero and spin-half particles

In order to get started, consider first the most general renormalizable interactions allowed for N interacting spin-zero particles. As outlined in Subsection 1.3.1, we may, without loss of generality, represent these particles with N real scalar fields, $\phi^i(x),\ i = 1, \ldots, N$.

We are instructed to write down a Lorentz-invariant Lagrangian density,

$$\mathcal{L}_s = \mathcal{L}_0 + \mathcal{L}_{\rm int} \tag{1.157}$$

where $\mathcal{L}_0$ is the free Lagrangian of Subsection 1.3.1 and $\mathcal{L}_{\rm int}$ is the interaction term that is by definition not quadratic or linear in the fields. $\mathcal{L}_{\rm int}$ is to be constructed solely from $\phi^i(x)$ and $\partial_\mu \phi^i(x)$ subject to the requirement (renormalizability) that it involves interactions of at most dimension four in powers of mass. In order to do so it is necessary to compute the mass dimension of the fields, $\phi^i(x)$, themselves. This is easily done once the free Lagrangian is put into canonical form.

Comparing with standard form, Eq. (1.64), shows that the scalar field must have dimensions of M^1 (when $\hbar = c = 1$) if $\mathcal{L}_0$ is to have dimension M^4. This may then be used to infer the restrictions imposed on $\mathcal{L}_{\rm int}$ by renormalizability. It is easy to now show that the most general renormalizable interactions possible among N spin-zero particles are:

$$\begin{aligned}\mathcal{L}_s &= \mathcal{L}_0 - V(\phi) \\ &= -\frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \rho - v_i\phi^i - \frac{1}{2}\mu^2_{ij}\phi^i\phi^j - \frac{1}{3!}\xi_{ijk}\phi^i\phi^j\phi^k \\ &\quad -\frac{1}{4!}\lambda_{ijkl}\phi^i\phi^j\phi^k\phi^l \end{aligned} \tag{1.158}$$

The generalization to include also spin-half particles is again straightforward. Inspection of the canonically normalized kinetic term, Eq. (1.110), implies that a spinor field carries dimension $M^{\frac{3}{2}}$. This implies that the most general renormalizable Lagrangian involving spins zero and half must be:

$$\mathcal{L}_m = \mathcal{L}_s - \frac{1}{2}\overline{\psi}^n \partial\!\!\!/ \psi^n - \frac{1}{2}m_n\overline{\psi}^n\psi^n - g_{mni}\overline{\psi}^m\psi^n\phi^i - \mathrm{i}h_{mni}\overline{\psi}^m\gamma_5\psi^n\phi^i \tag{1.159}$$

Here $\mathcal{L}_s$ is as in Eq. (1.158) and the new spin-half/spin-zero interaction terms are known generically as *Yukawa couplings*.

1.5.2 Spin-one couplings: gauge invariance

We would like to write down a general set of renormalizable couplings involving particles from spins zero through one. It turns out not to be possible to do so for the massive spin-one particle (apart from one exception that is a special case of the general situation considered below). We turn therefore directly to the case of massless spin-one particles.

The straightforward thing to try is to couple massless spin-one particles to other particles by writing down interactions that involve the field-strength, $f_{\mu\nu}$. Dimension counting again shows that this is impossible because the free Lagrangian, Eq. (1.139), implies that $f_{\mu\nu}$ has dimensions of M^2. The lowest-dimension interaction possible would then be something like $\overline{\psi}\gamma^{\mu\nu}\psi f_{\mu\nu}$ which has dimension M^5 and so is not renormalizable.

The only remaining possibility then is to build couplings directly from the gauge potential, $A_\mu(x)$. This is somewhat delicate, because as we have seen, $A_\mu(x)$ does *not* transform as a four-vector – it is only a four-vector up to a gauge transformation: $A_\mu \to A_\mu + \partial_\mu\omega$. It follows that the interaction Lagrangian itself will only be Lorentz invariant provided that the interactions are required to be gauge invariant.

It is beyond the scope of this book to work out the requirements of gauge invariance in all of their detail. We content ourselves here with simply motivating the construction and then quoting the final results.

Suppose, then, that we write down an interaction term

$$\mathcal{L}_{\rm int} = A_\mu(x) J^\mu[\phi] \tag{1.160}$$

with $J^\mu[\phi]$ some four-vector function of the other fields and possibly their derivatives. Under a gauge transformation, $\delta A_\mu(x) = \partial_\mu \omega(x)$, if $\delta\phi^i = 0$, this interaction Lagrangian transforms to

$$\delta\mathcal{L}_{\rm int}(x) = \partial_\mu \omega(x) J^\mu[\phi(x)] \tag{1.161}$$

We need to cancel Eq. (1.161) with the contribution from another term in the Lagrangian. One can imagine doing so in one of two ways. Extra interaction terms can be added, and/or the transformation rules can be altered. The first of these options must fail in the present instance because the required term would have to be linear in the gauge potential in order to produce a variation like Eq. (1.161), and Eq. (1.160) is already the most general such Lagrangian.

The required transformation rule may be most easily seen by repeating the steps leading to Eq. (1.146) in the proof of Noether's theorem, with one alteration. In the previous section Noether's theorem was derived subject to the condition that the transformation parameter, ϵ^a, be independent of spacetime position, x^μ. In the present case, however, the transformation parameter, ω, cannot be spacetime independent because the gauge potential transforms into its gradient. Consider, then, the variation of the Lagrangian under a transformation as in Eq. (1.145)

$$\delta\phi^i(x) = \epsilon^a(x) F^i_a[\phi, \partial_\mu\phi; x] \tag{1.162}$$

but with the transformation parameter a function of x^μ. Suppose further that the Lagrangian would be invariant if ϵ^a had been chosen as constant. The Lagrangian in this case fails to be invariant with spacetime-dependent ϵ^a only because of its dependence on the derivatives, $\partial_\mu\phi^i$, of the fields. The variation of the Lagrangian therefore becomes

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi^i(x)} \epsilon^a(x) F^i_a[\phi, \partial_\mu\phi; x] + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))} \partial_\mu(\epsilon^a(x) F^i_a[\phi, \partial_\mu\phi; x]) \\ &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i(x))} F^i_a[\phi, \partial_\mu\phi; x] \partial_\mu\epsilon^a(x) \end{aligned}$$

$$= \quad -j^\mu_a(x)\partial_\mu\epsilon^a(x) \tag{1.163}$$

Comparing Eq. (1.161) with Eq. (1.163) shows that the gauge variation of the spin-one coupling can cancel against the variation of the spin-zero and spin-half "matter" Lagrangian if

(i) the coefficient function, $J^\mu_a[\phi]$, is identified with the conserved current,

$$J^\mu_a[\phi] = j^\mu_a(x) \tag{1.164}$$

associated with a symmetry of this matter Lagrangian, and

(ii) the gauge transformations are enlarged to include the transformation of the matter fields with respect to this symmetry with a spacetime-dependent parameter:

$$\delta A_\mu(x) \quad = \quad \partial_\mu\omega(x) \tag{1.165}$$

$$\delta\phi^i(x) \quad = \quad \omega(x)F^i[\phi, \partial_\mu\phi; x] \tag{1.166}$$

This promotion of a spacetime-independent symmetry of the matter Lagrangian to a spacetime-dependent symmetry of the matter/spin-one Lagrangian is called the *gauging* of the symmetry. The corresponding spin-one particles are known as *gauge bosons*.

More generally, if there are more than one spin-one fields, and if the symmetries involved transform one spin-one particle into another, then the conserved current, $j^\mu_a(x)$, will itself depend on the $A^a_\mu(x)$s. This leads to self-couplings of the gauge bosons amongst themselves. Such a symmetry is called a *non-abelian* symmetry, and will require a generalization of the above discussion. We here summarize the results of such a generalization.

Consider a (renormalizable) Lagrangian, $\mathcal{L}_m[\phi]$, depending on a collection of spin-zero and spin-half "matter" fields. Suppose that $\mathcal{L}_m$ is invariant with respect to the following *global* (i.e. spacetime-independent) symmetry transformations:

$$\delta\phi^i(x) = i\omega^a(T_a)^i_j\phi^j(x) \tag{1.167}$$

In general, repetition of several symmetry transformations produces further symmetries so the transformations, Eq. (1.167), form a Lie algebra and the matrices ${(T_a)^i}_j$ necessarily satisfy the *commutation relations* (see Appendix B):

$$[T_a, T_b] = if^c_{ab}T_c \tag{1.168}$$

where the coefficients f^c_{ab} are a set of numbers that are characteristic of the algebra involved. The good news is that all of the algebras of this type that are of physical interest have been found and are cataloged once and for all.

The most general renormalizable way to couple this Lagrangian to a bunch of spin-one particles is given by the following prescription.

(i) Associate each spin-one particle, $A^a_\mu(x)$, with one of the generators, (T_a), of the symmetry algebra.

(ii) Replace ordinary spacetime derivatives everywhere in $\mathcal{L}_m$ with the following *covariant derivatives*:

$$\mathrm{D}_\mu \phi^i(x) \equiv \partial_\mu \phi^i(x) - \mathrm{i} A^a_\mu(x)(T_a)^i_j \phi^j(x) \tag{1.169}$$

(iii) Add the following gauge-boson Lagrangian

$$\mathcal{L}_g \equiv -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{1.170}$$

with the covariant field strength, $F^a_{\mu\nu}(x)$, defined by

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu \tag{1.171}$$

The total Lagrangian is then given by the sum: $\mathcal{L} = \mathcal{L}_m[\phi, \mathrm{D}_\mu \phi] + \mathcal{L}_g$. It is invariant (in fact $\mathcal{L}_m$ and $\mathcal{L}_g$ are separately invariant) under the *local* or gauged generalization of transformation, Eq. (1.167):

$$\delta A^a_\mu(x) = \partial_\mu \omega^a(x) - f^a_{bc} \omega^b(x) A^c_\mu(x), \tag{1.172}$$

$$\delta \phi^i(x) = i\omega^a(x)(T_a)^i_j \phi^j(x) \tag{1.173}$$

1.6 Some illustrative examples

Before proceeding it is useful to consider a few illustrative examples.

1.6.1 Quantum electrodynamics: an abelian gauge theory

Consider, first, the theory describing physics at scales below the mass of the muon, $m_\mu = 106$ MeV. The elementary particles in this energy range are the electron and the neutrinos, represented by a Dirac spinor field, $e(x)$, and three Majorana spinor fields, $\nu_i(x)$; and the photon, represented by the gauge potential, $A_\mu(x)$. We wish to write down the most general renormalizable interactions of these particles, which should furnish a reasonable description of their behavior at energies much less than $2m_\mu$.

From the previous discussion, the coupling of the photon must be to some conserved current – in this case electric charge. The current is

$$J^\mu_{\mathrm{em}}(x) = -\mathrm{i} e \overline{e} \gamma^\mu e(x) \tag{1.174}$$

(where unfortunately the electric coupling and the electron field have the same symbol e and must be told apart by context), and the corresponding local symmetry transformation is therefore

$$\begin{aligned} \delta e(x) &= -\mathrm{i}e\omega(x)e(x) \\ \delta\nu_i(x) &= 0 \\ \delta A_\mu(x) &= \partial_\mu\omega(x) \end{aligned} \tag{1.175}$$

The most general renormalizable interaction must therefore be

$$\mathcal{L} = -\overline{e}(\not{D} + m_e)e - \overline{\nu}_i(\not{\partial} + m_{\nu_i})\nu_i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.176}$$

in which

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ \text{and} \qquad \mathrm{D}_\mu e(x) &= \partial_\mu e(x) + \mathrm{i}eA_\mu(x)e(x) \end{aligned} \tag{1.177}$$

Equation (1.176) has two features that are worth remarking on here. The first is that the Lagrangian has broken up into the sum of two terms: $\mathcal{L} = \mathcal{L}_{\rm QED} + \mathcal{L}_\nu$ in which $\mathcal{L}_{\rm QED}$ is independent of the neutrino fields and $\mathcal{L}_\nu$ depends only on the neutrino fields. Since $\mathcal{L}_\nu$ is quadratic this implies that the neutrinos cannot interact at all with the other particles through renormalizable interactions. This is the major part of the present understanding of why it is that neutrinos couple so feebly to the rest of matter. The other observation is that the part of the Lagrangian, $\mathcal{L}_{\rm QED}$, that depends on electrons and photons is precisely the standard Lagrangian for quantum electrodynamics (QED). This Lagrangian is indeed known to give an extremely precise description of the interactions of electrons and photons. We here have the beginnings of an explanation of why it must have the form that it does. To the extent that any theory at higher energies has the observed spectrum of particles and preserves the conservation of electric charge, it must reproduce QED at energies, E, well below the mass of the muon, up to non-renormalizable corrections that are suppressed by powers of (E/m_μ).

1.6.2 Scalar electrodynamics: spontaneous symmetry breaking

The gauge-invariant Lagrangian of the previous sections appears to have the serious drawback that it can only describe the interactions of massless spin-one particles. This turns out not to be true in general, as we shall demonstrate using a less orthodox example, called the abelian Higgs model.

The theory consists of a single charged spinless particle, with complex field $\phi(x) = (\phi_{re} + i\phi_{im})/\sqrt{2}$, coupled to electromagnetism, $A_\mu(x)$.

The most general renormalizable matter Lagrangian that is invariant under the global rephasing (or $U(1)$) symmetry $\phi \to \mathrm{e}^{\mathrm{i}e\omega}\phi$ (and is analytic in ϕ) is

$$\mathcal{L}_\phi = -\partial_\mu\phi^*\partial^\mu\phi - a(\phi^*\phi)^2 - b(\phi^*\phi) - c \tag{1.178}$$

Gauging this symmetry and coupling to the photon gives the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mathrm{D}_\mu\phi^*\mathrm{D}^\mu\phi - a(\phi^*\phi)^2 - b(\phi^*\phi) - c \tag{1.179}$$

in which

$$\mathrm{D}_\mu\phi = \partial_\mu\phi - \mathrm{i}e\, A_\mu\phi \tag{1.180}$$

and the field strength is as in Eq. (1.177). Although stability implies that the real constant $a = \lambda^2$ must be non-negative, the sign of b is arbitrary.

We wish to extract the spectrum of this theory for weak couplings $e \ll 1$ and $\lambda^2 \ll 1$. There are two qualitatively different possibilities, depending on the sign of b. If $b = \mu^2$ is positive, then the unperturbed Lagrangian simply consists of those terms that are quadratic in the fields. The spectrum for this unperturbed theory was worked out in the previous sections and consists of a massless spin-one photon and a charged, spinless particle with mass $m_\phi^2 = \mu^2$ (see the sentence following Eq. (1.69)).

Things are different if it should happen that $b = -\mu^2$ were negative. In this case a naive repetition of the steps outlined earlier would have us identify the quadratic part of Eq. (1.179) as the unperturbed Lagrangian. One sign that this cannot be quite right is that the mass of the spinless particle in this unperturbed theory would then be imaginary: $m_\phi^2 = -\mu^2$. A tachyonic mass such as this is the sign that the assumed ground-state field configuration – in this case $\phi = 0$—is unstable, since a negative squared-mass implies that the field modes with $|\mathbf{p}| < \mu$ have a complex energy: $E = \sqrt{\mathbf{p}^2 - \mu^2} = E_r - \mathrm{i}E_i$, and so have a runaway time dependence: $\exp(-\mathrm{i}Et) = \exp[+E_\mathrm{i}t - \mathrm{i}E_\mathrm{r}t]$.

More properly, since we are interested in the energies of the lowest excitations about the ground state, i.e. the vacuum, we must first check that we have properly identified the ground state. The weak-coupling limit we are interested in may be used to justify doing so semiclassically. In the semiclassical limit the ground state is just described by its classical field configuration. Being a ground state, this configuration must by definition minimize the energy. Furthermore, the energy of the configuration is semiclassically dominated by the classical energy which is easily computable from the system's Lagrangian. In the present instance the energy density is

$$\mathcal{H} = \frac{\partial \phi^*}{\partial t}\frac{\partial \phi}{\partial t} + \mathrm{D}\phi^* \cdot \mathrm{D}\phi + \lambda^2 (\phi^* \phi)^2 + b\, \phi^* \phi + c + \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \tag{1.181}$$

Here $E_i = F_{i0}$ and $B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$. Since this is a sum of non-negative terms, it is minimized by minimizing each term separately. The electromagnetic field energy is minimized at zero field, $\mathbf{B} = \mathbf{E} = 0$, and the gradient terms in the scalar energy are smallest for constant fields, $\partial\phi/\partial t = \nabla\phi = 0$. If $b \geq 0$ then the potential energy is also minimized by zero field, $\phi = 0$, as was implicitly assumed above. If $b = -\mu^2$, however, then the scalar-field energy is minimized when $\phi^*\phi \equiv v^* v = \mu^2/(2\lambda^2)$. This value v which the scalar field takes in vacuum is called its vacuum expectation value, or *v.e.v.*.

The low-energy excitations are found semiclassically by perturbing about this stable field configuration. The unperturbed system consists of all terms that are quadratic or less in the fluctuations about the minimum-energy field configuration. Since the ground-state constructed in this way is by construction stable, tachyonic modes never appear in such an expansion. When $b \geq 0$ and the ground-state configuration is zero, this agrees with the naive treatment outlined earlier.

For $b < 0$ we must expand instead in powers of the difference: $\varphi \equiv \phi - v$. Doing so with the Lagrangian of Eq. (1.179) gives the following unperturbed result:

$$\begin{aligned} \mathcal{L}_0 \;=\; & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu \varphi^* \partial^\mu \varphi + \mathrm{i}e\, A_\mu (v \partial^\mu \varphi^* - v^* \partial^\mu \varphi) - e^2 v^* v\, A_\mu A^\mu \\ & -V_0 - \lambda^2 (v^* \varphi + v \varphi^*)^2 \end{aligned} \tag{1.182}$$

The constant V_0 contains all of the φ-independent terms and so represents the ground-state energy density.

Unfortunately, because of the terms that mix the vector with scalar fields, we cannot directly use the results of the previous sections to read off the particle spectrum. Happily enough, gauge invariance now comes to our aid. Recall that the Lagrangian, and so all of the physics, is unchanged by the gauge transformation

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu \omega(x) \\ \phi(x) &\to \exp[\mathrm{i}e\omega(x)]\phi(x) \end{aligned} \tag{1.183}$$

We may therefore use this freedom to redefine fields to put the Lagrangian into a particularly convenient form. A useful choice for the present purposes is to use the transformations of Eq. (1.183) to make the scalar field everywhere real, $\phi^*(x) = \phi(x)$ for all x. The utility of this choice arises from the observation that the $A_\mu \partial^\mu \varphi$ cross terms then vanish.

The spectrum may now be directly read off as before. The quadratic terms in the electromagnetic potential describe a spin-one particle with mass $M_A^2 = 2e^2 v^2$. The photon is no longer massless! The spin-zero sector now consists of a single real scalar of mass $m_\varphi^2 = 4\lambda^2 v^2 = 2\mu^2$. Since the gauge condition completely eliminates the imaginary part of the scalar field, an entire scalar degree of freedom has been "removed" from the spectrum. This degree of freedom has re-emerged as the longitudinal spin state of the massive spin-one particle. This process, in which a vector field "eats" a scalar one in the process of becoming massive, is known as the *Higgs mechanism*.

The process of using the gauge freedom to impose conditions on the fields is known as "choosing a gauge." The choice made here is known as "unitary" or "physical" gauge since it makes the spectrum of the theory easy to identify.

The lesson to be learned is that a gauge symmetry need not imply that the corresponding spin-one gauge particle need be massless. This is the second time we have encountered an exception to a general symmetry consequence for the particle spectrum. The circumstances here are similar to those described in Subsection 1.4.3. In both cases the root cause lies in the fact that the ground state is not invariant under the symmetry in question, and it is this non-invariance that ruins the symmetry predictions for the spectrum of fluctuations about that ground state. This is again the phenomenon of spontaneous symmetry breaking.

To see that the ground state indeed breaks the relevant symmetry in the present example, notice that any ground state field configuration $\phi = v$ is not invariant under the transformations of Eq. (1.183). This condition is intimately related to what was our working definition of spontaneous symmetry breaking in Subsection 1.4.3. There we defined it by the condition that the conserved charge, Q, not annihilate the ground state, $Q|\Omega\rangle \neq 0$. The one condition is a consequence of the other, since $\langle\Omega|\phi|\Omega\rangle = v \neq 0$ implies that the commutator $\langle\Omega|[Q, \phi]|\Omega\rangle$ cannot be zero as would be required if $Q|\Omega\rangle = 0$.

1.6.3 QCD: an $SU(3)$ gauge theory

To a good approximation, the theory of nuclei and their constituents is quantum chromodynamics (QCD), a gauge theory with group $SU_c(3)$. We review it here in some detail, because it is a good lesson in how non-abelian gauge theories work, as well as being directly a component of the standard model.

The theory of QCD contains several types of Dirac fermions called *quarks*, labeled $u, d, s, \ldots$ for up, down, strange, ... (There are six altogether, named u, d, s, c, b, and t, but only u, d, s are light.) However, when we say there is "a" quark u, we really mean there are three quark fields, written $u_{\rm r}$, $u_{\rm g}$, and $u_{\rm b}$ (rgb for "red," "green," and "blue"), which have exactly the same mass; similarly, $d, s, \ldots$ are replicated in triplicate, also labeled r, g, b. It is convenient to group these three fields in a column vector, $[u_{\rm r},\ u_{\rm g},\ u_{\rm b}]^T$, or u_a in index notation. It is customary when possible to suppress this index (matrix notation), and it is important to appreciate that the index a is not the spinorial index we have already met – each $u_{\rm r}$, $u_{\rm g}$, $u_{\rm b}$ has four spinor components. When one writes $\overline{u}u$, it really means $\overline{u}_a u_a$ with the a sum implicit and where the spinor indices are summed over for each color separately (spinorial and color indices are independent). The free Lagrangian for the up quarks is,

$$\mathcal{L}_{0,u} = -\overline{u}_a(\partial\!\!\!/ + m_u)u_a \equiv -\overline{u}(\partial\!\!\!/ + m_u)u \tag{1.184}$$

and the Lagrangians for the d, s quarks are similar.

At the free theory level, nothing would change if we made the replacement, $[u_{\rm r},\ u_{\rm g},\ u_{\rm b}]^T \to [u_{\rm g},\ -u_{\rm r},\ u_{\rm b}]^T$, exchanging the role of red and green quarks. More generally, nothing is changed by making an arbitrary unitary rotation $u_a \to \tilde{U}_{ab}u_b$, $\tilde{U}^\dagger = \tilde{U}^{-1}$, under which the free Lagrangian changes to

$$-\overline{u}_a(\partial\!\!\!/ + m_u)u_a \to -\overline{u}_b\tilde{U}^\dagger_{ba}(\partial\!\!\!/ + m_u)\tilde{U}_{ac}u_c = -\overline{u}_a(\partial\!\!\!/ + m_u)u_a \tag{1.185}$$

At the free level, the theory has a symmetry under $U(3)$ (3×3 unitary matrix) rotations between the u quarks – and separately under independent $U(3)$ rotations of each other quark type. This $U(3)$ matrix can be decomposed as $\tilde{U} = \mathrm{e}^{\mathrm{i}\theta}U$, with $U \in SU(3)$ a *special unitary matrix*, that is, unitary matrix of determinant 1. Any $SU(3)$ matrix can be exponentiated as $U = \exp(\mathrm{i}M)$, with M a traceless, 3×3 Hermitian matrix. An $N \times N$ complex Hermitian matrix has N^2 independent entries, and the tracelessness condition removes one, so there are eight independent parameters to describe M. Such matrices can always be written in terms of a standard basis of traceless Hermitian matrices, $U = \exp(\mathrm{i}\omega_\alpha\lambda_\alpha/2)$, with ω_α some coefficients and λ_α the Gell-Mann matrices, explicitly,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{1.186}$$

chosen to satisfy $\operatorname{tr} \lambda_\alpha \lambda_\beta = 2\delta_{\alpha\beta}$. (Do not confuse the index α with the indices a, b earlier: the α index runs over the eight such independent matrices, while a, b are row and column indices for these matrices and run over three values.) The Gell-Mann matrices satisfy an algebra,

$$\left[\frac{\lambda_\alpha}{2}, \frac{\lambda_\beta}{2}\right] = \mathrm{i} f^\gamma_{\alpha\beta} \frac{\lambda_\gamma}{2} \tag{1.187}$$

where ${f^\gamma}_{\alpha\beta}$, the *structure constants* of the group $SU(3)$, are real and antisymmetric in all three indices.

QCD is defined as the interacting theory for which the rotations in which $u, d, s, \ldots$ are each rotated by the same $SU(3)$ matrix are gauged. (These gauge interactions break all the remaining symmetries except each $U(1)$ symmetry associated with separate phase rotations for each quark species. We return to this issue in Section 2.5.) Now let us see in this example why the conditions, Eq. (1.169) – Eq. (1.171), are necessary. For the u field kinetic term to be invariant under symmetry transformations, it must involve the covariant derivative,

$$\mathcal{L}_u = -\overline{u}(\not{D} + m_u)u, \qquad \mathrm{D}_\mu = \partial_\mu - \mathrm{i} g_3 G^\alpha_\mu \frac{\lambda_\alpha}{2} \tag{1.188}$$

Here G^α_μ are eight spin-1 gauge fields, called *gluon fields*, with the sum on α implicit and g_3 a coupling constant analogous to the electric charge of QED, called the *strong coupling* (frequently written as g_s). (Remember that we suppress matrix indices; λ is a 3×3 matrix multiplying the column vector u, so $\lambda_\alpha u$ means $(\lambda_\alpha)_{ab} u_b$.) However, the invariance of this expression also requires a specific transformation rule for the field G^α_μ. Under an infinitesimal gauge transformation,

$$u \to \left(1 + \mathrm{i} g_3 \omega^\alpha \frac{\lambda_\alpha}{2}\right) u \tag{1.189}$$

and taking G^α_μ to change to $G^\alpha_\mu + \delta G^\alpha_\mu$ under gauge transformations, this Lagrangian changes to

$$\mathcal{L}_u \quad \to \quad \overline{u} \left(1 - \mathrm{i} g_3 \omega^\alpha \frac{\lambda^\dagger_\alpha}{2}\right) \left[m_u + \gamma^\mu \left(\partial_\mu - \mathrm{i} g_3 (G^\beta_\mu + \delta G^\beta_\mu) \frac{\lambda_\beta}{2}\right)\right]$$

$$
\begin{aligned}
&\times\left(1+\mathrm{i}g_3\omega^\gamma\frac{\lambda_\gamma}{2}\right)u \\
=\ &\overline{u}\left[m_u+\gamma^\mu\left(\partial_\mu-\mathrm{i}g_3G^\alpha_\mu\frac{\lambda_\alpha}{2}-\mathrm{i}g_3\delta G^\alpha_\mu\frac{\lambda_\alpha}{2}\right.\right. \\
&\left.\left.+\mathrm{i}g_3(\partial_\mu\omega^\alpha)\frac{\lambda_\alpha}{2}+\mathrm{i}g_3^2 f^\alpha_{\beta\gamma}G^\beta_\mu\omega^\gamma\frac{\lambda_\alpha}{2}\right)\right]u \qquad (1.190)
\end{aligned}
$$

(at linear order in infinitesimal ω), which is unchanged only if we identify the change under gauge transformations of the field G as

$$\delta G^\alpha_\mu=\partial_\mu\omega^\alpha-gf^\alpha_{\beta\gamma}\omega^\beta G^\gamma_\mu \qquad (1.191)$$

reproducing Eq. (1.172). The combination $\partial_\mu G^\alpha_\nu-\partial_\nu G^\alpha_\mu$ transforms quite non-trivially under this gauge transformation rule, and is not the correct object to identify as a field strength. However, the combination (compare with Eq. (1.171))

$$G^\alpha_{\mu\nu}\equiv\partial_\mu G^\alpha_\nu-\partial_\nu G^\alpha_\mu+gf^\alpha_{\beta\gamma}G^\beta_\mu G^\gamma_\nu \qquad (1.192)$$

transforms as

$$G^\alpha_{\mu\nu}\to G^\alpha_{\mu\nu}-f^\alpha_{\beta\gamma}\omega^\beta G^\gamma_{\mu\nu} \qquad (1.193)$$

and therefore the combination $G^\alpha_{\mu\nu}G^{\alpha\mu\nu}$ is invariant, and may appear in the Lagrangian. The full Lagrangian of QCD is therefore

$$\mathcal{L}_{QCD}=-\sum_q\bar{q}(\not{D}+m)q-\frac{1}{4}G^\alpha_{\mu\nu}G^{\alpha\mu\nu} \qquad (1.194)$$

where $q=u,d,s$.

The physics of this theory is quite non-trivial and occupies Chapter 8 and Chapter 9.

1.7 Problems

[1.1] **Identities for Majorana spinors**

Prove the following useful relations for Majorana spinors ψ_1, ψ_2,

$$
\begin{aligned}
\overline{\psi}_1\psi_2 &= +\overline{\psi}_2\psi_1 \\
\overline{\psi}_1\gamma_5\psi_2 &= +\overline{\psi}_2\gamma_5\psi_1 \\
\overline{\psi}_1\gamma^\mu\psi_2 &= -\overline{\psi}_2\gamma^\mu\psi_1 \\
\overline{\psi}_1\gamma^\mu\gamma_5\psi_2 &= +\overline{\psi}_2\gamma^\mu\gamma_5\psi_1 \\
\overline{\psi}_1[\gamma^\mu,\gamma^\nu]\psi_2 &= -\overline{\psi}_2[\gamma^\mu,\gamma^\nu]\psi_1
\end{aligned}
$$

Hint: It is possible to invert the order of a series of matrices which contract a column vector on the right and row vector on the left, $cM_1M_2v = v^TM_2^TM_1^Tc^T$, for instance. However, since the operators ψ_1, ψ_2 are anti-commuting objects, there is a factor of -1 when doing so here; so $\overline{\psi}_1\psi_2 = -\psi_2^T\overline{\psi}_1^T$. Use this manipulation, and the identities in Eq. (1.93) and Eq. (1.94).

Next, show that for any spinors, Hermitian conjugation takes the form,

$$\begin{aligned}
\left(\overline{\psi}_1\psi_2\right)^* &= +\overline{\psi}_2\psi_1 \\
\left(\overline{\psi}_1\gamma_5\psi_2\right)^* &= -\overline{\psi}_2\gamma_5\psi_1 \\
\left(\overline{\psi}_1\gamma^\mu\psi_2\right)^* &= -\overline{\psi}_2\gamma^\mu\psi_1 \\
\left(\overline{\psi}_1\gamma^\mu\gamma_5\psi_2\right)^* &= -\overline{\psi}_2\gamma^\mu\gamma_5\psi_1 \\
\left(\overline{\psi}_1[\gamma^\mu,\, \gamma^\nu]\psi_2\right)^* &= -\overline{\psi}_2[\gamma^\mu\, ,\, \gamma^\nu]\psi_1
\end{aligned}$$

by using repeatedly Eq. (1.89)and Eq. (1.91). Note that Hermitian conjugation involves a reversal of the order of operators, so $(\psi_1^\dagger\psi_2)^\dagger = \psi_2^\dagger\psi_1$ without a minus sign.

Combine these to get the following relations for Majorana spinors:

$$\begin{aligned}
\left(\overline{\psi}_1\psi_2\right)^* &= +\overline{\psi}_1\psi_2 \\
\left(\overline{\psi}_1\gamma_5\psi_2\right)^* &= -\overline{\psi}_1\gamma_5\psi_2 \\
\left(\overline{\psi}_1\gamma^\mu\psi_2\right)^* &= +\overline{\psi}_1\gamma^\mu\psi_2 \\
\left(\overline{\psi}_1\gamma^\mu\gamma_5\psi_2\right)^* &= -\overline{\psi}_1\gamma^\mu\gamma_5\psi_2 \\
\left(\overline{\psi}_1[\gamma^\mu, \gamma^\nu]\psi_2\right)^* &= +\overline{\psi}_1[\gamma^\mu, \gamma^\nu]\psi_2
\end{aligned}$$

Use these to justify the requirements on the coefficients A, B, C, D, and E mentioned under Eq. (1.102).

[1.2] $\boldsymbol{\mathcal{O}(N)}$ scalar theories

The kinetic term $\frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i$ for N real scalar fields is invariant under a symmetry $\varphi_i \to \mathcal{O}_{ij}\varphi_j$, where $\mathcal{O}^T\mathcal{O} = 1,\ i, j = 1, \ldots, N$. These form the group of $N \times N$ real orthogonal matrices $\mathcal{O}(N)$. When N is even, $\mathcal{O}(N)$ contains as a subgroup the group of $(N/2) \times (N/2)$ complex unitary matrices, $U(N/2)$. When the interactions respect only this subgroup rather than the full $\mathcal{O}(N)$ group, it is often convenient to use complex fields.

[1.2.1] Example 1: $N = 2$.

(i) Write down the most general renormalizable Lagrangian for two real scalar fields, φ_1 and φ_2, subject to the discrete symmetries $\varphi_1 \to -\varphi_1$, $\varphi_2 \to \varphi_2$ and $\varphi_1 \to \varphi_1$, $\varphi_2 \to -\varphi_2$.

(ii) Re-express this Lagrangian in terms of the complex variables $\psi = \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$ and $\psi^* = \frac{1}{\sqrt{2}}(\varphi_1 - \mathrm{i}\varphi_2)$.

(iii) In this case the groups $\mathcal{O}(2)$ and $U(1)$ are equivalent to one another. If the $\mathcal{O}(2)$ transformations are written

$$\mathcal{O}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

find the transformation rules for ψ and ψ^*.

(iv) What further restrictions are placed on the Lagrangian by requiring that it be $\mathcal{O}(2)$ invariant (including interaction terms)? Write the resulting Lagrangian in terms of both the variables (φ_1, φ_2) and (ψ, ψ^*).

(v) Assuming the coupling to be weak, allowing a semiclassical approximation, what is the ground state (i.e. background value for the fields) and spectrum (i.e. masses) of this $\mathcal{O}(2)$-symmetric model if the coefficient of the quadratic term of the potential is positive? What are the ground states and spectrum if the coefficient of the quadratic term is negative? Which field is massless (such a massless field is called a Goldstone boson)?

[1.2.2] Example 2: $N = 4$.

(i) What is the most general form for a renormalizable theory of four real scalars, (assuming as above invariance under separate reflections of each field)?

(ii) In this case the maximal symmetry group is $\mathcal{O}(4)$ which consists of 4×4 real orthogonal matrices. These by definition are the group that leaves $\phi^T\phi = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 + (\phi_4)^2$ invariant. As a group $\mathcal{O}(4)$ is equivalent to $SU(2) \times SU(2)$. This can be seen as follows: Define the complex fields $\varphi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2)$ and $\psi = \frac{1}{\sqrt{2}}(\phi_3 + \mathrm{i}\phi_4)$ together with their complex conjugates and construct the 2×2 matrix whose columns are $\chi \equiv \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ and $\bar{\chi} \equiv \varepsilon\chi^* = \begin{pmatrix} \psi^* \\ -\varphi^* \end{pmatrix}$, i.e.

$$\Phi = \begin{pmatrix} \varphi & \psi^* \\ \psi & -\varphi^* \end{pmatrix}$$

Then Φ satisfies

$$\bar{\Phi} \equiv \varepsilon \Phi^* \varepsilon = \Phi \tag{1.195}$$

$$\det \Phi = -(\varphi^* \varphi + \psi^* \psi) = -\frac{1}{2} \sum_{i=1}^{4} \phi_i^2 = -\frac{1}{2} \phi^T \phi \tag{1.196}$$

The group $\mathcal{O}(4)$ can therefore be described as those linear transformations of Φ that preserve Eq. (1.195) and Eq. (1.196). Show that these conditions are satisfied by

(a) $\Phi \to U\Phi$; or
(b) $\Phi \to \Phi V$

for U and V arbitrary 2×2 complex unitary matrices with unit determinant. Transformations (a) and (b) each form an $SU(2)$ group and $\mathcal{O}(4) \approx SU(2) \times SU(2)$.

(iii) The complex variable $\chi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ is convenient if invariance under only one of the $SU(2)$s is required. Choosing this to be the $SU(2)$ formed by multiplication on the left, χ and $\bar{\chi}$ transform as doublets: $\chi \to U\chi, \bar{\chi} \to U\bar{\chi}$. Construct the most general renormalizable Lagrangian consistent with invariance under a single $SU(2)$. Did you include the invariant term

$$\sum_{a=1}^{3} (\chi^\dagger \tau^a \chi)(\chi^\dagger \tau^a \chi)$$

with τ^a being the Pauli matrices? Should you? Which terms, if any, are not invariant under the "other" $SU(2)$?

(iv) For the $SU(2)$-invariant model, give the ground state and spectrum in the semiclassical approximation for both choices of sign for the coefficient of the quadratic term of the potential. When the background field is non-zero, what subgroup of the original invariance group leaves the background fields invariant? What is the dimension of this subgroup? What is the dimension of the original symmetry group? How many massless states are there?

[1.3] **Vacuum energies**

Consider the model consisting of one free Majorana fermion and one complex scalar field:

$$\mathcal{L} = -\frac{1}{2} \bar{\psi}(\partial\!\!\!/ + m)\psi - (\partial_\mu \varphi)^*(\partial^\mu \varphi) - \mu^2 \varphi^* \varphi$$

The Hamiltonian density for this model is (defining $\dot{\varphi} = \partial_t \varphi$)

$$\mathcal{H}(x) = \dot{\varphi}^* \dot{\varphi} \,+\, (\nabla\varphi)^*(\nabla\varphi) \,+\, \mu^2 \varphi^* \varphi + \frac{1}{2}\bar{\psi}(\gamma \cdot \nabla + m)\psi$$

Use the mode expansions

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\varphi^{(1)} + i\varphi^{(2)}\right]$$

$$\varphi^{(i)}(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\left[\mathrm{e}^{\mathrm{i}px} a_{\mathbf{p}}^{(i)} + \mathrm{e}^{-\mathrm{i}px} a_{\mathbf{p}}^{*(i)}\right]$$

$$\psi(x) = \sum_{\sigma=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\left[u_{\mathbf{p},\sigma}\,\mathrm{e}^{\mathrm{i}px} b_{\mathbf{p},\sigma} + v_{\mathbf{p},\sigma}\,\mathrm{e}^{-\mathrm{i}px} b^*_{\mathbf{p},\sigma}\right]$$

to express the total energy, H, in terms of the creation and annihilation operators $a_{\mathbf{p}}^{(i)}$ and $b_{\mathbf{p},\sigma}$. What is the zero-point energy in this theory? What is the zero-point energy when $\mu = m$? Assume the standard ordering convention: $AB \to \frac{1}{2}(AB + BA) \equiv \frac{1}{2}\{A,\, B\}$ when quantizing. Also assume the standard relations:

$$[a_{\mathbf{p}}^{(i)}, a_{\mathbf{p}'}^{*(j)}] = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')\delta^{ij}$$

$$\left\{b_{\mathbf{p},\sigma}, b^*_{\mathbf{p}',\sigma'}\right\} = \delta_{\sigma,\sigma'} 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')$$

and

$$\left[a_{\mathbf{p}}^{(i)}, b_{\mathbf{p}',\sigma}\right] = 0 \text{ etc.}$$

[1.4] **Symmetries and Yukawa interactions**

Consider a theory with one Majorana fermion, and two real scalar fields φ, χ subject to the symmetry:

$$\begin{aligned} \delta\psi &= \mathrm{i}\omega\gamma_5\psi \\ \delta\varphi &= 2\omega\chi \\ \delta\chi &= -2\omega\varphi \end{aligned}$$

for ω an infinitesimal, spatially constant parameter.

[1.4.1] Write down the most general renormalizable Lagrangian coupling the scalars to each other and to the fermion. Identify the vacuum field configuration and mass spectrum both in the broken and unbroken phases (i.e. for both choices of sign for the coefficient of the quadratic term of the potential).

[1.4.2] Couple a spin-one particle to this symmetry; i.e., write down covariant derivatives for the fields ψ, χ, and φ and construct an action invariant with respect to these transformations with $\partial_\mu \omega \neq 0$. Again identify the spectrum in both broken and unbroken phases.

[1.5] **Spinor identities**

Derive the following formulae concerning the spin-half wave function:

$$u(\mathbf{p},\sigma)\bar{u}(\mathbf{p},\sigma) = \frac{1}{2}(m - \mathrm{i}\not{p})(1 + \mathrm{i}\gamma_5 \not{s})$$

$$\text{and} \sum_{\sigma=\pm\frac{1}{2}} u(\mathbf{p},\sigma)\bar{u}(\mathbf{p},\sigma) = (m - \mathrm{i}\not{p})$$

in which p^μ is the particle four-momentum and s^μ is a four-pseudovector whose components in the rest frame are $s^0 = 0$, $\mathbf{s} = 2\sigma\mathbf{e}$ where $\mathbf{e}$ is the unit vector in the direction along which the spin components, $\sigma = \pm\frac{1}{2}$, are measured. (Choose $\mathbf{e}$ to lie along the positive x^3-axis.) Notice these imply the frame-independent conditions: $s^\mu s_\mu = +1$ and $s^\mu p_\mu = 0$. Recall, also that $(i\not{p} + m)u(\mathbf{p},\sigma) = 0$ and $p^\mu p_\mu = -m^2$.

Hint: Since $u(\mathbf{p},\sigma)\bar{u}(\mathbf{p},\sigma)$ is a 4×4 matrix, expand it in terms of the basis matrices S, P, V, A, and T, defined as $S = \mathbf{1}$, $P = \gamma_5$, $V = \gamma^\mu$, $A = \gamma^\mu\gamma_5$, and $T = [\gamma^\mu, \gamma^\nu]$. Since $u\bar{u}$ transforms covariantly under Lorentz transformations, the coefficients of these matrices are scalars, pseudoscalars, vectors, etc.Evaluate the coefficients by taking traces after multiplying by an appropriate matrix. It may prove convenient to evaluate those coefficients that transform as vectors and tensors in the rest frame of the particle.

Is the resulting expression well behaved in the zero-mass limit?

[1.6] **Fermion mass matrix diagonalization**

Prove the theorem that, for any complex, symmetric matrix, A, there exists a unitary matrix, U, for which

$$U^T A U = M$$

is real, diagonal and non-negative. (Recall we used this theorem to show that the spin-half mass matrix could always be put into standard form.)

Hint: Notice that this would be trivial if $[A, A^\dagger] = 0$ because then if we break A into its real and imaginary parts, $A = R + \mathrm{i}S$ for R, S real and symmetric, we see that $[A, A^\dagger] = 2\mathrm{i}[S, R] = 0$. Since S and R are both real and symmetric and commute, they can both be diagonalized by the same real orthogonal matrix, $\mathcal{O}$. This implies that $\mathcal{O}^T A \mathcal{O} = \mathrm{diag}$ and we could define $U = \mathcal{O}D$ with D being a diagonal matrix whose elements are phases that can be chosen to make the entries of M real and non-negative. In the general case when $[A, A^\dagger] \neq 0$, we know that $A^\dagger A$ is Hermitian and so is diagonalizable by a unitary matrix, V. Define the new matrix $B \equiv V^T A V$ and show that $B = B^T$ and $[B,\ B^\dagger] = 0$.

[1.7] **A Dirac matrix identity**

Prove the identity which shows that $\gamma_{\mu\nu}\gamma_5$ is not linearly independent of $\epsilon_{\mu\nu\lambda\rho}\gamma^{\lambda\rho}$

$$\epsilon_{\mu\nu\lambda\rho}\gamma^{\lambda\rho} = 2\mathrm{i}\gamma_{\mu\nu}\,\gamma_5 \tag{1.197}$$

Here $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

[1.8] **More useful identities:** Prove the following identities:

[1.8.1]

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho P_{\mathrm{R}} &= (\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\lambda\nu} - \mathrm{i}\epsilon^{\mu\nu\lambda\rho})P_{\mathrm{R}} \\ &+(\eta^{\mu\nu}\gamma^{\lambda\rho} - \eta^{\mu\lambda}\gamma^{\nu\rho} + \eta^{\mu\rho}\gamma^{\nu\lambda} - \eta^{\nu\lambda}\gamma^{\rho\mu} + \eta^{\nu\rho}\gamma^{\lambda\mu} - \eta^{\lambda\rho}\gamma^{\nu\mu})P_{\mathrm{R}}\end{aligned} \tag{1.198}$$

Here $P_{\mathrm{R}} = \frac{1}{2}(1-\gamma_5)$ is the usual projection matrix onto right-handed spinors and $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ is half of the commutator of two Dirac matrices.

[1.8.2] For X and Y any product of an *odd* number of gamma matrices prove the following trace formula:

$$\mathrm{tr}[XYP_{\mathrm{R}}] = \frac{1}{2}\,\mathrm{tr}[X\gamma^\mu P_{\mathrm{R}}]\,\mathrm{tr}[Y\gamma_\mu P_{\mathrm{L}}] \tag{1.199}$$

P_{R} is as before and $P_{\mathrm{L}} = \frac{1}{2}(1+\gamma_5)$.

[1.9] **Fiertz rearrangements**

The sixteen Dirac matrices I, γ_5, γ^μ, $\gamma^\mu\gamma_5$, and $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ provide a basis in terms of which any 4×4 complex matrix can be expressed (prove this). Given this property, show that this provides the following useful way to rewrite a dyadic product of two anticommuting spinors:

$$\begin{aligned}P_{\mathrm{L}}[\psi_1\overline{\psi}_2]P_{\mathrm{R}} &= -\frac{1}{2}[\overline{\psi}_2\gamma^\mu P_{\mathrm{L}}\,\psi_1]\gamma_\mu P_{\mathrm{R}} \\ P_{\mathrm{R}}[\psi_1\overline{\psi}_2]P_{\mathrm{L}} &= -\frac{1}{2}[\overline{\psi}_2\gamma^\mu P_{\mathrm{R}}\,\psi_1]\gamma_\mu P_{\mathrm{L}} \\ P_{\mathrm{L}}[\psi_1\overline{\psi}_2]P_{\mathrm{L}} &= -\frac{1}{2}[\overline{\psi}_2 P_{\mathrm{L}}\,\psi_1]P_{\mathrm{L}} - \frac{1}{8}[\overline{\psi}_2\gamma^{\mu\nu}P_{\mathrm{L}}\,\psi_1]\ \gamma_{\mu\nu}P_{\mathrm{L}} \\ P_{\mathrm{R}}[\psi_1\overline{\psi}_2]P_{\mathrm{R}} &= -\frac{1}{2}[\overline{\psi}_2 P_{\mathrm{R}}\,\psi_1]\ P_{\mathrm{R}} - \frac{1}{8}[\overline{\psi}_2\gamma^{\mu\nu}P_{\mathrm{R}}\,\psi_1]\ \gamma_{\mu\nu}P_{\mathrm{R}}\end{aligned} \tag{1.200}$$

2

The standard model: general features

The last chapter developed the general principles for writing down a relativistic quantum field theory. It showed what types of fields are possible, and explained that spin-one fields can only appear in an interacting, renormalizable theory if they are coupled via the gauge principle.

In this chapter, we write down specifically what the field content of the standard model is. The interactions will then follow as the most general set of renormalizable interactions, compatible with that field content. We then explore what the vacuum and the particle content are, and write down the complete interaction Hamiltonian in the particle basis.

We will not attempt to motivate theoretically why the particle content of the standard model is what it is. We have no deep understanding of why the gauge group is $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$, for instance. We just take the field content as observed fact, and present it. Note however that the field content of the standard model is not completely arbitrary; once the gauge group is known, the fermionic field content is somewhat constrained by the requirement of anomaly cancellation, which we discuss at the end of the chapter.

2.1 Particle content

The strong, weak, and electromagnetic interactions are understood as arising due to the exchange of various spin-one bosons amongst the spin-half particles that make up matter. The gauged symmetry group of the standard model is $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$. The specific gauge bosons associated with the generators of the algebra of the group are:

$$\begin{array}{ccccc} SU_{\rm c}(3) & \times & SU_{\rm L}(2) & \times & U_Y(1) \\ \downarrow & & \downarrow & & \downarrow \\ 8\ G^\alpha_\mu & & 3\ W^a_\mu & & B_\mu \\ \alpha = 1, \ldots, 8 & & a = 1, 2, 3 & & \end{array} \tag{2.1}$$

The eight spin-one particles, $G^\alpha_\mu(x)$, associated with the factor $SU_{\rm c}(3)$ are called *gluons* and the associated subscript "c" is meant to denote "color." Gluons are thought to be massless. Any particle that transforms with respect to this factor of the gauge group, and so which couples to the gluons, is said to be colored or to carry color. This interaction is also called the "strong interaction," and any particle which couples to the gluons is said to be "strongly interacting." Three spin-one particles, $W^a_\mu(x)$, are associated with the factor $SU_{\rm L}(2)$, and one, $B_\mu(x)$, with the factor $U_Y(1)$. The subscript "L" is meant to indicate that only the left-handed fermions turn out to carry this quantum number. The subscript "Y" is meant to distinguish the group associated with the quantum number (defined below) of *weak hypercharge*, denoted Y, from the group associated with ordinary electric charge, denoted Q. The electromagnetic group will be written as $U_{\rm em}(1)$. The four spin-one bosons associated with the factors $SU_{\rm L}(2) \times U_Y(1)$ are related to the physical bosons that mediate the weak interactions, $W^\pm$ and Z^0, and the familiar photon from QED, in a way we will explain in Section 2.3.

Apart from spin-one particles we are aware of a number of fundamental spin-half particles and one fundamental spin-zero particle. Our knowledge to date about the character of the interactions of these fields may be compactly summarized by giving their transformation properties with respect to the gauge group $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$. The fermions transform in a relatively complicated way with respect to this symmetry group. There are three copies (families or generations) of particles, each copy of which couples identically to all spin-one particles.

Leptons are, by definition, those spin-half particles which do not take part in the strong interactions. Six leptons are known to date. They are denoted individually by $e, \mu, \tau, \nu_e, \nu_\mu$, and ν_τ, and collectively by ℓ.

Hadrons, on the other hand, are defined as those particles that do take part in the strong interactions. The spectrum of known hadrons is rich and varied but, as we shall see, appears to be accounted for as the bound states of six quarks u, d, s, c, b, and t, denoted collectively as q.

Because of the relatively large number of spin-half fields involved, a few words on notation may be appropriate. Spinors written in capital letters L, E, D, U, Q, or script letters $\mathcal{E}, \mathcal{U}, \mathcal{D}$, and neutrinos ν_i are taken as Majorana spinors. The left- and right-handed components of these spinors are denoted by subscripts L, R. Spinors written in lower case Roman letters l_i, u_i, d_i, e, u, c, t, d, s, b, or by μ, τ are Dirac spinors, which we will introduce in turn.

For example, the electron field is represented in quantum electrodynamics by the Dirac spinor, $e(x)$. Denote the left- and right-handed components of this spinor by e_{L} and e_{R} respectively:

$$e = \begin{pmatrix} e_{\mathrm{L}} \\ e_{\mathrm{R}} \end{pmatrix} \tag{2.2}$$

In the standard model, however, the electron is represented by two Majorana fields, $\mathcal{E}(x)$ and $E(x)$, that are defined to contain the left- and right- handed parts of $e(x)$ respectively. That is,

$$\mathcal{E} = \begin{pmatrix} e_{\mathrm{L}} \\ \epsilon e_{\mathrm{L}}^* \end{pmatrix}, \quad E = \begin{pmatrix} -\epsilon e_{\mathrm{R}}^* \\ e_{\mathrm{R}} \end{pmatrix} \tag{2.3}$$

where the 2×2 matrix ϵ is defined in Eq. (1.79). The Dirac spinor, e, is therefore related to the Majorana fields, $\mathcal{E}$ and E, by projecting onto the left- or right-handed part:

$$e = P_{\mathrm{L}}\mathcal{E} + P_{\mathrm{R}}E \tag{2.4}$$

The "left-handed" electron field, $\mathcal{E}$, itself appears within an $SU_{\mathrm{L}}(2)$-doublet with the field, ν, whose left-handed part contains the left-handed electron-neutrino. This doublet is denoted $L(x)$:

$$P_{\mathrm{L}} L = \begin{pmatrix} P_{\mathrm{L}}\,\nu \\ P_{\mathrm{L}}\,\mathcal{E} \end{pmatrix} \tag{2.5}$$

The notation here is somewhat confusing; the matrix structure shown for L above does not show spinorial matrix structure, but shows matrix structure under the group $SU_{\mathrm{L}}(2)$; each component, ν and $\mathcal{E}$, is a 4-component Majorana spinor. Generally, when possible spinorial structure is suppressed in what follows.

Members of successive generations are denoted by a generation index, m, that runs from 1 to 3. The generations are numbered in increasing order with respect to the mass of the corresponding charged lepton:

$$\begin{aligned} &\nu_m \text{ denotes } \nu_1 = \nu_e,\ \ \nu_2 = \nu_\mu,\ \ \nu_3 = \nu_\tau \\ &e_m \text{ denotes } e_1 = e,\ \ e_2 = \mu,\ \ e_3 = \tau \\ &u_m \text{ denotes } u_1 = u,\ \ u_2 = c,\ \ u_3 = t \end{aligned}$$

$$\text{and} \qquad d_m \text{ denotes } d_1 = d, \;\; d_2 = s, \;\text{ and } d_3 = b \tag{2.6}$$

The transformation properties of the fermions and scalar are summarized by giving the representation of the gauge group in which they transform. A standard way to label the representations of $SU_{\rm L}(2)$ and $SU_{\rm c}(3)$ is with their dimension. So the two-dimensional spinor representation of $SU_{\rm L}(2)$ is written **2** (familiar from the physics of spin as the spin-half representation) and the two three-dimensional representations of $SU_{\rm c}(3)$ would be **3** or $\overline{\mathbf{3}}$. The trivial (invariant) representation is written as **1**. The transformation properties with respect to $U_Y(1)$ may be specified by giving the corresponding eigenvalue of the generator, Y, called the *weak hypercharge*. Y is normalized so that the action of $U_Y(1)$ on a field with eigenvalue y is given by $\psi \to \exp[\mathrm{i}\omega(x)y]\psi$.

With these conventions the fermionic particle content of the standard model may be summarized as follows:

$$\begin{aligned}
P_{\rm L}L_m &= \begin{pmatrix} P_{\rm L}\nu_m \\ P_{\rm L}\mathcal{E}_m \end{pmatrix} \quad \text{transforms as} \quad && \left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right) \\
P_{\rm R}E_m & && \left(\mathbf{1}, \mathbf{1}, -1\right) \\
P_{\rm L}Q_m &= \begin{pmatrix} P_{\rm L}\mathcal{U}_m \\ P_{\rm L}\mathcal{D}_m \end{pmatrix} && \left(\mathbf{3}, \mathbf{2}, +\frac{1}{6}\right) \\
P_{\rm R}U_m & && \left(\mathbf{3}, \mathbf{1}, +\frac{2}{3}\right) \\
P_{\rm R}D_m & && \left(\mathbf{3}, \mathbf{1}, -\frac{1}{3}\right)
\end{aligned} \tag{2.7}$$

Here the first number represents the $SU_{\rm c}(3)$ representation, the second number is the $SU_{\rm L}(2)$ representation and the final number is the eigenvalue of the weak hypercharge, Y. In the case of $SU_{\rm L}(2)$ doublets, we have named their upper and lower $SU_{\rm L}(2)$components, $L_m = (P_{\rm L}\,\nu_m P_{\rm L}\,\mathcal{E}_m)^T$ and $Q_m = (P_{\rm L}\,\mathcal{U}_m P_{\rm L}\,\mathcal{D}_m)^T$. We could in principle do this for the three separate colors of the Q, U, and D fields; but it turns out to be useful to do so for the $SU_{\rm L}(2)$ content but not for the $SU_{\rm c}(3)$ content.

Since the left- and right-handed pieces of a Majorana spinor are the complex conjugates of one another, they must transform in complex-conjugate representations. It follows then that

$$P_{\rm R}L_m = \begin{pmatrix} P_{\rm R}\nu_m \\ P_{\rm R}\mathcal{E}_m \end{pmatrix} \quad \text{transforms as} \quad \left(\mathbf{1}, \mathbf{2}, +\frac{1}{2}\right)$$

$$\begin{aligned}
&P_{\rm L}E_m && \left(\mathbf{1},\mathbf{1},+1\right)\\
&P_{\rm R}Q_m = \begin{pmatrix} P_{\rm R}\mathcal{U}_m \\ P_{\rm R}\mathcal{D}_m \end{pmatrix} && \left(\overline{\mathbf{3}},\mathbf{2},-\frac{1}{6}\right)\\
&P_{\rm L}U_m && \left(\overline{\mathbf{3}},\mathbf{1},-\frac{2}{3}\right)\\
&P_{\rm L}D_m && \left(\overline{\mathbf{3}},\mathbf{1},+\frac{1}{3}\right)
\end{aligned} \tag{2.8}$$

We note in passing that if the standard model were to be supplemented to include a right-handed neutrino field, N_m, this field would be a singlet,

$$P_{\rm R}N_m \qquad \text{transforms as:} \quad (\mathbf{1},\mathbf{1},0) \tag{2.9}$$

with respect to the gauge group $SU_{\rm c}(3)\times SU_{\rm L}(2)\times U_Y(1)$. We will discuss such a singlet some more in Chapter 10, see also Problem 2.3.

Apart from fermions, the Lagrangian must also involve the fields representing the spin-one gauge bosons. These fields and their transformation rules are denoted as follows:

$$\begin{aligned}
&G^\alpha_\mu \qquad \text{transforms as:} && (\mathbf{8},\mathbf{1},0)\\
&W^a_\mu && (\mathbf{1},\mathbf{3},0)\\
&B_\mu && (\mathbf{1},\mathbf{1},0)
\end{aligned} \tag{2.10}$$

Lastly, the theory contains a scalar field, which contains the physical degree of freedom which becomes the celebrated Higgs boson. The Higgs field ϕ transforms as

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad \text{transforms as} \left(\mathbf{1},\mathbf{2},\frac{1}{2}\right). \tag{2.11}$$

As discussed in Appendix B, if we multiply the conjugate of ϕ, ϕ^*, by the antisymmetric tensor ϵ (acting on its $SU_{\rm L}(2)$ indices), the result is also a valid $SU_{\rm c}(3)\times SU_{\rm L}(2)\times U_Y(1)$ representation, which we call $\tilde{\phi}$:

$$\tilde{\phi} \equiv \begin{pmatrix} \phi^{0*} \\ -\phi^{+*} \end{pmatrix} = \varepsilon\phi^* \quad \text{transforms as} \left(\mathbf{1},\mathbf{2},-\frac{1}{2}\right) \tag{2.12}$$

which is the same representation as $P_{\rm L}L$. It is a matter of convention whether one considers the field ϕ as fundamental and $\tilde{\phi}$ as derived from it, or vice versa; we follow the almost universal convention to do the former. As we shall see, although ϕ contains four real components, only one of them manifests as a scalar particle, due to the *Higgs mechanism*, which we will discuss in Section 2.3.

The representation content we have presented is merely a short form for the invariance of the Lagrangian under the following symmetries:

$$
\begin{aligned}
\delta\phi &= \left[\left(\frac{\mathrm{i}}{2}\omega_1(x) + \frac{\mathrm{i}}{2}\omega_2^a(x)\tau_a\right)\right]\phi \\
\delta L_m &= \left[\left(-\frac{\mathrm{i}}{2}\omega_1(x) + \frac{\mathrm{i}}{2}\omega_2^a(x)\tau_a\right)P_{\mathrm{L}} + \left(\frac{\mathrm{i}}{2}\omega_1(x) - \frac{\mathrm{i}}{2}\omega_2^a(x)\tau_a^*\right)P_{\mathrm{R}}\right]L_m \\
\delta E_m &= [\mathrm{i}\omega_1(x)P_{\mathrm{L}} - \mathrm{i}\omega_1(x)P_{\mathrm{R}}]E_m \\
\delta Q_m &= \left[\left(\frac{\mathrm{i}}{6}\omega_1(x) + \frac{\mathrm{i}}{2}\omega_2^a(x)\tau_a + \frac{\mathrm{i}}{2}\omega_3^\alpha(x)\lambda_\alpha\right)P_{\mathrm{L}} + \right. \\
&\qquad \left. + \left(-\frac{\mathrm{i}}{6}\omega_1(x) - \frac{\mathrm{i}}{2}\omega_2^a(x)\tau_a^* - \frac{\mathrm{i}}{2}\omega_3^\alpha(x)\lambda_\alpha^*\right)P_{\mathrm{R}}\right]Q_m \\
\delta U_m &= \left[\left(-\frac{2\mathrm{i}}{3}\omega_1(x) - \frac{\mathrm{i}}{2}\omega_3^\alpha(x)\lambda_\alpha^*\right)P_{\mathrm{L}} + \left(\frac{2\mathrm{i}}{3}\omega_1(x) + \frac{\mathrm{i}}{2}\omega_3^\alpha(x)\lambda_\alpha\right)P_{\mathrm{R}}\right]U_m \\
\delta D_m &= \left[\left(\frac{\mathrm{i}}{3}\omega_1(x) - \frac{\mathrm{i}}{2}\omega_3^\alpha(x)\lambda_\alpha^*\right)P_{\mathrm{L}} + \left(-\frac{\mathrm{i}}{3}\omega_1(x) + \frac{\mathrm{i}}{2}\omega_3^\alpha(x)\lambda_\alpha\right)P_{\mathrm{R}}\right]D_m \\
\delta G_\mu^\alpha &= \partial_\mu\omega_3^\alpha(x) - f^\alpha_{\beta\gamma}\omega_3^\beta(x)\, G_\mu^\gamma \\
\delta W_\mu^a &= \partial_\mu\omega_2^a(x) - \epsilon^{abc}\omega_2^b(x)\, W_\mu^c \\
\delta B_\mu &= \partial_\mu\omega_1(x)
\end{aligned}
\tag{2.13}
$$

In these expressions the generators of $SU_{\mathrm{L}}(2)$ have been explicitly written as $T_a = \frac{1}{2}\tau_a$ where $\tau_a, a = 1, 2, 3$ denotes the 2×2 *Pauli* matrices that act on the $SU_{\mathrm{L}}(2)$-indices

$$
\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.14}
$$

(The same matrices appeared in discussing the spin structure of fermions in Section 1.3. We use the notation τ_i when they act on $SU_{\mathrm{L}}(2)$ indices and σ_i when they act on spinorial indices.) Similarly, the generators of $SU_{\mathrm{c}}(\mathbf{3})$ (when acting on the **3** representation) are given explicitly by $T_\alpha = \frac{1}{2}\lambda_\alpha$ where $\lambda_\alpha, \alpha = 1, \ldots, 8$ denote the 3×3 Gell-Mann matrices given in Eq. (1.186).

