4-3-2-8-7-6

Dan Freed

Joint work (some in progress) with Constantin Teleman

Builds on work with Mike Hopkins and Jacob Lurie

Dedicated to Graeme, whose influence is evident throughout

The definition of conformal field theory

Graeme Segal

I shall propose a definition of 2-dimensional conformal field theory which I believe is equivalent to that used by Friedan et al. The idea arises from joint work with Quillen.

§1 The definition

The category $\mathcal{C}$ is defined as follows. There is a sequence of objects $\{C_n\}_{n \geq 0}$, where C_n is the disjoint union of a set of n

Great theorems are awesome; great definitions are transformational.

The Definition of Quantum Field Theory

Definition (Segal): A 2d conformal field theory is a homomorphism

$$F\colon \mathrm{Bord}^{\mathrm{conf}}_{\langle 1,2\rangle} \longrightarrow \mathbf{Vect}^{\mathrm{top}}_{\mathbb{C}}$$

Definition (Atiyah): An n-dimensional topological quantum field theory is a homomorphism

$$F\colon \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} \longrightarrow \mathbf{Vect}$$

$\mathcal{X}(n)$: an n-dimensional (topological) tangential structure

Remark: $\mathrm{Bord}^{SO}_{\langle n-1,n\rangle}$ categorifies the classical bordism group Ω^{SO}_{n-1}
TQFT categorifies classical bordism invariants, e.g.

$$\mathrm{Sign}\colon \Omega^{SO}_{4k} \to \mathbb{Z}$$

Invertible Field Theories

$\alpha\colon \mathrm{Bord}_{\langle n-1,n\rangle} \to \mathbf{Vect}$ is invertible if $\alpha(M)$ is invertible for every M

$\alpha(X) \in \mathbb{C}^{\times}$ for X^n closed
$\alpha(Y)$ is a line for Y^{n-1} closed

Example: There is a 3-dimensional TQFT $\alpha\colon \mathrm{Bord}^{\mathrm{framed}}_{\langle 2,3\rangle} \to \mathbf{Vect}$ such that $\alpha(X) = \mu^{\Theta(X)}$, where $\mu = e^{2\pi i/24}$ and $\Theta(X) \in \Omega_3^{\mathrm{framed}} \cong \mathbb{Z}/24\mathbb{Z}$

Replace codomain with $\mathcal{C}$, a symmetric monoidal category
$y \in \mathcal{C}$ is invertible if there exist $y' \in \mathcal{C}$ and $(y \otimes y' \xrightarrow{\cong} 1_{\mathcal{C}}) \in \mathcal{C}$

Remark: Topological invertible theories factor to a map of spectra

$$\begin{array}{ccc} \mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle} & \xrightarrow{\alpha} & C \\ \downarrow & & \uparrow \\ |\mathrm{Bord}^{\mathcal{X}(n)}_{\langle n-1,n\rangle}| & \xrightarrow{\tilde{\alpha}} & C^{\times} \end{array}$$

Extended field theories

Early '90s in connection with topological Chern-Simons theory. Also of interest in non-topological theories, e.g. 2d conformal theories.

Domain: n-category Bord_n of bordisms
n composition laws on n-manifolds with corners

Codomain: Arbitrary n-category (or (∞, n)-category)

Example: $\mathcal{C} = \mathrm{Alg}$ 2-category of algebras, bimodules, intertwiners
$\mathrm{Alg} \longrightarrow \mathbf{Cat}$ maps $A \longmapsto {}_A\mathrm{Mod}$
+(category number) via algebra structure: $\mathrm{Alg} = E_1(\mathbf{Vect})$
A invertible $\iff$ A central simple

Example: G finite group. $A = \mathrm{Map}(G, \mathbb{C})$ under convolution.

$$\alpha\colon \mathrm{Bord}_2 \longrightarrow \mathrm{Alg}$$

with $\alpha(\mathrm{pt}) = A$. Then $\alpha(S^1) =$ class functions. N.B.: α is not invertible.
This is an extended version of 2d Dijkgraaf-Witten theory.

