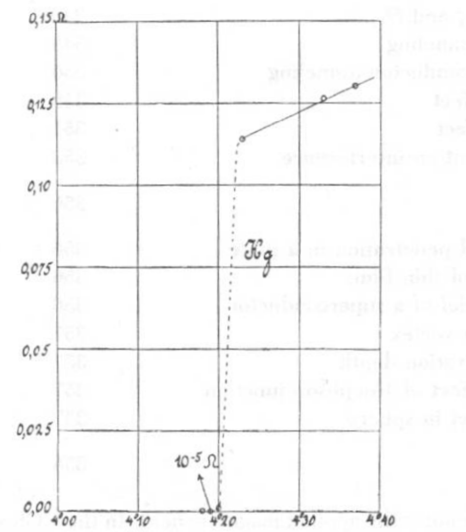
Superconductivity



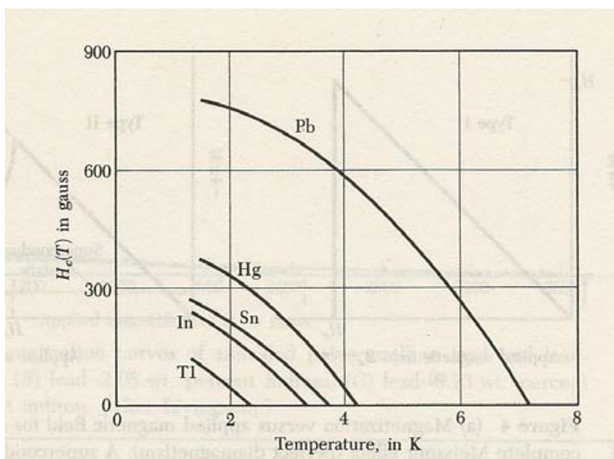
Superconductivity was first observed by HK Onnes in 1911 in mercury at T ~ 4.2 K (Fig. 1).

The temperature at which the resistivity falls to zero is the critical temperature, T_c.

Superconductivity occurs in many metallic elements (highest T_c in Nb at 9.5 K), alloys (e.g. Nb_3Ge with T_c = 23 K) and perovskite cuperates (e.g. $La_{2-x}Ba_xCuO_4$ with T_c ~30 K, $YBa_2Cu_3O_7$ with T_c = 90 K, and $Bi_2Sr_2CaCu_2O_8$ with T_c = 95 K) The perovskite superconductors are termed high-T_c superconductors because their T_c is > 30 K, which exceeds the upper limit for T_c according to the theory by Bardeen, Cooper and Schrieffer (Nobel 1972). Müller and Bednorz won the Nobel Prize in Physics in 1987 for discovering the $La_{2-x}Ba_xCuO_4$ material that had stimulated a surge in the research of high-T_c superconductors. The highest T_c is currently found in $HgBa_2Ca_2Cu_3O_x$ (T_c = 133 K).

1. Meissner Effect

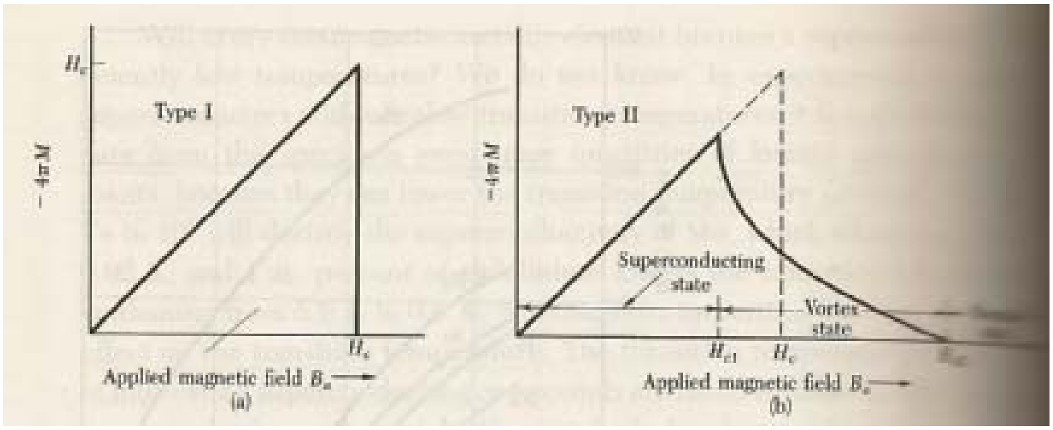
Another signature of superconductivity is the **Meissner Effect**. In observing this effect, as a superconductor is cooled in a constant applied magnetic field, magnetic flux is completely expulsed from the superconductor (so B=0 inside the superconductor) when the Tc is reached. In other words, superconductors exhibit perfect diagmagnetism. Note that the Meissner Effect is not expected for perfect conductors. It is because according to the Faraday Law, a perpetual current will flow to maintain the magnetic field inside the conductor, not to expulse it.



