Model categories

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To Dan Kan

Contents

Preface

Model categories, first introduced by Quillen in [**Qui67**], form the foundation of homotopy theory. The basic problem that model categories solve is the following. Given a category, one often has certain maps (weak equivalences) that are not isomorphisms, but one would like to consider them to be isomorphisms. One can always formally invert the weak equivalences, but in this case one loses control of the morphisms in the quotient category. If the weak equivalences are part of a model structure, however, then the morphisms in the quotient category from X to Y are simply homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y.

Because this idea of inverting weak equivalences is so central in mathematics, model categories are extremely important. However, so far their utility has been mostly confined to areas historically associated with algebraic topology, such as homological algebra, algebraic K-theory, and algebraic topology itself. The author is certain that this list will be expanded to cover other areas of mathematics in the near future. For example, Voevodksy's work [**Voe97**] is certain to make model categories part of every algebraic geometer's toolkit.

These examples should make it clear that model categories are really fundamental. However, there is no systematic study of model categories in the literature. Nowhere can the author find a definition of the category of model categories, for example. Yet one of the main lessons of twentieth century mathematics is that to study a structure, one must also study the maps that preserve that structure.

In addition, there is no excellent source for information about model categories. The standard reference [**Qui67**] is difficult to read, because there is no index and because the definitions are not ideal (they were changed later in [**Qui69**]). There is also [**BK72**, Part II], which is very good at what it does, but whose emphasis is only on simplicial sets. More recently, there is the expository paper [**DS95**], which is highly recommended as an introduction. But there is no mention of simplicial sets in that paper, and it does not go very far into the theory.

The time seems to be right for a more careful study of model categories from the ground up. Both of the books [**DHK**] and [**Hir97**], unfinished as the author writes this, will do this from different perspectives. The book [**DHK**] overlaps considerably with this one, but concentrates more on homotopy colimits and less on the relationship between a model category and its homotopy category. The book [**Hir97**] is concerned with localization of model categories, but also contains a significant amount of general theory. There is also the book [**GJ97**], which concentrates on simplicial examples. All three of these books are highly recommended to the reader.

This book is also an exposition of model categories from the ground up. In particular, this book should be accessible to graduate students. There are very few prerequisites to reading it, beyond a basic familiarity with categories and functors, and some familiarity with at least one of the central examples of chain complexes, simplicial sets, or topological spaces. Later in the book we do require more of the reader; in Chapter 7 we use the theory of homotopy limits of diagrams of simplicial sets, developed in [**BK72**]. However, the reader who gets that far will be well equipped to understand [**BK72**] in any case. The book is not intended as a textbook, though it might be possible for a hard-working instructor to use it as one.

This book is instead the author's attempt to understand the theory of model categories well enough to answer one question. That question is: when is the homotopy category of a model category a stable homotopy category in the sense of [**HPS97**]? I do not in the end answer this question in as much generality as I would like, though I come fairly close to doing so in Chapter 7. As I tried to answer this question, it became clear that the theory necessary to do so was not in place. After a long period of resistance, I decided it was my duty to develop the necessary theory, and that the logical and most useful place to do so was in a book which would assume almost nothing of the reader. A book is the logical place because the theory I develop requires a foundation which is simply not in the literature. I think this foundation is beautiful and important, and therefore deserves to be made accessible to the general mathematician.

We now provide an overview of the book. See also the introductions to the individual chapters. The first chapter of this book is devoted to the basic definitions and results about model categories. In highfalutin language, the main goal of this chapter is to define the 2-category of model categories and show that the homotopy category is part of a pseudo-2-functor from model categories to categories. This is a fancy way, fully explained in Section 1.4, to say that not only can one take the homotopy category of a model category, one can also take the total derived adjunction of a Quillen adjunction, and the total derived natural transformation of a natural transformation between Quillen adjunctions. Doing so preserves compositions for the most part, but not exactly. This is the reason for the word "pseudo". In order to reach this goal, we have to adopt a different definition of model category from that of [**DHK**], but the difference is minor. The definition of [**DHK**], on the other hand, is considerably different from the original definition of [**Qui67**], and even from its refinement in [**Qui69**].

After the theoretical material of the first chapter, the reader is entitled to some examples. We consider the important examples of chain complexes over a ring, topological spaces, and chain complexes of comodules over a commutative Hopf algebra in the second chapter, while the third is devoted to the central example of simplicial sets. Proving that a particular category has a model structure is always difficult. There is, however, a standard method, introduced by Quillen [**Qui67**] but formalized in [**DHK**]. This method is an elaboration of the small object argument and is known as the theory of cofibrantly generated model categories. After examining this theory in Section 2.1, we consider the category of modules over a Frobenius ring, where projective and injective modules coincide. This is perhaps the simplest nontrivial example of a model category, as every object is both cofibrant and fibrant. Nevertheless, the material in this section has not appeared in print before. Then we consider chain complexes of modules over an arbitrary ring. Our treatment differs somewhat from the standard one in that we do not assume our chain complexes are bounded below. We then move on to topological spaces.

Here our treatment is the standard one, except that we offer more details than are commonly provided. The model category of chain complexes of comodules over a commutative Hopf algebra, on the other hand, has not been considered before. It is relevant to the recent work in modular representation theory of Benson, Carlson, Rickard and others (see, for example [**BCR96**]), as well as to the study of stable homotopy over the Steenrod algebra [**Pal97**]. The approach to simplicial sets given in the third chapter is substantially the same as that of [**GJ97**].

In the fourth chapter we consider model categories that have an internal tensor product making them into closed monoidal categories. Almost all the standard model categories are like this: chain complexes of abelian groups have the tensor product, for example. Of course, one must require the tensor product and the model structure to be compatible in an appropriate sense. The resulting monoidal model categories play the same role in the theory of model categories that ordinary rings do in algebra, so that one can consider modules and algebras over them. A module over the monoidal model category of simplicial sets, for example, is the same thing as a simplicial model category. Of course, the homotopy category of a monoidal model category is a closed monoidal category in a natural way, and similarly for modules and algebras. The material in this chapter is all fairly straightforward, but has not appeared in print before. It may also be in [**DHK**], when that book appears.

The fifth and sixth chapters form the technical heart of the book. In the fifth chapter, we show that the homotopy category of any model category has the same good properties as the homotopy category of a simplicial model category. In our highfalutin language, the homotopy pseudo-2-functor lifts to a pseudo-2-functor from model categories to closed Ho **SSet**-modules, where Ho **SSet** is the homotopy category of simplicial sets. This follows from the idea of framings developed in [**DK80**]. This chapter thus has a lot of overlap with [**DHK**], where framings are also considered. However, the emphasis in [**DHK**] seems to be on using framings to develop the theory of homotopy colimits and homotopy limits, whereas we are more interested in making Ho **SSet** act naturally on the homotopy category. There is a nagging question left unsolved in this chapter, however. We find that the homotopy category of a monoidal model category is naturally a closed algebra over Ho **SSet**, but we are unable to prove that it is a *central* closed algebra.

In the sixth chapter we consider the homotopy category of a pointed model category. As was originally pointed out by Quillen [**Qui67**], the apparently minor condition that the initial and terminal objects coincide in a model category has profound implications in the homotopy category. One gets a suspension and loop functor and cofiber and fiber sequences. In the light of the fifth chapter, however, we realize we get an entire closed Ho $\mathbf{SSet}_*$-action, of which the suspension and loop functors are merely specializations. Here Ho $\mathbf{SSet}_*$ is the homotopy category of pointed simplicial sets. We prove that the cofiber and fiber sequences are compatible with this action in an appropriate sense, as well as reproving the standard facts about cofiber and fiber sequences. We then get a notion of pre-triangulated categories, which are closed Ho $\mathbf{SSet}_*$-modules with cofiber and fiber sequences satisfying many axioms.

The seventh chapter is devoted to the stable situation. We define a pre-triangulated category to be *triangulated* if the suspension functor is an equivalence of categories. This is definitely not the same as the usual definition of triangulated categories, but it is closer than one might think at first glance. We also argue that it is a better definition. Every triangulated category that arises in nature is the homotopy category of a model category, so will be triangulated in our stronger sense. We also consider generators in the homotopy category of a pointed model category. These generators are extremely important in the theory of stable homotopy categories developed in [**HPS97**]. Our results are not completely satisfying, but they do go a long way towards answering our original question: when is the homotopy category of a model category a stable homotopy category?

Finally, we close the book with a brief chapter containing some unsolved or partially solved problems the author would like to know more about.

I would like to acknowledge the help of several people in the course of writing this book. I went from knowing very little about model categories to writing this book in the course of about two years. This would not have been possible without the patient help of Phil Hirschhorn, Dan Kan, Charles Rezk, Brooke Shipley, and Jeff Smith, experts in model categories all. I wish to thank John Palmieri for countless conversations about the material in this book. Thanks are also due Gaunce Lewis for help with compactly generated topological spaces, and Mark Johnson for comments on early drafts of this book. And I wish to thank my family, Karen, Grace, and Patrick, for the emotional support so necessary in the frustrating enterprise of writing a book.

CHAPTER 1

Model categories

In this first chapter, we discuss the basic theory of model categories. It very often happens that one would like to consider certain maps in a category to be isomorphisms when they are not. For example, these maps could be homology isomorphisms of some kind, or homotopy equivalences, or birational equivalences of algebraic varieties. One can always invert these "weak equivalences" formally, but there is a foundational problem with doing so, since the class of maps between two objects in the localized category may not be a set. Also, it is very difficult to understand the maps in the resulting localized category. In a model category, there are weak equivalences, but there are also other classes of maps called cofibrations and fibrations. This extra structure allows one to get precise control of the maps in the category obtained by formally inverting the weak equivalences.

Model categories were introduced by Quillen in [**Qui67**], as an abstraction of the usual situation in topological spaces. This is where the terminology came from as well. Quillen's definitions have been modified over the years, by Quillen himself in [**Qui69**] and, more recently, by Dwyer, Hirschhorn, and Kan [**DHK**]. We modify their definition slightly to require that the functorial factorizations be part of the structure. The reader may object that there is now more than one different definition of a model category. That is true, but the differences are slight: in practice, a structure that satisfies one definition satisfies them all. We present our definition and some of the basic facts about model categories in Section 1.1.

At this point, the reader would certainly like some interesting examples of model categories. However, that will have to wait until the next chapter. The axioms for a model category are very powerful. This means there one can prove many theorems about model categories, but it also means that it is hard to check that any particular category is a model category. We need to develop some theory first, before we can construct the many examples that appear in Chapter 2.

In Section 1.2 we present Quillen's results about the homotopy category of a model category. This is the category obtained from a model category by inverting the weak equivalences. The material in this section is standard, as the approach of Quillen has not been improved upon.

In Section 1.3 we study Quillen functors and their derived functors. The most obvious requirement to make on a functor between model categories is that it preserve cofibrations, fibrations, and weak equivalences. This requirement is much too stringent however. Instead, we only require that a Quillen functor preserve half of the model structure: either cofibrations and trivial cofibrations, or fibrations and trivial fibrations, where a trivial cofibration is both a cofibration and a weak equivalence, and similarly for trivial fibrations. This gives us left and right Quillen functors, and could give us two different categories of model categories. However, in practice functors of model categories come in adjoint pairs. We therefore define a morphism of model categories to be an adjoint pair, where the left adjoint is a left Quillen functor and the right adjoint is a right Quillen functor. Of course, we still have to pick a direction, but it is now immaterial which direction we pick. We choose the direction of the left adjoint.

A Quillen functor will induce a functor on the homotopy categories, called its total (left or right) derived functor. This operation of taking the derived functor does not preserve identities or compositions, but it does do so up to coherent natural isomorphism. We describe this precisely in Section 1.3, as is also done in [**DHK**] but has never been done explicitly in print before that.

This observation leads naturally to 2-categories and pseudo-2-functors, which we discuss in Section 1.4. The category of model categories is not really a category at all, but a 2-category. The operation of taking the homotopy category and the total derived functor is not a functor, but instead is a pseudo-2-functor. The 2-morphisms of model categories are just natural transformations, so this section really just points out that there is a convenient language to talk about these kind of phenomena, rather than introducing any deep mathematics. This language is convenient for the author, who will use it throughout the book. However, the reader who prefers not to use it should skip this section and refer back to it as needed.

1.1. The definition of a model category

In this section we present our definition of a model category, and derive some basic results. As mentioned above, our definition is different from the original definition of Quillen and is even slightly different from the modern refinements of [**DHK**]. The reader is thus advised to look at the definition we give here and read the comments following it, even if she is familiar with model categories.

Other sources for model categories and basic results about them include the original source [**Qui67**], the very readable [**DS95**], and the more modern [**DHK**] and [**Hir97**].

We make some preliminary definitions.

Given a category $\mathcal{C}$, we can form the category $\operatorname{Map}\mathcal{C}$ whose objects are morphisms of $\mathcal{C}$ and whose morphisms are commutative squares.

Definition 1.1.1. Suppose $\mathcal{C}$ is a category.

1. A map f in $\mathcal{C}$ is a *retract* of a map $g \in \mathcal{C}$ if f is a retract of g as objects of $\operatorname{Map}\mathcal{C}$. That is, f is a retract of g if and only if there is a commutative diagram of the form
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$
where the horizontal composites are identities.
2. A *functorial factorization* is an ordered pair (α, β) of functors $\operatorname{Map}\mathcal{C} \to \operatorname{Map}\mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \operatorname{Map}\mathcal{C}$. In particular, the domain of $\alpha(f)$ is the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$, and the codomain of $\beta(f)$ is the codomain of f.

Definition 1.1.2. Suppose $i\colon A \to B$ and $p\colon X \to Y$ are maps in a category $\mathcal{C}$. Then i has the *left lifting property with respect to* p and p has the *right lifting* *property with respect to* i if, for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

there is a lift $h\colon B \to Y$ such that $hi = f$ and $ph = g$.

DEFINITION 1.1.3. A *model structure* on a category $\mathcal{C}$ is three subcategories of $\mathcal{C}$ called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties:

1. (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that gf is defined and two of f, g and gf are weak equivalences, then so is the third.
2. (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.
3. (Lifting) Define a map to be a *trivial cofibration* if it is both a cofibration and a weak equivalence. Similarly, define a map to be a *trivial fibration* if it is both a fibration and a weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.
4. (Factorization) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

DEFINITION 1.1.4. A *model category* is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.

This definition of a model category differs from the definition in [**Qui67**] in the following ways. Recall that Quillen distinguished between model categories and closed model categories. That distinction has not proved to be important, so recent authors have only considered closed model categories. We therefore drop the adjective closed. In addition, Quillen only required finite limits and colimits to exist. All of the examples he considered where only such colimits and limits exist are full subcategories of model categories where all small colimits and limits exist. Since it is technically much more convenient to assume all small colimits and limits exist, we do so. Quillen also assumed the factorizations merely exist, not that they are functorial. However, in all the examples they can be made functorial.

The changes we have discussed so far are due to Kan and appear in [**DHK**]. We make one further change in that we make the functorial factorizations part of the model structure, rather than merely assuming they exist. This is a subtle difference, necessary for various constructions to be natural with respect to maps of model categories.

We always abuse notation and refer to a model category $\mathcal{C}$, leaving the model structure implicit. We will discuss several examples of model categories in the next two chapters. We can give some trivial examples now.

EXAMPLE 1.1.5. Suppose $\mathcal{C}$ is a category with all small colimits and limits. We can put three different model structures on $\mathcal{C}$ by choosing one of the distinguished subcategories to be the isomorphisms and the other two to be all maps of $\mathcal{C}$. There are then obvious choices for the functorial factorizations, and this gives a model structure on $\mathcal{C}$. For example, we could define a map to be a weak equivalence if and only if it is an isomorphism, and define every map to be both a cofibration and a fibration. In this case, we define the functors α and δ to be the identity functor, and define $\beta(f)$ to be the identity of the codomain of f and $\gamma(f)$ to be the identity of the domain of f.

EXAMPLE 1.1.6. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories. Then $\mathcal{C} \times \mathcal{D}$ becomes a model category in the obvious way: a map (f, g) is a cofibration (fibration, weak equivalence) if and only if both f and g are cofibrations (fibrations, weak equivalences). We leave it to the reader to define the functorial factorizations and verify that the axioms hold. We could do this with any set of model categories. We refer to the model structure just defined as the *product model structure*.

REMARK 1.1.7. A very useful property of the axioms for a model category is that they are self-dual. That is, suppose $\mathcal{C}$ is a model category. Then the opposite category $\mathcal{C}^{\mathrm{op}}$ is also a model category, where the cofibrations of $\mathcal{C}^{\mathrm{op}}$ are the fibrations of $\mathcal{C}$, the fibrations of $\mathcal{C}^{\mathrm{op}}$ are the cofibrations of $\mathcal{C}$, and the weak equivalences of $\mathcal{C}^{\mathrm{op}}$ are the weak equivalences of $\mathcal{C}$. The functorial factorizations also get inverted: the functor α of $\mathcal{C}^{\mathrm{op}}$ is the opposite of the functor δ of $\mathcal{C}$, the functor β of $\mathcal{C}^{\mathrm{op}}$ is the opposite of the functor γ of $\mathcal{C}$, the functor γ of $\mathcal{C}^{\mathrm{op}}$ is the opposite of the functor β of $\mathcal{C}$, and the functor δ of $\mathcal{C}^{\mathrm{op}}$ is the opposite of the functor α of $\mathcal{C}$. We leave it to the reader to check that these structures make $\mathcal{C}^{\mathrm{op}}$ into a model category. We denote it by $D\mathcal{C}$, and refer to $D\mathcal{C}$ as the dual model category of $\mathcal{C}$. Note that $D^2\mathcal{C} = \mathcal{C}$ as model categories. In practice, this duality means that every theorem about model categories has a dual theorem.

If $\mathcal{C}$ is a model category, then it has an initial object, the colimit of the empty diagram, and a terminal object, the limit of the empty diagram. We call an object of $\mathcal{C}$ *cofibrant* if the map from the initial object 0 to it is a cofibration, and we call an object *fibrant* if the map from it to the terminal object 1 (or $*$) is a fibration. We call a model category (or any category with an initial and terminal object) *pointed* if the map from the initial object to the terminal object is an isomorphism.

Given a model category $\mathcal{C}$, define $\mathcal{C}_*$ to be the category under the terminal object $*$. That is, an object of $\mathcal{C}_*$ is a map $* \xrightarrow{v} X$ of $\mathcal{C}$, often written (X, v). We think of (X, v) as an object X together with a *basepoint* v. A morphism from (X, v) to (Y, w) is a morphism $X \longrightarrow Y$ of $\mathcal{C}$ that takes v to w.

Note that $\mathcal{C}_*$ has arbitrary limits and colimits. Indeed, if $F\colon \mathcal{I} \longrightarrow \mathcal{C}_*$ is a functor from a small category $\mathcal{I}$ to $\mathcal{C}_*$, the limit of F as a functor to $\mathcal{C}$ is naturally an element of $\mathcal{C}_*$ and is the limit there. The colimit is a little trickier. For that, we let $\mathcal{J}$ denote $\mathcal{I}$ with an extra initial object $*$. Then F defines a functor $G\colon \mathcal{J} \longrightarrow \mathcal{C}$, where $G(*) = *$, and G of the map $* \longrightarrow i$ is the basepoint of $F(i)$. The colimit of G in $\mathcal{C}$ then has a canonical basepoint, and this defines the colimit in $\mathcal{C}_*$ of F. For example, the initial object, the colimit of the empty diagram, in $\mathcal{C}_*$ is $*$, and the coproduct of X and Y is $X \vee Y$, the quotient of $X \amalg Y$ obtained by identifying the basepoints. In particular, $\mathcal{C}_*$ is a pointed category.

There is an obvious functor $\mathcal{C} \longrightarrow \mathcal{C}_*$ that takes X to $X_+ = X \amalg *$, with basepoint $*$. This operation of adding a disjoint basepoint is left adjoint to the forgetful functor $U\colon \mathcal{C}_* \longrightarrow \mathcal{C}$, and defines a faithful (but not full) embedding of $\mathcal{C}$ into the pointed category $\mathcal{C}_*$. If $\mathcal{C}$ is already pointed, these functors define an equivalence of categories between $\mathcal{C}$ and $\mathcal{C}_*$.

PROPOSITION 1.1.8. *Suppose* $\mathcal{C}$ *is a model category. Define a map* f *in* $\mathcal{C}_*$ *to be a cofibration* (*fibration, weak equivalence*) *if and only if* Uf *is a cofibration* (*fibration, weak equivalence*) *in* $\mathcal{C}$. *Then* $\mathcal{C}_*$ *is a model category.*

PROOF. It is clear that weak equivalences in $\mathcal{C}_*$ satisfy the two out of three property, and that cofibrations, fibrations, and weak equivalences are closed under retracts. Suppose i is a cofibration in $\mathcal{C}_*$ and p is a trivial fibration. Then Ui has the left lifting property with respect to Up; it follows that i has the left lifting property with respect to p, since any lift must automatically preserve the basepoint. Similarly, trivial cofibrations have the left lifting property with respect to fibrations. If $f = \beta(f) \circ \alpha(f)$ is a functorial factorization in $\mathcal{C}$, then it is also a functorial factorization in $\mathcal{C}_*$; we give the codomain of $\alpha(f)$ the basepoint inherited from α, and then $\beta(f)$ is forced to preserve the basepoint. Thus the factorization axiom also holds and so $\mathcal{C}_*$ is a model category. $\square$

Note that we could replace the terminal object $*$ by any object A of $\mathcal{C}$, to obtain the model category of objects under A. In fact, we could also consider the category of objects over A, whose objects consist of pairs (X, f), where $f\colon X \longrightarrow A$ is a map in $\mathcal{C}$. A similar proof as in Proposition 1.1.8 shows that this also forms a model category. Finally, we could iterate these constructions to form the model category of objects under A and over B. We leave the exact statements and proofs to the reader.

Note that by applying the functors β and α to the map from the initial object to X, we get a functor $X \mapsto QX$ such that QX is cofibrant, and a natural transformation $QX \xrightarrow{q_X} X$ which is a trivial fibration. We refer to Q as the *cofibrant replacement functor* of $\mathcal{C}$. Similarly, there is a *fibrant replacement functor* RX together with a natural trivial cofibration $X \longrightarrow RX$.

The following lemma is often useful when dealing with model categories.

LEMMA 1.1.9 (The Retract Argument). *Suppose we have a factorization* $f = pi$ *in a category* $\mathcal{C}$, *and suppose that* f *has the left lifting property with respect to* p. *Then* f *is a retract of* i. *Dually, if* f *has the right lifting property with respect to* i, *then* f *is a retract of* p.

PROOF. First suppose f has the left lifting property with respect to p. Write $f\colon A \to C$ and $i\colon A \to B$. Then we have a lift $r\colon C \to B$ in the diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "p"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"'] & A \arrow[d, "f"] \\ C \arrow[r, "r"'] & B \arrow[r, "p"'] & C \end{tikzcd}$$

displays f as a retract of i. The proof when f has the right lifting property with respect to i is similar. $\square$

The retract argument implies that the axioms for a model category are overdetermined.

LEMMA 1.1.10. *Suppose $\mathcal{C}$ is a model category. Then a map is a cofibration (a trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a map is a fibration (a trivial fibration) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations).*

PROOF. Certainly every cofibration does have the left lifting property with respect to trivial fibrations. Conversely, suppose f has the left lifting property with respect to trivial fibrations. Factor $f = pi$, where i is a cofibration and p is a trivial fibration. Then f has the left lifting property with respect to p, so the retract argument implies that f is a retract of i. Therefore f is a cofibration. The trivial cofibration part of the lemma is proved similarly, and the fibration and trivial fibration parts follow from duality. □

In particular, every isomorphism in a model category is a trivial cofibration and a trivial fibration, as is also clear from the retract axiom.