The electric charge Q of a field is defined in terms of the hypercharge Y and the $SU_{\mathrm{L}}(2)$ charge's T_3 component, according to $Q = T_3 + Y$. Note that the electromagnetic group is *not* directly the $U_Y(1)$ component of the standard model gauge group, and electric charge Q is *not* one of the basic charges particles carry under $SU_{\mathrm{c}}(3) \times SU_{\mathrm{L}}(2) \times U_Y(1)$; rather it is a derived quantity.

2.2 The Lagrangian

Now we write the most general renormalizable Lagrangian involving these fields. We will break the Lagrangian into two parts, those terms which do not contain the Higgs field ϕ and those terms which do. The Lagrangian takes the form

$$\mathcal{L}_{\rm SM} = \mathcal{L}_{fg} + \mathcal{L}_{\rm Higgs} \tag{2.15}$$

$$\begin{aligned}\mathcal{L}_{fg} = & -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu} - \frac{1}{4}W^{a\mu\nu}W^{a}_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{g_3^2\Theta_3}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}G^{\alpha\mu\nu}G^{\alpha\lambda\rho} \\ & - \frac{g_2^2\Theta_2}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}W^{a\mu\nu}W^{a\lambda\rho} - \frac{g_1^2\Theta_1}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}B^{\mu\nu}B^{\lambda\rho} - \frac{1}{2}\overline{L}_m \not{D} L_m \\ & - \frac{1}{2}\overline{E}_m \not{D} E_m - \frac{1}{2}\overline{Q}_m \not{D} Q_m - \frac{1}{2}\overline{U}_m \not{D} U_m - \frac{1}{2}\overline{D}_m \not{D} D_m\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathcal{L}_{\rm Higgs} = & -({\rm D}_\mu\phi)^\dagger({\rm D}^\mu\phi) - V(\phi^\dagger\phi) \\ & -(f_{mn}\bar{L}_m P_{\rm R} E_n \phi + h_{mn}\bar{Q}_m P_{\rm R} D_n \phi + g_{mn}\bar{Q}_m P_{\rm R} U_n \tilde{\phi} + \text{h.c.})\end{aligned} \tag{2.17}$$

$$\begin{aligned}V(\phi^\dagger\phi) &= \lambda\left[\phi^\dagger\phi - \mu^2/2\lambda\right]^2 \\ &= \lambda(\phi^\dagger\phi)^2 - \mu^2\phi^\dagger\phi + \mu^4/4\lambda\end{aligned} \tag{2.18}$$

in which the gauge field-strengths are given by

$$G^{\alpha}_{\mu\nu} = \partial_\mu G^{\alpha}_{\nu} - \partial_\nu G^{\alpha}_{\mu} + g_3 f^{\alpha}{}_{\beta\gamma} G^{\beta}_{\mu} G^{\gamma}_{\nu} \tag{2.19}$$

$$W^{a}_{\mu\nu} = \partial_\mu W^{a}_{\nu} - \partial_\nu W^{a}_{\mu} + g_2 \epsilon_{abc} W^{b}_{\mu} W^{c}_{\nu} \tag{2.20}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2.21}$$

The gauge-covariant derivatives are

$$\begin{aligned}{\rm D}_\mu L_m = & \ \partial_\mu L_m + \left[\frac{\rm i}{2}g_1 B_\mu - \frac{\rm i}{2}g_2 W^a_\mu \tau_a\right] P_{\rm L}\, L_m \\ & + \left[-\frac{\rm i}{2}g_1 B_\mu + \frac{\rm i}{2}g_2 W^a_\mu \tau^*_a\right] P_{\rm R}\, L_m\end{aligned} \tag{2.22}$$

$${\rm D}_\mu E_m = \partial_\mu E_m + {\rm i}g_1 B_\mu (P_{\rm R}\, E_m) - {\rm i}g_1 B_\mu (P_{\rm L}\, E_m) \tag{2.23}$$

$$\begin{aligned}{\rm D}_\mu Q_m = & \ \partial_\mu Q_m + \left[-\frac{\rm i}{2}g_3 G^\alpha_\mu \lambda_\alpha - \frac{\rm i}{2}g_2 W^a_\mu \tau_a - \frac{\rm i}{6}g_1 B_\mu\right] P_{\rm L}\, Q_m \\ & + \left[\frac{\rm i}{2}g_3 G^\alpha_\mu \lambda^*_\alpha + \frac{\rm i}{2}g_2 W^a_\mu \tau^*_a + \frac{\rm i}{6}g_1 B_\mu\right] P_{\rm R} Q_m\end{aligned} \tag{2.24}$$

$$\begin{aligned}{\rm D}_\mu U_m = & \ \partial_\mu U_m + \left[-\frac{\rm i}{2}g_3 G^\alpha_\mu \lambda_\alpha - \frac{2{\rm i}}{3}g_1 B_\mu\right] P_{\rm R} U_m \\ & + \left[\frac{\rm i}{2}g_3 G^\alpha_\mu \lambda^*_\alpha + \frac{2{\rm i}}{3}g_1 B_\mu\right] P_{\rm L} U_m\end{aligned} \tag{2.25}$$

$$\begin{aligned} \mathrm{D}_\mu D_m &= \partial_\mu D_m + \left[-\frac{\mathrm{i}}{2} g_3 G_\mu^\alpha \lambda_\alpha + \frac{\mathrm{i}}{3} g_1 B_\mu\right] P_{\mathrm{R}} D_m \\ &\quad + \left[+\frac{\mathrm{i}}{2} g_3 G_\mu^\alpha \lambda_\alpha^* - \frac{\mathrm{i}}{3} g_1 B_\mu\right] P_{\mathrm{L}} D_m \end{aligned} \tag{2.26}$$

$$\mathrm{D}_\mu \phi = \partial_\mu \phi - \frac{\mathrm{i}}{2} g_2 W_\mu^a \tau_a \phi - \frac{\mathrm{i}}{2} g_1 B_\mu \phi \tag{2.27}$$

Unitarity requires that the constants λ and μ^2 be real and stability demands that λ be positive.

It is worth emphasizing at this point why certain terms do *not* appear in $\mathcal{L}_{fg}$. In particular, only the $\mu^2 \phi^\dagger \phi$ term can be interpreted as a conventional mass term; there are no mass terms for the gauge fields, nor for the fermionic fields. The reason is that only terms which are singlets under $SU_{\mathrm{c}}(3) \times SU_{\mathrm{L}}(2) \times U_Y(1)$ can appear in the Lagrangian – otherwise it would not respect gauge invariance, that is, it would change under a gauge transformation. The rules for telling if a combination of fields is a singlet under $SU_{\mathrm{c}}(3)$ or $SU_{\mathrm{L}}(2)$ are summarized in appendix B; basically the rule is that all color and $SU_{\mathrm{L}}(2)$ indices must “tie off” against each other. The rule for $U_Y(1)$ is even easier; the charges of the fields must add to zero.

Consider for instance the would-be mass term for the E field,

$$\mathcal{L}_{\text{would-be}} = -\frac{m_{mn}}{2} \overline{E}_m E_n$$

Write $\overline{E}_m E_n = \overline{E}_m P_{\mathrm{L}} E_n + \overline{E}_m P_{\mathrm{R}} E_n$, and just consider the P_{R} term. $P_{\mathrm{R}} E$ has hypercharge -1. The hypercharge of $\overline{E} P_{\mathrm{R}}$ is also -1. To see this, note that

$$\overline{E} P_{\mathrm{R}} = E^\dagger \beta P_{\mathrm{R}} = E^\dagger P_{\mathrm{L}} \beta \tag{2.28}$$

is actually the conjugate field of $P_{\mathrm{L}} E$, and has the opposite charge as $P_{\mathrm{L}} E$. Therefore, the combination $\overline{E} P_{\mathrm{R}} E$ is hypercharge -2 and is not a gauge singlet. The combination $\overline{E} P_{\mathrm{L}} E$ is hypercharge $+2$ and is also not allowed. One can quickly check that no combination of two spinor fields is hypercharge neutral, so no such mass is permitted. The kinetic terms *are* invariant because $P_{\mathrm{L}} \gamma^\mu = \gamma^\mu P_{\mathrm{R}}$; so the left-handed component of a field couples to the Hermitian conjugate of the left-handed component and the gauge dependence does cancel.

For the case $\mu^2 < 0$, the minimum energy is obtained when $\phi = 0$, and the spectrum may be analyzed by perturbing in the gauge couplings, g_i, $i = 1, 2, 3$. (We return to the accuracy of this approximation in more detail later.) The unperturbed part of the Lagrangian becomes in this case those terms that are quadratic in the fields. The spectrum of this unperturbed theory is therefore that of a system of free spin-zero, spin-half, and spin-one particles, as was described in the previous chapter. Following the discussion leading up to Eq. (1.67)–Eq. (1.125), the scalar is massive with mass $m_{\rm H}^2 = -\mu^2$, and all spin-half and spin-one fields are massless!

Since the perturbative semiclassical analysis should apply to at least the electroweak part of the theory, we should instead consider the case $\mu^2 > 0$. Indeed, this is the reason for our convention choice in introducing μ^2. As we will see in the following sections, this choice gives rise to a spectrum of massive particles which is in good agreement with experiment.

The following general features of $\mathcal{L}_{\rm SM}$ bear special mention.

(i) $\mathcal{L}_{fg}$, $\mathcal{L}_{\rm Higgs}$ and $\mathcal{L}_{\rm SM}$ are the most general Lagrangian consistent with the given particle content and invariance under $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$. If the predictions made from such an $\mathcal{L}$ are wrong, then either the particle-content or renormalizability or the gauge group is wrong.

(ii) Because of $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$ invariance, all masses vanish in the absence of $\mathcal{L}_{\rm Higgs}$.

(iii) There are six parameters in $\mathcal{L}_{fg}$ of which only four enter into physical predictions (since Θ_1 and Θ_2 turn out to have no physical effects, for reasons we will not discuss). $\mathcal{L}_{\rm Higgs}$, on the other hand, contains no less than 15 parameters (as we shall see these may be taken to be the ten masses, the Higgs self-coupling, and the four Kobayashi–Maskawa angles). In this sense $\mathcal{L}_{\rm Higgs}$ parameterizes most of our ignorance and is the part of the theory that is the least understood. All of the couplings also turn out to be small (modulo some restrictions to which we return for g_3), allowing the use of perturbation theory to calculate the predictions of $\mathcal{L}$.

(iv) The terms on the first line of Eq. (2.17) could be equally well written in terms of $\tilde{\phi}$, rather than ϕ. The terms on the second line are most easily written as shown, and emphasize the importance that the Higgs field can enter the Hamiltonian either in the form ϕ or the form $\tilde{\phi}$. No term $\tilde{\phi}^\dagger \phi$ can occur, because this combination is identically zero!

2.3 The perturbative spectrum

The first step in analyzing the consequences of the standard model is to find its spectrum. We do so semiclassically, following the procedure of Subsection 1.6.2. For these purposes it is convenient here, as it was there, to use the gauge freedom to transform to unitary gauge. In the present context unitary gauge is defined by the following condition:

$$\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix} \tag{2.29}$$

where $H(x)$ is a real field and v is a real constant that minimizes the scalar potential. It may be shown that it is always possible to reach Eq. (2.29) from an arbitrary initial field configuration via a gauge transformation. The motivation for this gauge choice is that it ensures that no vector-scalar cross terms survive in the quadratic terms once we expand about the ground state. It is worth noting in passing that the gauge, Eq. (2.29), does not fix those gauge invariances that leave the Higgs *v.e.v.* invariant. In the present context, as is shown later in this section, this means that the electromagnetic gauge invariance still remains to be fixed.

v is determined by minimizing the potential in Eq. (2.18) and satisfies

$$v^2 = \mu^2/\lambda \tag{2.30}$$

In order to read off the particle masses we must identify the unperturbed Lagrangian, $\mathcal{L}_0$. This is equal to that part of $\mathcal{L}_{\rm SM}$ that is quadratic in the fluctuations. The expansion of $\mathcal{L}_{fg}$ is trivial and just contributes the spin-half and spin-one kinetic terms to $\mathcal{L}_0$. Everything else comes from the expansion of $\mathcal{L}_{\rm Higgs}$. Using the following result,

$$\mathrm{D}_\mu \phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} - \frac{\mathrm{i}}{2\sqrt{2}} \begin{pmatrix} g_2 W^3_\mu + g_1 B_\mu & g_2 W^1_\mu - \mathrm{i} g_2 W^2_\mu \\ g_2 W^1_\mu + \mathrm{i} g_2 W^2_\mu & -g_2 W^3_\mu + g_1 B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v+H \end{pmatrix} \tag{2.31}$$

the expansion of the scalar-field kinetic term becomes:

$$\begin{aligned} -(\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) \;=\; & -\frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{8}(v+H)^2 g_2^2 (W^1_\mu - \mathrm{i}W^2_\mu)(W^{1\mu} + \mathrm{i}W^{2\mu}) \\ & -\frac{1}{8}(v+H)^2(-g_2 W^{3\mu} + g_1 B^\mu)(-g_2 W^3_\mu + g_1 B_\mu) \end{aligned} \tag{2.32}$$

The scalar potential term contributes

$$\begin{aligned} V &= \frac{\lambda}{4}\Big[(v+H)^2 - \mu^2/\lambda\Big]^2 \\ &= \frac{\lambda}{4}(2vH + H^2)^2 \\ &= \lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4 \end{aligned} \tag{2.33}$$

The Yukawa couplings may be expanded in an identical way:

$$\begin{aligned} \bar{L}_m P_{\mathrm{R}}\, E_n \phi &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\nu}_m \\ \bar{\mathcal{E}}_m \end{pmatrix}^T P_{\mathrm{R}}\, E_n \begin{pmatrix} 0 \\ v+H \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (v+H) \bar{\mathcal{E}}_m P_{\mathrm{R}}\, E_n \end{aligned} \tag{2.34}$$

and similarly for Q, d, and D, and

$$\begin{aligned} \bar{Q}_m P_{\mathrm{R}}\, U_n \tilde{\phi} &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\mathcal{U}}_m \\ \bar{\mathcal{D}}_m \end{pmatrix}^T P_{\mathrm{R}}\, U_n \begin{pmatrix} v+H \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (v+H) \bar{\mathcal{U}}_m P_{\mathrm{R}} U_n \end{aligned} \tag{2.35}$$

Combining all of these results gives the expansion of $\mathcal{L}_{\mathrm{Higgs}}$ to be

$$\begin{aligned} \mathcal{L}_{\mathrm{Higgs}} = & -\frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{\lambda}{4} H^4 \\ & -\frac{1}{8} g_2^2 (v+H)^2 |W_\mu^1 - \mathrm{i} W_\mu^2|^2 \\ & -\frac{1}{8} (v+H)^2 (-g_2 W_\mu^3 + g_1 B_\mu)^2 \\ & -\frac{1}{\sqrt{2}} (v+H) [f_{mn} \bar{\mathcal{E}}_m P_{\mathrm{R}}\, E_n + \text{ h.c.}] \\ & -\frac{1}{\sqrt{2}} (v+H) [g_{mn} \bar{\mathcal{U}}_m P_{\mathrm{R}}\, U_n + \text{ h.c.}] \\ & -\frac{1}{\sqrt{2}} (v+H) [h_{mn} \bar{\mathcal{D}}_m P_{\mathrm{R}}\, D_n + \text{ h.c.}] \end{aligned} \tag{2.36}$$

2.3.1 Boson masses

$\mathcal{L}_{\mathrm{Higgs}}$ contains all of the mass terms, although some of these are not diagonal. They are, in more detail

2.3.1.1 Spin-zero particles

Comparing the H^2 term of $\mathcal{L}_{\mathrm{Higgs}}$ with the standard form, $-\frac{1}{2} m_{\mathrm{H}}^2 H^2$, gives

$$m_{\mathrm{H}}^2 = 2\lambda v^2 = 2\mu^2 \tag{2.37}$$

2.3.1.2 Spin-one particles

The relevant terms in this case are:

$$-\frac{1}{8} g_2^2 v^2 |W_\mu^1 - \mathrm{i} W_\mu^2|^2 - \frac{1}{8} v^2 (- g_2 W_\mu^3 + g_1 B_\mu)^2 \tag{2.38}$$

The fields W_μ^1 and W_μ^2 only appear in the combination $W_1^\mu W_{1\mu} + W_2^\mu W_{2\mu}$ and do not mix with any other fields. Their masses can therefore be read by inspection. Comparing this term to

$$-\frac{1}{2}M_1^2 W_\mu^1 W^{1\mu} - \frac{1}{2}M_2^2 W_\mu^2 W^{2\mu} \tag{2.39}$$

gives the masses

$$M_1^2 = M_2^2 = \frac{g_2^2 v^2}{4} \tag{2.40}$$

It is not an accident that these masses are equal. They are equal because the particles W_1 and W_2 are related by a symmetry that is not spontaneously broken, even when $v \neq 0$. To see this, consider performing a constant gauge transformation, $\partial_\mu \omega^a = 0$. The ground-state scalar field configuration then transforms as

$$\begin{aligned} \delta\begin{pmatrix}0\\ v\end{pmatrix} &= \frac{\mathrm{i}}{2}\omega_2^a \tau_a \begin{pmatrix}0\\ v\end{pmatrix} + \frac{\mathrm{i}}{2}\omega_1 \begin{pmatrix}0\\ v\end{pmatrix} \\ &= \frac{\mathrm{i}}{2}\begin{pmatrix}[\omega_2^1 - \mathrm{i}\omega_2^2]v\\ [\omega_1 - \omega_2^3]v\end{pmatrix} \end{aligned} \tag{2.41}$$

which vanishes provided that $\omega_2^1 = \omega_2^2 = 0$ and $\omega_1 = \omega_2^3 \equiv \omega$. This particular combination of $SU_{\mathrm{L}}(2) \times U_Y(1)$-transformations is therefore a symmetry of the ground state.

Under this symmetry the W fields transform according to Eq. (2.13):

$$\delta W_\mu^a = -\epsilon^{abc}\omega_2^b W_\mu^c, \qquad \text{or}, \qquad \delta\begin{pmatrix}W_\mu^1\\ W_\mu^2\end{pmatrix} = \omega\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}W_\mu^1\\ W_\mu^2\end{pmatrix} \tag{2.42}$$

This shows that W_μ^1 and W_μ^2 transform into one another under this symmetry. The condition $\omega_2^3 = \omega_1$ implies that the generator of this unbroken symmetry is $T_3 + Y$. Now, we saw earlier that the electric charge, Q, of a field is related to the $SU_{\mathrm{L}}(2) \times U_Y(1)$-generators by $Q = T_3 + Y$. It is precisely the electromagnetic gauge invariance, $U_{\mathrm{em}}(1)$, which is unbroken by the vacuum. W_μ^1 and W_μ^2 must therefore correspond to the two degrees of freedom associated with the distinct particle and antiparticle states required for an electrically charged particle. It is convenient in these cases to deal with fields that diagonalize the generator of electric charge. This corresponds, in the present case, to writing W_1 and W_2 as the real and imaginary parts of a complex, charged field:

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}}(W_\mu^1 \mp \mathrm{i}W_\mu^2) \tag{2.43}$$

which satisfies $\delta W^{\pm}_{\mu} = \pm i\omega W^{\pm}_{\mu}$ under electromagnetic gauge transformations, Eq. (2.42).

The mass term appropriate to such a charged field is $-M^2_W W^+_\mu W^{-\mu}$. Comparing with the Lagrangian, Eq. (1.121), therefore gives the $W^{\pm}$ mass to be

$$M_W = M_1 = M_2 = \frac{g_2 v}{2} \tag{2.44}$$

The remaining vector fields that appear in the mass term are W^3_μ and B_μ. They also only appear in one particular combination, $g_1 B_\mu - g_2 W^3_\mu$. We may normalize this combination (in order not to alter the standard form for the kinetic terms) to define the mass eigenstate:

$$\begin{aligned} Z_\mu &\equiv \frac{-g_1 B_\mu + g_2 W^3_\mu}{\sqrt{g_1^2 + g_2^2}} \\ &\equiv W^3_\mu \cos\theta_{\rm W} - B_\mu \sin\theta_{\rm W} \end{aligned} \tag{2.45}$$

This last equation defines the weak-mixing angle or Weinberg angle, $\theta_{\rm W}$, given by

$$\begin{aligned} \cos\theta_{\rm W} &= \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \\ \sin\theta_{\rm W} &= \frac{g_1}{\sqrt{g_1^2 + g_2^2}} \end{aligned} \tag{2.46}$$

In terms of this field the mass term, Eq. (1.124), is

$$-\frac{1}{8} v^2 (g_1^2 + g_2^2) Z_\mu Z^\mu \tag{2.47}$$

from which the mass may be read off:

$$M^2_Z = \frac{1}{4}(g_1^2 + g_2^2) v^2 \tag{2.48}$$

The final mass eigenstate is the combination of W^3_μ and B_μ that is orthogonal to Z_μ:

$$A_\mu = W^3_\mu \sin\theta_{\rm W} + B_\mu \cos\theta_{\rm W} = \frac{g_1 W^3_\mu + g_2 B_\mu}{\sqrt{g_1^2 + g_2^2}} \tag{2.49}$$

This is massless, as are the gluons, G^α_μ, that gauge $SU_{\rm c}(3)$. The masslessness of A_μ corresponds to the fact that the linear combination $Q = T_3 + Y$ is unbroken even when $v \neq 0$. A_μ is the corresponding massless gauge boson required for this unbroken symmetry. Since Q is the electric charge, we expect A_μ to have the couplings of the usual photon.

To summarize the relations between field bases, writing $c_{\rm W} \equiv \cos\theta_{\rm W}$ and $s_{\rm W} \equiv \sin\theta_{\rm W}$,

$$
\begin{array}{rclcrcl}
W^3_\mu &=& c_{\rm W} Z_\mu + s_{\rm W} A_\mu & \qquad & Z_\mu &=& c_{\rm W} W^3_\mu - s_{\rm W} B_\mu \\
B_\mu &=& -s_{\rm W} Z_\mu + c_{\rm W} A_\mu & & A_\mu &=& s_{\rm W} W^3_\mu + c_{\rm W} B_\mu \\
\\
\sqrt{2} W^+_\mu &=& W^1_\mu - {\rm i} W^2_\mu & & \sqrt{2} W^1_\mu &=& W^+_\mu + W^-_\mu \\
\sqrt{2} W^-_\mu &=& W^1_\mu + {\rm i} W^2_\mu & & \sqrt{2} W^2_\mu &=& {\rm i} W^+_\mu - {\rm i} W^-_\mu \\
\\
\sqrt{g_2^2 + g_1^2}\, W^3_\mu &=& g_2 Z_\mu + g_1 A_\mu & & \sqrt{g_2^2 + g_1^2}\, Z_\mu &=& g_2 W^3_\mu - g_1 B_\mu \\
\sqrt{g_2^2 + g_1^2}\, B_\mu &=& -g_1 Z_\mu + g_2 A_\mu & & \sqrt{g_2^2 + g_1^2}\, A_\mu &=& g_1 W^3_\mu + g_2 B_\mu
\end{array}
\tag{2.50}
$$

2.3.2 The custodial $SU(2)$

Notice that there is a relation amongst the three quantities M_W, M_Z, and $\theta_{\rm W}$

$$
\frac{M_W}{M_Z} = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \cos\theta_{\rm W} \tag{2.51}
$$

It is natural to ask how much this relation depends on the details of how the symmetry $SU_{\rm L}(2) \times U_Y(1)$ is broken, since any information that can restrict the arbitrariness in the symmetry breaking sector is welcome. Consider therefore the most general form for the spin-one mass matrix that is consistent with the symmetry-breaking pattern $SU_{\rm L}(2) \times U_Y(1) \to U_{\rm em}(1)$:

$$
\begin{pmatrix}
M_W^2 & & & \\
& M_W^2 & & \\
& & M_3^2 & m^2 \\
& & m^2 & M_0^2
\end{pmatrix}
\tag{2.52}
$$

This form has a simple explanation. As we saw above, unbroken electromagnetic gauge invariance dictates that the upper left 2×2 block of the matrix be proportional to the unit matrix: $M_W^2 I_{2\times 2}$. It similarly implies that the upper-right and the lower-left blocks must vanish. The lower-right 2×2 block is a priori an arbitrary symmetric matrix, subject to the one constraint that one of its eigenvalues must vanish. The vanishing of one of the eigenvalues corresponds to the masslessness of the photon, and is a general consequence of the fact that the electromagnetic gauge invariance is unbroken.

The requirement that one eigenvalue be zero is equivalent to the vanishing of the determinant:

$$\det\begin{pmatrix} M_3^2 & m^2 \\ m^2 & M_0^2 \end{pmatrix} = M_3^2 M_0^2 - m^4 = 0\,, \tag{2.53}$$

implying the condition $m^2 = \pm|M_0 M_3|$. (In the standard model, m^2 as defined here is negative.) The corresponding zero eigenvector may be written as

$$\begin{pmatrix} \mp\sin\theta_{\rm W} \\ \cos\theta_{\rm W} \end{pmatrix} \tag{2.54}$$

Equation (2.54) defines the mixing angle, $\theta_{\rm W}$, in the general case. We may now eliminate M_0^2 in favor of $\theta_{\rm W}$. The required relation is

$$\tan\theta_{\rm W} = \frac{\pm m^2}{M_3^2} = \left|\frac{M_0}{M_3}\right| \tag{2.55}$$

The non-zero eigenvalue, M_Z, is then given in terms of M_3 and $\theta_{\rm W}$ by

$$\begin{aligned} M_Z^2 &= \operatorname{tr}\begin{pmatrix} M_3^2 & m^2 \\ m^2 & M_0^2 \end{pmatrix} \\ &= M_0^2 + M_3^2 = M_3^2(1+\tan^2\theta_{\rm W}) = M_3^2\,\sec^2\theta_{\rm W} \end{aligned} \tag{2.56}$$

The mass relation implied by the symmetry breaking pattern $SU_{\rm L}(2)\times U_Y(1) \to U_{\rm em}(1)$ is therefore $M_3 = M_Z\cos\theta_{\rm W}$. An alternative way of expressing the mass formula, Eq. (2.51), is therefore $M_1 = M_2 = M_3 = M_W$.

The equality of M_3 and M_W within the standard model is a consequence of using a scalar $SU_{\rm L}(2)$-doublet, ϕ, to break $SU_{\rm L}(2)\times U_Y(1)$. The connection arises because of an *accidental symmetry* of the scalar self-couplings that determine the symmetry-breaking pattern that in turn determines the gauge boson mass matrix. The Higgs doublet, ϕ, may be thought of as four real scalar fields, corresponding to the real and imaginary parts of ϕ^0 and ϕ^+ in Eq. (2.11). An alternative way to write these four real fields would be as a column vector:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \tag{2.57}$$

As we saw in Subsection 1.3.1, the kinetic terms for four real scalar fields can be written as $\partial_\mu\Phi^T\partial^\mu\Phi$ and so is always invariant under the multiplication of Φ by an arbitrary 4×4 orthogonal matrix, $\mathcal{O} \in O(4)$. Now, in general the interaction terms of the Lagrangian break this symmetry completely. However, for the standard model, the two requirements of gauge invariance and renormalizability imply that the only possible scalar self-couplings are of the form $V = V(\phi^\dagger \phi) = V(\Phi^T \Phi)$. Even though it was not required to be so, this potential is therefore also invariant under these general $O(4)$ transformations. Any such global symmetry that appears as a simple consequence of gauge invariance and renormalizability is known as an accidental symmetry.

Once Φ develops a *v.e.v.*,

$$\langle \Phi \rangle = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.58}$$

this $O(4)$-invariance gets broken to the 3×3 orthogonal, $O(3)$, transformations that shuffle the lower three components amongst themselves. Since this $O(3)$ is unbroken, it constrains the form that the mass matrix may take. The ϕ gauge couplings that ultimately produce the gauge boson mass matrix are also invariant under these $O(3)$ transformations if the W^a_μs transform as a three-dimensional vector. Invariance of the mass matrix under this 3×3 transformation therefore implies that the upper-left 3×3 block of the spin-one mass matrix, Eq. (2.52), must be proportional to the unit matrix, implying $M_3 = M_1 = M_2 = M_W$ as required.

Since the group $O(3)$ is locally isomorphic to the group $SU(2)$, it is said that the symmetry-breaking sector has an accidental *custodial* $SU(2)$ invariance that is responsible for the mass formula, Eq. (2.51).

The utility of having such a symmetry understanding of this mass formula is that it points to the circumstances under which it might be altered and to how big the corrections might be. In fact, some of the interactions in the standard model, like the $\phi - B_\mu$ coupling and the Yukawa couplings, do *not* respect this custodial symmetry. We may expect, then, that radiative (quantum) corrections that involve these interactions can alter the mass relation. This is discussed in Section 7.5. Experimental verification of this relation is clearly of great importance since deviations point to detailed effects within the standard model, and potentially to indications of new physics.

2.3.3 Fermion masses

The terms quadratic in the fermion fields come from the Yukawa couplings after the shifting of the scalar field by v. The relevant terms are

$$\mathcal{L} = -\frac{v}{\sqrt{2}}[f_{mn}\bar{\mathcal{E}}_m P_{\rm R}\, E_n + g_{mn}\bar{\mathcal{U}}_m P_{\rm R} U_n + h_{mn}\bar{\mathcal{D}}_m P_{\rm R} D_n + \text{h.c.}] \qquad (2.59)$$

(It now becomes clear why it was convenient to label separately the different $SU_{\rm L}(2)$ components of the fields L and Q; the fact that the *v.e.v.* of the Higgs field breaks $SU_{\rm L}(2)$ symmetry means that a Yukawa coupling introduces a mass which picks out one or the other component.)

The mass terms induced by the Yukawa couplings of fermions to the Higgs *v.e.v.* are in general not diagonal in the generation indices, m and n. They may be diagonalized following the procedure outlined in Subsection 1.3.2. To this end, redefine the spin-half fields as follows:

$$\begin{array}{ll} P_{\rm L}\mathcal{E}_m = U^{(e)}_{mn} P_{\rm L}\,\mathcal{E}'_n & P_{\rm R} E_m = V^{(e)}_{mn} P_{\rm R}\, E'_n \\ P_{\rm L}\mathcal{U}_m = U^{(u)}_{mn} P_{\rm L}\,\mathcal{U}'_n & P_{\rm R} U_m = V^{(u)}_{mn} P_{\rm R}\, U'_n \\ P_{\rm L}\mathcal{D}_m = U^{(d)}_{mn} P_{\rm L}\,\mathcal{D}'_n & P_{\rm R} D_n = V^{(d)}_{mn} P_{\rm R}\, D'_n \end{array} \qquad (2.60)$$

where the matrices $U^{(e)}, U^{(u)}, U^{(d)}, V^{(e)}, V^{(u)}, V^{(d)}$ act on the generation indices (e.g. connect e to μ to τ) and must be unitary in order to preserve the canonical form for the kinetic terms.

As argued in Subsection 1.3.2, it is always possible to choose $U^{(e)} = V^{(e)*}, U^{(u)} = V^{(u)*}, U^{(d)} = V^{(d)*}$, and then choose $U^{(e)}$ to ensure that the new mass matrices are diagonal:

$$U^{(e)^\dagger} f V^{(e)} = V^{(e)^T} f V^{(e)} = \text{ diag}(f_e, f_u, f_\tau) \qquad (2.61)$$

with f_e, f_μ, f_τ real and non-negative. The same may be done for $V^{(u)^T} g V^{(u)}$ and $V^{(d)^T} h V^{(d)}$. The resulting mass terms then become (dropping the primes on the new fields)

$$\mathcal{L} = -\frac{1}{\sqrt{2}} v[f_m \bar{\mathcal{E}}_m P_{\rm R} E_m + g_m \bar{\mathcal{U}}_m P_{\rm R} U_m + h_m \bar{\mathcal{D}}_m P_{\rm R} D_m + \text{h.c.}] \qquad (2.62)$$

This has a simple expression in terms of the Dirac spinors, e_m, d_m, and u_m, defined as

$$\begin{array}{lll} e_m & \equiv & P_{\rm L}\,\mathcal{E}_m + P_{\rm R}\, E_m \\ d_m & \equiv & P_{\rm L}\,\mathcal{D}_m + P_{\rm R}\, D_m \\ u_m & \equiv & P_{\rm L}\,\mathcal{U}_m + P_{\rm R}\, U_m \end{array} \qquad (2.63)$$

To see this, use

$$\begin{aligned}\bar{\mathcal{E}}_m P_{\rm R} E_m + \text{ h.c.} &= \bar{\mathcal{E}}_m P_{\rm R} E_m + \bar{\mathcal{E}}_m P_{\rm L} E_m \\ &= \bar{\mathcal{E}}_m P_{\rm R} E_m + \bar{E}_m P_{\rm L} \mathcal{E}_m \\ &= \bar{e}_m P_{\rm R} e_m + \bar{e}_m P_{\rm L} e_m \\ &= \bar{e}_m e_m \end{aligned} \tag{2.64}$$

(The derivation of the identities used here was the subject of Problem 1 of Chapter 1.)

In terms of these Dirac spinors, the final form for the mass terms is

$$\mathcal{L} = -\frac{1}{\sqrt{2}} v (f_m \bar{e}_m e_m + g_m \bar{u}_m u_m + h_m \bar{d}_m d_m) \tag{2.65}$$

which, when compared to the standard mass term, $-m\bar{\psi}\psi$, gives the fermion masses as

$$m_n^{(e)} = \frac{1}{\sqrt{2}} f_n v, \qquad m_n^{(u)} = \frac{1}{\sqrt{2}} g_n v, \qquad m_n^{(d)} = \frac{1}{\sqrt{2}} h_n v \tag{2.66}$$

Notice that there is a separate Yukawa parameter, f_n, for every independent mass, m_n, so there are no mass formulae along the lines of Eq. (2.51) for the fermions. The numerical values of these fermion masses are presented in Appendix A.

Note that no mass term for the neutrinos is generated. If only renormalizable interactions and the minimal field content of the standard model are included, then this is exactly true, not just at the semiclassical level. A neutrino mass could appear if we extended the theory to include right-handed neutrinos N_m, because this would allow another Yukawa matrix between L and N. However, nothing forbids a mass term $m_m \bar{N}_m N_m$ for such right-handed neutrinos. One interpretation of the recent evidence for neutrino masses is that such right-handed neutrinos exist but their mass is very heavy. This is discussed in more detail in Chapter 10 and in Problem 2.3.

2.3.4 Hadrons

What we have just presented is the *perturbative* spectrum, that is, the spectrum assuming all interactions are weak. As we will discuss in Section 7.4, this is a valid approximation except for the $SU_{\rm c}(3)$ ("strong") interactions, which become strong at scales of order 500 MeV. The result is that quarks and gluons do not appear as actual particles of the spectrum. Rather, the particles we observe are bound states of quarks and gluons, in appropriate combinations to be color singlets. Such bound states are called *hadrons*. This is discussed in much more detail in Chapter 8. Here, we will just briefly explain the results and the nomenclature.

There are three ways to form a colorless combination of quarks and gluons. One is to have a bound state made purely of two or more gluons, called a "glueball." It is believed that such states should be heavy and highly unstable, making their identification difficult. The next way is to have a bound state made up of a quark and an antiquark, $q\bar{q}$ (possibly together with gluons and more $q\bar{q}$ pairs). Such bound states exist and are called *mesons*; the lightest meson is the pion, made up of a $u\bar{d}$ (π^+), a $d\bar{u}$ (π^-), or $(u\bar{u} - d\bar{d})/\sqrt{2}$ (π^0). The final way is to have a bound state of three quarks (possibly together with gluons and more $q\bar{q}$ pairs). Such a three-quark state is called a *baryon*, and its antiparticle, with three antiquarks, is an antibaryon. The lightest two baryons are the familiar proton and neutron, made up of uud and udd respectively. There is no straightforward way to relate the masses of the hadrons to the masses of the constituent quarks and gluons, because the binding energies involved are of order 500 MeV. In the case of the b- and c-containing hadrons, however, the mass is dominated by the mass of the heavy quark, making possible simpler relations between hadron and quark masses.

When energies are large compared to the hadronic binding energy, the language of quarks and gluons can be appropriate – within limits. For instance, in computing Z boson decays in Chapter 4, we will see that the total rate of decay into hadrons is given, up to small corrections, by the rate of decay into quarks; how the quarks stick together into hadrons determines what the actual final state is, but not the likelihood for the Z boson to create the quarks. Similarly, when a hadron is one of the particles participating in a collision, then at high energies we can often describe the collision in terms of the quarks and gluons residing within the hadron, as discussed in Chapter 9.

2.4 Interactions

We have determined the particle masses in terms of the various parameters of the Lagrangian. The predictive nature of the theory only appears once we identify how these parameters determine the strengths of particle interactions and compare the interactions we see with those that are predicted.

This section is largely bookkeeping. The most important parts to understand are the charged and neutral current interactions and the necessity of the Kobayashi–Maskawa matrix. Most of the content of this section is summarized by the Feynman rules presented in Section 5.4.

2.4.1 Higgs couplings

The couplings of the Higgs boson are found in the expansion of the Higgs Lagrangian, $\mathcal{L}_{\text{Higgs}}$, of Eq. (2.36):

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} &= -\frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \lambda v H^3 - \frac{1}{4}\lambda H^4 \\ &\quad -\frac{1}{8}g_2^2 (v+H)^2 |W_\mu^1 - \mathrm{i}W_\mu^2|^2 \\ &\quad -\frac{1}{8}(v+H)^2(-g_2 W_\mu^3 + g_1 B_\mu)^2 \\ &\quad -\frac{1}{\sqrt{2}}(v+H)\left[f_{mn}\bar{\mathcal{E}}_m P_{\mathrm{R}} E_n + \text{ h.c.}\right] \\ &\quad -\frac{1}{\sqrt{2}}(v+H)\left[g_{mn}\bar{\mathcal{U}}_m P_{\mathrm{R}} U_n + \text{ h.c.}\right] \\ &\quad -\frac{1}{\sqrt{2}}(v+H)\left[h_{mn}\bar{\mathcal{D}}_m P_{\mathrm{R}} D_n + \text{ h.c.}\right]\end{aligned}$$

This Lagrangian completely specifies the Higgs couplings to other particles.

2.4.1.1 Higgs self-couplings

The couplings of the Higgs to itself are easily read from the potential in Eq. (2.36):

$$\begin{aligned}\mathcal{L}_{\text{H-H}} &= -\lambda v H^3 - \frac{1}{4}\lambda H^4 \\ &= -\frac{m_{\mathrm{H}}^2}{2v} H^3 - \frac{m_{\mathrm{H}}^2}{8v^2} H^4 \end{aligned} \tag{2.67}$$

2.4.1.2 Higgs–gauge-boson couplings

The Higgs–gauge boson couplings are similarly given by

$$\begin{aligned}\mathcal{L}_{\mathrm{H-g}} &= -\frac{1}{8}g_2^2(2vH + H^2)|W_\mu^1 - \mathrm{i}W_\mu^2|^2 - \frac{1}{8}(2vH + H^2)(-g_2 W_\mu^3 + g_1\, B_\mu)^2 \\ &= -\left(\frac{H}{v} + \frac{H^2}{2v^2}\right)\left(2M_W^2 W_\mu^+ W^{-\mu} + M_Z^2 Z_\mu Z^\mu\right)\end{aligned} \tag{2.68}$$

2.4.1.3 Higgs–fermion couplings

The final Higgs interactions consist of Yukawa couplings between the Higgs scalar and the various fermions:

$$\begin{aligned}\mathcal{L}_{\rm H-f} &= -\frac{1}{\sqrt{2}}H(f_m\bar{e}_m e_m + g_m\bar{u}_m u_m + h_m\bar{d}_m d_m)\\ &= -\sum_f \frac{m_f}{v}\bar{f}fH \end{aligned} \tag{2.69}$$

Here and in the following we use f (for fermion) to run over the nine Dirac and three Majorana species labels e_i, u_i, d_i, ν_i; but the m_ν are zero.

Several points about these couplings are worth noting.

(i) Notice first that all other particles couple to the Higgs boson with strength m/v, in which m is the mass of the particle in question and v (which turns out to equal 246 GeV) is the symmetry-breaking vacuum expectation value. This ratio is small provided that $m \ll v$, which is true for all known particles, though only marginally so for the top quark, t. H must therefore couple weakly to all of the particles that have been discovered to date, and must furthermore couple preferentially to the heavier particles.

(ii) The Higgs-fermion couplings are automatically flavor-diagonal when expressed in terms of mass eigenstates. That is to say, the act of emission of a Higgs particle by a fermion does not convert one type (or "flavor") of fermion into another. This is an important property of the model since there are very strong limits on the existence of any transitions of this type. The only known interactions that can change fermion flavor are the $W^\pm$ interactions we meet later. The strongest limits on these types of flavor-changing couplings arise for those that involve the strange quark, $H\bar{s}d$ for example. Such an interaction would contribute to the extremely well measured mass difference, $m_{K_L} - m_{K_S} = (3.490 \pm 0.006) \times 10^{-12}$ MeV, between the two neutral kaons, K_L and K_S, or to *flavor-changing neutral-current* processes such as the decay $K_L \to \mu^+e^-$, which has never been observed to occur. More quantitatively, this last process is known to happen less frequently than once in every 5×10^{12} $K_{\rm L}$ decays.

(iii) As will be shown in Section 2.5, these Higgs couplings also conserve the discrete symmetries of charge conjugation, C, parity, P, and time reversal, T. This property is also not a general feature of more complicated symmetry-breaking sectors.

(iv) The Higgs boson has recently been discovered with a mass (as of late

2012) of about 126 GeV. The Higgs self-coupling is related to the mass, $2\lambda = (m_{\mathrm{H}}/v)^2$, so for the physical value of the Higgs mass, the self-couplings are perturbative but relatively large. Since the Higgs self-coupling terms are fixed by the now-known Higgs mass, measuring Higgs self-interactions would be a good way to test this sector of the model. As of this writing, only rather poor experimental limits exist on the Higgs self-coupling strengths.

2.4.2 Strong interactions

The strong interactions are by definition those that involve the spin-one gluons. The relevant terms in $\mathcal{L}$ are

$$\begin{aligned}\mathcal{L}_{\mathrm{strong}} \quad = \quad & -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu} - \frac{g_3^2\Theta_3}{64\pi^2}\epsilon_{\mu\nu\lambda\rho}G^{\alpha\mu\nu}G^{\alpha\lambda\rho} \\ & -\frac{1}{2}\overline{Q}_m \not{D} Q_m - \frac{1}{2}\overline{U}_m \not{D} U_m - \frac{1}{2}\overline{D}_m \not{D} D_m \end{aligned} \tag{2.70}$$

2.4.2.1 Gluon self-couplings

The $G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu}$ term describes the couplings of the gluons among themselves:

$$\mathcal{L}_{\mathrm{gl-gl}} = -\frac{1}{4}\mathcal{G}^{\alpha}_{\mu\nu}\mathcal{G}^{\alpha\mu\nu} - \frac{g_3}{2}f^{\alpha}_{\beta\gamma}\mathcal{G}^{\alpha}_{\mu\nu}G^{\beta\mu}G^{\gamma\nu} - \frac{g_3^2}{4}f^{\alpha}_{\beta\gamma}f^{\alpha}_{\delta\epsilon}\,G^{\beta}_{\mu}G^{\gamma}_{\nu}\,G^{\delta\mu}G^{\epsilon\nu} \tag{2.71}$$

plus the Θ_3 term which we have not written out. Here, $\mathcal{G}^{\alpha}_{\mu\nu}$ denotes the linearized field strength, $\partial_\mu G^{\alpha}_{\nu} - \partial_\nu G^{\alpha}_{\mu}$. The Θ_3 term has almost no impact in the following, because it has no effect on any perturbative calculation, and because Θ_3 is numerically almost exactly zero. This is a mystery, discussed in Subsection 11.4.2.

2.4.2.2 Gluon–fermion couplings

The couplings between gluons and fermions may be read from Eq. (2.70),

$$\mathcal{L}_{\mathrm{gl-f}} = +\frac{\mathrm{i}g_3}{2}\sum_q G^{\alpha}_{\mu}\,\bar{q}\gamma^{\mu}\lambda_{\alpha}q \tag{2.72}$$

in which the sum is over the six Dirac spinors representing the different flavors of quarks, $q = u_m, d_m$.

The emission of a gluon by a fermion causes a transition in the fermion's color quantum numbers. We return to these couplings in more detail later. In the meantime some features of these couplings bear comment.

(i) Because the standard model gauge group, $SU_{\mathrm{c}}(3) \times SU_{\mathrm{L}}(2) \times U_{Y}(1)$, is the *product* of a strong-interaction factor, $SU_c(3)$, with an electroweak factor, $SU_L(2) \times U_Y(1)$, all of the particles of the theory can be divided into two classes according to whether or not they carry strong-interaction quantum numbers. Quarks and gluons do and electrons, neutrinos, the Higgs particle, and the electroweak gauge bosons, W, Z, A, do not. This is the origin of the classification of elementary particles as *hadrons* or *leptons*. Hadrons involve the quarks and gluons and so participate in the strong interactions. For historical reasons only the spin-half particles that do not interact strongly are called leptons, and these therefore consist of the electron-type and neutrino-type fermions.

(ii) Gluon interactions are called "strong," as will be pursued in more detail in subsequent chapters, because unlike the electroweak interactions, the spectrum of strongly interacting particles cannot be described perturbatively in the gluon coupling, g_3. The observed hadrons consist of bound states of the more elementary quarks and gluons. This greatly complicates the interpretation of interactions that involve hadrons as initial or final particles. As we shall see, it turns out that it is nevertheless possible to accurately describe some carefully chosen observables in hadron collisions at sufficiently high energies within perturbation theory.

(iii) Just as was the case for the Higgs–fermion couplings, the emission of a gluon by a fermion can never change the flavor of the fermion. This may be seen from the above expressions, since the gluon–fermion interactions always have the form $G\bar{q}q$ and never involve two different types of quark, such as $G\bar{q}q'$. As a result, flavor type is conserved by the strong interactions. This has important consequences for the interactions and spectrum of all strongly-interacting particles, which will be explored in more detail later.

(iv) Apart from the Θ_3 term, the strong interactions as given above are invariant with respect to all three of the discrete symmetries, C, P, and T. (This conclusion is justified in more detail in Section 2.5.) The present evidence for the invariance of the strong interactions under these discrete symmetries (principally the current upper bound on the neutron's intrinsic electric dipole moment) implies that the *strong-CP parameter*, $|\Theta_3|$, must be smaller than $\approx 10^{-9}$. The potential significance of this CP-violating parameter is taken up in more detail in Subsection 11.4.2.

(v) The strength of all strong interactions is governed by a single coupling constant, g_3, so the strong interactions have a *universal* strength that is independent of the particle type that is participating in the interaction. This is an important experimental fact that is explained here as the natural consequence of the observation that the gluons are gauge bosons, and that all of the strongly-interacting fermions fall into the same representation (in this case triplets or antitriplets) of the gauge group $SU_c(3)$.

2.4.3 Electroweak interactions

We next turn to the couplings that involve the electroweak gauge bosons – those spin-one particles that correspond to the $SU_L(2) \times U_Y(1)$ factor of the gauge group. These come in two basic types. There are self-couplings that arise due to the non-linear terms in the gauge potentials within the $SU_L(2) \times U_Y(1)$ field strengths, and there are couplings with other particles that arise due to the use of gauge covariant derivatives in the kinetic-energy terms of the Lagrangian. We consider each of these in turn.

2.4.3.1 Electroweak boson self-interactions

There are both cubic and quartic self-couplings of the spin-one electroweak gauge bosons. Both arise from the non-linear terms in the $SU_L(2)$ gauge boson field strength

$$\mathcal{L} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} \tag{2.73}$$

The cubic terms are

$$\mathcal{L}_{\text{cubic}} = -\frac{1}{2} g_2 \epsilon_{abc} \mathcal{W}^a_{\mu\nu} W^{b\mu} W^{c\nu} = \mathcal{L}_{WW\gamma} + \mathcal{L}_{WWZ} \tag{2.74}$$

with the W-photon and W–Z trilinear couplings given in terms of the mass eigenstates, $W^1_\mu = \frac{1}{\sqrt{2}}(W^+_\mu + W^-_\mu)$, $W^2_\mu = \frac{-\mathrm{i}}{\sqrt{2}}(W^-_\mu - W^+_\mu)$, and $W^3_\mu = Z_\mu \cos\theta_W + A_\mu \sin\theta_W$, by

$$\mathcal{L}_{WW\gamma} = \mathrm{i} g_2 \sin\theta_W \Big[W^+_{\mu\nu} W^{-\mu} A^\nu - W^-_{\mu\nu} W^{+\mu} A^\nu + W^+_\mu W^-_\nu F^{\mu\nu} \Big] \tag{2.75}$$

$$\mathcal{L}_{WWZ} = \mathrm{i} g_2 \cos\theta_W \Big[W^+_{\mu\nu} W^{-\mu} Z^\nu - W^-_{\mu\nu} W^{+\mu} Z^\nu + W^+_\mu W^-_\nu Z^{\mu\nu} \Big] \tag{2.76}$$

In these expressions, $\mathcal{W}^a_{\mu\nu}$, $W^\pm_{\mu\nu}$, $Z_{\mu\nu}$ and $F_{\mu\nu}$ are the linear curls of the gauge potentials W^a_μ, $W^\pm_\mu$, Z_μ and A_μ respectively, *eg*, $W^\pm_{\mu\nu} = \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu$.

The interaction terms that are quartic in these fields are

$$\mathcal{L}_{\text{quartic}} = -\frac{1}{4} g_2^2 \epsilon_{abc} \epsilon_{ade} W^b_\mu W^c_\nu W^{d\mu} W^{e\nu}$$

$$= -\frac{1}{4}g_2^2\Big[(W^a_\mu W^\mu_a)^2 - W_{a\mu}W^a_\nu W^\mu_b W^{b\nu}\Big]$$
$$= \mathcal{L}_{WWWW} + \mathcal{L}_{WWZZ} + \mathcal{L}_{WW\gamma\gamma} + \mathcal{L}_{WWZ\gamma} \quad (2.77)$$

which, using the relation $W^a_\mu W^a_\nu = W^-_\mu W^+_\nu + W^+_\mu W^-_\nu + W^3_\mu W^3_\nu$ with $W^3_\mu = Z_\mu \cos\theta_\mathrm{W} + A_\mu \sin\theta_\mathrm{W}$, gives,

$$\begin{aligned}\mathcal{L}_{\text{quartic}} &= -\frac{1}{2}g_2^2\left[(W^+_\mu W^{-\mu})^2 - (W^+_\mu W^{+\mu})(W^-_\nu W^{-\nu})\right] \\ &\quad -g_2^2\Big[(W^+_\mu W^{-\mu})W^3_\nu W^{3\nu} - (W^+_\mu W^{3\mu})(W^-_\nu W^{3\nu})\Big] \quad (2.78)\end{aligned}$$

so

$$\mathcal{L}_{WWWW} = -\frac{1}{2}g_2^2\Big[(W^+_\mu W^{-\mu})^2 - (W^+_\mu W^{+\mu})(W^-_\nu W^{-\nu})\Big] \quad (2.79)$$
$$\mathcal{L}_{WWZZ} = -g_2^2\cos^2\theta_\mathrm{W}\Big[(W^+_\mu W^{-\mu})Z_\nu Z^\nu \; - \; (W^+_\mu Z^\mu)(W^-_\nu Z^\nu)\Big] \quad (2.80)$$
$$\mathcal{L}_{WW\gamma\gamma} = -g_2^2\sin^2\theta_\mathrm{W}\Big[(W^+_\mu W^{-\mu})(A_\nu A^\nu) \; - \; (W^+_\mu A^\mu)(W^-_\nu A^\nu)\Big] \quad (2.81)$$
$$\begin{aligned}\mathcal{L}_{WWZ\gamma} &= -g_2^2\sin\theta_\mathrm{W}\cos\theta_\mathrm{W}\Big[2(W^+_\mu W^{-\mu})(Z_\nu A^\nu) - \; (W^+_\mu A^\mu)(W^-_\nu Z^\nu) \\ &\quad -(W^+_\mu Z^\mu)(W^-_\nu A^\nu)\Big] \quad (2.82)\end{aligned}$$

Some brief comments.

(i) These self-interactions have been probed by the LEP-II experiments at the 2–3% level. However, compared to the precision with which the electroweak interactions of the fermions have been probed, these measurements are comparatively poor.

(ii) The interactions of the W particles with the massless A boson only involve the particular combination of couplings $g_2 \sin\theta_\mathrm{W}$. As will become clear once the remainder of the A couplings are presented, this combination has the interpretation of being the electromagnetic coupling constant, e, as is appropriate for the interactions of the photon, A, with a particle of electric charge 1.

(iii) These interactions preserve C, P, and T (see Section 2.5).

2.4.3.2 "Charged-current" fermion interactions

The only other electroweak interactions in the theory are the couplings between the electroweak bosons and spin-half and spin-zero particles. Since the couplings with the Higgs boson are given in Subsection 2.4.1, they need not be reconsidered again here.

The W^a_μ and B_μ–fermion couplings arise from the following kinetic terms,

$$\mathcal{L} = -\frac{1}{2}\overline{L}_m \not{D} L_m - \frac{1}{2}\overline{E}_m \not{D} E_m - \frac{1}{2}\overline{Q}_m \not{D} Q_m - \frac{1}{2}\overline{U}_m \not{D} U_m - \frac{1}{2}\overline{D}_m \not{D} D_m \quad (2.83)$$

Expanding each field in terms of the mass eigenstates gives

$$\begin{aligned} \mathcal{L}_{\rm ew} \;=\; & +\frac{\rm i}{4}\begin{pmatrix}\bar{\nu}_m \\ \bar{\mathcal{E}}_m\end{pmatrix}^T \gamma^\mu P_{\rm L} \begin{pmatrix} -g_1 B_\mu + g_2 W^3_\mu & g_2(W^1_\mu - {\rm i}W^2_\mu) \\ g_2(W^1_\mu + {\rm i}W^2_\mu) & -g_1 B_\mu - g_2 W^3_\mu \end{pmatrix} \begin{pmatrix}\nu_m \\ \mathcal{E}_m\end{pmatrix} \\ & +\frac{\rm i}{4}\begin{pmatrix}\bar{\mathcal{U}}_m \\ \bar{\mathcal{D}}_m\end{pmatrix}^T \gamma^\mu P_{\rm L} \begin{pmatrix} \frac{1}{3}g_1 B_\mu + g_2 W^3_\mu & g_2(W^1_\mu - {\rm i}W^2_\mu) \\ g_2(W^1_\mu + {\rm i}W^2_\mu) & \frac{1}{3}g_1 B_\mu - g_2 W^3_\mu \end{pmatrix} \begin{pmatrix}\mathcal{U}_m \\ \mathcal{D}_m\end{pmatrix} \\ & +\frac{\rm i}{3} g_1 B_\mu \bar{U}_m \gamma^\mu P_{\rm R} U_m - \frac{\rm i}{6} g_1 B_\mu \bar{D}_m \gamma^\mu P_{\rm R}\, D_m \\ & -\frac{\rm i}{2} g_1 B_\mu \bar{E}_m \gamma^\mu P_{\rm R} E_m + \text{ h.c.} \end{aligned} \quad (2.84)$$

The couplings between fermions and the charged spin-one particle, W^+_μ, are called the *charged-current* interactions. Because these interactions always involve projection operators $P_{\rm L}$ or $P_{\rm R}$, we may replace the Majorana fermions $\mathcal{U}, \mathcal{D}, \mathcal{E}$ with the Dirac fermions u, d, e (since the additional U, D, E fields introduced in this substitution are removed by the projection operator), giving

$$\mathcal{L}_{\rm cc} = \frac{{\rm i}g_2}{\sqrt{2}}\Big[W^+_\mu(\bar{\nu}_m\gamma^\mu P_{\rm L}\, e_m + \bar{u}_m\gamma^\mu P_{\rm L} d_m) + W^-_\mu(\bar{e}_m\gamma^\mu P_{\rm L}\,\nu_m + \bar{d}_m\gamma^\mu P_{\rm L} u_m)\Big] \quad (2.85)$$

Unfortunately, as written this expression is correct in the generation basis we had before making the field redefinitions described in Subsection 2.3.3. To learn what the interactions are in terms of the mass basis, we must perform the same transformations, $e_m = U^{(e)}_{mn} e'_n$, $u_m = U^{(u)}_{mn} u'_n$, and $d_m = U^{(d)}_{mn} d'_n$, on this expression. Since there is no mass term for neutrinos, we are free to also redefine the neutrino field by $\nu_m = U^{(e)}_{mn}\nu'_n$, since this does not alter their mass or kinetic terms (see, however, Chapter 10). Defining

$$V_{mn} = (U^{(u)^\dagger} U^{(d)})_{mn} \quad (2.86)$$

and introducing $e_W \equiv g_2/2\sqrt{2}$, gives the following expression:

$$\begin{aligned} \mathcal{L}_{\rm cc} \;=\; & {\rm i}e_W\Big[W^+_\mu(\bar{\nu}'_m\gamma^\mu(1{+}\gamma_5)e'_m + V_{mn}\bar{u}'_m\gamma^\mu(1{+}\gamma_5)d'_n) \\ & +W^-_\mu(\bar{e}'_m\gamma^\mu(1{+}\gamma_5)\nu'_m + (V^\dagger)_{mn}\bar{d}'_m\gamma^\mu(1{+}\gamma_5)u'_n)\Big] \end{aligned} \quad (2.87)$$

V_{mn} is a 3×3 unitary matrix called the *Kobayashi–Maskawa* (KM) – or sometimes the *Cabbibo–Kobayashi–Maskawa* (CKM)–matrix. It arises due to the necessity to perform different field redefinitions for up- and down-type quarks when diagonalizing masses. Since the matrix V_{mn} is 3×3 and unitary, it is described by nine real parameters. Not all of these nine parameters can be physically significant, however, because they may be changed by performing a field redefinition which has no other effects on the standard model Lagrangian. The only field redefinitions which can alter V_{mn} but which do not affect any other terms in the Lagrangian consist of multiplication of the various quark fields, u'_n and d'_n by a phase. Notice that since an overall rotation of all quarks by a common phase is a symmetry of the entire Lagrangian, and so leaves V_{mn} unchanged, this freedom to redefine fields allows the removal of at most five phases from V_{mn}. This would leave only four parameters of potential physical significance.

The choice of how to use these phase redefinitions to rotate the KM matrix is somewhat arbitrary. Partly for this reason, there are several different conventional ways in which to parameterize the KM matrix. The principal three are listed here for convenience. The parameterization advocated by the Particle Data Group is:

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{2.88}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\,\mathrm{e}^{-\mathrm{i}\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\,\mathrm{e}^{\mathrm{i}\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{2.89}$$

in which c_{ij} and s_{ij} are shorthand for $\cos\theta_{ij}$ and $\sin\theta_{ij}$ respectively, and the *mixing angles*, θ_{ij}, are experimentally known to satisfy $\theta_{13} \ll \theta_{23} \ll \theta_{12} \ll 1$. This implies that (for unknown reasons) charged-current interactions that link fermions of differing generation are highly suppressed in the standard model and so in particular V_{mn} is very close be being a unit matrix. We return to the experimental constraints on the matrix V_{mn} shortly.

There are two other parameterizations of the KM matrix that are commonly used in the literature. Many of the older sources parameterize the KM matrix in terms of the Euler angles of an $O(3)$ rotation together with one phase:

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix}\begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -c_2 s_1 & c_1 c_2 c_3 + s_2 s_3 e^{i\delta} & c_1 c_2 s_3 - c_3 s_2 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 - s_3 c_2 e^{i\delta} & c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} \tag{2.90}$$

Again $c_i (= \cos\theta_i)$ and $s_i (= \sin\theta_i)$ denote trigonometric functions of the Euler angles.

The third common parameterization is the *Wolfenstein* parameterization, which indicates the size of each matrix element in a particularly simple way. It is given, up to fourth order in the small quantity λ, by:

$$V = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{2.91}$$

The utility of this parameterization is that, since λ is found experimentally to be a small quantity, $\lambda \approx 0.2$, and A and $\rho^2 + \eta^2$ are $O(1)$, Eq. (2.91) summarizes the small size and hierarchy of the off-diagonal elements of V_{mn}.

It turns out (see Subsection 2.5.1) that these interactions preserve time reversal symmetry, T (or equivalently, CP) if the KM matrix can be made real by suitably redefining fields. Hence, it is interesting to know under which circumstances this is possible. In the generic case in which V_{mn} does not take any special form this can be decided by comparing the number of parameters available in a real versus a complex unitary matrix.

It is instructive to make the argument for the case of N generations of fermions. The counting goes as follows. The KM matrix is an $N \times N$ unitary matrix and so generically contains N^2 real parameters. If the KM matrix were real then it would be an orthogonal matrix, which can be described in terms of $\frac{1}{2}N(N-1)$ real parameters. The difference between these numbers, $N^2 - \frac{1}{2}N(N-1) = \frac{1}{2}N(N+1)$, is therefore the number of complex "phases" contained in V_{mn}. Not all of these phases, however, are physically significant, since some may be removed by absorbing phases into the various quark fields. Since such a redefinition does not affect any other term in the Lagrangian, any phase that can be removed in this way cannot cause any physical effects. Even though there are $2N$ species of quark fields, only $2N - 1$ phases may be removed in this way, since the overall multiplication of all quark fields by a common phase is a symmetry of the Lagrangian and does not change V_{mn}. The number of remaining physical phases is therefore

$$\begin{aligned} P &= \left[N^2 - \frac{1}{2}N(N-1)\right] - (2N-1) \\ &= \frac{1}{2}(N-1)(N-2) \end{aligned} \tag{2.92}$$

Notice that if there were only two generations, then $P = 0$ and so the KM matrix could be chosen to be a real 2×2 orthogonal matrix:

$$V_{mn} = \begin{pmatrix} \cos\theta_{\rm c} & \sin\theta_{\rm c} \\ -\sin\theta_{\rm c} & \cos\theta_{\rm c} \end{pmatrix} \tag{2.93}$$

It happens that the experimental values for the angles in the full KM matrix are such that those parts of it that mix the first two generations are very close to being of the form of Eq. (2.93). For historical reasons the first few components of the KM matrix are therefore sometimes written in this way. The *Cabbibo angle* is accordingly defined by: $\cos\theta_{\rm c} = V_{ud}$ and $\sin\theta_{\rm c} = V_{us}$.

Some comments.

(i) The charged-current interactions are the only ones within the model that connect fermions with differing flavors. In the absence of these charged-current interactions, the lightest species of fermion of any flavor would be absolutely stable, since flavor would be conserved. As a result, the charged-current interactions are the ones responsible for the majority of particle decays that have been observed.

(ii) Since there is no Kobayashi–Maskawa matrix in the leptonic component of the charged-current interactions, all leptons participate in these interactions with equal strength, determined by g_2. Just as was the case with the strong interactions, this result follows theoretically from the spin-one and hence gauge nature of the W boson, and the fact that all leptons that couple to the W boson are in doublets of $SU_{\rm L}(2)$. The experimentally observed property that all leptons participate in charged-current weak interactions with equal strength is called *weak universality*.