Superconductivity can be destroyed by applying a magnetic field above Hc, the critical field, Hc, which decreases with increasing T (see Fig. 2 at right).

2. Type I and II Superconductors

There are two types of superconductors, I and II, characterized by the behavior in an applied magnetic field (see Fig. 3 below).



In type I superconductors, there is always perfect diagmagnetism. Pure specimens of many materials exhibit this behavior.

In type II superconductors, when the applied field exceeds a value called H_{c1}, magnetic field lines start to penetrate the superconductor. Superconductivity persists until H_{c2} is reached. Between H_{c1} and H_{c2}, the superconductor is in the vortex state. A field H_{c2} of 41 T has been attained in an alloy of Nb, Al and Ge at 4.2K. Type II superconductors tend to be alloys or transition metals, which has high resistivity in the normal state (i.e., mean free path in the normal state is short). Commercial solenoids wound with a type II superconductor can produce magnetic fields of up to 16 to 30 T at 4.2 K depending on the material used.

3. Heat Capacity

The electronic part of the heat capacity of superconductor is found to be ~exp($-E_g/2kT$), where E_g is a constant. This is characteristic of the presence of an energy gap. E_g is the energy gap of the superconductor.

In insulators, the gap is caused by the periodic potential produced by the ions in the lattice. In superconductors, the gap has a different origin. The interactions between electrons (mediated by the lattice) causes the formation of Cooper pairs that together form a superconducting condensate that has a free energy lower than that of the normal state.

The superconductivity transition is second order. That is, there is a discontinuity in d^2F/dT^2 ~ heat capacity at the transition. ($dF = -PdV - SdT$. $S = -dF/dT$. So, $C = dQ/dT = TdS/dT$ is proportional to the second derivative of F.)

4. Isotope Effect

The Tc varies with the isotopic mass, M. When M is decreased, Tc is increased.

$M^{\alpha}Tc$ = constant, where $\alpha = 0.32$ to 0.5.

This shows that electron-lattice interactions are deeply involved in the phenomenon of superconductivity. The original BCS theory gave Tc ~ θ_{Debye} ~ $M^{-1/2}$, so $\alpha = 0.5$. With inclusion of coulomb interactions between electrons changes the relation.

5. London Equation

In 1935, Fritz and Heinz London postulated the following relation between the current and vector potential in and around a superconductor:

$$\vec{j} = -\frac{1}{\mu_0 \lambda_L^2}\vec{A}., \tag{1}$$

where $\lambda_L = (\varepsilon_0 mc^2/nq^2)^{1/2}$. By selecting the Coulomb gauge: $\nabla\cdot\boldsymbol{A} = 0$, and taking the curl of (1), one gets:

$$\nabla\times\vec{j} = -\frac{1}{\mu_0 \lambda_L^2}\vec{B}. \tag{2}$$

We shall later derive the London equation. First, we examine how it leads to the Meissner effect.

