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Aristotle's Syllogistic and Core Logic

by

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I use the Corcoran-Smiley interpretation of Aristotle's syllogistic as my starting point for an examination of the syllogistic from the vantage point of modern proof theory. I aim to show that fresh logical insights are afforded by a proof-theoretically more systematic account of all four figures. First I regiment the syllogisms in the Gentzen–Prawitz system of natural deduction, using the universal and existential quantifiers of standard first-order logic, and the usual formalizations of Aristotle's sentence-forms. I explain how the syllogistic is a fragment of my (constructive and relevant) system of Core Logic. Then I introduce my main innovation: the use of binary quantifiers, governed by introduction and elimination rules. The syllogisms in all four figures are re-proved in the binary system, and are thereby revealed as all on a par with each other. I conclude with some comments and results about grammatical generativity, ecthesis, perfect validity, skeletal validity and Aristotle's chain principle.

1. Introduction: the Corcoran-Smiley interpretation of Aristotle's syllogistic as concerned with *deductions*

Two influential articles, *Corcoran 1972* and *Smiley 1973*, convincingly argued that Aristotle's syllogistic logic anticipated the twentieth century's systematizations of logic in terms of natural deductions. They also showed how Aristotle in effect advanced a completeness proof for his deductive system.

In this study, I shall introduce a different modern perspective on Aristotle's syllogistic. I am less concerned to advance an interpretation that is faithful to Aristotle's text and more concerned to reveal certain logical insights afforded by a proof-theoretically more systematic account of Aristotle's syllogisms of all four figures. For this reason it is enough to acknowledge the Corcoran-Smiley re-interpretation as my point of departure, from which a fresh new line of inquiry can fruitfully be followed. I shall not be detained by more recent secondary articles challenging the accuracy of the Corcoran-Smiley re-interpretation in all minute respects. (*Martin 1997* is an example in this regard. The reader interested in following up on other papers in this secondary literature will find *Corcoran 2009* a useful source.)

It would be mistaken to think of the current undertaking as one of re-visiting Aristotle's syllogistic equipped with all the more sophisticated techniques of modern proof theory, with the aim of revealing the weaknesses and limitations of Aristotle's system. To quote *Lear 1980*, at p. ix:

> The very possibility of proof-theoretic inquiry emerges with Aristotle, for such study requires that one have a system of formal inferences that can be subjected to mathematical scrutiny. Before the syllogistic there was no such formalization that could be a candidate for proof-theoretic investigation.

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There is, however, a certain fascination in seeing what light can be shed on the syllogistic using techniques that could eventually have arisen only because of the groundwork Aristotle laid, but which were not yet available to him. It is a test of their fruitfulness to see what account those techniques can render of Aristotle's own subject matter.

1.1. *Deductivist reconstructions of Aristotle's syllogistic*

For both Corcoran and Smiley, the sentences

'Every F is G' and 'Some F is not G'

are defined by Aristotle to be *contradictories* of one another, as are the sentences

'Some F is G' and 'No F is G'.

Using my notation to effect the comparison, we have Corcoran writing, at p. 696:

> ['Every F is G'] and ['No F is G'] are defined to be *contradictories of* ['Some F is not G'] and ['Some F is G'] respectively (and vice versa) ... ,

and Smiley writing to the same effect, at p. 141:

> ['Every F is G'] and ['Some F is not G'] will be said to be each other's *contradictory*; likewise ['No F is G'] and ['Some F is G'].

Corcoran uses the notation $C(d)$ for the contradictory of a sentence d; and Smiley uses the notation $\overline{\mathrm{P}}$ for the contradictory of P. I shall borrow Smiley's notation here. Note that for both these authors, the contradictory of the contradictory of P is P itself:[1]

$$\overline{\overline{\mathrm{P}}} = \mathrm{P}.$$

Both Corcoran and Smiley present Aristotle's syllogistic as a deductive system, based on a certain limited choice of 'perfect' syllogisms (which are in effect two-premise rules of inference), plus certain other deductive rules. By means of these rules, all other syllogisms can be derived from the perfect ones. (At this stage, the reader unaquainted with Aristotle's syllogistic needs to have it mentioned that it is built into the notion of a syllogism that it is or contains a *valid* argument, and one enjoying a certain *form*. More explanation of these features is to be found in §2.)

Let us concentrate on Smiley's presentation of such a system. Smiley stresses Aristotle's *chain principle* as 'absolutely fundamental' to his syllogistic. This is 'the principle that the premisses of a syllogism must form a chain of predications linking the terms of the conclusion'. It is because of this principle that Aristotle is able to show that

> ... every valid syllogistic inference, regardless of the number of premisses, can be carried out by means of a succession of two-premiss syllogisms. ...
>
> One is thus led to ask what account of implication, if any, will harmonise with Aristotle's chain principle for syllogisms. The question invites a logical rather than a historical answer ...
>
> I shall offer my answer to my question in the shape of a formal system in which I shall put into practice the idea of treating syllogisms as deductions, and which is intended to match as closely as possible Aristotle's own axiomatisation of the syllogistic by means of conversion, reductio ad impossibile, and the two universal moods of the first figure.
>
> (*Smiley 1973*, at p. 140.)

[1] Thus the contradictory of a sentence is not the result of attaching something to it (for example, prefixing it with a negation sign).

Smiley's logical system, on behalf of Aristotle, and in this logically reconstructive spirit, has the following basic rules of inference, by means of which one can form deductions.

(1) The syllogism Every M is P, Every S is M; so, Every S is P (Barbara).
(2) The syllogism No M is P, Every S is M; so, No S is P (Celarent).
(3) The conversion No F is G; so, No G is F.
(4) The conversion Every F is G; so, Some G is F.
(5) The classical rule of *reductio ad impossibile*: a deduction of P from $\overline{\text{Q}}$ (the contradictory of Q), along with a deduction of $\overline{\text{P}}$, entitles one to form a deduction of Q from the combined premises – other than $\overline{\text{Q}}$ – of those two deductions.[2]

Aristotle's system prioritized Barbara and Celarent, the first two syllogisms of his first figure. Corcoran and Smiley were faithful to this feature in their respective accounts of Aristotle's method. Note that (5) is the only *strictly classical* (i.e. non-constructive) rule, and the only rule that involves discharge of assumptions made 'for the sake of argument'.

1.2. *The different inferential approach of this study*

We shall state some altogether new rules for Aristotle's quantifying expressions. *Each* of those expressions is governed by at least one basic rule that involves discharge of assumptions; and *every* basic rule is constructive. These rules also maintain *relevance* of premises to conclusions; and they are far more basic than any of the syllogisms themselves.[3] They have been devised with an eye only to the need to provide a deduction for each of Aristotle's syllogisms in all four figures; yet the rules do not call for any modification or extension in order to provide deductions for all arguments whose validity is determined by the quantifying expressions that the rules govern. Moreover, the rules furnish deductions for Aristotle's syllogisms in a beautifully uniform fashion: every two-premise syllogism has a deduction containing only three basic steps.[4] Thus no syllogism, or pair of syllogisms, is prioritized over the others.

2. What is a syllogism?

An *argument* is what modern logicians also call a (single-conclusion) *sequent*: a (finite) set Δ of sentences, called *premises*, followed by a single sentence φ, called the *conclusion*. Sequents are often represented as of the form $\Delta : \varphi$. Sequents are *valid* or *invalid*; it is the logician's job to

[2]The reader can easily verify that this is an equivalent way of stating Smiley's inductive clause (iii) in his definition of formal deduction, *loc. cit.*, at p. 142 *supra*. The clause in question is

> If $\langle \ldots P\rangle$ is a deduction of P from $X_1, \overline{Q}$, and $\langle \ldots \overline{P}\rangle$ is a deduction of $\overline{P}$ from X_2, then $\langle \ldots P, \ldots \overline{P}, Q\rangle$ is a deduction of Q from X_1, X_2.

[3]*Cf. Smiley 1994*, at p. 30:

> Aristotle's case for the chain condition is redolent of relevance – the need for some overt connection of meaning between premises and conclusion as a prerequisite for deduction.

This may be thought of as a 'macro' point on Aristotle's behalf. I shall be concerned to preserve it at the micro-level of the rules for the binary quantifiers to be proposed in §4.1 – which, to be sure, are not those of Aristotle himself.

[4]For those syllogisms that, from the point of view of a modern advocate of 'universally free' logic, require an extra existential premise, the deductions consist of *four* steps. (For a system of universally free logic, see *Tennant 1978*, Ch. 7.) But it should also be noted that modern systems of 'unfree', *many-sorted* logic, such as the various systems used in *Smiley 1962*, *Parry 1966* and *Gupta 1980*, have been suggested as ways of accommodating Aristotle's syllogistic. In such systems, the sortal variables a are assumed to range over non-empty sorts A, thereby making each such sort A analogous to the single domain presumed by the system of standard first-order logic. The latter system is *un*free and *single*-sorted, and in it we have $\forall xFx \vdash \exists xFx$. In an unfree many-sorted system, analogously, one has, for each sort A, both $\vdash \forall aAa$ and $\vdash \exists aAa$. When using a many-sorted system to regiment the syllogistic, there is no need for any extra existential premises of the form $\exists aAa$; as Smiley put it (*loc. cit.*, p. 66)

> . . . this is something implicit rather than explicit – the existence of the various As is a pre-condition of the successful application of the system rather than an assumption formulated or even formulable within the system.

See also footnote 8.

classify them as such. The valid ones are those that 'preserve truth from their premises to their conclusions'. This idea is usually explicated as follows:

Definition 2.1: $\Delta : \varphi$ is valid if and only if every interpretation of the non-logical vocabulary (in the sentences involved) that makes every sentence in Δ true makes φ true also.

One can read the colon in a sequent $\Delta : \varphi$ as the word 'so', or 'therefore', or '*ergo*'. In doing so, however, one must bear in mind that the sequent itself is a *complex singular term denoting an argument*. If (and *only* if) the argument were to be presented argumentatively, this would involve the arguer making it clear that the premises were thereby being *supposed*, or taken as *hypotheses*, or *assumed for the sake of argument*; and that the conclusion was being drawn from those premises. In general, of course, one could denote arguments by means of sequents which one would be unwilling to present argumentatively – indeed, which one would be concerned to point out should never be endorsed, because they are invalid.

What has come to be called an *Aristotelian syllogism* is a *valid* argument with two premises and a single conclusion.[5] These sentences are, moreover, of particular forms. Each of them involves two *non-logical terms* (predicates) from a trio of such terms, along with a *quantifying expression*, which is one of a, e, i or o – see below. In addition, the occurrences of the non-logical terms of the trio within the argument have to exhibit certain patterns, called *figures*. There are four such figures, explained below. Aristotle investigated only the first three of them.

It is now common practice to use the abbreviations in the left column below for the four logical forms of sentences of which Aristotle treats. Their English readings are given in the right column.

aFG	Every F is G
eFG	No F is G
iFG	Some F is G
oFG	Some F is not G

We use the sortal variable q, possibly with a numerical subscript, to stand for any of the quantifier expressions a, e, i or o in what follows. I shall call the forms in the foregoing display *Aristotelian forms* (of sentences).

The *major term* of an argument is the right-most one in its conclusion. It is rendered as P in the following. The major term occurs in exactly one of the two premises, which is accordingly called the *major premise*. The other premise is called the *minor premise*. The major premise is always listed first when an Aristotelian argument is stated:

$$\frac{\begin{array}{l}\text{Major premise}\\ \text{Minor premise}\end{array}}{\text{Conclusion}}$$

We shall ambiguously denote the quantifier expression of the major premise by q_1; that of the minor premise by q_2; and that of the conclusion by q_3. Remember, there are four possible values for these q_j: a, i, e and o.

One of the three terms of the argument, called the *middle term*, occurs in *both* of the premises, but *not* in the conclusion. It is rendered as M in the following. Since the major premise contains both the major term P and the middle term M, it is of the form q_1MP or q_1PM.