(iii) Weak universality does not hold for charged-current interactions involving quarks, because of the appearance there of the Kobayashi–Maskawa matrix, although there will be relationships amongst various hadronic charged-current interactions that follow from the unitarity of the KM matrix.

(iv) As is shown in Section 2.5, the charged-current interactions violate both C and P, since they involve only the left-handed components of the various fermion fields. They can only violate T if the KM matrix cannot be made real by a suitable choice of fields. It follows that all charged-current lepton interactions must preserve T and that the hadronic charged-current interactions can violate T only in a very specific way and only if there are at least three generations. At this time (2013), this source of T-violation is consistent with all of the experimental evidence.

(v) Although the lepton sector of the standard model does not involve a KM matrix and so cannot violate CP, this would not be so if the model were enlarged in such a way as to generate a neutrino mass matrix. As discussed in Chapter 10, very small neutrino masses are in fact observed. These suggest that CP violation in the neutrino sector may be observable. The observation of CP violation is a major experimental goal of modern neutrino physics.

2.4.3.3 "Neutral-current" fermion interactions

It remains to write out the couplings of the two neutral gauge bosons, A_μ, Z_μ, of the electroweak gauge group, $SU_{\rm L}(2) \times U_Y(1)$. Using the expressions

$$\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\rm W} & \sin\theta_{\rm W} \\ -\sin\theta_{\rm W} & \cos\theta_{\rm W} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \tag{2.94}$$

we see that these couplings are flavor-diagonal and of the form

$$\mathcal{L}_{\rm nc} = \sum_f - \left[\bar f \gamma^\mu P_{\rm L}\left(-{\rm i}g_2 W^3_\mu T_3 - {\rm i}g_1 B_\mu Y_{\rm L}\right) f + \bar f \gamma^\mu P_{\rm R}\left(-{\rm i}g_1 B_\mu Y_{\rm R}\right) f\right] \tag{2.95}$$

where $Y_{\rm L}$ is the hypercharge of the left-handed fermion and $Y_{\rm R}$ is that of the right-handed one, e.g., $Y_{\rm L} = -1/2$ for $P_{\rm L}\, e_m = P_{\rm L}\,\mathcal{E}_m$ and $Y_{\rm R} = -1$ for $P_{\rm R} e_m = P_{\rm R} E_m$ etc. Notice that $Y_{\rm R}$ agrees with the electric charge, Q, since all right-handed fields are singlets under $SU_{\rm L}(2)$ and so have $T_3 = 0$. This then implies $Q = T_3 + Y_{\rm L} = Y_{\rm R}$.

Now, define the combination of Dirac matrices, gauge potentials and group generators, T_3 and $Y_{\rm L,R}$, that appear in Eq. (2.95) above as M_μ. It may be reexpressed in the following form:

$$\begin{aligned} M_\mu &\equiv P_{\rm L}\, g_2 W^3_\mu T_3 + P_{\rm L}\, g_1 B_\mu Y_{\rm L} + P_{\rm R}\, g_1 B_\mu Y_{\rm R} \\ &= P_{\rm L}\, g_2 W^3_\mu T_3 + P_{\rm L}\, g_1 B_\mu (Q - T_3) + P_{\rm R}\, g_1 B_\mu Q \\ &= T_3 P_{\rm L}(g_2 W^3_\mu - g_1 B_\mu) + g_1 B_\mu Q \\ &= T_3 P_{\rm L}\left[g_2(Z_\mu \cos\theta_{\rm W} + A_\mu \sin\theta_{\rm W}) - g_1(A_\mu \cos\theta_{\rm W} - Z_\mu \sin\theta_{\rm W})\right] \\ &\qquad + g_1(A_\mu \cos\theta_{\rm W} - Z_\mu \sin\theta_{\rm W}) Q \end{aligned} \tag{2.96}$$

This simplifies further if we use the following relations among the coupling constants

$$g_2 = \cos\theta_{\rm W}\sqrt{g_1^2 + g_2^2} \quad \text{and} \quad g_1 = \sin\theta_{\rm W}\sqrt{g_1^2 + g_2^2}$$

so

$$g_1 \cos\theta_W = g_2 \sin\theta_W \equiv e$$

and

$$g_2 \cos\theta_W + g_1 \sin\theta_W = \sqrt{g_1^2 + g_2^2} = \frac{e}{\sin\theta_W \cos\theta_W}$$

Therefore,

$$M_\mu = \frac{e}{\sin\theta_W \cos\theta_W} \left[T_3 P_L - Q \sin^2\theta_W \right] Z_\mu + eQA_\mu \tag{2.97}$$

It is easily verified that the form of these interactions are not changed by the process of rotating to a basis of mass eigenstates for the fermion fields.

We may read from this the fermion couplings with the Z-boson and the massless photon, A. The photon–fermion coupling is

$$\mathcal{L}_{em} = \sum_f ieA_\mu \bar{f}\gamma^\mu Q f \tag{2.98}$$

in which the sum is over all fermion types, $f = e_m, \nu_m, d_m, u_m$, weighted by their electric charge, Q. Since the neutrino is electrically neutral it does not appear in the electromagnetic interactions. Comparing the interaction of Eq. (2.98) with that of QED in Eq. (1.176), we see that it is the combination $e = g_1 \cos\theta_W = g_2 \sin\theta_W = \sin\theta_W \cos\theta_W \sqrt{g_1^2 + g_2^2}$ that plays the role of the electromagnetic coupling constant – i.e. the absolute value of the electron charge – in this theory.

The Z_μ – or *neutral-current* – couplings are similarly given by

$$\begin{aligned} \mathcal{L}_{nc} &= \frac{ie}{\sin\theta_W \cos\theta_W} \sum_f Z_\mu \bar{f}\gamma^\mu \left[P_L T_3 - Q \sin^2\theta_W \right] f \\ &= \frac{ie}{\sin\theta_W \cos\theta_W} \sum_f Z_\mu \bar{f}\gamma^\mu (g_V + \gamma_5 g_A) f \end{aligned} \tag{2.99}$$

in which $g_V = \frac{1}{2}T_3 - Q\sin^2\theta_W$ and $g_A = \frac{1}{2}T_3$. Here T_3 refers to the charge, under the third generator of $SU_L(2)$, of the left-handed constituent of f, that is, $\mathcal{E}$, $\mathcal{D}$, or $\mathcal{U}$. The values of the charges g_V, g_A are given in Table 2.1.

These interactions share several noteworthy properties.

(i) The couplings of the massless spin-one particle are precisely those of quantum electrodynamics, justifying its identification with the photon. This is not an accident, but follows as a result of the requirement that the symmetry-breaking order parameter not break the gauge symmetry generated by the electric charge, Q.

Table 2.1. *Neutral-current charges of the fermions*

Fermion type	T_3	Q	g_V	g_A
ν_e, ν_μ, ν_τ	$+\frac{1}{2}$	0	+0.25	+0.25
e, μ, τ	$-\frac{1}{2}$	−1	−0.0189	−0.25
u, c, t	$+\frac{1}{2}$	$+\frac{2}{3}$	+0.0959	+0.25
d, s, b	$-\frac{1}{2}$	$-\frac{1}{3}$	−0.1730	−0.25

(ii) The neutral-current interactions that couple fermions to Z-bosons never involve fermions of more than one flavor at a time and so cannot change flavor. As was indicated earlier for the Higgs and strong interactions, the experimental absence of such flavor-changing neutral currents was a strong clue to the structure of the standard model and was even used to predict the existence of the fourth type of quark, c!

(iii) Electromagnetic interactions all conserve P, C, and T separately.

(iv) The neutral-current interactions, on the other hand, violate both P and C but do not break T (see Section 2.5 for details).

This concludes the tabulation of the interactions that are contained in the standard model Lagrangian.

2.5 Symmetry properties*

When exploring the consequences for experiment of any potential theoretical model, it is always necessary to make use of various approximation schemes. It is therefore of crucial importance to understand which of the predictions of the model are of general validity, and which depend on more details of the approximation scheme used. For this reason, the first step to take in exploring any model is to identify the symmetries that it predicts, since these can be used to draw exact conclusions concerning the existence of conservation laws and of systematics (such as degeneracies) in the spectrum of particles. Therefore, we will now discuss at some length the symmetries of the standard model, and what exact conservation laws they predict.

One of the most beautiful features of the standard model is its success in reproducing precisely the conservation laws and symmetries that had been distilled from experiment over the several decades before the discovery of

* This section, while good for your teeth and bones, is not necessary for most of the development of this book, and can be skipped in whole or in part if necessary.

the model. This accomplishment is all the more remarkable in light of the fact that the standard model is the most general theory consistent with a few very general principles, together with the given particle content and the requirement of renormalizability. As a result, none of the properties to be discussed in this section are built into the model as assumptions, and so they may be understood as general consequences of the basic principles of Section 1.2, together with the explicit particle content of the model.

Symmetries such as these, that are simply consequences of gauge invariance, particle content and renormalizability, are known as *accidental* symmetries. One example that has already been encountered is the custodial $SU(2)$ of the symmetry-breaking sector of Subsection 2.3.2.

2.5.1 Discrete symmetries

There are three discrete transformations that naturally arise within the quantum mechanics of any relativistic system. Two of these – parity, P, and time reversal, T – are related to (i.e. *automorphisms* of) the Lorentz group itself. The third discrete transformation – charge conjugation, C – consists of the interchange of every particle with its antiparticle.

It turns out that *none* of these are symmetries of the standard model, although the combined symmetry CPT is (and, in fact, is a symmetry of any quantum field theory which satisfies the basic principles laid out in Section 1.2). Nevertheless, we will take some time to discuss them. The reasons for doing so are, first, that the combined symmetry CP (or equivalently T) is *almost* a symmetry of the standard model, broken by very small subtle effects; and, second, that while C and P are very far from being symmetries of the standard model, at low energies $E \ll M_W$ they turn out to be accidental symmetries, as we will discuss in Section 7.3.

2.5.1.1 Definitions: P and T

The existence of the operations of parity and time reversal is related to the connectedness of the Lorentz group itself. The Lorentz group is reviewed in Appendix C. We show there that not all coordinate transformations permitted in special relativity can be built infinitesimally from the identity. In particular, two transformations of coordinates cannot: the parity transformation,

$$x^\mu \to P^\mu_\nu x^\nu \,, \qquad P^\mu_\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{2.100}$$

which reflects each space coordinate, and the time reversal transformation,

$$x^\mu \to T^\mu{}_\nu\,, \qquad T^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \tag{2.101}$$

which reverses the sign of time (see Appendix C).

Transformations P and T need not be symmetries of a given theory. If they are symmetries, and if their representations in the theory's Hilbert space are denoted by $\mathcal{P}$ and $\mathcal{T}$ respectively, then $\mathcal{P}$ can always be chosen to be a unitary operator and although $\mathcal{T}$ cannot be made unitary, it may always be chosen to be anti-unitary (that is, an operator which flips the sign of i). The reason $\mathcal{T}$ is antiunitary is that H must transform under the symmetry into an operator which still has a positive spectrum; this will be satisfied if $\mathcal{P}H\mathcal{P}^* = H$ and $\mathcal{T}H\mathcal{T}^* = H$. On the other hand, time evolution by a positive amount of time t, $\mathrm{e}^{-\mathrm{i}Ht}$, should be carried under time reversal to time evolution by a negative amount of time $-t$, $\mathcal{T}\,\mathrm{e}^{-\mathrm{i}Ht}\mathcal{T}^* = \mathrm{e}^{\mathrm{i}Ht}$. The only way that both of these can be true is if $\mathcal{T}$ is an anti-unitary operator, reversing the sign of i.

2.5.1.2 Definition: $\mathcal{C}$

Charge conjugation is defined as the interchange of every particle with its antiparticle. The unitary operator that represents this interchange in the Hilbert space will be denoted by $\mathcal{C}$.

Notice that the condition that a theory be charge-conjugation invariant is stronger than the condition of crossing symmetry discussed in Section 1.2. Crossing symmetry is a general consequence of relativistic quantum mechanics; it states that particles and antiparticles must appear in the action only in the schematic combination $(a + \bar{a}^*)$. This ensures that particles and antiparticles appear in all interactions with the same strength but does *not* imply that all interactions must be invariant with respect to interchange of a with $\bar{a}$.

It is a theorem, though, that the combined action of all three of these discrete transformations, CPT, must be a symmetry in any Lorentz invariant, local field theory.

2.5.1.3 Transformation rules

The action of $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{C}$ on particle states and on fields is determined (up to a conventionally fixed freedom to redefine fields) by their transformation properties under Lorentz transformations. Their action on a state,

$|\mathbf{p},\sigma\rangle$, that describes a particle of three-momentum $\mathbf{p}$, total spin j, and third component of angular momentum σ, may be chosen to be

$$\begin{aligned} \mathcal{P}|\mathbf{p},\sigma\rangle &= \alpha_p|-\mathbf{p},\sigma\rangle \\ \mathcal{T}|\mathbf{p},\sigma\rangle &= \alpha_t(-)^{j-\sigma}|-\mathbf{p},-\sigma\rangle \\ \mathcal{C}|\mathbf{p},\sigma\rangle &= \alpha_c\overline{|\mathbf{p},\sigma\rangle} \end{aligned} \tag{2.102}$$

In these expressions, α_p, α_t, and α_c are phases that are characteristic of each particle type, and the state $\overline{|\cdots\rangle}$ denotes the antiparticle for the state $|\cdots\rangle$. The transformation properties of the corresponding creation and annihilation operators are determined by those of the particle states

$$\begin{aligned} \mathcal{P}a^*_{\mathbf{p},\sigma}\mathcal{P}^* &= \alpha_p a^*_{-\mathbf{p},\sigma} \\ \mathcal{T}a^*_{\mathbf{p},\sigma}\mathcal{T}^* &= \alpha_t(-)^{j-\sigma}a^*_{-\mathbf{p},-\sigma} \\ \mathcal{C}a^*_{\mathbf{p},\sigma}\mathcal{C}^* &= \alpha_c\bar{a}^*_{\mathbf{p},\sigma} \end{aligned} \tag{2.103}$$

The transformation rules for the fields are then determined by their expansions in terms of creation and annihilation operators. Since these have the generic form

$$\phi \sim \sum_{\mathbf{p},\sigma}[u(\mathbf{p},\sigma)a_{\mathbf{p},\sigma} + v(\mathbf{p},\sigma)\bar{a}^*_{\mathbf{p},\sigma}] \tag{2.104}$$

the transformation rules for fields representing spin-zero particles become

$$\begin{aligned} \mathcal{P}\phi(x)\mathcal{P}^* &= \alpha_p^*\phi(x_p) \\ \mathcal{C}\phi(x)\mathcal{C}^* &= \alpha_c^*\phi^*(x) \end{aligned} \tag{2.105}$$

in which $x_p = (-\mathbf{x},t)$ denotes the image of $x = (\mathbf{x},t)$ under parity. (Since invariance of the theory under the combination CPT is guaranteed on general grounds, T-invariance is equivalent to CP-invariance. For this reason it suffices to have explicit expressions for the transformation rules under C and P in order to determine its symmetry properties.)

For spinor fields we have instead,

$$\begin{aligned} \mathcal{P}\psi(x)\mathcal{P}^* &= \alpha_p^*\beta\psi(x_p) \\ \mathcal{C}\psi(x)\mathcal{C}^* &= \alpha_c^*C\overline{\psi}^T(x) \end{aligned} \tag{2.106}$$

in which β and C are the matrices defined in Eq. (1.85) and Eq. (1.93) respectively. (The factor β exchanges left- and right-handed components and is necessary because parity flips handedness.)

Finally, for spin-one gauge potentials, V_a^μ, that correspond to the gauge generator, t_a, we have (up to gauge transformations)

$$\begin{aligned}\mathcal{P}[t_a V_a^\mu(x)]\mathcal{P}^* &= P^\mu{}_\nu[t_a V_a^\nu(x_p)] \\ \mathcal{C}[t_a V_a^\mu(x)]\mathcal{C}^* &= -[t_a V_a^\mu(x)]^* \end{aligned} \tag{2.107}$$

The phase in the transformation rule for the gauge potentials is fixed by the requirement that the covariant derivative, $\mathrm{D} = \partial - \mathrm{i}T_a V_a$, transform properly.

2.5.1.4 Invariance of the model

Using these transformation rules, we can test the standard model interactions of the previous section for invariance under the three independent symmetries of C, P, and CP.

The typical interaction Lagrangian density is the sum of several local operators, $\mathcal{O}_n(x)$, with some constant coefficients, c_n: $\mathcal{L}_{\rm int} = \sum_n c_n \mathcal{O}_n(x)$. The transformation properties of the operators, $\mathcal{O}_n(x)$, can be inferred in terms of those of the various fields of the theory in terms of which they are expressed. The resulting transformation rule for the interaction Lagrangian is

$$\begin{aligned}\mathcal{P}\mathcal{L}_{\rm int}\mathcal{P}^* &= \sum_n (\alpha_n)_p c_n \mathcal{O}_n(x_p) \\ \mathcal{C}\mathcal{L}_{\rm int}\mathcal{C}^* &= \sum_n (\alpha_n)_c c_n \mathcal{O}_n^*(x) \\ (\mathcal{C}\mathcal{P})\mathcal{L}_{\rm int}(\mathcal{C}\mathcal{P})^* &= \sum_n (\alpha_n)_p (\alpha_n)_c c_n \mathcal{O}_n^*(x_p)\end{aligned} \tag{2.108}$$

The phases $(\alpha_n)_p$ and $(\alpha_n)_c$ are products of the phases associated with the transformation of each field.

Since the action is given by the integral of $\mathcal{L}(x)$ over spacetime, the condition $\mathcal{P}\mathcal{L}(x)\mathcal{P}^* = \mathcal{L}(x_p)$ suffices to ensure that the action is invariant. The condition for parity invariance is therefore that there exist a choice of phases, α_ps, for each of the fields for which

$$(\alpha_n)_p = 1 \qquad\qquad \text{for all } n \tag{2.109}$$

This is a nontrivial condition because there can be more interactions, $\mathcal{O}_n$, than there are fields appearing within them.

The Lagrangian is also required by unitarity to be Hermitian, so the following relation among the operators is also true: $\sum_n c_n^* \mathcal{O}_n^* = \sum_n c_n \mathcal{O}_n$. The action is therefore charge-conjugation invariant provided that there exists a choice of charge-conjugation phases, α_cs, for each of the fields for which the coefficient of $\mathcal{O}_n^*$ is unchanged:

$$(\alpha_n)_c c_n = c_n^* \qquad \text{for all } n \tag{2.110}$$

CP-invariance is similarly ensured if phases can be chosen such that

$$(\alpha_n)_c(\alpha_n)_p c_n = c_n^* \qquad \text{for all } n \tag{2.111}$$

If we apply this formalism to the standard model Lagrangian then we find the results quoted in Section 2.4. The Higgs interactions, gluon interactions, and electromagnetic interactions all respect each of the three discrete symmetries, C, P, and CP. The neutral current couplings of the fermions to the neutral Z boson break both C and P but in such a way that the combination CP is unbroken. Finally, the charged-current coupling of the fermions to the W boson not only violates C and P, but can also violate CP, provided that there is not sufficient freedom to make the Kobayashi–Maskawa matrix real. As an illustration we show the manipulations for the charged-current quark interactions,

$$\mathcal{L} = \frac{\mathrm{i}g_2}{2\sqrt{2}}\Big[V_{mn}W_\mu^+\bar{u}_m\gamma^\mu(1+\gamma_5)d_n + (V^\dagger)_{mn}W_\mu^-\bar{d}_m\gamma^\mu(1+\gamma_5)u_n\Big] \tag{2.112}$$

In this case the transformation rules for the spin-one fields become $\mathcal{C}W_\mu^\pm\mathcal{C}^* = -W_\mu^\mp$ and $\mathcal{P}W_\mu^\pm\mathcal{P}^* = P^\nu{}_\mu W_\nu^\pm$. Then, under charge conjugation, we have

$$\begin{aligned}\mathcal{C}\,\mathcal{L}\,\mathcal{C}^* \;=\; \frac{\mathrm{i}g_2}{2\sqrt{2}}\Big\{&(\alpha_{u_m})_c(\alpha_{d_n})_c^*V_{mn}W_\mu^-\left[\bar{d}_n\gamma^\mu(1-\gamma_5)u_m\right]^* \\ &+(\alpha_{u_n})_c^*(\alpha_{d_m})_c(V^\dagger)_{mn}W_\mu^+[\bar{u}_n\gamma^\mu(1-\gamma_5)d_m]^*\Big\}\end{aligned} \tag{2.113}$$

and under parity transformations we get

$$\begin{aligned}\mathcal{P}\,\mathcal{L}\,\mathcal{P}^* \;=\; \frac{\mathrm{i}g_2}{2\sqrt{2}}\Big[&(\alpha_{u_m})_p(\alpha_{d_n})_p^*V_{mn}W_\mu^+\bar{u}_m\gamma^\mu(1-\gamma_5)d_n \\ &+(\alpha_{d_m})_p^*(\alpha_{u_n})_p(V^\dagger)_{mn}W_\mu^-\bar{d}_m\gamma^\mu(1-\gamma_5)u_n\Big]\end{aligned} \tag{2.114}$$

It is clear that there is no choice of phases for which the Lagrangian is parity or charge-conjugation invariant, because any choice that would make the term involving γ^μ invariant would make the $\gamma_5\gamma^\mu$ term not invariant (and vice versa). The point is that each operation replaces the projector $P_{\mathrm{L}} = (1+\gamma_5)/2$ with the projector $P_{\mathrm{R}} = (1-\gamma_5)/2$.

Combining both transformations, however, gives the following result:

$$\begin{aligned}(\mathcal{CP})\mathcal{L}(\mathcal{CP})^* &= \frac{\mathrm{i}g_2}{2\sqrt{2}} \\ \times\Big\{&(\alpha_{u_m})_c(\alpha_{d_n})_c^*(\alpha_{u_m})_p(\alpha_{d_n})_p^*V_{mn}W_\mu^-\left[\bar{d}_n\gamma^\mu(1+\gamma_5)u_m\right]^*\end{aligned}$$

$$+(\alpha_{u_n})_c^*(\alpha_{d_m})_c(\alpha_{u_n})_p^*(\alpha_{d_m})_p(V^\dagger)_{mn}W_\mu^+[\bar{u}_n\gamma^\mu(1+\gamma_5)d_m]^*\Big\} \quad (2.115)$$

If the phases can be chosen to satisfy $(\alpha_{u_m})_c(\alpha_{d_n})_c^*(\alpha_{u_m})_p(\alpha_{d_n})_p^* = 1$, and the KM matrix can be simultaneously chosen to be real, then this last equation would be precisely the complex conjugate of the original Lagrangian. Inspection of the other terms in the Lagrangian confirms that the phase choice can be made provided that the KM matrix may be chosen to be real. Therefore, as claimed, the standard model fails to conserve CP invariance only in that the KM matrix cannot be made purely real.

2.5.2 Continuous symmetries

It is of considerable interest to determine the continuous global symmetries of the standard model Lagrangian. The purpose of this section is to identify the exact, and some approximate, symmetries of this Lagrangian.

The starting point is the class of symmetries of the Lagrangian in the absence of all interactions or mass terms. This will give the maximum possible symmetry group which could exist, given the particle content of the model. The interactions of the theory will not respect all of this symmetry. We will consider each interaction in turn and see how it cuts down the size of the actual symmetry group, until we find what symmetries remain.

As is discussed in Chapter 1, when the basis of fields is chosen to be real (or Majorana), this class consists of a general independent orthogonal rotation among all of the bosonic fields of a given spin, as well as a unitary rotation amongst the left-handed fermions. For the standard model the group of all such transformations is $G_{\max} = O(4) \times O(12) \times U(45)$, corresponding to the four real scalar fields, 12 gauge potentials and three generations of fermions each containing 15 different species of fermion (one E, two from L, three each from U and D, and six from Q). We will write this group as $G_{\max} = G_0 \times G_{\frac{1}{2}} \times G_1$, with $G_0 = O(4)$ the group of scalar transformations, $G_{\frac{1}{2}} = U(45)$ the group of fermionic transformations, and $G_1 = O(12)$ the group of gauge-field transformations.

We wish to determine what subgroup of this group of transformations is preserved once the interactions are turned on. One immediate subgroup of this type is the group of gauge transformations themselves: $G_g \equiv SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1) \subset G$.

2.5.2.1 Gauge self-interactions

We next describe conditions G must satisfy if it is not to be broken by the gauge interactions.

Consider first the self-interactions of the twelve gauge bosons. As is discussed in more detail in Chapter 1, the free kinetic terms for these fields are invariant under the replacement of each field by an arbitrary linear combination of the fields, $\delta V^a_\mu = M^a_b V^b_\mu$, provided that the 12×12 matrix M^a_b is antisymmetric (and so its exponential, $[\exp(M)]^a_b$, is orthogonal). The group formed by these transformations is the group $G_1 = O(12)$. We wish to determine what subgroup of these transformations are also symmetries of the gauge boson self-interactions. In order to be an invariance of these interactions, a candidate symmetry transformation must preserve the structure constants of the gauge group

$$M^b_a c^c{}_{bd} + M^b_d c^c{}_{ab} = M^c_b c^b{}_{ad} \tag{2.116}$$

The algebra of infinitesimal symmetry transformations of the gauge boson self interactions is given by that subalgebra of G_1 that satisfies Eq. (2.116). This subalgebra must include the Lie algebra of the gauge group itself, because infinitesimal gauge rotations, $\delta V^a_\mu = \epsilon^b c^a{}_{bc} V^c_\mu$, automatically satisfy Eq. (2.116) by virtue of the Jacobi identity that is satisfied by the structure constants, c^a_{bc}.

An immediate consequence of Eq. (2.116) is that if the gauge group consists of several mutually commuting factors, $G_g = H_1 \times H_2 \times \cdots$ (as is the case for the standard model), then $M^b_a = 0$ unless both a and b correspond to generators that are in the same factor of G_g. It is a theorem of the theory of compact semisimple Lie groups that the only Lie subgroup of G_1 that satisfies Eq. (2.116) is the gauge subgroup itself (i.e. $G_1 \sim G_g$ consists of the group of *inner* automorphisms of G_g). As a result, there are no accidental global symmetries within the gauge boson sector of the theory.

2.5.2.2 Scalar–gauge and scalar self-couplings

The next simplest case is the scalar sector of the model. The Higgs doublet consists of four real scalar fields, $\phi_i = \phi_i^*$, and so the free kinetic terms of these fields are invariant under arbitrary $G_0 = O(4)$ rotations, $\delta\phi_i = iR^i_j \phi^j$ with $R + R^T = 0$, of these fields into one another. As discussed in Subsection 2.5.2, this symmetry is not broken by the scalar self-interactions as described by the scalar potential. We wish to know which subgroup of G_0 is also a symmetry of the scalar-gauge interactions. Our answer to this question is not specific to the example $O(4)$ but applies more generally for larger symmetry groups, G_0.

Consider a group G_R of symmetry transformations, with group generators we will designate as R. If the generators of the gauge transformations are t_a, then the condition for the group of symmetry transformations to be unbroken by the gauge transformations is

$$[t_a, R] = N_a^b t_b \tag{2.117}$$

for each R and each t_a of the gauge group. The coefficients N_a^b represent a rotation among the gauge potentials of the theory that might be necessary to compensate for the effects of the scalar rotation, R. For our application, we are interested in the case where G_R is a subgroup of G_0.

Note that R and t_a are all generators of the group G_0; so Eq. (2.117) is a special case of the Lie algebra of G_0. Choose a basis for the generators of G_0 such that the structure constants $f^A{}_{BC}$ are totally antisymmetric. Then $[t_a, R] = f_{aR}^B g_B$, with g_B one of the generators of G_0. For Eq. (2.117) to hold, either f_{aR}^B vanishes, or g_B must be one of the t_b. But that would imply that $[t_a, t_b] \propto R$, which cannot be – the t_a must be a subgroup of G_0, so their algebra should be closed. Therefore, R must either be a generator of the group of gauge transformations, or it must commute with all of the generators of the gauge group.

Since the solution in which R is a gauge transformation generator does not represent a new, accidental, symmetry, we focus on the alternative for which R commutes with all of the gauge transformations in G_g. By Schur's lemma, this implies that the transformations, R, cannot mix fields that transform in different irreducible representations of the gauge group. The resulting symmetry of the gauge interactions then becomes a product of orthogonal groups, $O(N_1) \times O(N_2) \times \cdots$ in which each factor describes the rotations of the N_i fields that all transform in the common representation, r_i, of the gauge group.

Since only a single irreducible representation of scalar fields appears in the standard model, and since there is no other subgroup of $O(4)$ which commutes with the $SU_{\rm L}(2) \times U_Y(1)$ subgroup, there are no accidental global symmetries of the scalar gauge couplings. It is purely the $U_Y(1)$ gauge couplings that break the potential $O(4)$ symmetry of the scalar sector. One way to see this is to notice that the Lie algebra of $O(4)$ is isomorphic to that of the algebra $SU(2) \times SU(2)$, of which one of the $SU(2)$ factors may be taken to be the gauge group $SU_{\rm L}(2)$. In the absence of the $U_Y(1)$ gauge couplings to the scalars, there would therefore be an entire $SU(2)$ subgroup of G_0 that commutes with the gauge group. This is the origin of the custodial $SU(2)$ symmetry of Subsection 2.3.2.

Although the standard model is not invariant under the full $O(4)$ invariance, conclusions based on this symmetry do become correct in the limit that the $U_Y(1)$ gauge coupling—and, as we shall see, the Yukawa couplings – vanish. Since this coupling is known to be experimentally small, it follows that the $O(4)$ symmetry is a good *approximate symmetry* of the standard model. Such approximate symmetries can be almost as useful as exact symmetries if the non-invariant couplings are sufficiently small.

2.5.2.3 Fermion–gauge couplings

The only place left to look for accidental global symmetries is inside the group $G_{\frac{1}{2}} = U(45)$ of transformations between the 45 species of left-handed fermions. (The number 45 arises as three generations times one E, two L, three U, three D, and six Q fields per generation. A quark species counts for three because of its three colors, L and Q count double because of the two flavors in each.)

If we work with a basis of fermions which are in definite representations of the gauge group—as opposed to being mass eigenstates – the condition that the symmetry transformations be preserved by the fermion gauge interactions is a direct analog of Eq. (2.117). It follows that a subgroup of $G_{\frac{1}{2}}$ preserves the fermion–gauge interactions if it is either the subgroup of the gauge transformations themselves, or it commutes with this gauge subgroup. Since the 15 fermion species of a given generation transform under the gauge group

$$\begin{aligned} SU_{\rm c}(3) \quad \times \quad & SU_{\rm L}(2) \times U_Y(1) \text{ as } \left(\mathbf{3}, \mathbf{2}, +\frac{1}{6}\right) \oplus \left(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3}\right) \oplus \left(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3}\right) \\ \oplus \quad & \left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right) \oplus (\mathbf{1}, \mathbf{1}, +1) \end{aligned}$$

and since none of these irreducible representations is big enough to admit an internal potential symmetry that commutes with the gauge group, there are no accidental symmetry transformations relating the fermions within a single generation.

The accidental symmetries of the fermion–gauge couplings are therefore

$$G_f \equiv U_Q(3) \times U_U(3) \times U_D(3) \times U_L(3) \times U_E(3) \subset G_{\frac{1}{2}} \tag{2.118}$$

Each factor of this group corresponds to a unitary rotation in generation space of the five types of irreducible $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$ representations of the model's fermion content.

2.5.2.4 Yukawa interactions

From the previous paragraphs, the only exact non-gauge symmetries of the gauge interactions of the standard model are $G_f = [U(3)]^5$, representing independent transformations, in generation space, of each type of fermion fields. The final issue is to determine which of these potential symmetries also preserves the Yukawa interactions of the theory.

The conditions that must be satisfied in order for these transformations to preserve the form of the Yukawa couplings of Eq. (2.17) are

$$\begin{aligned}(U_L^T f U_E)_{mn} &= f_{mn} \\ (U_Q^T g U_U)_{mn} &= g_{mn} \\ (U_Q^T h U_D)_{mn} &= h_{mn}\end{aligned} \tag{2.119}$$

These equations imply that the potential symmetry transformations must also satisfy the following additional conditions, which each involve only left-handed or only right-handed unitary transformations:

$$\begin{aligned}(U_E^\dagger f^\dagger f U_E)_{mn} &= (f^\dagger f)_{mn} \\ (U_L^T f f^\dagger U_L^*)_{mn} &= (f f^\dagger)_{mn} \\ (U_U^\dagger g^\dagger g U_U)_{mn} &= (g^\dagger g)_{mn} \\ (U_Q^T g g^\dagger U_Q)_{mn} &= (g g^\dagger)_{mn} \\ (U_D^\dagger h^\dagger h U_D)_{mn} &= (h^\dagger h)_{mn} \\ (U_Q^T h h^\dagger U_Q)_{mn} &= (h h^\dagger)_{mn}\end{aligned} \tag{2.120}$$

In order to analyze the implications of these conditions, it is convenient to work with a basis of fields for which the fermion mass matrix, and so also the Yukawa coupling matrices, are real and diagonal. (Since the transformation to this basis introduces the Kobayashi–Maskawa matrix into the charged-current fermion gauge couplings, these couplings must be re-examined for invariance at the end.)

In this basis, and taking the experimental information that none of the eigenvalues of the Yukawa coupling matrices f_{mn}, g_{mn}, and h_{mn} vanish or are degenerate, Eq. (2.120) implies that each of the unitary matrices must be diagonal with phases along their diagonals. This reduces the candidate symmetry group for the fermions to the multiplication of the left- and right-handed parts of each mass eigenstate by an independent $U(1)$ phase.

Using this form for the unitary transformations in the original condition of Eq. (2.119) implies that the left- and right-handed transformations must be equal for each type of fermion; that is $U_Q = U_U^* = U_D^*$ and $U_L = U_E^*$.

For leptons this is the end of the story, implying that the accidental symmetry of the lepton sector is $U_e(1) \times U_\mu(1) \times U_\tau(1)$:

$$U_L = U_E^* = \begin{pmatrix} e^{i\theta_e} & & \\ & e^{i\theta_\mu} & \\ & & e^{i\theta_\tau} \end{pmatrix} \tag{2.121}$$

For quarks, we must also check that these phase transformations preserve the form for the charged-current gauge interactions when written in terms of mass eigenstates as in Eq. (2.114). To be invariant, the candidate transformation must therefore commute with the KM matrix. For a generic unitary KM matrix the only combination of such transformations are those that are proportional to the unit matrix in generation space, and so which rotate all quarks by a common phase. Therefore, there is only a single $U(1)$ transformation left:

$$U_Q = U_U^* = U_D^* = \begin{pmatrix} e^{i\theta_B/3} & & \\ & e^{i\theta_B/3} & \\ & & e^{i\theta_B/3} \end{pmatrix} \tag{2.122}$$

The corresponding group is $U_B(1)$. The factor of 1/3 is chosen so that the charge of a quark under this $U(1)$ is 1/3. Since bound states of quarks always contain a multiple of 3 quarks (see Chapter 8), they have integer charge (0 or ± 1) under this symmetry.

The accidental global symmetry group of the standard model is therefore

$$G = U_e(1) \times U_\mu(1) \times U_\tau(1) \times U_B(1) \tag{2.123}$$

Each of the four generators of this symmetry group corresponds to a quantum number that appears to be experimentally conserved. They are:

(i) *electron number*: $L_e(e^-) = L(\nu_e) = +1$, $L_e(e^+) = L(\overline{\nu_e}) = -1$, $L_e = 0$ for all others;
(ii) *muon number*: $L_\mu(\mu^-) = L(\nu_\mu) = +1$, $L_\mu(\mu^+) = L(\overline{\nu}_\mu) = -1$, $L_\mu = 0$ for all others;
(iii) *tau number*: $L_\tau(\tau^-) = L(\nu_\tau) = +1$, $L_\tau(\tau^+) = L(\overline{\nu}_\tau) = -1$, $L_\tau = 0$ for all others;
(iv) *baryon number*: $B(q) = \frac{1}{3}$ for all quarks, q, $B(\overline{q}) = -\frac{1}{3}$ for antiquarks, and $B = 0$ for all others.

The sum $L = L_e + L_\mu + L_\tau$ is also known as *lepton number*. It is one of the triumphs of the standard model that its accidental symmetries correspond exactly with those conserved quantum numbers that had been experimentally observed.

Conservation of these quantum numbers immediately implies the stability of the lightest particles that carry nonzero values for them. Given that the neutrinos are massless and the charged leptons are not, we conclude that all neutrino types are absolutely stable in this theory. Similarly, the lightest baryon, which turns out to be the proton, is also predicted never to decay. The electron is similarly stable because it is the lightest particle in the theory that carries electric charge.

These conservation laws similarly forbid processes such as the reaction $\mu \to e\gamma$, since these do not conserve L_e or L_μ. This agrees with the current experimental upper bound on this decay, which at present indicates that it must occur less frequently than once in every 10^{11} μ decays.

In fact, there is now evidence that the separate lepton numbers are *not* conserved, and that neutrinos are not perfectly massless – though the effects which violate lepton number are tiny and are of no bearing in most particle physics experiments. Chapter 10 discusses the evidence for this violation, together with some of its implications. At present, experiments do not provide evidence for $L = L_e + L_\mu + L_\tau$ violation.

As it happens, one of the puzzling features of the standard model is the small size of the Yukawa couplings for almost all of the fermions of the theory. An equivalent way to phrase the same puzzle is to ask why the fermion masses (apart from that of the top quark) are all so small in comparison to, say, the masses of the W and the Z. To the extent that these Yukawa couplings can be ignored, there is a larger approximate flavor symmetry, $[U_{\rm L}(3) \times U_E(3)]$ for leptons and $[U_Q(2) \times U_U(2) \times U_D(3)]$ for quarks. A related $\mathcal{U}_L(3) \times \mathcal{U}_R(3)$ approximate symmetry emerges when electroweak interactions are turned off, and is very useful for analyzing the low-energy properties of the strongly interacting quark sector in which the implications of such a *chiral* $U_{\rm L}(3) \times U_{\rm R}(3)$ symmetry provides otherwise unobtainable information about the spectrum of the light strongly interacting particles. These approximate symmetries are considered in much more detail in Chapter 8.

2.5.3 Anomalies

The discussion of the previous sections has dealt exclusively with the symmetries of the *classical* action of the model and has neglected quantum considerations. We devote this section to a discussion of the potential complications that arise when considering symmetries within a quantum, as opposed to classical, field theory.

In order to outline the issue at stake, recall that there are several uses to which symmetries are applied. The most important place is in the coupling of light spin-one particles. Here it was argued that these interactions could only be Lorentz invariant and unitary if they were also invariant under local gauge transformations. Another application was to use the existence of global (or local) symmetries to infer the existence of local conservation laws and symmetry relations amongst the energy eigenvalues of the system concerned.

The logic used in all of these applications has been: (i) The invariance of the classical action under a particular symmetry transformation ensures, by Noether's theorem, the existence of a set of currents, j^μ_a, whose conservation, $\partial_\mu j^\mu_a = 0$, follows from the equations of motion for the fields; (ii) these conserved currents may be used to construct conserved charges, $Q_a = \int j^0_a \, \mathrm{d}^3x$, for which the equations of motion for the fields imply $[H, Q_a] = 0$.

Unfortunately, such classical arguments do not always hold in a quantum theory. The process of quantizing a given classical theory introduces ambiguities associated with the ordering of operators in the quantum theory. In a field theory this operator-ordering ambiguity is intimately related with the divergences at short distances, since operators only fail to commute when their spacelike separations tend to zero. Since different operator orderings for the system Hamiltonian give rise to different equations of motion, and since the conservation of the Noether current depends on these equations of motion, the form taken by the conserved current will in general depend on how these operator-ordering issues are resolved.

It could potentially happen that there is no operator ordering under which all would-be currents are conserved, even if they should be conserved at the classical level. That is to say, it might happen that the existence of a symmetry of the classical action might not be sufficient for the existence of a conserved quantum charge operator. Should this occur, we would lose the exact results we hoped to derive from the existence of the symmetry. The discovery that classical symmetries can fail in this way was so surprising when it was discovered that this failure of a symmetry to survive quantization was termed an *anomaly*. The purpose of the remainder of this section is to summarize under what circumstances a symmetry is "anomalous" in this way.

Precisely such an anomaly can indeed occur for a current if the symmetry at issue involves transformations on Majorana fermions. Since the distinction between right- and left-handed fields is essential here, the anomaly is termed the *chiral anomaly*. While it is beyond the scope of this book to derive how such an anomaly arises, the condition for the absence of a chiral anomaly may be fairly simply stated. Suppose that the generators of a classical symmetry acting on left-handed spinor fields are denoted by T_a. Then, as is discussed in Section 2.1, the action of the symmetry on a Majorana spinor becomes $\delta\psi^m = i\epsilon^a[(T_a)^m_n P_{\mathrm{L}} - (T^*_a)^m_n P_{\mathrm{R}}]\psi^n$. The classical symmetry survives quantization, and so is called *anomaly free*, if the *anomaly coef-*

ficients, A_{abc}, vanish for all a, b, and c. These coefficients are completely symmetric under permutations of the indices a, b, and c, and are defined by

$$A_{abc} = \mathrm{tr}(T_a\{T_b, T_c\}) \tag{2.124}$$

The curly brackets in this equation denote the anticommutator, $\{T_b, T_c\} \equiv T_bT_c + T_cT_b$, and the trace means that a sum is to be taken over all types of fermions, e.g. every color of every flavor of quark and every lepton, in each generation, with T denoting the action of the symmetry on that particular particle type (so if T_a represents the action of one of the color generators, it is $\lambda_a/2$ in color space when acting on a quark, and 0 when acting on a lepton, since leptons are colorless and do not change under a color rotation).

In particular, when the anomaly coefficient A_{abc} does not vanish and the indices b, c correspond to gauge symmetries, then the conservation of the current J^μ_a is violated by

$$\partial_\lambda J^\lambda_a = \frac{A_{abc}}{64\pi^2}\epsilon^{\mu\nu\alpha\beta} gF^b_{\mu\nu} gF^c_{\alpha\beta} \tag{2.125}$$

with F the field strength corresponding to symmetry b and g the associated gauge coupling.

A consequence of the structure of Eq. (2.124) is that there are no anomalies for *real* (or pseudoreal) fermion representations. A (pseudo-) real representation is defined to be one for which the generators $\mathrm{i}T_a$ are real up to a similarity transformation: $T^*_a = -ST_aS^{-1}$ for some invertible matrix S. To see that this ensures freedom from anomalies, notice that since the generators T_a are Hermitian it follows that $T^T_a = T^*_a$. Then

$$\begin{aligned} A_{abc} &= \mathrm{tr}(T_a\{T_b, T_c\}) \\ &= \mathrm{tr}[(T_a\{T_b, T_c\})^T] \\ &= \mathrm{tr}(\{T^T_c, T^T_b\}T^T_a) \\ &= \mathrm{tr}(\{T^*_c, T^*_b\}T^*_a) \\ &= -\mathrm{tr}(S\{T_c, T_b\}T_aS^{-1}) \\ &= -\mathrm{tr}(\{T_c, T_b\}T_a) \\ &= -A_{abc} = 0 \end{aligned} \tag{2.126}$$

This will make the calculation of several anomaly coefficients much easier.

An important special case of this last result occurs when fermion number is conserved and when the left- and right-handed fermions (as opposed to antifermions) transform in the same representation, t_a say, of the group of interest. In this case the generator of this group acting on *all* of the left-handed spinors (for fermions *and* antifermions) may be written in the block-diagonal form

$$T_a = \begin{pmatrix} t_a & 0 \\ 0 & -t_a^* \end{pmatrix} \tag{2.127}$$

This is manifestly pseudoreal since $T_a^* = -ST_aS^{-1}$. It follows that any symmetry that is left-right symmetric in this way must be anomaly-free.

Because of the central role symmetries play in field theory, we must check two things.

(i) First, since the gauge symmetries of the standard model are chiral in the sense just described, we must verify that they are anomaly-free, that is, that all anomalies involving three gauge symmetries vanish. Otherwise, the gauge fields will not couple to conserved currents, and the gauge interactions will not be simultaneously Lorentz-invariant and unitary. Since these are both basic principles of quantum field theory, a theory with anomalous gauge symmetries *does not exist* (is not a valid theory).

(ii) Next, we must see whether the exact and approximate global "accidental" symmetries of the standard model have anomalies or not. No issues of consistency need arise if they do have anomalies, since these symmetries are not associated with the couplings of any spin-one particles. It is nevertheless important to understand which are anomalous, since anomalies negate the argument that would allow these classical symmetries to imply the existence of exact conservation laws or spectral relations.

These two issues are the topics of the following two sections.

2.5.3.1 Cancellation of gauge anomalies

Let us verify that the anomaly coefficient, A_{abc}, vanishes in the standard model when all of the indices, a, b, and c, correspond to gauge group generators. As we shall see, this *anomaly cancellation* relies on the detailed quantum numbers of the standard model fermions and requires all of the members of a complete generation in order to work.

We consider each combination of generators in turn. We will use the notation "A(3, 3, 3)" for the anomaly coefficient involving three generators etc.We demonstrate that the contribution to the anomaly coefficient from each generation separately vanishes.

(i) A(3, 3, 3): The $SU_c(3)$ representations are all left-right symmetric. This anomaly coefficient must therefore vanish for the general reasons given above.

(ii) A(3, 3, 2): These coefficients are all proportional to the trace of the Pauli matrices since these furnish the two-dimensional $SU_{\mathrm{L}}(2)$ representations. Since the Pauli matrices are all traceless this anomaly coefficient must vanish.

(iii) A(3, 3, 1): The three-dimensional $SU_{\mathrm{c}}(3)$ generators are given by the Gell-Mann matrices, $\lambda_\alpha/2$, of Eq. (1.186). These are all tracefree and satisfy the following property:

$$\{\lambda_\alpha, \lambda_\beta\} = \frac{4}{3}\delta_{\alpha\beta} + 2d_{\alpha\beta\gamma}\lambda_\gamma$$

The trace over colors of $\delta_{\alpha\beta}$ will give 3, while the trace over $d_{\alpha\beta\gamma}\lambda_\gamma$ gives zero; so A(3,3,1) is therefore proportional to the trace over all left-handed colored fields (i.e. quarks) of the $U_Y(1)$ generator—weak hypercharge, Y. The anomaly coefficient therefore is

$$\begin{aligned} \mathrm{A}(3,3,1) = \sum_{\text{quarks}} Y &= 3(2y_{Q_{\mathrm{L}}} + y_{U_{\mathrm{L}}} + y_{D_{\mathrm{L}}}) \\ &= 3\left[2\left(\frac{1}{6}\right) + \left(-\frac{2}{3}\right) + \left(\frac{1}{3}\right)\right] \\ &= 0 \end{aligned} \tag{2.128}$$

The overall factor of 3 is the number of generations. The factor of 2 on $y_{Q_{\mathrm{L}}}$ is because of the two $SU_{\mathrm{L}}(2)$ flavors.

(iv) A(3, *X*, *Y*): This coefficient vanishes for X and Y equal to either 2 or 1 since it is proportional to the trace of a Gell-Mann matrix, which vanishes.

(v) A(2, 2, 2): As observed above, the only nontrivial $SU_{\mathrm{L}}(2)$ representations that appear within the standard model are doublets, and so the generators are represented by the Pauli matrices. Since all three Pauli matrices satisfy the following identity, $\tau_a^* = -\tau_2\tau_a\tau_2$, it follows that this representation is pseudoreal, and so the anomaly coefficient must vanish by the general argument of Eq. (2.124).

(vi) A(2, 2, 1): The Pauli matrices satisfy an identity similar to that satisfied by the Gell-Mann matrices: $\{\tau_a/2, \tau_b/2\} = \delta_{ab}/2$, which is doubled when summed over a doublet. This anomaly coefficient is therefore the sum over $SU_{\mathrm{L}}(2)$ doublets of the weak hypercharge, Y:

$$\begin{aligned} \mathrm{A}(2,2,1) = \sum_{\text{doublets}} Y &= 3(y_{L_{\mathrm{L}}} + 3y_{Q_{\mathrm{L}}}) \\ &= 3\left[\left(-\frac{1}{2}\right) + 3\left(\frac{1}{6}\right)\right] \\ &= 0 \end{aligned} \tag{2.129}$$

The factor of 3 on the Q contribution arises from the trace on colors.

(vii) A(2, 1, 1): This coefficient vanishes simply because it is proportional to the the trace of a single Pauli matrix, which is zero.

(viii) A(1, 1, 1): This coefficient is proportional to the sum over all left-handed fermions of the cube of the weak hypercharge:

$$\begin{aligned} \mathrm{A}(1,1,1) &= 2\sum_{\text{all}} Y^3 = 6(2y_{L_\mathrm{L}}^3 + y_{E_\mathrm{L}}^3 + 6y_{Q_\mathrm{L}}^3 + 3y_{U_\mathrm{L}}^3 + 3y_{D_\mathrm{L}}^3) \\ &= 6\left(2\left(-\frac{1}{2}\right)^3 + (+1)^3 + 6\left(\frac{1}{6}\right)^3 + 3\left(-\frac{2}{3}\right)^3 + 3\left(\frac{1}{3}\right)^3\right) \\ &= 0 \end{aligned} \tag{2.130}$$

It is clear that anomaly cancellations in the standard model require non-trivial relationships between the number of species of and the quantum numbers for the quarks and leptons. It is also clear that the values of the hypercharges of the different species are not accidental. The relations $Y_{E_\mathrm{L}} + Y_{L_\mathrm{L}} = 1/2$, $Y_{D_\mathrm{L}} + Y_{Q_\mathrm{L}} = 1/2$, and $Y_{U_\mathrm{L}} + Y_{Q_\mathrm{L}} = -1/2$ are enforced by the requirement that the Yukawa interaction terms be hypercharge-invariant. However, until now, the fact that $Y_{E_\mathrm{L}} = 1$ and not, say, $1+\epsilon$, has been a mystery. This is important; if it were $1+\epsilon$, the neutrinos would possess electric charges of $-\epsilon$. Similarly, Y_{D_L} could be $1/3+\delta$ rather than $1/3$, in which case the neutron would be charged, and the electron and proton charges would differ. (The proton charge is $2Q_u + Q_d$.) In fact, limits on neutrino and neutron charges and on proton–electron charge differences are very strong; for instance, the electron and proton charges differ in absolute value by no more than a part in 10^{21}. The reason is that Eq. (2.129) and Eq. (2.130) only sum to zero if $\epsilon = \delta = 0$. Therefore the equality of the proton charge and the electron charge, and the vanishing of the neutrino and neutron charges, are exact identities within the standard model.

We next consider the potential anomalies that could arise in the Lorentz algebra. The Lorentz group has been treated here as a global rather than a gauge symmetry and so might be treated in the following section. However, the introduction of gravitational interactions requires it to be gauged, so if gauge-Lorentz anomalies exist, then the theory of gravitation would be inconsistent. Therefore we consider it here.

The only standard model particles that are in complex representations of the Lorentz group are the fermions. Since the Lorentz generators on fermions (c.f. Subsection 1.3.2) are essentially equivalent to $SU(2)$ transformations, the anomaly cancellation arguments are similar to those for an $SU(2)$ gauge group. It follows that the only anomaly coefficient that does not vanish immediately due to the properties of the Pauli matrices is A(J,J,1), in which J generically denotes the Lorentz generators. The condition that this anomaly coefficient be zero is that the trace of the weak hypercharge over all left-handed fermions vanish:

$$\begin{aligned}\operatorname*{tr}_{\text{all}} Y &= 3(2y_{L_\text{L}} + y_{E_\text{L}} + 6y_{Q_\text{L}} + 3y_{U_\text{L}} + 3y_{D_\text{L}}) \\ &= 3\left[2\left(-\frac{1}{2}\right) + (+1) + 6\left(\frac{1}{6}\right) + 3\left(-\frac{2}{3}\right) + 3\left(\frac{1}{3}\right)\right] \\ &= 0 \end{aligned} \tag{2.131}$$

2.5.3.2 Anomalies in global symmetries

We next compute the anomalies for the accidental global symmetries and for some of the approximate global symmetries that were identified in the previous sections.

For baryon number, B, the anomaly coefficients are:

$$\begin{aligned} A(3,3,B) &= \sum_{\text{quarks}} B = 6\left(\frac{1}{3}\right) + 3\left(-\frac{1}{3}\right) + 3\left(-\frac{1}{3}\right) = 0 \\ A(2,2,B) &= \sum_{\text{doublets}} B = 9\left(\frac{1}{3}\right) = 3 \\ A(1,1,B) &= \sum_{\text{all}} 2Y^2 B \\ &= 36\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right) + 18\left(-\frac{2}{3}\right)^2\left(-\frac{1}{3}\right) + 18\left(\frac{1}{3}\right)^2\left(-\frac{1}{3}\right) = -3; \\ A(1,B,B) &= \sum_{\text{all}} 2YB^2 \\ &= 36\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)^2 + 18\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)^2 + 18\left(\frac{1}{3}\right)\left(-\frac{1}{3}\right)^2 = 0; \\ A(B,B,B) &= \sum_{\text{all}} 2B^3 = 2(36-18-18)/27 = 0; \\ A(J,J,B) &= \sum_{\text{all}} B = (12-6-6) = 0 \end{aligned} \tag{2.132}$$

For lepton numbers, L_e, L_μ, L_τ, each of these charges gets contributions only from its own generation, so the factors of 3 from the generation sum in the baryon results will be absent. It suffices to compute the anomalies for one of them since the results are identical for the others. Anomalies between lepton symmetries vanish.

$$A(2,2,L_e) = \sum_{\text{doublets}} L_e = 1$$

$$
\begin{aligned}
A(1,1,L_e) &= \sum_{\text{all}} 2Y^2 L_e = 4\left(-\frac{1}{2}\right)^2 (+1) + 2\,(+1)^2\,(-1) = -1 \\
A(1,L_e,L_e) &= \sum_{\text{all}} 2Y L_e^2 = 4\left(-\frac{1}{2}\right)(+1)^2 + 2\,(+1)\,(-1)^2 = 0; \\
A(L_e,L_e,L_e) &= \sum_{\text{all}} 2L_e^3 = 4(+1)^3 + 2(-1)^3 = 2 \\
A(J,J,L_e) &= \sum_{\text{all}} L_e = 2(+1) + 1(-1) = 1 \qquad (2.133)
\end{aligned}
$$

Chiral $U(3)$ is an approximate symmetry under which the three lightest left- and right-handed quarks get shuffled amongst one another, $U_{q\mathrm{L}}(3) \times U_{q\mathrm{R}}(3)$. It will be of interest in Chapter 8, where we will need to know how much of this approximate symmetry group is anomaly-free. We consider here only the quark sector since this is the case that is of most direct interest in subsequent chapters. For brevity, we consider only the left-handed case explicitly here. Denote a general $U_{q\mathrm{L}}(3)$ generator by T_a and denote its specific 3×3 representation by t_a. Then

$$
\begin{aligned}
A(3,3,T_a) &\propto \mathrm{tr}\, t_a \\
A(2,2,T_a) &\propto \mathrm{tr}\ t_a \\
A(1,1,T_a) &\propto \mathrm{tr}\ t_a \\
A(1,T_a,T_b) &\propto \mathrm{tr}(t_a t_b) \propto \delta_{ab} \\
A(T_a,T_b,T_c) &\propto \mathrm{tr}(t_a\{t_b,t_c\}) \\
A(J,J,T_a) &\propto \mathrm{tr}\ t_a \qquad (2.134)
\end{aligned}
$$

Some comments.

(i) Perhaps the most basic observation about these anomaly coefficients is that they are not zero. It follows that the naive conclusions that are based on the corresponding symmetries can break down and so must be treated with caution. It turns out, however, that for physics at temperatures low compared to the W boson mass, any violation of the corresponding conservation laws due to quantum effects are proportional to $\exp(-8\pi^2/g^2)$ and so are negligibly small for weak couplings ($g \ll 1$). The same arguments indicate that those global symmetries that have anomalies due to any strong interactions are strongly broken, and so should not provide good approximations to the dynamics of the full quantum theory.

As a result, all of the consequences of the exact global symmetries are expected to hold for the standard model to an extremely good approximation. However, those symmetries having $SU_c(3)$ anomalies are expected to be strongly broken.

(ii) The only anomaly-free global symmetries of the standard model are found by taking appropriate linear combinations of the anomalous symmetries given above. The symmetries free of all anomalies, including gravitational anomalies, are $L_e - L_\mu$, $L_e - L_\tau$, and $L_\mu - L_\tau$ (which is linearly dependent on the first two).

(iii) Notice that all of the $SU_L(3) \times SU_L(2) \times U_Y(1)$ anomalies are the same for baryon number, B, as they are for the total lepton number, $L = L_e + L_\mu + L_\tau$. The Lorentz, B^3, and L^3 anomalies would also agree if the model were to be supplemented by a right-handed neutrino field per generation. This suggests that the combination B–L would be anomaly free, including gravitational effects, in the presence of right-handed neutrinos.

(iv) It is clear from Eq. (2.134) that all of the chiral $U(3)$ transformations have anomalies of one type or another. Only those with a non-vanishing trace receive $SU_c(3)$ anomalies, however, so the traceless ones would be bona fide symmetries to the extent that the electroweak interactions are negligible. Now, since the group $U(3)$ is generated by arbitrary 3×3 Hermitian matrices, and since any such matrix may always be decomposed as a linear combination of traceless Gell-Mann matrices and the unit matrix, it follows that the Lie algebra for $U(3)$ is equivalent to that of the product $SU(3) \times U(1)$. Since only the $U(1)$ generator has a non-vanishing trace, only it suffers from an $SU_c(3)$ anomaly. As a result, the strong interactions break the approximate symmetry $U_{q\mathrm{L}}(3) \times U_{q\mathrm{R}}(3)$ down to its subgroup $SU_{q\mathrm{L}}(3) \times SU_{q\mathrm{R}}(3) \times U_B(1)$. The unbroken $U_B(1)$ is that combination of the $U(1)$s that acts equally on left- and right-handed quark fields, and so may be recognized simply as baryon number.

2.6 Problems

[2.1] Anomaly cancellation and charge assignments

Complete the proof that anomaly cancellation fixes the charges of the standard model fermions.

First, take the hypercharge of the Higgs field ϕ to be exactly $+1/2$. This can be considered as the definition of the normalization of g_1. Then, write the hypercharges of $P_L L$ and $P_L Q$ as $q_L \equiv -1/2 - \epsilon$ and $q_Q \equiv 1/6 - \delta$.

Show that the hypercharges of the E, U, and D fields are fixed by the requirement that the Yukawa interactions be gauge invariant, and find expressions for q_E, q_D, and q_U, the hypercharges of $P_L E$, $P_L D$, and $P_L U$.

Then find expressions for the two anomaly conditions, Eq. (2.129) and Eq. (2.130), in terms of δ and ϵ. Show that the only simultaneous solution to both equations is $\epsilon = \delta = 0$.

[2.2] **Muon decay**

The muon μ decays via the reaction

$$\mu^- \to e^- \nu_\mu \bar{\nu}_e$$

However, the decay

$$\mu^- \to e^- \gamma$$

with γ a photon has never been observed. Explain in terms of symmetries why there is no obstacle in principle to the first decay, but the second decay is forbidden and is expected to have a rate in the standard model of zero.

[2.3] **Right-handed neutrinos**

Suppose a right-handed neutrino for each generation (invariant under $SU_c(3) \times SU_L(2) \times U_Y(1)$) is added to the standard model.

[2.3.1] Show that the only new renormalizable terms that can appear in the Lagrangian are (also rewriting the kinetic term for the left handed leptons):

$$\mathcal{L} = -\frac{1}{2}\bar{L}_m \not{D} L_m - \frac{1}{2}\bar{N}_m \not{\partial} N_m - \frac{1}{2} M_m \bar{N}_m N_m - (k_{mn} \bar{L}_m P_R N_n \tilde{\phi} + \text{h.c.})$$

where N_m is the Majorana spinor whose right-handed piece is the right-handed neutrino and L_m is the usual lepton doublet. M_m is a real mass parameter and k_{mn} are Yukawa coupling constants.

[2.3.2] Do any combinations of electron-number, muon-number and tau-number remain conserved in the presence of these terms?

[2.3.3] Argue that these new terms induce a neutrino mass. Specializing to the case of one generation for simplicity, write down the neutrino mass matrix and identify the basis of fields in which it is diagonal and positive.

[2.3.4] Express the lepton–Higgs and lepton–gauge-boson interactions in terms of these mass eigenstates. (It is most convenient to keep using Majorana spinors here because the mass matrix does not take a simple form in terms of Dirac spinors.)

[2.4] **Two Higgs doublet models**

Suppose the Higgs doublet of the standard model is supplemented by a second complex doublet, ψ, transforming as $(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$ under $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$.

[2.4.1] If ψ is written $\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix}$, what are the electric charges of the component fields χ and ξ?

[2.4.2] Write out the covariant derivative $\mathrm{D}_\mu \psi$ explicitly in terms of the gauge fields G^α_μ, W^a_μ and B_μ.

[2.4.3] Assuming the potential must be a function of the invariants $a = \phi^\dagger\phi$, $b = \psi^\dagger\psi$, and $c = \phi^T \varepsilon \psi$, where ϕ is the usual Higgs doublet, what is the most general renormalizable form? How many independent real parameters does it contain? Need the parameters appearing in the potential be real? Is the combination $d = \phi^\dagger\psi \quad SU_{\rm L}(2) \times U_Y(1)$ invariant?

[2.4.4] Suppose the parameters of the potential are such that it is minimized when

$$\phi = \phi_{\rm min} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$\psi = \psi_{\rm min} = \begin{pmatrix} \frac{1}{\sqrt{2}}(u + \mathrm{i}w) \\ 0 \end{pmatrix}$$

u, v, w all real. Do these values break the electromagnetic group $U_{\rm em}(1)$ generated by the electric charge $Q = T_3 + Y$? Identify the terms in the Lagrangian that are quadratic in the gauge fields and find their masses in terms of u, v, and w. Call the mass eigenstates $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp \mathrm{i} W^2_\mu)$, $Z_\mu = W^3_\mu \cos\theta - B_\mu \sin\theta$, and $A_\mu = B_\mu \cos\theta + W^3_\mu \sin\theta$. Express $\cos\theta$ in terms of the gauge couplings g_1 and g_2. Is the standard model mass relation $M_W = M_Z \cos\theta$ also true for this model?