Consider the Maxwell's equation:

$$\nabla\times\vec{B} = \mu_0\vec{j}$$
$$\nabla\times\nabla\times\vec{B} = -\nabla^2\vec{B} = \mu_0\nabla\times\vec{j}$$

Sub. (2) in this, one gets:

$$\nabla^2\boldsymbol{B} = \boldsymbol{B}/\lambda_L^2. \tag{3}$$

The solution is:

$$B(x) = B(0)\exp(-x/\lambda_L). \tag{4}$$

It shows that the magnetic field decays exponentially into the superconductor over a penetration depth of $\lambda_L = [\varepsilon_0 mc^2/(nq^2)]^{1/2} = [m/(\mu_0 nq^2)]^{1/2}$ By substituting typical values for the parameters, one finds that λ_L is on the order of 10^{-7} m. Such a small value of the penetration depth explains the Meissner effect.

The Meissner effect is intimately tied to the fact that the electrons (forming pairs) in the superconducting state behave as bosons and so tend to be locked down at the lowest energy in exactly the same state. (There is more amplitude to go into the same state than into an unoccupied state by the factor $N^{1/2}$, where N is the occupancy of the lowest state.) This fact may itself account for why there is no resistance. In ordinary flow of current, electrons get knocked out of the regular flow leading to deterioration of the general momentum. But in superconductors, to get one electron away from what all the others are in is very hard because of the tendency of all Bose particles to go in the same state. A current once started, tends to keep on going forever.

Since all the electrons in a superconducting condensate are in the same state and there are many of them (~10^{22} per cm^3), it is reasonable to suppose that the number density of electrons, $n(\boldsymbol{r})$ is $\approx |\psi(\boldsymbol{r})|^2$, where $\psi(\boldsymbol{r})$ is the electron wave function. So, we may write

$$\psi(\boldsymbol{r}) = n(\boldsymbol{r})^{1/2} e^{i\theta(\boldsymbol{r})} \tag{5}$$

where $\theta(\mathbf{r})$ is a phase factor. Recall that the particle current density $\boldsymbol{J}^{\text{particle}}$ in the presence of a vector potential $\boldsymbol{A}$ is:

$$\vec{J}^{\,particle} = \frac{1}{2}\left\{\left[\frac{-i\hbar\vec{\nabla} - q\vec{A}}{m}\psi\right]^*\psi + \psi^*\left[\frac{-i\hbar\vec{\nabla} - q\vec{A}}{m}\psi\right]\right\}.$$

Substitute eqn. (5) in this equation and multiply the RHS by q to turn the equation into one for the charge current density, $\boldsymbol{J}$, one gets:

$$\vec{J} = \frac{\hbar q}{m}\left(\vec{\nabla}\theta - \frac{q}{\hbar}\vec{A}\right)n. \tag{6}$$

This equation says that $\boldsymbol{J}$ has two pieces. One comes from the gradient of the phase. The other comes from the vector potential $\boldsymbol{A}$. With the Coulomb gauge, i.e., $\nabla\cdot\boldsymbol{A} = 0$, eqn. (6) gives

$$(\hbar q/m)\nabla^2\theta = \nabla\cdot\boldsymbol{J} = d\rho/dt = 0,$$

where $\rho \equiv qn$ is the charge density. This means that the phase of the wave function is a constant everywhere in the superconductor and so cannot contribute to $\boldsymbol{J}$. Equation (6) becomes:

$$\vec{J} = -\frac{q^2 n}{m}\vec{A} = -\frac{1}{\mu_0 \dfrac{m}{\mu_0 n q^2}}\vec{A} = -\frac{1}{\mu_0 \lambda_L^2}\vec{A}$$

This is the same equation postulated by London and London.

6. Coherence Length

Another fundamental length that characterizes a superconductor is the coherence length ξ. It is a measure of the distance within which the superconducting electron concentration cannot change drastically in a spatially-varying B field.

Consider a wave function with a strong modulation:

$$\varphi(x) = 2^{-1/2}[\exp(i(k+q)x)+\exp(ikx)].$$

The probability density in space is

$$\varphi^*\varphi = 1 + \cos qx$$

For plane wave wave-function, $\psi(x) = \exp(ikx)$ (where a normalization condition of $\psi^*\psi = 1$ is adopted).