The third term of the trio is called the *minor term*. It is rendered as S in the following. The

[5]As Nicholas Rescher has pointed out, he and various other scholars think that 'for Aristotle himself a syllogism was simply a pair of duly related premises. The conclusion was left as a problem and did not serve as a constituent part of the syllogism.' (Personal correspondence.) But Corcoran has pointed out that both he and Smiley emphasize that Aristotle did not limit syllogisms to two-premise arguments, and that some of Aristotle's syllogisms contain propositions other than the premises and conclusion. (Personal correspondence.)
I do not intend to resolve this disagreement here. We just need a clear technical term for the purposes of this study. Accordingly, an 'Aristotelian syllogism' will be a valid two-premise argument of the restricted form specified in the text. It will reside in one of the four figures.

conclusion contains the minor term S left-most. It follows from what we have specified that the minor premise contains the middle term M and the minor term S. So it is of the form q_2SM or q_2MS.

It also follows from the foregoing stipulations that the conclusion of an argument must have the form q_3SP. Thus arguments can vary in their patterns of term-occurrences only in respect of their major and minor premises, and the way the latter combine the major term P, the middle term M and the minor term S. The major premise contains P and M (in either order). The minor premise contains M and S (in either order). So, bearing in mind that the major premise is always listed first, the only possible combinations of terms within the two premises are the following four, called *figures*:[6]

	First Figure	Second Figure	Third Figure	Fourth Figure
Major premise	q_1MP	q_1PM	q_1MP	q_1PM
Minor premise	q_2SM	q_2SM	q_2MS	q_2MS

The *mood* of an argument is defined as its ordered triple

$$\langle q_1, q_2, q_3 \rangle.$$

Clearly, there are $4^3 = 64$ possible moods within any one figure. Matters can be simplified, though. It is obvious that both e and i are 'symmetric' quantifier expressions. That is, eFG is equivalent to eGF, and iFG is equivalent to iGF. Thus there are really only 16 moods within any one figure that deserve serious independent attention.

It is the combination of figure and mood, for two-premise arguments involving sentences of the above forms, under the constraints of major-, minor- and middle-term occurrences within premises and conclusion, that determine the *logical form* of the argument in question. An argument of mood $\langle q_1, q_2, q_3 \rangle$ and of figure k will be said to have the form $\langle q_1, q_2, q_3 \rangle$-$k$. Aristotle gave each (valid) syllogistic form a name – see below. The systematic notation $\langle q_1, q_2, q_3 \rangle$-$k$, however, is completely specific and is a helpful mnemonic – and is therefore preferable. In deference to tradition, however, we also give the medieval scholastics' names below.[7] Note how the vowels in the names codify the moods of the syllogisms.

Remember: a syllogism is a *valid* argument. Some of Aristotle's syllogisms, it turns out, need to be supplemented with certain existential premises in order to make them valid in the system on offer here. These supplementations are supplied below, without further comment. They all take the form 'There is some F'. Aristotle presumed that every non-logical term (i.e., monadic predicate) has a non-empty extension. Modern logicians, by contrast, allow for empty extensions.[8]

[6] Aristotle himself, in his *Prior Analytics*, investigated only syllogisms in the first three Figures. His pupil Theophrastus added the Fourth Figure, which has also been attributed to Galen. It is strange that Aristotle omitted the Fourth Figure, given his usual systematic thoroughness. There has been debate over many centuries as to whether Aristotle ought to have recognized the Fourth Figure, or whether its arguments can really be assigned, 'indirectly', to the First. The Fourth Figure was recognized by Peter of Mantua in 1483, and was argued for by Peter Tartaret (ca. 1480), by Richard Crackenthorpe in 1622, and by Antoine Arnauld in 1662. For a scrupulously scholarly account of these matters, see *Rescher 1966*. Another useful source is *Henle 1949*. Here we follow the post-*Port Royal*, English tradition of giving the Fourth its due. *Smiley 1994* §III offers an intriguing explanation for Aristotle's exclusion of the Fourth Figure, by suggesting 'a connection between it and the role of Platonic division in shaping syllogistic logic'.

[7] The mnemonic poem

Barbara, Celarent, Darii, Ferio que prioris;
Cesare, Camestres, Festino, Baroko secundae;
Tertia, Darapti, Disamis, Datisi, Felapton, Bokardo, Ferison, habet;
Quarta in super addit Bramantip, Camenes, Dimaris, Fesapo, Fresison

is of uncertain provenance, but is thought to be at least as old as William of Sherwood (1190–1249).

[8] We note, however, that *Smiley 1962* proposed a modern formalization of Aristotelian forms using *sortal quantifiers*. Just as in standard (unfree!) logic the domain is taken to be non-empty, so too in sortal quantification theory the various sortal domains are taken to be non-empty. An Aristotelian form such as 'All As are Bs' is translated, 'sortally', as $\forall a\, B(a)$, where

Hence their need to make certain existential premises explicit.

2.1. *Syllogisms of the* First Figure

These have the form $\dfrac{\begin{array}{l} q_1\mathrm{MP} \\ q_2\mathrm{SM} \end{array}}{q_3\mathrm{SP}}$. They are:

$\langle a, a, a\rangle$-1 (Barbara)
$$\frac{\begin{array}{l}\text{Every M is P}\\ \text{Every S is M}\end{array}}{\text{Every S is P}}$$

$\langle e, a, e\rangle$-1 (Celarent)
$$\frac{\begin{array}{l}\text{No M is P}\\ \text{Every S is M}\end{array}}{\text{No S is P}}$$

$\langle a, i, i\rangle$-1 (Darii)
$$\frac{\begin{array}{l}\text{Every M is P}\\ \text{Some S is M}\end{array}}{\text{Some S is P}}$$

$\langle e, i, o\rangle$-1 (Ferio)
$$\frac{\begin{array}{l}\text{No M is P}\\ \text{Some S is M}\end{array}}{\text{Some S is not P}}$$

$\langle a, a, i\rangle$-1 (Barbari)
$$\frac{\begin{array}{l}\text{There is some S}\\ \text{Every M is P}\\ \text{Every S is M}\end{array}}{\text{Some S is P}}$$

$\langle e, a, o\rangle$-1 (Celaront)
$$\frac{\begin{array}{l}\text{There is some S}\\ \text{No M is P}\\ \text{Every S is M}\end{array}}{\text{Some S is not P}}$$

2.2. *Syllogisms of the* Second Figure

These have the form $\dfrac{\begin{array}{l} q_1\mathrm{PM} \\ q_2\mathrm{SM} \end{array}}{q_3\mathrm{SP}}$. They are:

$\langle e, a, e\rangle$-2 (Cesare)
$$\frac{\begin{array}{l}\text{No P is M}\\ \text{Every S is M}\end{array}}{\text{No S is P}}$$

a is now a sortal variable ranging over the *A*s – of which there must be at least one. As Smiley notes (p. 66),

> Since the interpretation of our many-sorted logic demands that all the relevant domains of individuals shall be non-empty, there is a sense in which all the wff., whether cast in affirmative or negative form, have an existential import. But this is something implicit rather than explicit – the existence of the various *A*s is a pre-condition of the successful application of the system rather than an assumption formulated or even formulable within the system.

Since, in the formal reconstruction here of the syllogistic, we are not sortalizing after Smiley's fashion, we can and must on occasion make existential import explicit in order to secure the validity of certain syllogisms.
The sortalizing strategy was employed also by *Parry 1966*.

$\langle a, e, e\rangle$-2 (Camestres)
$$\frac{\text{Every P is M} \quad \text{No S is M}}{\text{No S is P}}$$

$\langle e, i, o\rangle$-2 (Festino)
$$\frac{\text{No P is M} \quad \text{Some S is M}}{\text{Some S is not P}}$$

$\langle a, o, o\rangle$-2 (Baroco)
$$\frac{\text{Every P is M} \quad \text{Some S is not M}}{\text{Some S is not P}}$$

$\langle e, a, o\rangle$-2 (Cesaro)
$$\frac{\text{There is some S} \quad \text{No P is M} \quad \text{Every S is M}}{\text{Some S is not P}}$$

$\langle a, e, o\rangle$-2 (Camestros)
$$\frac{\text{There is some S} \quad \text{Every P is M} \quad \text{No S is M}}{\text{Some S is not P}}$$

2.3. *Syllogisms of the* Third Figure

These have the form $\frac{q_1\text{MP} \quad q_2\text{MS}}{q_3\text{SP}}$. They are:

$\langle a, a, i\rangle$-3 (Darapti)
$$\frac{\text{There is some M} \quad \text{Every M is P} \quad \text{Every M is S}}{\text{Some S is P}}$$

$\langle e, a, o\rangle$-3 (Felapton)
$$\frac{\text{There is some M} \quad \text{No M is P} \quad \text{Every M is S}}{\text{Some S is not P}}$$

$\langle i, a, i\rangle$-3 (Disamis)
$$\frac{\text{Some M is P} \quad \text{Every M is S}}{\text{Some S is P}}$$

$\langle a, i, i\rangle$-3 (Datisi)
$$\frac{\text{Every M is P} \quad \text{Some M is S}}{\text{Some S is P}}$$

$\langle o, a, o\rangle$-3 (Bocardo)
$$\frac{\text{Some M is not P} \quad \text{Every M is S}}{\text{Some S is not P}}$$

$\langle e, i, o\rangle$-3 (Ferison)

$$\frac{\text{No M is P} \quad \text{Some M is S}}{\text{Some S is not P}}$$

2.4. *Syllogisms of the* Fourth Figure

These have the form $\frac{q_1\text{PM} \quad q_2\text{MS}}{q_3\text{SP}}$. They are:

$\langle a, a, i\rangle$-4 (Bramantip)

$$\frac{\text{There is some P} \quad \text{Every P is M} \quad \text{Every M is S}}{\text{Some S is P}}$$

$\langle a, e, e\rangle$-4 (Camenes)

$$\frac{\text{Every P is M} \quad \text{No M is S}}{\text{No S is P}}$$

$\langle i, a, i\rangle$-4 (Dimaris)

$$\frac{\text{Some P is M} \quad \text{Every M is S}}{\text{Some S is P}}$$

$\langle a, e, o\rangle$-4 (Calemos)

$$\frac{\text{There is some S} \quad \text{Every P is M} \quad \text{No M is S}}{\text{Some S is not P}}$$

$\langle e, a, o\rangle$-4 (Fesapo)

$$\frac{\text{There is some M} \quad \text{No P is M} \quad \text{Every M is S}}{\text{Some S is not P}}$$

$\langle e, i, o\rangle$-4 (Fresison)

$$\frac{\text{No P is M} \quad \text{Some M is S}}{\text{Some S is not P}}$$

3. Natural Deductions in the style of Gentzen and Prawitz, for Aristotle's syllogisms

Natural deductions in the style of *Gentzen 1934, 1935* and *Prawitz 1965* employ *serial* forms of elimination rules for conjunction ($\wedge$), the conditional ($\rightarrow$), and the universal quantifier ($\forall$). These rules find frequent application in proofs of Aristotle's syllogisms, as the reader will see below. The extra existential premises of the form 'There is some F' are now translated into Aristotle's notation as iFF. In conventional notation this is of course rendered as $\exists xFx$.

In the table below we provide for each syllogism its conventional translation into modern logical notation, and a natural deduction in the style of Gentzen and Prawitz. It is no surprise that each syllogism admits of such proof. The system of proof in question is, after all, complete for *all* valid arguments expressible in the language, which (if it contains any non-monadic predicate symbols) is much more extensive than just the monadic fragment that suffices for formulating the premises and conclusions of Aristotelian syllogisms.

The Gentzen-Prawitz deductive system is a hammer to crack these Aristotelian walnuts. Once cracked, however, each walnut is beautiful inside. We lay out the results below for two purposes. First, there ought to be some place in the literature to which the beginning student of logic can turn in order to see proofs of all of Aristotle's syllogisms laid out in the complete formal detail afforded by a modern logical system. Secondly, the resulting extensive but surveyable 'database' helps the more advanced theorist by suggesting to his or her inspecting eye certain metalogical themes that can be explored with great profit. These remarks apply also to the collection of proofs laid out in §5, using rules governing the binary quantifiers to be introduced in §4.