The Cobordism Hypothesis

Applies to fully extended *topological* theories

Bord_n^{SO}: (∞, n)-category of oriented bordisms
$\mathcal{C}$: symmetric monoidal (∞, n)-category

$$F\colon\ \mathrm{Bord}_n^{SO} \longrightarrow \mathcal{C}$$

Cobordism hypothesis (Baez-Dolan-Hopkins-Lurie):
F is determined by $F(\mathrm{pt}_+)$. Furthermore, any n-dualizable, SO_n-invariant object $c \in \mathcal{C}$ determines a theory F with $F(\mathrm{pt}_+) = c$.

n-dualizability: data attached to Morse handles exists
SO_n-invariance: extra data on c

Example: $n = 2$, $\mathcal{C} = \mathrm{Alg}_k$ the Morita 2-category of algebras
$A \in \mathcal{C}$ is 2-dualizable if it is finite dimensional semisimple
SO_2-invariance data: Frobenius structure (nondegenerate trace)

Criteria for Invertibility

Theorem: Suppose $\alpha\colon \mathrm{Bord}_n^{SO} \to \mathcal{C}$. Then if either

1. $\alpha(S^k)$ is invertible for some $k \leqslant n/2$; or
2. $\alpha(S^n)$ is invertible and $\alpha(S^p \times S^{n-1-p})$ is invertible for all p,

then α is invertible.

This is a kind of localization theorem for Bord_n^{SO}: e.g., 1 says if we invert S^k then we invert every bordism.

Example: $n = 2$, $k = 1$, $\mathcal{C} = \mathrm{Alg}_k$. If A is a 2-dualizable (finite dimensional, semisimple) Frobenius algebra, then it defines $\alpha\colon \mathrm{Bord}_2^{SO} \to \mathrm{Alg}_k$ with $\alpha(S^1)$ equal to the center of A. So α is invertible if the center of A is k.

Proof Sketch

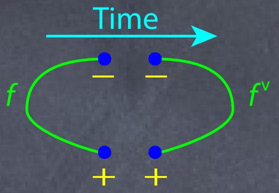
First, by the cobordism hypothesis (easy part) it suffices to prove that $\alpha(\mathrm{pt}_+)$ is invertible; '+' denotes the orientation. We omit 'α' and simply say 'pt_+ is invertible'.

We aim to prove that the 0-manifolds pt_+ and pt_- are inverse:

$$S^0 = \mathrm{pt}_+ \amalg \mathrm{pt}_- = \mathrm{pt}_+ \otimes \mathrm{pt}_- \cong \varnothing^0 = 1$$

with inverse isomorphisms given by

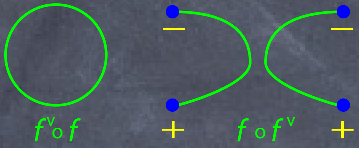
$$f = D^1 : 1 \longrightarrow S^0$$
$$f^\vee = D^1 : S^0 \longrightarrow 1$$



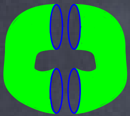
We are reduced to a statement about 1d bordisms: the compositions

$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$

must be proved to be identity.



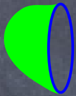
Let's now consider $n = 2$ where we assume that S^1 is invertible. We apply an easy algebraic lemma which asserts that invertible objects are dualizable and the dualization data is invertible. For S^1 these data are dual cylinders, and so the composition $S^1 \times S^1$ is also invertible.



Lemma: Suppose $\mathcal{D}$ is a symmetric monoidal category, $x \in \mathcal{D}$ is invertible, and $g\colon 1 \to x$ and $h\colon x \to 1$ satisfy $h \circ g = \mathrm{id}_1$. Then $g \circ h = \mathrm{id}_x$ and so each of g, h is an isomorphism.

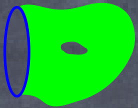
Proof: x^{-1} is a dual of x, $\quad g^\vee = x^{-1}g\colon x^{-1} \to 1$, $h^\vee = x^{-1}h\colon 1 \to x^{-1}$, $\quad$ so the lemma follows from $(h \circ g)^\vee = \mathrm{id}_1$.

Apply the lemma to the 2-morphisms



$$g = D^2\colon 1 \longrightarrow S^1$$

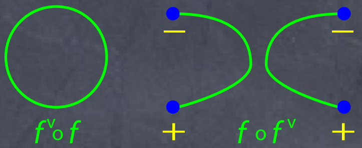
$$h = S^1 \times S^1 \backslash D^2\colon S^1 \longrightarrow 1$$



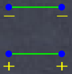
Conclude that $S^1 \cong 1$ and $S^2 = g^\vee \circ g$ is invertible.
Also, $g \circ g^\vee = \mathrm{id}_{S^1} \otimes S^2$, a simple surgery.