The kinetic energy (KE) of $\psi(x)$ is $\hbar^2k^2/2m$. For the modulate w.f., the KE is

$$KE = \int dx\varphi^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\varphi = \frac{1}{2}\left(\frac{\hbar^2}{2m}\right)\left[(k+q)^2+k^2\right] \cong \left(\frac{\hbar^2}{2m}\right)(k^2+kq),$$

given $q \ll k$.

The increase of energy required to modulate is $\hbar^2kq/2m$. If this exceeds E_g, superconductivity will be destroyed. This sets the critical value for q_0:

$$\hbar^2kq/2m = E_g$$

Define an intrinsic coherence length ξ_0 to be $1/q_0$.

So, $$\xi_0 = \hbar^2k_F/2mE_g = \hbar v_F/2E_g$$

BCS theory predicts that:

$$\xi_0 = 2\hbar v_F/\pi E_g$$

ξ_0 is typically 380 to 16,000 Å and λ_L is 340 to 1,100 Å.

In impure materials and in alloys, $\xi < \xi_0$ since the wave function is already wiggled to begin with. In these materials, the mean free path is also shorter than the intrinsic value. It has been shown that

$$\xi = (\xi_0 l)^{1/2}. \quad \text{and} \quad \lambda = \lambda_L(\xi_0/l)^{1/2},$$

where l is a constant less than ξ_0.

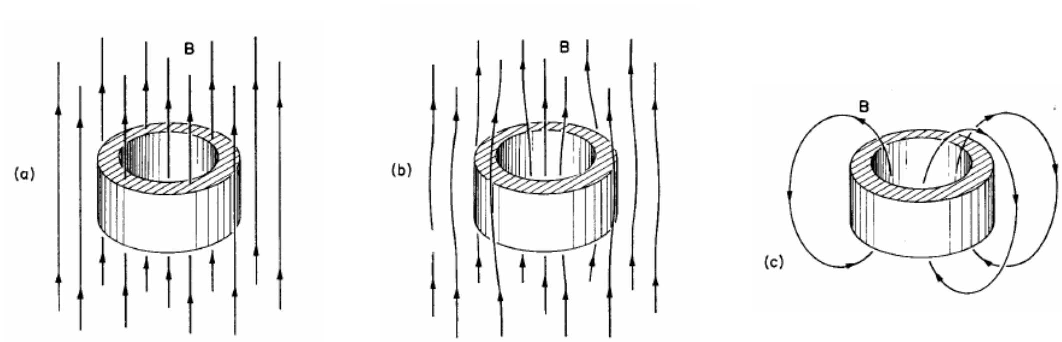
So, $$\lambda/\xi = \lambda_L / l.$$

If $\lambda/\xi < 1$, the formation of vortex is not feasible so the superconductor is type I.

Conversely, if $\lambda/\xi <1$, the superconductor is type II.

7. Flux Quantization

London made another interesting prediction about the phenomenon of superconductivity. Consider a ring made of a superconductor with thickness larger than λ. Suppose we start with a magnetic field through the ring then cool it to the superconducting state, and afterward turn the magnetic field off. The sequence of events is sketched below.



In the normal state, there will be a field in the body of the ring as sketched in part (a). When the ring is superconducting, the field is forced outside the ring. But there is still some flux through the hole of the ring as shown in part (b). Upon turning the magnetic field off, the lines of field going through the hole are trapped as shown in part (c). The flux Φ through the hole cannot decrease because dΦ/dt must be equal to the line integral of ***E*** around the ring, which is zero in a superconductor. As the external field is removed, a super current (essentially an eddy current) starts to flow around the ring to keep the flux through the ring a constant. But pertinent to the Meissner effect, it only persists over a penetration depth near the surface of the ring.