3.1. *Gentzen–Prawitz Natural Deductions for Syllogisms of the First Figure*

$\langle a,a,a\rangle$-1 (Barbara)	Every M is P Every S is M Every S is P	aMP aSM aSP	$\forall x(Mx\to Px)$ $\forall x(Sx\to Mx)$ $\forall x(Sx\to Px)$	$\dfrac{\dfrac{\dfrac{\dfrac{\overline{Sa}^{(1)} \quad \dfrac{\forall x(Sx\to Mx)}{Sa\to Ma}}{Ma} \quad \dfrac{\forall x(Mx\to Px)}{Ma\to Pa}}{Pa}}{Sa\to Pa}{\scriptstyle(1)}}{\forall x(Sx\to Px)}$
$\langle e,a,e\rangle$-1 (Celarent)	No M is P Every S is M No S is P	eMP aSM eSP	$\neg\exists x(Mx\wedge Px)$ $\forall x(Sx\to Mx)$ $\neg\exists x(Sx\wedge Px)$	$\dfrac{\dfrac{\overline{\exists x(Sx\wedge Px)}^{(1)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Sa\wedge Pa}^{(2)}}{Sa} \quad \dfrac{\forall x(Sx\to Mx)}{Sa\to Ma}}{Ma} \quad \dfrac{\overline{Sa\wedge Pa}^{(2)}}{Pa}}{Ma\wedge Pa}}{\exists x(Mx\wedge Px)} \quad \neg\exists x(Mx\wedge Px)}{\bot}}{\bot}{\scriptstyle(2)}}{\neg\exists x(Sx\wedge Px)}{\scriptstyle(1)}$
$\langle a,i,i\rangle$-1 (Darii)	Every M is P Some S is M Some S is P	aMP iSM iSP	$\forall x(Mx\to Px)$ $\exists x(Sx\wedge Mx)$ $\exists x(Sx\wedge Px)$	$\dfrac{\exists x(Sx\wedge Mx) \quad \dfrac{\dfrac{\dfrac{\overline{Sa\wedge Ma}^{(1)}}{Sa} \quad \dfrac{\dfrac{\overline{Sa\wedge Ma}^{(1)}}{Ma} \quad \dfrac{\forall x(Mx\to Px)}{Ma\to Pa}}{Pa}}{Sa\wedge Pa}}{\exists x(Sx\wedge Px)}}{\exists x(Sx\wedge Px)}{\scriptstyle(1)}$
$\langle e,i,o\rangle$-1 (Ferio)	No M is P Some S is M Some S is not P	eMP iSM oSP	$\neg\exists x(Mx\wedge Px)$ $\exists x(Sx\wedge Mx)$ $\exists x(Sx\wedge\neg Px)$	$\dfrac{\exists x(Sx\wedge Mx) \quad \dfrac{\dfrac{\dfrac{\overline{Sa\wedge Ma}^{(1)}}{Sa} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Sa\wedge Ma}^{(1)}}{Ma} \quad \overline{Pa}^{(2)}}{Ma\wedge Pa}}{\exists x(Mx\wedge Px)} \quad \neg\exists x(Mx\wedge Px)}{\bot}}{\neg Pa}{\scriptstyle(2)}}{Sa\wedge\neg Pa}}{\exists x(Sx\wedge\neg Px)}}{\exists x(Sx\wedge\neg Px)}{\scriptstyle(1)}$
$\langle a,a,i\rangle$-1 (Barbari)	There is some S Every M is P Every S is M Some S is P	iSS aMP aSM iSP	$\exists xSx$ $\forall x(Mx\to Px)$ $\forall x(Sx\to Mx)$ $\exists x(Sx\wedge Px)$	$\dfrac{\exists xSx \quad \dfrac{\dfrac{\overline{Sa}^{(1)} \quad \dfrac{\dfrac{\overline{Sa}^{(1)} \quad \dfrac{\forall x(Sx\to Mx)}{Sa\to Ma}}{Ma} \quad \dfrac{\forall x(Mx\to Px)}{Ma\to Pa}}{Pa}}{Sa\wedge Pa}}{\exists x(Sx\wedge Px)}}{\exists x(Sx\wedge Px)}{\scriptstyle(1)}$

$\langle e,a,o\rangle$-1 (Celaront)	There is some S No M is P Every S is M Some S is not P	iSS eMP aSM oSP	$\exists xSx$ $\neg\exists x(Mx\wedge Px)$ $\forall x(Sx\to Mx)$ $\exists x(Sx\wedge\neg Px)$

$$\dfrac{\exists xSx \quad \dfrac{\dfrac{\overline{Sa}^{(1)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Sa}^{(1)} \quad \dfrac{\forall x(Sx\to Mx)}{Sa\to Ma}}{Ma} \quad \overline{Pa}^{(2)}}{Ma\wedge Pa}}{\exists x(Mx\wedge Px)} \quad \neg\exists x(Mx\wedge Px)}{\bot}}{\neg Pa}{}^{(2)}}{Sa\wedge\neg Pa}}{\exists x(Sx\wedge\neg Px)}}{\exists x(Sx\wedge\neg Px)}{}^{(1)}$$

3.2. *Gentzen–Prawitz Natural Deductions for Syllogisms of the Second Figure*

$\langle e,a,e\rangle$-2 (Cesare)	No P is M Every S is M No S is P	ePM aSM eSP	$\neg\exists x(Px\wedge Mx)$ $\forall x(Sx\to Mx)$ $\neg\exists x(Sx\wedge Px)$

$$\dfrac{\dfrac{\overline{\exists x(Sx\wedge Px)}^{(2)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{Sa\wedge Pa}^{(1)}}{Pa} \quad \dfrac{\dfrac{\overline{Sa\wedge Pa}^{(1)}}{Sa} \quad \dfrac{\forall x(Sx\to Mx)}{Sa\to Ma}}{Ma}}{Pa\wedge Ma}}{\exists x(Px\wedge Mx)} \quad \neg\exists x(Px\wedge Mx)}{\bot}}{\bot}{}^{(1)}}{\neg\exists x(Sx\wedge Px)}{}^{(2)}$$

$\langle a,e,e\rangle$-2 (Camestres)	Every P is M No S is M No S is P	aPM eSM eSP	$\forall x(Px\to Mx)$ $\neg\exists x(Sx\wedge Mx)$ $\neg\exists x(Sx\wedge Px)$

$$\dfrac{\dfrac{\overline{\exists x(Sx\wedge Px)}^{(1)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{Sa\wedge Pa}^{(2)}}{Sa} \quad \dfrac{\dfrac{\overline{Sa\wedge Pa}^{(2)}}{Pa} \quad \dfrac{\forall x(Px\to Mx)}{Pa\to Ma}}{Ma}}{Sa\wedge Ma}}{\exists x(Sx\wedge Mx)} \quad \neg\exists x(Sx\wedge Mx)}{\bot}}{\bot}{}^{(2)}}{\neg\exists x(Sx\wedge Px)}{}^{(1)}$$

$\langle e,i,o\rangle$-2 (Festino)	No P is M Some S is M Some S is not P	ePM iSM oSP	$\neg\exists x(Px\wedge Mx)$ $\exists x(Sx\wedge Mx)$ $\exists x(Sx\wedge\neg Px)$

$$\dfrac{\exists x(Sx\wedge Mx) \quad \dfrac{\dfrac{\dfrac{\overline{Sa\wedge Ma}^{(1)}}{Sa} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{Pa}^{(2)} \quad \dfrac{\overline{Sa\wedge Ma}^{(1)}}{Ma}}{Pa\wedge Ma}}{\exists x(Px\wedge Mx)} \quad \neg\exists x(Px\wedge Mx)}{\bot}}{\neg Pa}{}^{(2)}}{Sa\wedge\neg Pa}}{\exists x(Sx\wedge\neg Px)}}{\exists x(Sx\wedge\neg Px)}{}^{(1)}$$

$\langle a,o,o\rangle$-2 (Baroco)	Every P is M Some S is not M Some S is not P	aPM oSM oSP	$\forall x(Px\to Mx)$ $\exists x(Sx\wedge\neg Mx)$ $\exists x(Sx\wedge\neg Px)$

$$\dfrac{\exists x(Sx\wedge\neg Mx) \quad \dfrac{\dfrac{\dfrac{\overline{Sa\wedge\neg Ma}^{(1)}}{Sa} \quad \dfrac{\dfrac{\dfrac{\overline{Pa}^{(2)} \quad \dfrac{\forall x(Px\to Mx)}{Pa\to Ma}}{Ma} \quad \dfrac{\overline{Sa\wedge\neg Ma}^{(1)}}{\neg Ma}}{\bot}}{\neg Pa}{}^{(2)}}{Sa\wedge\neg Pa}}{\exists x(Sx\wedge\neg Px)}}{\exists x(Sx\wedge\neg Px)}{}^{(1)}$$

$\langle e, a, o\rangle$-2 (Cesaro)

There is some S	iSS	$\exists x Sx$
No P is M	ePM	$\neg\exists x(Px \land Mx)$
Every S is M	aSM	$\forall x(Sx \to Mx)$
Some S is not P	oSP	$\exists x(Sx \land \neg Px)$

$$\dfrac{\exists x Sx \qquad \dfrac{\dfrac{[Sa]^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Pa]^{(2)} \qquad \dfrac{[Sa]^{(1)} \qquad \dfrac{\forall x(Sx \to Mx)}{Sa \to Ma}}{Ma}}{Pa \land Ma}}{\exists x(Px \land Mx)} \qquad \neg\exists x(Px \land Mx)}{\bot}}{\neg Pa}\,(2)}{Sa \land \neg Pa}}{\exists x(Sx \land \neg Px)}}{\exists x(Sx \land \neg Px)}\,(1)$$

$\langle a, e, o\rangle$-2 (Camestros)

There is some S	iSS	$\exists x Sx$
Every P is M	aPM	$\forall x(Px \to Mx)$
No S is M	eSM	$\neg\exists x(Sx \land Mx)$
Some S is not P	oSP	$\exists x(Sx \land \neg Px)$

$$\dfrac{\exists x Sx \qquad \dfrac{\dfrac{[Sa]^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Sa]^{(1)} \qquad \dfrac{[Pa]^{(2)} \qquad \dfrac{\forall x(Px \to Mx)}{Pa \to Ma}}{Ma}}{Sa \land Ma}}{\exists x(Sx \land Mx)} \qquad \neg\exists x(Sx \land Mx)}{\bot}}{\neg Pa}\,(2)}{Sa \land \neg Pa}}{\exists x(Sx \land \neg Px)}}{\exists x(Sx \land \neg Px)}\,(1)$$

3.3. *Gentzen–Prawitz Natural Deductions for Syllogisms of the Third Figure*

$\langle a, a, i\rangle$-3 (Darapti)

There is some M	iMM	$\exists x Mx$
Every M is P	aMP	$\forall x(Mx \to Px)$
Every M is S	aMS	$\forall x(Mx \to Sx)$
Some S is P	iSP	$\exists x(Sx \land Px)$

$$\dfrac{\exists x Mx \qquad \dfrac{\dfrac{\dfrac{[Ma]^{(1)} \qquad \dfrac{\forall x(Mx \to Sx)}{Ma \to Sa}}{Sa} \qquad \dfrac{[Ma]^{(1)} \qquad \dfrac{\forall x(Mx \to Px)}{Ma \to Pa}}{Pa}}{Sa \land Pa}}{\exists x(Sx \land Px)}}{\exists x(Sx \land Px)}\,(1)$$

$\langle e, a, o\rangle$-3 (Felapton)