Recall that we must prove that the compositions

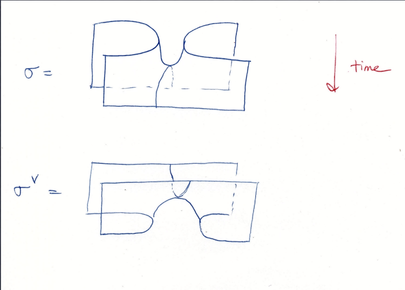
$$f^\vee \circ f = S^1 : 1 \longrightarrow 1$$
$$f \circ f^\vee \qquad : S^0 \longrightarrow S^0$$



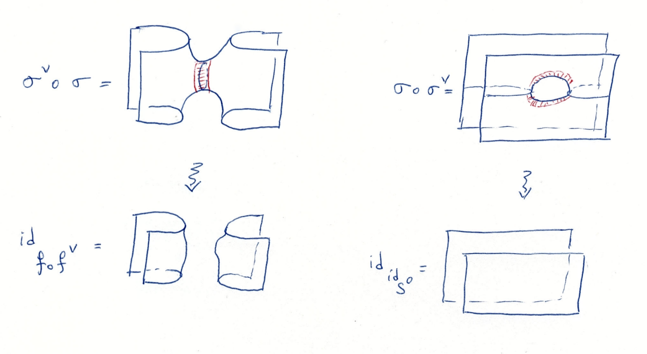
are the identity. We just did the first.

For the second, $\mathrm{id}_{S^0} =$ 

and we will show that the saddle $\sigma \colon f \circ f^\vee \to \mathrm{id}_{S^0}$ is an isomorphism with inverse $\sigma^\vee \otimes S^2$.



The saddle σ is diffeomorphic to $D^1 \times D^1$, which is a manifold with corners. Its dual $\sigma^\vee$ is the time-reversed bordism.



Inside each composition $\sigma^\vee \circ \sigma$ and $\sigma \circ \sigma^\vee$ we find a cylinder $\mathrm{id}_{S^1} = D^1 \times S^1$, which is $(S^2)^{-1} \otimes g \circ g^\vee = (S^2)^{-1} \otimes (S^0 \times D^2)$ by a previous argument. Making the replacement we get the desired isomorphisms to identity maps.

This completes the proof of the theorem in $n = 2$ dimensions.

In higher dimensions a new ingredient—a dimensional reduction argument—also appears. This uses the Cartesian product on bordisms.

4: An Invertible Topological Theory

Let A be a braided tensor category with braiding $\beta_{x,y} \colon x \otimes y \to y \otimes x$

Modular tensor category (quantum group): finiteness and nondegeneracy
Müger and others prove nondegeneracy $\iff$

$$\begin{aligned} \{x \in A : \beta_{y,x} \circ \beta_{x,y} = \mathrm{id}_{x \otimes y} \text{ for all } y \in A\} &= \{\text{multiples of } 1 \in A\} \\ &= \mathbf{Vect} \end{aligned} \qquad (*)$$

Braided tensor categories form the objects of a 4-category:

object	category #
element of $\mathbb{C}$	-1
$\mathbb{C}$-vector space	0
$\mathbf{Vect}_{\mathbb{C}}$	1
$\mathbf{Cat}_{\mathbb{C}}$	2
$\mathbf{Cat}^{\otimes}_{\mathbb{C}} = \mathbf{E_1}(\mathbf{Cat}_{\mathbb{C}})$	3
$\mathbf{Cat}^{\beta\otimes}_{\mathbb{C}} = \mathbf{E_2}(\mathbf{Cat}_{\mathbb{C}})$	4

MTC A: 4-dualizable and SO_4-invt
$\alpha_A \colon \mathrm{Bord}_4^{SO} \to \mathbf{Cat}^{\beta\otimes}_{\mathbb{C}}$ (cob. hyp.)
$\alpha_A(\mathrm{pt}_+) = A$
$(*) \implies \alpha_A(S^2) = \mathbf{Vect}$ invertible
Thm $\implies \alpha_A$ invertible
Crane-Yetter theory
$\alpha_A(X^4) = \mu^{\mathrm{Sign}(X)}, \; \mu = \mu(A) \in \mathbb{C}^\times$

Corollary: A modular tensor category $A \in \mathbf{Cat}^{\beta\otimes}_{\mathbb{C}}$ is invertible.