COROLLARY 1.1.11. *Suppose $\mathcal{C}$ is a model category. Then cofibrations (trivial cofibrations) are closed under pushouts. That is, if we have a pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"'] \\ B \arrow[r] & D \end{tikzcd}$$

where f is a cofibration (trivial cofibration), then g is a cofibration (trivial cofibration). Dually, fibrations (trivial fibrations) are closed under pullbacks.

PROOF. Because g is a pushout of f, if f has the left lifting property with respect to a map h, so does g. □

An extremely useful result about model categories is Ken Brown's Lemma.

LEMMA 1.1.12 (Ken Brown's lemma). *Suppose $\mathcal{C}$ is a model category and $\mathcal{D}$ is a category with a subcategory of weak equivalences which satisfies the two out of three axiom. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which takes trivial cofibrations betwen cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

PROOF. Suppose $f\colon A \to B$ is a weak equivalence of cofibrant objects. Factor the map $(f, 1_B)\colon A \amalg B \to B$ into a cofibration $A \amalg B \xrightarrow{q} C$ followed by a trivial fibration $C \xrightarrow{p} B$. The pushout diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \amalg B \end{tikzcd}$$

shows that the inclusion maps $A \xrightarrow{i_1} A \amalg B$ and similarly $B \xrightarrow{i_2} A \amalg B$ are cofibrations. By the two out of three axiom, both $q \circ i_1$ and $q \circ i_2$ are weak equivalences, hence trivial cofibrations (of cofibrant objects). By hypothesis, we then have that both $F(q \circ i_1)$ and $F(q \circ i_2)$ are weak equivalences. Since $F(p \circ q \circ i_2) = F(1_B)$ is also a weak equivalence, we conclude from the two out of three axiom that $F(p)$ is a weak equivalence, and hence that $F(f) = F(p \circ q \circ i_1)$ is a weak equivalence, as required. We leave the dual argument to the reader. □

1.2. The homotopy category

In this section, we follow the standard approach to define and study the homotopy category of a model category $\mathcal{C}$. There is nothing new in this section: all authors follow the original approach of Quillen [**Qui67**] with only slight modifications. The particular modifications we use all come from [**DS95**] or [**DHK**]. The basic result is that the localization $\operatorname{Ho}\mathcal{C}$ of a model category $\mathcal{C}$ obtained by inverting the weak equivalences is equivalent to the quotient category $\mathcal{C}_{cf}/\sim$ of the cofibrant and fibrant objects by the homotopy relation. To those readers less familiar with model categories, I wish to emphasize that $\operatorname{Ho}\mathcal{C}$ is not the same category as $\mathcal{C}_{cf}/\sim$, merely equivalent to it. This point confused the author for quite some time when he was learning about model categories.

Definition 1.2.1. Suppose $\mathcal{C}$ is a category with a subcategory of weak equivalences $\mathcal{W}$. Define the homotopy "category" $\operatorname{Ho}\mathcal{C}$ as follows. Form the free category $F(\mathcal{C}, \mathcal{W}^{-1})$ on the arrows of $\mathcal{C}$ and the reversals of the arrows of W. An object of $F(\mathcal{C}, \mathcal{W}^{-1})$ is an object of $\mathcal{C}$, and a morphism is a finite string of composable arrows $(f_1, f_2, \dots, f_n)$ where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i of W. The empty string at a particular object is the identity at that object, and composition is defined by concatenation of strings. Now, define $\operatorname{Ho}\mathcal{C}$ to be the quotient category of $F(\mathcal{C}, \mathcal{W}^{-1})$ by the relations $1_A = (1_A)$ for all objects A, $(f, g) = (g \circ f)$ for all composable arrows f, g of $\mathcal{C}$, and $1_{\operatorname{dom} w} = (w, w^{-1})$ and $1_{\operatorname{codom} w} = (w^{-1}, w)$ for all $w \in \mathcal{W}$. Here $\operatorname{dom} w$ is the domain of w and $\operatorname{codom} w$ is the codomain of w.

The notation $\operatorname{Ho}\mathcal{C}$ is certainly not ideal for this "category". The right notation is $\mathcal{C}[\mathcal{W}^{-1}]$. Our excuse for not adopting the right notation is that we will always take $\mathcal{C}$ to be a subcategory of a model category and take $\mathcal{W}$ to be the weak equivalences in $\mathcal{C}$.

Note that this definition makes it clear that $\operatorname{Ho} D\mathcal{C} = (\operatorname{Ho}\mathcal{C})^{\mathrm{op}}$ if $\mathcal{C}$ is a model category. One can also check that if $\mathcal{C}$ and $\mathcal{D}$ are model categories and we give $\mathcal{C} \times \mathcal{D}$ the product model structure, then $\operatorname{Ho}(\mathcal{C} \times \mathcal{D})$ is isomorphic to $\operatorname{Ho}\mathcal{C} \times \operatorname{Ho}\mathcal{D}$. This is also true if we have more than two factors.

The reason for the quotes around "category" is that $\operatorname{Ho}\mathcal{C}(A, B)$ may not be a set in general. So $\operatorname{Ho}\mathcal{C}$ may not exist until we pass to a higher universe. We will make this passage to a higher universe implicitly until we prove that it is in fact not necessary if $\mathcal{C}$ is a model category.

Note that there is a functor $\mathcal{C} \xrightarrow{\gamma} \operatorname{Ho}\mathcal{C}$ which is the identity on objects and takes morphisms of $\mathcal{W}$ to isomorphisms. The category $\operatorname{Ho}\mathcal{C}$ is characterized by a universal property.

Lemma 1.2.2. *Suppose $\mathcal{C}$ is a category with a subcategory $\mathcal{W}$.*

(i) *If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor that sends maps of $\mathcal{W}$ to isomorphisms, then there is a unique functor $\operatorname{Ho} F\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ such that $(\operatorname{Ho} F) \circ \gamma = F$.*

(ii) *Suppose* $\delta\colon \mathcal{C} \to \mathcal{E}$ *is a functor that takes maps of* $\mathcal{W}$ *to isomorphisms and enjoys the universal property of part* (*i*). *Then there is a unique isomorphism* $\operatorname{Ho}\mathcal{C} \xrightarrow{F} \mathcal{E}$ *such that* $F\gamma = \delta$.

(iii) *The correspondence of part* (*i*) *induces an isomorphism of categories between the category of functors* $\operatorname{Ho}\mathcal{C} \to \mathcal{D}$ *and natural transformations and the category of functors* $\mathcal{C} \to \mathcal{D}$ *which take maps of* $\mathcal{W}$ *to isomorphisms and natural transformations.*

Proof. For part (i), we must define $\operatorname{Ho} F$ to be F on objects and morphisms of $\mathcal{C}$, and define $(\operatorname{Ho} F)(w^{-1}) = (Fw)^{-1}$. This is indeed a functor by the presentation of $\operatorname{Ho}\mathcal{C}$ as a quotient of a free category. Part (ii) follows in the standard way. That is, if $\delta\colon \mathcal{C} \to \mathcal{E}$ also enjoys the universal property of $\operatorname{Ho}\mathcal{C}$, then there is a unique functor $F\colon \operatorname{Ho}\mathcal{C} \to \mathcal{E}$ such that $F\gamma = \delta$ and a unique functor $G\colon \mathcal{E} \to \operatorname{Ho}\mathcal{C}$ such that $G\delta = \gamma$. Then both GF and the identity functor of $\operatorname{Ho}\mathcal{C}$ preserve γ, so must be equal. Similarly, both FG and the identity functor of $\mathcal{E}$ preserve δ, so must be equal. Thus F is an isomorphism.

For part (iii), given a functor $F\colon \mathcal{C} \to \mathcal{D}$ which takes weak equivalences to isomorphisms, we associate to it $\operatorname{Ho} F$. Given a natural transformation $\tau\colon F \to G$, we associate to it $\operatorname{Ho}\tau\colon \operatorname{Ho} F \to \operatorname{Ho} G$, where $\operatorname{Ho}\tau_X = \tau_X$. The transformation $\operatorname{Ho}\tau$ is natural on $\operatorname{Ho}\mathcal{C}$ because it is natural with respect to weak equivalences, so is forced to be natural with respect to their inverses as well. Composition of natural transformations is obviously preserved. The inverse of this functor takes a functor $G\colon \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ to $G \circ \gamma$, and a natural transformation τ to $\tau \circ \gamma$, where $(\tau \circ \gamma)_X = \tau_X$. □

Proposition 1.2.3. *Suppose* $\mathcal{C}$ *is a model category. Let* $\mathcal{C}_c$ (*resp.* $\mathcal{C}_f$,$\mathcal{C}_{cf}$) *denote the full subcategory of cofibrant* (*resp. fibrant, cofibrant and fibrant*) *objects of* $\mathcal{C}$. *Then the inclusion functors induce equivalences of categories* $\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$ *and* $\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_f \to \operatorname{Ho}\mathcal{C}$.

Proof. We prove that $\operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$ is an equivalence, leaving the other cases to the reader. Certainly $\mathcal{C}_c \xrightarrow{i} \mathcal{C}$ preserves weak equivalences, so does induce a functor $\operatorname{Ho} i\colon \operatorname{Ho}\mathcal{C}_c \to \operatorname{Ho}\mathcal{C}$. The inverse is induced by the cofibrant replacement functor Q. Recall that QX is cofibrant and there is a natural trivial fibration $QX \xrightarrow{q_X} X$. In particular, Q preserves weak equivalences and so induces a functor $\operatorname{Ho} Q\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}_c$. The natural transformation q can be thought of as a natural weak equivalence $Q \circ i \to 1_{\mathcal{C}_c}$ or $i \circ Q \to 1_{\mathcal{C}}$. On the homotopy category, $\operatorname{Ho} q$ is therefore a natural isomorphism $\operatorname{Ho} i \circ \operatorname{Ho} Q \to 1_{\operatorname{Ho}\mathcal{C}_c}$ and a natural isomorphism $\operatorname{Ho} Q \circ \operatorname{Ho} i \to 1_{\operatorname{Ho}\mathcal{C}}$, so $\operatorname{Ho} Q$ and $\operatorname{Ho} i$ are inverse equivalences of categories. □

We now summarize the standard alternative construction of $\operatorname{Ho}\mathcal{C}_{cf}$ which makes it clear that $\operatorname{Ho}\mathcal{C}_{cf}$, and hence $\operatorname{Ho}\mathcal{C}$, is really a category without having to pass to a higher universe.

Definition 1.2.4. Suppose $\mathcal{C}$ is a model category, and $f, g\colon B \to X$ are two maps in $\mathcal{C}$.

1. A *cylinder object* for B is a factorization of the fold map $\nabla\colon B \amalg B \to B$ into a cofibration $B \amalg B \xrightarrow{i_0+i_1} B'$ followed by a weak equivalence $B' \xrightarrow{s} B$.
2. A *path object* for X is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence $X \xrightarrow{r} X'$ followed by a fibration $X' \xrightarrow{(p_0,p_1)} X \times X$.

3. A *left homotopy* from f to g is a map $H\colon B' \to X$ for some cylinder object B' for B such that $Hi_0 = f$ and $Hi_1 = g$. We say that f and g are *left homotopic*, written $f \overset{\ell}{\sim} g$, if there is a left homotopy from f to g.
4. A *right homotopy* from f to g is a map $K\colon B \to X'$ for some path object X' for X such that $p_0K = f$ and $p_1K = g$. We say that f and g are *right homotopic*, written $f \overset{r}{\sim} g$, if there is a right homotopy from f to g.
5. We say that f and g are *homotopic*, written $f \sim g$, if they are both left and right homotopic.
6. f is a *homotopy equivalence* if there is a map $h\colon X \to B$ such that $hf \sim 1_B$ and $fh \sim 1_X$.

Note that a path object for B in $\mathcal{C}$ is the same thing as a cylinder object for B in $D\mathcal{C}$, the dual model category. Similarly, a right homotopy between f and g in $\mathcal{C}$ is the same thing as a left homotopy between f and g in $D\mathcal{C}$. Thus we need only prove results about left homotopies and cylinder objects, and the dual statements will automatically hold for right homotopies and path objects.

We get a functorial cylinder object $B \times I$ for B by applying the functorial factorization to the fold map $B \amalg B \to B$. This cylinder object has the additional property that the map $B \times I \xrightarrow{s} B$ is a trivial fibration. Dually, we get a functorial path object X^I for X by applying the functorial factorization to the diagonal map, and in this case the map $X \xrightarrow{r} X^I$ is a trivial cofibration. Note that, if B' is an arbitrary cylinder object for B, there is a weak equivalence $B' \to B \times I$ compatible with the structure maps (i_0, i_1) and s. Indeed, such a map is given by a lift in the diagram

$$\begin{tikzcd} B \amalg B \arrow[r, "(i_0,i_1)"] \arrow[d, "(i_0,i_1)"'] & B \times I \arrow[d, "s"] \\ B' \arrow[r, "s"] & B \end{tikzcd}$$

Similarly, given a path object X' for X, there is a map $X^I \to X'$ compatible with the structure maps r and (p_0, p_1).

The following proposition sums up the properties of the left homotopy relation, and dually, the right homotopy relation. This proposition is standard and comes originally from [**Qui67**].

Proposition 1.2.5. *Suppose $\mathcal{C}$ is a model category, and $f, g\colon B \to X$ are two maps of $\mathcal{C}$.*

(i) *If $f \overset{\ell}{\sim} g$ and $h\colon X \to Y$, then $hf \overset{\ell}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $h\colon A \to B$, then $fh \overset{r}{\sim} gh$.*

(ii) *If X is fibrant, $f \overset{\ell}{\sim} g$, and $h\colon A \to B$, then $fh \overset{\ell}{\sim} gh$. Dually, if B is cofibrant, $f \overset{r}{\sim} g$, and $h\colon X \to Y$, then $hf \overset{r}{\sim} hg$.*

(iii) *If B is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(B, X)$. Dually, if X is fibrant, then right homotopy is an equivalence relation in $\mathcal{C}(B, X)$.*

(iv) *If B is cofibrant and $h\colon X \to Y$ is a trivial fibration or a weak equivalence of fibrant objects, then h induces an isomorphism*

$$\mathcal{C}(B, X)/\overset{\ell}{\sim} \xrightarrow{\cong} \mathcal{C}(B, Y)/\overset{\ell}{\sim}\ .$$

Dually, if X is fibrant and $h\colon A \longrightarrow B$ is a trivial cofibration or a weak equivalence of cofibrant objects, then h induces an isomorphism

$$\mathcal{C}(B,X)/\overset{r}{\sim} \xrightarrow{\cong} \mathcal{C}(A,X)/\overset{r}{\sim}\,.$$

(v) *If B is cofibrant, then $f \overset{\ell}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if X' is any path object for X, there is a right homotopy $K\colon B \longrightarrow X'$ from f to g. Dually, if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{\ell}{\sim} g$, and there is a left homotopy from f to g using any cylinder object for B.*

Proof. We only need prove the claims about left homotopies, by duality. Part (i) is straightforward, and we leave it to the reader. For part (ii), suppose X is fibrant, $f \overset{\ell}{\sim} g$, and $h\colon A \longrightarrow B$. Suppose $H\colon B' \longrightarrow X$ is a left homotopy from f to g, where $B \amalg B \xrightarrow{i} B' \xrightarrow{s} B$ is a cylinder object for B. Because X is fibrant, we can assume that the map $B' \xrightarrow{s} B$ is a trivial fibration. Indeed, we can factor the weak equivalence s into a trivial cofibration $B' \longrightarrow B''$ followed by a trivial fibration $B'' \xrightarrow{s'} B$. Then B'' is also a cylinder object for B, and because X is fibrant, there is an extension of the homotopy $H\colon B' \longrightarrow X$ to a homotopy $H'\colon B'' \longrightarrow X$. We will therefore assume that $s\colon B' \longrightarrow B$ is a trivial fibration.

Now, suppose $A \amalg A \xrightarrow{j} A' \xrightarrow{t} A$ is a cylinder object for A. Consider the commutative diagram

$$\begin{array}{ccc} A \amalg A & \xrightarrow{i\circ(f\amalg f)} & B' \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ A' & \xrightarrow{ft} & B \end{array}$$

We can find a lift $k\colon A' \longrightarrow B'$ in this diagram, and Hk is the desired left homotopy from fh to gh.

We now prove part (iii). The left homotopy relation is always reflexive and symmetric, no matter what B is. Indeed, if $B \amalg B \longrightarrow B' \xrightarrow{s} B$ is a cylinder object for B, then fs is a left homotopy from f to f. Suppose $H\colon B' \longrightarrow X$ is a left homotopy from f to g. Then we can make a new cylinder object B'' for B by simply switching i_0 and i_1. Then $H\colon B'' \longrightarrow X$ is a left homotopy from g to f. We are left with proving the left homotopy relation is transitive, and for this we need to assume B is cofibrant. Suppose $H\colon B' \longrightarrow X$ is a left homotopy from f to g, and $H'\colon B'' \longrightarrow X$ is a left homotopy from g to h. Let C be the pushout in the diagram

$$\begin{array}{ccc} B & \xrightarrow{i_1} & B' \\ {\scriptstyle i_0'}\big\downarrow & & \big\downarrow \\ B'' & \longrightarrow & C \end{array}$$

We have a factorization $B \amalg B \xrightarrow{j_0+j_1} C \xrightarrow{t} B$ of the fold map. Indeed, define j_0 as the composite $B \xrightarrow{i_0} B' \longrightarrow C$, define j_1 as the composite $B \xrightarrow{i_1'} B'' \longrightarrow C$, and define s as the map $C \longrightarrow B$ induced by s and s'. The map $j_0 + j_1$ may not be a cofibration, but, because B is cofibrant, the map t is a weak equivalence. Indeed, $i_1\colon B \longrightarrow B'$ is a trivial cofibration, so the map $B'' \longrightarrow C$ is also a trivial cofibration. Since s' is also a weak equivalence, so is t.

Now, we have a map $C \xrightarrow{K} X$ induced by H and H', such that $Kj_0 = f$ and $Kj_1 = g$. This is not a left homotopy, but it can be made into one by factoring $j_0 + j_1$ into a cofibration followed by a trivial cofibration $C' \to C$. Then C' is a cylinder object for A, and the composite $C' \to C \xrightarrow{K} X$ is a left homotopy between f and h.

We now prove part (iv). The case when $h\colon X \to Y$ is a weak equivalence of fibrant objects follows from the trivial fibration case and Ken Brown's lemma. So suppose h is a trivial fibration, and consider the map

$$F\colon \mathcal{C}(B,X)/\overset{\ell}{\sim} \to \mathcal{C}(B,Y)/\overset{\ell}{\sim}$$

which makes sense by parts (i) and (iii). We first show that F is surjective. So suppose $f'\colon B \to Y$ is a map. Then, since B is cofibrant, we can find a map $f\colon B \to X$ such that $hf = f'$. So the map F is surjective even without taking homotopy classes. Now suppose $hf \overset{\ell}{\sim} hg$, and choose a left homotopy $H\colon B' \to Y$ from hf to hg. Then we can find a lift $K\colon B' \to X$ in the diagram

$$\begin{tikzcd} B \amalg B \arrow[r, "f+g"] \arrow[d, "i_0+i_1"'] & X \arrow[d, "h"] \\ B' \arrow[r, "H"] & Y \end{tikzcd}$$

and the map K is a left homotopy from f to g. Hence F is injective as well.

Finally, we prove part (v). Suppose B is cofibrant and $f \overset{\ell}{\sim} g\colon B \to X$ by a left homotopy $H\colon B' \to X$. Then the map $i_0\colon B \to B'$ is a trivial cofibration. Suppose $X \xrightarrow{r} X' \xrightarrow{(p_0,p_1)} X \times X$ is a path object for X. Then we can find a lift $J\colon B' \to X'$ in the diagram

$$\begin{tikzcd} B \arrow[r, "rf"] \arrow[d, "i_0"'] & X' \arrow[d, "(p_0,p_1)"] \\ B' \arrow[r, "(fs,H)"] & X \times X \end{tikzcd}$$

Then $K = Ji_1$ is a right homotopy from f to g, as required. □

We have two immediate corollaries.

Corollary 1.2.6. *Suppose $\mathcal{C}$ is a model category, B is a cofibrant object of $\mathcal{C}$, and X is a fibrant object of $\mathcal{C}$. Then the left homotopy and right homotopy relations coincide and are equivalence relations on $\mathcal{C}(B,X)$. Furthermore, if $f \sim g\colon B \to X$, then there is a left homotopy $H\colon B' \to X$ from f to g using any cylinder object B' for B. Dually, there is a right homotopy $K\colon B \to X'$ from f to g using any path object X' for B.*

Corollary 1.2.7. *The homotopy relation on the morphisms of $\mathcal{C}_{cf}$ is an equivalence relation and is compatible with composition. Hence the category $\mathcal{C}_{cf}/\sim$ exists.*

The functor $\mathcal{C}_{cf} \to \mathcal{C}_{cf}/\sim$ inverts the homotopy equivalences in $\mathcal{C}_{cf}$. We would like it to invert the weak equivalences.

Proposition 1.2.8. *Suppose $\mathcal{C}$ is a model category. Then a map of $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

PROOF. Suppose first that $f\colon A \to B$ is a weak equivalence of cofibrant and fibrant objects. Then, by Ken Brown's lemma and Proposition 1.2.5, if X is also cofibrant and fibrant we have an isomorphism $f_*\colon (\mathcal{C}_{cf}/\sim)(X, A) \to (\mathcal{C}_{cf}/\sim)(X, B)$. Taking $X = B$, we find a map $g\colon B \to A$, unique up to homotopy, such that $fg \sim 1_B$. In particular $fgf \sim f$, so taking $X = A$, we find that $gf \sim 1_A$. Thus f is a homotopy equivalence.

Conversely, suppose f is a homotopy equivalence between cofibrant and fibrant objects. Factor f into a trivial cofibration $A \xrightarrow{g} C$ followed by a fibration $p\colon C \to B$. Then C is also cofibrant and fibrant, so g is a homotopy equivalence, as we have just proved. We will show that p is a weak equivalence. To do so, let $f'\colon B \to A$ be a homotopy inverse for f, and let $H\colon B' \to B$ be a left homotopy from ff' to 1_B. Let $H'\colon B' \to C$ be a lift in the commutative square

$$\begin{tikzcd} B \arrow[r, "gf'"] \arrow[d, "i_0"'] & C \arrow[d, "p"] \\ B' \arrow[r, "H"] & B \end{tikzcd}$$

and let $q = H'i_1\colon B \to C$. Then $pq = 1_B$, and H' is a left homotopy from gf' to q. Now, let $g'\colon C \to A$ be a homotopy inverse for g. Then $p \sim pgg' \sim fg'$. Hence $qp \sim (gf')(fg') \sim 1_C$.