There is some M	iMM	$\exists x Mx$
No M is P	eMP	$\neg\exists x(Mx \land Px)$
Every M is S	aMS	$\forall x(Mx \to Sx)$
Some S is not P	oSP	$\exists x(Sx \land \neg Px)$

$$\dfrac{\exists x Mx \qquad \dfrac{\dfrac{\dfrac{[Ma]^{(1)} \qquad \dfrac{\forall x(Mx \to Sx)}{Ma \to Sa}}{Sa} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Ma]^{(1)} \qquad [Pa]^{(2)}}{Ma \land Pa}}{\exists x(Mx \land Px)} \qquad \neg\exists x(Mx \land Px)}{\bot}}{\neg Pa}\,(2)}{Sa \land \neg Pa}}{\exists x(Sx \land \neg Px)}}{\exists x(Sx \land \neg Px)}\,(1)$$

$\langle i, a, i\rangle$-3 (Disamis)

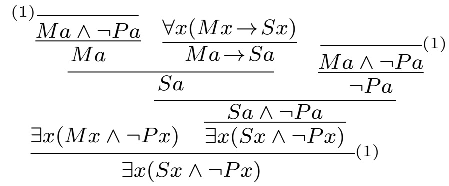
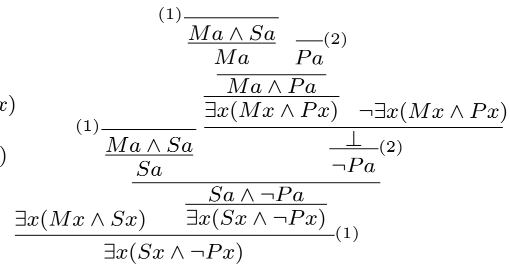
Some M is P	iMP	$\exists x(Mx \land Px)$
Every M is S	aMS	$\forall x(Mx \to Sx)$
Some S is P	iSP	$\exists x(Sx \land Px)$

$$\dfrac{\exists x(Mx \land Px) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[Ma \land Pa]^{(1)}}{Ma} \qquad \dfrac{\forall x(Mx \to Sx)}{Ma \to Sa}}{Sa} \qquad \dfrac{[Ma \land Pa]^{(1)}}{Pa}}{Sa \land Pa}}{\exists x(Sx \land Px)}}{\exists x(Sx \land Px)}\,(1)$$

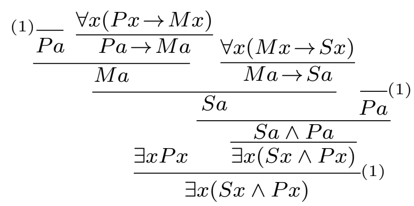
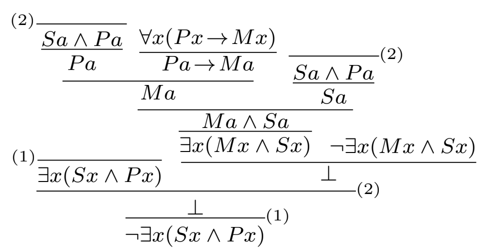
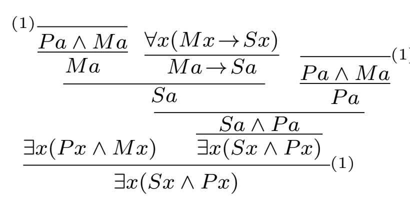
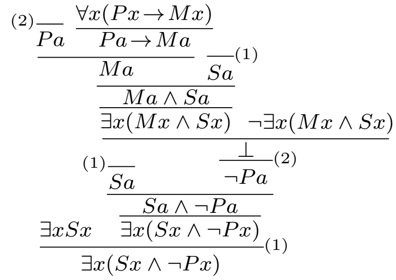
$\langle a, i, i\rangle$-3 (Datisi)

Every M is P	aMP	$\forall x(Mx \to Px)$
Some M is S	iMS	$\exists x(Mx \land Sx)$
Some S is P	iSP	$\exists x(Sx \land Px)$

$$\dfrac{\exists x(Mx \land Sx) \qquad \dfrac{\dfrac{\dfrac{[Ma \land Sa]^{(1)}}{Sa} \qquad \dfrac{\dfrac{[Ma \land Sa]^{(1)}}{Ma} \qquad \dfrac{\forall x(Mx \to Px)}{Ma \to Pa}}{Pa}}{Sa \land Pa}}{\exists x(Sx \land Px)}}{\exists x(Sx \land Px)}\,(1)$$

$\langle o, a, o\rangle$-3 (Bocardo)	Some M is not P Every M is S Some S is not P	oMP aMS oSP	$\exists x(Mx \wedge \neg Px)$ $\forall x(Mx \rightarrow Sx)$ $\exists x(Sx \wedge \neg Px)$	
$\langle e, i, o\rangle$-3 (Ferison)	No M is P Some M is S Some S is not P	eMP iMS oSP	$\neg\exists x(Mx \wedge Px)$ $\exists x(Mx \wedge Sx)$ $\exists x(Sx \wedge \neg Px)$	

3.4. *Gentzen–Prawitz Natural Deductions for Syllogisms of the Fourth Figure*

$\langle a, a, i\rangle$-4 (Bramantip)	There is some P Every P is M Every M is S Some S is P	iPP aPM aMS iSP	$\exists x Px$ $\forall x(Px \rightarrow Mx)$ $\forall x(Mx \rightarrow Sx)$ $\exists x(Sx \wedge Px)$	
$\langle a, e, e\rangle$-4 (Camenes)	Every P is M No M is S No S is P	aPM eMS eSP	$\forall x(Px \rightarrow Mx)$ $\neg\exists x(Mx \wedge Sx)$ $\neg\exists x(Sx \wedge Px)$	
$\langle i, a, i\rangle$-4 (Dimaris)	Some P is M Every M is S Some S is P	iPM aMS iSP	$\exists x(Px \wedge Mx)$ $\forall x(Mx \rightarrow Sx)$ $\exists x(Sx \wedge Px)$	
$\langle a, e, o\rangle$-4 (Calemos)	There is some S Every P is M No M is S Some S is not P	iSS aPM eMS oSP	$\exists x Sx$ $\forall x(Px \rightarrow Mx)$ $\neg\exists x(Mx \wedge Sx)$ $\exists x(Sx \wedge \neg Px)$	

$\langle e,a,o\rangle$-4 (Fesapo)	There is some M No P is M Every M is S Some S is not P	iMM ePM aMS oSP	$\exists xMx$ $\neg\exists x(Px\wedge Mx)$ $\forall x(Mx\to Sx)$ $\exists x(Sx\wedge\neg Px)$	$\dfrac{\exists xMx \quad \dfrac{\dfrac{\dfrac{\overline{Ma}^{(1)} \quad \dfrac{\forall x(Mx\to Sx)}{Ma\to Sa}}{Sa} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{Pa}^{(2)}\quad \overline{Ma}^{(1)}}{Pa\wedge Ma}}{\exists x(Px\wedge Mx)}\quad \neg\exists x(Px\wedge Mx)}{\bot}}{\neg Pa}{\scriptstyle(2)}}{Sa\wedge\neg Pa}}{\exists x(Sx\wedge\neg Px)}}{\exists x(Sx\wedge\neg Px)}{\scriptstyle(1)}$
$\langle e,i,o\rangle$-4 (Fresison)	No P is M Some M is S Some S is not P	ePM iMS oSP	$\neg\exists x(Px\wedge Mx)$ $\exists x(Mx\wedge Sx)$ $\exists x(Sx\wedge\neg Px)$	$\dfrac{\exists x(Mx\wedge Sx) \quad \dfrac{\dfrac{\dfrac{\overline{Ma\wedge Sa}^{(1)}}{Sa} \quad \dfrac{\dfrac{\dfrac{\dfrac{\overline{Pa}^{(2)}\quad \dfrac{\overline{Ma\wedge Sa}^{(1)}}{Ma}}{Pa\wedge Ma}}{\exists x(Px\wedge Mx)}\quad \neg\exists x(Px\wedge Mx)}{\bot}}{\neg Pa}{\scriptstyle(2)}}{Sa\wedge\neg Pa}}{\exists x(Sx\wedge\neg Px)}}{\exists x(Sx\wedge\neg Px)}{\scriptstyle(1)}$

The reader will no doubt have noticed that the number of steps within a Gentzen–Prawitz syllogistic proof (i.e., the number of applications of primitive rules of inference, or the number of furcations within the proof tree) varies from syllogism to syllogism. Each proof is either six, seven or nine steps long. This is because the Gentzen–Prawitz system uses *serial* elimination rules for $\wedge$, $\to$ and $\forall$; and because some of the syllogisms involve an extra (third) premise of 'existential import'.

More important, from a theoretical point of view, is that every syllogism has been furnished above with a *constructive* Gentzen–Prawitz natural deduction (in normal form). No use has been made of strictly classical negation rules. Inspection reveals also that (i) no use has been made of *Ex Falso Quodlibet*, and (ii) all applications of discharge rules occasioned *non-vacuous* discharge of the assumptions that had been made for the sake of argument. Thus the natural deductions in question are all *relevant*.

Deductions in normal form that are both constructive and relevant comprise what the author calls *Core Logic*.[9] We may summarize the rather operose catalog above as follows: *Aristotle's syllogistic is part of Core Logic*. The Gentzen–Prawitz system of natural deduction, however, is not the best way to present Core Logic. Its best presentation involves the use of *parallelized* elimination rules, whose major premises *stand proud*, with no proof-work above them. More on that in due course.

4. Aristotelian syllogistic with binary quantifiers

The modern logical notation employed above uses the standard *unary* quantifiers $\exists$ and $\forall$ to translate 'restricted' quantifications from English into logical notation:

*a*FG	Every F is G	$\forall x(Fx\to Gx)$
*e*FG	No F is G	$\neg\exists x(Fx\wedge Gx)$
*i*FG	Some F is G	$\exists x(Fx\wedge Gx)$
*o*FG	Some F is not G	$\exists x(Fx\wedge\neg Gx)$

The extra binary connectives – namely, $\to$ and $\wedge$ – that appear to be occasioned by this use of conventional logical notation cannot be discerned within the English forms thus translated.

[9]See Tennant (2012).

They are complications that are forced upon us by our choice of the unary quantifiers $\exists$ and $\forall$ to translate restricted quantifications of English into logical notation.

An alternative translation method employs *binary quantifiers*.[10] Unlike Aristotle's a, e, i and o, these binary quantifiers bind variables, the way that the unary Fregean ones do. But the binary quantifiers have no need for auxiliary connectives to construct a satisfactory formal sentence representing the logical form of a given English sentence. Using uppercase A, E, I and O for our more 'Fregean' *variable-binding* binary quantifiers, in order to distinguish them from Aristotle's quantifiers, our logical forms would be the ones in the right-most column:

aFG	Every F is G	$Ax(Fx, Gx)$
eFG	No F is G	$Ex(Fx, Gx)$
iFG	Some F is G	$Ix(Fx, Gx)$
oFG	Some F is not G	$Ox(Fx, Gx)$

(One needs to be very careful not to read the *negative* existential $Ex(Fx, Gx)$ as 'There exists an F that is G'! That would be an easy mistake to make, since logicians so often use 'E' as an existential quantifier.)

We owe to Frege the device of variable-binding in connection with quantifiers. We owe to Gentzen the device of Introduction and Elimination rules for logical operators in a system of natural deduction (and, correlatively, the device of 'right' and 'left' logical rules, respectively, in the sequent calculus). In this post-Fregean and post-Gentzenian age it would be anachronistic indeed to suggest that Aristotle himself could have had anything like these suggested binary quantifiers A, E, I and O in mind when formulating his syllogistic – let alone the logical rules that will be stated below, governing each of these new quantifiers. We shall see, however, that the new quantifier-rules afford dramatic simplifications in the deductions of Aristotle's syllogisms. And these rules have been formulated in response to the modest task of furnishing all of Aristotle's (and Theophrastus's) syllogisms with deductions.

4.1. *Introduction Rules and Parallelized Elimination Rules for the new Binary Quantifiers in a system of Natural Deduction*

In order for $Qx(\varphi, \psi)$ $(Q = A, E, I, O)$ to be a sentence, both φ and ψ must have x as their only free variable.

φ^x_t is the result of replacing all free occurrences of the variable x in φ by an occurrence of the *closed* singular term t. (A singular term is closed just in case it has no free occurrences of variables.) Use of the notation φ^x_t indicates that x enjoys free occurrences in φ, so that the substitution of t for such free occurrences is genuine.