Relative Quantum Field Theory

Definition: Fix an integer $n \geqslant 0$ and let α be an extended $(n+1)$-dimensional quantum field theory. A quantum field theory F relative to α is a homomorphism

$$F\colon \mathbf{1} \longrightarrow \tau_{\leqslant n}\alpha$$

or

$$\widetilde{F}\colon \tau_{\leqslant n}\alpha \longrightarrow \mathbf{1}$$

Truncation: Restrict domain(α) along $\mathrm{Bord}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Bord}_{\langle n-1,n,n+1\rangle}$

$$F(X^n) \in \alpha(X^n) \qquad \text{OR} \qquad \widetilde{F}(X^n;\xi) \in \mathbb{C}, \qquad \xi \in \alpha(X^n)$$

$$\widetilde{F}(Y^{n-1};\mu) \in \mathbf{Vect}_{\mathbb{C}}^{\mathrm{top}}, \qquad \mu \in \alpha(Y^{n-1})$$

Representation:

$$\begin{aligned} \alpha(Y^{n-1}) &\longrightarrow \mathbf{Vect}_{\mathbb{C}}^{\mathrm{top}} \\ \mu &\longmapsto \widetilde{F}(Y^{n-1};\mu) \end{aligned} \qquad \alpha(Y^{n-1}) \in \mathbf{Cat}_{\mathbb{C}}^{\mathrm{top}}$$

Examples of Relative Theories

Definition: If α is invertible we say F is anomalous with anomaly α

$F(X^n) \in \alpha(X^n)$ takes values in a line (Lagrangian anomaly)
$F(Y^{n-1}) \in \alpha(Y^{n-1})$ takes values in a gerbe (Hamiltonian anomaly)

In $n = 2$ conformal field theory $\alpha(X^2) =$ determinant line of Riemann surface X. Dual picture: $\widetilde{F}(X; \xi) \in \mathbb{C}$ for $\xi \in \alpha(X)$.

Example: $n = 1$, $\mathcal{C} = \mathrm{Alg}_k$, $\alpha(\mathrm{pt}) = A = \mathrm{Map}(G, \mathbb{C})$

V	finite dimensional representation of G
$F(\mathrm{pt})\colon \mathbf{1} \to A$	V as a left A-module
$F(S^1) \in \alpha(S^1)$	character of V

Canonical choice: $V = A$ with left action (regular representation)

The 2d theory α only depends on the Morita class of A, but the relative theory depends on A itself.

3-4: Chern-Simons as a Relative Theory

$A \in \mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$	modular tensor category
α_A	4d invertible theory with values in $\mathbf{Cat}_{\mathbb{C}}^{\beta\otimes} = \mathbf{E_2}(\mathbf{Cat}_{\mathbb{C}})$
F_A	3d relative theory: A as left A-module

$\alpha_A\colon \mathrm{Bord}_4^{SO} \to \mathbf{Cat}_{\mathbb{C}}^{\beta\otimes}$ Crane- Yetter theory discussed earlier

$F_A\colon \mathbf{1} \to \alpha_A$ Chern-Simons theory as a relative oriented theory

Remarks: F_A is defined on the 3d oriented bordism category Bord_3^{SO}
α_A is the usual framing anomaly of Chern-Simons
Finiteness properties of A prove that F_A exists
Reshetikhin-Turaev recast in terms of cobordism hypothesis
Walker has a similar picture

Problem: To proceed to chiral Wess-Zumino-Witten as a 2-3 theory we would like

$$\alpha_A \xrightarrow{\cong} \mathbf{1}$$

so that $\chi\colon \mathbf{1} \xrightarrow{F_A} \alpha_A \xrightarrow{\cong} \mathbf{1}$ is an absolute 3d theory