So far everything works the same way as discussed above, but there is an essential difference. The argument made above that θ must be a constant in a solid piece of superconductor does not apply to all regions of a ring. Well inside the body of the ring, the current density ***J*** is zero. So, eqn. (6) gives

$$\hbar\nabla\theta = q\boldsymbol{A} \tag{7}$$

Now, consider a line integral of ***A*** taken along a path that goes around the ring near the center of its cross-section where the current density is zero. From eqn. (7),

$$\hbar\oint\nabla\theta\cdot d\vec{s} = q\oint\vec{A}\cdot d\vec{s}\ . \tag{8}$$

But the RHS is just $q\Phi$, where Φ is the flux $\boldsymbol{B}$ ($\equiv\nabla\times\boldsymbol{A}$) through the hole. So, we have

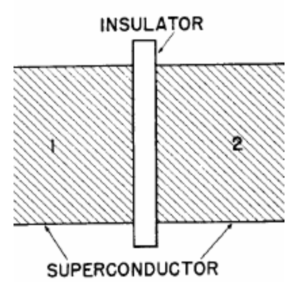
$$\oint\nabla\theta\cdot d\vec{s} = \frac{q}{\hbar}\Phi\ . \tag{9}$$

The LHS is the change in phase, $\Delta\theta$, of the wave function upon going around the ring, along the aforementioned path. Because the start and end points of this path are the same, the value of the wave function, $\psi(\boldsymbol{r}) = n(\boldsymbol{r})^{1/2}e^{i\theta(\boldsymbol{r})}$, must not change. This is possible only if $\Delta\theta$ = integer $\cdot\ 2\pi$, where n is an integer. Substitute this in (9), ones gets

$$\Phi = \text{integer}\cdot 2\hbar/q = \text{integer}\cdot h/q. \tag{10}$$

This result shows that the flux trapped in the hole must be quantized, equal to an integer times h/q. The independent experiments of Deaver and Fairbank and Doll and Nabauer (1961) showed that the units of flux quantum was $h/(2e)$, a half of what London expected when he derived eqn. (10). But now it is understood that q should be $2e$ because of the pairing of electrons according to the BCS theory.

8. The Josephson Junction



Consider two superconductors that are connected by a thin layer of insulating material as show at right. Such an arrangement is called a Josephson junction. The insulating layer is thin enough that electrons from one side of the junction can tunnel through the layer to the other side. We denote the amplitude (i.e., the wave function) to find an electron on one side, ψ_1, and that on the other, ψ_2. We further simplify the problem by assuming that the superconducting materials on the two sides are the same, and there is no magnetic field. The two amplitudes should be related in the following way:

$$i\hbar\frac{\partial\psi_1}{\partial t} = U_1\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = U_2\psi_2 + K\psi_1$$

The constant K is a characteristic of the junction. If K were zero, these equations would describe the lowest energy state – with energy U – of each superconductor. When K is nonzero, there is coupling between the two sides and electrons can leak from one side to the other. By the assumption that the two superconductors are the same, U_1 should be equal to U_2. But suppose now we connect the two superconducting regions to the two terminals of a battery so that there is a potential difference V across the junction. Then $U_1 - U_2 = qV$. For convenience, we define the zero of energy to be halfway between, then the above two equations are

$$i\hbar\frac{\partial\psi_1}{\partial t} = \frac{qV}{2}\psi_1 + K\psi_2$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = -\frac{qV}{2}\psi_2 + K\psi_1 \tag{11}$$

We write the wave functions on the two sides to be of the form given in eqn. (5):

$$\psi_1 = n_1^{1/2}e^{i\theta 1}$$

$$\psi_2 = n_2^{1/2}e^{i\theta 2}$$

where $n_1 \approx n_2$ are the electron density on the two sides. Substituting these in the two equations (11), one gets four equations by equating the real and imaginary parts in each case. Writing $\theta_2 - \theta_1 = \delta$, the result is

$$\dot{n}_1 = +\frac{2}{\hbar}K\sqrt{n_1 n_2}\sin\delta,$$

$$\dot{n}_2 = -\frac{2}{\hbar}K\sqrt{n_1 n_2}\sin\delta, \tag{12}$$

$$\dot{\theta}_1 = +\frac{K}{\hbar}\sqrt{\frac{n_2}{n_1}}\cos\delta - \frac{qV}{2\hbar},$$

$$\dot{\theta}_2 = +\frac{K}{\hbar}\sqrt{\frac{n_1}{n_2}}\cos\delta + \frac{qV}{2\hbar}. \tag{13}$$