It follows that qp is a weak equivalence. Indeed, if $K\colon C' \to C$ is a left homotopy from 1_C to qp, then $Ki_0 = 1_C$ is a weak equivalence, as is i_0, so K is a weak equivalence. Thus $Ki_1 = qp$ is also a weak equivalence. Now, the commutative diagram

$$\begin{tikzcd} C \arrow[r, "1_C"] \arrow[d, "p"'] & C \arrow[r, "1_C"] \arrow[d, "qp"] & C \arrow[d, "p"] \\ B \arrow[r, "q"] & C \arrow[r, "p"] & B \end{tikzcd}$$

shows that p is a retract of qp. Hence p is a weak equivalence, as required, and so f is too. $\square$

COROLLARY 1.2.9. *Suppose $\mathcal{C}$ is a model category. Let $\gamma\colon \mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}_{cf}$ and $\delta\colon \mathcal{C}_{cf} \to \mathcal{C}_{cf}/\sim$ be the canonical functors. Then there is a unique isomorphism of categories $\mathcal{C}_{cf}/\sim \xrightarrow{j} \operatorname{Ho}\mathcal{C}_{cf}$ such that $j\delta = \gamma$. Furthermore j is the identity on objects.*

PROOF. We show that $\mathcal{C}_{cf}/\sim$ has the same universal property that $\operatorname{Ho}\mathcal{C}_{cf}$ enjoys (see Lemma 1.2.2). The functor δ takes homotopy equivalences to isomorphisms, and hence takes weak equivalences to isomorphisms by Proposition 1.2.8. Now suppose $F\colon \mathcal{C}_{cf} \to \mathcal{D}$ is a functor that takes weak equivalences to isomorphisms. Let $A \amalg A \xrightarrow{i_0+i_1} A' \xrightarrow{s} A$ be a cylinder object for A. Then $si_0 = si_1 = 1_A$, and so, since s is a weak equivalence, we have $Fi_0 = Fi_1$. Thus if $H\colon A' \to B$ is a left homotopy between f and g, we have $Ff = (FH)(Fi_0) = (FH)(Fi_1) = Fg$, and so F identifies left (or, dually, right) homotopic maps. Thus there is a unique functor $G\colon \mathcal{C}_{cf}/\sim \to \mathcal{D}$ such that $G\delta = F$. Indeed, G is the identity on objects and takes the equivalence class of a map f to Ff. Lemma 1.2.2 then completes the proof. $\square$

Finally, we get what must be considered the fundamental theorem about model categories.

THEOREM 1.2.10. *Suppose* $\mathcal{C}$ *is a model category. Let* $\gamma\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ *denote the canonical functor, and let* Q *denote the cofibrant replacement functor of* $\mathcal{C}$ *and* R *denote the fibrant replacement functor.*

(i) *The inclusion* $\mathcal{C}_{cf} \to \mathcal{C}$ *induces an equivalence of categories* $\mathcal{C}_{cf}/\sim \xrightarrow{\cong} \operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}$.

(ii) *There are natural isomorphisms*

$$\mathcal{C}(QRX, QRY)/\sim \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(RQX, RQY)/\sim$$

In addition, there is a natural isomorphism $\operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(QX, RY)/\sim$*, and, if* X *is cofibrant and* Y *is fibrant, there is a natural isomorphism* $\operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(X, Y)/\sim$*. In particular,* $\operatorname{Ho}\mathcal{C}$ *is a category without moving to a higher universe.*

(iii) *The functor* $\gamma\colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ *identifies left or right homotopic maps.*

(iv) *If* $f\colon A \to B$ *is a map in* $\mathcal{C}$ *such that* γf *is an isomorphism in* $\operatorname{Ho}\mathcal{C}$*, then* f *is a weak equivalence.*

PROOF. The first part is a combination of Proposition 1.2.3 and Corollary 1.2.9. The inverse of the equivalence $\operatorname{Ho}\mathcal{C}_{cf} \to \operatorname{Ho}\mathcal{C}$ is given by $\operatorname{Ho}Q \circ \operatorname{Ho}R$ (or $\operatorname{Ho}R \circ \operatorname{Ho}Q$). This gives us the natural isomorphisms $\mathcal{C}(QRX, QRY)/\sim \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(RQX, RQY)$ of part (ii). The rest of part (ii) follows from Proposition 1.2.5 and the natural weak equivalences $QX \to X \to RX$.

Part (iii) was proved already in the proof of Corollary 1.2.9. Finally, for part (iv), suppose $f\colon A \to B$ is a map in $\mathcal{C}$ such that γf is an isomorphism in $\operatorname{Ho}\mathcal{C}$. Then QRf is an isomorphism in $\mathcal{C}_{cf}/\sim$, from which it follows easily that QRf is a homotopy equivalence. By Proposition 1.2.8, we see that QRf is a weak equivalence. Then, using the fact that both the natural transformations $QX \to X$ and $X \to RX$ are weak equivalences, we find that f must be a weak equivalence. □

We will frequently abbreviate $\operatorname{Ho}\mathcal{C}(X, Y)$ and $\operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ by $[X, Y]$ in the sequel.

1.3. Quillen functors and derived functors

In this section, we study morphisms of model categories. We call such morphisms Quillen adjunctions or Quillen functors, and we show that a Quillen functor induces a functor of the homotopy categories. This process of associating a derived functor to a Quillen functor is not itself functorial, but it is functorial up to natural isomorphism, as we indicate in this section. Occasionally this derived functor is an equivalence of categories when the original Quillen functor is not. We call such functors Quillen equivalences, and characterize them.

In thinking about the results of this section, the author was heavily influenced by [**DHK**], and there is a great deal of overlap between this section and some of [**DHK**].

1.3.1. Quillen functors.

We begin with the definition of a Quillen functor.

DEFINITION 1.3.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

1. We call a functor $F\colon \mathcal{C} \to \mathcal{D}$ a *left Quillen functor* if F is a left adjoint and preserves cofibrations and trivial cofibrations.

2. We call a functor $U: \mathcal{D} \to \mathcal{C}$ a *right Quillen functor* if U is a right adjoint and preserves fibrations and trivial fibrations.
3. Suppose (F, U, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$. That is, F is a functor $\mathcal{C} \to \mathcal{D}$, U is a functor $\mathcal{D} \to \mathcal{C}$, and φ is a natural isomorphism $\mathcal{D}(FA, B) \to \mathcal{C}(A, UB)$ expressing U as a right adjoint of F. We call (F, U, φ) a *Quillen adjunction* if F is a left Quillen functor.

Note that Ken Brown's lemma 1.1.12 implies that every left Quillen functor preserves weak equivalences between cofibrant objects, and that every right Quillen functor preserves weak equivalences between fibrant objects. For most of the results of this section, we could assume only that our left adjoints preserve cofibrant objects and weak equivalences between them, and dually that our right adjoints preserve fibrant objects and weak equivalences between them. But experience has taught us that little is gained by such generality, and that simplicity is lost.

In practice, Quillen adjunctions are almost always referred to by their left (or right) adjoint alone, but the actual adjunction is always what is meant. Given a Quillen adjunction (F, U, φ), we always denote the unit map $X \to UFX$ by η and the counit map $FUX \to X$ by ε.

At this point the reader will have to take for granted that Quillen adjunctions abound. We will see many examples of Quillen adjunctions in the next chapter. The most famous example is probably the Quillen adjunction from simplicial sets to topological spaces whose left adjoint is the geometric realization and whose right adjoint is the singular complex. We can give the following simple examples now.

EXAMPLE 1.3.2. Suppose $\mathcal{C}$ is a model category and I is a set. A product functor $\mathcal{C}^I \to \mathcal{C}$ is a right adjoint to the diagonal functor $c: \mathcal{C} \to \mathcal{C}^I$. By definition of the product model structure (see Example 1.1.6), the product preserves fibrations and trivial fibrations, and the diagonal functor preserves cofibrations and trivial cofibrations. Hence the diagonal functor and the product functor define a Quillen adjunction $\mathcal{C} \to \mathcal{C}^I$. Similarly, a coproduct functor is a left adjoint to the diagonal functor, and defines a Quillen adjunction $\mathcal{C}^I \to \mathcal{C}$.

EXAMPLE 1.3.3. If $\mathcal{C}$ is a model category, the disjoint basepoint functor $\mathcal{C} \to \mathcal{C}_*$ (see Proposition 1.1.8) is part of a Quillen adjunction, where the right adjoint is the forgetful functor. Indeed, it is clear that the forgetful functor is a right Quillen functor. Lemma 1.3.4 implies that the disjoint basepoint functor is a left Quillen functor.

We have the following simple lemma, which explains why we did not need to require that U be a right Quillen functor in the definition of a Quillen adjunction.

LEMMA 1.3.4. *Suppose $(F, U, \varphi): \mathcal{C} \to \mathcal{D}$ is an adjunction, and $\mathcal{C}$ and $\mathcal{D}$ are model categories. Then (F, U, φ) is a Quillen adjunction if and only if U is a right Quillen functor.*

PROOF. Use adjointness to show that Ff has the left lifting property with respect to p if and only if f has the left lifting property with respect to Up. Then use the characterization of cofibrations, trivial cofibrations, fibrations, and trivial fibrations by lifting properties. □

We can of course compose left (resp. right) Quillen functors to get a new left (resp. right) Quillen functor. We can also compose adjunctions. If $(F, U, \varphi): \mathcal{C} \to$

$\mathcal{D}$ and $(F', U', \varphi')\colon \mathcal{D} \to \mathcal{E}$ are adjunctions, we can define their composition to be the adjunction $(F' \circ F, U \circ U', \varphi \circ \varphi')\colon \mathcal{C} \to \mathcal{E}$. Here $\varphi \circ \varphi'$ is the composite

$$\mathcal{E}(F'FA, B) \xrightarrow[\cong]{\varphi'} \mathcal{D}(FA, U'B) \xrightarrow[\cong]{\varphi} \mathcal{C}(A, UU'B)$$

Composition of adjunctions is associative and has identities. The identity adjunction of a category $\mathcal{C}$ is the identity functor together with the identity adjointness isomorphism. The composition of Quillen adjunctions is a Quillen adjunction.

We can therefore define several different notions of a category of model categories, using as our morphisms left Quillen functors, right Quillen functors, or Quillen adjunctions. The author's choice, based on experience, is to define a morphism of model categories to be a Quillen adjunction. Note that, whatever choice one makes for a morphism of model categories, such a morphism never has to preserve the functorial factorizations. Hence if we take a model category $\mathcal{C}$ and use the same category, cofibrations, fibrations, and weak equivalences, but choose different functorial factorizations to form a new model category $\mathcal{C}'$, the identity functor will be an isomorphism of model categories between them. Thus the choice of functorial factorizations has no effect on the isomorphism class of the model category.

Notice however that none of these categories are categories in the strict sense of the word. Indeed, a category is supposed to have a class of objects, and between any two objects, a set of morphisms. But every class defines a model category, where the only morphisms are identities and they are weak equivalences, cofibrations, and fibrations. So the collection of all model categories contains the collection of all classes, which is certainly not a class. Similarly, the collection of all functors from one model category to another need not be a set.

There are several possible solutions to this problem. One idea is to restrict to small model categories, where the objects are required to form a set. However, there are no nontrivial small model categories, since model categories are required to have all small limits and colimits. So that idea fails. Another idea is to ascend to a higher universe, as we have already implicitly done in forming the category of model categories. However, a better idea is to consider the collection of model categories, Quillen adjunctions, and natural transformations as a 2-category. We will explain this further in the next section.

Also note that, if (F, U, φ) is a Quillen adjunction, then

$$D(F, U, \varphi) = (U, F, \varphi^{-1})\colon D\mathcal{D} \to D\mathcal{C}$$

is a Quillen adjunction between the dual model categories. Note also that D preserves identities and composition (in the opposite order), so that D is a contravariant functor (in a higher universe) such that D^2 is the identity functor.

Another example of a functor from model categories to themselves is provided by the correspondence $\mathcal{C} \mapsto \mathcal{C}_*$.

Proposition 1.3.5. *A Quillen adjunction $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ induces a Quillen adjunction $(F_*, U_*, \varphi_*)\colon \mathcal{C}_* \to \mathcal{D}_*$ between the model categories of Proposition 1.1.8. Furthermore, $F_*(X_+)$ is naturally isomorphic to $(FX)_+$. This correspondence is functorial.*

Proof. Define U_* by $U_*(X, v) = (Ux, Uv)$, which makes sense since U preserves the terminal object. Then U_* obviously preserves fibrations and trivial fibrations, so will be a right Quillen functor if it has a left adjoint. We define $F_*(X, v)$

by the pushout diagram

$$\begin{tikzcd} F(*) \arrow[r, "Fv"] \arrow[d] & FX \arrow[d] \\ * \arrow[r] & F_*(X,v) \end{tikzcd}$$

We leave it to the reader to verify that this is the left adjoint of U_*. Let V denote the functor that forgets the basepoint. Then $VU_* = UV$, so, by adjointness, $F_*(X_+)$ is naturally isomorphic to $(FX)_+$. The functoriality is clear, at least up to the choice of pushouts. This really means that the correspondence that takes F to F_* is functorial up to natural isomorphism, a concept that we discuss more fully in the next section. □

1.3.2. Derived functors and naturality. For the rest of this section we study the functors on the homotopy category induced by Quillen functors.

Definition 1.3.6. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

1. If $F\colon \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, define the *total left derived functor* $LF\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ to be the composite

$$\operatorname{Ho}\mathcal{C} \xrightarrow{\operatorname{Ho}Q} \operatorname{Ho}\mathcal{C}_c \xrightarrow{\operatorname{Ho}F} \operatorname{Ho}\mathcal{D}$$

   Given a natural transformation $\tau\colon F \to F'$ of left Quillen functors, define the *total derived natural transformation* $L\tau$ to be $\operatorname{Ho}\tau \circ \operatorname{Ho}Q$, so that $(L\tau)_X = \tau_{QX}$.
2. If $U\colon \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, define the *total right derived functor* $RU\colon \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$ of U to be the composite

$$\operatorname{Ho}\mathcal{D} \xrightarrow{\operatorname{Ho}R} \operatorname{Ho}\mathcal{D}_f \xrightarrow{\operatorname{Ho}U} \operatorname{Ho}\mathcal{C}$$

   Given a natural transformation $\tau\colon U \to U'$ of right Quillen functors, define the *total derived natural transformation* $R\tau$ to be $\operatorname{Ho}\tau \circ \operatorname{Ho}R$, so that $R\tau_X = \tau_{RX}$.

In practice, a functor is almost never both a left Quillen functor and a right Quillen functor, so can just refer to its total derived functor, leaving out the direction.

This definition is the reason we have assumed that the functorial factorizations are part of the structure of a model category. Otherwise, in order to define LF, we would have to choose a functorial cofibrant replacement, so we would not be able to define LF in a way that depends only on the model category $\mathcal{C}$. Also note that we can define LF even if F is not a left Quillen functor, but just a functor that takes weak equivalences between cofibrant objects to weak equivalences. Dually, we can define RU if U is any functor that takes weak equivalences between fibrant objects to weak equivalences.

As one would expect, given a set I and a model category $\mathcal{C}$, the total right derived functor of a product functor $\mathcal{C}^I \to \mathcal{C}$ is a product functor $(\operatorname{Ho}\mathcal{C})^I \cong \operatorname{Ho}\mathcal{C}^I \to \operatorname{Ho}\mathcal{C}$. We will see this later, after we have discussed derived adjunctions.

Note that the total derived natural transformation is functorial. That is, if $\tau\colon F \to F'$ and $\tau'\colon F' \to F''$ are natural transformations between weak left Quillen functors, then $L(\tau' \circ \tau) = (L\tau') \circ (L\tau)$, and of course $L(1_F) = 1_{LF}$. We have a dual statement for natural transformations between right Quillen functors.

Note, on the other hand, that the operation of taking the total derived functor is clearly not functorial, since for example $L(1_{\mathcal{C}}) = \operatorname{Ho} Q$, the functor that takes X to QX. Nonetheless, the total derived functor is almost functorial, in the following precise sense.

THEOREM 1.3.7. *For every model category $\mathcal{C}$, there is a natural isomorphism $\alpha\colon L(1_{\mathcal{C}}) \to 1_{\operatorname{Ho}\mathcal{C}}$. Also, for every pair of left Quillen functors $F\colon \mathcal{C} \to \mathcal{D}$ and $F'\colon \mathcal{D} \to \mathcal{E}$, there is a natural isomorphism $m = m_{F'F}\colon LF' \circ LF \to L(F' \circ F)$. These natural isomorphisms satisfy the following properties.*

1. *An associativity coherence diagram commutes. That is, if $F\colon \mathcal{C} \to \mathcal{C}'$, $F'\colon \mathcal{C}' \to \mathcal{C}''$, and $F''\colon \mathcal{C}'' \to \mathcal{C}'''$ are left Quillen functors, then the following diagram commutes.*

$$\begin{tikzcd}
(LF'' \circ LF') \circ LF \arrow[r, "m_{F''F'} \circ LF"] \arrow[d, equal] & L(F'' \circ F') \circ LF \arrow[r, "m_{(F''\circ F')F}"] & L((F'' \circ F') \circ F) \arrow[d, equal] \\
LF'' \circ (LF' \circ LF) \arrow[r, "LF'' \circ m_{F'F}"] & LF'' \circ L(F' \circ F) \arrow[r, "m_{F''(F'\circ F)}"] & L(F'' \circ (F' \circ F))
\end{tikzcd}$$

2. *A left unit coherence diagram commutes. That is, if $F\colon \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, then the following diagram commutes.*

$$\begin{tikzcd}
L1_{\mathcal{D}} \circ LF \arrow[r, "m"] \arrow[d, "\alpha \circ LF"'] & L(1_{\mathcal{D}} \circ F) \arrow[d, equal] \\
1_{\operatorname{Ho}\mathcal{D}} \circ LF \arrow[r, equal] & LF
\end{tikzcd}$$

3. *A right unit coherence diagram commutes. That is, if $F\colon \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, then the following diagram commutes.*

$$\begin{tikzcd}
LF \circ L1_{\mathcal{C}} \arrow[r, "m"] \arrow[d, "LF \circ \alpha"'] & L(F \circ 1_{\mathcal{C}}) \arrow[d, equal] \\
LF \circ 1_{\operatorname{Ho}\mathcal{C}} \arrow[r, equal] & LF
\end{tikzcd}$$

PROOF. We define $\alpha\colon L(1_{\mathcal{C}}) \to 1_{\operatorname{Ho}\mathcal{C}}$ to be $\operatorname{Ho} q$, where $q\colon QX \to X$ is the natural trivial fibration from the cofibrant replacement QX to X. We define $m_{F'F}$ to be the map

$$m_{F'F}\colon (LF')(LF)X = F'QFQX \xrightarrow{F'q_{FQX}} F'FQX = L(F'F)X.$$

Then $m_{F'F}$ is obviously natural on $\mathcal{C}$, but since both the source and target of $m_{F'F}$ preserve weak equivalences, $m_{F'F}$ is also natural on $\operatorname{Ho}\mathcal{C}$. Since F preserves cofibrant objects, $q_{FQX}\colon QFQX \to FQX$ is a weak equivalence between cofibrant objects. Thus $m_{F'F} = F'q_{FQX}$ is still a weak equivalence, and hence an isomorphism in $\operatorname{Ho}\mathcal{C}$.

To show that the associativity coherence diagram commutes, we need only show that

$$(F''F'q_{FQX}) \circ (F''q_{F'QFQX}) = (F''q_{F'FQX}) \circ (F''QF'q_{FQX})$$

as maps $F''QF'QFQX \to F''F'FQX$. This follows from the naturality of q. The left unit coherence diagram commutes by definition, as both maps are

$$q_{FQX}\colon QFQX \to FQX.$$

To show that the right unit coherence diagram commutes, we must show that

$$Fq_{QX} = FQq_X \colon FQQX \to FQX.$$

This is not true in $\mathcal{C}$ itself, but it is true in $\operatorname{Ho}\mathcal{C}$. Indeed, it suffices to show these two maps are equal for cofibrant X, since every X is isomorphic in $\operatorname{Ho}\mathcal{C}$ to a cofibrant object. The naturality of q implies that $q_X \circ q_{QX} = q_X \circ Qq_X \colon QQX \to X$. Of course q_X is a weak equivalence between cofibrant objects if X is cofibrant, so Fq_X is also a weak equivalence. It follows that, in $\operatorname{Ho}\mathcal{C}$, we have $Fq_{QX} = (Fq_X)^{-1} = FQq_X$. Thus the right unit coherence diagram commutes for cofibrant X, and hence for all X. □

We would also like to claim that m, and not just each $m_{F'F}$, is natural. In order to do this, we need to recall the obvious fact that one can compose natural transformations horizontally as well as vertically.

Definition 1.3.8. Suppose $\sigma\colon F \to G$ is a natural transformation of functors $\mathcal{C} \to \mathcal{D}$, and $\tau\colon F' \to G'$ is a natural transformation of functors $\mathcal{D} \to \mathcal{E}$. The *horizontal composition* $\tau * \sigma$ is the natural transformation $F' \circ F \to G' \circ G$ given by $(\tau * \sigma)_X = \tau_{GX} \circ F'\sigma_X = G'\sigma_X \circ \tau_{FX}$.

Lemma 1.3.9. *Suppose $\sigma\colon F \to G$ is a natural transformation of weak left Quillen functors $\mathcal{C} \to \mathcal{D}$, and $\tau\colon F' \to G'$ is a natural transformation of weak left Quillen functors $\mathcal{D} \to \mathcal{E}$. Let m be the composition isomorphism of Theorem 1.3.7. Then the following diagram commutes.*

$$\begin{tikzcd} LF' \circ LF \arrow[r, "m"] \arrow[d, "L\tau * L\sigma"'] & L(F' \circ F) \arrow[d, "L(\tau*\sigma)"] \\ LG' \circ LG \arrow[r, "m"] & L(G' \circ G) \end{tikzcd}$$

Proof. This is just a matter of unravelling the definitions. The map $L(\tau * \sigma) \circ m\colon F'QFQX \to G'GQX$ is the composite $\tau_{GQX} \circ F'\sigma_{QX} \circ F'q_{FQX}$. We can use the naturality of q to rewrite this composite as $\tau_{GQX} \circ F'q_{GQX} \circ F'Q\sigma_{QX}$. We can then use the naturality of τ to rewrite this as $G'q_{GQX} \circ \tau_{QGQX} \circ F'Q\sigma_{QX}$, which is the definition of $m \circ (L\tau * L\sigma)$. □

Of course, there are versions of Theorem 1.3.7 and Lemma 1.3.9 for right Quillen functors as well. We can summarize these results, in the language of the next section, by saying that the homotopy category, total derived functor, and total derived natural transformation define a pseudo-2-functor from the 2-category of model categories, left (resp. right) Quillen functors, and natural transformations to the 2-category of categories.

We would like to make the same claim for adjunctions, so we need to show that the total derived functor preserves adjunctions.

Lemma 1.3.10. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories and $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. Then LF and RU are part of an adjunction $L(F, U, \varphi) = (LF, RU, R\varphi)$, which we call the* derived adjunction.