[2.4.5] What are the possible Yukawa couplings of the spin zero fields, ϕ and ψ, to the fermions? Suppose the Lagrangian is required to be invariant under the symmetry:

$$P_{\rm R} E_m \to \mathrm{e}^{\mathrm{i}\theta} P_{\rm R} E_m,\ P_{\rm R} U_m \to \mathrm{e}^{\mathrm{i}\theta} P_{\rm R} U_m,\ P_{\rm R} D_m \to \mathrm{e}^{\mathrm{i}\theta} P_{\rm R} D_m$$

$$\phi \to \mathrm{e}^{-\mathrm{i}\theta}\phi \quad \text{and} \quad \psi \to \mathrm{e}^{-\mathrm{i}\theta}\psi$$

with θ a real constant and all other fields being invariant. What are the resulting restrictions on the Yukawa couplings and Higgs potential, $V(\phi,\ \psi)$?

[2.5] **Adjoint Higgs fields**

Suppose that the standard model is supplemented by a second complex Higgs field that transforms as a triplet of $SU_{\rm L}(2)$ rather than as a doublet; i.e.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

and

$$\delta_2\psi = {\rm i}\omega_2^a t_a \psi$$

with

$$t_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; t_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -{\rm i} & 0 \\ {\rm i} & 0 & -{\rm i} \\ 0 & {\rm i} & 0 \end{pmatrix}, \; t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(You can verify that t_1, t_2, t_3 satisfy the algebra of $SU_{\rm L}(2)$ generators.) Suppose also that the hypercharge, Y, of the field ψ is zero.

[2.5.1] What is the electric charge of each component field, ψ_1, ψ_2, and ψ_3?

[2.5.2] Suppose the potential for ψ and the usual Higgs field, ϕ, is minimized when

$$\phi = \phi_{\rm min} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$\psi = \psi_{\rm min} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(u + {\rm i}w) \\ 0 \end{pmatrix}$$

Do these values respect the electromagnetic gauge group $U_{\rm em}(1)$ generated by the electric charge $Q = T_3 + Y$?

[2.5.3] Find the masses of the spin-one fields $W_\mu^\pm$, Z_μ, and A_μ, where, as usual, $Z_\mu = W_\mu^3 \cos\theta - B_\mu \sin\theta$ and $A_\mu = B_\mu \cos\theta + W_\mu^3 \sin\theta$. What is $\cos\theta$ in terms of the gauge couplings? Is the mass relation $M_W = M_Z \cos\theta$ still valid?

[2.6] **Gauged *B*–*L* coupling**

Suppose the standard model is extended to contain an extra $U(1)$ symmetry $U(1)'$, with gauge boson F_μ and gauge coupling g_4. Suppose that the Higgs boson has charge 0 under this gauge boson, but the left-handed lepton doublet $P_{\rm L}L$ has charge -1.

Also assume a complex scalar field χ, of charge $+1$ under the new symmetry but uncharged under hypercharge, is added to the Lagrangian. Write its effective potential as

$$V(\chi) = \lambda_\chi \left(\chi^* \chi - \frac{\mu^2}{2} \right)^2 \tag{2.135}$$

so that when $\mu^2 > 0$, it develops a vacuum expectation value. (There can also be an interaction term between the Higgs boson and χ, but assume that such a term is absent.)

[2.6.1] Revisit Problem 2.3, where a right-handed neutrino N is added to the standard model. What is the charge of $P_\mathrm{R} N$ under $U(1)'$, and is the Majorana neutrino mass $M\bar{N}N$ still allowed?

[2.6.2] Based on the requirement that the Yukawa couplings preserve $U(1)'$ symmetry, and that all gauge anomalies cancel (in particular, the $(3,3,1')$, $(2,2,1')$, $(1,1,1')$, $(1,1',1')$, and $(1',1',1')$ anomaly coefficients are non-trivial), what must be the charges of the standard model fermions? Show that anomaly cancellation actually *demands* that the theory possess an N field.

[2.6.3] What linear combinations of baryon number and the three lepton numbers remain conserved? Are there any Yukawa couplings involving the χ field?

[2.6.4] Argue that if $\mu^2 < 0$ so the χ field has no condensate, the F field is massless. In analogy with the Coulomb interaction mediated by the electromagnetic A field between charged particles, argue that there will be a Coulomb-like interaction between the electron and the neutron. Is it attractive or repulsive? How might it be observed or (very tightly!) constrained?

[2.6.5] Suppose that $\mu^2 > 0$. What is the spectrum of bosons? Does the normal relation between W and Z boson masses hold? Is there any mixing between F_μ and Z_μ, A_μ?

[2.7] **Colored scalar fields**

Suppose the standard model is extended to include a complex scalar field $\tilde{D}$, transforming under the $(\mathbf{3}, \mathbf{1}, -\frac{1}{3})$ representation of $SU_\mathrm{c}(3) \times SU_\mathrm{L}(2) \times U_Y(1)$;

$$\delta \tilde{D} = \left(\frac{-\mathrm{i}g_1}{3} \omega_1 + \frac{\mathrm{i}g_3}{2} \lambda_\alpha \omega_3^\alpha \right) \tilde{D}\,, \quad \mathrm{D}_\mu \tilde{D} = \left(\partial_\mu - \frac{\mathrm{i}g_3}{2} G_\mu^\alpha \lambda_\alpha + \frac{\mathrm{i}g_1}{3} B_\mu \right) \tilde{D}$$

(This is the same as the transformation property of $P_\mathrm{R}\, D$.)

[2.7.1] Show that $\tilde{D}^*$ transforms under the $(\overline{\mathbf{3}}, \mathbf{1}, +\frac{1}{3})$ representation of $SU_{\rm c}(3)\times SU_{\rm L}(2)\times U_Y(1)$(which is the same as the transformation rule for $P_{\rm L} D$, see Eq. (2.13) and Eq. (2.26)), and that $\tilde{D}^\dagger \tilde{D}$ (with the contraction over the color indices implicit; the † means that $\tilde{D}^*$ is written as a row vector) is an $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$ invariant.

[2.7.2] Show that the following renormalizable interactions are allowed for the $\tilde{D}$ field: a kinetic and gauge interaction term,

$$-(\mathrm{D}_\mu \tilde{D})^\dagger (\mathrm{D}^\mu \tilde{D})$$

a mass term,

$$-M^2_{\tilde{D}} \tilde{D}^\dagger \tilde{D}$$

the following scalar interaction terms,

$$-\lambda' (\tilde{D}^\dagger \tilde{D})^2 - \lambda'' \phi^\dagger \phi \, \tilde{D}^\dagger \tilde{D}$$

and the following new Yukawa interactions:

$$-x_{mn} \bar{Q}_m P_{\rm R} L_n \tilde{D} - y_{mn} \epsilon_{rst} \bar{U}^r_m P_{\rm R} \, D^s_n \tilde{D}^t - z_{mn} \bar{U}_m P_{\rm L} \, E_n \tilde{D} + \text{h.c.}$$

in which x_{mn}, y_{mn}, and z_{mn} are new (complex 3×3 matrix) Yukawa couplings, r, s, t are color indices, ϵ_{rst} is the totally antisymmetric tensor on color indices, and color indices are implicitly summed in the other two terms.

Argue that there are no other renormalizable interactions which are gauge invariant and satisfy all of the basic principles.

[2.7.3] What is the mass squared of $\tilde{D}$, including both the explicit effects of its mass term and the effects of v the *v.e.v.* of the Higgs boson? Is the mass of $\tilde{D}$ determined by its coupling to the Higgs boson, or is it an independent free parameter of the model?

[2.7.4] Argue that there is *no* assignment of lepton or baryon number to the $\tilde{D}$ field which leaves either B or L symmetry unbroken. Hence, the addition of such a scalar field generically leads to the violation of B and L symmetries.

Show, however, that if the Lagrangian is required to be invariant under a discrete symmetry, $\tilde{D} \to -\tilde{D}$ with all other fields unaffected, then none of the Yukawa couplings are permitted and conserved baryon and lepton numbers can again be defined. Further, show that in this case there is a new global $U(1)$ symmetry $\tilde{D} \to \mathrm{e}^{\mathrm{i}\theta_D} \tilde{D}$ which ensures that the number of $\tilde{D}$ particles is conserved.

[2.8] **Adjoint representation fermions**

Suppose that two Majorana fermions were added to the standard model;

$\tilde{W}$, a triplet under $SU_{\rm L}(2)$, transforming as $(\mathbf{1}, \mathbf{3}, 0)$, and $\tilde{G}$, an octet under $SU_{\rm c}(3)$, transforming as $(\mathbf{8}, \mathbf{1}, 0)$. That is, the transformation properties are,

$$\delta P_{\rm L}\tilde{W}^a = -\epsilon_{abc}\omega_2^b P_{\rm L}\tilde{W}^c\,, \qquad {\rm D}_\mu P_{\rm L}\tilde{W}^a = \left(\partial_\mu\delta_{ac} + g_2\epsilon_{abc}W_\mu^b\right) P_{\rm L}\tilde{W}^c\,,$$

and

$$\delta P_{\rm L}\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta P_{\rm L}\tilde{G}^\gamma\,, \qquad {\rm D}_\mu P_{\rm L}\tilde{G}^\alpha = \left(\partial_\mu\delta_{\alpha\gamma} + g_3 f_{\alpha\beta\gamma}G_\mu^\beta\right) P_{\rm L}\tilde{G}^\gamma$$

[2.8.1] Show that the reality of ϵ_{abc} and $f_{\alpha\beta\gamma}$ cause $P_{\rm R}\,\tilde{W}$ and $P_{\rm R}\,\tilde{G}$ to have the same transformation properties as $P_{\rm L}\,\tilde{W}$ and $P_{\rm L}\,\tilde{G}$.

[2.8.2] Show that, contrary to what happened with the fermions of the standard model, the new fields $\tilde{W}$ and $\tilde{G}$ *do* have $SU_{\rm c}(3) \times SU_{\rm L}(2) \times U_Y(1)$ invariant mass terms,

$$-\frac{m_{\tilde{W}}}{2}\,\overline{\tilde{W}}\tilde{W} - \frac{m_{\tilde{G}}}{2}\,\overline{\tilde{G}}\tilde{G}$$

Therefore, these particles may possess masses independent of their coupling to the Higgs boson.

[2.8.3] Show that the only new Yukawa interaction is

$$y_m\overline{L}_m\tau_a\tilde{\phi}P_{\rm R}\tilde{W}_a + {\rm h.c.}$$

3
Cross sections and lifetimes

Most of the applications of the standard model to experimental situations are concerned with processes in which almost free particles interact briefly and over short distances. These processes could be the collisions of various elementary particles within an accelerator (Chapter 6 and Chapter 9) or they could be the decay of an unstable elementary particle in flight (Chapter 4 and Chapter 5). Scattering (S-matrix) theory is the formalism that has been devised to study these systems.

This chapter presents a whirlwind review of the quantum theory of scattering. The purpose is to gather into one place all of the results that are required in order to use the Lagrangian of Chapter 2 to predict the outcomes of experiments. The first section sets up the notion of scattering states, which are meant to represent in a precise way the idea that the particles involved do not interact except for a short time interval. This is followed by a review of the calculation of scattering amplitudes using time-dependent perturbation theory.

In later chapters this formalism is finally used to compute the Feynman rules that describe the interactions contained within the standard model Lagrangian.

Readers in a hurry, or who find themselves bogged down in this section, should try to understand Section 3.2 and will need to learn the results at the end of Section 3.3, particularly Eq. (3.40) and Eq. (3.43).

3.1 Scattering states and the S-matrix

In a real scattering (or decay) process, the particles involved only interact briefly because they physically move apart from one another. For instance, in a scattering experiment, the initial particles are initially well separated from one another, but moving with velocities which bring them into mutual contact. From the perspective of quantum mechanics, this means that these initial states cannot be exact momentum eigenstates, since such states are not spatially localized at all. Similarly, they cannot be exact energy eigenstates to the extent that their profiles in position space change with time (as opposed to simply being multiplied by an overall phase e^{-iEt}). Instead, the initial particles are usually given by wave packets which are somewhat localized in both position and momentum (in a way which is consistent with the uncertainty relations), with the packets describing the relative approach of initially well-separated particles.

To the extent that the initially colliding particles are not correlated with one another and that the reactions do not depend on the environment within which they occur, one expects the probability of any given reaction to factorize into the product of the probability for the particles to meet, times the probability for the reaction to occur given that the meeting has taken place. Of these, the first factor can be expected to depend on the details of the wave packets which describe the initial state, since this controls things like how many particles are present and how quickly they approach one another. The second factor, however, might be expected to be independent of the details of the initial state and instead be more of an intrinsic property of the interactions involved. Indeed, these expectations are borne out in practice for collisions, and motivate the definition of initial-state-independent quantities, like *cross sections*, which describe the part of the reaction which does not depend on the details of how a particular reaction has been set up.

It is the inference of quantities like cross sections from experimental measurements which is of practical interest, since these directly bear the information about the underlying interactions like those described in earlier chapters. Because they are largely insensitive to the details of the wave packets describing the initial states, it turns out to be possible to compute quantities like cross sections directly in the limit that these initial states become energy and momentum eigenstates, even though this is not the limit within which real experiments take place. The idealized energy eigenstates to which one is led in this way are called *scattering states*, and their definition is the topic of this section.

Suppose, then, that the complete Hamiltonian, H, can be broken into two pieces, $H = H_0 + V$, in such a way that H_0 describes the evolution of the initial and final wave packets before and after the scattering. In the simplest instance H_0 might describe just the kinetic energy of moving free particles, with all of the interactions being put into V. But more complicated divisions of H are also possible, such as by including the strong and/or electromagnetic interactions in H_0 while placing the weak interactions into V.

In general the Hilbert space, $\mathcal{H}$, for the full system divides into two parts,

$$\mathcal{H} = \mathcal{B} \oplus \mathcal{S} \tag{3.1}$$

for which $\mathcal{S}$ contains those states of the full system whose evolution in time using H is well approximated at late or early times by evolution using H_0. That is, $\mathcal{S}$ are the states (particles) of the theory with Hamiltonian H_0. It is useful to define the origin of time so that the initial and final wave packets of the interacting particles are sufficiently widely separated that H_0 evolution suffices outside of a region $-T < t < T$, for some appropriately large and positive T. Not all states need reside in $\mathcal{S}$, and those which do not live in $\mathcal{B}$, which we loosely call bound states. For example, if our system consisted of electrons and protons interacting electromagnetically, then $\mathcal{S}$ might contain freely-moving electrons and protons, but $\mathcal{B}$ might contain bound hydrogen atoms.

Let us denote the eigenstates of H_0 by $|\alpha\rangle$, with α collectively denoting all of the labels which are required to describe single- and many-particle states and $H_0|\alpha\rangle = E_\alpha|\alpha\rangle$. We write a wave packet of such states as

$$|\phi_g\rangle \equiv \int \mathrm{d}\alpha \; g(\alpha)|\alpha\rangle \tag{3.2}$$

where $g(\alpha)$ defines an appropriately normalizable packet. The label α here is treated as a continuous variable because we envisage it to include (possibly among other labels) the momenta of the various particles included in the state. We assume that H_0 has the same spectrum on $\mathcal{H}$ as H does on $\mathcal{S}$, so the same labels, α, and energies, E_α, may be used to describe the eigenstates of the full system, $H|\alpha\rangle\!\rangle = E_\alpha|\alpha\rangle\!\rangle$ (where the double angle $\rangle\!\rangle$ is used to denote an eigenstate of H).

To describe scattering processes we work within the Schrödinger picture, where the burden of time evolution is carried by the state of the system. In a scattering problem we imagine that the time evolution of states prepared in appropriate wave packets, $|\phi_g\rangle$, have essentially the same evolution in the remote past and the remote future, $|t| \gg T$, using either H or H_0. That is, we require that there must exist an *out* state, $|\phi_g\rangle\!\rangle_{\mathrm{o}}$, which at late times evolves under H in the same way as does any properly normalizable packet $|\phi_g\rangle$ under H_0:

$$\lim_{t \gg T} \mathrm{e}^{-\mathrm{i}Ht}|\phi_g\rangle\!\rangle_{\mathrm{o}} = \lim_{t \gg T} \mathrm{e}^{-\mathrm{i}H_0 t}|\phi_g\rangle \tag{3.3}$$

There must similarly exist an *in* state, $|\phi_g\rangle\!\rangle_{\mathrm{i}}$, – in general different than

$|\phi_g\rangle\rangle_{\rm o}$ – whose evolution under H agrees with the evolution of a packet $|\phi_g\rangle$ under H_0 in the remote past:

$$\lim_{t\ll -T} {\rm e}^{-{\rm i}Ht}|\phi_g\rangle\rangle_{\rm i} = \lim_{t\ll -T} {\rm e}^{-{\rm i}H_0 t}|\phi_g\rangle \tag{3.4}$$

By choosing the limiting case of appropriately peaked wave packets, $g(\alpha)$, we may also formally define in this way idealized scattering eigenstates of the full Hamiltonian, $|\alpha\rangle\rangle_{\rm o,i}$, which satisfy

$$\lim_{t\gg T} {\rm e}^{-{\rm i}Ht}|\alpha\rangle\rangle_{\rm o} = \lim_{t\gg T} {\rm e}^{-{\rm i}H_0 t}|\alpha\rangle \quad \text{and} \quad \lim_{t\ll -T} {\rm e}^{-{\rm i}Ht}|\alpha\rangle\rangle_{\rm i} = \lim_{t\ll -T} {\rm e}^{-{\rm i}H_0 t}|\alpha\rangle \tag{3.5}$$

In terms of these states a scattering event corresponds to the transition from a state resembling a packet $|\phi_g\rangle$ at asymptotically early times to one resembling a different packet $|\phi_f\rangle$ at asymptotically late times. From the above definitions the amplitude for a such a process is given by the overlap

$$_{\rm o}\langle\langle\phi_f|\phi_g\rangle\rangle_{\rm i} \tag{3.6}$$

Any such scattering event may therefore be found from the limiting amplitude for the ideal process where the initial and final state are approximately energy eigenstates, and the matrix of all possible such amplitudes,

$$S_{\beta\alpha} := {}_{\rm o}\langle\langle\beta|\alpha\rangle\rangle_{\rm i} \tag{3.7}$$

therefore plays an important role, and is called the S-matrix. It is also convenient to define the operator, S, whose matrix elements between H_0 eigenstates, $|\alpha\rangle$, reproduce these transition amplitudes:

$$\langle\beta|S|\alpha\rangle := S_{\beta\alpha} \tag{3.8}$$

Our goal is to provide an explicit expression for S in terms of the known operators H_0 and V. A step towards this end is the definition of the Møller wave operators

$$\Omega(t) := {\rm e}^{{\rm i}Ht}\,{\rm e}^{-{\rm i}H_0 t} \tag{3.9}$$

in terms of which we have

$$|\alpha\rangle\rangle_{\rm o} = \lim_{t\gg T} \Omega(t)|\alpha\rangle \qquad \text{and} \qquad |\alpha\rangle\rangle_{\rm i} = \lim_{t\ll -T} \Omega(t)|\alpha\rangle \tag{3.10}$$

Since $|\alpha\rangle$ and $|\alpha\rangle\rangle_{\rm o,i}$ are normalized, $\Omega^{\pm} = \lim_{t\to\pm\infty}\Omega(t)$ are isometric operators. Notice, however, that the states $|\alpha\rangle\rangle_{\rm o,i}$ only span $\mathcal{S}$, while $|\alpha\rangle$ span $\mathcal{H}$, so $\Omega^{\pm}$ can only be unitary if $\mathcal{B} = \emptyset$ (i.e. there are no bound states).

These operators are useful because the S-matrix can be constructed from them using

$$S = \lim_{t\to\infty} \lim_{t'\to-\infty} \Omega^*(t)\Omega(t') = (\Omega^+)^*\Omega^- \tag{3.11}$$

The limit $t \to \mp\infty$ must of course be defined with some care, using appropriately normalized wave packets. This complication is ignored here with the understanding that a more careful treatment justifies the formal manipulations we present.

3.2 Time-dependent perturbation theory

We now derive an approximate expression for S as powers of the interaction V. In order to express S in a form that lends itself to such a perturbative approximation, we rewrite the operator $\Omega^*(t)\Omega(t')$ by re-expressing it as a solution to a first-order differential equation in the variable t. That is, $\Omega^*(t)\Omega(t')$ satisfies

$$\begin{aligned} \Omega^*(t)\Omega(t') &= \mathrm{e}^{\mathrm{i}H_0 t}\,\mathrm{e}^{-\mathrm{i}Ht}\mathrm{e}^{\mathrm{i}Ht'}\,\mathrm{e}^{-\mathrm{i}H_0 t'} \\ &= \mathrm{e}^{\mathrm{i}H_0 t}\,\mathrm{e}^{-\mathrm{i}H(t-t')}\,\mathrm{e}^{-\mathrm{i}H_0 t'} \end{aligned} \tag{3.12}$$

Evidently,

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left[\Omega^*(t)\Omega(t')\right] &= \mathrm{e}^{\mathrm{i}H_0 t}(H - H_0)\,\mathrm{e}^{-\mathrm{i}H(t-t')}\,\mathrm{e}^{-\mathrm{i}H_0 t'} \\ &= (\mathrm{e}^{\mathrm{i}H_0 t}V\,\mathrm{e}^{-\mathrm{i}H_0 t})\,\Omega^*(t)\Omega(t') \\ &= V(t)\,\Omega^*(t)\Omega(t') \end{aligned} \tag{3.13}$$

where this last equality defines the interaction picture V operator at time t, $V(t) := \mathrm{e}^{\mathrm{i}H_0 t}V\,\mathrm{e}^{-\mathrm{i}H_0 t}$.

Solutions of this differential equation, together with the initial condition $\Omega^*(t')\Omega(t') = 1$, are equivalent to solutions of the integral equation

$$\Omega^*(t)\Omega(t') = 1 - \mathrm{i}\int_{t'}^{t} \mathrm{d}\tau\; V(\tau)\Omega^*(\tau)\Omega(t') \tag{3.14}$$

This has the obvious iterative solution

$$\Omega^*(t)\Omega(t') = \sum_{n=0}^{\infty}(-\mathrm{i})^n \int_{t'}^{t}\mathrm{d}\tau_1 \int_{t'}^{\tau_1}\mathrm{d}\tau_2 \cdots \int_{t'}^{\tau_{n-1}}\mathrm{d}\tau_n\; V(\tau_1)V(\tau_2)\cdots V(\tau_n) \tag{3.15}$$

The S-matrix becomes

$$S = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} \Omega^*(t)\Omega(t')$$

$$= \sum_{n=0}^{\infty}(-\mathrm{i})^n \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \int_{-\infty}^{\tau_1} \mathrm{d}\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} \mathrm{d}\tau_n \, V(\tau_1)V(\tau_2)\cdots V(\tau_n) \tag{3.16}$$

One of our goals is to make Lorentz invariance as manifest as possible, so to this end it is desirable to rewrite this expression in a form where the temporal integration is over the same range as any spatial integrations, i.e. from $-\infty$ to ∞. This can be done via the following trick. Define the *time-ordering* operation by

$$\begin{aligned} T\left[V(t_1)\cdots V(t_n)\right] &\equiv V(t_{\mathrm{latest}})\cdots V(t_{\mathrm{earliest}}) \\ &= \sum_{P_n} V(t_{P_1})\cdots V(t_{P_n})\theta(t_{P_1}-t_{P_2})\cdots\theta(t_{P_{n-1}}-t_{P_n}) \end{aligned} \tag{3.17}$$

The sum here is over all permutations of the n times $t_1, \ldots, t_n$, and the Heaviside step function,

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.18}$$

ensures that only the permutation in which $t_{P_1} > t_{P_2} > \cdots > t_{P_n}$ contributes. Consider, then, the integral,

$$\begin{aligned} I &\equiv \int_{-\infty}^{\infty}\mathrm{d}\tau_1\cdots\int_{-\infty}^{\infty}\mathrm{d}\tau_n T[V(\tau_1)\cdots V(\tau_n)] \\ &= \sum_{P_n}\int_{-\infty}^{\infty}\mathrm{d}\tau_{P_1}\cdots\int_{-\infty}^{\infty}\mathrm{d}\tau_{P_n}V(\tau_{P_1})\cdots V(\tau_{P_n})\theta(\tau_{P_1}-\tau_{P_2})\cdots\theta(\tau_{P_{n-1}}-\tau_{P_n}) \\ &= n!\int_{-\infty}^{\infty}\mathrm{d}\tau_1\cdots\int_{-\infty}^{\tau_{n-1}}\mathrm{d}\tau_n V(\tau_1)\cdots V(\tau_n) \end{aligned} \tag{3.19}$$

Comparing the last line with the iterative expression for S, given above, implies that

$$S = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int_{-\infty}^{\infty}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_n \, T\left[V(\tau_1)\cdots V(\tau_n)\right] \tag{3.20}$$

This will be the final form for the perturbative expansion of the S-matrix in time-dependent perturbation theory.

Equation (3.20) has a particularly pretty form if the interaction Hamiltonian is given as an integral over a local Hamiltonian density,

$$V(t) = \int \mathrm{d}^3\mathbf{x}\, \mathcal{H}_I(\mathbf{x}, t) \tag{3.21}$$

since in this case the S-matrix becomes

$$S = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int_{-\infty}^{\infty}\mathrm{d}^4x_1\cdots\mathrm{d}^4x_n \, T[\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)] \tag{3.22}$$

This last equation is one of the main results of this chapter.

If we use energy and momentum eigenstates it is convenient to use the identity:

$$\langle\beta|\mathcal{O}(x)|\alpha\rangle = \langle\beta|\mathrm{e}^{-\mathrm{i}P\cdot x}\mathcal{O}(x=0)\,\mathrm{e}^{\mathrm{i}P\cdot x}|\alpha\rangle = \mathrm{e}^{\mathrm{i}(p_\alpha - p_\beta)\cdot x}\langle\beta|\mathcal{O}(x=0)|\alpha\rangle \tag{3.23}$$

to factor an overall energy-momentum conserving factor out of the S-matrix:

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \mathrm{i}\mathcal{M}_{\beta\alpha}(2\pi)^4\delta^4(p_\beta - p_\alpha) \tag{3.24}$$

The quantity $\mathcal{M}_{\beta\alpha}$ is called the *matrix element* for the transition from state α to state β. It is also conventional to define the *T-matrix element*, in which only the energy conserving delta function is factored out:

$$S_{\beta\alpha} = \delta_{\beta\alpha} - \mathrm{i}T_{\beta\alpha}2\pi\delta(p^0_\beta - p^0_\alpha) \tag{3.25}$$

We can read off the first few terms in the expansion of $\mathcal{M}$ directly from Eq. (3.22):

$$\mathcal{M}_{\beta\alpha} = \langle\beta|\mathcal{H}_I(x=0)|\alpha\rangle + \frac{-\mathrm{i}}{2!}\int \mathrm{d}^4x\langle\beta|T\left[\mathcal{H}_I(x)\mathcal{H}_I(x=0)\right]|\alpha\rangle + \cdots \tag{3.26}$$

This is an important result because it gives the S-matrix in terms of quantities that we know, namely the matrix elements of the interaction Hamiltonian density.

Equation (3.22) or Eq. (3.26) do not quite appear Lorentz-invariant, for two reasons. One reason is the appearance of the time-ordering operation, which leads to the functions $\theta(t_i - t_j)$ whose values may differ in different frames. (Recall that different Lorentz observers can disagree on the ordering in time of spacelike separated events.) This turns out not to be important because the operator ordering is only relevant for operators which do not commute. Locality ensures that commutators vanish for spacelike separated points; it is only for timelike or lightlike separated operators that the time ordering operation is important, and for such operators the time ordering is the same in all frames. Therefore, the time-ordering operation on a product of local operators is Lorentz invariant in a local theory, and this is not an obstacle to the Lorentz invariance of Eq. (3.22).

The other reason to doubt the Lorentz invariance of the S-matrix is because the integral of the Hamiltonian density need not be Lorentz-invariant. Note, however, that it is only the interaction part of the Hamiltonian density which appears in the above formulae, and to the extent that this does not involve derivatives of the fields it is typically related to the interaction part of the Lagrangian density by $\mathcal{H} = -\mathcal{L}$. When this is so we see that the

Lorentz invariance of the S-matrix and of $\mathcal{M}$ is manifest, since we know that $\int d^4x \; \mathcal{L}$ is Lorentz invariant by construction. As we see in later chapters Lorentz invariance also holds for interactions involving derivatives of fields, although this invariance arises in a more subtle way.

We shall use these equations – Eq. (3.23), Eq. (3.24), and Eq. (3.26) – extensively throughout what follows.

3.3 Decay rates and cross sections

The expressions obtained above for the S-matrix are proportional to an energy-conserving (and possibly to a momentum-conserving) delta function when expressed in terms of energy eigenstates rather than wave packets. This means that the square of S-matrix elements – the transition probabilities – are proportional to $\delta(0)$ and so must diverge. Physically, this divergence reflects the fact discussed earlier that scattering processes necessarily involve wave packets and *cannot* involve energy eigenstates. (It is also related to the difficulty, in infinite volume, of correctly normalizing an energy eigenstate.) If the initial and final states are energy and momentum eigenstates then their interactions never really turn on and off, because their wave functions spread throughout all of space, which prevents their influence on one another from changing over time. As a result, if we insist on using such eigenstates to compute the S-matrix (as we shall for convenience of calculation), we must more carefully sort out the relationship between physical quantities and the S-matrix elements we find. This is the purpose of the present section.

3.3.1 Wave packets

If the initial state is described by a wave packet, $|\phi_g\rangle\!\rangle_{\rm i} = \int d\alpha \; g(\alpha)|\alpha\rangle\!\rangle_{\rm i}$, then the probability of finding the system in the final state labeled by β becomes

$$P_g(\beta) = |{}_{\rm o}\langle\!\langle\beta|\phi_g\rangle\!\rangle_{\rm i}|^2 \quad = \int d\alpha\, d\alpha' \; g^*(\alpha')g(\alpha)\, {}_{\rm o}\langle\!\langle\beta|\alpha\rangle\!\rangle_{\rm i}\; {}_{\rm i}\langle\!\langle\alpha'|\beta\rangle\!\rangle_{\rm o} \qquad (3.27)$$

In most cases of practical interest, the initial state is prepared in such a way that the function $g(\alpha)$ is peaked about some value $\overline{\alpha}$, and the width of the wave packet is classical in the sense that the resolution of initial position and momentum measurements are much too large to push the limits of the uncertainty relations. It is also usually true that support of the initial wave packet is chosen to be over a region of α, over which $S_{\beta\alpha}$ depends only weakly on α. For instance, the energy width of a wave packet is usually small compared to the energy dependence of the scattering cross section or particle decay width. (Otherwise the experiment does a poor job in measuring the S-matrix, because it uses an inadequately resolved initial state.)

Under these circumstances (and assuming β is distinguishable from all of the α in the support of $g(\alpha)$, so we may write $S_{\beta\alpha} = -\mathrm{i}T_{\beta\alpha}2\pi\delta(E_\beta - E_\alpha)$), then Eq. (3.27) is approximately given by

$$P_g(\beta) \approx |T_{\beta\overline{\alpha}}|^2 \int \mathrm{d}\hat{\alpha}\,\mathrm{d}\hat{\alpha}'\; g^*(\alpha')g(\alpha) \tag{3.28}$$

In this expression $\mathrm{d}\alpha\,2\pi\delta(E_\alpha - E_\beta) = \mathrm{d}\hat{\alpha}$, and we use the fact that $T_{\beta\alpha}$ is approximately independent of α within the domain of support of $g(\alpha)$ to bring it outside of the integral. Notice that the energy-conserving delta functions are no longer a problem since they are used to perform part of the integration over α and α'.

We see that the probability in this case factorizes into a reaction dependent factor ($|T_{\beta\overline{\alpha}}|^2$) and a factor depending on the details of the experimental set-up. Our interest in the remainder of this section is in precisely identifying a convenient quantity which captures the initial-condition-independent factor.

3.3.2 The finite-volume trick

For the present purposes the important consequence of the previous section is Eq. (3.28), which expresses how reaction probabilities factorize in the situations of common practical interest. Since our interest is in finding a convenient way to identify the $|T_{\beta\overline{\alpha}}|^2$ factor in a calculation of $S_{\beta\alpha}$ based on energy and momentum eigenstates, we may feel free to use any old specification of the initial state, provided it captures this factorization (involves narrow ranges of energy and momentum). Obviously we should choose one which makes the calculations convenient.

A particularly simple way of specifying states, and seeing how to handle the subtleties associated with the delta functions in $S_{\beta\alpha}$, is to imagine the system being inside a box having large but finite volume Ω, and allowing the interactions to last only over a large but finite time interval, T. In this case we may simply use energy and momentum eigenstates, with the knowledge that the divergences associated with squaring delta functions are regularized by T and Ω. Once the regularization dependence cancels in the final physical quantities of interest, we may drop the temporary theoretical contrivance of the box.

In a finite-volume box we use particles in momentum states, $|\mathbf{p}]$, that are normalized to 1 in the box,

$$[\mathbf{p}|\mathbf{p}'] = \delta_{\mathbf{p},\mathbf{p}'} \tag{3.29}$$

which satisfy the completeness relation

$$\sum_{\mathbf{p}} |\mathbf{p}][\mathbf{p}| = 1 \tag{3.30}$$

This is to be distinguished from the continuum normalization we use in the infinite-volume limit,

$$\langle \mathbf{p}|\mathbf{p}'\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') \tag{3.31}$$

for which completeness is expressed by

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3}|\mathbf{p}\rangle\langle\mathbf{p}| = 1 \tag{3.32}$$

For a cubic box of volume Ω, subject to periodic boundary conditions on the walls, momentum eigenvalues take discrete values. There is one state for each cube of volume $(2\pi)^3/\Omega$ in momentum space. In the limit $\Omega \to \infty$, the spacing between momentum levels goes to zero and sums over momenta go to integrals according to

$$\frac{1}{\Omega}\sum_{\mathbf{p}} f(\mathbf{p}) \to \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} f(\mathbf{p}) \tag{3.33}$$

Here $f(\mathbf{p})$ represents an arbitrary function that satisfies the boundary conditions at the edge of the box. Comparison with the completeness relations shows that the states $|\mathbf{p}\rangle = (2E\Omega)^{1/2}\,|\mathbf{p}]$ are the ones which have the desired normalization for large Ω.

For a state, $|\alpha\rangle$, involving N_α particles this implies $|\alpha\rangle = (2E\Omega)^{N_\alpha/2}|\alpha]$. The box-normalized matrix element $S^{\square}_{\beta\alpha} \equiv [\beta|S|\alpha]$ is therefore related to the continuum-normalized $S_{\beta\alpha} = \langle\beta|S|\alpha\rangle$ by

$$S_{\beta\alpha} = (2E\Omega)^{(N_\alpha+N_\beta)/2} S^{\square}_{\beta\alpha} \tag{3.34}$$

When particle energies differ, $(2E)^{(N_\alpha+N_\beta)/2}$ is to be interpreted as the square root of the product of the energies of the particles in the in and out states.

At finite volume, in translationally invariant theories, the T-matrix is

$$T_{\beta\alpha} \equiv \mathcal{M}_{\beta\alpha}(2\pi)^3\delta^3_{\Omega}(\mathbf{p}_\beta - \mathbf{p}_\alpha) \tag{3.35}$$

so the S-matrix is given by

$$S_{\beta\alpha} = \delta(\beta-\alpha) - i(2\pi)^4\delta^4_{\Omega T}(p_\beta - p_\alpha)\mathcal{M}_{\beta\alpha} \tag{3.36}$$

The delta functions express energy and momentum conservation and appear in the form,

$$(2\pi)^3\delta^3_\Omega(\mathbf{p}_\alpha - \mathbf{p}_\beta) = \int_\Omega \mathrm{d}^3x\ \mathrm{e}^{\mathrm{i}(\mathbf{p}_\alpha - \mathbf{p}_\beta)\cdot\mathbf{x}} \tag{3.37}$$

$$(2\pi)^4\delta^4_{\Omega T}(p_\alpha - p_\beta) = \int_{\Omega T} \mathrm{d}^4x\ \mathrm{e}^{\mathrm{i}(p_\alpha - p_\beta)\cdot x} \tag{3.38}$$

The spatial integration is over the volume, Ω, and the temporal integration is from $-T/2$ to $+T/2$ respectively. As $\Omega T \to \infty$, $\delta_{\Omega T}$ goes to the standard delta-function but for finite T and Ω, $(2\pi)^4\delta_{\Omega T}(0) = \Omega T$.

In a time-translationally invariant theory it is the transition probability per unit time, or the transition rate, which is independent of time and so is well behaved as $T \to \infty$. Similarly, as $\Omega \to \infty$ the number of states in any finite momentum range diverges, making the probability of a transition to a specific state go to zero. It is therefore the rate, $\mathrm{d}\Gamma$, for the state $|\alpha\rangle$ to make a transition into any state in a small number, $\Delta\beta$, of states in the vicinity of $|\beta\rangle$ that is well behaved as $\Omega T \to \infty$. Since the density of states in momentum space is $\Omega/(2\pi)^3$, the number of states in an interval $\mathrm{d}\beta$ for an N_β-particle state is $\Delta\beta = (2E\Omega)^{N_\beta}\,\mathrm{d}\beta$. Here we have absorbed the powers of 2π into the measure on $\mathrm{d}\beta$, so that $\mathrm{d}\beta \equiv \prod \mathrm{d}^3k/[(2\pi)^3 2E_{\mathbf{k}}]$. With this notational convention, the rate becomes

$$\begin{aligned}
\mathrm{d}\Gamma(\alpha \to \beta) &= \frac{\mathrm{d}P(\alpha \to \beta)}{T} \\
&= \frac{|S^{\square}_{\beta\alpha}|^2}{T}\Delta\beta \\
&= \left[\frac{|S_{\beta\alpha}|^2}{T}\left(\frac{1}{2E\,\Omega}\right)^{(N_\alpha+N_\beta)}\right]\Delta\beta \\
&= \frac{1}{T}(2\pi)^4\delta^4_{\Omega T}(p_\beta - p_\alpha)(2\pi)^4\delta^4_{\Omega T}(0)\,|\mathcal{M}_{\beta\alpha}|^2\left(\frac{1}{2E\Omega}\right)^{N_\alpha+N_\beta}\Delta\beta \\
&= \Omega(2\pi)^4\delta^4_{\Omega T}(p_\alpha - p_\beta)\frac{1}{(2E\Omega)^{N_\alpha}}|\mathcal{M}_{\beta\alpha}|^2\,\mathrm{d}\beta \\
&= \Omega^{1-N_\alpha}\left[\prod_{i\epsilon\alpha}\frac{1}{2E_i}\right]|\mathcal{M}_{\beta\alpha}|^2(2\pi)^4\delta^4_{\Omega T}(p_\alpha - p_\beta)\,\mathrm{d}\beta
\end{aligned} \tag{3.39}$$

where the product means a product over the particles in the initial state. Notice that the δ-function ensures that the final integral over β runs over a *finite* range of integration and so can never diverge unless $\mathcal{M}_{\beta\alpha}$ is singular for some momenta.

Consider now the cases of most present interest, with $N_\alpha = 1$, $N_\alpha = 2$, and $N_\alpha > 2$.

3.3.2.1 Decay processes: $N_\alpha = 1$

In the limit $\Omega \to \infty$ and $T \to \infty$ the decay rate for a single particle is explicitly independent of Ω and T, and is given by

$$\begin{aligned} \mathrm{d}\Gamma(\alpha \to \beta) &= \frac{1}{2E_\alpha} |\mathcal{M}_{\beta\alpha}|^2 (2\pi)^4 \delta^4(P_\alpha - P_\beta)\, \mathrm{d}\beta\,, \\ \mathrm{d}\beta &\equiv \prod_{f\in\beta} \frac{\mathrm{d}^3\mathbf{k}_f}{2E_{\mathbf{k}_f}(2\pi)^3} \end{aligned} \tag{3.40}$$

This result is not quite Lorentz-invariant, because of the $1/(2E_\alpha)$ in front. But indeed, it should not be Lorentz-invariant, since a fast-moving particle's lifetime should be extended by time dilation; the $1/(2E_\alpha)$ factor precisely generates this time dilation effect.

3.3.2.2 Two-body scattering: $N_\alpha = 2$

When $N_\alpha = 2$, $\mathrm{d}\Gamma$ is proportional to Ω^{-1}. Since the single-particle states are normalized with $\int_\Omega \mathrm{d}^3x |\psi(x)|^2 = 1$, the number density of particles in the box as seen by an incident particle is $n = \Omega^{-1}$. The fact that $\mathrm{d}\Gamma$ is inversely proportional to the volume reflects the property that in the absence of initial-state coherence the reaction rate is proportional to the number density of target particles.

It is convenient and conventional to remove this dependence on the number of particles by dividing out a factor proportional to the incident flux of particles. Define, then, the cross section, $\mathrm{d}\sigma$, by

$$\mathrm{d}\sigma(\alpha \to \beta) = \frac{\mathrm{d}\Gamma}{F}(\alpha \to \beta) \tag{3.41}$$

In this expression the denominator, F, is fixed by requiring that (a) $\mathrm{d}\sigma$ be Lorentz invariant; and (b) F, when evaluated in the rest-frame of either of the particles, equals the particle flux: $nv_{\mathrm{rel}} = v_{\mathrm{rel}}/\Omega$.

Our next task is to find the function, F, determined by these conditions. Condition (a) implies that F must transform the same way as $\mathrm{d}\Gamma$ does under Lorentz transformations. Because of our choice of state normalization and integration measure $\mathrm{d}\beta$, the final-state factors are already Lorentz-invariant. Invariance of the cross section is therefore ensured if $F = f/(4E_1E_2\Omega)$, where E_k denotes the energy of the particles in the initial two-particle state, $|\alpha\rangle$, and f is a Lorentz-invariant function chosen to satisfy condition (b). Since the relative velocity of two particles,

$$v_{\mathrm{rel}} = \sqrt{1 - \frac{m_1^2 m_2^2}{(p_1 \cdot p_2)^2}} \tag{3.42}$$

is Lorentz-invariant and the scalar $-p_1 \cdot p_2$ equals $E_1 E_2$ in the particle rest frame the solution is $f = -4V_{\rm rel}(p_1 \cdot p_2)$.

We are led in this way to the following expression for the two-body cross section:

$$\mathrm{d}\sigma(\alpha \to \beta) = \frac{|\mathcal{M}_{\beta\alpha}|^2}{f}(2\pi)^4\delta^4(p_\alpha - p_\beta)\,\mathrm{d}\beta \tag{3.43}$$

$$\text{with} \quad f = (-4p_1 \cdot p_2)v_{\rm rel} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \tag{3.44}$$

To be completely explicit, and for later convenience, we pause here to calculate the factor $(2\pi)^4\delta^4(p_\alpha - p_\beta)\,\mathrm{d}\beta$ for a two-body final state, $N_\beta = 2$, in the center-of-mass frame of the two bodies. Denote the final-state quantum numbers by primes. In an arbitrary frame $\delta^4(p_\alpha - p_\beta)\,\mathrm{d}\beta$ is

$$\begin{aligned}
(2\pi)^4\delta^4(p_\alpha - p_\beta)\,\mathrm{d}\beta &= (2\pi)^4\delta^4(p_\alpha - p_1' - p_2')\frac{\mathrm{d}^3\mathbf{p}_1'\mathrm{d}^3\mathbf{p}_2'}{(2\pi)^6 4E_1'E_2'} \\
&= 2\pi\delta(E_\alpha - E_1' - E_2')\frac{\mathrm{d}^3\mathbf{p}_1'}{(2\pi)^3 4E_1'E_2'}\bigg|_{\mathbf{p}_2' = \mathbf{p}_\alpha - \mathbf{p}_1'} \\
&= \frac{p_1'^2\,\mathrm{d}^2\Omega_1'}{(2\pi)^2 4E_1'E_2'|\mathrm{d}(E_1' + E_2')/\mathrm{d}p_1'|} \\
&= \frac{p_1'^3\,\mathrm{d}^2\Omega_1'}{16\pi^2(E_2'\mathbf{p}_1' - E_1'\mathbf{p}_2')\cdot\mathbf{p}_1'}
\end{aligned} \tag{3.45}$$

$\mathrm{d}^2\Omega_1' = \sin\theta'\mathrm{d}\,\theta'\,\mathrm{d}\phi'$ is the element of solid angle where θ' and ϕ' give the direction of the vector $\mathbf{p}_1'$. In the center-of-mass frame, $\mathbf{p}_1' = -\mathbf{p}_2'$ and $E_1' + E_2' = E_\alpha$, so

$$(2\pi)^4\delta^4(p_\alpha - p_\beta)\,\mathrm{d}\beta = \frac{p_1' d^2\Omega_1'}{16\pi^2 E_\alpha} \quad \text{(c.m.)} \tag{3.46}$$

In this case, the final-state integral consists of the sum over the direction of one of the two final-state particles.

3.3.2.3 Many-body collisions: $N_\alpha > 2$

The reaction rate per unit volume, $\mathrm{d}\Gamma/\Omega$, is proportional to Ω^{-N_α}. For N_α distinct particles in the initial state this again represents the incident-particle density that is expected for incoherent scattering:

$$\Omega^{-N_\alpha} = \prod_{i=1}^{N_\alpha} n_i \tag{3.47}$$

In this case the reaction rate per unit volume becomes

$$\frac{\mathrm{d}\Gamma(\alpha \to \beta)}{\Omega} = \prod_{i=1}^{N_\alpha} \left[\frac{n_i}{2E_i} \right] |\mathcal{M}_{\beta\alpha}|^2 (2\pi)^4 \delta^4(p_\alpha - p_\beta) \, \mathrm{d}\beta \tag{3.48}$$

Part II

Applications: leptons

4

Elementary boson decays

We wish to put the formalism of the previous chapters to use to describe the properties of the standard-model particles. Since many of the properties of the theory are simpler at higher energies we choose to do this by starting with the properties of the heavy bosons of the theory and then working our way down in energy towards more familiar particles. We also choose to focus here on the properties of the elementary bosons since these furnish among the simplest examples of the scattering formalism of the previous chapter.

Among the most basic particle properties are their masses and lifetimes. The masses of the gauge bosons of the theory are dealt with in previous (and in subsequent) chapters, so we concentrate here on their lifetimes.

4.1 Z^0 decay

4.1.1 Z^0 decay: preliminaries

We wish to compute within the standard-model the decay lifetime of the neutral electroweak gauge boson, Z^0, as a function of the parameters of the model. We do so using the perturbative framework of Chapter 3. The basic result of that chapter, for the present purposes, is given by Eq. (3.24) and Eq. (3.26),

$$\begin{aligned} S_{\beta\alpha} &= \delta_{\beta\alpha} - \mathrm{i}\,(2\pi)^4 \delta^4(p_\beta - p_\alpha)\mathcal{M}_{\beta\alpha}, \quad \text{with} \\ \mathcal{M}_{\beta\alpha} &= \langle\beta|\mathcal{H}_I(0)|\alpha\rangle + \frac{-\mathrm{i}}{2!}\int \mathrm{d}^4x \langle\beta|T\left[\mathcal{H}_I(x)\mathcal{H}_I(0)\right]|\alpha\rangle + \cdots \end{aligned} \quad (4.1)$$

We see that, in the absence of other effects, the dominant contribution to Z^0 decay will come from any interactions of the model for which the matrix element

$$\langle\beta|\mathcal{H}_I(0)|Z^0\rangle \neq 0 \quad (4.2)$$

for some final state $|\beta\rangle$ into which the Z^0 may kinematically decay. If there is no such final state or interaction then the dominant contribution must instead be second order, i.e.,

$$\frac{(-\mathrm{i})^2}{2!}\int \mathrm{d}^4x \langle\beta|T[\mathcal{H}_I(x)\mathcal{H}_I(x=0)]|Z^0\rangle \neq 0 \tag{4.3}$$

We must continue in this way until a nonzero result is eventually obtained.

If the Z^0 boson is to decay, it cannot appear in the final state. It follows that, in order to contribute to the matrix element of Eq. (4.2), any candidate interaction must be strictly linear in the field $Z_\mu(x)$. Inspection of the Z^0 couplings of Section 2.4 shows that there are only a few candidate interactions of this type. The candidates are $\mathcal{L}_{WWZ}$ of Eq. (2.76), $\mathcal{L}_{WWZ\gamma}$ of Eq. (2.82), and $\mathcal{L}_{nc}$ of Eq. (2.99). These would respectively describe the processes $Z^0 \to W^+W^-$, $Z^0 \to W^+W^-\gamma$, and $Z \to f\bar{f}$. Conservation of four-momentum implies that the sum of the masses in any candidate final state, $|\beta\rangle$, must be less than the mass of the Z^0. This rules out the first two processes, leaving only the decay of the Z^0 into a fermion–antifermion pair through a neutral-current weak interaction.

We now compute the resulting Z^0 decay rate. We do so in some detail in this section in order to develop some of the computational tools that are useful for general calculations of this sort. The first step is to identify the interaction Hamiltonian that corresponds to $\mathcal{L}_{nc}$. Since this term of the Lagrangian does not involve any time derivatives it is tempting to conclude that $\mathcal{H}_{nc} = -\mathcal{L}_{nc}$. This is not quite true in the present instance, however, because of the appearance of the time component of the gauge potential, $Z_0(x)$. The additional terms in $\mathcal{H}_{nc}$ that arise from this source are the analogs of the contact Coulomb interaction of quantum electrodynamics and are not even Lorentz invariant. At this point one might sensibly worry that they could potentially ruin the Lorentz invariance of the S-matrix being computed. Happily, their effect turns out to precisely cancel another source of Lorentz non-invariance that is encountered in Section 5.2. The upshot is that the naive relation, $\mathcal{H}_I = -\mathcal{L}_I$, may be used after all, so these terms are therefore ignored in all of what follows.

The interaction Hamiltonian density therefore is

$$\mathcal{H}_I = -\mathcal{L}_{nc} = -\mathrm{i}e_Z Z_\mu \overline{f}\gamma^\mu(g_V + g_A\gamma_5)f \tag{4.4}$$

in which the coupling constant is $e_Z = e/(\sin\theta_\mathrm{W}\cos\theta_\mathrm{W})$. The desired matrix element then becomes

$$\mathcal{M}(Z \to f\bar{f}) \;=\; \langle f(\mathbf{p},\sigma); \bar{f}(\mathbf{q},\zeta)|\mathcal{H}_I(0)|Z(\mathbf{k},\lambda)\rangle$$

$$= -\mathrm{i}e_Z\langle f(\mathbf{p},\sigma);\bar{f}(\mathbf{q},\zeta)|\overline{f}\gamma^\mu(g_V+g_A\gamma_5)fZ_\mu|Z(\mathbf{k},\lambda)\rangle$$
$$= -\mathrm{i}e_Z\langle 0|b_{p,\sigma}\bar{b}_{q,\zeta}\overline{f}\gamma^\mu(g_V+g_A\gamma_5)fZ_\mu a^*_{k,\lambda}|0\rangle \qquad (4.5)$$

This matrix element may be evaluated once the fields appearing within the interaction Hamiltonian are expressed in terms of creation and annihilation operators. These are given in Chapter 1 by Eq. (1.116) and Eq. (1.82):

$$Z_\mu(x) = \sum_{\lambda'=-1}^{1}\int\frac{\mathrm{d}^3k'}{2E_{\mathbf{k}'}(2\pi)^3}\left[\epsilon_\mu(\mathbf{k}',\lambda')a_{\mathbf{k}',\lambda'}\,\mathrm{e}^{\mathrm{i}k'x}+\mathrm{h.c.}\right] \qquad (4.6)$$

$$\psi(x) = \sum_{\sigma'=\pm\frac{1}{2}}\int\frac{\mathrm{d}^3p'}{2E_{\mathbf{p}'}(2\pi)^3}\left[u(\mathbf{p}',\sigma')b_{\mathbf{p}'\sigma'}\,\mathrm{e}^{\mathrm{i}p'x}+v(\mathbf{p}',\sigma')\bar{b}^*_{\mathbf{p}'\sigma'}\mathrm{e}^{-\mathrm{i}p'x}\right] \qquad (4.7)$$

The matrix element, Eq. (4.5), clearly gets contributions only from those terms in the expansion of the fields, Eq. (4.6) and Eq. (4.7), in which the destruction operator, a, appearing in $Z_\mu(x)$ destroys the incoming Z^0 boson, and the creation operators, b^* from $\bar{f}(x)$ and $\bar{b}^*$ from $f(x)$, create the fermion–antifermion pair. The matrix element then is

$$\mathcal{M}(Z\to f\bar{f}) = -\mathrm{i}e_Z\epsilon_\mu(\mathbf{k},\lambda)\bar{u}(\mathbf{p},\sigma)\gamma^\mu(g_V+g_A\gamma_5)v(\mathbf{q},\zeta) \qquad (4.8)$$

The differential decay rate is related to this result by Eq. (3.40):

$$2E_Z\mathrm{d}\Gamma[Z(\mathbf{k},\lambda)\to f\bar{f}] = \left|\mathcal{M}(Z\to f\bar{f})\right|^2(2\pi)^4\delta^4(k-p-q)\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{4E_{\mathbf{p}}E_{\mathbf{q}}(2\pi)^6}$$
$$= e_Z^2|\epsilon_\mu\bar{u}\gamma^\mu(g_V+g_A\gamma_5)v|^2$$
$$\times(2\pi)^4\delta^4(k-p-q)\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{4E_{\mathbf{p}}E_{\mathbf{q}}(2\pi)^6} \qquad (4.9)$$

The next step we must take is to evaluate the square of the matrix elements, $|\epsilon_\mu\bar{u}\gamma^\mu(g_V+g_A\gamma_5)v|^2$, that arise in this last expression. The evaluation proceeds differently depending on whether the particles involved are polarized or unpolarized. We consider the two cases of polarized and unpolarized initial Z^0 bosons separately.

4.1.2 Unpolarized Z^0 decay

Consider the decay of a sample of Z^0s that have no net polarization. We take the initial density matrix in the 3×3 spin space of the Z^0 meson to be the unit matrix:

$$\rho = \frac{1}{3}\sum_{\lambda=-1}^{1}|Z(\mathbf{k},\lambda)\rangle\langle Z(\mathbf{k},\lambda)| \qquad (4.10)$$

In order to proceed we need to generalize the S-matrix formalism slightly to include the case for which the initial state is not a pure state, $|\alpha\rangle$, but is rather described by a density matrix, ρ. In this case the probability of there being a transition to a final state, $|\beta\rangle$, is given by the trace

$$p(\beta) = \mathrm{tr}(\rho P_\beta) \tag{4.11}$$

in which $P_\beta = |\beta\rangle\langle\beta|$ is the projection operator onto the subspace of Hilbert space that is spanned by $|\beta\rangle$. In the special case where the initial state is a pure state, $\rho = |\alpha\rangle\langle\alpha|$, this reduces to the squared amplitude $|\langle\beta|\alpha\rangle|^2$. More generally, if the initial system could be in state $|i\rangle$ with probability P_i, then $\rho = \sum_i P_i|i\rangle\langle i|$ and $p(\beta) = \sum_i P_i|\langle\beta|i\rangle|^2$.

Using this expression, the differential decay rate for a sample of Z^0s that is described by the density matrix of Eq. (4.10) is then given by averaging the result of Eq. (4.9) over the initial Z^0 spin, λ. If, as is usually the case, the spins of the final fermions are not measured in the detector, then we must also sum over all possible final-state polarizations:

$$\mathrm{d}\Gamma[Z(\mathbf{k}) \to f\bar{f}] = \frac{1}{3}\sum_{\lambda=-1}^{1}\sum_{\sigma=\pm\frac{1}{2}}\sum_{\zeta=\pm\frac{1}{2}} \mathrm{d}\Gamma[Z(\mathbf{k},\lambda) \to f\bar{f}] \tag{4.12}$$

The spin sums may be evaluated using the polarization vector identity given by Eq. (1.119) and the spinor identities given in Eq. (1.99) and Eq. (1.100).

That part of the squared amplitude which involves the Z^0 polarization then becomes

$$\begin{aligned}
&\sum_{\lambda=-1}^{1} |\epsilon_\mu \bar{u}\gamma^\mu(g_V + g_A\gamma_5)v|^2 \\
&= \sum_{\lambda=-1}^{1} \epsilon_\mu(\mathbf{k},\lambda)\epsilon_\nu^*(\mathbf{k},\lambda)[\bar{u}\gamma^\mu(g_V + g_A\gamma_5)v][\bar{u}\gamma^\nu(g_V + g_A\gamma_5)v]^* \\
&= \left[\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}\right][\bar{u}\gamma^\mu(g_V + g_A\gamma_5)v][\bar{u}\gamma^\nu(g_V + g_A\gamma_5)v]^*
\end{aligned} \tag{4.13}$$

A similar manipulation may be performed for the fermion spinors, u and v, once the trick of rewriting the spinor product as a trace over Dirac matrices is used:

$$\bar{u}Mu = \sum_{ij} \bar{u}_i M_{ij} u_j = \mathrm{tr}[M(u\bar{u})] \tag{4.14}$$

In this last expression, $(u\bar{u})$ denotes the dyadic matrix whose matrix elements are given by $(u\bar{u})_{ij} = u_i\bar{u}_j$. Using this trick gives

$$[\bar{u}\gamma^\mu(g_V + g_A\gamma_5)v][\bar{u}\gamma^\nu(g_V + g_A\gamma_5)v]^*$$

$$
\begin{aligned}
&= -[\bar{u}\gamma^\mu(g_V + g_A\gamma_5)v][\bar{v}\gamma^\nu(g_V + g_A\gamma_5)u] \\
&= -\operatorname{tr}[\gamma^\mu(g_V + g_A\gamma_5)v\bar{v}\gamma^\nu(g_V + g_A\gamma_5)u\bar{u}] \qquad (4.15)
\end{aligned}
$$

The utility of this way of writing things is that the dyadics $u\bar{u}$ and $v\bar{v}$ have simple expressions, given by Eq. (1.99) and Eq. (1.100) respectively, when both of the spinors in the dyadic refer to the same particle. Performing the fermion spin sums using these expressions gives

$$\sum_{\sigma=\pm\frac{1}{2}} u(\mathbf{p},\sigma)\bar{u}(\mathbf{p},\sigma) = (m_f - i\not{p}) \qquad (4.16)$$

$$\sum_{\sigma=\pm\frac{1}{2}} v(\mathbf{q},\zeta)\bar{v}(\mathbf{q},\zeta) = (-m_f - i\not{q}) \qquad (4.17)$$

so summing the result of Eq. (4.15) over the fermion spins then gives

$$
\begin{aligned}
&\sum_{\sigma,\zeta=\pm\frac{1}{2}} [\bar{u}\gamma^\mu(g_V + g_A\gamma_5)v][\bar{u}\gamma^\nu(g_V + g_A\gamma_5)v]^* \\
&= \operatorname{tr}[\gamma^\mu(g_V + g_A\gamma_5)(m_f + i\not{q})\gamma^\nu(g_V + g_A\gamma_5)(m_f - i\not{p})] \qquad (4.18)
\end{aligned}
$$

4.1.3 Evaluating Dirac traces

Further progress requires the evaluation of various traces over Dirac matrices, of form $\operatorname{tr}[\gamma_{\mu_1}\dots\gamma_{\mu_n}]$ or $\operatorname{tr}[\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_n}]$. (Traces involving multiple γ_5 can always be handled by anti-commuting a γ_5 across the γ_μ which separate it from another, and using $\gamma_5\gamma_5 = \mathbf{1}$.)

There are two procedures for evaluating such traces. One procedure is to use repeatedly the identity, Eq. (C.56) from Appendix C and the cyclicity of the trace. Here we will present an alternative, in some respects more powerful, approach. Namely, we take advantage of their transformation properties under the (improper) Lorentz group.

The key observation is that the Dirac gamma-matrices, γ_μ, satisfy the following property:

$$D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu_\nu\gamma^\nu \qquad (4.19)$$

in which Λ^μ_ν is an arbitrary Lorentz transformation whose representation on spinor fields – c.f. Eq. (1.72) – is denoted $D(\Lambda)$. This implies that a trace over n gamma matrices is an *invariant tensor* of the Lorentz group. That is,

$$\Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}\operatorname{tr}[\gamma^{\nu_1}\dots\gamma^{\nu_n}] = \operatorname{tr}[\gamma^{\mu_1}\dots\gamma^{\mu_n}] \qquad (4.20)$$

for all Lorentz transformations. A trace that includes a factor of the matrix

γ_5 is similarly an invariant Lorentz *pseudotensor*:

$$\Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} \operatorname{tr}[\gamma_5 \gamma^{\nu_1} \ldots \gamma^{\nu_n}] = \det(\Lambda) \operatorname{tr}[\gamma_5 \gamma^{\mu_1} \ldots \gamma^{\mu_n}] \tag{4.21}$$

Now comes the main point: *any* such invariant tensor of the Lorentz group may be constructed from products of the invariant metric tensor, $\eta_{\mu\nu}$. Similarly, any invariant pseudotensor may be constructed from products of the metric tensor and an odd power of the completely antisymmetric *Levi–Civita symbol*, $\epsilon^{\mu\nu\lambda\rho}$. This last tensor is an invariant pseudotensor by virtue of the following identity that is satisfied by any 4×4 matrix:

$$\Lambda^{\mu_1}_{\nu_1} \Lambda^{\mu_2}_{\nu_2} \Lambda^{\mu_3}_{\nu_3} \Lambda^{\mu_4}_{\nu_4} \epsilon^{\nu_1\nu_2\nu_3\nu_4} = \det(\Lambda) \epsilon^{\mu_1\mu_2\mu_3\mu_4} \tag{4.22}$$

The traces may therefore be evaluated up to an overall multiplicative factor by writing down the most general combinations of metric and Levi–Civita tensors that has the same number and symmetry of indices. The multiplicative factor may then be chosen by evaluating the trace for a particularly simple choice of indices. This procedure may be illustrated as follows.

(i)

$$\operatorname{tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0 \qquad \text{if } n \text{ is odd.} \tag{4.23}$$

This is so because the result must be expressed as a combination of metrics and Levi–Civita symbols. However, each of these has an even number of indices. They cannot be combined into an object with an odd number of indices, so the result must vanish.

(ii)

$$\operatorname{tr}[\gamma_5 \gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0 \qquad \text{if } n \text{ is odd} \tag{4.24}$$

This result is an immediate consequence of the previous one since $\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ involves an even number of gamma matrices.

(iii)

$$\operatorname{tr}[\gamma^\mu \gamma^\nu] = 4\eta^{\mu\nu} \tag{4.25}$$

There is only one invariant second-rank symmetric tensor: the metric itself, $\eta^{\mu\nu}$. This establishes Eq. (4.25) up to the value of the proportionality constant. To fix this constant, choose the special case where $\mu = \nu = 1$, for which $\operatorname{tr}[(\gamma^1)^2] = \operatorname{tr}[1] = 4 = 4\eta^{11}$.

(iv)

$$\operatorname{tr}[\gamma_5 \gamma^\mu \gamma^\nu] = 0 \tag{4.26}$$

To see this, note that $\gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5$. The γ^μ may then be moved to the end by cyclicity of the trace, proving that the result must be antisymmetric in μ, ν. But the only second-rank invariant tensor is symmetric, so the answer must be zero.