φ^a_x is the result of replacing all occurrences of the parameter a in φ by an occurrence of the variable x. Use of the notation φ^a_x indicates that a occurs in φ, so that the substitution of x for a is genuine.

In the rules of natural deduction that follow, the parameter a may occur only where indicated. Note that the elimination rules are all in *parallelized* form. Their major premises stand proud. Hence all deductions in this system are in normal form.

A-Introduction

$$\dfrac{\begin{array}{c}\overline{\varphi}^{\,(i)}\\ \vdots\\ \psi\end{array}}{Ax(\varphi^a_x, \psi^a_x)}\,(i)$$

A-Elimination

$$\dfrac{Ax(\varphi, \psi) \qquad \varphi^x_t \qquad \begin{array}{c}\overline{\psi^x_t}^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

[10]See *Altham and Tennant 1975* for the usefulness of such binary quantifiers in the logic of plurality. They can be independently motivated by grammatical and logical considerations in the regimentation of quantifier expressions in natural language. It is not at all *ad hominem* to suggest their fruitful use in regimenting premises and conclusions in Aristotelian syllogisms.

E-Introduction
$$\dfrac{\begin{array}{c}\underbrace{{}^{(i)}\overline{\varphi^x_a}\ ,\ \overline{\psi^x_a}^{(i)}}\\ \vdots\\ \bot\end{array}}{Ex(\varphi,\psi)}\,(i)$$

E-Elimination
$$\dfrac{Ex(\varphi,\psi)\qquad \varphi^x_t\qquad \psi^x_t}{\bot}$$

I-Introduction
$$\dfrac{\varphi^x_t\qquad \psi^x_t}{Ix(\varphi,\psi)}$$

I-Elimination
$$\dfrac{Ix(\varphi,\psi)\qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{\varphi^x_a}\ ,\ \overline{\psi^x_a}^{(i)}}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

O-Introduction
$$\dfrac{\varphi^x_t\qquad \begin{array}{c}\overline{\psi^x_t}^{(i)}\\ \vdots\\ \bot\end{array}}{Ox(\varphi,\psi)}\,(i)$$

O-Elimination
$$\dfrac{Ox(\varphi,\psi)\qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{\varphi^x_a}\ ,\ \dfrac{\psi^x_a}{\bot}(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

Note how *O*-Elimination allows one to discharge an *inference* of the form 'ψ^x_a, so $\bot$'. The latter is the inferential equivalent of the negation of ψ^x_a.

With the application of a rule in natural deduction, the left-right ordering of displayed assumptions and of immediate subproofs is immaterial. Hence we see immediately that our rules confer the same logical force on $Qx(\varphi,\psi)$ and $Qx(\psi,\varphi)$, for Q taking the value E or I.

There is no rule of *Ex Falso Quodlibet*. There are no strictly classical negation rules. The system of binary-quantifier deduction is both constructive and relevant. It is part of Core Logic.

4.2. *Right and Left Logical Rules for the new Binary Quantifiers in a Sequent Calculus*

In all the sequent rules to follow, the parameter a occurs nowhere in the conclusion-sequent. A sequent 'empty on the right', i.e. of the form '$\Delta :$ ' states the non-satisfiability of Δ. So one could also write it as '$\Delta : \bot$'.

A-Right $\quad \dfrac{\Delta,\varphi^x_a : \psi^x_a}{\Delta : Ax(\varphi,\psi)}$ $\qquad$ *A*-Left $\quad \dfrac{\Delta : \varphi^x_t \qquad \psi^x_t,\Gamma : \theta}{Ax(\varphi,\psi),\Delta,\Gamma : \theta}$

E-Right $\quad \dfrac{\Delta,\varphi^x_a,\psi^x_a :}{\Delta : Ex(\varphi,\psi)}$ $\qquad$ *E*-Left $\quad \dfrac{\Delta : \varphi^x_t \qquad \Gamma : \psi^x_t}{\Delta,\Gamma,Ex(\varphi,\psi) :}$

I-Right $\quad \dfrac{\Delta : \varphi^x_t \qquad \Gamma : \psi^x_t}{\Delta,\Gamma : Ix(\varphi,\psi)}$ $\qquad$ *I*-Left $\quad \dfrac{\Delta,\varphi^x_a,\psi^x_a : \theta}{\Delta, Ix(\varphi,\psi) : \theta}$

O-Right $\quad \dfrac{\Delta : \varphi^x_t \qquad \Gamma,\psi^x_t :}{\Delta,\Gamma : Ox(\varphi,\psi)}$ $\qquad$ *O*-Left $\quad \dfrac{\Delta,\varphi^x_a : \psi^x_a}{\Delta, O(\varphi,\psi) :}$

Applications of the rules with just one premise-sequent (*A*-Right, *E*-Right, *I*-Left, *O*-Left) give rise to *monofurcations* within the proof-tree. Applications of the other rules give rise to *bifurcations*.

5. Proofs of syllogisms using binary-quantifier rules

Our binary quantifier rules afford significantly shorter deductions for Aristotle's syllogisms. I give both natural deductions and the sequent proofs that correspond to them. The extra existential premises of the form 'There is some F' are translated into the new formal notation as $Ix(Fx, Fx)$.

It is important to realize that the system of natural deduction with parallelized elimination rules whose major premises stand proud produces deductions that are, in effect, isomorphic to the corresponding sequent proofs. Moreover, *the order of elimination steps can be reversed*, so that there are two distinct formal proofs (two normal natural deductions, and two cut-free sequent proofs) for each standard syllogism with two premises. I shall give the alternatives for the first two examples (Barbara and Celarent); and leave it as an exercise for the reader to discover similar alternatives to each of the natural deductions and sequent proofs that we provide for the remaining syllogisms.

The reader should be aware that the re-orderability of elimination rules allows even this simple deductive system for syllogistic to provide different proofs of the same sequent. The proofs in question have rich enough inferential structure to be identified as the outcomes of *different strategies of proof-search*. This is a strong point in favor of using parallelized elimination rules rather than serial ones. For, with serial rules, those different strategies of proof-search will produce, as their results, one and the same proof.

5.1. *Binary-Quantifier Deductions for Syllogisms of the First Figure*

(Barbara)

$$\dfrac{\dfrac{Ax(Mx,Px)\qquad \dfrac{Ax(Sx,Mx)\quad \overline{Sa}^{(1)}\quad \overline{Ma}^{(3)}}{Ma}{\scriptstyle (3)}\qquad \overline{Pa}^{(2)}}{Pa}{\scriptstyle (2)}}{Ax(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{\dfrac{Sa:Sa\qquad Ma:Ma}{Ax(Sx,Mx),Sa:Ma}\qquad Pa:Pa}{Ax(Sx,Mx),Ax(Mx,Px),Sa:Pa}}{Ax(Sx,Mx),Ax(Mx,Px):Ax(Sx,Px)}$$

Alternatively:

$$\dfrac{\dfrac{Ax(Sx,Mx)\qquad {}^{(1)}\overline{Sa}\qquad \dfrac{Ax(Mx,Px)\quad \overline{Ma}^{(2)}\quad \overline{Pa}^{(3)}}{Pa}{\scriptstyle (3)}}{Pa}{\scriptstyle (2)}}{Ax(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa\qquad \dfrac{Ma:Ma\qquad Pa:Pa}{Ax(Mx,Px),Ma:Pa}}{Ax(Sx,Mx),Ax(Mx,Px),Sa:Pa}}{Ax(Sx,Mx),Ax(Mx,Px):Ax(Sx,Px)}$$

(Celarent)

$$\dfrac{\dfrac{Ex(Mx,Px)\qquad \dfrac{Ax(Sx,Mx)\quad \overline{Sa}^{(1)}\quad \overline{Ma}^{(2)}}{Ma}{\scriptstyle (2)}\qquad \overline{Pa}^{(1)}}{\bot}}{Ex(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{\dfrac{Sa:Sa\qquad Ma:Ma}{Ax(Sx,Mx),Sa:Ma}\qquad Pa:Pa}{Ex(Mx,Px),Ax(Sx,Mx),Sa,Pa:}}{Ex(Mx,Px),Ax(Sx,Mx):Ex(Sx,Px)}$$

Alternatively:

$$\dfrac{\dfrac{Ax(Sx,Mx)\qquad {}^{(1)}\overline{Sa}\qquad \dfrac{Ex(Mx,Px)\quad \overline{Ma}^{(2)}\quad \overline{Pa}^{(1)}}{\bot}}{\bot}{\scriptstyle (2)}}{Ex(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa\qquad \dfrac{Ma:Ma\qquad Pa:Pa}{Ex(Mx,Px),Ma,Pa:}}{Ex(Mx,Px),Ax(Sx,Mx),Sa,Pa:}}{Ex(Mx,Px),Ax(Sx,Mx):Ex(Sx,Px)}$$

(Darii)

$$\dfrac{Ix(Sx,Mx)\qquad \dfrac{{}^{(1)}\overline{Sa}\qquad \dfrac{Ax(Mx,Px)\quad \overline{Ma}^{(1)}\quad \overline{Pa}^{(2)}}{Pa}{\scriptstyle (2)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa\qquad \dfrac{Ma:Ma\qquad Pa:Pa}{Ma,Ax(Mx,Px):Pa}}{Sa,Ma,Ax(Mx,Px):Ix(Sx,Px)}}{Ix(Sx,Mx),Ax(Mx,Px):Ix(Sx,Px)}$$

(Ferio)

$$\dfrac{Ix(Sx,Mx) \qquad \dfrac{\overline{Sa}^{(1)} \qquad \dfrac{Ex(Mx,Px) \quad \overline{Ma}^{(1)} \quad \overline{Pa}^{(2)}}{\bot}}{Ox(Sx,Px)}{\scriptstyle (2)}}{Ox(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa \qquad \dfrac{Ma:Ma \qquad Pa:Pa}{Ma, Pa, Ex(Mx,Px):}}{Sa, Pa, Ex(Mx,Px) : Ox(Sx,Px)}}{Ix(Sx,Px), Ex(Mx,Px) : Ox(Sx,Px)}$$

(Barbari)

$$\dfrac{Ix(Sx,Sx) \qquad \dfrac{\overline{Sa}^{(1)} \qquad \dfrac{Ax(Mx,Px) \qquad \dfrac{Ax(Sx,Mx) \quad \overline{Sa}^{(1)} \quad \overline{Ma}^{(3)}}{Ma}{\scriptstyle (3)} \qquad \overline{Pa}^{(2)}}{Pa}{\scriptstyle (2)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa \qquad \dfrac{\dfrac{Sa:Sa \qquad Ma:Ma}{Sa, Ax(Sx,Mx) : Ma} \qquad Pa:Pa}{Sa, Ax(Mx,Px), Ax(Sx,Mx) : Pa}}{Sa, Ax(Mx,Px), Ax(Sx,Mx) : Ix(Sx,Px)}}{Ix(Sx,Sx), Ax(Mx,Px), Ax(Sx,Mx) : Ix(Sx,Px)}$$

(Celaront)

$$\dfrac{Ix(Sx,Sx) \qquad \dfrac{\overline{Sa}^{(1)} \qquad \dfrac{Ex(Mx,Px) \qquad \dfrac{Ax(Sx,Mx) \quad \overline{Sa}^{(1)} \quad \overline{Ma}^{(3)}}{Ma}{\scriptstyle (3)} \qquad \overline{Pa}^{(2)}}{\bot}}{Ox(Sx,Px)}{\scriptstyle (2)}}{Ox(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa \qquad \dfrac{\dfrac{Sa:Sa \qquad Ma:Ma}{Sa, Ax(Sx,Mx) : Ma} \qquad Pa:Pa}{Sa, Ex(Mx,Px), Ax(Sx,Mx), Pa :}}{Sa, Ex(Mx,Px), Ax(Sx,Mx) : Ox(Sx,Px)}}{Ix(Sx,Sx), Ex(Mx,Px), Ax(Sx,Mx) : Ox(Sx,Px)}$$