Proof. The desired adjointness isomorphism $R\varphi$ must be a natural isomorphism $R\varphi\colon \operatorname{Ho}\mathcal{D}(FQX, Y) \to \operatorname{Ho}\mathcal{C}(X, URY)$. Note that $\operatorname{Ho}\mathcal{D}(FQX, Y)$ is naturally isomorphic to $\mathcal{D}(FQX, RY)/\sim$, and similarly $\operatorname{Ho}\mathcal{C}(X, URY)$ is naturally isomorphic to $\mathcal{C}(QX, URY)/\sim$. Thus we only need verify that φ respects the homotopy relation. So suppose A is cofibrant in $\mathcal{C}$ and B is fibrant in $\mathcal{D}$. Suppose $f, g\colon FA \to B$ are homotopic. Then there is a path object B' for B and a right homotopy $H\colon FA \to B'$ from f to g. Because U preserves products, fibrations, and weak equivalences between fibrant objects, UB' is a path object for UB. Hence $\varphi H\colon A \to UB'$ is a right homotopy from φf to φg. Conversely, suppose φf and φg are homotopic. Then there is a cylinder object A' for A and a left homotopy $H\colon A' \to UB$ from φf to φg. Since F preserves coproducts, cofibrations, and weak equivalences between cofibrant objects, FA' is a cylinder object for FA. Hence $\varphi^{-1}H\colon FA' \to B$ is a left homotopy from $\varphi^{-1}\varphi f = f$ to g. ☐

EXAMPLE 1.3.11. Suppose I is a set and $\mathcal{C}$ is a model category. We have seen in Example 1.3.2 that we have a Quillen adjunction $\mathcal{C} \to \mathcal{C}^I$, where $\mathcal{C}^I$ is given the product model structure. The left adjoint is the diagonal functor c and the right adjoint is a product functor. We have $(Lc)(X) = c(QX)$. Hence Lc is naturally isomorphic to the diagonal functor c' on $\operatorname{Ho}\mathcal{C}$, under the isomorphism $\operatorname{Ho}\mathcal{C}^I \cong (\operatorname{Ho}\mathcal{C})^I$. It follows that the total right derived functor of a product functor on $\mathcal{C}$ is a product functor on $\operatorname{Ho}\mathcal{C}$. Similarly, the total left derived functor of a coproduct functor on $\mathcal{C}$ is a coproduct functor on $\operatorname{Ho}\mathcal{C}$.

Example 1.3.11 shows that the homotopy category of a model category has all small coproducts and products, and thus has more structure than a random category. We will see that this is only the tip of the iceberg in the rest of this book.

1.3.3. Quillen equivalences.

Sometimes $L(F, U, \varphi)$ is an adjoint equivalence of categories even when (F, U, φ) is not. We now investigate this question.

DEFINITION 1.3.12. A Quillen adjunction $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is called a *Quillen equivalence* if and only if, for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f\colon FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\varphi(f)\colon X \to UY$ is a weak equivalence in $\mathcal{C}$.

PROPOSITION 1.3.13. *Suppose $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. Then the following are equivalent*:

(a) *(F, U, φ) is a Quillen equivalence.*
(b) *The composite $X \xrightarrow{\eta} UFX \xrightarrow{Ur_{FX}} URFX$ is a weak equivalence for all cofibrant X, and the composite $FQUX \xrightarrow{Fq_{UX}} FQX \xrightarrow{\varepsilon} X$ is a weak equivalence for all fibrant X.*
(c) *$L(F, U, \varphi)$ is an adjoint equivalence of categories.*

PROOF. We first show that (a)⇒(b). If (F, U, φ) is a Quillen equivalence and X is cofibrant, then $\varphi r_{FX}\colon X \to URFX$ is a weak equivalence, adjoint to the weak equivalence $r_{FX}\colon FX \to RFX$. In terms of the unit η of φ, we have $\varphi r_{FX} = Ur_{FX} \circ \eta$. Similarly, if X is fibrant, $\varepsilon \circ Fq_{UX} = \varphi^{-1}q_{UX}$ is a weak equivalence adjoint to $q_{UX}\colon QUX \to UX$.

Conversely, suppose (F, U, φ) satisfies (b). Given a weak equivalence $f\colon FX \to Y$, where X is cofibrant and Y is fibrant, φf is the composite $X \xrightarrow{\eta} URX \xrightarrow{Uf} UY$.

We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, equal] & UFX \arrow[r, "Uf"] \arrow[d, "Ur_{FX}"'] & UY \arrow[d, "Ur_Y"'] \\
X \arrow[r] & URFX \arrow[r, "URf"] & URY
\end{tikzcd}$$

Since f is a weak equivalence, so is Rf. Since U preserves weak equivalences between fibrant objects, URf is a weak equivalence. Thus the bottom horizontal composite is a weak equivalence, as is the right vertical map. It follows that the top horizontal composite φf is a weak equivalence. Similarly, if $\varphi f\colon X \to UY$ is a weak equivalence, then we have a commutative diagram

$$\begin{tikzcd}
FQX \arrow[r, "FQ(\varphi f)"] \arrow[d, "Fq_X"'] & FQUY \arrow[r] \arrow[d, "Fq_{UY}"'] & Y \arrow[d, equal] \\
FX \arrow[r, "F(\varphi f)"] & FUY \arrow[r, "\varepsilon"] & Y
\end{tikzcd}$$

The bottom horizontal composite is f, and both the top horizontal composite and the left vertical map are weak equivalences, so f is a weak equivalence.

To see that (b)$\Leftrightarrow$(c), note that the unit of $R\varphi$ is the map $X \xrightarrow{q_X^{-1}} QX \xrightarrow{Ur_{FQX}\circ\eta} URFQX$. Thus, by Theorem 1.2.10, the unit of $R\varphi$ is an isomorphism if and only if $Ur_{FQX}\circ\eta$ is a weak equivalence for all X. But this holds if and only if $Ur_{FX}\circ\eta$ is a weak equivalence for all cofibrant X. The proof of this uses the fact that F preserves weak equivalences between cofibrant objects, the fact that U preserves weak equivalences between fibrant objects, and the commutative diagram

$$\begin{tikzcd}
QX \arrow[r, "\eta"] \arrow[d, "q_X"'] & UFQX \arrow[r, "Ur_{FQX}"] \arrow[d, "UFq_X"'] & URFQX \arrow[d, "URFq_X"'] \\
X \arrow[r, "\eta"] & UFX \arrow[r, "Ur_X"] & URFX
\end{tikzcd}$$

Dually, the counit of $R\varphi$ is an isomorphism if and only if $\varepsilon \circ Fq_{UX}$ is a weak equivalence for all fibrant X. $\square$

Proposition 1.3.13 has a couple of useful corollaries.

Corollary 1.3.14. *Suppose (F, U, φ) and (F, U', φ') are Quillen adjunctions from $\mathcal{C}$ to $\mathcal{D}$. Then (F, U, φ) is a Quillen equivalence if and only if (F, U', φ') is so. Dually, if (F', U, φ'') is another Quillen adjunction, then (F, U, φ) is a Quillen equivalence if and only if (F', U, φ'') is so.*

Proof. The adjunction (F, U, φ) is a Quillen equivalence if and only if the derived adjunction $(LF, RU, R\varphi)$ is an adjoint equivalence of categories. But this will be true if and only if LF is an equivalence of categories, for then its adjoint RU is automatically also an equivalence of categories. As is well known and easy to prove, a functor G is an equivalence of categories if and only if it is full, faithful, and essentially surjective on objects (*i.e.* if for every object Y in the codomain, there is an object X in the domain and an isomorphism $GX \cong Y$). The dual statement is similar. $\square$

Because of Corollary 1.3.14, we usually speak of a Quillen equivalence F, omitting the rest of the adjunction.

COROLLARY 1.3.15. *Suppose* $F\colon \mathcal{C} \to \mathcal{D}$ *and* $G\colon \mathcal{D} \to \mathcal{E}$ *are left (resp. right) Quillen functors. Then if two out of three of* F, G, *and* GF *are Quillen equivalences, so is the third.*

PROOF. Recall that $L(GF)$ is naturally isomorphic to $LG \circ LF$ by Theorem 1.3.7. In view of Proposition 1.3.13, it suffices to check that equivalences of categories satisfy the two out of three property. We leave it to the reader to check this well-known fact. □

This corollary suggests that we should think of the category of model categories as itself something like a model category, with the weak equivalences being the Quillen equivalences. We do not know if it is possible to make this intuition rigorous. Note, however, that a Quillen adjunction which is a retract of a Quillen equivalence is itself a Quillen equivalence, as the intuition suggests. The easiest way to check this is to use Theorem 1.4.3.

We now give the most useful criterion for checking when a given Quillen adjunction is a Quillen equivalence. Recall that a functor is said to *reflect* some property of morphisms if, given a morphism f, if Ff has the property so does f.

COROLLARY 1.3.16. *Suppose* $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ *is a Quillen adjunction. The following are equivalent*:

(a) (F, U, φ) *is a Quillen equivalence.*
(b) F *reflects weak equivalences between cofibrant objects and, for every fibrant* Y, *the map* $FQUY \to Y$ *is a weak equivalence.*
(c) U *reflects weak equivalences between fibrant objects and, for every cofibrant* X, *the map* $X \to URFX$ *is a weak equivalence.*

PROOF. Suppose first that F is a Quillen equivalence. We have already seen in Proposition 1.3.13 that the map $X \to URFX$ is a weak equivalence for all cofibrant X and that the map $FQUY \to Y$ is a weak equivalence for all fibrant Y. Now suppose $f \to X \to Y$ is a map between cofibrant objects such that Ff is a weak equivalence. Then, since F preserves weak equivalences between cofibrant objects, FQf is also a weak equivalence. Thus $(LF)f$ is an isomorphism. Since LF is an equivalence of categories, this implies that f is an isomorphism in the homotopy category, and hence a weak equivalence. Thus F reflects weak equivalences between cofibrant objects. The dual argument implies that U reflects weak equivalences between fibrant objects. Thus (a) implies both (b) and (c).

To see that (b) implies (a), we will show that $L(F, U, \varphi)$ is an equivalence of categories. The counit map $(LF)(RU)X \to X$ is an isomorphism by hypothesis. We must show that the unit map $X \to (RU)(LF)X$ is an isomorphism. But $(LF)X \to (LF)(RU)(LF)X$ is inverse to the counit map of $(LF)X$, so is an isomorphism. Since F reflects weak equivalences between cofibrant objects, this implies that $QX \to QURFQX$ is a weak equivalence for all X. Since Q reflects all weak equivalencs, this implies that $X \to URFQX = (RU)(LF)X$ is a weak equivalence, as required. A similar proof shows that (c) implies (a). □

As an example, consider the following proposition.

PROPOSITION 1.3.17. *Suppose* $F\colon \mathcal{C} \to \mathcal{D}$ *is a Quillen equivalence, and suppose in addition that the terminal object* $*$ *of* $\mathcal{C}$ *is cofibrant and that* F *preserves the terminal object. Then* $F_*\colon \mathcal{C}_* \to \mathcal{D}_*$ *is a Quillen equivalence.*

PROOF. Let U denote the right adjoint of F, and recall that $U_*(X,v) = (Ux, Uv)$. Since U reflects weak equivalences between fibrant objects, U_* does as well. We must check that, if (X,v) is cofibrant, the map $(X,v) \to U_*RF_*(X,v)$ is a weak equivalence. Recall that (X,v) is cofibrant if and only if $v\colon * \to X$ is a cofibration: since $*$ is cofibrant, this implies in particular that X is cofibrant. Let V denote the functor that forgets the basepoint. Then we must show that the map $X \to VU_*RF_*X = UVRF_*(X,v)$ is a weak equivalence. The fibrant replacement functor R on $\mathcal{D}_*$ is defined to be the fibrant replacement functor on $\mathcal{D}$ together with the induced basepoint (see Proposition 1.1.8), so $UVRF_*(X,v) = URVF_*X$. Since $F(*) \cong *$, there is a natural isomorphism $F_*(X,v) \cong (FX, Fv)$. The result follows. □

This section has mostly been about when the total derived functor of a Quillen functor is an equivalence of categories. We could also ask when the total derived natural transformation is a natural isomorphism.

LEMMA 1.3.18. *Suppose $\tau\colon F \to G$ is a natural transformation between left (resp. right) Quillen functors. Then $L\tau$ ($R\tau$) is a natural isomorphism if and only if τ_X is a weak equivalence for all cofibrant (resp. fibrant) X.*

PROOF. Assume F and G are left Quillen functors. Then $(L\tau)_X = \tau_{QX}$, so $L\tau$ is a natural isomorphism if and only if τ_{QX} is a weak equivalence for all X. Since F and G preserve weak equivalences between cofibrant objects, this is true if and only if τ_X is a weak equivalence for all cofibrant X. We leave the dual statement to the reader. □

We think of natural transformations such that τ_X is a weak equivalence for all cofibrant X as "2-weak equivalences". They also satisfy the appropriate two out of three property and are closed under retracts.

1.4. 2-categories and pseudo-2-functors

In this section, we give a summary of the basic language of 2-categories. Since many of the theorems in this book assert that a certain correspondence is a pseudo-2-functor between 2-categories, the language in this section is used throughout the book. However, it is only language. The reader who is uninterested in the language can skip this section and refer to it as needed. References for 2-categories include [**KS74**] and [**Gra74**].

A 2-category will have objects, morphisms, and 2-morphisms, or morphisms between morphisms. The basic example of a 2-category is the 2-category of categories. An object is a category, a morphism is a functor, and a 2-morphism is a natural transformation. We use this example to discover and motivate the general definition of a 2-category.

We first make a technical point. We follow the usual convention that the objects of a category may form a proper class, but the morphisms between two given objects form a set. This means that, in order to consider the 2-category of categories, we must allow the objects of a 2-category to form a "superclass", and the morphisms between any two objects to form a class. Here a superclass is to a class as a class is to a set. One might think that we could require the 2-morphisms between any two morphisms to form a set, but there is an example of two functors such that the natural transformations between them form a proper class. Hence we must allow the 2-morphisms between any two morphisms to form a proper class as well. One could avoid all this by only considering the 2-category of all small categories, but that would eliminate all model categories from consideration, so we do not do so.

Now, in the 2-category of categories, we can certainly compose functors. There are two different ways to compose natural transformations, however. Given two natural transformations $\sigma\colon F \to G$ and $\tau\colon G \to H$, we can form the vertical composition $\tau \circ \sigma\colon F \to H$, where $(\tau \circ \sigma)_X = \tau_X \circ \sigma_X$. The vertical composition is associative, and there is an identity natural transformation 1_F for each functor F. The vertical composition makes functors from $\mathcal{C}$ to $\mathcal{D}$ into a category.

On the other hand, given natural transformations $\sigma\colon F \to F'\colon \mathcal{C} \to \mathcal{D}$ and $\tau\colon G \to G'\colon \mathcal{D} \to \mathcal{E}$, we can form the horizontal composition $\tau * \sigma\colon G \circ F \to G' \circ F'\colon \mathcal{C} \to \mathcal{E}$ defined in Definition 1.3.8. Horizontal composition is associative and allows us to form a category whose objects are categories and whose morphisms are natural transformations. The identity at $\mathcal{C}$ is the identity natural transformation of the identity functor.

With this example in mind, we can now give the rather long definition of a 2-category.

Definition 1.4.1. A 2-*category* $\mathbf{K}$ consists of a superclass $\mathbf{K}_0$ called the objects of $\mathbf{K}$, a superclass $\mathbf{K}_1$ called the morphisms of $\mathbf{K}$, a superclass $\mathbf{K}_2$ called the 2-morphisms of $\mathbf{K}$, two maps $\mathbf{K}_0 \to \mathbf{K}_1$ and $\mathbf{K}_1 \to \mathbf{K}_2$ each called the identity and denoted 1, two maps $\mathbf{K}_1 \to \mathbf{K}_0$ and $\mathbf{K}_2 \to \mathbf{K}_1$ called domains and denoted d, two maps $\mathbf{K}_1 \to \mathbf{K}_0$ and $\mathbf{K}_2 \to \mathbf{K}_1$ called codomains and denoted c, and three composition maps $\circ\colon \mathbf{K}_1 \times_{\mathbf{K}_0} \mathbf{K}_1 \to \mathbf{K}_1$, vertical composition $\circ\colon \mathbf{K}_2 \times_{\mathbf{K}_1} \mathbf{K}_2 \to \mathbf{K}_2$, and horizontal composition $*\colon \mathbf{K}_2 \times_{\mathbf{K}_0} \mathbf{K}_2 \to \mathbf{K}_2$ satisfying the properties below. The pullbacks used to define the compositions $\circ$ are taken over the domain and codomain maps, and the pullback used to define $*$ is taken over the composition d^2 of the domain maps and the composition c^2 of the codomain maps. The properties these structures must satisfy are the following, where we use roman capital letters to denote objects of $\mathbf{K}$, roman lower-case letters to denote morphisms of $\mathbf{K}$, and lower-case greek letters to denote 2-morphisms of $\mathbf{K}$.

1. We have $d^2 = dc$ and $c^2 = cd$ as maps $\mathbf{K}_2 \to \mathbf{K}_0$.
2. We have $d1_A = c1_A = A$ for all objects A, and $d1_f = c1_f = f$ for all morphisms f.
3. If $dg = cf$ so the composition is defined, then $d(g \circ f) = df$ and $c(g \circ f) = cg$. Similarly, if $d\tau = c\sigma$ so the vertical composition is defined, then $d(\tau \circ \sigma) = d\sigma$, and $c(\tau \circ \sigma) = c\tau$. Also, if $d^2\tau = c^2\sigma$ so the horizontal composition is defined, we have $d(\tau * \sigma) = d\tau \circ d\sigma$ and $c(\tau * \sigma) = c\tau \circ c\sigma$.
4. Composition is unital. That is, we have $1_{cf} \circ f = f = f \circ 1_{df}$ for all 1-morphisms f. Similarly, we have $1_{c\tau} \circ \tau = \tau = \tau \circ 1_{d\tau}$ for all 2-morphisms τ. And we have $1_{1_{cc\tau}} * \tau = \tau * 1_{1_{dd\tau}}$ for all 2-morphisms τ.
5. Composition is associative. That is, if $dg = cf$ and $dh = cg$, then $h \circ (g \circ f) = (h \circ g) \circ f$. Similarly, if $d\sigma = c\rho$ and $d\tau = c\sigma$, then $\tau \circ (\sigma \circ \rho) = (\tau \circ \sigma) \circ \rho$. Finally, if $d^2\sigma = c^2\rho$ and $d^2\tau = c^2\sigma$, then $\tau * (\sigma * \rho) = (\tau * \sigma) * \rho$.
6. The two different compositions of 2-morphisms are compatible. That is, if we have four 2-morphisms τ', σ', τ and σ such that $d\tau = c\sigma$, $d\tau' = c\sigma'$, and $d^2\tau' = c^2\tau$, then $(\tau' \circ \sigma') * (\tau \circ \sigma) = (\tau' * \tau) \circ (\sigma' * \sigma)$.
7. Given two objects A and B, the collection of all morphisms f with $df = A$ and $cf = B$ is a class. Similarly, given two morphisms f and g, the collection of all 2-morphisms τ such that $d\tau = f$ and $c\tau = g$ is a class.

Of course, if f is a morphism in a 2-category such that $df = A$ and $cf = B$, we write $f\colon A \to B$. And if τ is a 2-morphism such that $d\tau = f$ and $c\tau = g$, we write $\tau\colon f \to g$. An invertible 2-morphism is called a 2-isomorphism, and an invertible morphism is called an isomorphism. We have an obvious notion of a 2-*functor* between 2-categories as well, which is simply a correspondence that preserves identities, domains, codomains, and all compositions.

We leave to the reader the easy check that categories, functors, and natural transformations do form a 2-category. An even more important example for us is the 2-category of categories, adjunctions, and natural transformations. Here an adjunction is a triple (F, U, φ) as in Definition 1.3.1, and a 2-morphism from (F, U, φ) to (F', U', φ') is just a natural transformation $F \to F'$. We leave it the reader to verify that this does form a 2-category, which we denote by $\mathbf{Cat}_{ad}$. Similarly, we get a 2-category of model categories, Quillen adjunctions, and natural transformations, which we denote by $\mathbf{Mod}$.

Examples of 2-functors include the obvious forgetful 2-functors from $\mathbf{Mod}$ to $\mathbf{Cat}_{ad}$ and from $\mathbf{Cat}_{ad}$ to the 2-category of categories and functors. A more interesting example is the contravariant duality 2-functor on $\mathbf{Cat}_{ad}$ and $\mathbf{Mod}$. Here we define $D\mathcal{C} = \mathcal{C}^{\mathrm{op}}$, given the opposite model structure if we are in $\mathbf{Mod}$. Given an adjunction (F, U, φ), we define $D(F, U, \varphi) = (U, F, \varphi^{-1})$. Given adjunctions (F, U, φ) and (F', U', φ') and a natural transformation $\tau\colon F \to F'$, we define $D\tau\colon U' \to U$ as the composite

$$U'X \xrightarrow{\eta_{U'X}} UFU'X \xrightarrow{U\tau_{U'X}} UF'U'X \xrightarrow{U\varepsilon'_X} UX.$$

where η is the unit of φ and ε' is the counit of φ'. Note that τ and $D\tau$ are compatible, in the sense that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{D}(F'A, B) \arrow[r, "\tau_A^*"] \arrow[d, "\varphi'"'] & \mathcal{D}(FA, B) \arrow[d, "\varphi"'] \\
\mathcal{C}(A, U'B) \arrow[r, "((D\tau)_B)_*"] & \mathcal{C}(A, UB)
\end{tikzcd}$$

Furthermore, the commutativity of this diagram characterizes $D\tau$. This defines contravariant 2-functors $D\colon \mathbf{Cat}_{ad} \to \mathbf{Cat}_{ad}$ and $D\colon \mathbf{Mod} \to \mathbf{Mod}$ such that D^2 is the identity 2-functor. Note in particular that τ is a natural isomorphism if and only if $D\tau$ is.

One of the guiding principles of category theory is that, in a category, the natural equivalence relation on objects is not equality, but isomorphism. Similarly, in a 2-category, the natural equivalence relation on morphisms is not equality, but isomorphism. That is, two morphisms $f, g\colon A \to B$ are isomorphic if there is an invertible 2-morphism $\tau\colon f \to g$. This induces an equivalence relation on objects, called equivalence. Two objects A and B are equivalent if there are morphisms $f\colon A \to B$ and $g\colon B \to A$ such that the compositions $g \circ f$ and $f \circ g$ are isomorphic (not equal) to the respective identities. Of course, in the 2-category of categories, this is just the usual notion of equivalence of categories.

If we apply this principle to functors, we are led to the following definition.

Definition 1.4.2. Suppose $\mathbf{K}$ and $\mathbf{L}$ are 2-categories. A *pseudo-2-functor* $F\colon \mathbf{K} \to \mathbf{L}$ is three maps of superclasses $\mathbf{K}_0 \to \mathbf{L}_0$, $\mathbf{K}_1 \to \mathbf{L}_1$, and $\mathbf{K}_2 \to \mathbf{L}_2$ all denoted F, together with 2-isomorphisms $\alpha\colon F(1_A) \to 1_{FA}$ for all objects A of

$\mathbf{K}$ and 2-isomorphisms $m_{gf}\colon Fg \circ Ff \longrightarrow F(g \circ f)$ for all (ordered) pairs (g, f) of morphisms of $\mathbf{K}$ such that gf makes sense, satisfying the following properties.