Trivializing the Anomaly

Suppose X is a closed oriented 3-manifold, and we write $X = \partial W$ for a compact oriented 4-manifold W with boundary.
The composition

$$\mathbf{1}(X) = \mathbf{1} \xrightarrow{F_A(X)} \alpha_A(X) \xrightarrow{\alpha_A(W)} \alpha_A(\varnothing^3) = \mathbf{1}$$

is multiplication by a number in $\mathbb{C}$

$W \rightsquigarrow W'$ multiplies this by $\lambda^{2\pi icn/8}$, where $n = \mathrm{Sign}(W' \cup_X W)$
$c =$ central charge

Signature structure (σ) makes sense on 1-, 2-, 3-, and 4-manifolds, and every (w_1, σ)-manifold of these dimensions bounds a (w_1, σ)-manifold. Therefore, we recover the Reshetikhin-Turaev 1-2-3-theory, defined on bordisms with a signature structure.

3: Chern-Simons as an Absolute Theory

A tangential structure using p_1 (Blanchet-Habegger-Masbaum-Vogel). It is, in fact, a stable tangential structure. If M is an oriented bordism, a p_1-structure is a lift of a classifying map of TM:

$$\begin{array}{ccccc} & & BO\langle w_1, p_1\rangle & & \\ & \nearrow & \downarrow & & \\ M & \xrightarrow{TM} & BO\langle w_1\rangle & \xrightarrow{p_1} & K(\mathbb{Z},4) \end{array}$$

(w_1, p_1)-bordism groups:

$$\Omega^{(w_1,p_1)}_{\{0,1,2,3,4\}} \cong \{\mathbb{Z}\ ,\ 0\ ,\ 0\ ,\ \mathbb{Z}/3\mathbb{Z}\ ,\ 0\}$$

To define an absolute theory χ on (w_1, p_1)-bordisms we:

(i) choose a cube root of $e^{2\pi i c/8}$

(ii) formally extend the theory to pt_+ and pt_- (not described here)

2-3: Chiral WZW as a Relative Theory

$\alpha_{CA}\colon \mathrm{Bord}_{\langle 1,2,3\rangle}^{(w_1,p_1,\mathrm{conf})} \longrightarrow \mathbf{Cat}_{\mathbb{C}}$ conformal anomaly (invertible)

$\chi\colon \mathrm{Bord}_{\langle 1,2,3\rangle}^{(w_1,p_1)} \longrightarrow \mathbf{Cat}_{\mathbb{C}}$ Chern-Simons (topological)

Chiral WZW is a relative 2d theory of maps $\mathrm{Bord}_2^{(w_1,p_1,\mathrm{conf})} \longrightarrow \mathbf{Cat}_{\mathbb{C}}^{\mathrm{top}}$:

$$F\colon \chi \longrightarrow \alpha_{CA}$$

This is, of course, a restatement of Segal's weakly conformal CFT.

Increasing specificity of modular tensor category A:

4: A up to E_2 Morita equivalence
3-4: A up to E_1 Morita equivalence
2-3: A with projective representation $A \longrightarrow \mathbf{Vect}_{\mathbb{C}}^{\mathrm{top}}$
(projective loop group representations)

4-3-2 for Tori

Let Π be a lattice and $G = T = \Pi \otimes \mathbb{R}/\mathbb{Z}$ the associated torus. A class $\lambda \in H^4(BT;\mathbb{Z})$ is an even symmetric bihomomorphism

$$b\colon \Pi \times \Pi \to \mathbb{Z},$$

assumed nondegenerate over $\mathbb{Q}$.

The kernel of the induced homomorphism $T \to T^*$ is a Pontrjagin self-dual finite abelian group π, with a quadratic function

$$q\colon \pi \longrightarrow \mathbb{Q}/\mathbb{Z}$$

(π, q) figures in the twisted equivariant K-theory (FHT) as Verlinde conjugacy classes, in special modular tensor categories (Frölich-Kerler, Quinn, ...), and in toral Chern-Simons (Belov-Moore, FHLT, ...)