(v)

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho] = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda}) \quad (4.27)$$

A fourth-rank invariant tensor (as opposed to pseudotensor) must be constructed from a sum of pairs of metric tensors. The three distinct pairs that are possible are those that appear on the right-hand side of Eq. (4.27). The coefficient of each of these terms is most easily determined by evaluating both sides with a simple choice for the indices. For example, the coefficient of the first term is determined to be 4 by the choice $\mu = \nu = 0$ and $\lambda = \rho = 1$. With this choice only the first term on the right-hand side is nonzero since the metric is diagonal, and the left-hand side becomes $\mathrm{tr}[(\gamma^0)^2(\gamma^1)^2] = \mathrm{tr}[-1] = -4 = 4\eta^{00}\eta^{11}$.

(vi)

$$\mathrm{tr}\,[\gamma_5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho] = 4\mathrm{i}\epsilon^{\mu\nu\lambda\rho} \quad (4.28)$$

The right-hand side of this result is again the unique fourth-rank invariant pseudotensor. Its coefficient is easily determined by the evaluating the choice $\mu = 0$, $\nu = 1$, $\lambda = 2$ and $\rho = 3$ for which the right-hand side is $4\mathrm{i}\epsilon^{0123} = 4\mathrm{i}$ (c.f. Eq. (1.33)) and the left-hand side is $\mathrm{tr}[\gamma_5\gamma^0\gamma^1\gamma^2\gamma^3] = \mathrm{tr}[\mathrm{i}(\gamma_5)^2] = 4\mathrm{i}$.

These results suffice for the present purposes. Traces involving more than four gamma matrices may be evaluated in a similar fashion.

4.1.4 Z^0 decay: formulae

With these results, we can evaluate the traces that arise in Eq. (4.18):

$$\begin{aligned}
& \mathrm{tr}[\gamma^\mu(g_V + g_A\gamma_5)(m_f + \mathrm{i}\not{q})\gamma^\nu(g_V + g_A\gamma_5)(m_f - \mathrm{i}\not{p})] \\
= \ & m_f^2\ \mathrm{tr}[\gamma^\mu(g_V + g_A\gamma_5)\gamma^\nu(g_V + g_A\gamma_5)] \\
& \quad + \mathrm{tr}[\gamma^\mu(g_V + g_A\gamma_5)\not{q}\gamma^\nu(g_V + g_A\gamma_5)\not{p}] \\
= \ & m_f^2\ \mathrm{tr}[\gamma^\mu(g_V^2 - g_A^2)\gamma^\nu] \\
& \quad + \mathrm{tr}[\gamma^\mu(g_V^2 + g_A^2 + 2g_Vg_A\gamma_5)\not{q}\gamma^\nu\not{p}] \\
= \ & 4m_f^2(g_V^2 - g_A^2)\eta^{\mu\nu} \\
& \quad + 4(g_V^2 + g_A^2)(q^\mu p^\nu + p^\mu q^\nu - \eta^{\mu\nu}p\cdot q) + 8\mathrm{i}g_Vg_A\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta \quad (4.29)
\end{aligned}$$

In going from the first to second expressions, we have dropped terms linear in m_f because they involve an odd number of gamma matrices, and therefore vanish in the trace. Between the second and third expressions, we have moved $(g_V + g_A\gamma_5)$ across either 1 or 2 intervening gamma matrices; as γ_5 anticommutes with each γ_α, its sign flips once for each intervening gamma matrix. The last step uses the trace identities numbered 3, 5, and 6 above.

Contracting against $[\eta_{\mu\nu} + k_\mu k_\nu / M_Z^2]$ from Eq. (4.13), and using (due to the δ function) $k_\mu = p_\mu + q_\mu$, the averaged matrix element squared $\overline{\mathcal{M}}^2 = \frac{1}{3}\sum_{\gamma\sigma\sigma} |\mathcal{M}|^2$ becomes

$$\overline{\mathcal{M}^2} = \frac{4e_Z^2}{3}\left[-2(g_V^2 + g_A^2)p\cdot q + 4m_f^2(g_V^2 - g_A^2) + \frac{4m_f^2}{M_Z^2} g_A^2 (m_f^2 - p\cdot q)\right] \quad (4.30)$$

This should be combined with Eq. (4.9) to give the polarization averaged differential decay rate,

$$\mathrm{d}\overline{\Gamma}[Z(\mathbf{k}) \to f\bar{f}] = \frac{1}{2k^0}\,\overline{\mathcal{M}^2}(2\pi)^4\delta^4(k-p-q)\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{2p^0 2q^0 (2\pi)^6} \quad (4.31)$$

Notice that this displays the proper Lorentz-transformation properties appropriate to a decay rate. All of the factors in Eq. (4.31) are manifestly Lorentz invariant except for the $1/2k^0$ prefactor. Since the Z^0-boson energy, k^0, is related to its rest mass, M_Z, and speed, v, by $k^0 = M_Z/\sqrt{1-v^2}$, it follows that in a general frame $\mathrm{d}\Gamma = \mathrm{d}\Gamma_{\text{rest}}\sqrt{1-v^2}$, implying the correct time dilation for the lifetime $\tau = 1/\Gamma$.

The decay rate in the Z^0 rest frame is found by making the substitution $k^\mu = (M_Z, \mathbf{0})$, which implies that $(2\pi)^4\delta^4(p+q-k) = 2\pi\delta(p^0 + q^0 - M_Z)$ $(2\pi)^3\delta^3(\mathbf{p}+\mathbf{q})$. It follows that the outgoing fermion and antifermion have a specific energy in the Z^0 rest frame. In this case, because the fermion and antifermion have equal masses, the outgoing fermion energies and momenta are

$$\begin{aligned} p^0 &= q^0 = M_Z/2 \\ |\mathbf{p}| &= |\mathbf{q}| = \sqrt{(p^0)^2 - m_f^2} = \frac{1}{2}\sqrt{M_Z^2 - 4m_f^2} \end{aligned} \quad (4.32)$$

This kind of delta-function distribution of outgoing-particle energies is characteristic of a two-body decay process.

The rest-frame differential decay rate may be simplified by using the delta functions to perform the integrals over $\mathbf{q}$ and $p = |\mathbf{p}|$. Suppose θ and ϕ are the polar angles that give the direction of the outgoing fermion in the Z^0 rest frame. Then the result $p\cdot q = -p^0q^0 + \mathbf{p}\cdot\mathbf{q} = -(p^0)^2 - \mathbf{p}^2 = m_f^2 - M_Z^2/2$ implies that the differential decay rate, $\mathrm{d}\Gamma$, for the decay of *unpolarized* Z^0s is independent of θ and ϕ. This is not surprising, as the initial state is rotationally symmetric.

The total and differential decay rate in the Z^0 rest frame is therefore,

$$\begin{aligned}\Gamma(Z \to f\bar{f}) &= 4\pi \frac{\mathrm{d}\Gamma}{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}(Z \to f\bar{f}) \\ &= \frac{e_Z^2}{12\pi} M_Z \left[(g_V^2 + g_A^2) + 2(g_V^2 - 2g_A^2)\frac{m_f^2}{M_Z^2} \right] \sqrt{1 - \frac{4m_f^2}{M_Z^2}} \quad (4.33)\end{aligned}$$

Before turning to the implications of this expression, a short aside is in order to compute the same quantity for a perfectly polarized sample of Z^0s.

4.1.5 Polarized Z^0 decay

The differential decay rate for polarized Z^0s is found using the same techniques. Assuming that the spin of the outgoing fermion and antifermion are not observed, the main difference is that there is in this case no sum over the initial Z^0 spin, and so the identity used in Eq. (4.13) is no longer available.

The differential decay rate is therefore still given by Eq. (4.31), with the difference that in this case

$$\begin{aligned}\overline{\mathcal{M}^2} \quad \to \quad \overline{\mathcal{M}^2}_{\mathrm{pol}} &\equiv e_Z^2 \sum_{\sigma,\zeta} |\epsilon_\mu \bar{u}\gamma^\mu (g_V + g_A\gamma_5) v|^2 \\ &= e_Z^2 \,\mathrm{tr}\,[\not{\epsilon}(g_V + g_A\gamma_5)(m_f + \mathrm{i}\not{q})\not{\epsilon}^*(g_V + g_A\gamma_5)(m_f - \mathrm{i}\not{p})] \quad (4.34)\end{aligned}$$

This trace may be evaluated using the techniques of Subsection 4.1.2. If the initial Z^0 is linearly polarized so that $\epsilon_\mu = \epsilon_\mu^*$ and $\epsilon \cdot \epsilon = 1$, then the result is

$$\overline{\mathcal{M}^2}_{\mathrm{pol}} = 4e_Z^2\{m_f^2(g_V^2 - g_A^2) - (g_V^2 + g_A^2)[p \cdot q - 2(\epsilon \cdot p)(\epsilon \cdot q)]\} \quad (4.35)$$

In the Z^0 rest frame, choose the direction of the Z^0 spin, ϵ_μ, to define the z-axis. Then taking the polar angles of the direction of the outgoing fermion to be (θ, ϕ), we have $\epsilon \cdot p = -\epsilon \cdot q = |\mathbf{p}| \cos\theta$. The resulting differential cross section is independent of ϕ, as is expected due to the axial symmetry of the initial state, but does depend on θ in the following way:

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\sin\theta\,\mathrm{d}\theta} &= 2\pi \frac{\mathrm{d}\Gamma}{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi} \\ &= \frac{e_Z^2 M_Z}{16\pi} \sqrt{1 - \frac{4m_f^2}{M_Z^2}} \\ &\quad \times \left[g_V^2 \left(1 - \cos^2\theta + \frac{4m_f^2}{M_Z^2}\cos^2\theta \right) + g_A^2 \left(1 - \frac{4m_f^2}{M_Z^2} \right)(1 - \cos^2\theta) \right] \\ &\qquad (4.36)\end{aligned}$$

As a check, notice that the integral of Eq. (4.36) over the interval $0 < \theta < \pi$ reproduces the same total decay rate as does the unpolarized result of Eq. (4.33), as it must.

4.1.5.1 The massless limit

Eq. (4.36) has a particularly simple physical interpretation in the limit of vanishing fermion mass, $m_f \to 0$. In this limit the differential decay rate becomes

$$\frac{d\Gamma}{\sin\theta\, d\theta} \approx \frac{e_Z^2}{16\pi} M_Z (g_V^2 + g_A^2)(1 - \cos^2\theta) \tag{4.37}$$

This result vanishes when the outgoing fermion comes out parallel or antiparallel to the initial Z^0 boson's polarization vector, ϵ_μ. This has a simple explanation in terms of the interplay between conservation of angular momentum and conservation of helicity (which is conserved in the limit of massless fermions).

The neutral-current interaction of Eq. (4.4) that is responsible for the Z^0 decay always pairs up fermions of definite helicity. That is, since this interaction Hamiltonian always involves the field combination $\bar{f}_L \gamma^\mu f_L$, it must always create a left-handed fermion together with the antiparticle to a left-handed fermion, which is a right-handed antifermion. The term which involves $\bar{f}_R \gamma^\mu f_R$ must similarly create a right-handed fermion and a left-handed antifermion. When the fermion or antifermion comes out along the direction of the initial Z^0 boson's polarization vector, then the total component of angular momentum along this direction is $J_z = \pm 1$. The angular momentum of the initial state along this direction is zero, however, so this decay configuration must be forbidden by conservation of angular momentum.

We return now to the main line of argument and explore the implications of Eq. (4.33) for the Z^0 decay width in the Z^0 rest frame.

4.1.6 Z^0 decay: applications

The rate for a Z^0 to decay into a particular species of fermion–antifermion pair, $f\bar{f}$, is given by Eq. (4.33):

$$\Gamma(Z \to f\bar{f}) \;=\; \frac{e_Z^2}{12\pi} M_Z \left[(g_V^2 + g_A^2) + 2(g_V^2 - 2g_A^2)\frac{m_f^2}{M_Z^2} \right] \sqrt{1 - \frac{4m_f^2}{M_Z^2}}$$

Table 4.1. *Fermion neutral-current coupling constants*

Fermion Type	T_3	Q	g_V	g_A	$(g_V^2+g_A^2)$
ν_e, ν_μ, ν_τ	$+\frac{1}{2}$	0	0.25	0.25	0.125
e, μ, τ	$-\frac{1}{2}$	–1	–0.0189	–0.25	0.0629
u, c, t	$+\frac{1}{2}$	$+\frac{2}{3}$	0.0959	0.25	0.0717
d, s, b	$-\frac{1}{2}$	$-\frac{1}{3}$	–0.1730	–0.25	0.0924

$$\approx \quad \frac{e_Z^2}{12\pi}(g_V^2 + g_A^2)M_Z \qquad \text{for } m_f^2 \ll M_Z^2 \tag{4.38}$$

The last line gives the approximate form for the decay rate to the extent that the mass ratio, m_f^2/M_Z^2, is negligible. This is a very good approximation for all of the fermions of the standard model except the top quark, which is anyway too heavy to appear as a decay product for the Z^0. The heaviest allowed decay product is the b quark, for which this mass ratio is $m_b^2/M_Z^2 \approx (5/90)^2 \approx 3 \times 10^{-3}$.

Given this formula for the Z^0 decay rate into differing fermion species, we may sum the contributions of all of the species of fermions in the standard model that are kinematically allowed to contribute, and thereby compute the total lifetime of the Z^0 within the standard model.

The coupling constants g_V and g_A in the standard model are given in terms of the third component of weak isospin, T_3, and electric charge, Q, by $g_V = \frac{1}{2}T_3 - Q\sin^2\theta_{\rm W}$ and $g_A = \frac{1}{2}T_3$. The corresponding couplings of the standard model fermions are tabulated in Table 4.1 (using $\sin^2\theta_{\rm W} = 0.2311$, see Appendix A.)

From this table it is straightforward to compute the total Z^0 lifetime within the standard model.

Rather than computing the decay rate for each species of fermion in the model, it is convenient to compute the total decay rate, $\Gamma_{\rm tot}$, and the fraction of Z^0 decays – or *branching fraction*, $B_f = \Gamma(Z \to f\bar{f})/\Gamma_{\rm tot}$ – that go into each particular fermion species. The reason for quoting results in this way is that the branching fraction is more reliably computable since it just depends on the numbers g_V and g_A and so is less subject to errors in the values of the experimentally determined couplings. The branching fractions are also much easier to measure experimentally.

Using the numerical values for parameters given in Appendix A, we find

$$\begin{aligned}\Gamma(Z \to f\bar{f}) &= \frac{\alpha}{3\sin^2\theta_{\rm W}\cos^2\theta_{\rm W}} M_Z(g_V^2 + g_A^2)N_{\rm c} \\ &= (1.336 \text{ GeV}) \cdot (g_V^2 + g_A^2)N_{\rm c}\end{aligned} \tag{4.39}$$

Table 4.2. *Computed and measured Z^0 branching fractions*

Fermion type	Computed	Measured
$\nu_e\bar{\nu}_e + \nu_\mu\bar{\nu}_\mu + \nu_\tau\bar{\nu}_\tau$	20.5%	$^{f1}(20.00 \pm 0.06)\%$
e^+e^-	3.45%	$(3.363 \pm 0.004)\%$
$\mu^+\mu^-$	3.45%	$(3.366 \pm 0.007)\%$
$\tau^+\tau^-$	3.45%	$(3.370 \pm 0.008)\%$
$b\bar{b}$	15.18%	$^{f2}(15.14 \pm 0.05)\%$
$u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c}$	54%	$^{f3}(54.76 \pm 0.06)\%$
Total width	2.44 GeV	(2.4952 ± 0.0023) GeV

$f1$: i.e. $Z \to$ unobserved final state.
$f2$: i.e. $Z \to B\bar{B}$.
$f3$: i.e. $Z \to$ non-$B\bar{B}$ hadrons.

The constant N_c here represents the number of colors that is appropriate to fermion type f. $N_c = 1$ must therefore be chosen when f is a lepton and $N_c = 3$ when f is a quark. $\alpha = e^2/(4\pi)$ denotes the electromagnetic fine-structure constant whose value we take at $\mu = M_Z$ to be $\alpha = 1/127.9$. The total Z^0 width then becomes

$$\begin{aligned} \Gamma_{\rm tot} &= (1.336 \text{ GeV})[3 \cdot (0.125) + 3 \cdot (0.0629) + 9 \cdot (0.0924) + 6 \cdot (0.0717)] \\ &= 2.44 \text{ GeV} \end{aligned} \tag{4.40}$$

The corresponding Z^0 lifetime is therefore

$$\tau(Z) = \frac{1}{\Gamma_{\rm tot}} = 2.69 \times 10^{-25} \text{ s} \tag{4.41}$$

Since even an ultra-relativistic particle can only travel around 10^{-18} m in this time, Z^0 particles decay well before they are seen, and so must be reconstructed in a detector from their decay products.

Some of the branching fractions are listed in Table 4.2.

There are several points to be made about these results.

(i) The factor $M_Z\sqrt{1 - (4m_f^2/M_Z^2)}$ in the decay rate has its origin in the integration over *phase space*. That is, it arises from the integration over the final-state momenta $\int \mathrm{d}^3q \; \mathrm{d}^3p$. For $m_f \ll M_Z$ this factor is $O(M_Z)$ since this is the typical size of the momentum available to the final-state particles. Since the total rate for a process is given by an integral over all of the final states that can take part, it is a rule of thumb that if two processes have equal-size couplings then the one with more available phase space (i.e. the one with more available final states) will have the larger rate.

The phase-space factor is proportional to the momentum available to the final fermions, and so tends to zero as m_f approaches $M_Z/2$, as is required by four-momentum conservation. In the event that m_f should be close to $M_Z/2$ this *phase-space suppression* can make the decay rate into fermion species f much smaller than might otherwise be expected.

(ii) The overall order of magnitude of the Z^0 decay rate can be estimated reasonably well without performing the entire detailed calculation. This may be done by keeping track of factors of coupling constants and the volume of phase space appropriate to the process of interest. Since factors of 2π are ubiquitous in these calculations, and since their omission can appreciably affect the size of the result, it is important also to keep track of these factors. There is a factor of $(2\pi)^4$ from the momentum-conserving delta function, a $(2\pi)^{-3}$ from each final state particle's momentum integration, and a (2π) from the $d\Omega$ angular integral for all but one of the final-state particles. (As will be seen later there can also be an additional factor of $(4\pi)^{-2}$ for each loop in the relevant Feynman diagram if such loops arise.)

The matrix element for Z^0 decay is clearly proportional to the coupling constants, $e_Z g_V$ and $e_Z g_A$, of the neutral-current interaction term in the Lagrangian. Since, for massless fermions, the total rate is found by *incoherently* adding the rate due to left-handed fermions to that for right-handed fermions, these two couplings must appear in the combination $e_Z^2(g_V^2+g_A^2)$ when fermion masses are neglected. The momentum integrals and squared matrix element therefore provide $\Gamma \sim [e_Z^2(g_V^2+g_A^2)/2\pi]X$. Here the phase-space volume, X, represents the result obtained by integrating over all final-state momenta, and whose value can be estimated by dimensional analysis. In the present example the volume of phase space is $O(M_Z)$ if m_f is not too close to M_Z, since M_Z is the typical energy available in the decay. Since Γ has dimensions of mass (in units with $\hbar = c = 1$), we get $\Gamma \sim [e_Z^2(g_V^2+g_A^2)/2\pi]M_Z$. Comparing this estimate with the full calculation, Eq. (4.38), shows that the estimate has only missed the purely numerical factor 1/6. This is typical of the accuracy of this type of simple order-of-magnitude estimate for two-body decays (see also Subsection 5.1.1).

(iii) The next feature of this result that bears remarking is that the decay width, Γ, is much smaller than the mass, M_Z, since $\Gamma/M_Z \approx$

$e_Z^2/(12\pi) \sim 10^{-2}$. This implies that the Z^0 is reasonably stable for a particle of its mass. As we shall see, Z^0s have been observed as a resonance in e^+e^- annihilation in high-energy electron–positron storage rings at CERN and at SLAC. The small size of the width of the Z^0 translates into the narrowness of the resulting resonance (see Subsection 6.4.1).

(iv) Inspection of the coupling constants, g_V and g_A, of the table shows that the neutrinos couple to the Z^0 with the largest strength of the fermions of the standard model. The vector coupling, g_V, of the remaining fermions is smaller due to a partial cancellation between $\frac{1}{2}T_3$ and $Q\sin^2\theta_{\rm W}$. This cancellation is most complete for the charged leptons, e, μ, and τ, and would be perfect if $\sin^2\theta_{\rm W}$ were exactly 0.25. As a result, the charged-lepton neutral-current couplings may be considered to first approximation as being purely axial in nature.

(v) Although the data measures the decay rate into hadrons, the decay rate we have computed is really the decay rate into a quark–antiquark pair. Since the observed hadrons are really bound states of the quarks and since no isolated quark has ever been directly detected, it is not immediately clear that the rate for producing quark–antiquark pairs should be related to the rate for Z^0 decays into hadrons.

The argument is discussed in more detail in later chapters, but the main point can be made schematically here. The key observation that makes this connection relies on the fact that the rate we have computed is an *inclusive* rate in the sense that only the total rate for producing hadrons is considered without trying to distinguish one type of hadron from another. The observable therefore has the form of a sum over all possible final hadronic states:

$$d\Gamma(Z \to \text{hadrons}) \propto \text{tr}[\rho(Z)P(\beta)] \tag{4.42}$$

in which $\rho(Z)$ denotes the density matrix that describes the sample of initial Z^0 bosons and $P(\beta)$ is the projection matrix within the Hilbert space onto the subspace spanned by the observables labeling the final state. For the total rate for producing hadrons this projection matrix is the projector, P_H, onto the entire subspace of strongly interacting particles: $P_H = \sum_h |h\rangle\langle h|$ for some basis of hadronic states, $|h\rangle$. Now an equally good basis for the subspace of hadronic states is formed by the set of color-neutral many-quark and - gluon states, $|q, g\rangle$, even though no particular hadron-mass eigenstate may be well approximated by any particular multi-quark and - gluon state. The projector that appears in Eq. (4.42) may therefore be written $P_H =$

$\sum_{q,g} |q,g\rangle\langle q,g|$. Once the projector is expressed in terms of a sum on quark and gluon states the calculation simplifies dramatically. This is because the strong coupling constant is small, $\alpha_3 = g_3^2/(4\pi) \approx 0.12$, when it is evaluated at the scale, $\mu \approx M_Z$, appropriate to a Z^0 decay. It follows that the contribution of each of the quark basis states to Eq. (4.42) is well approximated at these energies by perturbation theory. To lowest order the dominant quark states that contribute are precisely the quark–antiquark pairs for which we have performed the calculation.

This is the general pattern. Although rates that involve identifying specific strongly interacting final-state particles cannot be computed without detailed knowledge of the wavefunctions of these particles, inclusive quantities that simply involve a sum over all possible hadronic states (possibly with some prescribed value for a quantum number such as B that is conserved by the strong interactions) may be reliably calculated (at high energies) within perturbation theory.

(vi) The above table allows a comparison between the computed and observed widths for Z^0 decays into various final states. Since measurements of Z^0 properties have been made with great precision, this comparison furnishes a significant test of the standard model's accuracy. This is all the more true given the success of the model in describing other neutral-current phenomena (to be described in later chapters) using the same set of model parameters.

Before performing this comparison, however, we need to have an idea of the size of the potential corrections to the computed result. These corrections arise from processes that involve more than one power of the interaction Hamiltonian in Eq. (4.1). Corrections to the leading result can be expected to be suppressed in size by additional powers of the relevant coupling constants. For processes involving strongly interacting particles the typical size of a correction from additional strong interactions is $O(\alpha_3/(4\pi)) \approx 1\%$. All other things being equal, electroweak interactions can be expected to be even smaller since they are instead proportional to $O(\alpha/(4\pi \sin^2\theta_{\rm W})) \approx 3 \times 10^{-3}$.

This counting turns out to be modified somewhat in the case when the correction involves the exchange of a massless particle such as a photon or a gluon. Then the appearance of infrared mass singularities can introduce factors of the logarithm of a large mass ratio which can increase the size of the correction. For strongly interacting particles this kind of effect would increase the above estimate to $O\{[\alpha_3/(4\pi)] \log(M_Z^2/\Lambda_{\rm QCD}^2)\} \approx 5 \times 10^{-2}$. $\Lambda_{\rm QCD} \approx 150$ MeV is a scale that is typical of the strong interactions, and which is discussed at length in Chapter 8. The analogous estimate for the size of an electromagnetic correction is $O\{[\alpha/(4\pi)]\log(M_Z^2/m_f^2)\}$, which can be as large as 7×10^{-3} when f is an electron.

To summarize, we expect the uncertainty in the theoretical prediction to be in the neighborhood of around 5% for decays that involve strongly interacting quarks in the final state. Electromagnetic corrections should be the largest for Z^0 decays into electrons, for which they could be in the neighborhood of 1% of the lowest-order result. Electromagnetic corrections to decays into other final states should be smaller still. Since the neutrino does not interact strongly or electromagnetically, the prediction for the branching fraction into neutrino pairs should be accurate to within fractions of a percent.

These estimates would indicate that the uncertainty in the prediction, Eq. (4.41), for the total rate should be accurate to the level of roughly 0.13 GeV. The calculations of the hadronic branching fractions could also be in error at the few percent level. A real calculation of the size of these corrections is required in order to use the accuracy of the experiment to make a better test of the model. To date, such more precise comparisons between experiment and theory have been spectacularly successful. For instance, one of the best current experimental determinations of α_3 arises from the corrections it generates, in the width Γ_Z.

4.2 $W^\pm$ decays

The calculation of the decay properties of the charged electroweak boson, $W^\pm$, follows the same lines as did that for the Z^0. The total decay rate (but not necessarily the partial widths) of the W^+ and W^- are guaranteed to be equal to one another by the fact that CPT is a symmetry of the theory. The partial width $W^+\to\beta$ must also be equal to the conjugate process, $W^-\to\overline{\beta}$, to the extent that the relevant interactions preserve CP. Since our interest in this section is restricted to the dominant decays of the $W^\pm$ which are well-described within the Born approximation, which is CP perserving, it suffices to focus here on, say, the W^+.

4.2.1 $W^\pm$ decays: formulae

The first step is to identify the standard-model interactions for which the matrix element $\langle\beta|\mathcal{H}_I|W\rangle\neq 0$, since these can directly mediate the decay.

An argument that is identical to the one used for the Z^0 shows that the only such interaction is the charged-current fermion coupling of Eq. (2.87),

$$\begin{aligned}\mathcal{H}_I &= -\mathcal{L}_{\rm cc}\\ &= -\mathrm{i}e_W\Big[W^+_\mu(\bar\nu_m\gamma^\mu(1+\gamma_5)e_m + V_{nm}\bar u_m\gamma^\mu(1+\gamma_5)d_n)\\ &\quad +W^-_\mu(\bar e_m\gamma^\mu(1+\gamma_5)\nu_m + (V^\dagger)_{mn}\bar d_m\gamma^\mu(1+\gamma_5)u_n)\Big] \qquad (4.43)\end{aligned}$$

As before, e_W is the coupling constant $e_W \equiv \frac{g_2}{2\sqrt{2}} = \frac{e}{2\sqrt{2}\sin\theta_{\rm W}}$. The dominant $W^\pm$ decays are therefore predicted to be into fermion–antifermion pairs, like $W^+\to e^+\nu_e$, $W^+\to\bar s u$, etc.

4.2.1.1 Neglect of fermion masses

To the extent that all fermion masses may be neglected compared to M_W – an excellent approximation for the standard model given that the t quark is too heavy to allow the decay $W^+\to t\bar d, t\bar s$, or $t\bar b$ – no additional calculation is necessary to determine the differential rate for W^+ decays. This is because the differential decay rate for the process $W^+\to\bar f_m f_n$ may be directly lifted from the results of the previous section using the following translation table:

$$\begin{aligned} g_V, g_A &\to 1\\ M_Z &\to M_W\\ e_Z &\to e_W U_{nm}\\ \text{with}\qquad U_{nm} &= \begin{cases}\text{unit matrix, } \delta_{mn} & \text{if } f_m, f_n \text{ are leptons}\\ \text{KM matrix, } V_{nm} & \text{if } f_m, f_n \text{ are quarks}\end{cases} \qquad (4.44)\end{aligned}$$

The differential decay rate for the decay of a linearly polarized $W^\pm$ boson into a fermion–antifermion pair, $\bar f_m f_n$, (with the final fermion spins unmeasured) therefore is

$$\frac{\mathrm{d}\Gamma}{\sin\theta\mathrm{d}\theta}[W^+\to\bar f_m f_n] \approx \frac{e_W^2}{8\pi}|U_{nm}|^2 M_W N_{\rm c}(1-\cos^2\theta); \quad m_m^2, m_n^2 \ll M_W^2 \qquad (4.45)$$

The notation is the same as in the previous section. θ denotes the polar angle of the outgoing fermion in the rest frame of the decaying W^+ with the initial polarization direction chosen to define the z-axis.

The total (unpolarized) decay rate is similarly

$$\Gamma(W^+\to\bar f_m f_n) \approx \frac{e_W^2}{6\pi}|U_{nm}|^2 N_{\rm c} M_W; \quad m_m^2, m_n^2 \ll M_W^2 \qquad (4.46)$$

4.2.1.2 Non-vanishing fermion masses

Before exploring the implications of these expressions, we pause to generalize the above results to the case where the fermion masses are not neglected.

These generalizations may be straightforwardly proven using the techniques of the previous section.

The full expression for the differential decay rate for polarized W^+ bosons into an unpolarized fermion–antifermion pair, $\bar{f}_m f_n$, is

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\sin\theta\mathrm{d}\theta}[W^+ \to \bar{f}_m f_n] &= \frac{e_W^2}{8\pi}|U_{nm}|^2 N_\mathrm{c} M_W \sqrt{\left(1-\frac{\overline{m}^2}{M_W^2}\right)^2 - \frac{4m_m^2 m_n^2}{M_W^4}} \\ &\quad \times \left[1-\frac{\overline{m}^2}{M_W^2} - \left[\left(1-\frac{\overline{m}^2}{M_W^2}\right)^2 - \frac{4m_m^2 m_n^2}{M_W^4}\right]\cos^2\theta\right]\end{aligned} \tag{4.47}$$

in which $\overline{m}^2 = m_m^2 + m_n^2$ is the sum of the squared masses of the final spin-half particles.

Of particular interest is a special case of this last expression for which the mass of only one of the fermions is negligible. This is the result that is appropriate if f_n is a neutrino and the rate is desired as a function of the charged lepton mass. The results are

$$\begin{aligned}\frac{\mathrm{d}\Gamma}{\sin\theta\,\mathrm{d}\theta}[W \to \bar{f}_m f_n] &= \frac{e_W^2}{8\pi}|U_{nm}|^2 M_W N_\mathrm{c}\left[1-\left(1-\frac{m_m^2}{M_W^2}\right)\cos^2\theta\right] \\ &\quad \times \left(1-\frac{m_m^2}{M_W^2}\right)^2; \quad m_n^2 \ll m_m^2,\, M_W^2\end{aligned} \tag{4.48}$$

The total decay rate in this last case becomes

$$\Gamma(W \to \bar{f}_m f_n) = \frac{e_W^2}{6\pi}|U_{nm}|^2 M_W N_\mathrm{c}\left(1+\frac{m_m^2}{M_W^2}\right)\left(1-\frac{m_m^2}{M_W^2}\right)^2 \tag{4.49}$$

Since this is a two-body decay, four-momentum conservation implies that the spectrum of outgoing fermions has a delta-function distribution as a function of the energy of the outgoing fermions: $d\Gamma/dE \propto \delta(E-E_0)$. Since the fermions have equal and opposite three-momenta in the $W^\pm$ rest frame, it follows that their energies in this frame are given in terms of the particle masses by

$$\begin{aligned}p_m^0 &= \frac{M_W}{2} + \frac{m_m^2 - m_n^2}{2M_W} \\ p_n^0 &= \frac{M_W}{2} - \frac{m_m^2 - m_n^2}{2M_W}\end{aligned} \tag{4.50}$$

This process is clearly kinematically allowed provided only that the sum of fermion and antifermion masses is smaller than M_W.

4.2.2 $W^{\pm}$ decays: applications

The total decay rate and the branching fractions of the $W^{\pm}$ boson into fermion–antifermion pairs may now be computed within the standard model by applying these formulae.

It is convenient to normalize the total decay rate by the partial rate for the decay of a W^+ into an positron–neutrino pair:

$$\begin{aligned} \Gamma(W^+ \to e^+\nu_e) &= \frac{\alpha}{12\sin^2\theta_{\rm W}} M_W \\ &= (226 \text{ MeV}) \end{aligned} \quad (4.51)$$

In terms of this partial rate the total $W^{\pm}$ decay width therefore is

$$\begin{aligned} \Gamma_{\rm tot} &= \Gamma(W^+ \to e^+\nu_e)\left[3 + 3\sum_{n=1}^{2}\sum_{m=1}^{3}|V_{nm}|^2\right] \\ &= 9\Gamma(W^+ \to e^+\nu_e) \\ &= 2.04 \text{ GeV} \end{aligned} \quad (4.52)$$

The first factor of 3 in the square bracket in the first equation corresponds to the three families of leptons. The second 3 represents the three colors of each quark. The sum over 'up-type' quarks only runs over the first two generations because the top quark is too heavy to be a W^+ decay product. The second equality in Eq. (4.52) uses the unitarity of the Kobayashi–Maskawa matrix $\sum_{n=1}^{2}\sum_{m=1}^{3}|V_{nm}|^2 = \sum_{n=1}^{2}[VV^\dagger]_{nn} = 2$.

The decay width of Eq. (4.52) corresponds to a $W^{\pm}$ boson lifetime of

$$\tau(W^{\pm}) = \frac{1}{\Gamma_{\rm tot}} = 3.22 \times 10^{-25} \text{ s} \quad (4.53)$$

These particles clearly decay well before they may themselves be directly detected.

Many branching fractions are once again calculable as pure numbers, independent of model parameters. A few branching fractions are presented in Table 4.3.

The $W^{\pm}$ boson is again very long lived on the scale of its mass, and decays into leptons 33% of the time, and hadrons the rest of the time.

Since the strengths of the $W^{\pm}$ couplings to fermions do not depend on the fermions' electric charges or other such quantum numbers, to this approximation the only difference in the branching fractions into different species of particles is due to the existence of the Kobayashi–Maskawa matrix. As a result, the model predicts absolutely no difference among the decay rates into lepton pairs until masses and radiative (loop) corrections are included.

Table 4.3. *Computed and measured W^+ branching fractions*

Fermion type	Computed	Measured
$e^+\nu_e$	11.1%	$(10.75 \pm 0.13)\,\%$
$\mu^+\nu_\mu$	11.1%	$(10.57 \pm 0.15)\,\%$
$\tau^+\nu_\tau$	11.1%	$(11.25 \pm 0.20)\,\%$
Hadrons	66.7%	$(67.60 \pm 0.27)\,\%$
Total width	2.04 GeV	(2.085 ± 0.042) GeV

The size of these corrections are expected to be roughly the same size as for Z^0 decays – around a percent for electrons and less for μs and τs.

Like the Z^0 width, the $W^\pm$ width is larger than our theoretical estimate. This is because of (computable) positive $O(\alpha_3)$ corrections; the width is in good agreement with a more detailed calculation.

4.3 Higgs decays

The last massive elementary boson of the model to be considered here is the spinless Higgs particle. Here we compute its decay rate to two-body final states, which are in fact believed to dominate its decay rate. We will find, however, that the decay rate is suppressed, because the Higgs coupling is proportional to the relatively small mass of the final state particles. This means that a significant fraction of Higgs decays may occur via formally higher-order processes, which we will explore in the next chapter.

The interaction terms in the Lagrangian that are linear in the Higgs scalar, which can potentially mediate Higgs decay through a matrix element of the form $\langle\beta|\mathcal{H}_I|H\rangle \neq 0$, are of two types: the Higgs-fermion Yukawa couplings of Eq. (2.69), and the Higgs-electroweak boson interactions of Eq. (2.68):

$$\mathcal{H}_\mathrm{f} = -\mathcal{L}_\mathrm{Hf} = \sum_f \frac{m_f}{v}\bar{f}fH \tag{4.54}$$

and

$$\mathcal{H}_\mathrm{g} = -\mathcal{L}_\mathrm{H-g} = \frac{H}{v}\left(2M_W^2 W_\mu^+ W^{-\mu} + M_Z^2 Z_\mu Z^\mu\right) \tag{4.55}$$

The first of these can mediate the potential decay $H \to f\overline{f}$, and the second can mediate $H \to W^+W^-$ and $H \to Z^0Z^0$. Since we now know that $m_\mathrm{H} < 2M_W < 2M_Z$, the latter decay processes are not allowed, due to energy conservation; the final state energy is bounded below by the sum of the final particles' masses. Therefore we will focus exclusively on decays to fermions. Since the coupling strength is proportional to the mass of the fermion in question, the dominant decay is expected to be into the heaviest particle that is still light enough for the decay to be kinematically allowed, which is the b-quark. Therefore we expect Higgs decay to be predominantly to $b\bar{b}$ pairs.

The matrix element for this process may be evaluated using the expansion of the fields in terms of creation and annihilation operators, as in Subsection 4.1.3. The result is

$$\begin{aligned} \mathcal{M}(H \to f\bar{f}) &= \langle f(\mathbf{p},\sigma); \bar{f}(\mathbf{q},\zeta)|\mathcal{H}_f(0)|H(\mathbf{k})\rangle \\ &= \frac{m_f}{v}\langle f(\mathbf{p},\sigma); \bar{f}(\mathbf{q},\zeta)|\bar{f}fH|H(\mathbf{k})\rangle \\ &= \frac{m_f}{v}\bar{u}(\mathbf{p},\sigma)v(\mathbf{q},\zeta) \end{aligned} \tag{4.56}$$

Summing the square of this matrix element over final-state spins gives

$$\begin{aligned} \overline{\mathcal{M}^2}_f &\equiv \sum_{\sigma,\zeta}|\mathcal{M}(H \to f\bar{f})|^2 \\ &= -\frac{m_f^2}{v^2}\ \mathrm{tr}[(m_f - \mathrm{i}\not{p})(m_f + \mathrm{i}\not{q})] \\ &= \frac{4m_f^2}{v^2}(-p\cdot q - m_f^2) \end{aligned} \tag{4.57}$$

The differential decay rate is therefore

$$\begin{aligned} \mathrm{d}\Gamma(H \to f\bar{f}) &= \frac{1}{2k^0}\overline{\mathcal{M}^2}_f(2\pi)^4\delta^4(p+q-k)\frac{\mathrm{d}^3p\ \mathrm{d}^3q}{4p^0q^0(2\pi)^6} \\ &= \frac{2m_f^2}{v^2k^0}(-p\cdot q - m_f^2)(2\pi)^4\delta^4(p+q-k)\ \frac{\mathrm{d}^3p\ \mathrm{d}^3q}{4p^0q^0(2\pi)^6} \end{aligned} \tag{4.58}$$

In the Higgs rest frame the fermions clearly come out back-to-back and with energies equal to half the Higgs mass. Owing to the rotational symmetry of the problem, the decay probability in the rest frame is also independent of the direction of the outgoing fermion–antifermion pair. Including the potential sum over final-state color, the total Higgs decay rate into a particular flavor of fermion in the Higgs rest frame becomes

$$\begin{aligned} \Gamma(H \to f\bar{f}) &= \frac{m_\mathrm{H}}{8\pi}\left(\frac{m_f}{v}\right)^2 N_\mathrm{c}\left(1 - \frac{4m_f^2}{m_\mathrm{H}^2}\right)^{3/2} \\ &\approx \frac{m_\mathrm{H}}{8\pi}\left(\frac{m_f}{v}\right)^2 N_\mathrm{c} \quad \text{for } m_f \ll m_\mathrm{H} \end{aligned} \tag{4.59}$$

This decay rate is clearly very sensitive to the final-state fermion mass, and as advertised is largest for the heaviest fermions. Given the physical value of the Higgs mass, the largest contribution comes from the b quark. Neglecting the ratio $m_b^2/m_{\rm H}^2$ and using the present value of 126 GeV for $m_{\rm H}$ gives a Higgs partial width of

$$\Gamma(H \to b\bar{b}) = (3.5 \times 10^{-5})\, m_{\rm H} = 4.4\,\text{MeV} \tag{4.60}$$

which corresponds to a Higgs lifetime of $\tau(H) \simeq 1.5 \times 10^{-22}$ s. Such a particle would typically propagate less than 10^{-13} meters before decaying, which is far too short to be separated from the production point. Therefore the Higgs boson must be detected through its decay products.

4.4 Problems

[4.1] ***W* width at finite fermion mass**

Calculate the rate $\Gamma(W^- \to e^-\bar{\nu}_e)$ without assuming $m_e \ll M_w$. Use

$$\mathcal{L} = \mathrm{i}e_W W_\mu^- \bar{e}\gamma^\mu(g_V + g_A\gamma_5)\nu \;+\; \text{h.c.}\,, \qquad g_V = g_A = 1$$

[4.1.1] Show that the matrix element is

$$\langle e(p)\bar{\nu}_e(q)|\mathcal{H}|W(k)\rangle \;\;=\;\; -\mathrm{i}e_W\epsilon_\mu(k)\bar{u}(p)\gamma^\mu(g_V + g_A\gamma_5)v(q)$$

[4.1.2] Show that, if $m_\nu \neq 0$ and $m_e \neq 0$, we would get

$$\sum_{\sigma_1\sigma_2} |\epsilon_\mu(k)\bar{u}(p)\gamma^\mu(g_V + g_A\gamma_5)v(q)|^2 = \\ 4\left\{m_e m_\nu(g_V^2 - g_A^2) + (g_V^2 + g_A^2)\left[-p\cdot q + 2\epsilon\cdot p\;\epsilon\cdot q\right]\right\}$$

[4.1.3] Suppose the initial W^- is linearly polarized in the direction $\vec{e}$. In the W^- rest frame, show that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta} = \frac{e_W^2 M_W}{16\pi}(g_V^2 + g_A^2)\left[1 - \left(1 - \frac{m_e^2}{M_W^2}\right)\cos^2\theta\right]\left(1 - \frac{m_e^2}{M_W^2}\right)^2$$

where θ is the angle between $\vec{p}$ and $\vec{e}$. Assume $m_\nu = 0$.

[4.1.4] Show that the unpolarized rate is

$$\Gamma(W^- \to e^-\bar{\nu}_e) = \frac{e_W^2 M_W}{12\pi}(g_V^2 + g_A^2)\left(1 + \frac{m_e^2}{2M_W^2}\right)\left(1 - \frac{m_e^2}{M_W^2}\right)^2$$

[4.2] **Decay of the top quark**

Consider the top quark, with a mass of $m_t \simeq 173$ GeV.

[4.2.1] Identify the only interaction term in the Lagrangian which is linear in the top quark. Can a single insertion of this interaction term cause the top quark to decay? What are the decay products?

[4.2.2] Write an expression for the matrix element for the dominant top-quark decay process.

[4.2.3] Find a compact expression for the square of the matrix element, summing over final-state spin or helicity states and averaging over the initial top-quark helicity state.

[4.2.4] Compute the width of the top quark. Neglect the masses of any other fermions in comparison to the top-quark mass, but treat the masses of W and Z bosons as comparable to the top-quark mass. You should be able to find an analytic expression for the decay rate. Then, substitute in physical values and express the answer in GeV.

[4.3] **Heavy Higgs decay and the 'Equivalence Theorem'**

Before its discovery, experimentalists had to search for the Higgs boson in a wide mass range, including masses heavy enough to allow the decay to W^+W^- or to two Z bosons. This calculation is also instructive in what it teaches about how longitudinal spin-states of the gauge bosons behave in the limit of weak gauge couplings.

[4.3.1] Show that the matrix element for $H \to W^+W^-$ can be written

$$\begin{aligned}\mathcal{M}(H \to W^+W^-) &= \langle W^+(\mathbf{p},\sigma); W^-(\mathbf{q},\zeta)|\mathcal{H}_g(0)|H(\mathbf{k})\rangle \\ &= \frac{2M_W^2}{v}\epsilon^*_\mu(\mathbf{q},\zeta)\epsilon^{*\mu}(\mathbf{p},\sigma)\,. \end{aligned} \tag{4.61}$$

[4.3.2] Square and sum over gauge boson spins using Eq. (1.119), and show that the differential decay rate is

$$\begin{aligned} d\Gamma(H \to W^+W^-) &= \frac{1}{2k^0}\left(\frac{2M_W^2}{v}\right)^2\left[2 + \frac{(p\cdot q)^2}{M_W^4}\right] \times \\ &\qquad (2\pi)^4\delta^4(p+q-k)\,\frac{d^3p\, d^3q}{2p^0 2q^0(2\pi)^6}\,, \end{aligned} \tag{4.62}$$

and the total Higgs decay rate in the Higgs rest frame is

$$\begin{aligned}\Gamma(H \to W^+W^-) &= \frac{m_H}{16\pi}\,\frac{m_H^2}{v^2}\left[1-4\left(\frac{M_W^2}{m_H^2}\right)+12\left(\frac{M_W^2}{m_H^2}\right)^2\right]\sqrt{1-\frac{4M_W^2}{m_H^2}} \\ &\approx \frac{m_H}{16\pi}\,\frac{m_H^2}{v^2} \quad \text{for } M_W \ll m_H\,. \end{aligned} \tag{4.63}$$

An interesting feature about this result is that it is proportional to the square of the *Higgs* mass, $(m_H/v)^2$, rather than to the square of the mass of the final-state particle as was the case for Higgs decays into fermions, Eq. (4.59). Since the ratio $(m_H/v)^2$ is essentially the Higgs self coupling, λ, (*c.f.* Eq. (2.37)), this reflects the fact that the longitudinal component of the massive gauge bosons originate as components of the scalar doublet that are 'eaten' by the Higgs mechanism. This allows a simple interpretation for the two terms in the square bracket in the last equality of Eq. (4.62): The factor of 2 corresponds to the two transverse polarization states of the W meson which couple with a strength proportional to the gauge coupling, $g_2 \approx M_W/v$, and the remaining term represents the momentum-dependent coupling of the longitudinal 'Goldstone mode' that couples proportional to the Higgs self-coupling as above. This ability to compute the interactions of longitudinally polarized gauge bosons in terms of the scalars that they've eaten is sometimes called the "Equivalence Theorem."

[4.3.3] Compare the width you find to the width to decay to a $b\bar{b}$ pair, for a Higgs mass of 180 GeV.

[4.3.4] For the decay $H \to Z^0 Z^0$ the only difference is the statistics of the final two-boson state. Show that the final result is

$$\Gamma(H \to Z^0 Z^0) = \frac{1}{2}\Gamma(H \to W^+ W^-)\Big|_{M_W \to M_Z}, \tag{4.64}$$

so the rate for decay into Z^0's is the same as it is separately for each of the two states, W_1 or W_2, that make up the $W^\pm$ (with the substitution of M_Z for M_W).

Hint: Since the two final Z^0 particles are identical, there are three changes to be made:

(i) In the evaluation of the matrix element, $\mathcal{M}$, of Eq. (4.61), there is a factor of $\frac{1}{2}$ because the numerical coefficient of the interaction Hamiltonian, Eq. (4.55), is half as large for Z^0's as it is for $W^\pm$'s.

(ii) There is a factor of 2 because there are now two ways the fields, $Z_\mu Z^\mu$, can create the two particles in the final state.

(iii) Finally, the range of the final integration over the solid angle of the direction of one of the outgoing particles need only be 2π steradians rather than the usual 4π since it is impossible to distinguish which Z boson heads in which direction.

[4.3.5] What would be the observed final-state particles from these decay processes, considering that the W and Z bosons are themselves unstable? How might these decays be distinguished experimentally from other processes that produce the same final states?

[4.4] **Gamma-matrix identities**

Prove the following useful formulae involving gamma matrices. You should need only the relations, $\gamma^\mu\gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu$, and $\eta^\mu_\mu = 4$.

$$
\begin{aligned}
\not{k}\not{k} &= k^2 \\
\not{k}\not{p}\not{k} &= 2p\cdot k\not{k} - k^2\not{p} \\
\gamma^\mu\gamma_\mu &= 4 \\
\gamma^\mu\not{k}\gamma_\mu &= -2\not{k} \\
\gamma^\mu\not{p}\not{k}\gamma_\mu &= 4p\cdot k \\
\gamma^\mu\not{p}\not{k}\not{q}\gamma_\mu &= -2\not{q}\not{k}\not{p}
\end{aligned}
$$

5

Leptonic weak interactions: decays

The next simplest application of the standard model to understanding the properties of the observed elementary particles is to compute the decay lifetimes of the other weakly interacting particles of the model. The only remaining particles that do not participate in the strong interactions are the leptonic fermions. This chapter is devoted to a calculation of their decay properties.

The purpose of this chapter is threefold. Two of these are straightforward. Lepton decays furnish our first example of a "second-order" decay that proceeds via a virtual particle, and so provide a good motivation for a full description of the Feynman rules of the theory. This calculation also provides some insight into the observed properties of real leptons and so allows more contact with experimentally accessible quantities. Indeed, the weak decays of the known fundamental particles provide much of our current information concerning the electroweak couplings. The third and final motivation is to provide the first illustration of the utility of the technique of effective Lagrangians for computing the virtual effects of heavy particles.

5.1 Qualitative features

The six flavors of fundamental leptons are $e, \mu, \tau, \nu_e, \nu_\mu$, and ν_τ. Four of these are absolutely stable in the standard model by virtue of exact or extremely good approximate conservation laws of the model. The stable species are the three neutrino types and the electron. They are absolutely stable because they are each the lightest particles that carry nonzero values for a conserved quantum number. They cannot decay because any potential decay product would have to be lighter than the decaying particle, and would have to carry a nonzero value for the quantum number in question. No such particles exist by the very assumption that the original particle is the lightest one that carries this quantum number.

The neutrinos, being massless, are the lightest particles that carry the appropriate lepton number: L_e, L_μ, and L_τ. The electron is similarly stable since it is the lightest particle that carries electric charge. One might wonder whether the stability of the neutrinos might be suspect because of the anomalies in the conservation of L_e, L_μ, and L_τ discussed in Subsection 2.5.3. This turns out not to be so; there are three anomaly-free quantum numbers in the standard model, $L_e - L_\mu$, $L_\mu - L_\tau$, and $B - L$, which (together with the fact that the neutrinos are all lighter than any particle carrying a nonzero baryon number) are sufficient to ensure the stability of all three neutrinos. Note however that if the standard model is enlarged by relaxing renormalizability to allow dimension-5 operators, as discussed in Chapter 10, then lepton numbers are generically violated and the neutrinos may not be absolutely stable. However, estimates of their lifetimes are so long that the question of their stability is not experimentally interesting.

The decay properties of the remaining two leptons, μ and τ, are computed here. The first step is to determine which interactions are responsible for their decays. In this regard, notice that in the absence of the charged-current fermion interactions, the symmetry group of the leptonic sector of the standard model would be larger than $U_e(1)\times U_\mu(1)\times U_\tau(1)$. They would in particular include a separate symmetry under the rotation of the muon, say, by a phase that is independent of the muon neutrino. This would imply the separate conservation of the number of muon (minus antimuons) and muon neutrinos (minus muon antineutrinos), and so imply the stability of the μ (and similarly of the τ). It follows that any process which results in μ or τ decay must necessarily involve the lepton charged-current interaction at least once. The dominant contribution to the decay will be that one which involves the fewest interactions.

For definiteness consider τ^- decay. In order to involve the minimum number of interactions – one – there must be a potential decay product, $|\beta\rangle$, for which the matrix element

$$\langle\beta|\mathcal{H}_{\rm cc}(0)|\tau^-\rangle = -\frac{{\rm i}g_2}{2\sqrt{2}}\langle\beta|W_\mu^+\,\overline{\nu}_\tau\gamma^\mu(1+\gamma_5)\tau|\tau^-\rangle \neq 0 \qquad (5.1)$$

The only state, $|\beta\rangle$, for which this matrix element is not zero is $|\beta\rangle = |W^-;\nu_\tau\rangle$. This cannot be a decay product for a τ^- particle, since the W^- boson is more massive than is the τ^-.

It follows that τ^-- (and μ^--) decay must arise at at least second order in the perturbative expansion of Eq. (3.26). That is, the dominant contribution to a decay $\tau^- \to \beta$ must proceed via the matrix element

$$\mathcal{M}(\tau^- \to \beta) = \frac{-\mathrm{i}}{2!} \int \mathrm{d}^4x \langle \beta | T[\mathcal{H}_{\mathrm{I}}(x)\mathcal{H}_{\mathrm{I}}(0)] | \tau^- \rangle + \cdots \tag{5.2}$$

if not at higher order.

From the above considerations, the interaction term which destroys the τ^- particle must be the charged-current Hamiltonian appearing in Eq. (5.1). Besides destroying the τ^-, this interaction also creates W^- and ν_τ particles. The second interaction term must therefore destroy the created W^- particle, in order to produce a final state that involves only particles that are less massive than the initial τ^- lepton. As is demonstrated in some detail in Section 4.2, the only interaction that can cause a transition from a W^- particle to lighter particles is once again the charged-current fermion interaction. These interactions destroy the W^- and produce a fermion and an antifermion, for instance, $e^-\overline{\nu}_e$. The τ-neutrino that is produced when the τ lepton is destroyed must appear in the final state to carry off the nonzero L_τ of the original τ^-.

The dominant decay processes must therefore be three-body decays, of the form $\tau^- \to f_m\overline{f_n}\nu_\tau$, in which f_m and f_n are any two fermions that are related to one another through the charged-current interactions and which are lighter than the initial τ^-. The rate for this decay is given to first approximation by Eq. (5.2), in which the relevant terms in the interaction Hamiltonian are

$$\mathcal{H}_{\mathrm{I}} \subset -\mathrm{i}e_W \left(W_\mu^+ \overline{\nu}_\tau \gamma^\mu (1+\gamma_5)\tau + \sum_{mn} U_{mn}^* W_\mu^- \overline{f}_n \gamma^\mu (1+\gamma_5) f_m \right) \tag{5.3}$$

The matrix U_{mn} in this expression is the same as that used in Section 4.2 and represents the unit matrix, δ_{mn}, if f_m and f_n are leptons and the Kobayashi–Maskawa matrix, V_{mn} if they are quarks. As before, $e_W \equiv g_2/2\sqrt{2} = e/(2\sqrt{2}\,\sin\theta_{\mathrm{W}})$.

Many of the properties of μ^- and τ^- decays follow from these general observations before any detailed calculations are performed.

5.1.1 μ^- decays

(i) For μ^- decays there is only one combination of three fermions for which the sum of the masses is smaller than the μ^- mass itself, and is therefore kinematically allowed. The three particles in the final state are completely determined by the conservation laws for the decay. The final state must include the electron, e^-, since this is the only negatively charged particle that is lighter than the μ^-. Conservation of L_e and L_μ then dictate that the remaining two fermions must be ν_μ to carry off the initial muon number, and $\overline{\nu}_e$ to cancel the electron number of the final electron. The dominant decay must therefore be

$$\mu^- \to e^- \overline{\nu}_e \nu_\mu \tag{5.4}$$

(ii) Of the three particles in the final state, only the electron is detectable (without heroic efforts) since the neutrinos interact so weakly as to make them easily leave any detector without interacting at all. The observable of most interest is therefore the decay rate as a function of the final electron's quantum numbers. Since Eq. (5.4) is a three-body decay, the electron can emerge with a continuous range of energies, with energy conservation satisfied by having the remainder of the initial muon's energy shared by the remaining neutrinos. One of the goals of the next section is to compute the number of electrons of any given energy that emerge from a sample of decaying muons.

(iii) Counting the coupling constants and (2π)s associated with the decay rate allows a simple estimate of its size and so of the muon lifetime. This estimate compares reasonably well with the more detailed calculation to follow. The decay involves two insertions of $\mathcal{H}_I$, which is linear in g_2, so it follows that $\mathcal{M} \propto g_2^2$.

There is another factor that must be included as well, the suppression associated with the necessity to produce and destroy a virtual W^- boson. As is justified in more detail in what follows, this suppression is given by a factor of $1/M_W^2$ in the amplitude. This factor is the relativistic analog of the familiar energy denominators of non-relativistic quantum mechanical perturbation theory (c.f. Eq. (1.34) for example.)

Including this factor gives the estimate $\mathcal{M} \sim g_2^2/M_W^2$. Since the typical energy available to the final particles in the muon rest frame is m_μ, the integral of the squared matrix element over phase space may be estimated by including the appropriate power of m_μ. Since our estimate for $|\mathcal{M}|^2$ has dimension M^{-4} and the decay rate is dimension M, the power of m_μ required by dimensional analysis is m_μ^5.

It remains to find the power of (2π) arising from the phase space integration. Each of the three integrals over final particle momenta introduces $(2\pi)^{-3}$, but there is a $(2\pi)^4$ from the energy and momentum conserving delta function. There are two independent solid angle integrations, each contributing $\sim(2\pi)$. The total power of (2π) is therefore $(2\pi)^{-3}$.

The decay rate is therefore of order

$$\begin{aligned}\Gamma(\mu^- \to e^-\overline{\nu}_e\nu_\mu) &\sim \frac{|\mathcal{M}|^2 m_\mu^5}{(2\pi)^3}\\ &\sim \frac{g_2^4}{(2\pi)^3}\,\frac{m_\mu^5}{M_W^4}\\ &\sim \frac{2\alpha^2}{\pi\sin^4\theta_{\rm W}}\,\frac{m_\mu^5}{M_W^4}\\ &\sim 2\times 10^{-15}\, m_\mu\\ &\sim 2\times 10^{-16}\ \text{GeV}\end{aligned} \tag{5.5}$$

corresponding to a lifetime of $\tau(\mu) \sim 3\times10^{-9}$ s. Unlike in the previous chapter we take here the value $\alpha \approx 1/137$ for the electromagnetic fine structure constant that is appropriate to low energies compared to the weak scale. (The scale dependence of α is discussed in Section 7.4.)

This differs from the measured lifetime of $\tau_{\rm exp} = 2.2 \times 10^{-6}$ s by some three orders of magnitude, motivating the more careful calculation performed below. Notice that an extremely relativistic particle with a lifetime of a microsecond can travel several hundred meters before decaying. Muons therefore live long enough to escape the region immediately surrounding the interaction point and can enter the surrounding detector for observation.

(iv) The branching fractions for differing final states in μ^- decay may also be simply estimated. As argued above, the decay into $e\nu\overline{\nu}$ is the only one that may proceed to second order in the interactions of the model. This will therefore have a branching fraction of essentially $\approx$100%.

There will be other decay products available, and so deviations from the 100% branching fraction, to the extent that higher-order processes are possible. One such process that arises once three powers of the interaction Hamiltonian are allowed is the decay $\mu^- \to e^-\overline{\nu}_e\nu_\mu\gamma$, in which a photon is emitted by the initial muon or by the final electron. Apart from all of the factors of coupling constants that already arise for the purely three-body decay, the matrix element for emitting an additional photon involves an extra factor of the electromagnetic coupling, e, and the phase-space integration involves an extra $(2\pi)/(2\pi)^3$ (the numerator from an angular integration, the denominator from the momentum integration measure).

The estimate of the branching fraction for decays with an extra photon in the final state is therefore $B(\mu^- \to e\overline{\nu}\nu\gamma) \sim e^2/(2\pi)^2 \sim$

2×10^{-3}. Having the photon pair produce an electron–positron pair – $\mu^- \to e^- e^+ e^- \overline{\nu}\nu$ – would bear another factor of $e^2/(2\pi)^2$ of suppression, for a branching fraction $\sim 10^{-5}$. These are in rough agreement with the measured branching fractions

$$\begin{aligned} B_{\rm exp}(\mu^- \to e^- \overline{\nu}\nu) &= \sim 100\% \\ B_{\rm exp}(\mu^- \to e^- \overline{\nu}\nu\gamma) &= (1.4 \pm 0.4)\% \\ B_{\rm exp}(\mu^- \to e^- \overline{\nu}\nu e^+ e^-) &= (3.4 \pm 0.4) \times 10^{-5} \end{aligned} \tag{5.6}$$

The photon branching fraction is larger than our estimate because it turns out to be enhanced by a factor of $\log(m_\mu/m_e)$, for reasons we will discuss in Subsection 6.7.2.

5.1.2 τ^- decays

(i) The tau lepton differs from the muon only through the size of its mass. The arguments of the preceding section for muons therefore apply equally well to taus to the extent that they do not rely crucially on the value of the initial lepton's mass.

Because of its larger mass, the τ lepton can decay at second order in $\mathcal{H}_{\rm cc}$ to many more three-fermion final states than could the relatively light muon. It has two purely leptonic decays: $\tau^- \to e^- \overline{\nu}_e \nu_\tau$ and $\tau^- \to \mu^- \overline{\nu}_\mu \nu_\tau$. At the quark level it can also decay into either $\tau^- \to \overline{u} d \nu_\tau$ or $\tau^- \to \overline{u} s \nu_\tau$. All of the other quark combinations are ruled out by energy conservation. (The $\overline{c}s$ and $\overline{c}d$ combination superficially appears to be just possible since the charm and strange quark masses sum to a value just below the tau mass. The c quark nevertheless cannot contribute to τ decays because real hadrons are bound states of these quarks and all of these bound states that contain a single charmed quark are too heavy to be produced as a tau decay product.)

(ii) The decay rate for the τ is easily estimated given the decay rate of the muon. All of the estimates that lead to Eq. (5.5) apply equally well to tau decays and so the same result holds here. In particular the ratio of the tau decay rate to the muon decay rate must scale like the fifth power of the ratio of their masses. Using the experimentally observed muon lifetime therefore gives

$$\begin{aligned} \tau(\tau) &\sim \left(\frac{m_\mu}{m_\tau}\right)^5 \tau(\mu) \\ &\sim 1.6 \times 10^{-12} \text{ s} \end{aligned} \tag{5.7}$$

This estimate is just about right (to within the accuracy of the estimate) since the observed τ lifetime is $(2.906 \pm 0.010) \times 10^{-13}$ s. The factor of about 5 is expected because of the five allowed decay products for the τ (the $\overline{u}d$ counts as 3 because there are three available colors.) A relativistic particle with this lifetime can travel a tenth of a millimeter or more before decaying, which can be a visible displacement with the proper experimental setup.