5.2. *Binary-Quantifier Deductions for Syllogisms of the Second Figure*

(Cesare)

$$\dfrac{\dfrac{Ax(Sx,Mx) \qquad \overline{Sa}^{(1)} \qquad \dfrac{Ex(Px,Mx) \quad \overline{Pa}^{(1)} \quad \overline{Ma}^{(2)}}{\bot}}{\bot}{\scriptstyle (2)}}{Ex(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa \qquad \dfrac{Pa:Pa \qquad Ma:Ma}{Ex(Px,Mx), Pa, Ma :}}{Sa, Pa, Ex(Px,Mx), Ax(Sx,Mx) :}}{Ex(Px,Mx), Ax(Sx,Mx) : Ex(Sx,Px)}$$

(Camestres)

$$\dfrac{\dfrac{Ax(Px,Mx) \qquad \overline{Pa}^{(1)} \qquad \dfrac{Ex(Sx,Mx) \quad \overline{Sa}^{(1)} \quad \overline{Ma}^{(2)}}{\bot}}{\bot}{\scriptstyle (2)}}{Ex(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Pa:Pa \qquad \dfrac{Sa:Sa \qquad Ma:Ma}{Ex(Sx,Mx), Sa, Ma :}}{Pa, Sa, Ex(Sx,Mx), Ax(Px,Mx) :}}{Ex(Sx,Mx), Ax(Px,Mx) : Ex(Sx,Px)}$$

(Festino)

$$\dfrac{Ix(Sx,Mx) \qquad \dfrac{\overline{Sa}^{(1)} \qquad \dfrac{Ex(Px,Mx) \quad \overline{Ma}^{(1)} \quad \overline{Pa}^{(2)}}{\bot}}{Ox(Sx,Px)}{\scriptstyle (2)}}{Ox(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa \qquad \dfrac{Ma:Ma \qquad Pa:Pa}{Ma, Pa, Ex(Px,Mx) :}}{Sa, Ma, Ex(Mx,Px) : Ox(Sx,Px)}}{Ix(Sx,Mx), Ex(Mx,Px) : Ox(Sx,Px)}$$

(Baroco)

$$\dfrac{Ox(Sx,Mx) \qquad \dfrac{\overline{Sa}^{(1)} \qquad \dfrac{Ax(Px,Mx) \quad \overline{Pa}^{(2)} \quad \dfrac{\overline{Ma}^{(3)}}{\bot}{\scriptstyle (1)}}{\bot}{\scriptstyle (3)}}{Ox(Sx,Px)}{\scriptstyle (2)}}{Ox(Sx,Px)}{\scriptstyle (1)}$$

$$\dfrac{\dfrac{Sa:Sa \qquad \dfrac{Pa:Pa \qquad Ma:Ma}{Ax(Px,Mx), Pa : Ma}}{Ax(Px,Mx), Sa : Ma, Ox(Sx,Px)}}{Ox(Sx,Mx), Ax(Px,Mx) : Ox(Sx,Px)}$$

Note that the natural deduction on the left reveals our first use of an *inference to absurdity* ($\bot$) being discharged by an application of O-Elimination. The inference in question is from Ma to $\bot$ at the top right, labeled (1), which is discharged at the final step. In the sequent proof on the right, the same stratagem (of 'assuming the falsity of Ma') is effected by allowing the penultimate sequent to be a 'multiple-conclusion' sequent. This sequent has *two sentences* in its succedent (i.e., on the right of the colon), one of them being Ma. For the sequent-system theorist, 'multiplicity on the right' is a warning sign that one might be 'classicizing' the logic. In this instance, however, we are dealing with *constructively innocuous multiplicity* on the right. The great advantage of exploiting it is that we can avoid any explicit use of the negation sign, and work only with atomic sentences when the binary quantifiers are not involved.

(Cesaro)

$$\dfrac{Ix(Sx,Sx) \quad \dfrac{\overline{Sa}^{(1)} \quad \dfrac{Ax(Sx,Mx) \quad \overline{Sa}^{(1)} \quad \dfrac{Ex(Px,Mx) \quad \overline{Pa}^{(2)} \quad \overline{Ma}^{(3)}}{\bot}}{\bot}{}^{(3)}}{Ox(Sx,Px)}{}^{(2)}}{Ox(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{Sa:Sa \quad \dfrac{Sa:Sa \quad \dfrac{Pa:Pa \quad Ma:Ma}{Ex(Px,Mx),Pa,Ma:}}{Ax(Sx,Mx),Sa,Ex(Px,Mx),Pa:}}{Sa,Ax(Sx,Mx),Ex(Px,Mx):Ox(Sx,Px)}}{Ix(Sx,Sx),Ax(Sx,Mx),Ex(Px,Mx):Ox(Sx,Px)}$$

(Camestros)

$$\dfrac{Ix(Sx,Sx) \quad \dfrac{\overline{Sa}^{(2)} \quad \dfrac{Ax(Px,Mx) \quad \overline{Pa}^{(2)} \quad \dfrac{Ex(Sx,Mx) \quad \overline{Sa}^{(1)} \quad \overline{Ma}^{(3)}}{\bot}}{\bot}{}^{(3)}}{Ox(Sx,Px)}{}^{(2)}}{Ox(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{Sa:Sa \quad \dfrac{Pa:Pa \quad \dfrac{Sa:Sa \quad Ma:Ma}{Ex(Sx,Mx),Pa,Ma:}}{Ax(Px,Mx),Sa,Ex(Px,Mx),Pa:}}{Sa,Ax(Sx,Mx),Ex(Px,Mx):Ox(Sx,Px)}}{Ix(Sx,Sx),Ax(Sx,Mx),Ex(Px,Mx):Ox(Sx,Px)}$$

5.3. *Binary-Quantifier Deductions for Syllogisms of the Third Figure*

(Darapti)

$$\dfrac{Ix(Mx,Mx) \quad \dfrac{\dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(1)} \quad \overline{Sa}^{(2)}}{Sa}{}^{(2)} \quad \dfrac{Ax(Mx,Px) \quad \overline{Ma}^{(1)} \quad \overline{Pa}^{(3)}}{Pa}{}^{(3)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{\dfrac{Ma:Ma \quad Sa:Sa}{Ax(Mx,Sx),Ma:Sa} \quad \dfrac{Ma:Ma \quad Pa:Pa}{Ax(Mx,Px),Ma:Pa}}{Ma,Ax(Mx,Sx),Ax(Mx,Px):Ix(Sx,Px)}}{Ix(Mx,Mx),Ax(Mx,Sx),Ax(Mx,Px):Ix(Sx,Px)}$$

(Felapton)

$$\dfrac{Ix(Mx,Mx) \quad \dfrac{\dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(1)} \quad \overline{Sa}^{(3)}}{Sa}{}^{(3)} \quad \dfrac{Ex(Mx,Px) \quad \overline{Ma}^{(1)} \quad \overline{Pa}^{(2)}}{\bot}{}^{(3)}}{Ox(Sx,Px)}{}^{(2)}}{Ox(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{\dfrac{Ma:Ma \quad Sa:Sa}{Ax(Mx,Sx),Ma:Sa} \quad \dfrac{Ma:Ma \quad Pa:Pa}{Ex(Mx,Px),Ma,Pa:}}{Ma,Ax(Mx,Sx),Ax(Mx,Px):Ox(Sx,Px)}}{Ix(Mx,Mx),Ax(Mx,Sx),Ax(Mx,Px):Ox(Sx,Px)}$$

(Disamis)

$$\dfrac{Ix(Mx,Px) \quad \dfrac{\dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(1)} \quad \overline{Sa}^{(2)}}{Sa}{}^{(2)} \quad \overline{Pa}^{(1)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{\dfrac{Ma:Ma \quad Sa:Sa}{Ax(Mx,Sx),Ma:Sa} \quad Pa:Pa}{Ma,Pa,Ax(Mx,Sx):Ix(Sx,Px)}}{Ix(Mx,Px),Ax(Mx,Sx):Ix(Sx,Px)}$$

(Datisi)

$$\dfrac{Ix(Mx,Sx) \quad \dfrac{\overline{Sa}^{(1)} \quad \dfrac{Ax(Mx,Px) \quad \overline{Ma}^{(1)} \quad \overline{Pa}^{(2)}}{Pa}{}^{(2)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{Sa:Sa \quad \dfrac{Ma:Ma \quad Pa:Pa}{Ax(Mx,Px),Ma:Pa}}{Ma,Sa,Ax(Mx,Px):Ix(Sx,Px)}}{Ix(Mx,Sx),Ax(Mx,Px):Ix(Sx,Px)}$$

(Bocardo)

$$\dfrac{Ox(Mx,Px) \quad \dfrac{\dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(1)} \quad \overline{Sa}^{(3)}}{Sa}{}^{(3)} \quad \dfrac{\overline{Pa}^{(2)}}{\bot}{}^{(1)}}{Ox(Sx,Px)}{}^{(2)}}{Ox(Sx,Px)}{}^{(1)}$$

$$\dfrac{\dfrac{\dfrac{Ma:Ma \quad Sa:Sa}{Ax(Mx,Sx),Ma:Sa} \quad Pa:Pa}{Ax(Mx,Sx),Ma:Ox(Sx,Px),Pa}}{Ox(Mx,Px),Ax(Mx,Sx):Ox(Sx,Px)}$$

(Ferison)

$$\dfrac{Ix(Mx,Sx) \quad \dfrac{\overline{Sa}^{(1)} \quad \dfrac{Ex(Px,Mx) \quad \overline{Pa}^{(2)} \quad \overline{Ma}^{(1)}}{\bot}}{Ox(Sx,Px)}\,(2)}{Ox(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{Sa : Sa \quad \dfrac{Pa : Pa \qquad Ma : Ma}{Ex(Px,Mx), Pa, Ma :}}{Ma, Sa, Ex(Px,Mx) : Ox(Sx,Px)}}{Ix(Mx,Sx), Ex(Px,Mx) : Ox(Sx,Px)}$$

5.4. *Binary-Quantifier Deductions for Syllogisms of the Fourth Figure*

(Bramantip)

$$\dfrac{Ix(Px,Px) \quad \dfrac{\dfrac{Ax(Px,Mx) \quad \overline{Pa}^{(1)} \quad \dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(2)} \quad \overline{Sa}^{(3)}}{Sa}\,(3)}{Sa}\,(2) \quad \overline{Pa}^{(1)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{\dfrac{Pa : Pa \quad \dfrac{Ma : Ma \qquad Sa : Sa}{Ma, Ax(Mx,Sx) : Sa}}{Pa, Ax(Px,Mx), Ax(Mx,Sx) : Sa} \quad Pa : Pa}{Pa, Ax(Px,Mx), Ax(Mx,Sx) : Ix(Sx,Px)}}{Ix(Px,Px), Ax(Px,Mx), Ax(Mx,Sx) : Ix(Sx,Px)}$$

(Camenes)

$$\dfrac{\dfrac{Ex(Mx,Sx) \quad \dfrac{Ax(Px,Mx) \quad \overline{Pa}^{(1)} \quad \overline{Ma}^{(2)}}{Ma}\,(2) \quad \overline{Sa}^{(1)}}{\bot}}{Ex(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{\dfrac{Pa : Pa \qquad Ma : Ma}{Ax(Px,Mx), Pa : Ma} \qquad Sa : Sa}{Ex(Mx,Sx), Ax(Px,Mx), Sa, Pa :}}{Ex(Mx,Sx), Ax(Px,Mx) : Ex(Sx,Px)}$$

(Dimaris)

$$\dfrac{Ix(Px,Mx) \quad \dfrac{\dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(1)} \quad \overline{Sa}^{(2)}}{Sa}\,(2) \quad \overline{Pa}^{(1)}}{Ix(Sx,Px)}}{Ix(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{\dfrac{Ma : Ma \qquad Sa : Sa}{Ma, Ax(Mx,Sx) : Sa} \qquad Pa : Pa}{Pa, Ma, Ax(Mx,Sx) : Ix(Sx,Px)}}{Ix(Px,Mx), Ax(Mx,Sx) : Ix(Sx,Px)}$$