1. F preserves domains and codomains. That is, $dF(f) = F(df)$ and $cF(f) = F(cf)$ for all morphisms f of $\mathbf{K}$. Similarly, $dF(\tau) = F(d\tau)$ and $cF(\tau) = F(c\tau)$ for all 2-morphisms of $\mathbf{K}$.
2. F is functorial with respect to vertical composition. That is, $F(1_f) = 1_{Ff}$ for all morphisms f of $\mathbf{K}$, and $F(\tau \circ \sigma) = F(\tau) \circ F(\sigma)$ for all ordered pairs (τ, σ) of 2-morphisms of $\mathbf{K}$ such that $\tau \circ \sigma$ makes sense.
3. An associativity coherence diagram commutes. That is, for all ordered triples (h, g, f) of morphisms of $\mathbf{K}$ such that $h \circ g \circ f$ makes sense, the following diagram commutes.
$$\begin{tikzcd}
(Fh \circ Fg) \circ Ff \arrow[r, "m_{hg} * 1_{Ff}"] \arrow[d, equal] & F(h \circ g) \circ Ff \arrow[r, "m_{(h \circ g) f}"] & F((h \circ g) \circ f) \arrow[d, equal] \\
Fh \circ (Fg \circ Ff) \arrow[r, "1_{Fh} * m_{gf}"] & Fh \circ F(g \circ f) \arrow[r, "m_{h(g \circ f)}"] & F(h \circ (g \circ f))
\end{tikzcd}$$
4. A left unit coherence diagram commutes. That is, for all morphisms f of $\mathbf{K}$, the following diagram commutes
$$\begin{tikzcd}
F1_{cf} \circ Ff \arrow[r, "m"] \arrow[d, "\alpha * 1_{Ff}"'] & F(1_{cf} \circ f) \arrow[d, equal] \\
1_{F(cf)} \circ Ff \arrow[r, equal] & Ff
\end{tikzcd}$$
5. A right unit coherence diagram commutes. That is, for all morphisms f of $\mathbf{K}$, the following diagram commutes
$$\begin{tikzcd}
Ff \circ F1_{df} \arrow[r, "m"] \arrow[d, "1_{Ff} * \alpha"'] & F(f \circ 1_{df}) \arrow[d, equal] \\
Ff \circ 1_{F(df)} \arrow[r, equal] & Ff
\end{tikzcd}$$
6. m is natural with respect to horizontal composition. That is, if $\sigma\colon f \longrightarrow f'$ and $\tau\colon g \longrightarrow g'$ are 2-morphisms of $\mathbf{K}$ such that $g \circ f$ makes sense, then the following diagram commutes.
$$\begin{tikzcd}
Fg \circ Ff \arrow[r, "m_{gf}"] \arrow[d, "F\tau * F\sigma"'] & F(g \circ f) \arrow[d, "F(\tau * \sigma)"'] \\
Fg' \circ Ff' \arrow[r, "m_{g'f'}"] & F(g' \circ f')
\end{tikzcd}$$

Note that pseudo-2-functors need not preserve isomorphisms, but they do preserve equivalences. Similarly, if we apply a pseudo-2-functor to a commutative diagram of morphisms, the resulting diagram need no longer be commutative; but it is commutative up to natural isomorphism.

Then we can restate Theorem 1.3.7 and Lemma 1.3.9 in the following way.

THEOREM 1.4.3. *The homotopy category, derived adjunction, and derived natural transformation define a pseudo-2-functor* $\mathbf{Ho}\colon \mathbf{Mod} \longrightarrow \mathbf{Cat}_{ad}$ *from the* 2*-category of model categories and Quillen adjunctions to the* 2*-category of categories and adjunctions. Furthermore,* $\mathbf{Ho}$ *commutes with the duality* 2*-functor, in the sense that* $D \circ \mathbf{Ho} = \mathbf{Ho} \circ D$.

We leave the proof of the statement about duality to the reader. In particular, one must check that $D(L\tau) = R(D\tau)$ for a natural transformation τ of left Quillen functors. To check this, one uses the diagram that characterizes $D(L\tau)$.

COROLLARY 1.4.4. (a) *A Quillen adjunction (F, U, φ) is a Quillen equivalence if and only if $D(F, U, \varphi)$ is so.*

(b) *A natural transformation $\tau\colon F \to F'$ between left Quillen functors is a weak equivalence for all cofibrant X if and only if $(D\tau)_Y$ is a weak equivalence for all fibrant Y.*

PROOF. For part (a), use the characterization of Quillen equivalences as Quillen adjunctions (F, U, φ) such that $\mathbf{Ho}(F, U, \varphi)$ is an equivalence of categories, and Theorem 1.4.3. For part (b), use the analogous characterization in Lemma 1.3.18. $\square$

Another example of a pseudo-2-functor on **Mod** is the correspondence that takes a model category $\mathcal{C}$ to $\mathcal{C}_*$ (Proposition 1.1.8), and a Quillen adjunction (F, U, φ) to the Quillen adjunction (F_*, U_*, φ_*) (Proposition 1.3.5). This is not a 2-functor due to the choice of pushouts necessary to define F_*, but it is a pseudo-2-functor since any two choices of pushout are uniquely isomorphic.

CHAPTER 2

Examples

In this chapter we discuss four different examples of model categories: modules over a Frobenius ring, chain complexes of modules over a ring, topological spaces, and cochain complexes of comodules over a Hopf algebra. We defer the more involved discussion of the central example of simplicial sets to the next chapter. None of these examples is necessary for the theory that appears in later chapters, except that we need the model structure on (compactly generated) topological spaces in order to prove that simplicial sets form a model category. However, we will frequently use these categories as examples.

We use the method of cofibrantly generated model categories, which we discuss in Section 2.1. We have chosen to use arbitrary transfinite compositions in our study of cofibrantly generated model categories, though in fact we could get away with countable compositions in all the examples we consider. The reason we have done so is primarily one of taste: it seems artificial to restrict oneself to categories where countable composition will suffice. Furthermore, the localization process discussed in [**Hir97**] will almost always require transfinite compositions. This is already clear in [**Bou79**]. In practice, using transfinite compositions just means replacing induction arguments with arguments using Zorn's lemma. This is a simple enough switch that we feel no qualms in asking the reader to make it.

Our treatment of cofibrantly generated model categories is based on [**Hir97**] and [**DHK**]. The ideas behind cofibrantly generated model categories are already apparent in [**Qui67**], and were expanded in [**GZ67**]. We will use the notion of cofibrantly generated model categories throughout the rest of the book.

The simplest non-trivial example of a model category is probably the category of modules over a Frobenius ring. We discuss that example first. It does not seem to have been described before. Its homotopy category, in case the Frobenius ring is the group ring of a finite group, has been studied a great deal recently by group representation theorists. See for example [**BCR96**] and [**Ric**]. Next, we consider chain complexes of modules over an arbitrary ring, in Section 2.3. Most discussions of this example, as in [**Qui67**] and in [**DS95**], restrict attention to chain complexes concentrated in nonnegative degrees. This again seems like an artificial restriction, and we do not make it.

We then discuss the example of topological spaces in Section 2.4. The major difference between our treatment and the ones in [**Qui67**], [**DS95**], and [**GJ97**] is that we give complete proofs of results that are generally left to the reader. We find these results quite tricky to prove, so we hope that the reader will find our proofs valuable.

We conclude the chapter with Section 2.5, where we discuss chain complexes of comodules over a Hopf algebra. The homotopy category of this model category is the stable homotopy category discussed in [**HPS97**, Section 9.5]. So far as the author is aware, this model category has never been studied before. The main difference between it and chain complexes of modules over a ring is that the weak equivalences are homotopy isomorphisms, not homology isomorphisms.

2.1. Cofibrantly generated model categories

It tends to be quite difficult to prove that a category admits a model structure. The axioms are always hard to check. This section is devoted to minimizing the things we need to check. We have to do this in some form before constructing any interesting examples, unfortunately. This section will require the reader to fight through some thickets of abstraction. It may help the less experienced reader to assume that all ordinal and cardinal numbers are either finite or $\infty = \aleph_0$, the first infinite ordinal.

The author learned the results in this section from [**DHK**] and [**Hir97**], which also contain other results about this material which the reader may find useful.

The main tool in this section is the small object argument, which tells us how to construct functorial factorizations in categories. In order to develop this tool, we will need some results about infinite compositions in categories. This in turn will require some basic set theory, which we now begin with.

2.1.1. Ordinals, cardinals, and transfinite compositions. We require of the reader some basic familiarity with ordinals, cardinals, and transfinite induction. Recall that an ordinal is the well-ordered set of all smaller ordinals. Every ordinal λ has a successor ordinal $\lambda + 1$. We will often think of an ordinal as a category where there is a unique map from α to β if and only if $\alpha \leq \beta$.

We can use ordinals to define the notion of transfinite composition.

Definition 2.1.1. Suppose $\mathcal{C}$ is a category with all small colimits, and λ is an ordinal. A *λ-sequence* in $\mathcal{C}$ is a colimit-preserving functor $X\colon \lambda \to \mathcal{C}$, commonly written as

$$X_0 \to X_1 \to \dots \to X_\beta \to \dots.$$

Since X preserves colimits, for all limit ordinals $\gamma < \lambda$, the induced map

$$\operatorname{colim}_{\beta<\gamma} X_\beta \to X_\gamma$$

is an isomorphism. We refer to the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as the *composition* of the λ-sequence, though actually the composition is not unique, but only unique up to isomorphism under X, since the colimit is not unique. If $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$ and every map $X_\beta \to X_{\beta+1}$ for $\beta + 1 < \lambda$ is in $\mathcal{D}$, we refer to the composition $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ as a *transfinite composition* of maps of $\mathcal{D}$.

Of course, if $\lambda = \aleph_0$, a λ-sequence is just an ordinary sequence.

Our next goal is to define what it means for an object to be small. Essentially an object is small if a map from it to a long enough composition factors through some stage in the composition. To make this precise, we need to remind the reader of some facts about cardinals. Given a set A, define the cardinality of A, $|A|$, to be the smallest ordinal for which there is a bijection $|A| \to A$. A cardinal is an ordinal κ such that $\kappa = |\kappa|$.

Definition 2.1.2. Let γ be a cardinal. An ordinal α is *γ-filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \gamma$, then $\sup A < \alpha$.

An ordinal λ is γ-filtered if and only if the *cofinality* of λ is greater than γ [**Jec78**, p. 26]. Note that, if γ is finite, a γ-filtered ordinal is just a limit ordinal. Given an infinite cardinal γ, the smallest γ-filtered ordinal is the first cardinal γ_1 larger than γ. In general, any successor cardinal larger than an infinite cardinal γ is γ-filtered, where a successor cardinal is the first cardinal larger than some other cardinal. But there are also non-cardinal ordinals which are γ-filtered, such as $2\gamma_1$.

Now we define a small object.

Definition 2.1.3. Suppose $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$ and κ is a cardinal. We say that A is *κ-small relative to* $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \dots \to X_\beta \to \dots$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$ for $\beta+1<\lambda$, the map of sets

$$\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some κ. We say that A is *small* if it is small relative to $\mathcal{C}$ itself.

Note that, if $\kappa < \kappa'$ and A is κ-small relative to $\mathcal{D}$, then A is also κ'-small relative to $\mathcal{D}$, since every κ'-filtered ordinal is κ-filtered.

The simplest case of Definition 2.1.3 is when the cardinal κ is finite.

Definition 2.1.4. Suppose $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, and A is an object of $\mathcal{C}$. We say that A is *finite relative to* $\mathcal{D}$ if A is κ-small relative to $\mathcal{D}$ for a finite cardinal κ. We say A is *finite* if it is finite relative to $\mathcal{C}$ itself. In this case, maps from A commute with colimits of arbitrary λ-sequences, as long as λ is a limit ordinal.

Example 2.1.5. Every set is small. Indeed, if A is a set, we claim that A is $|A|$-small. To see this, suppose λ is a $|A|$-filtered ordinal, and X is a λ-sequence of sets. Given a map $A \xrightarrow{f} \operatorname{colim}_{\beta<\lambda} X_\beta$, we find for each $a \in A$ an index β_a such that $f(a)$ is in the image of X_{β_a}. Then we let γ be the supremum of the β_a. Because λ is $|A|$-filtered, $\gamma < \lambda$, and the map f will factor through a map $g\colon A \to X_\gamma$ as required. A similar argument shows that if two maps $A \to X_\beta$ and $A \to X_\gamma$ are equal in the colimit, they must be equal in some stage of the colimit. Note that a set is finite in the category of sets if and only if it is a finite set, whence the terminology.

Example 2.1.6. If R is a ring, every R-module is small. Indeed, suppose A is an R-module. Let $\kappa = |A|(|A|+|R|)$. Let λ be a κ-filtered ordinal, and let X be a λ-sequence of R-modules. By Example 2.1.5, the map $\operatorname{colim} R\text{-mod}(A, X_\beta) \to R\text{-mod}(A, \operatorname{colim} X_\beta)$ is injective, and any map $f\colon A \to \operatorname{colim} X_\beta$ factors as a map *of sets* through a map $g\colon A \to X_\alpha$ for some $\alpha < \lambda$. The map g may not be an R-module map, of course. Nevertheless, for each pair $(x,y) \in A \times A$, there is a $\beta_{(x,y)}$ such that $g(x+y) = g(x) + g(y)$ in $X_{\beta_{(x,y)}}$. Similarly, for each pair $(r,x) \in R \times X$, there is a $\beta_{(r,x)}$ such that $g(rx) = rg(x)$ in $X_{\beta_{(r,x)}}$. Let γ be the supremum of all this ordinals. Then $\gamma < \lambda$, and the map g defines a factorization of f through an R-module map $A \to X_\gamma$, as required. Note that finitely presented R-modules are finite.

2.1.2. Relative I-cell complexes and the small object argument. The main advantage of knowing that certain objects are small is that such knowledge allows us to construct functorial factorizations. We begin with some preliminary definitions.

Definition 2.1.7. Let I be a class of maps in a category $\mathcal{C}$.

1. A map is *I-injective* if it has the right lifting property with respect to every map in I. The class of I-injective maps is denoted I-inj.
2. A map is *I-projective* if it has the left lifting property with respect to every map in I. The class of I-projective maps is denoted I-proj.
3. A map is an *I-cofibration* if it has the left lifting property with respect to every I-injective map. The class of I-cofibrations is the class (I-inj)-proj and is denoted I-cof.
4. A map is an *I-fibration* if it has the right lifting property with respect to every I-projective map. The class of I-fibrations is the class (I-proj)-inj and is denoted I-fib.

If $\mathcal{C}$ is a model category, and I is the class of cofibrations, then I-inj is the class of trivial fibrations, and I-cof $= I$. Dually, if I is the class of fibrations, then I-proj is the class of trivial cofibrations, and I-fib $= I$.

Note that $I \subseteq I$-cof and $I \subseteq I$-fib. Also, we have (I-cof)-inj $=$ I-inj and (I-fib)-proj $=$ I-proj. Furthermore, if $I \subseteq J$, then I-inj $\supseteq$ J-inj and I-proj $\supseteq$ J-proj. Thus I-cof $\subseteq$ J-cof and I-fib $\subseteq$ J-fib.

The following lemma is often useful.

Lemma 2.1.8. *Suppose $(F, U\varphi)\colon \mathcal{C} \to \mathcal{D}$ is an adjunction, I is a class of maps in $\mathcal{C}$, and J is a class of maps in $\mathcal{D}$. Then*

(a) *$U(FI$-inj$) \subseteq I$-inj.*
(b) *$F(I$-cof$) \subseteq FI$-cof.*
(c) *$F(UJ$-proj$) \subseteq J$-proj.*
(d) *$U(J$-fib$) \subseteq UJ$-fib.*

Proof. For part (a), suppose $g \in FI$-inj, and $f \in I$. Then g has the right lifting property with respect to Ff, and so, by adjointness, Ug has the right lifting property with respect to f. Thus $Ug \in I$-inj, as required. For part (b), suppose $f \in I$-cof and $g \in FI$-inj. Then, by part (a), $Ug \in I$-inj, and so f has the left lifting property with respect to Ug. Adjointness implies that Ff has the left lifting property with respect to g, and so $Ff \in (FI$-inj)-proj $= FI$-cof. Parts (c) and (d) are dual. $\square$

In general, the maps of I-cof may have little to do with I. We single out a certain subclass of I-cof.

Definition 2.1.9. Let I be a set of maps in a category $\mathcal{C}$ containing all small colimits. A *relative I-cell complex* is a transfinite composition of pushouts of elements of I. That is, if $f\colon A \to B$ is a relative I-cell complex, then there is an ordinal λ and a λ-sequence $X\colon \lambda \to \mathcal{C}$ such that f is the composition of X and such that, for each β such that $\beta + 1 < \lambda$, there is a pushout square

$$\begin{array}{ccc} C_\beta & \longrightarrow & X_\beta \\ {\scriptstyle g_\beta}\big\downarrow & & \big\downarrow \\ D_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

such that $g_\beta \in I$. We denote the collection of relative I-cell complexes by I-cell. We say that $A \in \mathcal{C}$ is an *I-cell complex* if the map $0 \longrightarrow A$ is a relative I-cell complex.

Note that the identity map at A is the transfinite composition of the trivial 1-sequence A, so identity maps are relative I-cell complexes. In fact, if $f\colon A \longrightarrow B$ is an isomorphism, then f is also (another choice for) the composition of the 1-sequence A, so f is a relative I-cell complex.

LEMMA 2.1.10. *Suppose I is a class of maps in a category $\mathcal{C}$ with all small colimits. Then I-cell $\subseteq I$-cof.*

PROOF. It suffices to show that I-cof is closed under transfinite compositions and pushouts. Since I-cof is defined by a lifting property, this is straightforward. □

We need some basic results about relative I-cell complexes. We begin with the following technical lemma.

LEMMA 2.1.11. *Suppose λ is an ordinal and $X\colon \lambda \longrightarrow \mathcal{C}$ is a λ-sequence such that each map $X_\beta \longrightarrow X_{\beta+1}$ is either a pushout of a map of I or an isomorphism. Then the transfinite composition of X is a relative I-cell complex.*

PROOF. Define an equivalence relation $\sim$ on λ as follows. If $\alpha \le \beta$, define $\alpha \sim \beta$ if, for all γ such that $\alpha \le \gamma < \beta$, the map $X_\gamma \longrightarrow X_{\gamma+1}$ is an isomorphism. Then each equivalence class $[\alpha]$ under $\sim$ is a closed interval $[\alpha', \alpha'']$ of λ, and one can easily check that if $\alpha \le \beta$ and $\alpha \sim \beta$ then the map $X_\alpha \longrightarrow X_\beta$ is an isomorphism. The set of equivalence classes is itself a well-ordered set, and so is isomorphic to a unique ordinal μ. The functor X descends to a functor $Y\colon \mu \longrightarrow \mathcal{C}$, where $Y_{[\alpha]} = X_{\alpha'}$. Each map $Y_\beta \longrightarrow Y_{\beta+1}$ is a pushout of a map of I. Once can check that Y is a μ-sequence, since if $[\beta]$ is a limit ordinal of μ, then β' must be a limit ordinal of λ. Since the transfinite composition of Y is isomorphic to the transfinite composition of X, the proof is complete. □

LEMMA 2.1.12. *Suppose $\mathcal{C}$ is a category with all small colimits, and I is a set of maps of $\mathcal{C}$. Then I-cell is closed under transfinite compositions.*

PROOF. Suppose $X\colon \lambda \longrightarrow \mathcal{C}$ is a λ-sequence of relative I-cell complexes, so that each map $X_\beta \longrightarrow X_{\beta+1}$ is a relative I-cell complex. Then $X_\beta \longrightarrow X_{\beta+1}$ is the composition of a λ-sequence $Y\colon \gamma_\beta \longrightarrow \mathcal{C}$ of pushouts of maps of I. Consider the set S of all pairs of ordinals (β, γ) such that $\beta < \lambda$ and $\gamma < \gamma_\beta$. Put a total order on S by defining $(\beta, \gamma) < (\beta', \gamma')$ if $\beta < \beta'$ or if $\beta = \beta'$ and $\gamma < \gamma'$. Then S becomes a well-ordered set, so is isomorphic to a unique ordinal μ. We therefore get a functor $Z\colon \mu \longrightarrow \mathcal{C}$, which one can readily verify is a μ-sequence. Each map $Z_\alpha \longrightarrow Z_{\alpha+1}$ is either one of the maps $Y_\gamma \longrightarrow Y_{\gamma+1}$ or else is an isomorphism. Since a composition of X is also a composition of Z, Lemma 2.1.11 implies that a composition of X is a relative I-cell complex. □

Another useful property of relative I-cell complexes is that we can take the pushout over coproducts of maps of I rather than just maps of I.

LEMMA 2.1.13. *Suppose $\mathcal{C}$ is a category with all small colimits, and I is a set of maps of $\mathcal{C}$. Then any pushout of coproducts of maps of I is in I-cell.*

Proof. Suppose K is a set and $g_k\colon C_k \to D_k$ is a map of I for all k in K. Suppose f is the pushout in the diagram

$$\begin{tikzcd} \coprod C_k \arrow[r] \arrow[d, "\coprod g_k"'] & X \arrow[d, "f"] \\ \coprod D_k \arrow[r] & Y \end{tikzcd}$$

We must show that f is a relative I-cell complex. To do so, we may as well assume that K is an ordinal λ, since every set is isomorphic to an ordinal. We then form a λ-sequence by letting $X_0 = X$, by letting $X_{\beta+1}$ be the pushout $X_\beta \amalg_{C_\beta} D_\beta$ over g_β, and by letting $X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha$ for limit ordinals β. One can easily check that the transfinite composition $X \to X_\lambda$ of this λ-sequence is isomorphic to the map f, and hence that f is a relative I-cell complex. □

It is also easily checked that a pushout of a relative I-cell complex is a relative I-cell complex, though we do not need this result.

The reason for considering the theory of transfinite compositions and relative I-cell complexes is the small object argument, which we now present.

Theorem 2.1.14 (The small object argument). *Suppose $\mathcal{C}$ is a category containing all small colimits, and I is a set of maps in $\mathcal{C}$. Suppose the domains of the maps of I are small relative to I-cell. Then there is a functorial factorization (γ, δ) on $\mathcal{C}$ such that, for all morphisms f in $\mathcal{C}$, the map $\gamma(f)$ is in I-cell and the map $\delta(f)$ is in I-inj.*

Proof. Choose a cardinal κ such that every domain of I is κ-small relative to I-cell, and let λ be a κ-filtered ordinal. Given $f\colon X \to Y$, we will define a functorial λ-sequence $Z^f\colon \lambda \to \mathcal{C}$ such that $Z^f_0 = X$ and a natural transformation $Z^f \xrightarrow{\rho^f} Y$ factoring f. Each map $Z^f_\beta \to Z^f_{\beta+1}$ will be a pushout of a coproduct of maps of I. Then we will define γf to be the composition of Z^f, and δf to be the map $E_f = \operatorname{colim} Z^f \to Y$ induced by ρ^f. Of course, γ and δ will then also depend on a choice of colimit functor as well. It follows from Lemma 2.1.13 and Lemma 2.1.12 that γf is a relative I-cell complex.

We will define Z^f and $\rho^f\colon Z^f \to Y$ by transfinite induction, beginning with $Z^f_0 = X$ and $\rho^f_0 = f$. If we have defined Z^f_α and ρ^f_α for all $\alpha < \beta$ for some limit ordinal β, define $Z^f_\beta = \operatorname{colim}_{\alpha<\beta} Z^f_\alpha$, and define ρ^f_β to be the map induced by the ρ^f_α. Having defined Z^f_β and ρ^f_β, we define $Z^f_{\beta+1}$ and $\rho^f_{\beta+1}$ as follows. Let S be the set of all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & Z^f_\beta \arrow[d, "\rho^f_\beta"] \\ B \arrow[r] & Y \end{tikzcd}$$

where $g \in I$. For $s \in S$, let $g_s\colon A_s \to B_s$ denote the corresponding map of I. Define $Z^f_{\beta+1}$ to be the pushout in the diagram

$$\begin{tikzcd} \coprod_{s\in S} A_s \arrow[r] \arrow[d, "\coprod g_s"'] & Z^f_\beta \arrow[d] \\ \coprod_{s\in S} B_s \arrow[r] & Z^f_{\beta+1} \end{tikzcd}$$

Define $\rho^f_{\beta+1}$ to be the map induced by ρ^f_β.