(iii) As is noted above, the tau meson has more decay channels open to it than does the muon just by virtue of the fact that it is so much heavier. Predictions for the τ^- branching fractions may be made simply by counting the degrees of freedom available in each channel. These predictions are quite robust since they rely on few (if any) of the details of any potentially poorly-measured parameters of the model. One of these predictions, that follows simply from the observation that the tau decays via a virtual W boson and from the universal nature of the couplings of the W, is that the branching fraction for the two leptonic decays must be equal. This and other predictions are summarized in the following.

$$\begin{aligned}
B(\tau \to e\overline{\nu}\nu) &= B(\tau \to \mu\overline{\nu}\nu) \\
&\approx \frac{1}{2 + 3(|V_{ud}|^2 + |V_{us}|^2)} \\
&\approx 20\% \\
B(\tau \to \text{strange hadrons}) &\approx B(\tau \to \overline{u}s\nu) \\
&\approx \frac{3|V_{us}|^2}{2 + 3(|V_{ud}|^2 + |V_{us}|^2)} \\
&\approx 2\% \\
B(\tau \to \text{non-strange hadrons}) &\approx B(\tau \to \overline{u}d\nu) \\
&\approx \frac{3|V_{ud}|^2}{2 + 3(|V_{ud}|^2 + |V_{us}|^2)} \\
&\approx 58\%
\end{aligned} \tag{5.8}$$

By way of comparison, the corresponding experimental numbers are

$$\begin{aligned}
B_{\text{exp}}(\tau \to e\overline{\nu}\nu) &= (17.83 \pm 0.04)\% \\
B_{\text{exp}}(\tau \to \mu\overline{\nu}\nu) &= (17.41 \pm 0.04)\% \\
B_{\text{exp}}(\tau \to \text{strange hadrons}) &\approx (2.875 \pm 0.050)\% \\
B_{\text{exp}}(\tau \to \text{non-strange hadrons}) &\approx (61.85 \pm .11)\%
\end{aligned} \tag{5.9}$$

The agreement is within the accuracy of the estimate. Note that the branching fraction to hadrons is systematically higher than the leading-order prediction. A more detailed calculation turns out to show that the rates for the $\overline{u}d$ and $\overline{u}s$ decay modes receive a positive $O(\alpha_3)$ correction; so the difference from the naive branching fraction estimates allows a determination of the size of α_3. The size of α_3 determined in this way differs from the determination from the width of the Z^0 boson, discussed at the end of Subsection 4.1.6; but this is expected, as we will discuss in Subsection 7.4.1, and the difference turns out to agree with the predictions of the standard model.

5.2 The calculation

Consider, for simplicity, the decay $\tau^- \to \nu_\tau \overline{f}_m f_n$ in which none of the initial or final polarizations are measured. We must evaluate the matrix element, Eq. (5.2), using the interaction Hamiltonian, Eq. (5.3). The first term in the interaction Hamiltonian is responsible for destroying the initial τ^- meson and creating the ν_τ. The second term creates the final fermion–antifermion pair, $\overline{f}_m f_n$. Since the total amplitude is the product of two powers of the interaction Hamiltonian, there are two types of contributions, corresponding to which interaction Hamiltonian destroys the τ particle and which creates the $\overline{f}_m f_n$ pair. Each of these turns out to contribute equally to the total amplitude, and so we compute here only one of them and multiply the result by two.

It is convenient to write out the action of the interaction Hamiltonian separately for the τ, ν_τ, W-boson, and $\overline{f}_m f_n$ sectors of the Hilbert space. To this end write the initial and final states as

$$\begin{aligned} |\tau^-\rangle &= |\tau^-\rangle_\tau \otimes |0\rangle_W \otimes |0\rangle_f \\ |\nu_\tau; \overline{f}_m; f_n\rangle &= |\nu_\tau\rangle_\tau \otimes |0\rangle_W \otimes |\overline{f}_m; f_n\rangle_f \end{aligned} \tag{5.10}$$

The utility of writing this dependence out explicitly is that the desired matrix element, Eq. (5.2), then factorizes into three parts, which may be dealt with separately:

$$\begin{aligned} -\mathrm{i}\mathcal{M} &= \frac{(-\mathrm{i})^2}{2!} \int \mathrm{d}^4 x \langle \nu_\tau(\mathbf{l}); \overline{f}_m(\mathbf{q}); f_n(\mathbf{p}) | T\left[\mathcal{H}_{\mathrm{cc}}(x)\mathcal{H}_{\mathrm{cc}}(0)\right] | \tau(\mathbf{k})\rangle \\ &= 2\frac{(-\mathrm{i})^2}{2!}(-e_W^2 U_{mn}^*) \int \mathrm{d}^4 x \, A^\mu(\mathbf{k}, \mathbf{l}; x) G_{\mu\nu}(x) B^\nu(\mathbf{q}, \mathbf{p}) \end{aligned} \tag{5.11}$$

in which the factors A^μ, B^ν, and $G_{\mu\nu}$ are defined by

$$A^\mu(\mathbf{k}, \mathbf{l}; x) = {}_\tau\langle \nu_\tau(\mathbf{l})| \left[\overline{\nu}_\tau \gamma^\mu (1+\gamma_5)\tau\right](x) \, |\tau(\mathbf{k})\rangle_\tau$$

$$B^\nu(\mathbf{q},\mathbf{p}) = {}_f\langle \overline{f}_m(\mathbf{q}); f_n(\mathbf{p})| \left[\overline{f}_n \gamma^\nu (1+\gamma_5) f_m\right](0)|0\rangle_f$$
$$G_{\mu\nu}(x) = {}_W\langle 0|T\left[W^+_\mu(x)W^-_\nu(0)\right]|0\rangle_W \tag{5.12}$$

The first factor of 2 in the last line of Eq. (5.11) is the factor of discussed in the opening paragraph of this section, which corresponds to the two ways in which the interaction terms can destroy the τ: the τ can be destroyed by the interaction at spacetime point x, or by the one at 0.

These matrix elements are evaluated by expanding each field in terms of its creation and annihilation operators and then evaluating the resulting matrix elements of these operators. The evaluation of matrix elements A^μ and B^ν only involves initial or final states and so closely parallels the evaluation of those matrix elements performed in previous chapters. They give

$$A^\mu(\mathbf{k},\mathbf{l};x) = \overline{u}_\nu(\mathbf{l})\gamma^\mu(1+\gamma_5)u_\tau(\mathbf{k})\,\mathrm{e}^{\mathrm{i}(k-l)x}$$
$$B^\nu(\mathbf{q},\mathbf{p}) = \overline{u}_n(\mathbf{p})\gamma^\nu(1+\gamma_5)v_m(\mathbf{q}) \tag{5.13}$$

5.2.1 The W propagator

The matrix element $G_{\mu\nu}(x)$ of the W field operators that arises in Eq. (5.12) is called the W *propagator*. It is determined completely by the properties of the W bosons that contribute to it as intermediate states. Its evaluation requires a little more care and is the topic of the present aside.

$G_{\mu\nu}(x)$ may be evaluated by inserting a complete set of one-particle W-states. (For notational simplicity we drop the ubiquitous subscript "W" on the Hilbert-space state vectors in this section but it is implicit in all formulae.) Recalling the definition, Eq. (3.17), of the time-ordering operation, T, and inserting a complete set of 1-particle W-boson states between the operators, gives

$$\begin{aligned} G_{\mu\nu}(x) &= \langle 0|T\left[W^+_\mu(x)W^-_\nu(0)\right]|0\rangle \\ &= \sum_{\lambda=-1}^{1}\int \frac{\mathrm{d}^3r}{2E_{\mathbf{r}}(2\pi)^3}\Big[\langle 0|W^+_\mu(x)|W^+(\mathbf{r},\lambda)\rangle\langle W^+(\mathbf{r},\lambda)|W^-_\nu(0)|0\rangle\theta(x^0) \\ &\quad + \langle 0|W^-_\nu(0)|W^-(\mathbf{r},\lambda)\rangle\langle W^-(\mathbf{r},\lambda)|W^+_\mu(x)|0\rangle\theta(-x^0)\Big] \\ &= \sum_{\lambda=-1}^{1}\int \frac{\mathrm{d}^3r}{2E_r(2\pi)^3}\left[\,\epsilon_\mu\epsilon^*_\nu(\mathbf{r},\lambda)\,\mathrm{e}^{\mathrm{i}r'x}\theta(x^0) + \epsilon_\nu\epsilon^*_\mu(\mathbf{r},\lambda)\,\mathrm{e}^{-\mathrm{i}r'x}\theta(-x^0)\,\right] \end{aligned} \tag{5.14}$$

Here $\mathbf{r}$ is the momentum of the inserted state, and $E_{\mathbf{r}} = \sqrt{\mathbf{r}^2 + M_W^2}$ is its energy. The four-vector, r'_μ, that appears in the phase $\mathrm{e}^{\pm \mathrm{i}r'x}$ is defined with timelike component given by $r'^0 \equiv E_{\mathbf{r}}$.

$\theta(x)$ is the step function that is unity when its argument is positive and is zero when its argument is negative. For the present purposes the following integral representation proves convenient:

$$\theta(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x\omega}}{\omega - \mathrm{i}\epsilon'}\, \mathrm{d}\omega \tag{5.15}$$

ϵ' here denotes a positive infinitesimal that is to be taken to zero at the end of the calculation.

The spin sum may be evaluated using Eq. (1.119):

$$\begin{aligned} \Pi_{\mu\nu}(\mathbf{r}, E_r) &\equiv \sum_{\lambda=-1}^{1} \epsilon_\mu(\mathbf{r}, \lambda)\epsilon^*_\nu(\mathbf{r}, \lambda) \\ &= \eta_{\mu\nu} + r'_\mu r'_\nu / M_W^2 \end{aligned} \tag{5.16}$$

Putting this spin sum, and Eq. (5.15), into Eq. (5.14) gives the desired expression for the propagator. After performing a change of integration variables in this result in order to put the coefficient of both of the step functions, $\theta(x^0)$ and $\theta(-x^0)$, into a common form, the result becomes

$$\begin{aligned} G_{\mu\nu}(x) &= -\mathrm{i} \int \frac{\mathrm{d}^3 r}{(2\pi)^3} \frac{\mathrm{d}\omega}{2\pi} \Pi_{\mu\nu}(\mathbf{r}, E_{\mathbf{r}})\, \mathrm{e}^{\mathrm{i}\mathbf{r}\cdot\mathbf{x} - \mathrm{i}\omega x^0} \\ &\qquad \times \frac{1}{2E_{\mathbf{r}}} \left[\frac{1}{E_{\mathbf{r}} - \omega - \mathrm{i}\epsilon'} + \frac{1}{E_{\mathbf{r}} + \omega - \mathrm{i}\epsilon'} \right] \\ &= -\mathrm{i} \int \frac{\mathrm{d}^4 r}{(2\pi)^4} \frac{\Pi_{\mu\nu}(\mathbf{r}, E_{\mathbf{r}})}{r^2 + M_W^2 - i\epsilon}\, \mathrm{e}^{\mathrm{i}rx} \end{aligned} \tag{5.17}$$

The four-vector r_μ (as opposed to r'_μ of Eq. (5.14)) that appears in the last equality is defined with time component r^0 equal to the integration variable, ω (as opposed to $E_{\mathbf{r}}$). The infinitesimal, ϵ, appearing here has been rescaled from the original infinitesimal, ϵ', of Eq. (5.15) by $\epsilon \equiv 2E_{\mathbf{r}}\, \epsilon' > 0$.

5.2.1.1 Lorentz covariance: an aside

This last expression is almost, but not quite, covariant with respect to Lorentz transformations. The qualification comes because the polarization "tensor," $\Pi_{\mu\nu}(\mathbf{r}, E_{\mathbf{r}})$, depends on the variable $E_{\mathbf{r}}$ rather than the time component of r_μ: $r^0 = \omega$. This is something of an embarrassment since it would seem to imply a loss of Lorentz invariance for the S-matrix element that is being computed! Happily enough this particular failure of Lorentz invariance is just what is required to cancel another source that has been glossed over up until this point. (See, however, the discussion in Section 4.1.) This other source of Lorentz non-invariance would have arisen in Eq. (5.3) if the interaction Hamiltonian had been properly identified. The result that is implicit in this equation is that the charged-current interaction Hamiltonian, $\mathcal{H}_{\rm cc}$, is related to the interaction Lagrangian, $\mathcal{L}_{\rm cc}$, by $\mathcal{H}_{\rm cc} = -\mathcal{L}_{\rm cc}$ as would usually be the case for a non-derivative interaction. This relation does not hold for the couplings of gauge potentials, however, as is perhaps more familiar in quantum electrodynamics where the non-derivative coupling, $\mathcal{L}_{\rm I} = A_\mu J^\mu$, produces a non-covariant Coulomb contact interaction in the Hamiltonian.

It is beyond the scope of this book to detail how these two sources of Lorentz non-invariance cancel one another out (see, however, Problem 5.3 for an illustration of what is involved). The final result is simple to state, however. The full calculation is equivalent to neglecting the difference between $\mathcal{H}_{\rm I}$ and $-\mathcal{L}_{\rm I}$ and replacing the naively time-ordered W propagator of Eq. (5.14) through Eq. (5.17) by the covariant expression obtained by replacing $\Pi_{\mu\nu}(\mathbf{r}, E_{\mathbf{r}})$ in Eq. (5.17) by $\Pi_{\mu\nu}(r) \equiv \Pi_{\mu\nu}(\mathbf{r}, r^0)$:

$$\begin{aligned}\tilde{G}_{\mu\nu}(x) &= \langle 0|T^* \left[W_\mu^+(x) W_\nu^-(0)\right] |0\rangle \\ &= -{\rm i} \int \frac{{\rm d}^4 r}{(2\pi)^4} \Pi_{\mu\nu}(r) \frac{{\rm e}^{{\rm i}rx}}{r^2 + M_W^2 - {\rm i}\epsilon} \end{aligned} \tag{5.18}$$

The upshot of this aside is that $\tilde{G}_{\mu\nu}(x)$ of Eq. (5.18) must be used in the amplitude of Eq. (5.11) rather than $G_{\mu\nu}(x)$.

5.3 The large-mass expansion

The results for A^μ, B^ν, and $\tilde{G}_{\mu\nu}$ accumulated above, in Eq. (5.13) and Eq. (5.18), may now be combined in Eq. (5.11) for the matrix element $\mathcal{M}(\tau^- \to \nu_\tau \overline{f}_m f_n)$. The x integral may be performed and gives a momentum conserving delta function, $\int {\rm d}^4 x \ {\rm e}^{{\rm i}(k-l+r)x} = (2\pi)^4 \delta^4(k - l + r)$, giving the following result for $\mathcal{M}$:

$$\begin{aligned}\mathcal{M}(\tau \to \nu_\tau \overline{f}_m f_n) &= e_W^2 U_{mn}^* [\overline{u}_\nu(\mathbf{l}) \gamma^\mu (1+\gamma_5) u_\tau(\mathbf{k})][\overline{u}_n(\mathbf{p}) \gamma^\nu (1+\gamma_5) v_m(\mathbf{q})] \\ &\quad \times \left[\frac{\eta_{\mu\nu} + (k-l)_\mu (k-l)_\nu / M_W^2}{(k-l)^2 + M_W^2 - {\rm i}\epsilon}\right] \end{aligned} \tag{5.19}$$

All of the techniques of the previous sections may be brought to bear on this expression to evaluate the corresponding τ^- decay rate. A great simplification is possible at this point, however, if it is agreed to neglect any contributions that are suppressed relative to the dominant one by powers of the small quantity $m_\tau^2/M_W^2 \approx 5 \times 10^{-4}$. (The approximation is even better for the muon where this ratio is 300 times smaller.) In this case the entire W-boson propagator, as represented by the last square bracket in Eq. (5.19), may be expanded in inverse powers of M_W^2:

$$\frac{\eta_{\mu\nu} + (k-l)_\mu (k-l)_\nu / M_W^2}{(k-l)^2 + M_W^2 - i\epsilon} \approx \frac{\eta_{\mu\nu}}{M_W^2} \tag{5.20}$$

since in the rest frame of the tau meson all of the components of the four-momentum, $k-l$, are at most equal to m_τ.

Within this approximation, the matrix element simplifies to

$$\mathcal{M}(\tau \to \nu_\tau \overline{f}_m f_n) = \frac{e_W^2 U_{mn}^*}{M_W^2} [\overline{u}_\nu(\mathbf{l})\gamma^\mu(1+\gamma_5)u_\tau(\mathbf{k})][\overline{u}_n(\mathbf{p})\gamma_\mu(1+\gamma_5)v_m(\mathbf{q})] \tag{5.21}$$

It is conventional to denote the coupling combination e_W^2/M_W^2, that appears in this expression, by $G_{\rm F}/\sqrt{2}$. i.e.

$$\frac{G_{\rm F}}{\sqrt{2}} = \frac{e_W^2}{M_W^2} = \frac{g_2^2}{8M_W^2} = \frac{1}{2v^2} \tag{5.22}$$

The constant $G_{\rm F} = 1.1664 \times 10^{-5}$ $({\rm GeV})^{-2}$ obtained in this way is for historical reasons called the *Fermi coupling constant*. Indeed, it is the measurement of $G_{\rm F}$ through comparison of the predicted and measured muon lifetimes that fixes the value of the Higgs *v.e.v.*, v, that is quoted in Appendix A.

Returning to the matrix element, Eq. (5.21), averaging over the two initial spin states of the τ and summing over spins of the final fermions gives

$$\begin{aligned}\overline{\mathcal{M}^2} &= \frac{1}{2}\sum_{\rm spins} |M(\tau \to \nu_\tau \overline{f}_m f_n)|^2 \\ &= \frac{G_{\rm F}^2}{4}|U_{mn}|^2 M^{\mu\nu}(k,l) N_{\mu\nu}(p,q) \end{aligned} \tag{5.23}$$

in which the quantities $M^{\mu\nu}$ and $N^{\mu\nu}$ denote traces over Dirac matrices:

$$\begin{aligned} M^{\mu\nu}(k,l) &\equiv {\rm tr}[\gamma^\mu(1+\gamma_5)u_\tau\overline{u}_\tau\gamma^\nu(1+\gamma_5)u_\nu\overline{u}_\nu] \\ &= {\rm tr}[\gamma^\mu(1+\gamma_5)(m_\tau - {\rm i}\not{k})\gamma^\nu(1+\gamma_5)(-{\rm i}\not{l})] \\ &= 8\Big[(\eta^{\mu\nu} k\cdot l - k^\mu l^\nu - k^\nu l^\mu) - {\rm i}\epsilon^{\mu\nu\lambda\rho}k_\lambda l_\rho\Big] \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} N_{\mu\nu}(p,q) &\equiv {\rm tr}[\gamma_\mu(1+\gamma_5)v_m\overline{v}_m\gamma_\nu(1+\gamma_5)u_n\overline{u}_n] \\ &= -\,{\rm tr}\,[\gamma^\mu(1+\gamma_5)(m_m + {\rm i}\not{q})\gamma^\nu(1+\gamma_5)(m_n - {\rm i}\not{p})] \\ &= 8\Big[(\eta^{\mu\nu} p\cdot q - p^\mu q^\nu - p^\nu q^\mu) - {\rm i}\epsilon^{\mu\nu\lambda\rho}q_\lambda p_\rho\Big] \end{aligned} \tag{5.25}$$

Contracting $M^{\mu\nu}$ with $N_{\mu\nu}$ (using the identity $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\lambda\rho} = 2(\delta^\alpha_\rho\delta^\beta_\lambda - \delta^\alpha_\lambda\delta^\beta_\rho)$) finally gives the simple result,

$$M^{\mu\nu}(k,l)N_{\mu\nu}(q,p) = 256(l\cdot p)(k\cdot q) \tag{5.26}$$

Combining all of the above formulae gives the following differential decay rate:

$$d\Gamma(\tau\to\nu_\tau\overline{f}_m f_n) = \frac{64G_{\rm F}^2|U_{mn}|^2}{2k^0}(l\cdot p)(k\cdot q)(2\pi)^4\delta^4(p+q+l-k)\,\frac{d^3l\,d^3p\,d^3q}{8l^0p^0q^0(2\pi)^9} \tag{5.27}$$

This expression must now be integrated over phase space to produce the desired differential or total decay rate. We will perform the integration assuming that both the ν_τ and one of the other leptons is massless. This treatment is relevant for $\tau^-\to\mu^-\overline{\nu}_\mu\nu_\tau$ and for $\mu^-\to e^-\overline{\nu}_e\nu_\mu$, and is not too bad for decays of ν_τ into quarks (where g_3^2 effects which we will not compute are anyway more important than the up quark mass).

Consider for concreteness the case of a purely leptonic τ decay: $\tau^-\to\mu^-\overline{\nu}_\mu\nu_\tau$. We now compute this differential decay rate as a function of the final muon's energy and mass.

Since we do not observe the neutrino momenta l and q, it is convenient to integrate over them first. Moving everything else outside the q,l integrations leaves the integral,

$$I_{\mu\nu}(k,p) \equiv \int l_\mu q_\nu(2\pi)^4\delta^4(l+p+q-k)\frac{d^3l\,d^3q}{4l^0q^0(2\pi)^6} \tag{5.28}$$

which we must evaluate. $I_{\mu\nu}(k,p)$ as defined is a second-rank tensor that is a function of k^μ and p^μ only through the combination $w^\mu \equiv (k-p)^\mu$. The most general possible form for such a tensor is

$$I_{\mu\nu}(w) = Aw^2\eta_{\mu\nu} + Bw_\mu w_\nu \tag{5.29}$$

in which A and B can a priori be arbitrary functions of the Lorentz-invariant variable $w^2 = w^\mu w_\mu$. The scalar variables A and B are much more convenient to deal with than is the full tensor $I_{\mu\nu}$, since they are Lorentz-*in*variant (as opposed to being *co*variant), and so may be simply evaluated in the most convenient reference frame.

A and B turn out to be both determined in terms of a single scalar integral

$$I(w) \equiv \int (2\pi)^4\delta^4(l+q-w)\frac{d^3l\,d^3q}{4l^0q^0(2\pi)^6} \tag{5.30}$$

To see this, note first that

$$\eta^{\mu\nu}I_{\mu\nu}(w) \quad = \quad Aw^2\eta^{\mu\nu}\eta_{\mu\nu} + B\eta^{\mu\nu}w_\mu w_\nu = (4A+B)w^2$$

$$
\begin{aligned}
&= \int l\cdot q\,(2\pi)^4\delta^4(l+q-w)\frac{\mathrm{d}^3l\,\mathrm{d}^3q}{4l^0q^0(2\pi)^6} \\
&= \frac{w^2}{2}I(w)
\end{aligned}
\tag{5.31}
$$

In the last equality the identity $w^2 = (l+q)^2 = 2l\cdot q$ is used, which relies on four-momentum conservation as well as the masslessness of the neutrinos, $l^2 = q^2 = 0$. Similarly,

$$
\begin{aligned}
w^\mu w^\nu I_{\mu\nu}(w) &= Aw^2 w^\mu w^\nu \eta_{\mu\nu} + Bw^\mu w^\nu w_\mu w_\nu = (A+B)(w^2)^2 \\
&= \int (w\cdot q)(w\cdot l)(2\pi)^4\delta^4(l+q-w)\frac{\mathrm{d}^3l\,\mathrm{d}^3q}{4l^0q^0(2\pi)^6} \\
&= \int (l\cdot q)^2(2\pi)^4\delta^4(l+q-w)\frac{\mathrm{d}^3l\,\mathrm{d}^3q}{4l^0q^0(2\pi)^6} \\
&= \frac{(w^2)^2}{4}I(w)
\end{aligned}
\tag{5.32}
$$

which uses $w\cdot l = (l+q)\cdot l = l\cdot q = w\cdot q$. These equations may be solved for A and B, giving $B = 2A = I/6$.

It remains to evaluate $I(w)$. It is first convenient to perform the l integration using the following identity, which holds for any Lorentz-invariant integrand:

$$
\int \frac{\mathrm{d}^3l}{2l^0(2\pi)^3} = \int \frac{\mathrm{d}^4l}{(2\pi)^4}2\pi\delta(l^2+m^2)\theta(l^0)
\tag{5.33}
$$

for our case, $m^2 = 0$. Then

$$
\begin{aligned}
I(w) &= \int (2\pi)^4\delta^4(l+q-w)2\pi\delta(l^2)\theta(l^0)\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{\mathrm{d}^3q}{2q^0(2\pi)^3} \\
&= \int 2\pi\delta[(w-q)^2]\theta[w^0-q^0]\frac{\mathrm{d}^3q}{2q^0(2\pi)^3}
\end{aligned}
\tag{5.34}
$$

Notice that the four-vector, w^μ, is timelike, since

$$
w^2 = (k-p)^2 = (l+q)^2 = 2l\cdot q = -2|\mathbf{q}||\mathbf{l}|(1-\cos\theta) \le 0
\tag{5.35}
$$

θ here represents the angle between the three-vectors $\mathbf{l}$ and $\mathbf{q}$. There must therefore be a reference frame in which the three-vector components of w vanish, $\mathbf{w} = \mathbf{0}$. Define $w^0 = E$ in this frame. Then, also in the same frame, $w^2 = -E^2$ and $(w-q)^2 = w^2 - 2w\cdot q = -E^2 + 2Eq^0$. The last integral is most conveniently evaluated in this frame:

$$
I(w) = \frac{1}{2\pi}\int \delta[E^2 - 2Eq^0]\theta[E-q^0]q^0\,\mathrm{d}q^0
$$

Fig. 5.1. Differential $\tau \to \mu$ decay rate, as function of the muon energy

$$= \frac{1}{8\pi}\theta(-w^2) \tag{5.36}$$

where the θ function is because the integration only has support provided w is timelike. Clearly, then, $B = 2A = \theta(-w^2)/(48\pi)$, and

$$I_{\mu\nu}(w) = \frac{1}{96\pi}\Big[\eta_{\mu\nu}w^2 + 2w_\mu w_\nu\Big]\theta(-w^2) \tag{5.37}$$

Inserting this integral into the decay rate, Eq. (5.27), finally gives the differential decay rate as a function of the muon energy and mass (normalized to the tau mass so that $\varepsilon \equiv p^0/m_\tau$ and $\mu \equiv m_\mu/m_\tau$):

$$\begin{aligned}\frac{d\Gamma}{d\varepsilon}(\tau^- \to \mu^-\overline{\nu}_\mu\nu_\tau) &= \frac{G_{\rm F}^2 m_\tau^5}{4\pi^3}\left(\varepsilon - \frac{4\varepsilon^2}{3} + \varepsilon\mu^2 - \frac{2\mu^2}{3}\right)\sqrt{\varepsilon^2 - \mu^2} \\ &\approx \frac{G_{\rm F}^2 m_\tau^5}{4\pi^3}\varepsilon^2\left(1 - \frac{4\varepsilon}{3}\right) \quad \text{for } \mu \ll 1 \end{aligned} \tag{5.38}$$

The shape of this curve as a function of ε is plotted in Figure 5.1.

The kinematically allowed range for the muon energy is $0 < p^0 < (m_\tau^2 + m_\mu^2)/(2m_\tau)$. This may be seen from the condition that $k - p$ be timelike as seen in the tau rest frame: $(k - p)^2 = k^2 - 2k \cdot p + p^2 = -m_\tau^2 + 2m_\tau p^0 - m_\mu^2 < 0$. Integrating p^0 over this range then gives the total rate for τ^- decays into this channel (neglecting all fermion masses):

$$\Gamma(\tau^- \to \nu_\tau \overline{f}_m f_n) = \frac{G_{\rm F}^2 m_\tau^5}{192\pi^3} |U_{mn}|^2 \tag{5.39}$$

Some final comments about this result.

(i) Equation (5.39), applied to muon decay, may be compared to the estimate of Section 5.1 to see if the discrepancy of this estimate with the experimental value persists or is instead an artifact of the inaccuracy of the earlier estimate. The full and approximate results are

$$\begin{aligned} \Gamma_{\rm calc}(\mu^- \to e^- \overline{\nu}_e \nu_\mu) &= \frac{G_{\rm F}^2 m_\mu^5}{192\pi^3} \\ &= \frac{g_2^4}{3 \cdot 2^9 \cdot (2\pi)^3} \frac{m_\mu^5}{M_W^4} \\ \text{and} \quad \Gamma_{\rm est}(\mu^- \to e^- \overline{\nu}_e \nu \mu) &\sim \frac{g_2^4}{(2\pi)^3} \frac{m_\mu^5}{M_W^4} \end{aligned} \tag{5.40}$$

The approximate estimate has missed the numerical factor of $2^{-9} \cdot 3^{-1} = 1/1536$, which provides the missing three orders of magnitude. This illustrates a general *caveat* for the order-of-magnitude estimates: they are useful for judging the rough size of a rate but are not a substitute for a real calculation. The source of this large number in the full calculation is in the integration of the squared amplitude over phase space. The estimate of this integration using simple dimensional analysis and counting of 2πs is the weakest part of the arguments of Section 5.1. Although it furnishes reasonable accuracy for the two-body decays of the previous sections, it can be potentially more of a problem in decays which involve more final-state particles, since the rates for these processes involve a multidimensional integration over phase space.

The full result, Eq. (5.39), gives the following μ^- and τ^- decay rates into leptons:

$$\begin{aligned} \Gamma_{\rm tot}(\mu^-) &= \Gamma(\mu^- \to e^- \overline{\nu}_e \nu_\mu) \\ &= \frac{G_{\rm F}^2 m_\mu^5}{192\pi^3} \\ &= 3.009 \times 10^{-19} \text{ GeV} \\ \text{so} \quad \tau(\mu^-) &= 2.187 \times 10^{-6} \text{ s} \end{aligned} \tag{5.41}$$

and

$$\Gamma_{\rm tot}(\tau^-) = \Big[2 + 3(|V_{ud}|^2 + |V_{us}|^2)\Big] \Gamma(\tau^- \to e^- \overline{\nu}_e \nu_\mu)$$

$$
\begin{aligned}
&= 5 \cdot \frac{G_{\mathrm{F}}^2 m_\tau^5}{192\pi^3} \\
&= 2.025 \times 10^{-12}\ \mathrm{GeV} \\
\text{so} \quad \tau(\tau^-) &= 3.25 \times 10^{-13}\ \mathrm{s} \qquad (5.42)
\end{aligned}
$$

Corrections to the muon lifetime should be smaller than a percent or so since they are purely electromagnetic. The corrections to the τ lifetime might be somewhat larger since they can include strong-interaction corrections for the hadronic final states. For comparison, the measured lifetimes are

$$
\begin{aligned}
\tau_{\mathrm{exp}}(\mu^-) &= (2.196\,981\,1 \pm 0.000\,002\,2) \times 10^{-6}\ \mathrm{s} \\
\tau_{\mathrm{exp}}(\tau^-) &= (2.906 \pm 0.010) \times 10^{-13}\ \mathrm{s} \qquad (5.43)
\end{aligned}
$$

The comparison for the μ^- is not really fair, since the value of G_{F} is determined from this width. However, with G_{F} so determined, the τ^- width is fair game. Again we emphasize that the τ^- width is not that close to the prediction (10% discrepancy); this is because of strong interaction physics. The partial widths to leptons are in very good agreement with the predictions of our formulae.

(ii) The shape of the differential decay probability, $\mathrm{d}\Gamma/\mathrm{d}E$, for decays into leptons as a function of the charged-lepton energy, E, is given in Figure 5.1. It vanishes like E^2 as $E \to 0$ and rises monotonically to a maximum at the *endpoint*, i.e. the maximum energy that is kinematically available to the charged-lepton (roughly half of the mass of the decaying particle in the present case). The most probable energy for the outgoing charged-lepton is therefore its endpoint value. The E^2-dependence for small E is also easily understood. It arises partly from the phase space measure for relativistic fermions, $\mathrm{d}^3p/p^0 \approx E\,\mathrm{d}E$, and partly from the proportionality to E of the squared matrix element, $\overline{\mathcal{M}^2}$ of Eq. (5.27).

5.4 Feynman rules

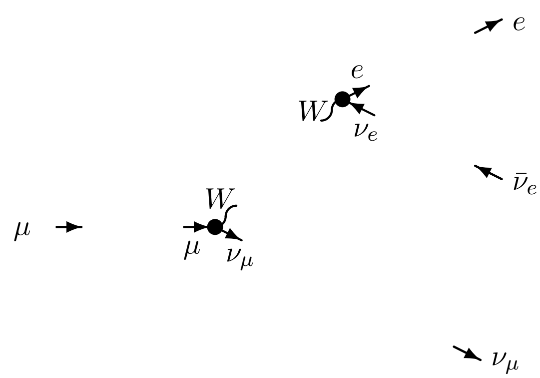
The general pattern for the perturbative evaluation of general scattering amplitudes and decay rates is similar to the examples that have been encountered up to this point. In each case the desired matrix element is found by expressing it in terms of the fields of the theory which are then themselves expressed in terms of the corresponding creation and annihilation operators. The resulting matrix elements of these operators may then be evaluated by applying the rules of Section 1.1.

In each case we pick up standard factors corresponding to the polarization vectors or spinors in the expansions of the fields, and to the coupling constants and Dirac matrices of the interaction Lagrangians. These rules may be very graphically summarized in a way that allows the corresponding amplitude to be straightforwardly written down. The procedure is to associate a line to the propagation of every particle in a particular matrix element. These lines end whenever the corresponding particle is created or destroyed by one of the creation or annihilation operators of the matrix element of interest. The lines drawn in this way form a *Feynman graph* (or *Feynman diagram*) that is associated with the given matrix element.

Explicitly, the relation between the procedure we have followed so far, and the drawing of a Feynman graph, is as follows. First, one chooses the initial and final states under consideration. For the case of μ^- decay, this consisted of a μ^- in the initial state, $|\mu^-\rangle$, and $e^-, \bar{\nu}_e, \nu_\mu$ in the final state, $\langle e^- \bar{\nu}_e \nu_\mu|$. These are represented graphically by putting the end of a line on the left-hand side of the graph for each initial state particle and the end of a line on the right-hand side for each final state particle, labeled with the particle type:

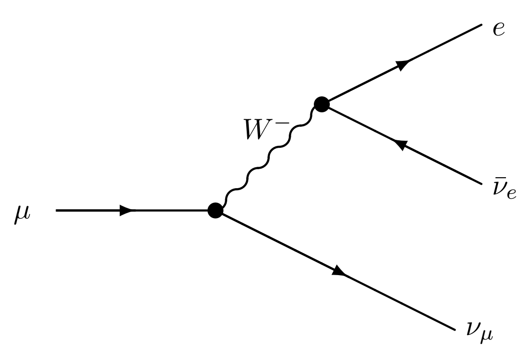
Fermions are given arrows pointing right for particles and left for antiparticles. Some people let time flow from the bottom to the top of the figure rather than from the left to the right.

Next, one must determine what insertions of the interaction Hamiltonian are involved. Each H_I insertion is represented by a dot (vertex) with the stubs of lines coming out, labeled as dictated by the fields appearing in H_I. Fermion fields are given an arrow entering the vertex if the field operator e, μ, ν is involved and an arrow leaving the vertex if the antifield operator $\bar{e}, \bar{\mu}, \bar{\nu}$ is involved. For the case of μ^- decay, the graph is now



The locations on the page of the vertices in the graph are arbitrary, and are usually chosen so that the final graph will look nice.

Finally, each field in an H_I insertion provides either a creation or an annihilation operator. These must be contracted either with initial or final state particles, or with each other. When an annihilation operator destroys an initial-state particle, a line is drawn between the incoming line end and the line stub on that vertex; similarly with creation operators and final state particles. These lines are called *incoming lines* and *outgoing lines*, respectively. Two fields in H_I insertions which combine to form a propagator are represented in the graph by attaching the line stubs on the vertices with an *internal line*. In all cases the line ends connected by a line must be of the same particle type, and for fermions they must have compatible arrow directions. By convention fermions are drawn with solid lines, electroweak gauge bosons are wiggly lines, and gluons are curly lines. We will draw scalars with dashed lines – conventions here are less uniform. The result is that the diagram corresponding to the μ^- decay process we have analyzed is



We emphasize again that the exact location on the page of the vertices and lines is arbitrary and is usually chosen to make the picture easy to draw.

Feynman graphs with a given initial and final state, and sets of $H_{\rm I}$ insertions and creation- and annihilation-operator pairings which can induce the transition from the initial and final state, are in one-to-one correspondence. Drawing Feynman graphs represents a particularly efficient and visual way of finding the possible ways in which an initial state can become a final state. Therefore, the possible processes contributing to a matrix element $\mathcal{M}$ for a process involving a given initial and final state may be found by drawing all possible Feynman graphs which have external lines appropriate to the initial and final states, and involving the vertices corresponding to the interactions of the theory of interest. Furthermore, $\mathcal{M}$ itself can be found from the graphs; it is the sum over each graph of an expression which can be determined by replacing each element (line and vertex) of the graph with an expression determined by the *Feynman rules* of the theory. Each graph must also be multiplied by a symmetry factor, which is precisely the graph theoretic symmetry factor of the Feynman graph.

We first present a table of the *Feynman rules* of the standard model. That is, we present a list of the proper factors that should be associated with each internal line, external line, and vertex in order for a graph to reproduce the corresponding standard-model amplitude. We then consider two examples for which the operator calculation has been done in earlier sections in order to illustrate how the graphical method correctly reproduces matrix elements.

5.4.1 External lines

This section lists the factors that are associated with the external lines of a Feynman graph – and so the initial or final particles of an amplitude.

5.4.1.1 Incoming lines (initial states)

The following factors give the (momentum-space) Feynman rules appropriate to an incoming spin-zero, spin-half, or spin-one particle. The arrows on the fermion lines indicate the direction of fermion-number flow, the dot indicates where the line attaches to an interaction vertex.

Spin-zero

$$1 \tag{5.44}$$

Spin-half fermion

$$u_i(\mathbf{p},\sigma) \tag{5.45}$$

Spin-half antifermion

$$\overline{v}_i(\mathbf{p}, \sigma) \tag{5.46}$$

Spin-one

$$\epsilon_\mu(\mathbf{p}, \sigma) \tag{5.47}$$

5.4.1.2 Outgoing lines (final states)

The momentum-space Feynman rules for an outgoing spin-zero, spin-half, or spin-one particle are similarly

Spin-zero

$$1 \tag{5.48}$$

Spin-half fermion

$$\overline{u}_i(\mathbf{p}, \sigma) \tag{5.49}$$

Spin-half antifermion

$$v_i(\mathbf{p}, \sigma) \tag{5.50}$$

Spin-one

$$\epsilon^*_\mu(\mathbf{p}, \sigma) \tag{5.51}$$

These are the factors needed to compute the matrix element $\mathcal{M}$. When integrating $\mathcal{M}^2$ over final state momenta one must use the phase space measure $\mathrm{d}^3p/(2p^0[2\pi]^3)$, and there is a factor of $1/(2p^0)$ from the state normalization of each incoming particle.

5.4.2 Internal lines

The momentum space description for an internal line is given by the *propagator* for the corresponding particle. The propagators for the three lowest-spin particles of interest are listed here.

Notice that the Dirac indices on the spin-half propagator are such that Dirac matrix multiplication orders propagators and vertices *oppositely* to the order they would have if they are ordered consecutively along a fermion line in the direction of fermion-number flow. Here and in the expressions for vertices, when some index (color, for instance) is not explicitly displayed, it is contracted with a δ function between the lines which carry that index.

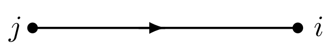
The spin-one propagator given here depends on whether the spin-one particle has a mass or not. For massive spin-one particles *unitary gauge* is presented as was used for the W and Z bosons in the text. This propagator is less useful for massless particles like the photon or gluons, however, since it has a singular limit as the particle mass tends to zero. For massless particles we use instead the propagator in what is called the renormalizable ξ *gauge*. We are free to choose a gauge that is different from unitary gauge for the photon and gluons because the unitary gauge condition, Eq. (2.29), does *not* fix the electromagnetic or $SU_c(3)$ gauge invariance. Notice that the ξ-gauge propagator tends to the unitary gauge result as ξ tends to infinity. The special cases $\xi = 1$ and $\xi = 0$ carry the special names of *Feynman gauge* and *Landau gauge* respectively.

ξ gauge is also useful for massive spin-one particles in higher-loop calculations because of its better behavior as $p^2 \to \infty$. In this case there are extra Feynman rules involving "unphysical scalars" and "ghosts" that must also be included. As these are not necessary for the tree-level computations that are encountered in this book, a description of the full ξ-gauge Feynman rules are reserved for Appendix D.

Spin-zero

$$-\,\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+m^2-\mathrm{i}\epsilon} \tag{5.52}$$

Spin-half



$$-\,\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\left[\frac{-\mathrm{i}\not{p}+m}{p^2+m^2-\mathrm{i}\epsilon}\right]_{ij} \tag{5.53}$$

Spin-one (unitary gauge)

$$-\,\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\eta_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}}{p^2+m^2-\mathrm{i}\epsilon} \tag{5.54}$$

Spin-one (ξ gauge)

$$-\,\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\eta_{\mu\nu}+(\xi-1)\frac{p_\mu p_\nu}{p^2+\xi m^2}}{p^2+m^2-\mathrm{i}\epsilon} \tag{5.55}$$

5.4.3 Vertices

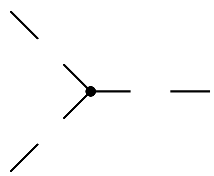
The Feynman rules that differentiate the standard model from any other theory of interacting spin-one, spin-half, and spin-zero particles are those that describe the *vertices* or interactions of the theory. There is a separate vertex for each type of interaction that is given in Section 2.4.

They are all tabulated here for completeness. All momenta are taken as being directed into the vertex. We do not include labels whenever they are connected by a delta function in an obvious way (for instance, color indices in the $Hf\bar{f}$ coupling), or for momentum assignments when the Feynman rule does not depend on them in a complicated way.

Some numerical coefficients in denominators in the following expressions are printed in boldface. We do this for diagrams where there are always multiple ways to attach the external lines to the vertex; in practice, these factors are almost always canceled by the combinatorics of the number of ways to form a graph. Many other references absorb these factors into the computation of the combinatorial factor for the diagram.

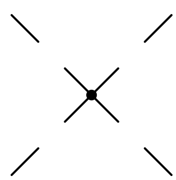
5.4.3.1 Higgs self-couplings

H^3 coupling



$$\left(-3\mathrm{i}\frac{m_{\mathrm{H}}^2}{\mathbf{6}v}\right)(2\pi)^4\delta^4(k+l+p) \tag{5.56}$$

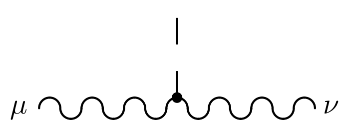
H^4 coupling



$$\left(-3\mathrm{i}\frac{m_{\mathrm{H}}^2}{\mathbf{24}v^2}\right)(2\pi)^4\delta^4(k+l+p+q) \tag{5.57}$$

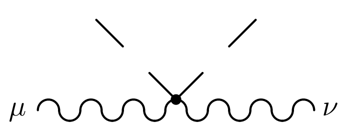
5.4.3.2 Higgs–gauge boson couplings

HW^+W^- coupling



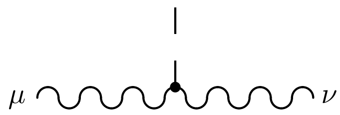
$$\left(-2\mathrm{i}\frac{M_W^2}{v}\right)\eta_{\mu\nu}(2\pi)^4\delta^4(k+l+p) \tag{5.58}$$

$H^2W^+W^-$ coupling



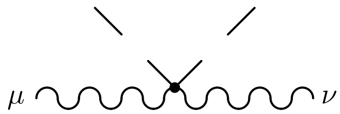
$$\left(-2\mathrm{i}\frac{M_W^2}{\mathbf{2}v^2}\right)\eta_{\mu\nu}(2\pi)^4\delta^4(k+l+p+q) \tag{5.59}$$

HZ^2 coupling



$$\left(-2\mathrm{i}\frac{M_Z^2}{\mathbf{2}v}\right)\eta_{\mu\nu}(2\pi)^4\delta^4(k+l+p) \tag{5.60}$$

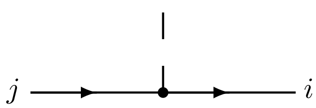
H^2Z^2 coupling



$$\left(-2\mathrm{i}\frac{M_Z^2}{\mathbf{4}v^2}\right)\eta_{\mu\nu}(2\pi)^4\delta^4(k+l+p+q) \tag{5.61}$$

5.4.3.3 Higgs–fermion couplings

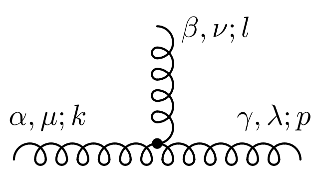
$Hf\overline{f}$ coupling



$$\left(-\mathrm{i}\frac{m_f}{v}\right)\delta_{ij}(2\pi)^4\delta^4(k+l+p) \tag{5.62}$$

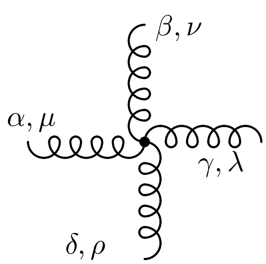
5.4.3.4 Gluon self-couplings

G^3 coupling



$$+\frac{g_3}{\mathbf{6}}f_{\alpha\beta\gamma}\left[(k-p)_\nu\eta_{\mu\lambda}+(l-k)_\lambda\eta_{\mu\nu} + (p-l)_\mu\eta_{\nu\lambda}\right](2\pi)^4\delta^4(k+l+p) \tag{5.63}$$

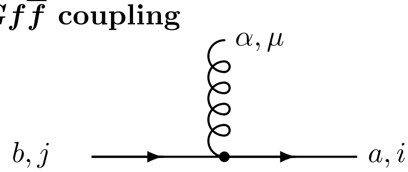
G^4 coupling



$$\begin{aligned}-\mathrm{i}\frac{g_3^2}{\mathbf{24}}\Big[&f_{\xi\alpha\beta}f_{\xi\gamma\delta}\left(\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}\right)\\ &+f_{\xi\alpha\gamma}f_{\xi\beta\delta}\left(\eta_{\mu\nu}\eta_{\lambda\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}\right)\\ &+f_{\xi\alpha\delta}f_{\xi\beta\gamma}\left(\eta_{\mu\nu}\eta_{\lambda\rho}-\eta_{\nu\rho}\eta_{\mu\lambda}\right)\Big]\\ \times(2\pi)^4&\delta^4(k+l+p+q)\end{aligned} \tag{5.64}$$

5.4.3.5 Gluon–fermion couplings

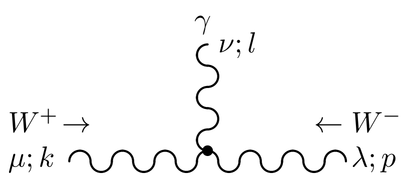
$Gf\bar{f}$ coupling



$$-\frac{g_3}{2}(\lambda_\alpha)_{ab}(\gamma^\mu)_{ij}(2\pi)^4\delta^4(k+l+p) \quad (5.65)$$

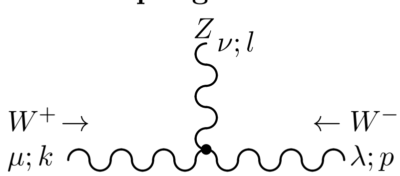
5.4.3.6 Electroweak boson self-couplings

$W^+W^-\gamma$ coupling



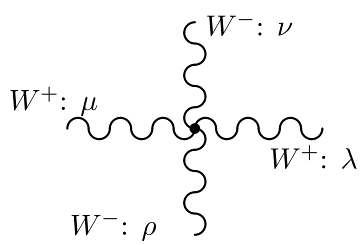
$$\mathrm{i}e\left[(k-p)_\nu\eta_{\mu\lambda} + (l-k)_\lambda\eta_{\mu\nu} + (p-l)_\mu\eta_{\nu\lambda}\right] \times(2\pi)^4\delta^4(k+l+p) \quad (5.66)$$

W^+W^-Z coupling



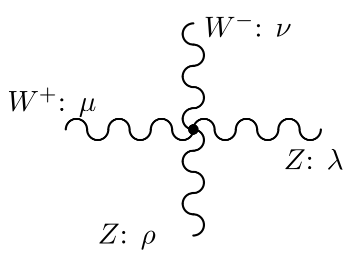
$$\mathrm{i}e\cot\theta_{\mathrm{W}}\Big[(k-p)_\nu\eta_{\mu\lambda} + (l-k)_\lambda\eta_{\mu\nu} + (p-l)_\mu\eta_{\nu\lambda}\Big](2\pi)^4\delta^4(k+l+p) \quad (5.67)$$

$W^+W^-W^+W^-$ coupling



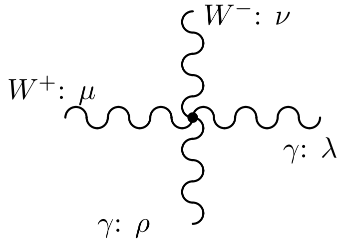
$$\frac{\mathrm{i}g_2^2}{\mathbf{4}}\left[2\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\rho}\right] \times(2\pi)^4\delta^4(k+l+p+q) \quad (5.68)$$

$W^+W^-Z^2$ coupling



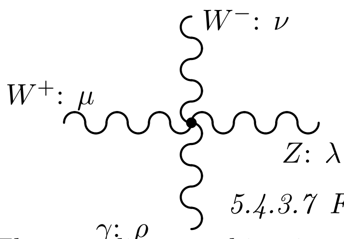
$$-\frac{\mathrm{i}e^2\cot^2\theta_{\mathrm{W}}}{\mathbf{2}}\left[2\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}\right] \times(2\pi)^4\delta^4(k+l+p+q) \quad (5.69)$$

$W^+W^-\gamma^2$ coupling



$$-\frac{\mathrm{i}e^2}{\mathbf{2}}\left[2\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}\right] \times (2\pi)^4\delta^4(k+l+p+q) \tag{5.70}$$

$W^+W^-Z\gamma$ coupling

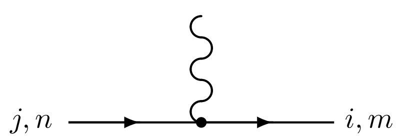


$$-\mathrm{i}e^2\cot\theta_{\mathrm{W}}\left[2\eta_{\mu\nu}\eta_{\lambda\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\lambda}\eta_{\nu\rho}\right] \times (2\pi)^4\delta^4(k+l+p+q) \tag{5.71}$$

5.4.3.7 Fermion electroweak couplings

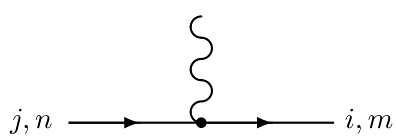
The coupling combinations e_W and e_Z used here are the same as those used elsewhere in the text: $e_Z = e/(\sin\theta_{\mathrm{W}}\cos\theta_{\mathrm{W}}) = \sqrt{g_1^2+g_2^2}$ and $e_W = \mathrm{e}/(2\sqrt{2}\sin\theta_{\mathrm{W}}) = g_2/(2\sqrt{2})$. g_V and g_A are the quantum number combinations $g_V = \frac{1}{2}T_3 - Q\sin^2\theta_{\mathrm{W}}$ and $g_A = \frac{1}{2}T_3$, listed explicitly in Table 2.1 and Table 4.1. The matrix U_{mn} is the unit matrix δ_{mn} when "m" and "n" are leptons and is the Kobayashi–Maskawa matrix, V_{mn}, when they are quarks.

$W^+f_n\overline{f}_m$ coupling



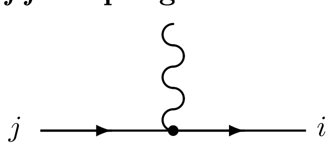
$$-e_W U_{mn}[\gamma^\mu(1+\gamma_5)]_{ij}(2\pi)^4\delta^4(k+l+p) \tag{5.72}$$

$W^-f_n\overline{f}_m$ coupling



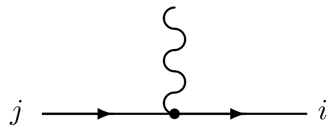
$$-e_W U^*_{nm}[\gamma^\mu(1+\gamma_5)]_{ij}(2\pi)^4\delta^4(k+l+p) \tag{5.73}$$

$Zf\overline{f}$ coupling



$$-e_Z[\gamma^\mu(g_V+g_A\gamma_5)]_{ij}(2\pi)^4\delta^4(k+l+p) \tag{5.74}$$

$\gamma f\overline{f}$ coupling



$$-\, eQ_f[\gamma^\mu]_{ij}(2\pi)^4\delta^4(k+l+p) \quad (5.75)$$

5.4.4 The rules

These rules allow a Feynman graph to be converted into an S-matrix element by the following steps.

(i) Draw all graphs that can connect the desired initial and final states using only those vertices that can contribute to the order of perturbation theory that is desired – recall that each vertex is proportional to the coupling constant of the corresponding interaction so each additional vertex costs extra powers of the couplings.

(ii) For each graph, replace each internal line, external line, and vertex by the expression given above.

(iii) Integrate the result over the four-momentum flowing through all of the internal lines (corresponding to summing over all virtual intermediate states), and sum over all Dirac and Lorentz indices.

(iv) If the graph contains n vertices then divide its contribution by $n!$ (c.f. the denominator of Eq. (3.22).) If there are p distinct ways of forming the given graph using the same set of interactions and initial and final states, then multiply the contribution of the graph by p. The product of these factors is called the *symmetry factor* of the graph.

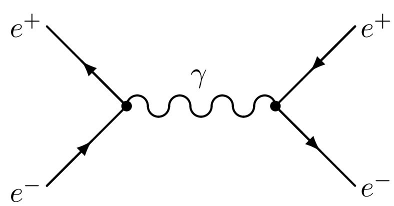
(v) Multiply the result by a factor of -1 for each closed fermion (or ghost, see Appendix D) loop in the graph.

(vi) When comparing different matrix elements for the same process, there can be a relative minus sign if the fermion lines connect together differently (see Section 6.6).

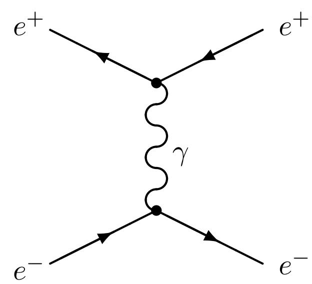
(vii) To convert the resulting S-matrix element to a matrix element $\mathcal{M}$, multiply by i and remove the overall energy-momentum conserving delta function $(2\pi)^4\delta^4(p_\alpha - p_\beta)$ (with p_α and p_β the sums over all incoming and outgoing momenta, respectively).

A few of these items demand some clarification. To clarify item (iv): if a graph contains, say, two vertices and they are of distinct types, for instance, a $W^+\bar{\nu}_\mu\mu$ vertex and a $W^-\bar{e}\nu_e$ vertex, then there is a factor of $1/2!$ from the multiple vertices and a factor of 2 from the different choices of which vertex generates the $W^+\bar{\nu}_\mu\mu$ interaction and which is the $W^-\bar{e}\nu_e$ interaction. An alternate way of expressing item (iv) is to say that, for each time the *same type* of vertex appears n times, there is a factor of $1/n!$; and the result is multiplied by the number of ways of constructing the graph out of the required types of vertices. Also, when a vertex has several of the same field operator, there are generally multiple ways that the graph can be formed, corresponding to different choices for which field operator does each job. For a simple (though hardly physically realizable!) example, consider the scattering process $HH \to HH$. The $HHHH$ vertex mediates this process, but any of the four H operators can annihilate the first H incoming state, any of the remaining three H operators can annihilate the other, and either of the two remaining H operators can create the first H final state, leading to $4\times3\times2\times1 = 24$ ways to build the graph, canceling the $1/\mathbf{24}$ denominator in the Feynman rule for the vertex. Therefore, we would get $\mathcal{M} = 3m_{\rm H}^2/v^2$ (plus the contribution of other diagrams).

To clarify item (vi): a relative minus sign arises whenever the pairing off of the creation and annihilation operators for fermions requires an odd number of anti-commutations of those operators. This is also the origin of the rule, item (v). The sure-fire way to determine whether the matrix element contributions associated with two diagrams have a relative minus sign, due to such fermionic operator anticommutation, is to draw one diagram next to the mirror image of the other, and connect the lines corresponding to the same external particles. Then count the number of fermionic loops in this picture. Next, do the same with either of the original diagrams and itself. If the number of fermion loops differs by an odd number, there is a relative -1 in the original diagrams' contribution to the matrix element. To give an example, consider the following two diagrams for the scattering process, $e^-e^+ \to e^-e^+$:



and



(5.76)

Is there a relative sign? To find out, we mirror-image the second element, and connect the lines for the final-state e^+ particles, the final-state e^- par-

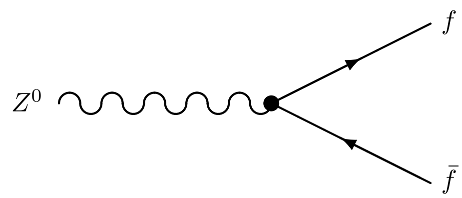
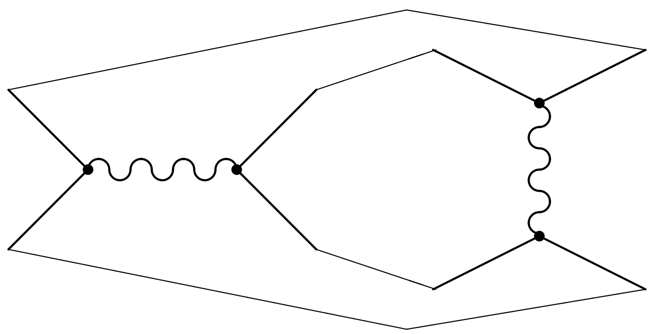


Fig. 5.2. The Feynman graph for $Z^0 \to f\overline{f}$.

ticles, the initial-state e^+ particles, and the initial-state e^- particles:



(5.77)

Then we count how many fermionic loops there are. The fermion lines form one big loop. Squaring either of the original diagrams gives two separate loops. Therefore, there is a (-1) relative factor between the diagrams, and when we compute the interference between these diagrams, we will have to include an extra factor of (-1) beyond what the Feynman rules otherwise provide.

5.4.4.1 Example: Z^0 decay

As an illustration of the application of these rules, we wish to recompute the matrix element for the Z^0 to decay into a fermion–antifermion pair. The corresponding graphs start with a single Z^0 boson external line and end with a fermion and antifermion external line.

The simplest set of vertices and internal lines that can connect these initial and final states is a single neutral current vertex, Eq. (5.74). The required graph therefore is as in Figure 5.2.

Using the above rules the S-matrix for this process becomes

$$\begin{aligned}\langle f\overline{f}|S|Z^0\rangle \quad = \quad & \frac{1}{1!}\Big[\epsilon_\mu(\mathbf{k},\lambda)\Big]\Big[\overline{u}(\mathbf{p},\sigma)\Big] \\ & \times\Big[-e_Z\gamma^\mu(g_V+g_A\gamma_5)(2\pi)^4\delta^4(p+q-k)\Big]\Big[v(\mathbf{q},\zeta)\Big]\end{aligned} \tag{5.78}$$

The number of independent ways of forming this graph with these vertices is

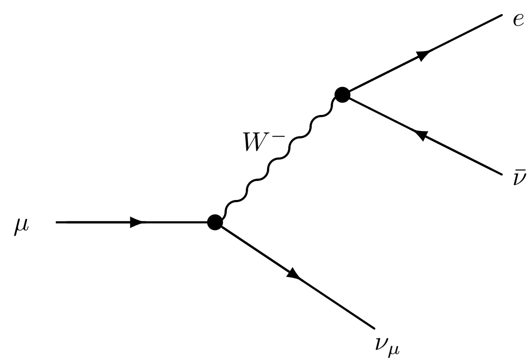


Fig. 5.3. The Feynman graph for the decay $\mu \to e\overline{\nu}\nu$.

in this case $p = 1$, and the denominator 1! corresponds to the graph having only a single vertex.

The matrix element $\mathcal{M}$ is obtained by stripping off the energy-momentum conserving delta function $(2\pi)^4\delta^4(p+q-k)$ from this expression and multiplying by i. The result obtained in this way is identical to Eq. (4.8) derived by directly evaluating the matrix element.

5.4.4.2 Example: μ^- decay

Perhaps a less trivial example is the amplitude for μ^- decay into $e^-\overline{\nu}_e\nu_\mu$. In this case the relevant graph has a single-fermion initial external line for the μ^- and has two fermion and one antifermion final external lines. Two charged-current vertices are also required. The graph therefore is as in Figure 5.3.

The S-matrix associated with this graph is

$$\begin{aligned}\langle e\overline{\nu}\nu|S|\mu\rangle \;\; = \;\; & \frac{2}{2!}\int \frac{\mathrm{d}^4 r}{(2\pi)^4}\overline{u}(\mathbf{l})\left[-e_W\gamma^\mu(1+\gamma_5)(2\pi)^4\delta^4(l+r-k)\right]u(\mathbf{k}) \\ & \times\overline{u}(\mathbf{p})\left[-e_W\gamma^\nu(1+\gamma_5)(2\pi)^4\delta^4(p+q-r)\right]v(\mathbf{q}) \\ & \times\left[\frac{-\mathrm{i}}{r^2+M_W^2-\mathrm{i}\epsilon}\left(\eta_{\mu\nu}+\frac{r_\mu r_\nu}{M_W^2}\right)\right] \qquad (5.79)\end{aligned}$$

Since this graph contains two vertices, the denominator in the first line of Eq. (5.79) is 2!. The numerator of the same term is 2, corresponding to the two equal contributions depending on which charged-current interaction vertex destroys the muon and which creates the electron–antineutrino pair.

Grouping terms and identifying the matrix element, $\mathcal{M}(\mu \to e\overline{\nu}\nu)$, from this S-matrix element then gives the same result as is found in Eq. (5.19) by operator methods.

5.5 Problems

[5.1] **Neutron lifetime**

Compute the lifetime of the neutron, in the approximation where the vertex between the W boson, the neutron, and the proton, is the same as the vertex between the W boson, a down quark (in the neutron), and an up quark (in the proton), except that the γ_5 factor is rescaled by a factor g_A.

[5.1.1] Making the approximations $M_W \gg m_p, m_n \gg Q \equiv m_n - m_p \sim m_e$, show that the neutron β-decay rate is given in the neutron rest-frame by

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_e} = \frac{G_{\mathrm{F}}^2 |V_{ud}|^2}{2\pi^3\, 16 m_n^2} \left[\mathcal{F} + \mathcal{G}\right] E_e \sqrt{E_e^2 - m_e^2}(Q - E_e)\sqrt{(Q - E_e)^2 - m_\nu^2}$$

Here m_ν is a hypothetical neutrino mass and the *Fermi* and *Gamow–Teller* terms $\mathcal{F}$ and $\mathcal{G}$ are defined by

$$\mathcal{F} = |\bar{u}_p(\mathbf{p})\gamma^0(g_V + g_A\gamma_5)u_n(\mathbf{p}')|^2$$

and

$$\mathcal{G} = |\bar{u}_p(\mathbf{p})\vec{\gamma}(g_V + g_A\gamma_5)u_n(\mathbf{p}')|^2$$

Assume that only the outgoing electron energy, E_e, is measured.

[5.1.2] Evaluate $\mathcal{F}$ and $\mathcal{G}$ in the approximation that the nucleon does not recoil; i.e., $\mathbf{p}_p = 0$ (or is $\ll m_n$) in the neutron rest frame.