4: Finite Path Integral

Eilenberg-MacLane: (π, q) determines a homotopy class of maps

$$q\colon K(\pi,2) \longrightarrow K(\mathbb{Q}/\mathbb{Z},4)$$

X^4 closed oriented, get $q_X : H^2(X;\pi) \longrightarrow \mathbb{Q}/\mathbb{Z}$. Path integral over π-gerbes reduces to a finite Gauss sum:

$$\begin{aligned}\alpha(X) &= \sum_{\mathcal{G}} \frac{\#H^0(X;F)}{\#H^1(X;F)}\, e^{2\pi i q_X(\mathcal{G})} \\ &= \frac{\#H^0(X;F)}{\#H^1(X;F)} \sqrt{\#H^2(X;F)}\ \exp\Big[2\pi i(\operatorname{sign} b)(\operatorname{Sign} X)/8\Big] \\ &= (\sqrt{\#F})^{\text{Euler}\,X}\, \mu^{(\operatorname{sign} b)(\operatorname{Sign} X)} \qquad\qquad (\mu = e^{2\pi i/8})\end{aligned}$$

Finite path integral gives fully extended 4d theory α (F, FHLT)

For example, $\alpha(Y^3)$ is the vector space of invariant functions $H^2(Y;\pi) \to \mathbb{C}$. An automorphism a of a π-gerbe $\mathcal{G} \to Y$ acts on $\mathbb{C}$ by $\exp\big(2\pi i \bar{b}(\mathcal{G}, a)\big)$, where

$$\bar{b}\colon H^2(Y;\pi) \times H^1(Y;\pi) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Since $\bar{b}$ is nondegenerate, invariant functions vanish away from trivial $\mathcal{G}$

It follows from 2 of theorem that α is invertible

Finite path integral takes values in "3-algebras", so will construct relative 3d theory using canonical A as left A-module

Chiral WZW for torus is essentially free, but with topological features, so potential construction using Heisenberg groups built from differential cohomology...

From 4-3-2 to 8-7-6

Arguments from string theory (Witten, Strominger) predict the existence of a 6d superconformal field theory, in some ways analogous to chiral WZW.

We call it Theory $\mathcal{X}$. (If nothing else, '$\mathcal{X}$' is for 'si$\mathcal{X}$'.)

Witten spoke about it at Segal 60.

It is a relative theory. We sketch ideas to approach 8 and 7-8. These are culled from physics, especially as explained to us by Greg Moore.

Construction of 6-7 in general will require new ideas.

8-7 and a Bit of 6

The same data (π, q) should give 8 and 7-8.

Starting point is homotopy class of maps

$$q\colon K(\pi, 4) \longrightarrow K(\mathbb{Q}/\mathbb{Z}, 8)$$

Analogous story (finite path integral, canonical module, ...) should exist, though many points yet to be understood. Perhaps a construction of non-interacting 6d Theory $\mathcal{X}$, analogous to 2d chiral WZW for tori.

Frenkel-Kac-Segal construction: Level 1 2d chiral WZW for ADE Lie groups is chiral WZW for maximal torus.

False analogy: This is *not* expected for the 6d Theory $\mathcal{X}$: there are interacting theories based on ADE Lie algebras

6-7: Theory $\mathcal{X}$

These are the expectations from physics arguments.

Data: A triple $(\mathfrak{g}, b, \Gamma)$ consisting of

- a real Lie algebra $\mathfrak{g}$ with an invariant inner product b such that all coroots have square length 2
- a full lattice $\Gamma \supset \Gamma'$ extending the coroot lattice Γ' such that the inner product is integral and even on Γ

Special cases: (1) ADE Lie algebra $\mathfrak{g}$ (2) (Π, b) with $\mathfrak{g} = \Pi \otimes \mathbb{R}$

As before, extract (π, q) from this data.

Expectations: (i) 7d topological QFT $\alpha_{\mathfrak{g}} = \alpha_{(\mathfrak{g},b,\Gamma)} = \alpha_{(\pi,q)}$

(ii) 6d superconformal QFT $\mathcal{X}_{\mathfrak{g}} = \mathcal{X}_{(\mathfrak{g},b,\Gamma)}$ relative to $\alpha_{\mathfrak{g}}$

The New Mother Lode

$F + *F = 0$	Theory $\mathcal{X}$
4d	6d
classical PDE	quantum field theory
Riemannian self-duality (dimensions $4, 8, \dots$)	Lorentzian self-duality (dimensions $2, 6, 10, \dots$)
(G, λ) compact Lie group + level	$(\mathfrak{g}, b, \Gamma)$ ADE Lie algebra
BPS, Hitchin, Nahm, …	4d $N = 4$ super Yang-Mills, … geometric representation theory, Khovanov homology, …