It remains to show that $\delta f = \operatorname{colim} \rho^f_\beta \colon E_f = \operatorname{colim} Z^f_\beta \longrightarrow Y$ has the right lifting property with respect to I. To see this, suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & E_f \arrow[d, "\delta f"] \\ B \arrow[r, "k"] & Y \end{tikzcd}$$

where g is a map of I. Since the domains of the maps of I are κ-small relative to I-cell, there is a $\beta < \lambda$ such that h is the composite $A \xrightarrow{h_\beta} Z^f_\beta \longrightarrow E_f$. By construction, there is a map $B \xrightarrow{k_\beta} Z^f_{\beta+1}$ such that $k_\beta g = i h_\beta$ and $k = \rho_{\beta+1} k_\beta$, where i is the map $Z^f_\beta \longrightarrow Z^f_{\beta+1}$. The composition $B \xrightarrow{k_\beta} Z^f_{\beta+1} \longrightarrow E_f$ is the required lift in our diagram. □

Corollary 2.1.15. *Suppose I is a set of maps in a category $\mathcal{C}$ with all small colimits. Suppose as well that the domains of I are small relative to I-cell. Then given $f \colon A \longrightarrow B$ in I-cof, there is a $g \colon A \longrightarrow C$ in I-cell such that f is a retract of g by a map which fixes A.*

Proof. The small object argument gives us a factorization $f = pg$, where $g \in I$-cell and $p \in I$-inj. Since f is in I-cof, f has the left lifting property with respect to p, and so the retract argument 1.1.9 completes the proof. □

Corollary 2.1.15 then implies the following result, which is due to Hirschhorn.

Proposition 2.1.16. *Suppose I is a set of maps in a category $\mathcal{C}$ which has all small colimits. Suppose the domains of I are small relative to I-cell, and A is some object which is small relative to I-cell. Then A is in fact small relative to I-cof.*

Proof. Suppose A is κ-small relative to I-cell. Suppose λ is a κ-filtered ordinal and $X \colon \lambda \longrightarrow \mathcal{C}$ is a λ-sequence of I-cofibrations. We construct a λ-sequence Y of relative I-cell complexes, and natural transformations $i \colon X \longrightarrow Y$ and $r \colon Y \longrightarrow X$ with $ri = 1$ by transfinite induction. Define $Y_0 = X_0$, and i_0 and r_0 to be the identity map. Having defined Y_β, i_β, and r_β, apply the functorial factorization of Theorem 2.1.14 to the composite $Y_\beta \xrightarrow{r_\beta} X_\beta \xrightarrow{f_\beta} X_{\beta+1}$ to obtain $g_\beta \colon Y_\beta \longrightarrow Y_{\beta+1}$ and $r_{\beta+1} \colon Y_{\beta+1} \longrightarrow X_{\beta+1}$, with $g_\beta \in I$-cell, $r_{\beta+1} \in I$-inj, and $r_{\beta+1} g_\beta = f_\beta r_\beta$. Then we have a commutative square

$$\begin{tikzcd} X_\beta \arrow[r, "g_\beta i_\beta"] \arrow[d, "f_\beta"'] & Y_{\beta+1} \arrow[d, "r_{\beta+1}"] \\ X_{\beta+1} \arrow[r, equal] & X_{\beta+1} \end{tikzcd}$$

Since $f_\beta \in I$-cof and $r_{\beta+1} \in I$-inj, there is a lift $i_{\beta+1} \colon X_{\beta+1} \longrightarrow Y_{\beta+1}$ in this diagram. For limit ordinals β, we define $Y_\beta = \operatorname{colim}_{\alpha<\beta}$, $i_\beta = \operatorname{colim}_{\alpha<\beta} i_\alpha$, and $r_\beta = \operatorname{colim}_{\alpha<\beta} r_\alpha$.

Now once can easily check that the map $\operatorname{colim} \mathcal{C}(W, X_\beta) \longrightarrow \mathcal{C}(W, \operatorname{colim} X_\beta)$ is a retract of the corresponding map for Y. Since W is κ-small relative to I-cell, the corresponding map for Y is an isomorphism. Therefore the map for X must also be an isomorphism, and so W is κ-small relative to I-cof as well. $\square$

2.1.3. Cofibrantly generated model categories. The small object argument gives us the tool we need to construct model categories. We begin by defining a cofibrantly generated model category, following [**DHK**].

DEFINITION 2.1.17. Suppose $\mathcal{C}$ is a model category. We say that $\mathcal{C}$ is *cofibrantly generated* if there are sets I and J of maps such that:

1. The domains of the maps of I are small relative to I-cell;
2. The domains of the maps of J are small relative to J-cell;
3. The class of fibrations is J-inj; and
4. The class of trivial fibrations is I-inj.

We refer to I as the set of *generating cofibrations*, and to J as the set of *generating trivial cofibrations*. A cofibrantly generated model category $\mathcal{C}$ is called *finitely generated* if we can choose the sets I and J above so that the domains and codomains of I and J are finite relative to I-cell.

Finitely generated model categories will be important in Section 7.4, but cofibrantly generated model categories will suffice until then.

The following proposition sums up the basic properties of cofibrantly generated model categories. Its proof follows from Corollary 2.1.15 and Proposition 2.1.16.

PROPOSITION 2.1.18. *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and generating trivial cofibrations J.*

(a) *The cofibrations form the class I-cof.*
(b) *Every cofibration is a retract of a relative I-cell complex.*
(c) *The domains of I are small relative to the cofibrations.*
(d) *The trivial cofibrations form the class J-cof.*
(e) *Every trivial cofibration is a retract of a relative J-cell complex.*
(f) *The domains of J are small relative to the trivial cofibrations.*

If $\mathcal{C}$ is finitely generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Most of the model categories in common use are cofibrantly generated, and are often finitely generated. One (possible) exception is the category of chain complexes of abelian groups, where the weak equivalences are chain homotopy equivalences. Similar model categories, such as the Hurewicz model category of topological spaces [**Str72**], where the weak equivalences are the homotopy equivalences, are also probably not cofibrantly generated.

Notice that the functorial factorizations in a cofibrantly generated model category need not be given by the small object argument, though those factorizations are always available.

Note that we could define fibrantly generated model categories as well. Indeed, a model category is fibrantly generated if and only if its dual is cofibrantly generated. However, most of the categories one comes across in practice have no "cosmall" objects, so this definition is much less useful. For example, in the category of sets the only cosmall objects are the empty set and the one-point set. Indeed, the two-point set is a retract of every other set, so it suffices to show the two-point set is not cosmall. If the two-point set were cosmall, then every map from a sufficiently large product to the two-point set would factor through a projection map of the product. But take the map which assigns to every point in the diagonal one of the two points and to every other point in the product the other. Then this map does not factor through any projection.

When we need to consider the 2-category of cofibrantly generated model categories, we will just use the full sub-2-category of the category of model categories. That is, we make no requirement that Quillen functors preserve the generators.

We now show how to construct cofibrantly generated model categories.

THEOREM 2.1.19. *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose $\mathcal{W}$ is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $\mathcal{W}$ as the subcategory of weak equivalences if and only if the following conditions are satisfied.*

1. *The subcategory $\mathcal{W}$ has the two out of three property and is closed under retracts.*
2. *The domains of I are small relative to I-cell.*
3. *The domains of J are small relative to J-cell.*
4. *J-cell $\subseteq \mathcal{W} \cap I$-cof.*
5. *I-inj $\subseteq \mathcal{W} \cap J$-inj.*
6. *Either $\mathcal{W} \cap I$-cof $\subseteq J$-cof or $\mathcal{W} \cap J$-inj $\subseteq I$-inj.*

PROOF. These conditions certainly hold in a cofibrantly generated model category. Conversely, suppose we have a category $\mathcal{C}$ with a subcategory $\mathcal{W}$ and sets of maps I and J satisfying the hypotheses of the theorem. Define a map to be a fibration if and only if it is in J-inj, and define a map to be a cofibration if and only if it is in I-cof. Then certainly the cofibrations and fibrations are closed under retracts. It follows from the hypotheses that every map in J-cell is a trivial cofibration, and hence that every map in J-cof is a trivial cofibration. It also follows that every map in I-inj is a trivial fibration.

Define functorial factorizations $f = \beta(f) \circ \alpha(f) = \delta(f) \circ \gamma(f)$ by using the small object argument on I and J respectively (choosing colimit functors and appropriate cardinals). Thus $\alpha(f)$ is in I-cell, and is hence a cofibration, $\beta(f)$ is in I-inj, and is hence a trivial fibration, $\gamma(f)$ is in J-cell, and is hence a trivial cofibration, and $\delta(f)$ is in J-inj, and is hence a fibration.

It remains to verify the lifting axiom. This is where the last hypothesis comes in, with its two cases. Suppose first that $\mathcal{W} \cap I$-cof $\subseteq J$-cof. Then every trivial cofibration is in J-cof, and so has the left lifting property with respect to the fibrations, which form the class J-inj. Now given a trivial fibration $p\colon X \to Y$, we need to show that p has the right lifting property with respect to all cofibrations, or equivalently, with respect to I. We can factor $p = \beta(p) \circ \alpha(p)$, where $\alpha(p)$ is a cofibration and $\beta(p) \in I$-inj. Since $\mathcal{W}$ has the two out of three property, $\alpha(p)$ is a trivial cofibration. Hence, by the half of the lifting axiom we have already proven, p has the right lifting property with respect to $\alpha(p)$. It follows from the retract argument 1.1.9 that p is a retract of $\beta(p)$, so $p \in I$-inj as required.

The other case, where we assume $\mathcal{W} \cap J$-inj $\subseteq I$-inj, is similar, and we leave it to the reader. ∎

There are many advantages to knowing that a model category is cofibrantly generated. One of them is that it is easier to check that functors are Quillen functors.

LEMMA 2.1.20. *Suppose* $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ *is an adjunction between model categories. Supppose as well that* $\mathcal{C}$ *is cofibrantly generated, with generating cofibrations* I *and generating trivial cofibrations* J*. Then* (F, U, φ) *is a Quillen adjunction if and only if* Ff *is a cofibration for all* $f \in I$ *and* Ff *is a trivial cofibration for all* $f \in J$*.*

PROOF. Obviously the conditions are necessary. For the converse, note that Lemma 2.1.8 says that $F(I\text{-cof}) \subseteq FI\text{-cof}$. Let K be the class of cofibrations in $\mathcal{D}$. Then, by hypothesis, $FI \subseteq K$, and so $FI\text{-cof} \subseteq K\text{-cof}$. But the axioms imply that $K\text{-cof} = K$. Therefore $F(I\text{-cof}) \subseteq K$, and so F preserves cofibrations. A similar argument shows that F preserves trivial cofibrations, and so F is a left Quillen functor. □

The following lemma will be useful below.

LEMMA 2.1.21. *Suppose* $\mathcal{C}$ *is a cofibrantly generated model category. Then the model category* $\mathcal{C}_*$ *of Proposition 1.1.8 is cofibrantly generated. If* $\mathcal{C}$ *is finitely generated, so is* $\mathcal{C}_*$*.*

PROOF. Suppose I and J are sets of generating cofibrations and trivial cofibrations for $\mathcal{C}$. We claim that I_+ and J_+ will serve as generating cofibrations and trivial cofibrations for $\mathcal{C}_*$. Indeed, adjointness immediately implies that J_+-inj is the class of fibrations and that I_+-inj is the class of trivial fibrations. It remains to show that the domains of I_+ are small relative to I_+-cell, and similarly for J_+. Since the forgetful functor U commutes with sequential colimits, adjointness implies that we need only show that the domains of I are small relative to $U(I_+\text{-cell})$. But the maps of $U(I_+\text{-cell})$ are cofibrations, so the result follows. A similar proof will show that $\mathcal{C}_*$ is finitely generated if $\mathcal{C}$ is so. □

2.2. The stable category of modules

Perhaps the simplest nontrivial example of a model category is the category of modules over a Frobenius ring R, given the stable model structure.

Given a ring R, let R-mod denote the category of left R-modules.

DEFINITION 2.2.1. A ring R is a (*left*) *Frobenius ring* if the projective and injective R-modules coincide.

Examples of Frobenius rings include the group ring $k[G]$ of a finite group G over a field k [**CR88**, Section 62], and a finite graded connected Hopf algebra over a field [**Mar83**, Section 12.2].

DEFINITION 2.2.2. Suppose R is a ring. Given maps $f, g\colon M \to N$, define f to be *stably equivalent* to g, written $f \sim g$, if $f - g$ factors through a projective module.

Although we have defined stable equivalence for arbitrary rings, we will be interested in it only for Frobenius rings.

LEMMA 2.2.3. *Stable equivalence is an equivalence relation which is compatible with composition. That is, if* $f \sim g$*, then* $hf \sim hg$ *and* $fk \sim gk$*, whenever these compositions make sense.*

We leave the proof of this lemma to the reader.

DEFINITION 2.2.4. Let R be a ring. The *stable category of R-modules* is the category whose objects are left R-modules and whose morphisms are stable equivalence classes of R-module maps. A map f of R-modules is a *stable equivalence* if it is an isomorphism in the stable category.

The goal of this section is to show that the stable category of R-modules is the homotopy category of a model structure on the category of R-modules, when R is a Frobenius ring.

Note that the stable equivalences are closed under retracts and satisfy the two out of three property. Furthermore, if P is projective, then the inclusion $M \longrightarrow M \oplus P$ is a stable equivalence, as is its stable inverse, the surjection $M \oplus P \longrightarrow P$.

In order to define a cofibrantly generated model structure on R-mod, we need a set I of generating cofibrations and a set J of generating trivial cofibrations.

DEFINITION 2.2.5. Suppose R is a Frobenius ring. Let I denote the set of inclusions $\mathfrak{a} \longrightarrow R$, where $\mathfrak{a}$ is a left ideal in R. Let J denote the set consisting of the inclusion $0 \longrightarrow R$. Define a map f of R-modules to be a *fibration* if it has the right lifting property with respect to J, and define f to be a *cofibration* if $f \in I$-cof.

We claim that the cofibrations, fibrations, and stable equivalences define a model structure on R-mod. We prove this using Theorem 2.1.19, whose hypotheses we verify in a series of propositions and lemmas.

The following simple lemma, which does not require that R be a Frobenius ring, is left to the reader.

LEMMA 2.2.6. *A map p in R-mod is a fibration if and only if it is surjective.*

We now investigate the trivial fibrations.

LEMMA 2.2.7. *Suppose R is a Frobenius ring. Then a map p in R-mod is a trivial fibration if and only if p is a surjection with projective kernel.*

PROOF. Certainly a surjection with injective kernel is a trivial fibration. Conversely, suppose p is a trivial fibration. Then Lemma 2.2.6 implies that $p\colon M \longrightarrow N$ is surjective. Let $q\colon N \longrightarrow M$ be a stable inverse for p. Then there is a projective module P and maps $i\colon N \longrightarrow P$ and $h\colon P \longrightarrow N$ such that $pq - 1_N = hi$. Consider the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] \arrow[d] & M \arrow[r, "f"] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Q \arrow[r] & M \oplus P \arrow[r, "f\oplus h"'] & N \arrow[r] & 0
\end{tikzcd}$$

The map $(q, -i)\colon N \longrightarrow M \oplus P$ defines a splitting of the lower short exact sequence. Furthermore, since the inclusion $M \longrightarrow M \oplus P$ is a stable equivalence, the two out of three property implies that $f \oplus h$ is a stable equivalence. Therefore, the inclusion $Q \longrightarrow M \oplus P$ is stably trivial, and so factors through a projective. Using the retraction $M \oplus P \longrightarrow Q$ coming from the splitting, we find that the identiity map of Q factors through a projective. Hence Q is projective. The snake lemma implies that $Q/\ker f \cong P$, and so $Q/\ker f$ is also projective. Thus $\ker f$ is a retract of Q, and so is projective as required. $\square$

We must now characterize surjections with injective kernel. This can be done over an arbitrary ring, and relies on the following standard lemma [**Jac89**, Proposition 3.15].

LEMMA 2.2.8. *An R-module Q is injective if and only if, for all left ideals $\mathfrak{a}$ in R, every homomorphism $\mathfrak{a} \to Q$ can be extended to a homomorphism $A \to Q$.*

PROPOSITION 2.2.9. *If R is an arbitrary ring, a map p is in I-inj if and only if p is a surjection with injective kernel. In particular, if R is a Frobenius ring, the trivial fibrations form the class I-inj.*

PROOF. The second statement follows from the first and Lemma 2.2.7. Now suppose p is a surjection with injective kernel, and suppose we have a commutative diagram

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "f"] \arrow[d, "i"'] & M \arrow[d, "p"] \\ A \arrow[r, "g"'] & N \end{tikzcd}$$

Let $j\colon \ker f \to M$ denote the inclusion of the kernel of p. Since the kernel is injective, there is a splitting $q\colon N \to M$ such that $pq = 1_N$. Consider the map $qgi - f \to \mathfrak{a} \to M$. This map is in the kernel of p, so defines a map $r\colon \mathfrak{a}\colon \ker f$, such that $jr = qgi - f$. Since $\ker f$ is injective, there is an extension $s\colon A \to \ker f$ such that $si = r$. Then the map $qg - js\colon A \to M$ is a lift in the diagram. Thus p is in I-inj.

Conversely, suppose $p \in I$-inj. Since $J \subseteq I$, it follows that $p \in J$-inj, and so p is surjective. We must show that the kernel of p is an injective R-module. So suppose $\mathfrak{a}$ is a left ideal of R and $f\colon \mathfrak{a} \to \ker f$ is a homomorphism. Then we have a commutative diagram

$$\begin{tikzcd} \mathfrak{a} \arrow[r, "jf"] \arrow[d] & M \arrow[d, "p"] \\ A \arrow[r, "0"'] & N \end{tikzcd}$$

Since p is in I-inj, there is a lift in this diagram. Such a lift defines an extension of f to a homomorphism $A \to \ker f$, and so Q is injective by Lemma 2.2.8. $\square$

We need corresponding facts about the cofibrations.

LEMMA 2.2.10. *Over an arbitrary ring R, a map i of R-modules is in I-cof if and only if i is an injection.*

PROOF. By Proposition 2.2.9, i is in I-cof if and only if i has the left lifting property with respect to all surjections with injective kernel. The proof that injections have the left lifting property with respect to all surjections with injective kernel is exactly the same as the proof of the first half of Proposition 2.2.9. Conversely, suppose $i\colon A \to B$ has the left lifting property with respect to all surjections with injective kernel. Choose an embedding $A \to Q$ where Q is injective. Since i has the left lifting property with respect to $Q \to 0$, there is an extension $B \to Q$. In particular, i must be injective. $\square$

LEMMA 2.2.11. *Over an arbitrary ring R, a map is in J-cof if and only if it is an injection with projective cokernel. In particular, the elements of J-cof are stable equivalences.*

PROOF. The proof of this lemma is dual to the proof of Proposition 2.2.9. We have already seen, in Lemma 2.2.6, that J-inj is the class of surjections. Suppose i is an injection with projective cokernel $j\colon B \longrightarrow C$ and we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & M \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & N \end{array}$$

where p is surjective. Since the cokernel of i is projective, there is a retraction $r\colon B \longrightarrow A$. The map $pfr - g\colon B \longrightarrow N$ satisfies $(pfr - g)i = 0$, and so factors through a map $s\colon C \longrightarrow N$. Since C is projective, there is a map $t\colon C \longrightarrow M$ such that $pt = s$. Then $fr - tj$ is the desired lift in the diagram. Hence $i \in J$-cof.

Conversely, suppose $i\colon A \longrightarrow B$ is in J-cof. In particular, $i \in I$-cof and so i is injective. Let $q\colon B \longrightarrow C$ denote the cokernel of i. We must show that C is projective. Suppose $f\colon C \longrightarrow N$ is a map and $p\colon M \longrightarrow N$ is a surjection. Then we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{0} & M \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[fq]{} & N \end{array}$$

A lift in this diagram is a map $h\colon B \longrightarrow M$ such that $hi = 0$ and $ph = fq$. It follows that h factors through a map $g\colon C \longrightarrow M$ lifting f. Therefore C is projective as required. □

It is now straightforward to prove that R-mod is a model category when R is a Frobenius ring.

THEOREM 2.2.12. *Suppose R is a Frobenius ring. Then there is a cofibrantly generated model structure on R-mod where the cofibrations are the injections, the fibrations are the surjections, and the weak equivalences are the stable equivalences. If R is Noetherian, then the model structure above is finitely generated.*

PROOF. Apply Theorem 2.1.19, using the sets I and J in Definition 2.2.5. We have already seen that every R-module is small in Example 2.1.6. In case R is Noetherian, then every ideal $\mathfrak{a}$ is finitely presented, and so the domains and codomains of I and J are finite. □

Note that if there is a cofibrantly generated model structure on R-mod with I as the set of generating cofibrations and J as the set of generating trivial cofibrations, then in fact R must be a Frobenius ring and the weak equivalences must be the stable equivalences. Indeed, if P is projective, then $0 \longrightarrow P$ is in J-cof, and so is a weak equivalence. But then $P \longrightarrow 0$ is a weak equivalence and a fibration, so must be in I-inj. Thus P is injective. The converse is similar, and so R must be a

Frobenius ring. The weak equivalences are determined by I and J, as composites of the form I-inj $\circ$ J-cof, so must be the stable equivalences.

Also note that R-mod is a very unusual model category, since every object is both cofibrant and fibrant. One can easily check that f and g are either left or right homotopic in R-mod if and only if f and g are stably equivalent.

Given a homomorphism $f\colon R \to S$ of Frobenius rings, we get an adjunction from R-mod to S-mod whose left adjoint is the induction functor that takes M to $B \otimes_A M$, and whose right adjoint is the restriction functor. This adjunction will be a Quillen adjunction if and only if f makes B into a flat right A-module. Indeed, if induction is to preserve cofibrations, f must clearly be flat. If f is flat, then induction preserves cofibrations, so restriction preserves trivial fibrations. Since restriction always preserves surjections, restriction is a right Quillen functor as required.

2.3. Chain complexes of modules over a ring

Another fairly simple algebraic example of a model category is the category of chain complexes $\mathrm{Ch}(R)$ of (say, left) modules over a ring R. This section is devoted to that example.

We begin with the standard definitions.

Definition 2.3.1. Let R be a ring. Define the category $\mathrm{Ch}(R)$ of chain complexes over R and chain maps as follows. An object of $\mathrm{Ch}(R)$ is a *chain complex* of left R-modules: *i.e.* a collection of R-modules X_n for each integer (positive or negative) n and a differential $d = \{d_n\colon X_n \to X_{n-1}\}$, where each d_n is an R-module map and $d_{n-1}d_n = 0$ for all n. A morphism $f\colon X \to Y$ of $\mathrm{Ch}(R)$ is a *chain map*: *i.e.* a collection of R-module maps $f_n\colon X_n \to Y_n$ such that $d_n f_n = f_{n-1} d_n$.

Note that the category $\mathrm{Ch}(R)$ has all small limits and colimits, which are taken degreewise. The initial and terminal object is the chain complex 0, which is 0 in each degree. The category $\mathrm{Ch}(R)$ is also an abelian category, where short exact sequences are defined degreewise.