[5.1.3] Plot the quantity

$$y = \left[\frac{\mathrm{d}\Gamma/\mathrm{d}E_e}{E_e\sqrt{E_e^2 - m_e^2}}\right]^{1/2}$$

as a function of the electron energy, E_e , for the two cases $m_\nu = 0$ and $m_\nu = 10$ eV. This is known as a Curie plot. How do the two graphs differ?

[5.1.4] Evaluate numerically the total life time of the neutron, neglecting m_ν and using the numerical values $g_V = 1$, $g_A = 1.2701$, $m_p = 938.272\,00$ MeV, $m_n = 939.565$ MeV, $m_e = 0.511\,00$ MeV; $(1\ \mathrm{fm})^{-1} = 197.326\,96$ MeV, 1 s $= 2.9979\times10^{23}$ fm.

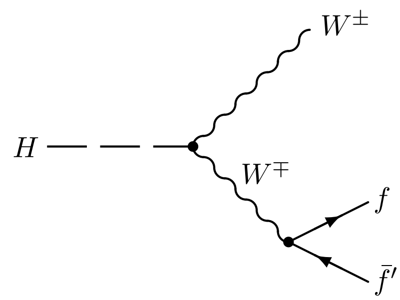
[5.2] **Higgs decay to $Wf\bar{f}$**

The Higgs boson is too heavy to decay to two (on-shell) W bosons, because $m_{\rm H} < 2M_W$. However, it can decay to one W boson and the decay products of another W boson, via an intermediate state (off-shell) W boson similar to the one involved in the muon decay process. This decay process is important because the decay width we found in Section 4.3, $H \to b\bar{b}$, has a very small width due to the very small b quark mass. (Furthermore, to date the only accelerators with enough energy to produce large Higgs samples have been hadron machines ($p - \overline{p}$ at the Tevatron and pp at the LHC). As we shall see, such collisions produce very large numbers of $q - \overline{q}$ pairs, so $b\overline{b}$ is not a very distinctive final state and is difficult to separate from the much larger rate of $b\overline{b}$ production from other processes.)

The process

$$H \to Wf\bar{f}'$$

occurs via the diagram



where f and $\bar{f}'$ are a pair of fermions which could result from the decay of a $W^\mp$ (that is, for W^- they are $e^-\bar{\nu}_e$, $\mu^-\bar{\nu}_\mu$, $\tau^-\bar{\nu}_\tau$, $d\bar{u}$, $s\bar{c}$). For this problem, you should systematically ignore the fermion masses (except m_t, which is so heavy that the top quark does not participate anyway). However, you cannot neglect the W-boson mass M_W. Label the initial momentum p, the final W-boson momentum as q, the momentum on the virtual W-boson propagator as r, and the final fermion and antifermion momenta as k and l (so $r = k + l$).

[5.2.1] **Matrix element**

Argue that exactly half the width, via this process, will be from the case with a W^+ in the final state, and half from the case with a W^-. (Is there a symmetry at play here?) Having made this argument, concentrate on the width when it is a W^+ appearing in the final state. Remember to multiply by 2 at the end of the problem.

Write down the matrix element for this process, before summing on the external state spins and polarizations.

[5.2.2] **Squared matrix element**

Evaluate the squared matrix element, summing over final state spins and polarizations. Carry out all Dirac traces to get an expression which is an algebraic function of the relevant particle four-momenta. It will turn out to be convenient *not* to contract all the Lorentz indices, however; leave the factors of the form $(\eta^{\mu\nu} + r^\mu r^\nu / M_W^2)$, from the W^- propagator, in this form.

[5.2.3] **Integration on fermionic momenta**

Write down the width as an integral over final-state momenta, of the squared matrix element.

Holding r fixed, carry out the integration over the final-state fermionic momenta. That is, perform the integrals over k, l. The integration is similar to the one we encountered for $I_{\mu\nu}(r)$ in the text. The resulting expression should be proportional to $(r^2\eta_{\mu\nu} - r_\mu r_\nu)$. It should now be straightforward to perform the rest of the Lorentz index contractions.

[5.2.4] **Total width**

Express the total width as a single integral. Re-write your answer by factoring all dimensionful quantities out of the integral, so it depends only on the dimensionless parameter $m_{\rm H}/M_W$ and the integration variable, which might for instance be p_W^0/M_W. If you cannot do the integral by hand, you will have to find some way of evaluating it numerically.

Compute the partial width $\Gamma(H \to W f\bar{f})$ for the values $m_{\rm H} = 126$ GeV and $m_W = 80.4$ GeV, and compare it to the partial width $\Gamma(H \to b\bar{b})$.

[5.2.5] **Z-pairs and experimental issues**

Repeat the calculation for the case $H \to Z f\bar{f}$ with $f\bar{f}$ a pair of fermions which can be produced by an off-shell Z. What is the partial width to Z bosons?

List common final states for the $W f\bar{f}$ and $Z f\bar{f}$ decays, considering that the W or Z also decays. What is the partial width of a Higgs boson to 4 leptons (electron and muon only), where one lepton pair reconstructs to the Z mass and the four leptons reconstruct to the Higgs mass? Can you explain why this is a particularly clean final state for study in hadron colliders?

[5.3] **The miracle of Lorentz invariance** Consider the following Lagrangian density for a real scalar field, $\phi(x)$, that is coupled to a classical background current, $J^\mu(x)$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - gJ^\mu\partial_\mu\phi \tag{5.80}$$

[5.3.1] Construct the canonical Hamiltonian density for this problem and show that it is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int}$$

with

$$\mathcal{H}_0 = \frac{1}{2}\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \tag{5.81}$$

$$\mathcal{H}_{\rm int} = gJ^0\pi + \frac{g^2}{2}J^0J^0 + g\mathbf{J}\cdot\nabla\phi = +gJ^\mu\partial_\mu\phi - \frac{g^2}{2}J^0J^0 \tag{5.82}$$

Here $\pi = \dot{\phi} - gJ^0$ is the canonical momentum. Notice in particular how the interaction Hamiltonian is *not* Lorentz-invariant.

[5.3.2] Find the propagator

$$G_{\mu\nu}(x,x') \equiv \langle 0|T\left[\partial_\mu\phi(x)\partial_\nu\phi(x')\right]|0\rangle$$

and show that it can be written in the following way:

$$G_{\mu\nu} = -{\rm i}\partial_\mu\partial'_\nu\int\frac{{\rm d}^4p}{(2\pi)^4}\frac{{\rm e}^{{\rm i}p\cdot(x-x')}}{p^2+\mu^2-i\epsilon} + \Delta_{\mu\nu}(x,x') \tag{5.83}$$

Explicitly compute the function $\Delta_{\mu\nu}(x,x')$ in this equation, and show that it is *not* Lorentz-covariant.

[5.3.3] Compute the vacuum transition-matrix element, $\langle 0|S|0\rangle$, to second order in the current, $J^\mu(x)$, and show that the above two sources of Lorentz-non-covariance cancel one another. This shows that the final Lorentz-invariant result is equivalent to what would have been obtained if we had simply used the naive expression $\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}$ and used the naive propagator $\tilde{G}_{\mu\nu} = G_{\mu\nu} - \Delta_{\mu\nu}$. Modified time ordering which produces this propagator is often called the "$T*$-ordering," and denoted $\tilde{G}_{\mu\nu}(x,x') \equiv \langle 0|T^*[\partial_\mu\phi(x)\partial_\nu\phi(x')]|0\rangle$.

6

Leptonic weak interactions: collisions

The only applications of the standard model discussed up to this point have been calculations of the decay rates for the unstable weakly interacting elementary particles of the model. These are important applications since much of what is known about the fundamental interactions of nature comes from the basic properties of the particles involved, including their decay products and lifetimes. As we have seen, the standard model is able to do a good job of accounting for these properties to within the accuracy of current measurements, at least within the leptonic sector.

There are other applications which the model must also describe, however. Prominent among these are reactions that are observed within particle accelerators. This is, after all, how these unstable particles are produced. This chapter is meant to present some of the standard model predictions for the results of elementary-particle collisions among leptons and electroweak bosons. We focus here on these particles since their collisions are understandable with the fewest complications. Hadronic collisions are the topic of Chapter 9.

e^+e^-–annihilation processes are the lepton collisions that have been of particular interest since these have been studied in great detail near and beyond the Z^0 resonance. The precision of these measurements has been used to test the model with exquisite precision. For this reason the reaction $e^+e^- \to f\overline{f}$ is examined in some detail.

Neutrino–electron scattering is another purely leptonic process of experimental interest. Beams of electron-type neutrinos (produced for instance in a nuclear reactor or the Sun), or muon-type neutrinos (produced by pion decay downstream of a target area within an accelerator), can be collided with electrons and the resulting collision rates compared with the predictions of the theory. Neutrino collisions with electrons also take place within the large neutrino observatories and must be understood in order to understand neutrino oscillations in solar, reactor, atmospheric, and beam experiments, and to understand neutrinos from supernovae, such as those observed from Supernova 1987A (and any more that are yet to come).

6.1 The Mandelstam variables

Before launching into a detailed calculation of the collision rates in various accelerators, some notational points must first be made. In any two-body scattering process in which only the momenta and energies of the scattering particles are observed (as opposed to their spins etc.) there are precisely two relativistically invariant variables on which Lorentz-invariant observables like cross sections can depend. There is a conventional choice for these variables that is outlined in this section.

Consider, then, a two-body process of the form $a + b \to c + d$ in which particles a, b, c, and d have four-momenta p_k^μ and masses $m_k^2 = -p_k^2$, with $k = a, b, c, d$. These four-momenta are arbitrary future-directed timelike (or possibly null) vectors that are subject only to the condition of four-momentum conservation:

$$p_a + p_b = p_c + p_d \tag{6.1}$$

If only momenta and energies are measured in this reaction then the cross section must depend only on the four four-momenta of the problem: p_a through p_d. Being Lorentz-invariant, the cross section $d\sigma(a+b \to c+d)$ can only depend on the independent Lorentz-invariant combinations that can be constructed from these momenta. Since the square of each of these four-vectors is a constant – being equal to the mass of the corresponding particle – the Lorentz-invariant combinations that contain the kinematic information (such as the directions traveled by each particle) are the six inner products: $p_k \cdot p_l$ with k and l running over particle types a to d with $k \neq l$. Since four-momentum conservation, Eq. (6.1), allows any one of the p_k to be eliminated in terms of the others only three of these inner products need a priori be considered as being distinct. If, for example, four-momentum conservation is chosen to eliminate p_d then the three inner products could be chosen to be $p_a \cdot p_b$, $p_a \cdot p_c$ and $p_b \cdot p_c$.

Instead of directly using these inner products, it is conventional to use the following equivalent three combinations, known as *Mandelstam variables* or *Mandelstam invariants*:

$$\begin{aligned} s &\equiv -(p_a + p_b)^2 \\ &= -2p_a \cdot p_b + m_a^2 + m_b^2 \end{aligned}$$

$$\begin{aligned} t &\equiv -(p_a - p_c)^2 \\ &= 2p_a \cdot p_c + m_a^2 + m_c^2 \\ u &\equiv -(p_a - p_d)^2 \\ &= 2p_a \cdot p_d + m_a^2 + m_d^2 \end{aligned} \tag{6.2}$$

These invariants may also be re-expressed in terms of the other four-momenta using four-momentum conservation:

$$\begin{aligned} s &= -(p_c + p_d)^2 \\ &= -2p_c \cdot p_d + m_c^2 + m_d^2 \\ t &= -(p_d - p_b)^2 \\ &= 2p_b \cdot p_d + m_b^2 + m_d^2 \\ u &= -(p_c - p_b)^2 \\ &= 2p_b \cdot p_c + m_b^2 + m_c^2 \end{aligned} \tag{6.3}$$

Now, given the masses of all of the particles involved, a two-body collision should be completely described in terms of two invariant parameters. These could be chosen to be the collision energy and scattering angle as seen in the center-of-mass frame, for example. There must therefore be a relationship amongst the three Mandelstam invariants. This relationship is easily derived if the definitions for s, t, and u in Eq. (6.2) are added to one another:

$$\begin{aligned} s + t + u &= -2p_a \cdot (p_b - p_c - p_d) + 3m_a^2 + m_b^2 + m_c^2 + m_d^2 \\ &= +2p_a^2 + 3m_a^2 + m_b^2 + m_c^2 + m_d^2 \\ &= m_a^2 + m_b^2 + m_c^2 + m_d^2 \end{aligned} \tag{6.4}$$

These variables can be related to the basic kinematic quantities in any given reference frame, such as the overall energy of the collision and the scattering angles, etc.There are two frames that are of the most practical interest. These are the *center-of-mass frame* – or CM frame for short – defined as the rest frame of the timelike four-vector $p_a + p_b$, and the *lab frame*, defined as the rest frame of particle "a." The lab frame terminology is appropriate for "fixed target" experiments in which a beam of high-energy particles impinge on a target at rest. Center-of-mass variables are useful both because they are frequently simpler, and because many modern experiments are beam-on-beam experiments where the center-of-mass frame is the same as the frame of the particle detector.

Lab frame: The lab frame is defined as the rest frame of particle "a":

$$E_a \equiv p_a^0 = m_a; \qquad \text{and} \qquad \mathbf{p}_a = 0 \tag{6.5}$$

In this frame inner products of four-vectors with p_a have a very simple form: $p_a \cdot p_b = -m_a E_b$. s, t, and u are therefore directly related to the energies of particles "b", "c" and "d" in this frame:

$$\begin{aligned} s &= +2m_a E_b + m_a^2 + m_b^2, && \text{lab frame} \\ t &= -2m_a E_c + m_a^2 + m_c^2, && \text{lab frame} \\ u &= -2m_a E_d + m_a^2 + m_d^2, && \text{lab frame} \end{aligned} \tag{6.6}$$

Once the energies E, and hence the magnitudes of the three-momenta, $|\mathbf{p}| = \sqrt{E^2 - m^2}$, are determined from these relations, the angular information may next be obtained from these same variables. Denote the angle between the direction of the incoming particle, $\mathbf{p}_b$, and the directions of the outgoing particles, $\mathbf{p}_c$ and $\mathbf{p}_d$, by θ_c^* and θ_d^*. Then the angular information is obtained from t and u as expressed in Eq. (6.3). To see this use $p_b \cdot p_c = -E_b E_c + \mathbf{p}_b \cdot \mathbf{p}_c = -E_b E_c + p_b p_c \cos\theta_c^*$. (The notation used here has $p_b \equiv |\mathbf{p}_b|$ with the understanding that the context will keep p_b defined in this way from being confused with the four-vector p_b.) The Mandelstam invariants t and u therefore become

$$\begin{aligned} t &= -2E_b E_d + 2p_b p_d \cos\theta_d^* + m_b^2 + m_d^2, && \text{lab frame} \\ &\approx -2E_b E_d (1 - \cos\theta_d^*) && \text{(ultra-relativistic)} \\ u &= -2E_b E_c + 2p_b p_c \cos\theta_c^* + m_b^2 + m_c^2, && \text{lab frame} \\ &\approx -2E_b E_c (1 - \cos\theta_c^*) && \text{(ultra-relativistic)} \end{aligned} \tag{6.7}$$

CM frame: The CM frame is defined as the frame in which the three-momenta of the initial particles (and so also of the final particles) are equal and opposite:

$$\mathbf{p}_a + \mathbf{p}_b = 0 \qquad \text{and so} \qquad E_a^2 - m_a^2 = E_b^2 - m_b^2 \tag{6.8}$$

In this frame the invariant s is simply the square of the total energy of the collision:

$$\begin{aligned} s &= (E_a + E_b)^2; && \text{CM frame} \\ &= (E_c + E_d)^2; && \text{CM frame} \end{aligned} \tag{6.9}$$

Clearly knowledge of s therefore completely determines the energies and the magnitudes of the three-momenta of all particles in this frame.

The directional information lies in t and u. Defining the angle θ

as the angle between the direction of the initial particle "a" and the direction of the outgoing particle "c" in the CM frame, we have

$$\begin{aligned} t &= -2E_aE_c + 2p_ap_c\cos\theta + m_a^2 + m_c^2; \qquad \text{CM frame} \\ &\approx -2E_aE_c(1-\cos\theta); \qquad \text{(ultra-relativistic)} \\ u &= -2E_aE_d + 2p_ap_d\cos(\pi-\theta) + m_a^2 + m_d^2 \\ &= -2E_aE_d - 2p_ap_d\cos\theta + m_a^2 + m_d^2; \qquad \text{CM frame} \\ &\approx -2E_aE_d(1+\cos\theta); \qquad \text{(ultra-relativistic)} \end{aligned} \tag{6.10}$$

These expressions also indicate the range of values over which s, t, and u may run. Inspection of Eq. (6.6) and Eq. (6.9) shows that s, t, and u must lie within the following kinematically allowed ranges:

$$\begin{aligned} s &\geq \max[m_a^2 + m_b^2; m_c^2 + m_d^2] \\ t &\leq \min[m_a^2 + m_c^2; m_b^2 + m_d^2] \\ u &\leq \min[m_a^2 + m_d^2; m_b^2 + m_c^2] \end{aligned} \tag{6.11}$$

6.2 e^+e^- annihilation: calculation

Consider now the collision process $e^+e^- \to f\overline{f}$. The cross section for this process is computed in this section for unpolarized initial electrons and with the spin of the final-state fermions unmeasured. This calculation is meant to provide an explicit illustration of how such cross sections are determined, as well as to derive formulae for the cross section that have applications in later sections and are of interest in themselves. Since most applications involve energies well in excess of 1 GeV – and the most interesting application is for $s \simeq M_Z \simeq 90$ GeV – the masses of the fermion final states are neglected to good approximation in this section.

Provided that the final state particles are not electrons or electron neutrinos, the standard model scattering amplitude is dominated by two Feynman diagrams, shown in Figure 6.1.

Should the final state particles be e^+e^- or $\nu_e\overline{\nu}_e$ then there are additional graphs such as those in Figure 6.2 that must also be included. Therefore, we postpone treatment of these final states to Section 6.5.

Using the Feynman rules of the previous chapter, the Z^0-exchange graph of Figure 6.1 has the following matrix element:

$$\begin{aligned} \mathcal{M}_{e^+e^-\to f\bar{f}} &= \frac{2(-e_Z)^2}{2!}\left[\overline{v}_e(\mathbf{p}')\gamma^\mu\Gamma_{Ze}u_e(\mathbf{p})\right]\left[\overline{u}_f(\mathbf{k})\gamma^\nu\Gamma_{Zf}v_f(\mathbf{k}')\right] \\ &\quad \times \left[\frac{1}{(p+p')^2 + M_Z^2 - \mathrm{i}\epsilon}\left(\eta_{\mu\nu} + \frac{(p+p')_\mu(p+p')_\nu}{M_Z^2}\right)\right] \end{aligned}$$

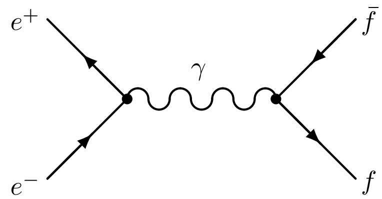


Fig. 6.1. The Feynman graphs for the process $e^+e^- \to f\overline{f}$.

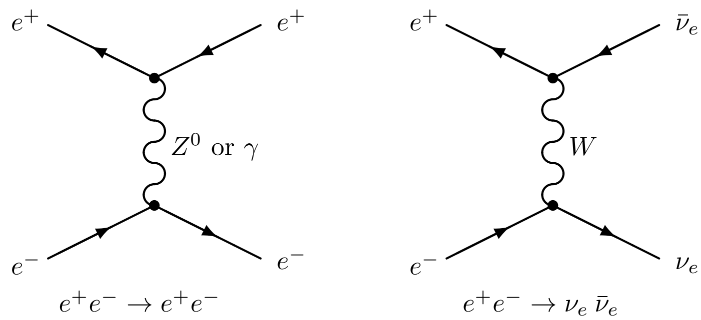


Fig. 6.2. Additional graphs for $e^+e^- \to e^+e^-$ and $e^+e^- \to \nu_e\overline{\nu}_e$.

(6.12)

In this equation, Γ_{Zf} denotes the Dirac matrix that specifies the Z-boson's neutral-current couplings to fermion type "f":

$$\begin{aligned}\Gamma_{Zf} &= g_V + g_A\gamma_5 \\ &= g_{\mathrm{L}}P_{\mathrm{L}} + g_{\mathrm{R}}P_{\mathrm{R}}\end{aligned} \tag{6.13}$$

P_{L} and P_{R} are the projection matrices onto left- and right-handed helicity as defined in Eq. (1.76) and Eq. (1.77). The coupling constants g_{L} and g_{R} are the more convenient combinations to use if the fermions involved are ultra-relativistic, since in this limit helicity is a conserved quantum number. They are given in terms of g_A and g_V by

$$g_{\mathrm{L}} = g_V + g_A = T_3 - Q\sin^2\theta_{\mathrm{W}} \tag{6.14}$$

$$g_{\mathrm{R}} = g_V - g_A = -Q\sin^2\theta_{\mathrm{W}} \tag{6.15}$$

The contribution of the photon-exchange graph is also easily obtained from Eq. (6.12) by making a few substitutions. First, the unitary-gauge Z^0 propagator must be replaced with the ξ-gauge photon propagator appropriate for a massless particle, $M_\gamma = 0$. Next the gauge coupling constant, e_Z, must be replaced by the electromagnetic one, $e_\gamma = e$. Also Γ_{Zf} must be replaced by $\Gamma_{\gamma f}$, which has the same form as in Eq. (6.13) with the numerical constants g_V and g_A of the neutral current replaced by the values $q_V = q_\mathrm{L} = q_\mathrm{R} = Q$ and $q_A = 0$ relevant for the electromagnetic current.

An immediate simplification of this amplitude is possible if the fermion masses are neglected relative to M_Z, as is the case here. This is because the $(p+p')_\mu(p+p')_\nu$ term in the Z-propagator may be dropped, since it contributes to the S-matrix an amount that is proportional to $g_A^2 m_\mathrm{f}^2/M_Z^2$. The same terms in the photon propagator may also be dropped for any fermion masses, since the axial couplings to the photon vanish, $q_A = 0$, for all fermion types.

The total matrix element, $\mathcal{M}(e^+e^- \to f\overline{f})$, is the sum of the photon- and Z-exchange contributions and therefore becomes:

$$\begin{aligned}\mathcal{M}(e^+e^- \to f\overline{f}) \;&=\; -\sum_{V=Z,\gamma} e_V^2\left[\overline{v}_e(\mathbf{p}')\gamma^\mu \Gamma_{Ve} u_e(\mathbf{p})\right]\left[\overline{u}_f(\mathbf{k})\gamma_\mu \Gamma_{Vf} v_f(\mathbf{k}')\right]\\ &\qquad \times\left[\frac{1}{(p+p')^2 + M_V^2 - \mathrm{i}\epsilon}\right]\end{aligned} \tag{6.16}$$

Averaging the square of this matrix element over the four initial spin states (two each for each incoming particle) and summing over the final spins (and, if necessary, colors) gives the following result:

$$\begin{aligned}\overline{\mathcal{M}^2} \;&=\; \frac{1}{4}\sum_{\text{spins}}\sum_{\text{colors}} |\mathcal{M}(e^+e^- \to f\overline{f})|^2\\ &=\; \frac{N_\mathrm{c}}{4}\sum_{V=Z,\gamma}\sum_{V'=Z,\gamma} e_V^2 e_V^2 \frac{K^{\mu\nu}(k,k')P_{\mu\nu}(p,p')}{(s-M_V^2)(s-M_{V'}^2)}\end{aligned} \tag{6.17}$$

in which $s = -(p+p')^2$ has been used, the "iϵ" terms have been dropped, and N_c is as usual $N_\mathrm{c} = 1$ if "f" is a lepton and $N_\mathrm{c} = 3$ if "f" is a quark. $K^{\mu\nu}$ and $P_{\mu\nu}$ represent the following Dirac traces:

$$\begin{aligned}P^{\mu\nu} \;&\equiv\; \sum_{\text{spins}} \mathrm{tr}\left[\gamma^\mu \Gamma_{Ve} u_e \overline{u}_e(\mathbf{p})\gamma^\nu \Gamma_{V'e} v_e \overline{v}_e(\mathbf{p}')\right]\\ &=\; -\,\mathrm{tr}\left[\gamma^\mu \Gamma_{Ve} \not{p} \gamma^\nu \Gamma_{V'e} \not{p}'\right]\\ &=\; -2\big[(g_{e\mathrm{L}}g'_{e\mathrm{L}} + g_{e\mathrm{R}}g'_{e\mathrm{R}})(p^\mu p'^\nu + p^\nu p'^\mu - p\cdot p'\eta^{\mu\nu})\\ &\qquad\qquad + \mathrm{i}(g_{e\mathrm{L}}g'_{e\mathrm{L}} - g_{e\mathrm{R}}g'_{e\mathrm{R}})\epsilon^{\mu\nu\lambda\rho}p_\lambda p'_\rho\big]\end{aligned} \tag{6.18}$$

and

$$\begin{aligned} K^{\mu\nu} &\equiv \sum_{\text{spins}} \text{tr}\big[\gamma^\mu \Gamma_{Vf} v_f \overline{v}_f(\mathbf{k}')\gamma^\nu \Gamma_{V'f} u_f \overline{u}_f(\mathbf{k})\big] \\ &= -\,\text{tr}\Big[\gamma^\mu \Gamma_{Vf} \not{k}' \gamma^\nu \Gamma_{V'f} \not{k}\Big] \\ &= -2\Big[(g_{f\text{L}}g'_{f\text{L}} + g_{f\text{R}}g'_{f\text{R}})(k^\mu k'^\nu + k^\nu k'^\mu - k\cdot k' \eta^{\mu\nu}) \\ &\qquad -\text{i}(g_{f\text{L}}g'_{f\text{L}} - g_{f\text{R}}g'_{f\text{R}})\epsilon^{\mu\nu\lambda\rho}k_\lambda k'_\rho\Big] \end{aligned} \tag{6.19}$$

The prime on the coupling constants g_{L} and g_{R} indicates it is the coupling appropriate to gauge boson V'.

Because the denominators involve the Mandelstam variable s, this process is conventionally referred to as an *s-channel* process.

Contracting these last results with one another gives the intermediate result

$$\begin{aligned} K^{\mu\nu}P_{\mu\nu} &= 16\Big[(g_{f\text{L}}g'_{f\text{L}}g_{e\text{L}}g'_{e\text{L}} + g_{f\text{R}}g'_{f\text{R}}g_{e\text{R}}g'_{e\text{R}})(p\cdot k')(p'\cdot k) \\ &\qquad +(g_{f\text{L}}g'_{f\text{L}}g_{e\text{R}}g'_{e\text{R}} + g_{f\text{R}}g'_{f\text{R}}g_{e\text{L}}g'_{e\text{L}})(p\cdot k)(p'\cdot k')\Big] \\ &= 4\Big[(g_{f\text{L}}g'_{f\text{L}}g_{e\text{L}}g'_{e\text{L}} + g_{f\text{R}}g'_{f\text{R}}g_{e\text{R}}g'_{e\text{R}})u^2 \\ &\qquad +(g_{f\text{L}}g'_{f\text{L}}g_{e\text{R}}g'_{e\text{R}} + g_{f\text{R}}g'_{f\text{R}}g_{e\text{L}}g'_{e\text{L}})t^2\Big] \end{aligned} \tag{6.20}$$

The last equality uses the ultra-relativistic approximation to Eq. (6.2) and Eq. (6.3) for the Mandelstam invariants as applied to this reaction: $s = -2p\cdot p' = -2k\cdot k'$, $t = 2p\cdot k = 2p'\cdot k'$, and $u = 2p\cdot k' = 2p'\cdot k$.

Combining these results gives the spin-averaged squared matrix element

$$\begin{aligned} \overline{\mathcal{M}^2} &= N_\text{c}\Bigg(\left|\sum_{V=Z,\gamma} e_V^2 \frac{g_{e\text{L}}g_{f\text{L}}}{s-M_V^2}\right|^2 u^2 + \left|\sum_{V=Z,\gamma} e_V^2 \frac{g_{e\text{R}}g_{f\text{R}}}{s-M_V^2}\right|^2 u^2 \\ &\qquad + \left|\sum_{V=Z,\gamma} e_V^2 \frac{g_{e\text{L}}g_{f\text{R}}}{s-M_V^2}\right|^2 t^2 + \left|\sum_{V=Z,\gamma} e_V^2 \frac{g_{e\text{R}}g_{f\text{L}}}{s-M_V^2}\right|^2 t^2 \Bigg) \end{aligned} \tag{6.21}$$

This last formula has a simple physical interpretation that might have been expected for massless – i.e. ultra-relativistic – fermions. Eq. (6.21) gives the rate for the collision process as the sum of the incoherent scattering rates in which the initial and final fermions have definite helicity.

Also, as is easily seen from Eq. (6.10), the limits where $u \to 0$ or $t \to 0$ correspond for ultra-relativistic fermions to the cases where the scattering angle, θ, between the directions of the incoming electron, e^-, and the outgoing fermion, f, in the CM frame approach zero ($t \to 0$) or $\pi(u \to 0)$. In this case the direction of motion of both the incident and final particles are parallel or antiparallel. An argument identical to that given in Subsection 4.1.5 then implies that the conservation of angular momentum along this common direction of motion is only consistent with conservation of helicity for specific choices for the relative helicities of the initial and final fermions. This is seen in the squared matrix element, Eq. (6.21), by the way that each of the terms for definite helicities vanishes either for $t = 0$ or for $u = 0$.

With Eq. (6.21) in hand, the cross section for $e^+e^- \to f\overline{f}$ is easily computed. From the definition, Eq. (3.41), of the differential cross section, we have

$$
\begin{aligned}
\mathrm{d}\sigma(e^+e^- \to f\overline{f}) &= \frac{1}{2p^0 2p'^0 f}\overline{\mathcal{M}^2}\,(2\pi)^4\delta^4(p+p'-k-k')\frac{\mathrm{d}^3k\,\mathrm{d}^3k'}{(2\pi)^6 2k^0 2k'^0} \\
\text{with} \qquad f &\equiv \frac{-p\cdot p'}{p^0p'^0}v_{\mathrm{rel}} \\
&\approx \frac{s}{2p^0p'^0} \qquad \text{(ultra-relativistic)}
\end{aligned}
\tag{6.22}
$$

so combining all of the above results gives

$$
\begin{aligned}
\mathrm{d}\sigma(e^+e^- \to f\overline{f}) &= \frac{8\pi^2\alpha^2}{s}N_{\mathrm{c}}\Big([|A_{\mathrm{LL}}(s)|^2 + |A_{\mathrm{RR}}(s)|^2]u^2 \\
&\quad + [|A_{\mathrm{LR}}(s)|^2 + |A_{\mathrm{RL}}(s)|^2]t^2\Big)\ \mathrm{d}\chi
\end{aligned}
\tag{6.23}
$$

in which the helicity amplitudes $A_{ij}(s)$, with $i, j =$L, R, are given by

$$
A_{ij} = \frac{1}{\sin^2\theta_{\mathrm{W}}\cos^2\theta_{\mathrm{W}}}\left(\frac{g_{ei}g_{fj}}{s - M_Z^2}\right) + \frac{Q_eQ_f}{s}
\tag{6.24}
$$

with g_{fi} the coupling strengths of left- and right-handed particles to the Z^0, and with $\mathrm{d}\chi$ denoting the Lorentz-invariant measure on phase-space,

$$
\begin{aligned}
\mathrm{d}\chi &\equiv (2\pi)^4\delta^4(p+p'-k-k')\frac{\mathrm{d}^3k\,\mathrm{d}^3k'}{(2\pi)^6 2k^0 2k'^0} \\
&= (2\pi)^4\delta^4(p+p'-k-k')\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}2\pi\delta(k'^2)\theta(k'^0)\frac{\mathrm{d}^4k'}{(2\pi)^4} \\
&= 2\pi\delta[(p+p'-k)^2]\theta(p^0+p'^0-k^0)\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0} \\
&= -\frac{1}{8\pi s}\delta(s+t+u-m_a^2-m_b^2-m_c^2-m_d^2)\ \mathrm{d}u\ \mathrm{d}t
\end{aligned}
\tag{6.25}
$$

This gives the final form for the invariant cross section:

$$
\frac{\mathrm{d}\sigma}{\mathrm{d}u\,\mathrm{d}t}(e^+e^- \to f\overline{f}) = -\frac{\pi\alpha^2}{s^2}N_{\mathrm{c}}\Big([|A_{\mathrm{LL}}(s)|^2 + |A_{\mathrm{RR}}(s)|^2]u^2
$$

$$+[|A_{\rm LR}(s)|^2 + |A_{\rm RL}(s)|^2]t^2\Big)\,\delta(s+t+u) \tag{6.26}$$

Evaluating this in the CM frame gives the differential cross section as a function of the angle between e^- and f directions, which is called the scattering angle θ:

$$\frac{d\sigma}{\sin\theta\, d\theta}(e^+e^- \to f\overline{f}) \;=\; \frac{\pi\alpha^2 s N_{\rm c}}{8}\Big\{[|A_{\rm LL}(s)|^2 + |A_{\rm RR}(s)|^2](1+\cos\theta)^2 + [|A_{\rm LR}(s)|^2 + |A_{\rm RL}(s)|^2](1-\cos\theta)^2\Big\} \tag{6.27}$$

This last result uses the relations $s = 4E^2$, $t = -2E^2(1-\cos\theta)$, and $u = -2E^2(1+\cos\theta)$ that connect the Mandelstam variables to θ. Integrating θ over its range $0 < \theta < \pi$ using $\int_0^\pi (1\pm\cos\theta)^2 \sin\theta \; d\theta = \frac{8}{3}$ gives the total rate:

$$\sigma(e^+e^- \to f\overline{f}) \;=\; \frac{\pi\alpha^2 s N_{\rm c}}{3}\Big(|A_{\rm LL}(s)|^2 + |A_{\rm RR}(s)|^2 + |A_{\rm LR}(s)|^2 + |A_{\rm RL}(s)|^2\Big) \tag{6.28}$$

6.3 e^+e^- annihilation: applications

The energy dependence of this cross section for electron–positron annihilation is largely governed by the s dependence of the polarized amplitudes $A_{ij}(s)$. There are naturally three regions to consider depending on the relative size of contributions due to photon- and Z^0-exchange. We consider each region successively in this section.

6.3.1 *Low energies: $e^+e^- \to$ hadrons*

For CM-frame energies that are very small compared to $M_Z = 90$ GeV – yet still large compared to the fermion masses, m_e and m_f – the amplitudes $A_{ij}(s)$ are well approximated by the contribution due to photon exchange:

$$A_{\rm LL} \approx A_{\rm LR} \approx A_{\rm RL} \approx A_{\rm RR} \approx \frac{Q_e Q_f}{s} \tag{6.29}$$

In this limit the electron–positron annihilation rate reduces to the form found in quantum electrodynamics,

$$\frac{d\sigma}{du\, dt}(e^+e^- \to f\overline{f})\Big|_{\gamma\text{-exchange}} = -\frac{2\pi\alpha^2}{s^2} Q_e^2 Q_f^2 N_{\rm c}\left(\frac{u^2+t^2}{s^2}\right)\;\delta(s+t+u) \tag{6.30}$$

which becomes, in the CM frame,

$$\frac{\mathrm{d}\sigma}{\sin\theta\,\mathrm{d}\theta}(e^+e^- \to f\overline{f})\Big|_{\gamma\text{-exchange}} = \frac{\pi\alpha^2}{2s}Q_e^2Q_f^2N_c(1+\cos^2\theta) \tag{6.31}$$

The total rate similarly reduces to the result familiar from QED,

$$\sigma(e^+e^- \to f\overline{f})|_{\gamma\text{-exchange}} = \frac{4\pi\alpha^2}{3s}Q_e^2Q_f^2N_c \tag{6.32}$$

6.3.1.1 $\mu^+\mu^-$ production

To get a feeling for the size of these numbers, consider $\mu^+\mu^-$ production at energies $\sqrt{s} = 1$ GeV. At these energies the ratio s/M_Z^2 is $s/M_Z^2 \approx 10^{-4}$ and $m_\mu^2/s \approx 10^{-2}$, so Eq. (6.32) provides a perfectly adequate description. In this case, using $\alpha = 1/137$ and $Q_e^2 = Q_\mu^2 = N_c = 1$ gives

$$\begin{aligned}\sigma(e^+e^- \to \mu^+\mu^-) &= \frac{4\pi\alpha^2}{3s}\\ &= 2.23\times 10^{-4}\ (\text{GeV})^{-2}\\ &= 87\ \text{nb}\end{aligned} \tag{6.33}$$

The units of the final line are *nanobarns* with a *barn* defined to be 10^{-24}cm^2. Now, an accelerator *luminosity* is defined as the rate at which the accelerator can deliver incident particles per unit area of beam. This is useful because the product of the cross section and luminosity gives the number of events which can be expected per unit time. Luminosity is usually quoted in inverse $\text{cm}^2\,\text{s}$; for instance, the LEP I experiment achieved $2.4\times 10^{31}/\text{cm}^2\,\text{s}$, but at a much higher energy than 1 GeV. A machine designed to study the 1 GeV energy range in detail, the VEPP-2000, has a luminosity of $10^{32}/\text{cm}^2\,\text{s} = 0.1/\text{nb.s}$, enough to produce about 9 $\mu^+\mu^-$ pairs per second.

For the purposes of comparison, a strong interaction cross section is roughly a typical strong interaction scale raised to the power that is dictated by dimensional analysis: $\sigma_{\text{str}} \sim \Lambda_{\text{QCD}}^{-2} \sim 40\ (\text{GeV})^{-2} \sim 20$ mbarn. We take the strong scale $\Lambda_{\text{QCD}} \approx 150$ MeV in this estimate.

6.3.1.2 Hadron production

There is an immediate application of these low-energy results that takes advantage of the fact that in this energy range the energy dependence of the cross section is the same for all particle types in the final state. To use this fact it is convenient to compute the cross section for producing hadrons in low-energy e^+e^- annihilations, normalized by the muon pair-production rate. The complication is that quarks and gluons interact very strongly with each other at low energies, as we discuss in Part III. In fact, the interactions are so strong that quarks and gluons are not valid external states for a reaction; instead they stick together into bound states called hadrons. However, at suitably high energies, $\sqrt{s} > 1$ GeV or so, the strong coupling is weaker and perturbation theory begins to be useful. At these energies the process of producing quarks and gluons and the process of their combining into hadronic bound states are approximately independent. Rather than summing over a complete set of hadronic final states, one can sum over color-neutral quark and gluon final states and ignore the question of how these project onto the hadronic states. For energies large enough to justify perturbation theory the dominant terms in the final-state sum are then the quark–antiquark pairs. The cross section for this process may therefore be computed by summing Eq. (6.32) over all quark flavors with masses small enough to allow pair production at the given CM energy, $\sqrt{s}$.

This gives the following expression for the ratio

$$\begin{aligned} R_{\rm H} &\equiv \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \\ &\approx \frac{3s}{4\pi\alpha^2} \sum_{q, 2m_q < \sqrt{s}} \sigma(e^+e^- \to q\bar{q}) \\ &= 3 \sum_{q, 2m_q < \sqrt{s}} Q_q^2 \end{aligned} \tag{6.34}$$

The overall factor of 3 here is due to the number of colors available to each quark type. The approximation that is used in the second line is the low-energy expression, Eq. (6.32), for the cross section in which fermion masses are neglected relative to $\sqrt{s}$. The neglect of fermion masses implies that Eq. (6.34) should not be expected to hold in the immediate vicinity of a threshold, $\sqrt{s} \approx 2m_q$.

Clearly $R_{\rm H}(s)$ is independent of energy between mass thresholds to the extent that photon exchange dominates the production cross section. Measurement of its value gives an indication of the number of quark degrees of freedom that are available at the given energy. It gives, in particular, an experimental indication of the number of colors, $N_{\rm c}$.

A plot of the experimental value for this ratio is given in Figure 6.3 (from data compiled and made freely available by the Particle Data Group). The solid lines in the figure represent Eq. (6.34) evaluated using u, d, s (three quarks), u, d, s, c (four quarks), and u, d, s, c, b (five quarks). At low energies, the hadronic character of the final state is important and the cross section has a number of peaks and troughs. Above this region, the cross section is

Fig. 6.3. The measured pair-production ratio, $R_{\rm H}$.

reasonably well approximated by Eq. (6.34) (provided $N_c = 3$), including steplike features at $s \simeq 2m_c$ and $s \simeq 2m_b$. (The figure does not show very high but, rather, narrow spikes at these points, which arise because of $c\overline{c}$ and $b\overline{b}$ bound states.)

6.3.2 Intermediate energies: asymmetries

The range of CM interaction energies in the neighborhood of 10 GeV or so is an intermediate range within which the neutral current contribution to the cross section is still small, but is large enough to be detectable. In this energy range we must keep the subdominant term in the expansion of the helicity amplitudes, $A_{ij}(s)$, in powers of s/M_Z^2:

$$A_{ij}(s) \approx \frac{Q_e Q_f}{s} - \frac{1}{\sin^2\theta_{\rm W}\cos^2\theta_{\rm W}} \frac{g_{ei}g_{fj}}{M_Z^2} \tag{6.35}$$

Since the small M_Z^{-2} effect of the neutral-current interaction must be picked out of a background of electromagnetic events it helps to focus on some sort of observable to which the electromagnetic interactions do not contribute at all. A natural choice for such an observable would be anything which measures either C or P violation, since this is a symmetry of the electromagnetic, but not of the neutral-current, interaction.

6.3.2.1 Left–right asymmetry

An example of this type of an observable is given by the comparison of cross sections as the helicity of the initial electron–positron pair is varied, since the amplitude only develops a dependence on helicity through the neutral-current couplings. A candidate example might be to take the difference between the cross section measured for left- and right-handed electrons colliding with unpolarized positrons:

$$\mathcal{A}_{\rm LR} = \frac{\sigma(e_{\rm L}) - \sigma(e_{\rm R})}{\sigma(e_{\rm L}) + \sigma(e_{\rm R})} \tag{6.36}$$

This is known as the *left–right asymmetry* and may be computed using Eq. (6.28) and Eq. (6.35). The leading contribution arises from the interference of the neutral-current and electromagnetic amplitudes:

$$\begin{aligned}\mathcal{A}_{\rm LR}(e^+e^- \to f\overline{f}) &= \frac{[|A_{\rm LL}|^2 + |A_{\rm LR}|^2] - [|A_{\rm RL}|^2 + |A_{\rm RR}|^2]}{[|A_{\rm RL}|^2 + |A_{\rm RR}|^2] + [|A_{\rm RL}|^2 + |A_{\rm RR}|^2]} \\ &\simeq -\left(\frac{s}{M_Z^2}\right)\left[\frac{(g_{e\rm L}^2 - g_{e\rm R}^2)(g_{f\rm L}^2 + g_{f\rm R}^2)}{2Q_eQ_f \sin^2\theta_{\rm W}\cos^2\theta_{\rm W}}\right]\end{aligned} \tag{6.37}$$

For $\sqrt{s} = 25$ GeV the ratio $s/M_Z^2 = 0.08$, so this asymmetry can be in the neighborhood of an 8% effect at these energies.

6.3.2.2 Forward–backward asymmetry

A similar kind of asymmetry that is also sensitive to C-violating interactions is the *forward–backward asymmetry*, $\mathcal{A}_{\rm FB}$. This is defined in the following way. Suppose the particle detector is imagined to be enclosed within a sphere which is centered at the collision point. If the direction of motion of the initial electron is taken to define the north pole of this sphere, then $\mathcal{A}_{\rm FB}$ is given by the difference in cross sections, call them $\sigma_\pm$, between those collisions for which the final fermion, f, emerges in the northern – $0 < \theta < \frac{\pi}{2}$ – and southern – $\frac{\pi}{2} < \theta < \pi$ – hemispheres of this sphere, normalized by the total cross section. That is to say, for

$$\sigma_\pm \equiv \pm \int_0^{\pm 1} \frac{{\rm d}\sigma}{{\rm d}\cos\theta}\,{\rm d}\cos\theta \tag{6.38}$$

we have

$$\begin{aligned}\mathcal{A}_{\rm FB} &= \frac{\sigma_+ - \sigma_-}{\sigma_+ + \sigma_-} \\ &= \frac{3}{4}\left(\frac{[|A_{\rm LL}(s)|^2 + |A_{\rm RR}(s)|^2] - [|A_{\rm LR}(s)|^2 + |A_{\rm RL}(s)|^2]}{[|A_{\rm LL}(s)|^2 + |A_{\rm RR}(s)|^2] + [|A_{\rm LR}(s)|^2 + |A_{\rm RL}(s)|^2]}\right)\end{aligned}$$

$$\approx -\left(\frac{s}{M_Z^2}\right)\left[\frac{3(g_{eL}^2 - g_{eR}^2)(g_{fL}^2 - g_{fR}^2)}{8Q_eQ_f \sin^2\theta_W \cos^2\theta_W}\right] \tag{6.39}$$

The approximation used in the third line here is the use of the approximate form, Eq. (6.35), for the helicity amplitudes, $A_{ij}(s)$.

Notice that although $\mathcal{A}_{LR}$ and $\mathcal{A}_{FB}$ are proportional at low energies to the squared difference of the left- and right-handed electron couplings, they each sample a different combination of the couplings of the pair-produced fermions. It was measurements of these asymmetries which first gave convincing evidence of the existence of the Z^0 boson, before any accelerator had sufficient energy to create one directly.

6.3.3 High energies: asymptotic forms

The next simplest limit takes $s \gg M_Z^2$, so the helicity amplitudes may be approximated by

$$A_{ij}(s) \approx \left[Q_eQ_f + \frac{g_{ei}g_{fj}}{\sin^2\theta_W \cos^2\theta_W}\right]\frac{1}{s} \tag{6.40}$$

In this limit the energy dependence of the cross section is precisely as it is in the photon-dominated case, but with the electromagnetic coupling constants Q_eQ_f replaced by the combination in the square bracket of Eq. (6.40) above.

At these energies the photon and Z^0 exchange graphs differ only in the strength of their couplings. This is the signature of electroweak unification; at high energies the weak and electromagnetic interactions are indeed very similar in form.

6.4 The *Z* boson resonance

Even a superficial inspection of Eq. (6.26), Eq. (6.27), or Eq. (6.28) indicates that there is a problem in the regime where the exchange momentum, $r \equiv (p + p')$, approaches the Z boson *mass shell*, $r^2 = -M_Z^2$, where the intermediate Z boson has the right four-momentum to be a real (as opposed to virtual) particle. As $s = -r^2$ approaches M_Z^2, the cross section apparently diverges. This indicates a failure of the perturbative expansion for the S-matrix. After all, the S-matrix elements are bounded by the general requirement of unitarity, so their perturbative approximations must also be bounded, or must be bad approximations. For this reason, in this section we will have to make a digression into the topic of higher-order perturbative corrections. We will find that these corrections are essential to resolving this puzzle.

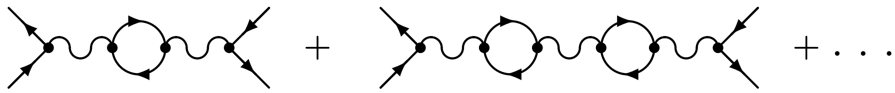


Fig. 6.4. Important corrections near $s = M_Z^2$.

There is a large body of knowledge concerning the higher-order perturbative corrections to the lowest-order expressions described up to this point throughout this book. As has already been seen, these corrections are an important part of the agreement of the standard model with experiment, particularly for the properties of the Z^0 boson, due to the accuracy of the experimental results that are now available. This is doubly true in the resonance regime, $s \simeq M_Z^2$, where the radiative corrections are already essential at leading order, and where exquisitely precise measurements, conducted by the LEP I experiment at CERN and the SLC experiment at SLAC, have thoroughly explored the physics and provided some of the highest-precision experiments, and tightest tests, of the standard model.

6.4.1 Corrections near resonance

From the fact that the perturbative S-matrix near the Z^0-mass shell diverges, it follows that there must be additional, supposedly higher-order, graphs that nevertheless contribute in an important way, when $s \simeq M_Z^2$. It is the purpose of this subsection to identify these contributions and to find their size, in order to have a good approximation to the lowest-order cross section near the pole.

The graphs that are the source of the difficulty are graphs of the form of Figure 6.4, which can be thought of as modifying the Z^0-boson propagator. Although these graphs are superficially suppressed relative to Figs. 6.2 by additional powers of the small electroweak coupling constants, they can be of comparable size for Z^0-boson four-momenta, r_μ, that lie within the immediate neighborhood (i.e. within $O(\alpha M_Z^2)$) of $r^2 = -M_Z^2$.

The reason that corrections are needed here, is because of a cancellation in the denominator of the propagator, $r^2 + M_Z^2 \simeq 0$, which renders the propagator much larger than the usual size $\sim 1/M_Z^2$. Each "loop" of the form shown in Figure 6.4 (the loop is the pair of fermionic propagators going in a circle, including the loop-momentum integration and the vertex factors) gives a contribution which is of the order αM_Z^2 (as can be argued on dimensional grounds). Each loop also leads to one more appearance of the gauge-boson propagator, introducing a factor of $1/(r^2+M_Z^2)$. When $r^2+M_Z^2 \sim \alpha M_Z^2$, the addition of a loop is not a suppressed correction. While there are other ways of adding loops, they do not lead to a new factor of $1/(s-M_Z^2)$ per loop, and we can therefore continue to neglect them.

Denote the contribution of one of these loops, after all polarization sums have been performed, as $M_Z^2\alpha(x+\mathrm{i}y)$, with x and y pure numbers of order 1. We know on dimensional grounds that this is the right general form for the correction. It is easy to see that the inclusion of these graphs does remove the divergence of the S-matrix near the Z^0-mass shell. The whole set of graphs with 0, 1, 2, ... "loops" inserted can be summed as a geometric series, and appears as a correction in the denominator of the Z^0-boson propagator:

$$\begin{aligned}(0\text{ loop}) + (1\text{ loop}) + \cdots \;\propto\; & \frac{1}{r^2+M_Z^2} + \frac{M_Z^2\alpha(x+\mathrm{i}y)}{(r^2+M_Z^2)^2} + \frac{[M_Z^2\alpha(x+\mathrm{i}y)]^2}{(r^2+M_Z^2)^3} + \cdots \\ = \; & \frac{1}{r^2 + M_Z^2(1-\alpha(x+\mathrm{i}y))} \end{aligned} \tag{6.41}$$

A radiative (loop) correction which can be understood as a correction of a propagator in this way is called a *self-energy correction*, because it represents a correction to the propagating particle's energy due to its interaction with the vacuum (with its own field).

The correction to the propagator is *complex*, and in particular it moves the singular point of the propagator away from the real point $r^2 + M_Z^2 = 0$ and out to a complex point:

$$r^2 = -M_Z^2[1 - \alpha(x + \mathrm{i}y)] \tag{6.42}$$

However, since the Z^0 boson four-momentum, r_μ, must necessarily be real, it can never satisfy Eq. (6.42), and so the corresponding source of the divergence of the S-matrix does not arise. Since these corrections are only significant for s within $O(\alpha M_Z^2)$ of $s = M_Z^2$, none of the discussions of the previous sections need be modified.

From the above considerations it is clearly the imaginary part of the contribution from diagrams like Figure 6.4 that is the most important. A real shift, e.g., αx in Eq. (6.42), can be re-interpreted as a shift in the Z^0 boson mass squared, $M_Z^2(\text{physical}) = M_Z^2(1 - \alpha x)$. In fact, it is the combination $M_Z^2(1 - \alpha x)$ which we "measure" as the true mass of the Z boson – a point we will return to in Subsection 7.4.1. An imaginary shift cannot be similarly absorbed. The next step is to determine how to compute the size of this shift reliably. The main conclusion to be argued is that the imaginary part of the shift in the position of the pole of the propagator is simply related to the mass and total decay width of the Z^0. Once this is established, the results of Chapter 4 may be used immediately to compute the size of the shift.

For these purposes we take advantage of the small range of momenta for which these corrections are appreciable. It is therefore a good approximation to take the corrections to the Z^0 propagator near the mass shell, due to Figure 6.4, to be independent of momentum. That is, we are neglecting the r^2 dependence of y above. It will be convenient to redefine y as $-\mathrm{i}M_Z^2 \alpha y \equiv -\mathrm{i}M_Z\Delta$. That is to say

$$\frac{1}{r^2 + M_Z^2 - \mathrm{i}\epsilon} \to \frac{1}{r^2 + M_Z^2 - \mathrm{i}M_Z\Delta} \tag{6.43}$$

where Δ is a constant with the dimensions of mass that is much smaller than the Z^0 mass itself: $\Delta \sim O(\alpha M_Z)$. The infinitesimal, ϵ, has been dropped since its role is to indicate how to avoid the singularity at $r^2 = -M_Z^2$ in the integration over the component r^0. This is no longer necessary since the additional term, $-\mathrm{i}M_Z\Delta$, shifts the singularity off of the (real) integration axis.

The remainder of the argument is to relate the parameter Δ to the properties of the particles "going around in the loop." The main observation is that the last two terms in the denominator of Eq. (6.43) may be written to lowest order in α as a perfect square:

$$M_Z^2 - \mathrm{i}M_Z\Delta = \left(M_Z - \frac{\mathrm{i}\Delta}{2}\right)^2 + O(\alpha^2) \tag{6.44}$$

which is completely equivalent to a shift of the Z^0 mass by a small imaginary part. If the arguments used to derive the propagator from the sum over virtual particle states in Subsection 5.2.1 are now run backwards, the shift of the pole of the propagator implies that the intermediate Z^0 states evolve in time with a small negative imaginary part for their mass.

Since the time dependence of these particle states, in the Z rest frame, is

$$\begin{aligned} |Z(t)\rangle &= \mathrm{e}^{-\mathrm{i}M_Z t}|Z(0)\rangle \\ &\to \mathrm{e}^{-\mathrm{i}M_Z t - \frac{1}{2}\Delta t}|Z(0)\rangle \end{aligned} \tag{6.45}$$

the probability that the Z^0 particle survives as a function of time therefore becomes

$$\begin{aligned} p(t) &= |\langle Z(t)|Z(0)\rangle|^2 \\ &= \mathrm{e}^{-\Delta t}p(0) \end{aligned} \tag{6.46}$$

This implies that Δ should be identified with the full decay width for the

Z^0 particle as computed in Section 4.1 (c.f. Eq. (4.38) and Eq. (4.39)):

$$\begin{aligned} \Delta &= \Gamma_Z \\ &= \frac{e_Z^2}{12\pi} M_Z \sum_f (g_V^2 + g_A^2) N_{\rm c} \end{aligned} \tag{6.47}$$

In a nutshell, then, the net effect of all of the higher-order graphs such as those of Figure 6.4 for the results of Section 6.2 is to replace the denominator $(s - M_Z^2)$ by

$$\frac{1}{s - M_Z^2} \to \frac{1}{s - M_Z^2 - {\rm i} M_Z \Gamma_Z} \tag{6.48}$$

Two comments concerning this replacement are in order.

(i) Notice that, as advertised, because Γ_Z is $O(\alpha M_Z)$, the difference between the corrected propagator and the original one is completely negligible *except* when $s - M_Z^2 = O(\alpha M_Z^2)$. This implies that none of the results of the previous sections are affected by this change (except at an order where there are other corrections anyway).

(ii) Although the S-matrix elements for $e^+e^- \to f\overline{f}$ no longer diverge after this substitution, they do become very large. In fact, for s precisely equal to M_Z^2, the Z^0-exchange contributions to the helicity amplitudes, $A_{ij}(s)$, are larger than those due to photon decay by a factor of $1/\alpha \sim 100$. This implies that there is an enormous enhancement of the Z^0-exchange amplitude for s in the immediate vicinity of M_Z^2, and so photon exchange may be neglected for these energies. The s-dependence of the squared helicity amplitudes then acquires the classic Lorentzian, or *Breit–Wigner* lineshape of a *resonance*:

$$\begin{aligned} {\rm d}\sigma(e^+e^- \to Z^0 \to f\overline{f}) &\propto \left| \frac{1}{s - M_Z^2 - {\rm i} M_Z \Gamma_Z} \right|^2 \\ &= \frac{1}{(s - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \end{aligned} \tag{6.49}$$

6.4.2 Application: $e^+e^- \to f\overline{f}$ near resonance

Using this replacement in Eq. (6.24) and neglecting photon exchange gives the following approximation for the helicity amplitudes, $A_{ij}(s)$, that holds near resonance ("near resonance" here means for $|\sqrt{s} - M_Z| \le O(\sqrt{\alpha M_Z^2}) \approx$ 10 GeV):

$$|A_{ij}(s)|^2 \approx \left| \frac{g_{ei} g_{fj}}{\sin^2\theta_{\rm W} \cos^2\theta_{\rm W}} \frac{1}{s - M_Z^2 - {\rm i} M_Z \Gamma_Z} \right|^2$$

$$= \frac{g_{ei}^2 g_{fj}^2}{\sin^4\theta_W \cos^4\theta_W} \frac{1}{(s - M_Z^2)^2 + M_Z^2\Gamma_Z^2} \tag{6.50}$$

The resulting expressions for the differential and integrated cross sections in the CM frame are

$$\begin{aligned} \frac{d\sigma}{\sin\theta\, d\theta}(e^+e^- \to Z^0 \to f\overline{f})\Big|_{\text{res}} &= \frac{\pi\alpha^2}{8\sin^4\theta_W\cos^4\theta_W} N_c \frac{s}{(s-M_Z^2)^2 + M_Z^2\Gamma_Z^2} \\ &\quad \times\Big([g_{eL}^2 g_{fL}^2 + g_{eR}^2 g_{fR}^2](1+\cos\theta)^2 \\ &\quad + [g_{eL}^2 g_{fR}^2 + g_{eR}^2 g_{fL}^2](1-\cos\theta)^2\Big) \end{aligned} \tag{6.51}$$

and

$$\begin{aligned} \sigma_{\text{res}}(e^+e^- \to Z^0 \to f\overline{f}) &= \frac{\pi\alpha^2}{3\sin^4\theta_W\cos^4\theta_W} N_c \frac{s}{(s-M_Z^2)^2 + M_Z^2\Gamma_Z^2} \\ &\quad \times (g_{eL}^2 + g_{eR}^2)(g_{fL}^2 + g_{fR}^2) \end{aligned} \tag{6.52}$$

A particularly clean prediction on resonance is possible for the asymmetries, $\mathcal{A}_{LR}$ and $\mathcal{A}_{FB}$, respectively defined by Eq. (6.36) and Eq. (6.38), since the common resonant energy dependence drops out of these cross section ratios:

$$\begin{aligned} \mathcal{A}_{LR}|_{\text{res}} &= \frac{g_{eL}^2 - g_{eR}^2}{g_{eL}^2 + g_{eR}^2} \\ &= \frac{\frac{1}{4} - \sin^2\theta_W}{\frac{1}{4} - \sin^2\theta_W + 2\sin^4\theta_W}, \\ \mathcal{A}_{FB}|_{\text{res}} &= \frac{(g_{eL}^2 - g_{eR}^2)(g_{fL}^2 - g_{fR}^2)}{(g_{eL}^2 + g_{eR}^2)(g_{fL}^2 + g_{fR}^2)} \\ &= \frac{(g_{fL}^2 - g_{fR}^2)}{(g_{fL}^2 + g_{fR}^2)} \mathcal{A}_{LR}|_{\text{res}} \end{aligned} \tag{6.53}$$

6.4.2.1 Factorization

Resonant amplitudes and cross sections have a particularly simple form when evaluated right on the central point of the resonance, $s = M_Z^2$. On resonance the Z^0 propagator may be simplified using the spin-sum identity, Eq. (1.119):

$$\frac{1}{r^2 + M_Z^2 - iM_Z\Gamma_Z}\left(\eta_{\mu\nu} + \frac{r_\mu r_\nu}{M_Z^2}\right) \to \frac{i}{M_Z\Gamma_Z}\sum_{\lambda=-1}^{1} \epsilon(\mathbf{r},\lambda)\epsilon^*(\mathbf{r},\lambda) \tag{6.54}$$

The scattering matrix element of Eq. (6.12) may be written as (c.f. Eq. (4.8) for the Z^0 decay matrix element)

$$\begin{aligned}
\mathcal{M}(e^+e^- \to f\overline{f})\Big|_{\text{res}} &= -\frac{\mathrm{i}e_Z^2}{M_Z\Gamma_Z}\sum_{\lambda=-1}^{1}\left[\overline{v}_e(\mathbf{p}')\gamma^\mu\Gamma_{Ze}u_e(\mathbf{p})\epsilon^*(\mathbf{p}+\mathbf{p}',\lambda)\right] \\
&\qquad\times\left[\overline{u}_f(\mathbf{k})\gamma^\nu\Gamma_{Zf}v_f(\mathbf{k}')\epsilon(\mathbf{p}+\mathbf{p}',\lambda)\right] \\
&= -\frac{\mathrm{i}}{M_Z\Gamma_Z}\sum_{\lambda=-1}^{1}\left[\mathrm{i}e_Z\overline{v}_e(\mathbf{p}')\gamma^\mu\Gamma_{Ze}u_e(\mathbf{p})\epsilon^*(\mathbf{p}+\mathbf{p}',\lambda)\right] \\
&\qquad\times\left[-\mathrm{i}e_Z\overline{u}_f(\mathbf{k})\gamma^\nu\Gamma_{Zf}v_f(\mathbf{k}')\epsilon(\mathbf{p}+\mathbf{p}',\lambda)\right] \\
&= +\frac{\mathrm{i}}{M_Z\Gamma_Z}\sum_{\lambda=-1}^{1}[\mathcal{M}(Z^0\to e^+e^-)]^*[\mathcal{M}(Z^0\to f\overline{f})]
\end{aligned} \tag{6.55}$$

That is, on resonance the cross section for $e^+e^- \to f\overline{f}$ *factorizes* into the product of the amplitude for the process $e^+e^- \to Z^0$ and the decay $Z^0 \to f\overline{f}$ for a Z^0 particle at rest, $E_Z = M_Z$. This is as would be expected if a real Z^0 is produced and then decays. The Z^0 so produced does not appear with its spin in a pure state but instead is prepared in a state that is described by a density matrix which has each of the three possible spin states equally weighted. Since this is indeed how real Z^0s are produced this justifies the choice made in Eq. (4.10) for the spin state of an unpolarized Z^0 sample.