(Calemos)

$$\dfrac{Ix(Sx,Sx) \quad \dfrac{\overline{Sa}^{(1)} \quad \dfrac{Ex(Mx,Sx) \quad \dfrac{Ax(Px,Mx) \quad \overline{Pa}^{(2)} \quad \overline{Ma}^{(3)}}{Ma}\,(3) \quad \overline{Sa}^{(1)}}{\bot}}{Ox(Sx,Px)}\,(2)}{Ox(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{Sa : Sa \quad \dfrac{\dfrac{Pa : Pa \qquad Ma : Ma}{Ax(Px,Mx), Pa : Ma} \qquad Sa : Sa}{Ax(Px,Mx), Ex(Mx,Sx), Pa, Sa :}}{Sa, Ax(Px,Mx), Ex(Mx,Sx) : Ox(Sx,Px)}}{Ix(Sx,Sx), Ax(Px,Mx), Ex(Mx,Sx) : Ox(Sx,Px)}$$

(Fesapo)

$$\dfrac{Ix(Mx,Mx) \quad \dfrac{\dfrac{Ax(Mx,Sx) \quad \overline{Ma}^{(1)} \quad \overline{Sa}^{(3)}}{Sa}\,(3) \quad \dfrac{Ex(Px,Mx) \quad \overline{Pa}^{(2)} \quad \overline{Ma}^{(1)}}{\bot}}{Ox(Sx,Px)}\,(2)}{Ox(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{\dfrac{Ma : Ma \qquad Sa : Sa}{Ax(Mx,Sx), Ma : Sa} \quad \dfrac{Pa : Pa \qquad Ma : Ma}{Ex(Px,Mx), Pa, Ma :}}{Ma, Ax(Mx,Sx), Ex(Px,Mx) : Ox(Sx,Px)}}{Ix(Mx,Mx), Ax(Mx,Sx), Ex(Px,Mx) : Ox(Sx,Px)}$$

(Fresison)

$$\dfrac{Ix(Mx,Sx) \quad \dfrac{\overline{Sa}^{(1)} \quad \dfrac{Ex(Px,Mx) \quad \overline{Pa}^{(2)} \quad \overline{Ma}^{(1)}}{\bot}}{Ox(Sx,Px)}\,(1)}{Ox(Sx,Px)}\,(1)$$

$$\dfrac{\dfrac{Sa : Sa \quad \dfrac{Pa : Pa \qquad Ma : Ma}{Ex(Px,Mx), Pa, Ma :}}{Ma, Sa, Ex(Px,Mx) : Ox(Sx,Px)}}{Ix(Mx,Sx), Ex(Px,Mx) : Ox(Sx,Px)}$$

Our binary rules afford proofs of a few more valid arguments (using Aristotelian forms) than just the syllogisms. For example, we can derive the sequent called (C1) in Corcoran's system (*loc. cit.*, p. 697) and called Rule 3 in Smiley's system (*loc. cit.*, p. 141), namely the sequent (in our notation)

$$Ex(Fx, Gx) : Ex(Gx, Fx).$$

Likewise, Corcoran's (C2), which is Smiley's Rule 4, can be derived, when it is supplied with the extra existential premise 'There is some F':

$$Ax(Fx, Gx), Ix(Fx, Fx) : Ix(Gx, Fx).$$

It will come as no surprise either that Corcoran's (C3):

$$Ix(Fx, Gx) : Ix(Gx, Fx)$$

can be derived. Finally, we can also derive

$$Ax(Fx, Gx), Fa : Ga,$$

thereby making formal provision for the hoary example 'All men are mortal; Socrates is a man; therefore, Socrates is mortal'.

We leave as an exercise for the reader the construction of sequent proofs of these four results, using the sequent rules furnished above for the binary quantifiers.

6. Features of Aristotle's syllogistic of special interest to the modern logician

6.1. *The grammar of Aristotelian forms is not generative*

The formal grammar for Aristotelian forms is very austere, as is the grammar for the forms involved in the language used by the proof system for the binary quantifiers. Recall that the rules of inference for binary quantifications involve not only major premises and conclusions of the form $Qx(Fx, Gx)$ ($Q = A, E, I$ or O), but also minor premises, or assumptions for the sake of argument, of the form Ft, Fa, etc. These latter additions are all that were needed in order to 'Gentzenize' Aristotle's syllogisms.

The grammar even for this slightly extended language allows one to generate, from a finite non-logical vocabulary of monadic predicates, only finitely many well-formed formulae. The language contains only one variable, x, and only one parameter, a. The rules of *Aristotelian grammar* are as follows.

(1) The variable x is a singular term, with itself as its only free occurrence of a variable.
(2) The parameter a is a singular term, with no free occurrences of variables.
(3) If P is a monadic predicate, and t is a singular term, then Pt is a formula whose free occurrences of variables are those of t.
(4) If φ, ψ are formulae with a free occurrence of x, then $Ax(\varphi, \psi)$, $Ex(\varphi, \psi)$, $Ix(\varphi, \psi)$, and $Ox(\varphi, \psi)$ are formulae whose free occurrences of variables are those of φ and ψ, save of x.
(5) [Closure clause] A thing is a singular term, or a formula, only if its being so follows from the preceding rules.

A *sentence* is a formula with no free ocurrences of variables.

Remark on terminology. I have taken care to say '*singular* term' in the foregoing definition even though the modern logician would simply say 'term'. This is because in the context of a discussion of Aristotle's syllogistic the term 'term' has come to mean *monadic predicate* rather than *singular term.*[11]

6.2. *Aristotelian* ecthesis *can be universal and can be existential*

It is remarkable that one needs *only one* parameter (here called a) in order to frame a complete set of deductive rules for the syllogistic. The parameter lends itself to construal as existential *or* universal, depending on its occurrences and non-occurrences within a sequent. *Lear 1980* (at p 4) advances the opinion that

> ... *ekthesis* is similar to the use of free variables in modern systems of natural deduction

and gives an example in which the particular instance 'corresponds to existential instantiation in natural deduction'. A case is made also by *Smith 1982* in favor of interpreting Aristotelian ecthesis as akin to the use of a parameter for existential elimination (instantiation). I submit, however, that there is a systematic deductive perspective – namely, that provided by our system of rules for the binary quantifiers – from which *all* uses of a parameter, be they for existential elimination or universal introduction, are cases of ecthesis; and that every case of ecthesis can be explicated by an appropriate use of a parameter. By instantiating with a parameter, one is setting out a deductive sub-problem to be solved, whose solution will vouchsafe a deduction for the overall problem, courtesy of a single application of a binary-quantifier rule. An interesting point that neither Lear nor Smith raise is that *only one* parameter is ever needed for Aristotle's syllogistic, even for *extended* syllogisms.

The need for more than one parameter will arise only when the formal language allows for sentences that have one quantifier-occurrence *within the scope of another.* But this does not happen in the formal 'Fregean' language that is expressively adequate for the syllogistic.

Lemma 6.1: *Every sentence allowed by the Aristotelian grammar has at most one quantifier-occurrence in it.*

Proof By inspection of the rules of the grammar, there are only two kinds of sentences: *atomic* sentences of the form Fa; and *quantified* sentences of the form $Qx(Fx, Gx)$, where both F and G are monadic predicates. □

The latter kind of sentences are the *Aristotelian forms.*

Observation 6.2 Aristotelian grammar does not permit iterative embeddings: no quantifier-occurrence can lie within the scope of another quantifier-occurrence.

This follows trivially from Lemma 6.1.

6.3. *Syllogisms are* perfectly *valid*

In this section and the next, two more demanding notions of semantic validity are introduced, and it will turn out that Aristotelian syllogisms are valid even in the more exigent of the two senses. A sustained case is about to be laid out for the claim that Aristotle's syllogistic is a fragment of Core Logic – and a very natural and instructive one at that.

Definition 6.3: An argument is *perfectly* valid just in case it is valid but every one of its proper sub-arguments is invalid.

Remark on terminology. This notion of perfect validity was defined and used in *Tennant 1984*, without any intended application to a study of Aristotle's syllogistic. It could easily have been called something like 'prime validity' instead, but the word 'perfect' was chosen back then, and it seems wise to stick with it, all things considered. The reader should be aware, however, that there

[11] I am indebted to an anonymous referee for stressing the need to make this clear.

is no intention here to suggest an analogy with Aristotle's notion of perfected proof or deduction. Any connection between Aristotle's proof-theoretic notion of perfect deduction and the above-defined semantic notion of perfect validity would have to be independently established.[12]

Lemma 6.4: *Given any Aristotelian form involving F and G (in either order), whatever non-empty extension is assigned to F, one can assign a non-empty extension to G so as to make that form true.*

Proof We need to consider the following forms. For each one we indicate how, in response to the assignment of an extension to F, an extension of G is to be chosen so as to make the form in question true.

$Ax(Fx, Gx)$		Let ext(G) = ext(F)
$Ax(Gx, Fx)$		Let ext(G) = ext(F)
$Ex(Fx, Gx)$, $Ex(Gx, Fx)$	equivalent	Let ext(G) be disjoint from ext(F)
$Ix(Fx, Gx)$, $Ix(Gx, Fx)$	equivalent	Let ext(G) overlap ext(F)
$Ox(Fx, Gx)$		Let ext(G) fail to include ext(F)
$Ox(Gx, Fx)$		Let ext(G) fail to be included in ext(F)

□

Lemma 6.5: *Given any Aristotelian form involving F and G (in either order), whatever non-empty extension is assigned to F, one can assign a non-empty extension to G so as to make that form false.*

Proof We now indicate how, in response to the assignment of an extension to F, an extension of G is to be chosen so as to make the form in question false.

$Ax(Fx, Gx)$		Let ext(G) fail to include ext(F)
$Ax(Gx, Fx)$		Let ext(G) fail to be included in ext(F)
$Ex(Fx, Gx)$, $Ex(Gx, Fx)$	equivalent	Let ext(G) overlap ext(F)
$Ix(Fx, Gx)$, $Ix(Gx, Fx)$	equivalent	Let ext(G) be disjoint from ext(F)
$Ox(Fx, Gx)$		Let ext(G) include ext(F)
$Ox(Gx, Fx)$		Let ext(G) be included in ext(F)

□

Corollary 6.6: *Let $Q\left[\genfrac{}{}{0pt}{}{F}{G}\right]$ be any of the above sentences using F and G as the two predicates. Ditto for $Q'\left[\genfrac{}{}{0pt}{}{G}{H}\right]$, using G and H as the two predicates. Then extensions can be fixed so as to make both $Q\left[\genfrac{}{}{0pt}{}{F}{G}\right]$ and $Q'\left[\genfrac{}{}{0pt}{}{G}{H}\right]$ true.*

Proof Fix any extension for F. By Lemma 6.4, fix an extension for G that makes $Q\left[\genfrac{}{}{0pt}{}{F}{G}\right]$ true. By Lemma 6.4 again, fix an extension for H that makes $Q'\left[\genfrac{}{}{0pt}{}{G}{H}\right]$ true. □

Corollary 6.7: *Let $Q_1\left[\genfrac{}{}{0pt}{}{F_1}{F_2}\right]$, $Q_2\left[\genfrac{}{}{0pt}{}{F_2}{F_3}\right]$, $\ldots$, $Q_{n-1}\left[\genfrac{}{}{0pt}{}{F_{n-1}}{F_n}\right]$, $Q_n\left[\genfrac{}{}{0pt}{}{F_n}{F_{n+1}}\right]$ be a chain $(n > 0)$. Then extensions can be fixed for $F_1, \ldots, F_{n+1}$ so as to make every member of that chain true.*

Proof By n applications of Corollary 6.6, the chain condition ensuring applicability. □

[12] I am indebted to an anonymous referee for stressing the need to make this clear.