Since we will be using the small object argument on $\mathrm{Ch}(R)$, the following lemma is useful.

Lemma 2.3.2. *Every object in* $\mathrm{Ch}(R)$ *is small. Every bounded complex of finitely presented* R*-modules is finite.*

Proof. Suppose $X \in \mathrm{Ch}(R)$. Let γ be an infinite cardinal larger than $|R \times \bigcup_n X_n|$, let λ be a γ-filtered ordinal, and let $Y\colon \lambda \to \mathrm{Ch}(R)$ be a λ-sequence. Denote the image of α under Y by Y^α. Suppose $f\colon X \to \operatorname{colim} Y$ is a chain map. Then, since the R-module X_n is γ-small by Example 2.1.6, and we chose γ to be infinite, there is an $\alpha < \lambda$ such that f factors through a map $g\colon X \to Y^\alpha$ of graded R-modules. The map g need not be a chain map, but for each homogeneous $x \in X$, there is a $\beta_x > \alpha$ such that $g(dx) = d(gx)$ in Y^{β_x}. Taking β to be the supremum of the β_x, we find that $\beta < \lambda$ and we get the desired factorization of f through a chain map $X \to Y^\beta$.

Similarly, if f and g are two chain maps $X \to Y^\alpha$ which are equal as maps to $\operatorname{colim} Y$, then for each $x \in X$ there is a $\beta_x > \alpha$ such that $fx = gx$ in Y^{β_x}. Taking β to be the supremum of the β_x, we find that $f = g$ as maps to Y^β, as required.

A similar argument shows that every bounded complex of finitely presented R-modules is finite. $\square$

We now define the standard model structure on $\mathrm{Ch}(R)$.

DEFINITION 2.3.3. Let R be a ring. Given an R-module M, define $S^n(M) \in \mathrm{Ch}(R)$ by $S^n(M)_n = M$ and $S^n(M)_k = 0$ if $k \neq n$. Similarly, define $D^n(M)$ by $D^n(M)_k = M$ if $k = n$ or $k = n-1$, and 0 otherwise. The differential d_n in $D^n(M)$ is the identity. We often denote $S^n(R)$ by simply S^n, and $D^n(R)$ by D^n. There is an evident injection $S^{n-1}(M) \to D^n(M)$. Now define the set I to consist of the maps $S^{n-1} \to D^n$, and define the set J to consist of the maps $0 \to D^n$. Define a map to be a *fibration* if it is in J-inj, and define a map to be a *cofibration* if it is in I-cof. Define a map f to be a *weak equivalence* if the induced map $H_n(f)$ on homology is an isomorphism for all n.

Of course, the homology $H_n X$ of chain complex X is defined by $H_n(X) = \ker d_n / \operatorname{im} d_{n+1}$. A chain complex X is called *acyclic* if $H_* X = 0$. Because homology is functorial, the weak equivalences are closed under retracts and satisfy the two out of three axiom. The only other thing we need about homology is that a short exact sequence of chain complexes induces a long exact sequence in homology.

We now characterize the fibrations. Before doing so, note that the functor D^n is left adjoint to the evaluation functor $\mathrm{Ev}_n\colon \mathrm{Ch}(R) \to R\text{-mod}$ that takes X to X_n. Similarly, the functor S^n is left adjoint to the cycle functor $Z_n\colon \mathrm{Ch}(R) \to R\text{-mod}$ that takes X to ZX_n, the kernel of d_n.

PROPOSITION 2.3.4. *A map $p\colon X \to Y$ in $\mathrm{Ch}(R)$ is a fibration if and only if $p_n\colon X_n \to Y_n$ is surjective for all n.*

PROOF. A diagram of the form

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

is equivalent to an element y in Y_n. A lift in this diagram is equivalent to an element x in X_n such that $px = y$. The lemma follows immediately. $\square$

We also characterize the trivial fibrations.

PROPOSITION 2.3.5. *A map $p\colon X \to Y$ in $\mathrm{Ch}(R)$ is a trivial fibration if and only if it is in I-inj.*

PROOF. The set of diagrams

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ D^n \arrow[r, "g"] & Y \end{tikzcd}$$

is in one-to-one correspondence with $\{(y, x) \in Y_n \oplus Z_{n-1}X \mid px = dy\}$. A lift in such a diagram is a class $z \in X_n$ such that $dz = x$ and $pz = y$.

Now suppose $p \in I$-inj. Take a cycle $y \in Z_n Y$. Then the pair $(y, 0)$ defines a diagram as above, so there is a class $z \in X_n$ such that $pz = y$ and $dz = 0$. Hence $Z_n p\colon Z_n X \to Z_n Y$ is surjective. It follows immediately that $H_n p$ is surjective. It also follows that p itself is surjective. Indeed, suppose y is an arbitrary element of Y_n. Then dy is a cycle, so there is a class $x \in Z_{n-1}X$ such that $px = dy$. The pair (y, x) corresponds to a diagram as above, so there is a class $z \in X_n$ such that $pz = y$. Hence p is surjective, so by Proposition 2.3.4, p is a fibration.

It remains to prove that $H_n p\colon H_n X \longrightarrow H_n Y$ is injective. Take an $x \in Z_n X$ such that $px = dy$ for some $y \in Y_{n+1}$. Then (y, x) also defines a diagram as above, so there is a class $z \in X_{n+1}$ such that $dz = x$. Thus $H_n p$ is injective as well, so p is a weak equivalence.

Conversely, suppose p is a trivial fibration. Given (y, x) such that $y \in Y_n$, $x \in Z_{n-1}X$, and $px = dy$, we must find a $z \in X_n$ such that $pz = y$ and $dz = x$. Since p is a fibration, it is surjective. Thus we have a short exact sequence

$$0 \longrightarrow K \longrightarrow X \xrightarrow{p} Y \longrightarrow 0.$$

Since p is a weak equivalence, we have $H_* K = 0$. First choose $w \in X_n$ such that $pw = y$. Then $dw - x \in Z_{n-1}K$, since $p(dw) = d(pw) = dy = px$, and $d(dw - x) = dx = 0$. Since $H_* K = 0$, there is a $v \in K_n$ such that $dv = dw - x$. Let $z = w - v$. Then $pz = y$ and $dz = x$, as required. □

Our next goal is to characterize the cofibrations. We begin with the cofibrant objects.

LEMMA 2.3.6. *Suppose R is a ring. If A is a cofibrant chain complex, then A_n is a projective R-module for all n. As a partial converse, any bounded below complex of projective R-modules is cofibrant.*

PROOF. Suppose $M \xrightarrow{q} N$ is a surjection of R-modules. Then we have a trivial fibration $D^n M \xrightarrow{p} D^n N$, which is q in degrees n and $n-1$ and 0 elsewhere. A map $A_{n-1} \xrightarrow{f} N$ induces a chain map $A \xrightarrow{g} D^n N$ which is f in degree $n-1$, fd in degree n, and 0 elsewhere. If A is cofibrant, there must be a lift $A \xrightarrow{h} D^n M$. Then $h_{n-1}\colon A_{n-1} \longrightarrow M$ is a lift of f. Thus A_{n-1} is projective.

Now suppose that A is a bounded below complex of projective R-modules, and $p\colon X \longrightarrow Y$ is a trivial fibration. Let K denote the kernel of p, and note $H_* K = 0$ since p is trivial. Suppose we are given $g\colon A \longrightarrow Y$. We must construct a lift $h\colon A \longrightarrow X$ of g. We construct h_n such that $p_n h_n = g_n$ and $dh_n = h_{n-1}d$ by induction. There is no trouble getting started since A is bounded below. So suppose we have defined h_k for all $k < n$ satisfying the conditions above. Since p_n is surjective and A_n is projective, there is a map $f\colon A_n \longrightarrow X_n$ such that $p_n f = g_n$. Consider the map $F\colon A_n \longrightarrow X_{n-1}$ defined by $F = df - h_{n-1}d$. Then one can check that $pF = dF = 0$, so that F is actually a map $X_n \xrightarrow{F} ZK_{n-1}$. Since K is acyclic, we have $ZK_{n-1} = BK_{n-1}$, where BK_{n-1} denotes the image of d_n in K_{n-1}. So, since A_n is projective, there is a map $G\colon A_n \longrightarrow K_n$ such that $dG = F$. Now define $h_n = f - G$. Then $ph_n = g_n$ and $dh_n = df - F = h_{n-1}d$, as required. □

REMARK 2.3.7. Not every complex of projective R-modules is cofibrant. To prove this, we will use the not yet proved fact that $\mathrm{Ch}(R)$ is a model category. Suppose $R = E(x)$, the exterior algebra on x over a field k. Let A be the complex which is R in every degree, and where the differential is multiplication by x. Then A is acyclic, so if A were also cofibrant, the map $0 \longrightarrow A$ would be a trivial cofibration. Now let X be the complex S^0, which is R in degree 0 and 0 elsewhere, and let Y be the complex which is k in degree 0 and 0 elsewhere. Then there is a fibration $p\colon X \longrightarrow Y$ which is the augmentation of R in degree 0. There is a map $A \xrightarrow{g} Y$ that is also the augmentation in degree 0. But there can be no lift $A \xrightarrow{h} X$, since such a lift would have to be the identity in degree 0 and that is not a chain map. Thus A cannot be cofibrant. In fact, the cofibrant objects correspond to the DG-projective complexes of [**AFH97**]. We leave it to the interested reader to check this.

Recall that two chain maps $f, g\colon X \to Y$ are said to be *chain homotopic* if there is a collection of R-module maps $D_n\colon X_n \to Y_{n+1}$ such that $dD_n + D_{n-1}d = f_n - g_n$ for all n.

LEMMA 2.3.8. *Suppose R is a ring, C is a cofibrant chain complex, and $H_*K = 0$. Then every map from C to K is chain homotopic to 0.*

PROOF. Let P be the chain complex defined by $P_n = K_n \oplus K_{n+1}$, with $d(x, y) = (dx, x - dy)$. Then there is an obvious surjection $p\colon P \to K$ that takes (x, y) to x. The kernel of p is just a shifted version of K. In particular, $H_*(\ker p) \cong H_{*-1}K = 0$. Thus p is a trivial fibration. Since C is cofibrant, there is a map $g = (f, D)\colon C \to P$ lifting f. Since g is a chain map, we must have $f - Dd = dD$, so D is the desired chain homotopy. □

With these two lemmas in hand, we can characterize the cofibrations.

PROPOSITION 2.3.9. *Suppose R is a ring. Then a map $i\colon A \to B$ in $\mathrm{Ch}(R)$ is a cofibration if and only if i is a dimensionwise split injection with cofibrant cokernel.*

PROOF. By definition, i is a cofibration if and only if it has the left lifting property with respect to surjections with acyclic kernel. Suppose first that i is a cofibration. Consider the map $A \to D^{n+1}A_n$ which is d in degree $n+1$ and the identity in degree n. Since $D^{n+1}A_n$ is acyclic, there is an extension of this map to a map $B \to D^{n+1}A_n$. In particular, i_n is a split monomorphism. The collection K-proj is always closed under pushouts, for any K in any category. Since the map $0 \to \operatorname{cok} i$ is the pushout of i through the map $A \to 0$, it follows that $\operatorname{cok} i$ is cofibrant.

Now suppose that i_n is an inclusion for all n and the cokernel C of i is cofibrant. Given a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

where p is a homology isomorphism and a dimensionwise surjection, we must find a lift $h\colon B \to X$ such that $ph = g$ and $hi = f$. Let $j\colon K \to X$ denote the kernel of p. We can write $B_n = A_n \oplus C_n$, since C_n is projective. Then the differential $d\colon B_n \to B_{n-1}$ is given by $d(a, c) = (da + \tau c, dc)$, where $\tau\colon C_n \to A_{n-1}$ can be an arbitrary map such that $d\tau + \tau d = 0$. The map g is then defined by $g(a, c) = pf(a) + \sigma(c)$, where the collection of maps $\sigma_n\colon C_n \to Y_n$ must satisfy $d\sigma = pf\tau + \sigma d$, since g is a chain map. A lift h in the diagram is then equivalent a collection of maps $\nu_n\colon C_n \to X_n$ such that $p\nu = \sigma$ and $d\nu = \nu d + f\tau$.

Using the fact that C_n is projective for all n, choose maps $G_n\colon C_n \to X_n$ such that $p_n G_n = \sigma_n$. Consider the map $r = dG - Gd - f\tau\colon C_n \to X_{n-1}$. Then $pr = 0$, so r defines a map $s\colon C_n \to K_{n-1}$ such that $js = r$. Furthermore, $dr = -dGd + f\tau d = -rd$, so $s\colon C \to \Sigma K$ is actually a chain map, where ΣK is the chain complex defined by $(\Sigma K)_n = K_{n-1}$ and $d_{\Sigma K} = -d_K$. By Lemma 2.3.8

, s is chain homotopic to 0. There is therefore a map $D\colon C_n \to K_n$ such that $-dD + Dd = s$, where the extra minus sign comes from the fact that the differential in ΣK is the negative of the differential in K. Let $\nu = G + jD$. Then $p\nu = pG = \sigma$, and $d\nu = \nu d + f\tau$. Therefore $h = (f, \nu)\colon B \to X$ is the desired lift in our diagram. □

The trivial cofibrations are a little simpler to understand.

Proposition 2.3.10. *Suppose R is a ring. Then a map $i\colon A \to B$ is in J-cof if and only if i is an injection whose cokernel is projective as a chain complex. In particular, every map in J-cof is a trivial cofibration.*

Proof. The proof of the first part is the same as the proof of Lemma 2.2.11. For the second part, we must show that a projective chain complex, which is obviously cofibrant, is also acyclic. Let C be projective. Let P be the complex defined by $P_n = C_n \oplus C_{n+1}$, where $d(x, y) = (dx, x - dy)$, as in Lemma 2.3.8. Then there is a surjection $P \to C$. Since C is projective, the identity map of C lifts to a map $C \to P$. The second component of this map is a collection of maps $D_n \to C_n \to C_{n+1}$ such that $dDx + Ddx = x$. In particular, if x is a cycle, then $dDx = x$, so x is also a boundary, and so C is acyclic. □

Since we have now verified all the hypotheses of Theorem 2.1.19, we have proved the following theorem.

Theorem 2.3.11. $\mathrm{Ch}(R)$ *is a finitely generated model category with I as its generating set of cofibrations, J as its generating set of trivial cofibrations, and homology isomorphisms as its weak equivalences. The fibrations are the surjections.*

It follows from Theorem 2.3.11 that every trivial cofibration is in J-cof, and so is an injection with projective cokernel. In particular, X is projective if and only if it is cofibrant and acyclic. Note that every chain complex is fibrant in this model structure. One can easily check that the right homotopy relation is precisely the chain homotopy relation. Indeed, given a chain complex X, a path object for X is given by the chain complex P, where $P_n = X_n \oplus X_n \oplus X_{n+1}$, with differential $d(x, y, z) = (dx, dy, -dz + x - y)$.

Th model structure we have just described is not the only commonly used model structure on $\mathrm{Ch}(R)$.

Definition 2.3.12. Let R be a ring. Define a map f in $\mathrm{Ch}(R)$ to be an *injective fibration* if f has the right lifting property with respect to all maps which are both injections and weak equivalences.

Theorem 2.3.13. *The injections, injective fibrations, and weak equivalences are part of a cofibrantly generated model structure, called the* injective model structure*, on* $\mathrm{Ch}(R)$*. The injective fibrations are the surjections with fibrant kernel; every fibrant object is a complex of injectives, and every bounded above complex of injectives is fibrant. The injective trivial fibrations are the surjections with injective kernel; a complex is injective if and only if it is fibrant and acyclic.*

To the author's knowledge, Theorem 2.3.13 has not appeared before. The injective model structure is usually used only with bounded above complexes, where one can use inductive arguments which are not available in the general case. Grodal [**Gro97**] has constructed the injective model structure using the results of [**AFH97**], but he does not prove that it is cofibrantly generated.

To prove this theorem, we need sets I' of generating injections and J' of generating injective weak equivalences. We do not construct such sets explicitly; rather, we just take all injective (trivial) cofibrations whose cardinality is not too large. This idea is sometimes referred to as the Bousfield-Smith cardinality argument, and is the tool used to construct localizations of model categories in [**Hir97**].

DEFINITION 2.3.14. Let R be a ring. Given a chain complex $X \in \mathrm{Ch}(R)$, define $|X|$ to be the cardinality of $\bigcup_n X_n$. Define γ to be the supremum of $|R|$ and ω. Define I' to be a set containing a map from each isomorphism class of injections $i\colon A \to B$ in $\mathrm{Ch}(R)$ such that $|B| \leq \gamma$. Define J' to be the set of all weak equivalences in I'.

We prove Theorem 2.3.13 in a series of propositions and lemmas.

PROPOSITION 2.3.15. *The class I'-cof is the class of injections, and the class I'-inj is the class of surjections whose kernel is injective as a chain complex.*

PROOF. Let us first note that, given a chain complex X and an element $x \in X_n$, there is a sub-chain complex Y containing x of cardinality at most γ. Indeed, we let Y_n be the submodule of X_n generated by x, Y_{n-1} be the submodule of X_{n-1} generated by dx, and $Y_k = 0$ otherwise. Then $|Y_n| \leq |R| \leq \gamma$, and similarly for Y_{n-1}.

Now, the class of injections is the class K-proj, where K is the class of surjections with injective kernel. The proof of this is very similar to the proof of Proposition 2.2.9, and so we leave it to the reader. In particular, since $I' \subseteq K$-proj, we have I'-cof $\subseteq$ (K-proj)-cof $= K$-proj, and so every I'-cofibration is an injection. Conversely, suppose $i\colon A \to B$ is injective. In order to show that i is an I'-cofibration, we must show that i has the left lifting property with respect to I'-inj. So suppose $p\colon X \to Y$ is in I'-inj, and we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

Let S be the partially ordered set of partial lifts in this diagram. That is, an element of S is a pair (C, h), where C is a sub-chain complex of B containing the image of i, and $h\colon C \to X$ is a chain map making the diagram commute. We give S the obvious partial ordering; $(C, h) \leq (C', h')$ if $C \subseteq C'$ and h' is an extension of h. Then S is nonempty and every chain in S has an upper bound. Therefore, Zorn's lemma applies and we can find a maximal element (M, h) of S. We claim that $M = B$. Indeed, suppose not, and choose a homogeneous $x \in B$ but not in M. Let Z be the sub-chain complex generated by x; then we have already seen that $|Z| \leq \gamma$. Let M' denote the sub-chain complex generated by M and x. Then we have a pushout diagram

$$\begin{array}{ccc} M \cap Y & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ M & \longrightarrow & M' \end{array}$$

Since the top horizontal map is in I', the bottom horizontal map is in I'-cof. Hence there is a lift h' in the diagram

$$\begin{array}{ccc} M & \xrightarrow{h} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ M' & \xrightarrow[g]{} & Y \end{array}$$

Then (M', h') is in S, but $(M', h') > (M, h)$. This contradiction shows that we must have had $M = B$, and therefore that i has the left lifting property with respect to I'-inj.

We have now proved that I'-cof is the class of injections. It follows that I'-inj consists of all maps which have the right lifting property with respect to all injections. This is easily seen to be the surjections with injective kernel, by a slight modification of Proposition 2.2.9. □

COROLLARY 2.3.16. *Every injective object is fibrant and acyclic. Every map in I'-inj is an injective fibration and a weak equivalence.*

PROOF. Any map in I'-inj has the right lifting property with respect to all injections, so is a in particular an injective fibration. We have just seen that if $p \in I'$-inj, then p is a surjection with injective kernel K. Let P be the complex defined by $P_n = K_n \oplus K_{n-1}$, with $d(x, y) = (dx + y, -dy)$. Then there is an obvious injection $K \longrightarrow P$. Since K is injective, the identity map extends to a map $P \xrightarrow{H} K$, where $H(x, y) = x + Dy$. Then D is a chain homotopy from the identity map of K to the zero map, and so K is acyclic. It follows that p is a homology isomorphism. □

We now need similar results for J'. We begin by characterizing the injective fibrations.

LEMMA 2.3.17. *Let R be a ring. If A is an injectively fibrant chain complex, then each A_n is an injective R-module. Any bounded above complex of injective R-modules is injectively fibrant.*

The proof of this lemma is very similar to the proof of Lemma 2.3.6, so we leave it to the reader.

REMARK 2.3.18. Just as in the projective case, not every complex of injective R-modules is injectively fibrant, and the injectively fibrant objects correspond to the DG-injective complexes of [**AFH97**]. The same example will work, assuming the yet to be proved fact that the injective model structure on $\mathrm{Ch}(R)$ is a model structure. Let k be a field, let $R = E(x)$, and let A be the complex with $A_n = R$ and d being multiplication by x. One can easily check that R is self-injective; indeed, R is a Frobenius ring. If the complex of injectives A were fibrant, then $A \longrightarrow 0$ would be an injective trivial fibration. Consider the inclusion $S^0(k) \longrightarrow S^0(R)$ that takes 1 to x. There is a map $S^0(k) \longrightarrow A$ which takes 1 to x in degree 0. The only possible extension to a map $S^0(R) \longrightarrow A$ is the identity in degree 0, but this is not a chain map. Thus A cannot be fibrant.

We also have the analogue of Lemma 2.3.8, which has the dual proof.

LEMMA 2.3.19. *Suppose R is a ring, C is an acyclic chain complex, and K is an injectively fibrant chain complex. Then every map from C to K is chain homotopic to* 0.

With these two lemmas in hand, the analogue of Proposition 2.3.9 also holds, with the dual proof.

PROPOSITION 2.3.20. *Suppose R is a ring. Then p is an injective fibration if and only if p is a dimensionwise split surjection with injectively fibrant kernel.*

We must now show that the J'-cofibrations are the injective weak equivalences. This is a little more difficult, so we begin with a lemma.

LEMMA 2.3.21. *Let R be a ring, and suppose $i\colon A \to B$ is an injective weak equivalence in* $\mathrm{Ch}(R)$. *For every sub-chain complex C of B with $|C| \le \gamma$, there is a sub-chain complex D of B with $|D| \le \gamma$ such that $i\colon D \cap A \to D$ is a weak equivalence.*

PROOF. For each element in $H_*(C/C\cap A)$ choose an element c in C representing it. Then there is a $b \in B$ such that $db - c$ is in A, since $H_*(B/A) = 0$. Form the complex $C + Rb + R(db)$, which also has size $\le \gamma$. If we iterate this construction for each of the $\le \gamma$ nontrivial classes in $H_*(C/C\cap A)$, we get a new sub-chain complex $FC \supseteq C$ with $|FC| \le \gamma$, such that the map $H_*(C/C\cap A) \to H_*(FC/FC\cap A)$ is zero.