The cross section for this process has a similar factorized form:

$$\begin{aligned}
\sigma(e^+e^- \to Z^0 \to f\overline{f}) &= \frac{e_Z^4}{48\pi}N_{\text{c}}\frac{1}{\Gamma_Z^2}(g_{e\text{L}}^2+g_{e\text{R}}^2)(g_{f\text{L}}^2+g_{f\text{R}}^2) \\
&= \frac{12\pi}{M_Z^2}\frac{\Gamma(Z\to e^+e^-)}{\Gamma_Z}\frac{\Gamma(Z\to f\overline{f})}{\Gamma_Z}
\end{aligned} \tag{6.56}$$

Here we used $g_{\text{L}}^2 + g_{\text{R}}^2 = 2(g_V^2 + g_A^2)$. Note that this result is precisely the Breit-Wigner result for (ultra-relativistic) scattering through a p-wave (spin-one) resonance,

$$\sigma = \frac{16\pi}{s}\frac{(2s_Z+1)}{(2s_e+1)(2s_{\overline{e}}+1)}\frac{\Gamma(Z\to e^+e^-)}{\Gamma_Z}\frac{\Gamma(Z\to f\overline{f})}{\Gamma_Z} \tag{6.57}$$

where $(2s_Z + 1) = 3$ is the number of spin states of the Z boson, and $(2s_e + 1) = (2s_{\overline{e}} + 1) = 2$ are the number of spin states of the incoming particles. This latter factor appears because we are computing the spin-averaged cross section; had we computed the cross section for a specific spin state of the e^+ and e^-, no such factor would appear.

Equation (6.56) has a natural physical interpretation. In order to bring this interpretation out, rewrite Eq. (6.56) in terms of the total cross section for Z^0 production on resonance, defined by summing the above result over all possible fermion–antifermion final states. Then, using $\sum_f \Gamma(Z \to f\overline{f}) = \Gamma_Z$, we find

$$\sigma(e^+e^- \to Z^0 \to f\overline{f}) = \sigma_{\text{tot}} B(Z \to f\overline{f}) \tag{6.58}$$

Here $B(Z \to f\overline{f}) = \Gamma(Z \to f\overline{f})/\Gamma_Z$ is the Z^0 branching fraction into the final fermion–antifermion pairs of flavor "f." The total cross section, σ_{tot}, is itself given explicitly by

$$\begin{aligned} \sigma_{\text{tot}}(e^+e^- \to Z^0) &= \frac{12\pi}{M_Z^2} B(Z \to e^+e^-) \\ &= 1.52 \times 10^{-4}\ (\text{GeV})^{-2} \\ &= 59.4\ \text{nb} \end{aligned} \tag{6.59}$$

where we used the experimental values for the width and branching fraction. Now both the total cross section, σ_{tot}, and the cross section $\sigma(e^+e^- \to Z^0 \to f\overline{f})$ give the rate per unit incident flux per unit time for the corresponding reactions. Also, the branching fraction, $B(Z^0 \to f\overline{f})$, gives the probability that any given Z^0, once produced, decays into an f–$\overline{f}$ pair. Equation (6.58) therefore declares that the probability for $f\overline{f}$ production is given on resonance by the product of the probability of creating a Z^0 with the probability of this Z^0 decaying into $f\overline{f}$.

Notice that the total cross section given by Eq. (6.59) translates into roughly one Z^0 produced per second using the luminosity of the LEP-I collider at CERN: $17 \times 10^{30}\ \text{cm}^{-2}\,\text{s}^{-1} = 1.7 \times 10^{-2}/\text{nb.s}$. We will see in Subsection 6.7.2 that Eq. (6.59) turns out to be missing a rather substantial correction, which brings the actual cross section down by about 28% from the one we have computed.

6.5 *t*-channel processes: crossing symmetry

Next, consider the process $e^- f \to e^- f$, with f any fermion other than e^+, e^-, $\bar{\nu}_e$, or ν_e. This type of scattering is dominantly mediated by the exchange of a virtual Z boson or by photon-exchange as in Figure 6.5.

There are a great many practical situations for which this cross-section is of interest. Some of these are: (i) elastic scattering of electrons and muons: $e^-\mu^- \to e^-\mu^-$; (ii) elastic muon–neutrino collisions with electrons, $e^-\nu_\mu \to e^-\nu_\mu$; or (iii) various hadronic processes considered in more detail in Chapter 9. Inelastic processes such as $e^-\nu_\mu \to \mu^-\nu_e$ may also be de-

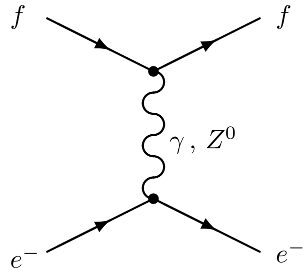


Fig. 6.5. The Feynman graph for $e^-f \to e^-f$.

scribed by the result derived below provided that all fermion masses may be neglected.

Consider therefore the evaluation of Figure 6.5 for the contribution of a vector boson V to the reaction $e^-(\mathbf{p})f(\mathbf{p}') \to e^-(\mathbf{k})f(\mathbf{k}')$. Inspection of the Feynman rules of the previous chapter gives a matrix element for this process of

$$\mathcal{M}(e^-f \to e^-f) = -\sum_{V=Z,\gamma} e_V^2[\overline{u}_e(\mathbf{k})\gamma^\mu\Gamma_{Ve}u_e(\mathbf{p})]\,[\overline{u}_f(\mathbf{k}')\gamma_\mu\Gamma_{Vf}u_f(\mathbf{p}')] \times\left[\frac{1}{(p-k)^2+M_V^2-\mathrm{i}\epsilon}\right] \quad (6.60)$$

This expression is very similar to Eq. (6.16). In fact, Eq. (6.60) may be obtained from Eq. (6.16) by the simple substitution of $\overline{u}_e(\mathbf{k})$ for $\overline{v}_e(\mathbf{p}')$ and $u_f(\mathbf{p}')$ for $v_f(\mathbf{k}')$, as well as with the replacements $p'_\mu \to -k_\mu$, $k \to k'$, and $k'_\mu \to -p'_\mu$. The signs of the four-momenta are reversed whenever an incoming particle becomes an outgoing particle or vice versa.

This correspondence allows the results of Section 6.2 to be used directly to give the spin-summed, squared matrix element for the present process. If we take the Mandelstam variables for elastic e^-f scattering (in the ultra-relativistic limit) as

$$\begin{aligned} s &= -(p+p')^2 \approx -2p\cdot p' \\ t &= -(p-k)^2 \approx +2p\cdot k \\ u &= -(p-k')^2 \approx +2p\cdot k' \end{aligned} \quad (6.61)$$

then the desired result is obtained from Eq. (6.21) with the following substitution:

$$\begin{aligned} s &= -2p\cdot p' \to +2p\cdot k = t \\ t &= +2p\cdot k \to +2p\cdot k' = u \end{aligned}$$

$$u = +2p \cdot k' \to -2p \cdot p' = s \tag{6.62}$$

The result therefore becomes

$$\overline{\mathcal{M}^2} = \left(\left| \sum_{V=Z,\gamma} e_V^2 \frac{g_{e\mathrm{L}} g_{f\mathrm{L}}}{t - M_V^2} \right|^2 s^2 + \left| \sum_{V=Z,\gamma} e_V^2 \frac{g_{e\mathrm{R}} g_{f\mathrm{R}}}{t - M_V^2} \right|^2 s^2 \right.$$
$$\left. + \left| \sum_{V=Z,\gamma} e_V^2 \frac{g_{e\mathrm{L}} g_{f\mathrm{R}}}{t - M_V^2} \right|^2 u^2 + \left| \sum_{V=Z,\gamma} e_V^2 \frac{g_{e\mathrm{R}} g_{f\mathrm{L}}}{t - M_V^2} \right|^2 u^2 \right) \tag{6.63}$$

This expression admits the same simple interpretation in terms of polarization amplitudes as did Eq. (6.21).

Also in analogy with the earlier treatment, the differential cross section becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}u\,\mathrm{d}t}(e^- f \to e^- f) = -\pi\alpha^2 \left([|A_{\mathrm{LL}}(t)|^2 + |A_{\mathrm{RR}}(t)|^2] \right.$$
$$\left. + [|A_{\mathrm{LR}}(t)|^2 + |A_{\mathrm{RL}}(t)|^2] \frac{u^2}{s^2} \right) \delta(s + t + u) \tag{6.64}$$

Here the helicity amplitudes are

$$A_{ij}(t) = \sum_V \frac{e_V^2}{e^2} \left(\frac{g_{ei} g_{fj}}{t - M_V^2} \right) \tag{6.65}$$

Because the Mandelstam variable t appears in the denominator, the process considered here is generally referred to as a *t-channel process*, as opposed to the s-channel process of the previous sections.

This cross section does not encounter problems when $t - M_V^2$ goes to zero, because the kinematically allowed range for t is $-s < t \equiv -Q^2 \leq 0$ for ultra-relativistic fermions. On the other hand, the cross-section does diverge as $t \to 0$. This is the familiar Coulomb divergence of the cross section, which again expresses a breakdown of an approximation we have made in deriving $\mathrm{d}\sigma/\mathrm{d}t$. Very small t corresponds to very small scattering angles, which classically would occur for large impact parameters. In principle, the long range of the Coulomb interaction ensures that multiple scatterings must be included to obtain an accurate determination of very-small-angle scattering amplitudes. These higher-order complications are typically not important when discussing the differential cross section as a function of angle, since it is typically true that the experiment of interest cannot distinguish scattering at sufficiently small angles from no scattering occurring at all.

In deriving the matrix element for this process, we were able to learn almost everything by recycling the results of the s-channel calculation of $e^+e^- \rightarrow f\bar{f}$. This recycling was possible because of a symmetry, called *crossing symmetry*, between processes with the same species, but where species move between initial and final states.

Suppose that we have computed the spin sum of $|\mathcal{M}|^2$ for some process. Form another process by making a series of exchanges, where an incoming particle/antiparticle is replaced with an outgoing antiparticle/particle or vice versa. Each external (incoming or outgoing) state in the original process is assigned to the corresponding external state in the new process. The new process is called a "crossing" of the old process, and its matrix element squared $|\mathcal{M}|^2$ is related to the old process's value by making the following substitutions:

(i) replace the momentum of each particle in the first process with the corresponding momentum of the analog particle in the second, with a minus sign if a particle has switched between incoming and outgoing;

(ii) multiply by (-1) for each fermion which switches between incoming and outgoing.

The origin of the minus sign is the following. When a fermion in the initial state is replaced with an anti-fermion in the final state, the squared matrix element goes from containing $\bar{u}(p)u(p) = -\mathrm{i}\not{p} + m$ to containing $\bar{v}(k)v(k) = -(\mathrm{i}\not{k} + m)$. This is not the same as we would get by the substitution $p \rightarrow -k$, it has in addition an overall minus sign. In the case we just considered, there were two such minus signs, so this rule did not matter.

We can quickly see that these rules give us the relation we already found between the matrix elements for $e^+(p)e^-(p') \rightarrow f(k)\bar{f}(k')$ and $e^-(p)f(p') \rightarrow e^-(k)f(k')$. The rule says we must multiply by $(-1)^2 = 1$ and make the substitutions, $p \rightarrow p$, $p' \rightarrow -k$, $k \rightarrow k'$, $k' \rightarrow -p'$ in the expression for $\mathcal{M}^2$ for the former process. Similarly, we can quickly find the matrix element for the process $e^-(p)\bar{f}(p') \rightarrow e^-(k)\bar{f}(k')$; we must multiply by $(-1)^2 = 1$ and make the substitutions, $p \rightarrow p$, $p' \rightarrow -k$, $k \rightarrow -p'$, and $k' \rightarrow k'$. In the ultra-relativistic limit, this changes the Mandelstam variables via

$$\begin{aligned} s &= -2p\cdot p' &\rightarrow&\quad 2p\cdot k = t \\ t &= 2p\cdot k &\rightarrow&\quad -2p\cdot p' = s \\ u &= 2p\cdot k' &\rightarrow&\quad 2p\cdot k' = u \end{aligned} \tag{6.66}$$

Applying these substitutions to Eq. (6.21) gives the differential cross section for $e^-\bar{f} \rightarrow e^-\bar{f}$.

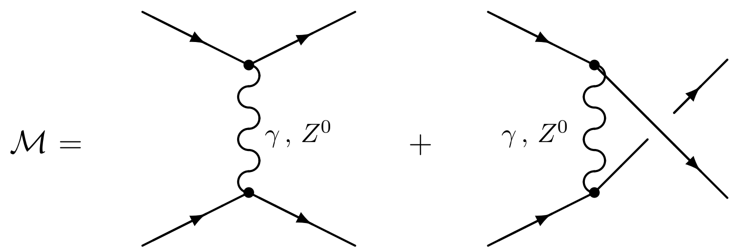


Fig. 6.6. The "uncrossed" and "crossed" graphs for e^-e^- scattering.

6.6 Interference: Møller scattering

Crossing symmetry makes it possible to recycle a small number of calculations into a complete list of desired matrix elements. However, not all matrix elements are as simple as $e^-e^+ \to f\bar{f}$. Two additional complications are possible; interference effects and bosons in external states. We will handle them in turn.

Consider the process $e^-e^- \to e^-e^-$, called Møller scattering. The new complication is that, because the initial and final particles are identical, there are two ways that the electron fields can create the final electrons, which must be summed over. This is equivalent to including both Feynman diagrams of Figure 6.6 – the *uncrossed* and *crossed* graphs – in the amplitude.

There are also two mutually compensating factors of two that arise. One is a factor of $\frac{1}{2}$ in the amplitude relative to the e^-f scattering result that arises because there are no longer two equivalent ways of evaluating the graph depending on whether the f-type vertex appears in the interaction at the spacetime point "x" or at the point $x = 0$. The other factor is a factor of 2 in the amplitude relative to the e^-f result that corresponds to the two different interaction operators that can now destroy each of the initial electrons.

The result is to simply add the contribution of the crossed graph to Eq. (6.60). The matrix element associated with the crossed graph is obtained from the matrix element found earlier for e^-f scattering by multiplying by an overall factor of -1 – due to the antisymmetry of fermi statistics for electrons – and then interchanging the four-momenta of the final particles: $k \leftrightarrow k'$. The easiest way to understand the factor of (-1) is to remember that a final state with two fermions in it is antisymmetric under exchange of the fermion labels, $|\mathbf{k}, \mathbf{k}'\rangle = -|\mathbf{k}', \mathbf{k}\rangle$, see Eq. (1.4).

This prescription may be equivalently formulated in terms of the helicity amplitudes, A_{ij}, that appear in the cross section of Eq. (6.65). For scattering between particles of identical helicity, the required substitution is given by $A_{\rm LL}(t) \to A_{\rm LL}(t) + A_{\rm LL}(u)$ and $A_{\rm RR}(t) \to A_{\rm RR}(t) + A_{\rm RR}(u)$. Since the amplitudes with mixed helicities, $A_{\rm LR} = A_{\rm RL}$, cannot interfere with one another, the correct replacement for them is

$$\begin{aligned} u^2 \left[\Big| A_{\rm LR}(t) \Big|^2 + \Big| A_{\rm RL}(t) \Big|^2 \right] &= 2u^2 \Big| A_{\rm LR}(t) \Big|^2 \\ &\to 2 \left[\Big| u A_{\rm LR}(t) \Big|^2 + \Big| t A_{\rm LR}(u) \Big|^2 \right] \end{aligned} \tag{6.67}$$

The differential cross section therefore is

$$\begin{aligned} \frac{{\rm d}\sigma}{{\rm d}u\,{\rm d}t}(e^-e^- \to e^-e^-) = -\frac{\pi\alpha^2}{s^2} \Big(&|A_{\rm LL}(t) + A_{\rm LL}(u)|^2 s^2 + |A_{\rm RR}(t) + A_{\rm RR}(u)|^2 s^2 \\ &+ |uA_{\rm LR}(t)|^2 + |tA_{\rm LR}(u)|^2 \Big)\ \delta(s+t+u) \end{aligned} \tag{6.68}$$

In the low-energy limit this expression simplifies to the QED result for ultra-relativistic *Møller scattering*:

$$\begin{aligned} \frac{{\rm d}\sigma}{{\rm d}u\,{\rm d}t}(e^-e^- \to e^-e^-) &= -\frac{2\pi\alpha^2}{s^2} \left(\left| \frac{s}{t} + \frac{s}{u} \right|^2 + \left| \frac{u}{t} \right|^2 + \left| \frac{t}{u} \right|^2 \right) \delta(s+t+u) \\ &= -\frac{2\pi\alpha^2}{s^2} \left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{ut} \right) \delta(s+t+u) \end{aligned} \tag{6.69}$$

which becomes, in the CM frame,

$$\begin{aligned} \frac{{\rm d}\sigma}{\sin\theta\,{\rm d}\theta}(e^-e^- \to e^-e^-) = \frac{\pi\alpha^2}{s} \Bigg[&\frac{1+\cos^4\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{1+\sin^4\left(\frac{\theta}{2}\right)}{\cos^4\left(\frac{\theta}{2}\right)} \\ &+ \frac{2}{\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)} \Bigg] \end{aligned} \tag{6.70}$$

Note however that, to determine the *total* cross section, one should either integrate over only half of available outgoing angles, or integrate over all angles and divide by two, to eliminate a double counting – the final state when an electron emerges (in the CM frame) with angle θ also has an electron emerging with angle $\pi - \theta$, and is therefore identical to the final state where the electron emerges with angle $\pi - \theta$.

We can easily use this result, together with crossing, to find the cross section for *Bhabha scattering*, $e^-e^+ \to e^-e^+$, see Problem 6.1.

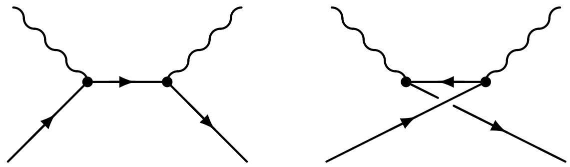


Fig. 6.7. Feynman graphs for Compton scattering.

6.7 Processes involving photons

So far we have only considered scattering processes in which all external lines are fermions. To be complete, we will briefly discuss what happens when an external line is a massless gauge boson. The reason we concentrate on a massless gauge boson is that, for the massive case, the polarization sum, Eq. (1.119), $\sum_\lambda \epsilon_\mu(p)\epsilon^*_\nu(p) = \eta_{\mu\nu} + p_\mu p_\nu/m^2$, is obviously Lorentz-covariant; but the corresponding massless sum, Eq. (1.132), is not obviously so. However, it turns out that there is a simple, Lorentz-invariant way to determine polarization averaged cross sections, thanks to gauge invariance.

6.7.1 Compton scattering, $e^-\gamma \to e^-\gamma$

Consider then the process $e^-\gamma \to e^-\gamma$. There are two Feynman graphs for this process, shown in Figure 6.7. Labeling the incoming electron momentum $\mathbf{p}_1$, the incoming photon momentum $\mathbf{p}_2$, the outgoing electron momentum $\mathbf{p}_3$, and the outgoing photon momentum $\mathbf{p}_4$, the corresponding matrix element is

$$\begin{aligned}\mathcal{M} &= e^2\epsilon_\mu(\mathbf{p}_2,\lambda)\epsilon_\nu(\mathbf{p}_4,\lambda') \\ &\quad\times\bar{u}(\mathbf{p}_3,\sigma')\left(\gamma^\nu\frac{-\mathrm{i}(\not{p}_1+\not{p}_2)+m_e}{(p_1+p_2)^2+m_e^2}\gamma^\mu+\gamma^\mu\frac{-\mathrm{i}(\not{p}_3-\not{p}_2)+m_e}{(p_3-p_2)^2+m_e^2}\gamma^\nu\right)u(\mathbf{p}_1,\sigma)\end{aligned} \tag{6.71}$$

We will eventually sum over polarizations for each photon, using Eq. (1.132),

$$\sum_\lambda \epsilon_\mu(\mathbf{p}_2,\lambda)\epsilon^*_\alpha(\mathbf{p}_2,\lambda) = \eta_{\mu\alpha} + p_\mu\overline{p}_\alpha + \overline{p}_\mu p_\alpha \tag{6.72}$$

At first sight this is worrying, since $\overline{p}^\mu$ is not uniquely defined; it is not obvious that the polarization-summed cross section will be Lorentz-invariant.

In fact, Lorentz invariance follows from the fact that, if we substitute

$\epsilon^\mu(p_2) \to p_2^\mu$ in the matrix element, we get zero:

$$\begin{aligned}
&\bar{u}(\mathbf{p}_3,\sigma')\left(\not{\epsilon}'\frac{-\mathrm{i}(\not{p}_1+\not{p}_2)+m_e}{(p_1+p_2)^2+m_e^2}\not{p}_2+\not{p}_2\frac{-\mathrm{i}(\not{p}_3-\not{p}_2)+m_e}{(p_3-p_2)^2+m_e^2}\not{\epsilon}'\right)u(\mathbf{p}_1,\sigma)\\
=\;&\bar{u}(\mathbf{p}_3,\sigma')\left(\not{\epsilon}'\frac{-\mathrm{i}(\not{p}_1+\not{p}_2)+m_e}{(p_1+p_2)^2+m_e^2}\Big[(\not{p}_1+\not{p}_2-\mathrm{i}m_e)-(\not{p}_1-\mathrm{i}m_e)\Big]\right)u(\mathbf{p}_1,\sigma)\\
&+\bar{u}(\mathbf{p}_3,\sigma')\left(\Big[(\not{p}_2-\not{p}_3+\mathrm{i}m_e)+(\not{p}_3-\mathrm{i}m_e)\Big]\frac{-\mathrm{i}(\not{p}_3-\not{p}_2)+m_e}{(p_3-p_2)^2+m_e^2}\not{\epsilon}'\right)u(\mathbf{p}_1,\sigma)\\
=\;&\bar{u}(\mathbf{p}_3,\sigma')\left(\not{\epsilon}'\frac{-\mathrm{i}(p_1+p_2)^2-\mathrm{i}m_e^2}{(p_1+p_2)^2+m_e^2}+\frac{\mathrm{i}(p_3-p_2)^2+\mathrm{i}m_e^2}{(p_3-p_2)^2+m_e^2}\not{\epsilon}'\right)u(\mathbf{p}_1,\sigma)\\
=\;&0
\end{aligned}\tag{6.73}$$

Here $\epsilon' \equiv \epsilon(p_4,\lambda')$. In passing from the second to third lines, we have used the Dirac equation, $(\not{p}_1 - \mathrm{i}m_e)u(\mathbf{p}_1) = 0$ and $\bar{u}(\mathbf{p}_3)(\not{p}_3 - \mathrm{i}m_e) = 0$.

Because of this relation, when we carry out the summation over spin states, the $\bar{p}_\mu p_\alpha$ terms in Eq. (6.72) will not contribute; we are therefore free to make the substitution

$$\sum_\lambda \epsilon_\mu(p,\lambda)\epsilon^*_\alpha(p,\lambda) \to \eta_{\mu\alpha} \tag{6.74}$$

The same holds for the final state photon, as we can quickly verify by substituting $\epsilon'_\mu \to (p_4)_\mu$ in Eq. (6.71). This not only provides a rather substantial simplification, but it also makes manifest the Lorentz invariance of the spin summed cross section.

The cancellation in Eq. (6.73), while at first sight rather remarkable, turns out to be an absolutely general property of external photon lines; when summing over all possible diagrams contributing to the matrix element, replacing $\epsilon_\mu(p) \to p_\mu$ always gives zero, so Eq. (6.74) may always be used. The physical origin of this property is gauge invariance. To ensure gauge invariance, it was necessary to ensure that the electromagnetic gauge field A_μ always couples to a conserved current; $\mathcal{L}_{\text{int}} = \int \mathrm{d}^4x\, A_\mu J^\mu$ with $\partial_\mu J^\mu = 0$ (see Subsection 1.5.2). In Fourier space, current conservation becomes $p_\mu J^\mu(p) = 0$, precisely the relation we need, because the current J^μ is what the polarization tensor ϵ_μ is contracted against.

Now we proceed with computing the spin averaged matrix element. To simplify expressions we will take the ultra-relativistic $m_e \to 0$ limit. Squaring Eq. (6.71), performing the spin sums, and using Eq. (6.74) for the polarization sums, gives

$$\overline{\mathcal{M}}^2 = \frac{e^4}{4}\,\mathrm{tr}\,\not{p}_1\Big(\gamma^\mu\frac{\not{p}_1+\not{p}_2}{(p_1+p_2)^2}\gamma^\nu+\gamma^\nu\frac{\not{p}_3-\not{p}_2}{(p_3-p_2)^2}\gamma^\mu\Big)\times$$

$$
\times\not{p}_3\left(\gamma_\nu\frac{\not{p}_1+\not{p}_2}{(p_1+p_2)^2}\gamma_\mu+\gamma_\mu\frac{\not{p}_3-\not{p}_2}{(p_3-p_2)^2}\gamma_\nu\right)
$$

$$
= \frac{e^4}{4}\,\mathrm{tr}\,\not{p}_1\left(\gamma^\mu\frac{\not{p}_1+\not{p}_2}{s}\gamma^\nu+\gamma^\nu\frac{\not{p}_3-\not{p}_2}{u}\gamma^\mu\right)\not{p}_3\left(\gamma_\nu\frac{\not{p}_1+\not{p}_2}{s}\gamma_\mu+\gamma_\mu\frac{\not{p}_3-\not{p}_2}{u}\gamma_\nu\right) \tag{6.75}
$$

Expanding gives four terms. The first term is

$$
\begin{aligned}
\frac{e^4}{4}\,\mathrm{tr}\,\not{p}_1\gamma^\mu\frac{\not{p}_1+\not{p}_2}{s}\gamma^\nu\not{p}_3\gamma_\nu\frac{\not{p}_1+\not{p}_2}{s}\gamma_\mu &= \frac{e^4}{s^2}\,\mathrm{tr}\,\not{p}_1(\not{p}_1+\not{p}_2)\not{p}_3(\not{p}_1+\not{p}_2)\,,\\
&= \frac{e^4}{s^2}\,\mathrm{tr}\,\not{p}_1\not{p}_2\not{p}_3\not{p}_2\\
&= \frac{8e^4}{s^2}(p_1\cdot p_2\;p_2\cdot p_3)\\
&= -\frac{2e^4}{s^2}(su) = -2e^4\frac{u}{s}
\end{aligned} \tag{6.76}
$$

where we use repeatedly the gamma matrix identities of Problem 4.4, and the approximation $p_1^2 = 0$. The second term is

$$
\begin{aligned}
\mathrm{tr}\left[\not{p}_1\gamma^\mu\frac{\not{p}_1+\not{p}_2}{s}\gamma^\nu\not{p}_3\gamma_\mu\frac{\not{p}_3-\not{p}_2}{u}\gamma_\nu\right] &= -2\,\mathrm{tr}\left[\not{p}_1\not{p}_3\gamma^\nu\frac{\not{p}_1+\not{p}_2}{s}\frac{\not{p}_3-\not{p}_2}{u}\gamma_\nu\right]\\
&= 8\frac{(p_1+p_2)\cdot(p_3-p_2)}{su}\,\mathrm{tr}\,\not{p}_1\not{p}_3\\
&= 0\,,
\end{aligned} \tag{6.77}
$$

because $(p_1+p_2)\cdot(p_3-p_2) = (t/2+s/2+u/2+0) = 0$. The third term is the Hermitian conjugate of the second, and also vanishes; the fourth term's evaluation is similar to the first. The result is

$$
\overline{\mathcal{M}^2}(e^-\gamma\to e^-\gamma) = -2e^4\left(\frac{u}{s}+\frac{s}{u}+\cdots\right) \tag{6.78}
$$

This is positive, because $u<0$ and $s>0$. In this expression the ellipses indicate terms which vanish as $m^2\to 0$ with s,t, and u fixed, which are required in order to properly capture the entire Compton-scattering cross section even in the ultra-relativistic limit (see Problem 6.6 for details). These additional terms are required because there are also ms hidden within the definitions of s,t, and u, and these conspire to ensure that the terms neglected in Eq. (6.78) compete with those that are included. This expression is more useful once it is used to obtain the cross section for e^+e^- annihilation, using crossing symmetry.

It is elementary to apply crossing symmetry to determine the annihilation rate $e^-e^+\to\gamma\,\gamma$. Labeling the incoming momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ and the final momenta $\mathbf{p}_3$ and $\mathbf{p}_4$, the momenta are reassigned via $p_3 \to -p_2$, $p_2 \to -p_3$. The Mandelstam variables are changed according to

$$\begin{aligned} s = -2p_1 \cdot p_2 \quad &\to \quad +2p_1 \cdot p_3 = t \\ t = +2p_1 \cdot p_3 \quad &\to \quad -2p_1 \cdot p_2 = s \\ u = +2p_1 \cdot p_4 \quad &\to \quad +2p_1 \cdot p_4 = u \end{aligned} \tag{6.79}$$

Furthermore, there is a factor of $(-1)^1$ because one fermion is reassigned from the final to the initial state. Therefore, the spin averaged matrix element squared is

$$\overline{\mathcal{M}^2}(e^- e^+ \to \gamma\,\gamma) = +2e^4 \left(\frac{t}{u} + \frac{u}{t} \right) \tag{6.80}$$

Since both t and u are negative, the (-1) ensures that the result is positive. This result manifestly has the right behavior if we interchange the two photon states, corresponding to $t \leftrightarrow u$. Using it to obtain the cross section leads to the correct ultra-relativistic limit

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t\,\mathrm{d}u} = -\frac{2\pi\alpha^2}{s^2} \left(\frac{t}{u} + \frac{u}{t} \right) \delta(s + t + u) \tag{6.81}$$

and so

$$\frac{\mathrm{d}\sigma}{\sin\theta\,\mathrm{d}\theta} = \frac{2\pi\alpha^2}{s} \left(\frac{1 + \cos^2\theta}{\sin^2\theta} \right) \tag{6.82}$$

Since the photons are identical particles, to find the total cross section one must integrate over only half of the space of final-state angles.

The interaction of quarks with gluons is almost the same as the electromagnetic processes we have considered here; the main added complication is the appearance of color factors. However, when the mutual interactions of gluons via the three or four gluon vertices, Eq. (5.63) and Eq. (5.64), are involved, then the polarization summation issues are more complicated and it is not permitted to make the substitution, Eq. (6.74). We postpone further discussion on this point to Chapter 9.

6.7.2 Radiated photons

Another application where photons appear in the final state is when one or more photons are radiated from a participating particle (initial or final) in a scattering process, for instance via the Feynman graph depicted in Figure 6.8.

Naively, such a process is suppressed by $\sim e^2/(2\pi)^2 = \alpha/\pi \ll 1$ with respect to the diagram without a photon emission, and should therefore be

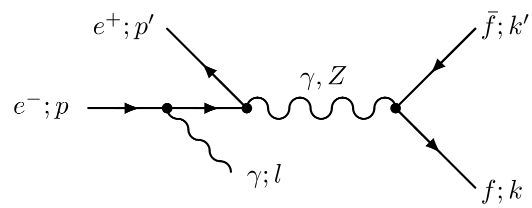


Fig. 6.8. Diagram for scattering with a photon emission.

negligibly small. However, this is not entirely correct. The probability to emit a photon in a scattering process is enhanced by *large logarithms*, which can make it quite important. In particular, at the end of this subsection, we will see that the on-resonance cross section for $e^+e^- \to Z \to f\bar{f}$ is reduced by about 1/4 due to these processes.

Consider the process shown in Figure 6.8. This is one of four diagrams which must be summed, and then squared, to find the photon emission rate. For general angles, when the center-of-mass-frame angle between the photon and any other particle is large, the naive estimate that the photon emission rate is suppressed by α/π, is correct. However this turns out not to be true in the special kinematic regime in which the photon is approximately collinear with another particle, for instance, when the angle between $\mathbf{p}$ and $\mathbf{l}$ in the CM frame is small.

In this regime, the four possible diagrams (with the photon attached to any of the four external states) all contribute, but they do not contribute equally. When the photon emerges from the e^-, the denominator of the intermediate electron propagator is $1/((p - l)^2 + m_e^2) = 1/(-2p \cdot l) \simeq 1/(2p^0 l^0 (1 - \cos\theta_{\mathbf{pl}}))$. Since $(1 - \cos\theta_{\mathbf{pl}}) \simeq 0$, this amplitude is enhanced, relative to the others. It turns out in this case that it is simplest to work in terms of the two transverse polarization states, in which case the amplitude in question is $\sim\sin(\theta_{\mathbf{pl}})/(1 - \cos\theta_{\mathbf{pl}}) \sim 1/\theta_{\mathbf{pl}}$, while the other amplitudes are not enhanced at small $\theta_{\mathbf{pl}}$. Therefore, it is permissible to drop the other amplitudes to determine the leading behavior in this small angle region.

Let us evaluate just the part of the square of the matrix element involving the photon emission, to compare with the case without photon emission. Where, without photon emission, we have the quantity $\sum_\sigma u(p,\sigma)\,\overline{u}\,(p,\sigma) =$

$(-i\not{p}+m)$, we now have the quantity

$$
\begin{aligned}
& -e^2 \sum_\lambda \frac{-i\not{p}+i\not{l}+m}{(p-l)^2+m^2} \not{\epsilon}^*(\lambda)(-i\not{p}+m)\not{\epsilon}(\lambda) \frac{-i\not{p}+i\not{l}+m}{(p-l)^2+m^2} \\
= \; & \frac{-e^2}{(2p\cdot l)^2} \sum_\lambda (-i\not{p}+i\not{l}+m)\Big\{\epsilon\cdot\epsilon^*(i\not{p}+m) - 2i\epsilon\cdot p\not{\epsilon}^*\Big\}(-i\not{p}+i\not{l}+m) \\
= \; & \frac{-e^2}{(2p\cdot l)^2} \sum_\lambda [-2i\epsilon\cdot\epsilon^* l\cdot p\not{l} \; - \; 2i\epsilon\cdot p\,(-i\not{p}+i\not{l}+m)\not{\epsilon}^*(-i\not{p}+i\not{l}+m)] \\
\simeq \; & \frac{ie^2\not{l}}{p\cdot l} + \sum_\lambda \frac{e^2|\epsilon\cdot p|^2}{(p\cdot l)^2}(-i\not{p}+i\not{l}+m)
\end{aligned}
\tag{6.83}
$$

where in the second step we used $l^2 = 0$ and $p^2 + m^2 = 0$, and in the last step we dropped a term $\propto (\epsilon\cdot p)(p\cdot l)/(p\cdot l)^2$, which is subleading in $\theta_{\mathbf{pl}}$. Since $\sum_\lambda |\epsilon\cdot p|^2 \simeq (p^0)^2\theta_{\mathbf{pl}}^2$ and $l\cdot p \simeq -l^0p^0\theta_{\mathbf{pl}}^2/2$, the remaining two terms are of the same order. Since l is almost collinear with p, for the rest of the calculation it is adequate to substitute $\not{l} \to (l^0/p^0)\not{p}$. Defining $l^0/p^0 \equiv x$, so x is the fraction of the electron's energy carried off by the photon, we find that the matrix element in the collinear limit is approximately given by

$$
(-i\not{p}) \to (-i\not{p})\frac{2e^2(1+(1-x)^2)}{x^2(p^0)^2\theta_{\mathbf{pl}}^2}
\tag{6.84}
$$

Being slightly more careful,

$$
-l\cdot p \simeq p^0l^0\left(1-\cos\theta_{\mathbf{pl}} + \frac{m_e^2}{2(p^0)^2}\right) \simeq \frac{(p^0)^2x}{2}\left(\theta_{\mathbf{pl}}^2 + \frac{m_e^2}{(p^0)^2}\right)
\tag{6.85}
$$

This correction at very small angles exists in both terms, and is important in cutting off the otherwise divergent angular integral. We can now do most of the integral over the photon momentum;

$$
\int \frac{d^3l}{(2\pi)^3 2l^0} = \frac{1}{8\pi^2}\int_{-1}^{1} d\cos\theta_{\mathbf{pl}}\, l^0 dl^0
\tag{6.86}
$$

The integral over the angle is

$$
\int_{-1}^{1} d\cos\theta_{\mathbf{pl}} \frac{1}{1+\frac{m^2}{2p^2}-\cos\theta_{\mathbf{pl}}} = \log\left(\frac{4p^2}{m_e^2}\right) + O(1) = \log\left(\frac{s}{m_e^2}\right) + O(1)
\tag{6.87}
$$

There is an unknown constant in this expression, because the approximations we have made break down at large angles. This constant could be found by making a more careful treatment, which included the interference between emissions from different lines at large angles. Nevertheless, our simplified treatment is sufficient to show that photon emission is logarithmically dominated by small opening angles, with the log cut off by the mass of the emitting particle; and it is sufficient to find the coefficient of that log.

The integration over l^0 is more delicate, because the emission of the photon potentially changes the kinematics of the rest of the diagram. One must recompute the remainder of the diagram, but changing the momentum carried by the incoming electron line from p to $(1-x)p$. For the case of emission from a final state line, f or $\bar{f}$, this does not matter. For the case of emission from an incoming line, e^+ or e^-, this *can* matter. In general, it matters if $x \sim 1$, so the kinematics is substantially disturbed. In the case of scattering through a resonance, such as the Z boson, an energy loss of $l^0 \sim \Gamma_Z$, or $x \sim \Gamma_Z/M_Z$, is important, as we discuss below. Nevertheless, for the present purposes we will ignore this complication. The likelihood to emit a photon with energy larger than ω is

$$\frac{e^2}{8\pi^2}\log\frac{s}{m_e^2}\int_{\omega/p}^{1}\frac{x\,\mathrm{d}x}{x^2}\left[1+(1-x)^2\right] \quad = \quad \frac{\alpha}{2\pi}\log\left(\frac{s}{m_e^2}\right)\left[\log\left(\frac{s}{4\omega^2}\right)-\frac{3}{2}\right] \tag{6.88}$$

This likelihood is not necessarily small, because it is amplified by the product of two logarithms which may each individually be large. We should also add the probabilities to emit from each of the other three legs, giving a total likelihood of

$$\frac{\sigma(\omega)}{\sigma} \simeq \frac{\alpha}{\pi}\left[\log\left(\frac{s}{m_e^2}\right)+Q_f^2\log\left(\frac{s}{m_f^2}\right)+O(1)\right]\left[\log\left(\frac{s}{4\omega^2}\right)-\frac{3}{2}\right] \tag{6.89}$$

The likelihood to emit a photon during a scattering can in fact be quite large. For instance, on the Z pole, for $\omega = m_e$, and considering only emissions from the incoming electrons, this evaluates to approximately 1.2.

Once again, we find a higher-order effect (more powers of α) which is just as large as the original effect. In fact, the result presented appears to be sick; if we ask for the rate to scatter with the emission of an arbitrarily soft photon, that rate is logarithmically divergent. It also appears that the total rate for the scattering to occur, after we add the possibility of this photon emission, is much larger than we had previously computed. The situation becomes still worse as we consider multiple-photon emission. This suggests that, as in Section 6.4, there may be some additional, formally higher order, diagrams which can compete with the lowest-order one and must be somehow resummed. This proves to be the case.

Consider the diagram of Figure 6.9. When it interferes with the diagram of Figure 6.1, this modifies the rate of scattering *without* photon emission. The interference of those diagrams is structurally very similar to the square

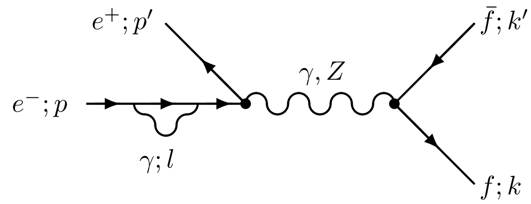


Fig. 6.9. A radiative correction to e^+e^- annihilation.

of the diagram of Figure 6.8. In particular, the photon in the loop, in Figure 6.9, contributes the same collinear and soft divergences as the real, emitted photon of Figure 6.8. There is one major difference, however; the sign turns out to be opposite. Therefore, the diagram of Figure 6.9 reduces the probability for the scattering process, without the photon emission.

Because the soft and collinear singularities are identical, the cross section to emit an extra photon, and the *reduction* of the cross section to scatter without a photon emission, are approximately the same size. Therefore, the *total* cross section, with or without photon emission, is unchanged up to $O(\alpha)$ correction, except for the correction to the kinematics because of the energy carried away by the photons.

The correct way to handle the computation of the total cross section, and the likelihood to emit any number of photons above some energy threshold, has been solved by Bloch and Nordsieck. One must introduce a cut-off frequency ω, below which a photon cannot be detected. (Realistically, all experiments have such a cut-off.) Then, the likelihood *not* to emit a collinear photon is given, approximately, by

$$\frac{\sigma(e^+e^- \to f\bar{f} + 0\gamma)}{\sum_n \sigma(e^+e^- \to f\bar{f} + n\gamma)} \simeq \exp(-\chi) \tag{6.90}$$

with

$$\chi = \frac{\alpha}{\pi}\left[\log\left(\frac{s}{m_e^2}\right) + Q_f^2 \log\left(\frac{s}{m_f^2}\right) + O(1)\right]\left[\log\left(\frac{s}{4\omega^2}\right) - \frac{3}{2}\right] \tag{6.91}$$

This is the expression for s channel exchange. In t channel exchange, it is t, not s, which controls the size of the collinear logarithm.

This phenomenon of photon emission from the initial state is referred to as *initial-state radiation*, or ISR. Let us see its effects on scattering processes. In a typical s channel scattering process, the kinematics is signifi-

Fig. 6.10. e^+e^- hadronic cross section near the Z^0 resonance.

cantly changed if one of the initial particles radiates a photon carrying $x \sim 1$ of the energy. In this case, the second bracketed quantity should be replaced with $\simeq$1. The change to the cross section can be several percent – not huge, but important when precision is required.

On the other hand, when the e^+e^- particles' energies are tuned to lie on the Z pole, the radiation of a photon with energy $\Gamma_Z/2$, the half-width of the resonance, is enough to move the scattering off resonance and reduce the cross section substantially. Therefore, the on-resonance cross section for $e^+e^- \to f\bar{f}$ is reduced by a factor of approximately

$$\begin{aligned} \frac{\sigma_{\text{tot}}(e^+e^- \to Z^0)}{\text{Eq. (6.59)}} &\simeq \exp\left(\frac{-\alpha}{\pi}\log\left(\frac{M_Z^2}{m_e^2}\right)\left[\log\left(\frac{M_Z^2}{\Gamma_Z^2}\right) - \frac{3}{2}\right]\right) \\ &\simeq 0.72 \end{aligned} \tag{6.92}$$

where in the numerical evaluation we used $\alpha \simeq 1/133$, a compromise between its value at the scale M_Z, where $\alpha \simeq 1/128$, and at m_e, where $\alpha \simeq 1/137$. Combining this with the fact that 30% of Z^0 decays are to leptons, the *hadronic* cross section on the Z^0 pole is expected to be only $\simeq$30 nb. The actual hadronic cross section on resonance is 30.5 nb, in agreement with a more careful calculation.

We illustrate the impact of initial state radiation on the cross section near the Z resonance in Figure 6.10. The initial-state-radiation corrected cross section is an integral over the energy lost to photons, of the probability for that energy loss due to initial-state radiation times the cross section at the reduced energy. The correction is very important in the agreement between theory and data, as shown by the inclusion of the cross section data from the four LEP experiments. Note that at high energies, the cross section actually exceeds the uncorrected value. This is because initial state radiation can lower the e^+e^- pair's energy to lie on resonance, enhancing the cross section.

Initial-state radiation also plays a prominent role in the physics of high-energy hadronic collisions, which we will discuss in Section 9.2, especially Subsection 9.2.3.

6.8 Problems

[6.1] **Crossing symmetry**

Use crossing symmetry to derive the cross section for Bhabha scattering in the ultra-relativistic limit, but still at $s \ll M_Z^2$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}u\,\mathrm{d}t}(e^-e^+ \to e^-e^+) = \frac{-2\pi\alpha^2}{s^2}\left(\left|\frac{u}{s}+\frac{u}{t}\right|^2 + \frac{t^2}{s^2}+\frac{s^2}{t^2}\right)\delta(s+t+u)$$

[6.2] **Electron–neutrino scattering**

[6.2.1] The process $e^-\bar{\nu}_e \to f_m\bar{f}_n$ (f neither e^- nor ν_e) proceeds via a single diagram. Draw that diagram and show that, taking all external states to be massless, it yields a cross section of

$$\frac{\mathrm{d}\sigma}{\mathrm{d}u\,\mathrm{d}t} = -\frac{G_\mathrm{F}^2}{\pi}N_c|U_{nm}|^2\frac{u^2}{s^2}\left(\frac{M_W^2}{s-M_W^2}\right)^2\delta(s+t+u)$$

Use crossing symmetry to find $\mathrm{d}\sigma/\mathrm{d}u\,\mathrm{d}t$ for $e^-\nu_\mu \to \mu^-\nu_e$ from this result.

[6.2.2] Compute the matrix element squared for $e^-\nu_e \to e^-\nu_e$. Careful: there are two diagrams, one involving Z exchange and one involving W exchange. Use crossing symmetry to find the $e^-\bar{\nu}_e \to e^-\bar{\nu}_e$ and $e^-e^+ \to \nu_e\bar{\nu}_e$ matrix elements. Beware; there are two diagrams, and a relative minus sign between them due to the different ways the initial- and final-state fermions connect to each other.

[6.3] **Supernova neutrinos**

Neutrinos and antineutrinos were observed in 1987 from the supernova in the nearby Large Magellanic Cloud. They were detected by observing their interactions with electrons and nuclei in a large tank of water. The neutrino energies were $\sim$ 10 MeV and in this energy range the relevant processes are elastic scattering with electrons and "quasi-elastic" (i.e. low energy) inverse beta decay: $\bar{\nu}_e + p \to n + e^+$ with the hydrogen nuclei.

[6.3.1] Why not consider the interactions with the oxygen nucleus, O^{16}?

$$\begin{aligned} \text{e.g.}: \quad & \bar{\nu}_e + {}^{16}_{8}\mathrm{O} \to {}^{16}_{7}\mathrm{N} + e^+ \\ \text{or}: \quad & \nu_e + {}^{16}_{8}\mathrm{O} \to {}^{16}_{9}\mathrm{F} + e^- \end{aligned}$$

(Hint: *Nucl. Phys.* **A 166** (1971) page 60.)

[6.3.2] In these reactions the interactions are detected by observing the Cherenkov radiation of the final charged particles. Given that the index of refraction of water is $n = 1.33$, what is the minimum energy that a particle of mass m must have to radiate Cherenkov light? Why not consider also elastic scattering of neutrinos by the hydrogen and oxygen nuclei?

[6.3.3] In the standard model, draw all of the tree level (i.e. no loops) Feynman graphs that contribute to the following processes:

(i) $\nu_\mu + e^- \to \nu_\mu + e^-$
(ii) $\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$
(iii) $\nu_e + e^- \to \nu_e + e^-$
(iv) $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$

[6.3.4] *Kinematics*

(i) Show that, for a neutrino of energy ω, the angle between the direction of the incident neutrino momentum and the scattered electron momentum in the lab frame, θ, is related to the same angle in the center-of-mass frame, φ, by

$$\tan\theta = \frac{\sin\varphi}{1+\cos\varphi}\frac{\sqrt{m^2+2m\omega}}{m+\omega}$$

This implies that the electron–neutrino interactions make electrons that tend to move away from the supernova (with an angular spread of $\Delta\theta \sim \sqrt{m/\omega} \sim 1/\sqrt{20}$ for 10 MeV neutrinos) regardless of the scattering probability $\mathrm{d}\sigma/\mathrm{d}\cos\varphi$ in the center-of-mass frame.

(ii) Show that the energy of the final electron in the lab frame, E, is given in terms of the incident neutrino energy and scattering angle by

$$E = m\left[\frac{(m+\omega)^2 + \omega^2\cos^2\theta}{(m+\omega)^2 - \omega^2\cos^2\theta}\right]$$

For fixed ω, what is the difference between the energy at $\theta = 0$ and that when $\theta = \sqrt{(m/\omega)} \ll 1$? This represents the energy range of the scattered electrons from this process.

[6.3.5] In the effective Fermi theory of weak interactions (basically, the limit $s, |t|, |u| \ll M_W^2$ so they can be dropped from the W and Z boson propagators), the relevant interaction Hamiltonian density is

$$\begin{aligned}\mathcal{H}_I &= \frac{G_{\rm F}}{\sqrt{2}}\Bigg\{[\bar{\nu}_e\gamma^\mu(1+\gamma_5)e][\bar{e}\gamma_\mu(1+\gamma_5)\nu_e] \\ &\qquad -\frac{1}{2}\sum_{m=e,\mu,\tau}[\bar{\nu}_m\gamma^\mu(1+\gamma_5)\nu_m][\bar{e}\gamma_\mu(\rho+\gamma_5)e]\Bigg\} \\ \text{where } \rho &= 1 - 4\sin^2\theta_{\rm W}.\end{aligned}$$

The first term can be rewritten, using the Fiertz identities of Problem 1.6, as

$$[\bar{\nu}\gamma^\mu(1+\gamma_5)e][\bar{e}\gamma_\mu(1+\gamma_5)\nu] = [\bar{\nu}_e\gamma^\mu(1+\gamma_5)\nu_e][\bar{e}\gamma_\mu(1+\gamma_5)e]$$

so

$$\mathcal{H}_I = \frac{G_{\rm F}}{\sqrt{2}}\sum_{m=e,\mu,\tau}[\bar{\nu}_m\gamma^\mu(1+\gamma_5)\nu_m][\bar{e}\gamma_\mu(h_V^{(m)} + \gamma_5 h_A^{(m)})e]$$

$$\begin{aligned}\text{where}: \ h_A^{(e)} &= 1 - \frac{1}{2} = \frac{1}{2} \\ h_A^{(\mu,\tau)} &= -\frac{1}{2} \\ \text{and}: \ h_V^{(e)} &= 1 - \frac{1}{2}\rho = \frac{1}{2} + 2\sin^2\theta_{\rm W} \\ h_V^{(\mu,\tau)} &= -\frac{1}{2}\rho = -\frac{1}{2} + 2\sin^2\theta_{\rm W}\end{aligned}$$

Use this result to show that the matrix element for neutrino–electron scattering is

$$\mathcal{M}(\nu e \to \nu e) = \frac{G_{\rm F}}{\sqrt{2}}\left[\bar{u}(q')\gamma^\mu(1+\gamma_5)u(q)\right]\left[\bar{u}(p')\gamma_\mu(h_V^{(m)} + h_A^{(m)}\gamma_5)u(p)\right]$$

and

$$\mathcal{M}(\bar{\nu}e \to \bar{\nu}e) = \frac{G_{\mathrm{F}}}{\sqrt{2}} \left[\bar{v}(q)\gamma^{\mu}(1+\gamma_5)v(q')\right] \left[\bar{u}(p')\gamma_{\mu}(h_V^{(m)} + h_A^{(m)}\gamma_5)u(p)\right]$$

[6.3.6] Averaging over the initial electron spin and summing over all final spins, show that

$$\begin{aligned}\overline{\mathcal{M}}^2 \;=\; 16G_{\mathrm{F}}^2 \Big\{&(h_V \pm h_A)^2(p\cdot q)(p'\cdot q') \\ &+(h_V \mp h_A)^2(p\cdot q')(p'\cdot q) + m^2(h_V^2 - h_A^2)(q\cdot q')\Big\}\end{aligned}$$

in which the upper sign corresponds to $\nu e \to \nu e$ and the lower sign to $\bar{\nu}e \to \bar{\nu}e$.

[6.3.7] Using

$$\mathrm{d}\sigma = \frac{1}{-2p\cdot q\, v_{\mathrm{rel}}}\overline{\mathcal{M}}^2(2\pi)^4\delta^4(p+q-p'-q')\frac{\mathrm{d}^3\mathbf{p}'\,\mathrm{d}^3\mathbf{q}'}{(2\pi)^6 2p'^0 2q'^0}$$

(i) Show that the differential cross-section in the center-of-mass frame is (neglecting the electron mass)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\varphi)} = \frac{G_{\mathrm{F}}^2}{2\pi}\omega_{\mathrm{cm}}^2\left\{(h_V \pm h_A)^2 + \frac{1}{4}(h_V \mp h_A)^2(1-\cos\varphi)^2\right\}$$

where ω_{cm} is the incident neutrino energy, and φ is the angle between the incident neutrino momentum and the final electron momentum, as before. What is the most probable center-of-mass scattering angle?

(ii) Show that the total cross section is (in terms of the lab energy of the neutrino)

$$\sigma = \frac{G_{\mathrm{F}}^2\omega m}{2\pi}\left[(h_V \pm h_A)^2 + \frac{1}{3}(h_V \mp h_A)^2\right](1 + O(m/\omega))$$

Using the values of the parameters $h_V^{(m)}, h_A^{(m)}$ given earlier and $\sin^2\theta_{\mathrm{W}} \approx 1/4$, calculate the ratios

$$\begin{aligned}&\sigma(\nu_e e \to \nu_e e) : \;\; \sigma(\bar{\nu}_e e \to \bar{\nu}_e e) : \;\; \sigma(\nu_\mu e \to \nu_\mu e) : \\ &\sigma(\bar{\nu}_\mu e \to \bar{\nu}_\mu e) : \;\; \sigma(\nu_\tau e \to \nu_\tau e) : \;\; \sigma(\bar{\nu}_\tau e \to \bar{\nu}_\tau e)\end{aligned}$$

[6.3.8] For nucleon–neutrino scattering ($\bar{\nu}_e + p^+ \to n + e^+$) at low energies, the weak current has matrix elements:

$$\langle n|J^{\mu}_{\mathrm{had}}|p\rangle = \bar{u}_n\gamma^{\mu}(g_V + g_A\gamma_5)u_p$$

with $g_V = 1$ and $g_A \simeq 1.270$. Using

$$\mathcal{H}_{\text{weak}} = \frac{G_{\text{F}}}{\sqrt{2}} V_{ud}[\bar{\nu}_e \gamma_\mu (1+\gamma_5) e] J^\mu_{\text{had}}$$

show that

$$\mathcal{M}(p^+ \bar{\nu}_e \to n e^+) = \frac{G_{\text{F}}}{\sqrt{2}} V_{ud} \bar{\upsilon}_{(\nu)}(\mathbf{q}) \gamma^\mu (1+\gamma_5) \upsilon_{(e)}(\mathbf{q}') \times \bar{u}_n(\mathbf{p}') \gamma_\mu (g_V + g_A \gamma_5) u_p(\mathbf{p})$$

where V_{ud} is the relevant Kobayashi–Maskawa matrix element.

[6.3.9] Treat the nucleon mass $m_p \gg \omega$ the neutrino energy.

(i) Neglecting $m_n - m_p$ and the electron mass, what is the lab energy of the final electron as a function of scattering angle and incident neutrino energy? (This is almost a trick question.)

(ii) Show that the differential scattering cross section as a function of the lab-frame scattering angle, θ, between the incident neutrino direction and the final positron direction is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} \approx |V_{ud}|^2 \frac{G_{\text{F}}^2 \omega^2}{2\pi} \Big\{ (g_V^2 + 3g_A^2) + (g_V^2 - g_A^2)\cos\theta \Big\}$$

(Neglect ω/m_N, m_e/ω, and $(m_n - m_p)/\omega$.) Notice the angular distribution of final electrons is different than that in the case of electron–neutrino scattering, and is effectively independent of θ (recall $(g_V^2 + 3g_A^2)/(g_V^2 - g_A^2) \approx -10$).

(iii) Calculate the total cross section as a function of the lab frame neutrino energy, ω.

[6.3.10] Using the given values for $h_A^{(m)}, h_V^{(m)}$, and g_V, g_A, and $\omega = 10$ MeV, calculate what percentage of observed events are expected to be due to $\bar{\nu}_e p$, $\nu_e e$, $\bar{\nu}_e e$, $\nu_\mu e$, $\bar{\nu}_\mu e$, $\nu_\tau e$, and $\bar{\nu}_\tau e$ scattering, assuming the supernova emits equal numbers of all types of neutrinos.

Don't forget that only two of the protons in a water molecule are in hydrogen! That is, there are two protons but ten electrons in a water molecule.

[6.3.11] ν_μ-*Matter interactions*: In accelerator based neutrino–matter scattering experiments, $\nu_\mu + e$ and $\nu_\mu +$ nucleon interactions are observed. In this case the neutrinos come from pion decay and so are much more energetic than from the supernova. Their energies are governed by the beam energy and are generally much greater than the nucleon mass in the nucleon rest frame. Consider the following four reactions:

$$\nu_\mu + e^- \to \nu_\mu + e^-$$
$$\nu_\mu + n \to p + \mu^-$$
$$\nu_\mu + n \to \nu_\mu + n$$
$$\nu_\mu + p \to \nu_\mu + p$$

in the limit of small momentum transfer. In the approximation that the nucleon (and lepton) masses can be neglected, what is the total cross-section for each of these processes in the center-of-mass frame? For the last two reactions use the neutral current interaction

$$\begin{aligned} \mathcal{H}_{\rm nc} &= -\frac{G_{\rm F}}{\sqrt{2}}[\bar{\nu}\gamma_\mu(1+\gamma_5)\nu]J^\mu_{\rm nc} \\ \text{and} \quad \langle N|J^\mu_{\rm nc}|N\rangle &= \frac{1}{(2\pi)^3}\bar{u}\gamma^\mu(k_V + k_A\gamma_5)u \end{aligned}$$

with $k_V = \frac{1}{2} - 2\sin^2\theta_{\rm W}$, $k_A = \frac{1}{2}$ if N is a proton and $k_V = -\frac{1}{2}$, $k_A = -\frac{1}{2}$ if N is a neutron.

[6.3.12] What is the ratio of their cross sections as a function of the neutrino energy in the lab frame? (Your answer should behave as $\sigma_N/\sigma_e \sim (m_N/m_e) \sim 10^3$ which shows why nucleon–neutrino scattering is so much easier to observe than electron–neutrino scattering.)

[6.4] **Higgsstrahlung**

At LEP II, the dominant mode used to search for the Higgs boson was “Higgsstrahlung,” $e^+e^- \to Z^* \to HZ$, where Z^* just means a Z intermediate state with an energy significantly different from the resonant energy M_Z. This might be an attractive mode for detailed Higgs studies at future electron colliders.

Compute the (integrated) cross section for this process as a function of the center-of-mass energy s, the Z-boson mass M_Z, and the Higgs-boson mass $m_{\rm H}$. Treat $m_e = 0$, but do not treat either M_Z or $m_{\rm H}$ as small compared to $\sqrt{s}$. Compare it to the cross section to go into $q\bar{q}$ final states.

List the most common final states (the Z and H bosons both subsequently decay). What features of the decay, if any, clearly distinguish it from a scattering $e^+e^- \to q\bar{q}$ with q any quark type?

Given the Higgs mass $m_{\rm H} = 126$ GeV, at what center of mass energy $\sqrt{s}$ does the cross-section obtain its maximum value? This would be an ideal energy for an e^+e^- collider designed to study Higgs bosons via this production process.

[6.5] **Resonances**

The $\Upsilon(4s)$ is a narrow resonance caused by a $b\bar{b}$ bound state. Its mass and width are $m_{\Upsilon(4s)} = 10.580$ GeV, $\Gamma_{\Upsilon(4s)} = 14$ MeV, with a branching fraction to electrons of $\mathrm{B}(\Upsilon(4s) \to e^+e^-) = 2.8 \times 10^{-5}$. It is experimentally useful because the $\Upsilon(4s)$ decays with almost 100% probability via $\Upsilon(4s) \to B\bar{B}$, with B a meson containing a $\bar{b}$ quark and $\bar{B}$ a meson containing a b quark. This gives a convenient way to produce B meson pairs approximately at rest, which has been exploited by the B-factories, BaBar and Belle.

What is the cross section for $e^+e^- \to B\bar{B}$ on the $\Upsilon(4s)$ resonance? Hint: the spin of the $\Upsilon(4s)$ is 1. Use Eq. (6.57).

What, approximately, is the correction to this cross section formula due to the radiation of soft photons from the e^+ and e^-?

[6.6] **Compton scattering**

[6.6.1] Repeat the calculation for unpolarized photon–electron scattering without neglecting the electron mass, to show that the spin-summed and averaged matrix element generalizes to

$$\overline{\mathcal{M}}^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} - 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right]$$

where electron (photon) four-momenta are denoted by $p(k)$ and final-state quantities carry a prime. Show that even though this naively approaches the result quoted in the main text, $-2e^4[(s/u) + (u/s)]$, if $m^2 \to 0$ with s, t, and u fixed, the kinematical relation between the initial and final photon energies and the final photon scattering angle θ (in the initial electron's rest frame) $k^0/k^{0'} = 1 + (k^0/m)(1 - \cos\theta)$, implies the nominally sub-dominant combination

$$m^2\left[\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right] = 1 - \cos\theta$$

contributes even when $k^0 \gg m$. Use this to show that the differential cross section in the electron rest frame is given by

$$\frac{\mathrm{d}\sigma}{\sin\theta\,\mathrm{d}\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{E'}{E}\right)^2\left[\frac{E'}{E} + \frac{E}{E'} - \sin^2\theta\right]$$

where $E = k^0$ and $E' = k^{0'}$.

[6.6.2] Repeat the calculation for photon–electron scattering, but this time without averaging (summing) over the initial (final) photon polarization. Denoting by $\varepsilon_{\rm i}$ and $\varepsilon_{\rm f}$ the polarization vector of the initial and final photon, show that the polarized cross section is given in the rest frame of the initial electron by the Klein–Nishina formula:

$$\frac{\mathrm{d}\sigma}{\sin\theta\,\mathrm{d}\theta} = \frac{\pi\alpha^2}{2m^2}\left(\frac{E'}{E}\right)^2\left[\frac{E'}{E} + \frac{E}{E'} + 4(\varepsilon_{\rm f}\cdot\varepsilon_{\rm i})^2 - 2\right]$$

Show that the sum over photon polarizations gives $\sum_{\rm if}(\varepsilon_{\rm f}\cdot\varepsilon_{\rm i})^2 = 1+\cos^2\theta$, and so reproduces the above result for the unpolarized cross section. In this form this result can be adapted to describe the important process of bremsstrahlung – the radiation of a photon by a charged fermion as it moves in the Coulomb field of a nucleus. The cross section for bremsstrahlung can be computed by replacing the initial photon of the above calculation with the appropriate Fourier component of the initial Coulomb field.

[6.6.3] Show that *regardless of the photon energy*, in the limit of small scattering angles ($\theta\to 0$) the polarized differential cross section reduces to the Thompson formula

$$\frac{\mathrm{d}\sigma}{\sin\theta\,\mathrm{d}\theta} = \frac{2\pi\alpha^2}{m^2}(\varepsilon_{\rm f}\cdot\varepsilon_{\rm i})^2$$

This is also the result for all angles in the limit $E \ll m$. The fact that this varies inversely with m^2 resolves a puzzle as to why ultra-relativistic muons and electrons behave so differently within detectors. After all, since electrons and muons have exactly the same gauge interactions within the standard model, any difference between their properties in a detector can only be due to their different mass and one might naively expect that this mass difference should become unimportant for ultra-relativistic particles. The above result shows that their small-angle Compton scattering (and bremsstrahlung) cross-sections differ dramatically even in the ultra-relativistic limit. Since it is the bremsstrahlung due to numerous small-angle scattering events which dominates the energy loss of an ultra-relativistic charged particle passing through matter, this proportionality of this cross section to $1/m^2$ ensures that muons lose their energy much less efficiently than electrons, and so are much more penetrating when they pass through a detector.