The first two equations say that $\dot{n}_1 = -\dot{n}_2$. They describe how, due to an imbalance between the collection of electrons and positive ion background, the densities would start to change and therefore describe the current that would begin to flow. The current from side 1 to side 2 would be $\dot{n}_1$ (or $-\dot{n}_2$) or

$$J = +\frac{2}{\hbar} K \sqrt{n_1 n_2} \sin\delta \, . \tag{14}$$

Because the two sides are connected by wires to a battery, the current that flows will not charge up region 2 or discharge region 1. When this factor is taken into account, n_1 and n_2 actually do not change. On the other hand, the current across the junction is still given by (14).

Since n_1 and n_2 remain constant and let's say it's equal to n_0, we may write eqn. (14) as

$$J = J_0 \sin\delta, \tag{15}$$

where $J_0 = 2Kn_0/\hbar$ is a characteristic of the junction.

The other pair of equations (13) tells us about θ_1 and θ_2. By taking the difference, we get:

$$\dot{\delta} = \dot{\theta}_2 - \dot{\theta}_1 = \frac{qV}{\hbar}. \tag{16}$$

That means we can write

$$\delta(t) = \delta_0 + \frac{q}{\hbar} \int V(t) dt, \tag{17}$$

where δ_0 is the value of δ at $t = 0$. We consider the cases when the voltage is DC and AC, respectively.

(i) DC Josephson effect

This corresponds to the case when V = 0. Equation (15) predicts that a spontaneous tunneling current can flow through the junction if the phase difference between the wave functions on the two sides is not zero. This surprising prediction has been observed in experiment.

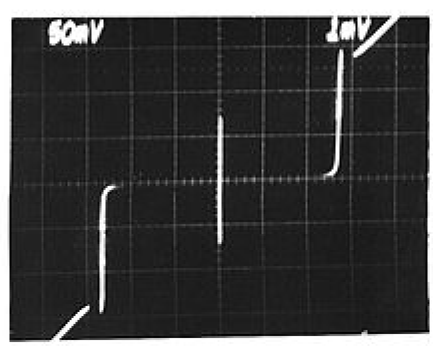


Figure at right: I vs. V characteristic of a Josephson junction. The current fluctuation seen at V = 0 is due to the DC Josephson effect. The vertical span of the fluctuation is $\pm J_0$*. The rise in current at V =* $\pm$ *2.8 mV is due to the superconductor bandgap. Each horizontal unit is 1 mV. Each vertical unit is 50* μ*A. (Wikipedia)*

(ii) AC Josephson effect

Suppose $V = V_0$, a DC voltage. Equations (17) and (15) give:

$$J = J_0 \sin(\delta_0 + qV_0 t/\hbar) \quad (15)$$

This predicts that there will be an AC current with amplitude J_0 and frequency $2eV_0/\hbar$. Because ħ is small, the frequency is rather high. For example, if $V_0 = 1$ V, the oscillation frequency is ~10^{15} Hz.

(iii) Inverse AC Josephson effect

Suppose the applied voltage is $V = V_0 + v\cos\omega t$, where $v << V$. Then

$$\delta(t) = \delta_0 + \frac{q}{\hbar} V_0 t + \frac{q}{\hbar}\frac{v}{\omega}\sin\omega t \quad (16)$$

For small dx, $\sin(x + dx) \approx \sin x + dx \cos x$.

Using this approximation for $\sin\delta$, one gets:

$$J = J_0\left[\sin\left(\delta_0 + \frac{q}{\hbar}V_0 t\right) + \frac{q}{\hbar}\frac{v}{\omega}\sin\omega t \cos\left(\delta_0 + \frac{q}{\hbar}V_0 t\right)\right].$$

The first term is zero on average, but the second term is not if

$$\omega = qV_0/\hbar.$$

There would be a DC current if the AC voltage has this frequency.