Corollary 6.8: *Let* $Q\Big[{F \atop G}\Big]$ *and* $Q'\Big[{G \atop H}\Big]$ *be as in Corollary 6.6. Then extensions can be fixed so as to make both* $Q\Big[{F \atop G}\Big]$ *and* $Q'\Big[{G \atop H}\Big]$ *false.*

Proof Fix any extension for F. By Lemma 6.5, fix an extension for G that makes $Q\Big[{F \atop G}\Big]$ false. By Lemma 6.5 again, fix an extension for H that makes $Q'\Big[{G \atop H}\Big]$ false. □

Corollary 6.9: *The single-premise argument that results by dropping any one premise from a syllogism is invalid.*

Proof Such a single-premise argument will have the form '$Q\Big[{F \atop G}\Big] : Q'\Big[{G \atop H}\Big]$', where $Q\Big[{F \atop G}\Big]$ and $Q'\Big[{G \atop H}\Big]$ are as in Corollary 6.6.

Fix any extension for F. By Lemma 6.4, fix an extension for G that makes $Q\Big[{F \atop G}\Big]$ true. By Lemma 6.5, fix an extension for H that makes $Q'\Big[{G \atop H}\Big]$ false. □

Corollary 6.10: *The argument '*$P, Q : \bot$*' that results by dropping the conclusion of a syllogism '*$P, Q : R$*' is invalid.*

Proof Immediate from Corollary 6.6. □

Theorem 6.11: *All syllogisms are perfectly valid.*

Proof Let '$P, Q : R$' be any syllogism. By Corollary 6.9, both '$P : R$' and '$Q : R$' are invalid. By Corollary 6.10, '$P, Q : \bot$' is invalid. So no proper sub-argument of '$P, Q : R$' is valid. □

6.4. *Syllogisms are* skeletally *valid*

Definition 6.12: A *substitution* is a mapping from predicate letters to predicate letters. A substitution induces in the natural way a mapping from sentences to sentences. Every substitution σ is of course *uniform* – that is, it replaces each predicate letter F at *all* its occurrences (within a given sentence or sequent) by an occurrence of the substituend σF. A *substitution on a sentence or on a sequent* is one whose domain contains all the predicate letters involved therein.

All subsequent talk of substitutions will be with implicit or explicit reference to a sequent that we have in mind. Thus one should think of the substitutions as restricted so as to deal only with predicate letters occurring in sentences within the sequent in question.

Definition 6.13: A substitution that maps each predicate letter to a predicate letter is called a *re-lettering.* If the mapping is also one-one, it is called a *permutation.* Thus a re-lettering that maps at least two distinct predicate letters to some same predicate letter is a *non-permutative re-lettering.*

A non-permutative re-lettering can induce important extra logical structure, because of repetitions of predicate letters that it can allow to occur. Consider, for example, the non-permutative re-lettering

$$\begin{matrix} F & G \\ \downarrow & \downarrow \\ F & F \end{matrix}$$

Applied to the invalid sequent $\emptyset : Ax(Fx, Gx)$, it produces the (perfectly) valid sequent $\emptyset : Ax(Fx, Fx)$. The latter, valid, sequent has a repetition of F that the former, invalid, sequent does not. Logical structure is not only a matter of patterns of placement of logical operators within sentences, but also a matter of patterns of *repetitions of predicate letters.* Indeed, it is easy to show (by induction on the complexity of finite sequents) that any finite valid sequent in the Aristotelian language involves a repetition of at least one predicate letter.

Definition 6.14: A substitution σ is *one-one* on a sequent if and only if the induced mapping $\theta \mapsto \sigma\theta$ is 1-1 on the sentences, i.e. the premises and conclusion(s), of that sequent.

Definition 6.15: A substitution σ is a *proper* substitution on a sequent in the Aristotelian language if and only if on that sequent, σ is one-one, and σ is a non-permutative re-lettering.

Definition 6.16: If a substitution σ is proper on $\Delta : \Phi$, then the resulting sequent $\sigma\Delta : \sigma\Phi$ is said to be a *proper substitution instance*, or *refinement*, of $\Delta : \Phi$, and $\Delta : \Phi$ is said to be a *proper suprasequent*, or *coarsening*, of $\sigma\Delta : \sigma\Phi$.

A proper substitution on $\Delta : \Phi$ can strictly increase its logical structure, by identifying formerly distinct predicate letters, thereby increasing 'repetition' of predicate letters within the sequent; but it does so without merging any distinct sentences in $\Delta : \Phi$ into a single sentence in the resulting sequent.

Example 6.17 The proper substitution

$$\begin{array}{ccc} F & G & H \\ \downarrow & \downarrow & \downarrow \\ F & G & F \end{array}$$

is one-one on the sequent

$$Q_1x(Fx, Gx), Q_2x(Gx, Hx) : Q_3x(Fx, Hx),$$

turning it into

$$Q_1x(Fx, Gx), Q_2x(Gx, Fx) : Q_3x(Fx, Fx),$$

without identifying any two distinct sentences.

By contrast, the substitution

$$\begin{array}{cccc} F & G & H & D \\ \downarrow & \downarrow & \downarrow & \downarrow \\ F & G & G & F \end{array},$$

which is a non-permutative re-lettering, fails to be one-one on the (invalid) sequent

$$Ix(Fx, Hx), Ix(Dx, Gx) : Ix(Gx, Fx),$$

because it turns it into the (valid) sequent

$$Ix(Fx, Gx) : Ix(Gx, Fx),$$

in which the formerly distinct premises have been merged into one.

Definition 6.18: An *entailment* is a substitution instance of a perfectly valid sequent.[13]

Definition 6.19: An argument is *skeletally* valid just in case it is perfectly valid and is not a proper substitution instance of any perfectly valid argument.

Theorem 6.20: *All syllogisms are skeletally valid.*

[13]This concept was introduced in *Tennant 1984*.

Proof Consider any syllogism Σ:

$$\frac{\begin{array}{c} Q_1\begin{bmatrix} P \\ M \end{bmatrix} \\ Q_2\begin{bmatrix} S \\ M \end{bmatrix} \end{array}}{Q_3\begin{bmatrix} S \\ P \end{bmatrix}}$$

By Theorem 6.11, Σ is perfectly valid. By inspection, any argument of which Σ is a proper substitution instance has the form

$$\frac{\begin{array}{c} Q_1\begin{bmatrix} P' \\ M_1 \end{bmatrix} \\ Q_2\begin{bmatrix} S' \\ M_2 \end{bmatrix} \end{array}}{Q_3\begin{bmatrix} S' \\ P' \end{bmatrix}}$$

Fix an extension for P'.

By Lemma 6.5, fix an extension for S' so as to make $Q_3\begin{bmatrix} S' \\ P' \end{bmatrix}$ false.

By Lemma 6.4, fix an extension for M_1 so as to make $Q_1\begin{bmatrix} P' \\ M_1 \end{bmatrix}$ true.

By Lemma 6.4, fix an extension for M_2 so as to make $Q_2\begin{bmatrix} S' \\ M_2 \end{bmatrix}$ true. □

6.5. *Forms of sequent-proofs of syllogisms*

The reader will have noted that the terminal step of every 3-step sequent proof of an Aristotelian syllogism is a *monofurcation*, and the two earlier steps are *bifurcations*. With 4-step sequent proofs (the ones for arguments requiring an extra existential premise), the last two steps are monofurcations, and the two earlier ones are bifurcations.

Our proofs of the syllogisms are all *core proofs*, in the sense of *Tennant of 2012*. They contain

- no 'vacuous discharges' of assumptions;
- no cuts;
- no dilutions (thinnings);
- no applications of strictly classical negation rules.

This last feature distinguishes my formal approach from those of both Corcoran and Smiley, which were explained at the outset. I avoid altogether the rule of *reductio ad impossibile*, which is conspicuously non-constructive.[14]

The proof-theoretic perspective offered here affords also fresh confirmation of an insight of Robin Smith. As I would be inclined to put it, Aristotle *could* have been the very first constructivist, in so far as he was concerned to be able to prove all his syllogisms; but, alas, he presupposed more than he needed to.[15]

Our proof-theoretic perspective affords another insight: Aristotle *was* indeed the very first relevantist, in so far as he was concerned to be able to prove all his syllogisms. But, alas, there was no irrelevantist orthodoxy for him to rebel against. He was laying the groundwork for such

[14] *Corcoran 2009* at p. 13, argues that 'in Aristotle's categorical syllogistic ... indirect deductions are indispensable'. This entails that the system of rules that Aristotle uses in order to perfect or complete deductions of arguments with more than two premises is inherently non-constructivizable. Thus the system of rules for the binary quantifiers opens up a possibility that was closed to Aristotle, *without* resorting to any essential extension of Aristotle's formal language.

[15] *Smith 1983* provides an 'ecthetic system' and shows that

> ... if ecthetic rules are added to a model for the syllogistic, indirect deductions may be dispensed with. That is ... for any consistent set S of categorical propositions and any proposition p, if S implies p then p can be deduced from S by a direct deduction in the ecthetic system. It is impossible to say whether Aristotle realized this or not ... (*loc. cit.*, p. 225)

Our system of rules for the binary quantifiers is of course ecthetic also, in the sense that certain of those rules allow (indeed, call for) the use of a (single) parameter a.

an orthodoxy eventually to arise, which it did by neglecting this signal feature (relevance) of his syllogistic – a feature now respected, and restored, by Core Logic.

6.6. *Aristotle's chain principle re-visited*

Our system of binary-quantifier rules, applied according to the foregoing constraints, affords a deeper understanding of why syllogisms can be 'strung together', with the conclusion of one being a premise of the next, so as to afford a licit deduction of the final conclusion from the ultimate premises. For the proof in *Tennant of 2012* of cut-elimination for Core Logic (for a standard first-order language) is easily adapted to our system for the binary quantifiers.

We shall say that the proof Π *connects with* the proof Σ just in case the conclusion (call it φ) of Π is a premise (i.e., an undischarged assumption) of Σ. So the two proofs may be rendered graphically as

$$\begin{array}{c} \Delta \\ \Pi \\ \varphi \end{array} \quad \text{and} \quad \begin{array}{c} \varphi, \Gamma \\ \Sigma \\ \theta \end{array} .$$

Here, φ is called the *cut sentence*. When φ is compound, we shall refer to its dominant operator as α. By virtue of φ's being displayed separately in the premises of the proof Σ, it is to be assumed that $\varphi \notin \Gamma$.

The *target sequent* is $\Delta, \Gamma : \theta$. Here, it is denoted by means of the premise-sets of the two proofs Π and Σ, and the conclusion of Σ.

We have furnished above, for every Aristotelian syllogism, a syllogistic deduction enjoying the form of a cut-free, thinning-free proof using only the rules for the binary quantifiers. By an *extended syllogistic deduction* we shall mean any cut-free, thinning-free proof that can be formed in accordance with those rules – not just those with two premises sharing only a middle term.

If Π and Σ are extended syllogistic deductions, and Π connects with Σ, then we can inquire after the result 'established' by placing a copy of Π over each premise-occurrence of φ within Σ. Call this construct

$$\begin{array}{c} \Pi \\ (\varphi) \\ \Sigma \end{array} .$$

Adding a little more detail:

$$\begin{array}{c} \Delta \\ \Pi \\ (\varphi)\ , \Gamma \\ \Sigma \\ \theta \end{array}$$

This construct might not count as an extended syllogistic deduction. This is because φ might be a major premise for a step of α-elimination at one of its premise-occurrences in Σ. Such a 'cut' would have to be eliminated, in order to turn the construct in question into a legitimate cut-free, thinning-free proof. The cut-elimination theorem tells us that this can be done. It says that there will be a reduct $[\Pi, \Sigma]$ whose premises form a subset (call it Θ) of $\Delta \cup \Gamma$, and whose conclusion is either θ or $\perp$.

This latter possibility – that the overall conclusion is $\perp$ – is, however, ruled out in the case where the members of Θ 'form a chain of predications linking the terms of' θ. For by satisfying that chain condition, the members of Θ are jointly satisfiable, by Corollary 6.7. Hence we know that the chain condition ensures that the reduct $[\Pi, \Sigma]$ is an extended syllogistic deduction that establishes the target sequent $\Theta : \theta$.

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