Now let D be the union of all the F^nC. Then $|D| \le \gamma$, and we claim that $H_*(D/D\cap A) = 0$. Indeed, a cycle in $D/D\cap A$ must be represented by an $x \in F_nC$ for some n, and this x will be a cycle in $F_nC/F_nC \cap A$. It follows that x is a boundary in $F_{n+1}C/F_{n+1}C \cap A$, and therefore also a boundary in $D/D\cap A$. Thus the inclusion $D \cap A \to D$ is a weak equivalence, as required. □

PROPOSITION 2.3.22. *The class J'-cof consists of the injective weak equivalences. The class J'-inj consists of the injective fibrations.*

PROOF. The second statement follows from the first. We first prove that maps in J'-cof are injective weak equivalences. Since $J' \subseteq I'$, J'-cof $\subseteq I'$-cof, so maps in J'-cof are injections. Suppose $i\colon A \to B$ is in J'-cof. We must show that i is a weak equivalence, or, equivalently, that the cokernel C of i is acyclic. Since J'-cof is closed under pushouts, the map $j\colon 0 \to C$ is in J'-cof. Since every map in J' is an injective weak equivalence, j has the left lifting property with respect to injective fibrations. Now let Q be an injective hull of C_n/dC_{n+1}. There is a map $D^nQ \to S^n(Q)$ which is the identity in degree n. This map is an injective fibration, since the kernel is a bounded above complex of injectives, and so is fibrant. There is a map $C \to S^n(Q)$ which is the composite $C_n \to C_n/dC_{n+1} \to Q$ in degree n. Since j has the left lifting property with respect to injective fibrations, there is a lift $C \to D^nQ$. This gives an extension of the injection $C_n/dC_{n+1} \to Q$ to a map $C_{n-1} \to Q$. In particular, the map $C_n/dC_{n+1} \to C_{n-1}$ must be injective, and so C has no homology. This shows that the maps in J'-cof are injective weak equivalences.

Now suppose $i\colon A \to B$ is an injective weak equivalence. To show that $i \in J'$-cof, we must show that i has the left lifting property with respect to J'-inj. So suppose $p \in J'$-inj and we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

Let S be the set of partial lifts (C, h), where $iA \subseteq C \subseteq B$, the injection $i\colon A \to C$ is a weak equivalence, and $h\colon C \to X$ is a partial lift in our diagram. Once again S is nonempty and every chain in S has an upper bound. So Zorn's lemma applies and we can find a maximal element (M, h) of S. Since $A \to M$ is a weak equivalence, so is the inclusion $M \to B$. If M is not all of B, choose an x in B but not in M, and let C denote the subcomplex generated by x. Then $|C| \leq \gamma$, and so Lemma 2.3.21 implies that there is a subcomplex D containing x with $|D| \leq \gamma$ and such that the inclusion $D \cap M \to D$ is a weak equivalence. Thus the pushout $M \to D + M$ of this inclusion through the inclusion $D \cap M \to M$ is in J'-cof. Since p is in J'-inj, we can find an extension of h to a partial lift h' on $D \cup M$. This violates the maximality of (M, h), and so we must have had $M = B$. Thus $i \in J'$-cof, as required. □

The recognition theorem 2.1.19 now implies that the injections, injective fibrations, and weak equivalences define a cofibrantly generated model structure on $\mathrm{Ch}(R)$, and this in turn implies that every complex that is injectively fibrant and acyclic is injective. This completes the proof of Theorem 2.3.13.

Note that the identity functor is a Quillen equivalence from the standard model structure on $\mathrm{Ch}(R)$ to the injective model structure. In general, a map of rings $f\colon R \to R'$ will induce a Quillen adjunction $\mathrm{Ch}(R) \to \mathrm{Ch}(R')$ of the standard model structures. The left adjoint, induction, takes X to $R' \otimes_R X$, and the right adjoint, restriction, is the forgetful functor. Restriction obviously preserves fibrations and trivial fibrations, so this is a Quillen adjunction. Since restriction preserves and reflects weak equivalences, induction is a Quillen equivalence if and only if the map $f\colon R \to R'$ is an isomorphism. On the other hand, if we give $\mathrm{Ch}(R)$ and $\mathrm{Ch}(R')$ the injective model structures, induction will be a Quillen functor if and only if f makes R' into a flat R-module. Again, this will be a Quillen equivalence if and only if f is an isomorphism.

Note that if M and N are R-modules, then $[S^n M, S^0 N] \cong \mathrm{Ext}^n_R(M, N)$. Indeed, a projective resolution of M is a cofibrant replacement for $S^0 M$ in $\mathrm{Ch}(R)$, and shifting it up n places gives us a cofibrant replacement for $S^n M$. Since $S^0 N$ is already fibrant, $[S^n M, S^0 N]$ is just chain homotopy classes of maps from a projective resolution of M to N, which is the usual definition of Ext.

2.4. Topological spaces

In this section we construct the standard model structure on **Top**. Our proof differs from the proofs in [**Qui67**] and [**DS95**] mostly in the level of detail. We give full proofs of the required smallness results, and we provide a careful proof that trivial fibrations have the right lifting property with respect to relative cell complexes. Both of these issues are completely avoided in both [**Qui67**] and [**DS95**]. We also briefly discuss the model categories of pointed topological spaces and compactly generated topological spaces.

Let **Top** denote the category of topological spaces and continuous maps. Basic facts about **Top** can be found in [**Mun75**]. In particular, **Top** is a symmetric monoidal category under the product. The product $-\times X$ does not commute with colimits in general. Given topological spaces X and Y, we can form the function space Y^X of continuous maps from X to Y, given the compact-open topology [**Mun75**, Section 7.5]. This function space is not well-behaved in general: however, if X is locally compact Hausdorff, then $(-)^X$ is right adjoint to the functor $-\times X\colon \mathbf{Top}\longrightarrow\mathbf{Top}$ (see [**Mun75**, Corollary 7.5.4]). Thus $-\times X$ does commute with colimits when X is locally compact Hausdorff.

We now recall the construction of colimits and limits in **Top**. If $F\colon \mathcal{I}\longrightarrow\mathbf{Top}$ is a functor, where $\mathcal{I}$ is a small category, a limit of F is obtained by taking the limit in the category of sets, then topologizing it as a subspace of the product $\prod F(i)$ for $i\in I$. The product is of course given the product topology. A colimit of F is obtained by taking the colimit $\operatorname{colim} F$ in the category of sets, and declaring a set U in $\operatorname{colim} F$ to be open if and only if $j_i^{-1}(U)$ is open in $F(i)$ for all $i\in I$, where $j_i\colon F(i)\longrightarrow\operatorname{colim} F$ is the structure map of the colimit.

Unlike the categories of sets, R-modules, and chain complexes of R-modules, not every object in **Top** is small. In fact, the Sierpinski space, consisting of two points where exactly one of them is open, is not small in **Top**, as was pointed out to the author by Stefan Schwede. To see this, given a limit ordinal λ, give the set $Y=\lambda\cup\{\lambda\}$ the order topology. Let X_α be $Y\times\{0,1\}$ modulo the equivalence relation $(x,0)\sim(x,1)$ if $x<\alpha$. Then the X_α define a λ-sequence in **Top**. The colimit X of the X_α is Y with an extra point $(\lambda,1)$ with exactly the same neighborhoods at λ. These two points define a continuous map from the Sierpinski space into X which does not factor continuously through any X_α. The same example shows that the indiscrete space on two points is not small.

The best we can do is the following lemma. Recall that an injective map $f\colon X\longrightarrow Y$ in **Top** is an *inclusion* if U is open in X if and only if there is a V open in Y such that $f^{-1}(V)=U$.

LEMMA 2.4.1. *Every topological space is small relative to the inclusions.*

PROOF. Suppose $X\colon \lambda\longrightarrow\mathbf{Top}$ is a λ-sequence of inclusions. This means that each map $X_\alpha\longrightarrow X_{\alpha+1}$ is an inclusion. However, it follows by transfinite induction that each map $X_\alpha\longrightarrow X_\beta$ is an inclusion for $\beta>\alpha$, and hence that the map $X_\alpha\longrightarrow\operatorname{colim} X$ is an inclusion. Thus, if we can factor a map $A\longrightarrow\operatorname{colim} X$ through a map *of sets* $A\longrightarrow X_\alpha$, then this map is automatically continuous. The lemma then follows from the fact that every set is small (Example 2.1.5). □

Lemma 2.4.1 is enough to let us use the small object argument. However, we also need a more refined smallness proposition. Define a map $f\colon X\longrightarrow Y$ to be a *closed T_1 inclusion* if f is a closed inclusion and if every point not in $Y\setminus f(X)$ is closed in Y.

PROPOSITION 2.4.2. *Compact topological spaces are finite relative to closed T_1 inclusions.*

PROOF. Let λ be a limit ordinal, and let $X\colon \lambda\longrightarrow\mathbf{Top}$ be a λ-sequence of closed T_1 inclusions. It follows that each map $X_\alpha\longrightarrow\operatorname{colim} X_\alpha$ is a closed T_1 inclusion. It suffices to show that, for all maps $f\colon A\longrightarrow\operatorname{colim} X_\alpha$, the image $f(A)\subseteq X_\alpha$ for some α. Suppose the image of f is not contained in X_α for any $\alpha<\lambda$. Then we can find a sequence of points $S = \{x_n\}_{n=1}^{\infty}$ in $f(A)$ and a sequence of ordinals $\{\alpha_n\}_{n=1}^{\infty}$ such that $x_n \in X_{\alpha_n} \setminus X_{\alpha_{n-1}}$. We take $\alpha_0 = 0$. Let μ be the supremum of the α_n. Then μ is a limit ordinal and $\mu \leq \lambda$. The intersection of any subset of S with any X_{α_n} is finite and avoids X_0, and is therefore closed in X_{α_n}. Since X_μ is the colimit in **Top** of the X_{α_n}, it follows that S has the discrete topology as a subspace of X_μ. Since $X_\mu \to \operatorname{colim} X_\alpha$ is a closed inclusion, S also has the discrete topology as a subset of the compact space $f(A) \subseteq \operatorname{colim} X_\alpha$. This is a contradiction, and so $f(A) \subseteq X_\alpha$ for some α as required. ◻

The symbol $\mathbf{R}$ will denote the topological space of real numbers. The symbol D^n will denote the unit disk in $\mathbf{R}^n$, and the symbol S^{n-1} will denote the unit sphere in $\mathbf{R}^n$, so that we have the boundary inclusion $S^{n-1} \to D^n$. In order for this to make sense when $n = 0$, we let $D^0 = \{0\}$ and $S^{-1} = \emptyset$.

Recall that two maps $f, g\colon X \to Y$ are *homotopic* if there is a map $H\colon X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. The map H is called a *homotopy* from f to g. Homotopy is an equivalence relation. In case X and Y are pointed spaces with basepoints x and y, respectively, and f and g are basepoint-preserving maps, then we define f and g to be homotopic if there is a homotopy H between them such that $H(x, t) = y$ for all $t \in I$. Choose $* = (1, 0, \ldots, 0)$ as the basepoint of S^n. Given a space X and a point $x \in X$, we denote the set of pointed homotopy classes of pointed maps from $(S^n, *)$ to (X, x) by $\pi_n(X, x)$, and refer to it as the *nth homotopy set of X at x*. For example, $\pi_0(X, x)$ is the set of path components of X. One can readily verify that the homotopy sets are functorial. Note that $\pi_n(X, x)$ is isomorphic to the set of pointed homotopy classes of pointed maps from $(I^n, \partial I^n)$ to (X, x), where I^n denotes the n-cube $I^{\times n}$, and ∂I^n denotes its boundary. Using this description, it is fairly straightforward to show that $\pi_n(X, x)$ is a group for $n \geq 1$, and that $\pi_n(f, x)$ is a group homomorphism for $n \geq 1$. If $f, g\colon (I^n, \partial I^n) \to (X, x)$ are maps, their product is defined by $(fg)(t_1, \ldots, t_n) = f(2t_1, t_2, \ldots, t_n)$ if $t_1 \leq \frac{1}{2}$, and $(fg)(t_1, \ldots, t_n) = g(2t_1 - 1, t_2, \ldots, t_n)$ if $t_1 \geq \frac{1}{2}$. This product is visibly not associative or unital, but it is so up to homotopy. If $n \geq 2$, we can use the second coordinate instead of the first to define a different multiplication; these two multiplications commute and have the same unit, so they must coincide. This implies that $\pi_n(X, x)$ is abelian for $n \geq 2$. See [**Spa81**, Section 7.2] for details.

If $f, g\colon X \to Y$ are homotopic by a homotopy that fixes a point $x \in X$, then one can easily check that $\pi_n(f, x) = \pi_n(g, x)$. However, if f and g are merely homotopic, then the trajectory of x defines a path $\alpha\colon I \to Y$. Conjugation by α defines an isomorphism $h_\alpha \pi_n(Y, f(x)) \to \pi_n(Y, g(x))$. In this case, one can check, by constructing explicit homotopies, that $\pi_n(g, x) = h_\alpha \circ \pi_n(f, x)$. In particular, $\pi_n(g, x)$ is an isomorphism if and only if $\pi_n(f, x)$ is an isomorphism. For more details, see [**Spa81**, Section 7.3].

We can now define the model structure on **Top**.

Definition 2.4.3. A map $f\colon X \to Y$ in **Top** is a *weak equivalence* if

$$\pi_n(f, x)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and all $x \in X$. Define the set of maps I' to consist of the boundary inclusions $S^{n-1} \to D^n$ for all $n \geq 0$, and define the set J to consist of the inclusions $D^n \to D^n \times I$ which take x to $(x, 0)$, for $n \geq 0$. The define the map f to be a *cofibration* if it is in I'-cof, and define f to be a *fibration* if it is in J-inj.

A map in I'-cell is usually called a *relative cell complex*; a relative CW-complex is a special case of a relative cell complex, where, in particular, the cells can be attached in order of their dimension. Note in particular that the maps of J are relative CW complexes, hence are relative I-cell complexes. Thus J-cof $\subseteq I'$-cof. A fibration is often known as a *Serre fibration* in the literature.

The comments immediately preceding Definition 2.4.3 imply that if f and g are homotopic, then f is a weak equivalence if and only if g is a weak equivalence. Recall that a map $f\colon X \to Y$ is called a *homotopy equivalence* if there is a map $g\colon Y \to X$ such that fg is homotopic to 1_Y and gf is homotopic to 1_X. Then every homotopy equivalence is a weak equivalence.

As usual, we need to verify the hypotheses of Theorem 2.1.19. We begin with the weak equivalences.

LEMMA 2.4.4. *The weak equivalences in* **Top** *are closed under retracts and satisfy the two out of three axiom.*

PROOF. This lemma is straightforward, except for the case where $f\colon X \to Y$ is a weak equivalence and $g\colon Y \to Z$ is a map such that $g \circ f$ is a weak equivalence. In this case, a given point $y \in Y$ may not be in the image of f. However, since $\pi_0(f)$ is an isomorphism, there is a point $x \in X$ and a path $\alpha\colon I \to Y$ from $f(x)$ to y. We then have a commutative diagram

$$\begin{array}{ccc} \pi_n(Y,y) & \longrightarrow & \pi_n(Z,g(y)) \\ \Big\downarrow & & \Big\downarrow \\ \pi_n(Y,f(x)) & \longrightarrow & \pi_n(Z,g(f(x))) \end{array}$$

where the left vertical map is conjugation by the path α, and the right vertical map is conjugation by the path $g \circ \alpha$. The bottom horizontal map is easily seen to be an isomorphism, and it follows that the top horizontal map is also an isomorphism, as required. $\square$

In view of Lemma 2.4.1, in order to apply the small object argument, we need to know that the maps of I'-cell are inclusions.

LEMMA 2.4.5. *Every map in I'-cell is a closed T_1 inclusion.*

PROOF. Since every map of I' is a closed T_1 inclusion, it suffices to verify that closed T_1 inclusions are closed under pushouts and transfinite compositions. Suppose we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & D \end{array}$$

where i is a closed T_1 inclusion. Then j is injective, so it suffices to show that, for every closed set V in C, its image $j(V)$ is closed in D. By definition of the topology on D, it suffices to show that $g^{-1}(j(V))$ is closed in B. But, since i is injective, $g^{-1}(j(V)) = i(f^{-1}(V))$. Since i is a closed inclusion, this is a closed set in B, and so j is a closed inclusion. Furthermore, if $d \in D$ is not in the image of j, then $g^{-1}(d)$ must be a single point not in the image of A. Hence $g^{-1}(d)$ is closed, and since $j^{-1}(d) = \emptyset$ is also closed, it follows that $\{d\}$ is closed in D. Thus j is a closed T_1 inclusion. We leave the proof that closed T_1 inclusions are closed under transfinite compositions to the reader; it is very similar to the proof used in Lemma 2.4.1. $\square$

COROLLARY 2.4.6. *Every map in I'-cof, and hence also every map in J-cof, is a closed T_1 inclusion.*

PROOF. Since we now know that the small object argument can be applied to I', Corollary 2.1.15 implies that we need only check that closed T_1 inclusions are closed under retracts. So suppose $f\colon A \longrightarrow B$ is a retract of the closed T_1 inclusion $g\colon X \longrightarrow Y$ by maps $i\colon A \longrightarrow X$ and $i\colon B \longrightarrow Y$, and corresponding retractions $r\colon X \longrightarrow A$ and $r\colon Y \longrightarrow B$. Then f is injective, so to show that f is a closed inclusion, we need only show that $f(C)$ is closed in B for all closed sets C in A. But one can easily check that $f(C) = i^{-1}gr^{-1}C$, so $f(C)$ is closed in B. Now suppose $b \in B \setminus f(A)$. Then $i(b) \in Y \setminus g(X)$. Indeed, if $i(b) = g(x)$, then $b = rg(x) = fr(x)$, which is impossible. Hence $i(b)$ is closed in Y, and so $b = i^{-1}(i(b))$ is closed in B, and so f is a closed T_1 inclusion. $\square$

For later use, we prove yet another smallness result for compact spaces mapping into cell complexes.

LEMMA 2.4.7. *Suppose λ is an ordinal and $X\colon \lambda \longrightarrow$ **Top** is a λ-sequence of pushouts of I' such that $X_0 = \emptyset$. Then every compact subset of $X_\lambda = \operatorname{colim} X_\alpha$ intersects the interiors of only finitely many cells.*

PROOF. Suppose K is a compact subset of X_λ, and suppose K intersect the interiors of infinitely many cells. Then we can find an infinite set S in K such that each point of S is in the interior of a different cell of X_λ. Let T be an arbitrary subset of S. We will show that T is closed in X_λ, and hence in K. It follows that S is an infinite subset of K with the discrete topology, a contradiction to the compactness of K. To see that T is closed in Y, we will show that $X_\alpha \setminus T$ is open in X_α by transfinite induction on α. The initial step of the induction is clear, as is the limit ordinal case. So suppose $X_\alpha \setminus T$ is open in X_α. Note that $X_{\alpha+1}$ is the union of the subspace X_α and the cell e_α, attached along the boundary of e_α. Hence $X_{\alpha+1} \setminus T$ is union of $X_\alpha \setminus T$ with either the interior of e_α of the interior of e_α minus one point. In either case, $X_{\alpha+1} \setminus T$ is open in $X_{\alpha+1}$. $\square$

We now show that every map in J-cof is a weak equivalence.

LEMMA 2.4.8. *Suppose λ is an ordinal, and $X\colon \lambda \longrightarrow$ **Top** is a λ-sequence of closed T_1 inclusions that are also weak equivalences. Then the map $X_0 \longrightarrow \operatorname{colim} X_\alpha$ is a weak equivalence (and a closed T_1 inclusion).*

PROOF. Let $X_\lambda = \operatorname{colim} X_\alpha$. We show that each map $i_\alpha\colon X_0 \longrightarrow X_\alpha$ is a weak equivalence by transfinite induction on α. This is obvious for $\alpha = 0$. The successor ordinal case of the induction is also clear. Now suppose β is a limit ordinal and i_α is a weak equivalence for all $\alpha < \beta$. Given a point $x \in X_0$ and a homotopy class $[f] \in \pi_n(X_\beta, x)$, Proposition 2.4.2 guarantees that $[f]$ is represented by a (necessarily pointed) map $g\colon (S^n, *) \longrightarrow (X_\alpha, x)$ for some $\alpha < \beta$. Hence $[f]$ is in the image of $\pi_n(X_\alpha, x)$ for some α. Since i_α is a weak equivalence, it follows that $\pi_n(X_0, x) \to \pi_n(X_\beta, x)$ is surjective for all $x \in X_0$.

To prove injectivity, suppose $f, g\colon (S^n, *) \to (X_0, x)$ become homotopic in X_β. Then there is a basepoint-preserving homotopy $H\colon S^n \times I \to X_\beta$ between $i_\beta f$ and $i_\beta g$. Proposition 2.4.2 again guarantees that H factors through a map $H'\colon S^n \times I \to X_\alpha$ for some $\alpha < \beta$. Since all the maps in the diagram X are injective, H' must be a basepoint-preserving homotopy between $i_\alpha f$ and $i_\alpha g$. Hence f and g represent the same element of $\pi_n(X_\alpha, x)$, and so must also represent the same element of $\pi_n(X_0, x)$. □

Proposition 2.4.9. *Every map in J-cof is a trivial cofibration.*

Proof. We have already seen that J-cof $\subseteq I'$-cof, so every map in J-cof is a cofibration. We must show that every map in J-cof is a weak equivalence. Recall that a map $i \to A \to B$ is an *inclusion of a deformation retract* if there is a homotopy $H\colon B \times I \to B$ such that $H(i(a), t) = i(a)$ for all $a \in A$, $H(b, 0) = b$ for all $b \in B$, and $H(b, 1) = ir(b)$ for some map $r\colon B \to A$. It follows that i is an inclusion map and r is a retraction of B onto A. The inclusion of a deformation retract is a homotopy equivalence, and hence a weak equivalence. Furthermore, each map of J is the inclusion of a deformation retract.

We now show that inclusions of deformation retracts are closed under pushouts. Suppose we have a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

where i is the inclusion of a deformation retract. Since I is locally compact Hausdorff, it follows that $D \times I$ is the pushout of $B \times I$ and $C \times I$ over $A \times I$. Let $K\colon B \times I \to B$ be a homotopy that makes i into the inclusion of a deformation retract. Then gK together with the map $C \times I \to D$ that takes (c, t) to jc together define a homotopy $H\colon D \times I \to D$. By construction, $H(c, t) = jc$ for all $c \in C$, and $H(d, 0) = d$ for all $d \in D$. Since $K(b, 1) \in iA$ for all $b \in B$, it follows that $H(d, 1) \in jC$ for all $d \in D$. Since j is an inclusion map, H is a deformation retraction, as required.

We now know that pushouts of maps of J are weak equivalences. They are also closed T_1 inclusions, by Corollary 2.4.6. Lemma 2.4.8 then guarantees that transfinite compositions of pushouts of maps of J are weak equivalences. Hence every map in J-cell is a weak equivalence. Since weak equivalences are closed under retracts, every map in J-cof is a weak equivalence as well. □

We now turn our attention to the fibrations.

Proposition 2.4.10. *Every map in I'-inj is a trivial fibration.*

Proof. Since J-cof $\subseteq I'$-cof, every map in I'-inj is a fibration. Suppose $p\colon X \to Y$ is in I'-inj, and $x \in X$. We must show that the map $\pi_n(p, x)\colon \pi_n(X, x) \to \pi_n(Y, p(x))$ is an isomorphism for all n. Note first that the map $* \to S^n$, as the pushout of the map $S^{n-1} \to D^n$, is in I'-cof. Thus, given a map $g\colon (S^n, *) \to (Y, p(x))$, there is a lift $f\colon (S^n, *) \to (X, x)$ such that $pf = g$. Hence $\pi_n(p, x)$ is surjective. To prove that it is injective, suppose we have two maps $f, g\colon (S^n, *) \to$

(X, x) such that pf and pg represent the same element of $\pi_n(Y, p(x))$. Then there is a homotopy $H\colon S^n \times I \to Y$ such that $H(x, 0) = p(f(x))$, $H(x, 1) = p(g(x))$, and $H(*, t) = p(x)$. The maps f and g define a map $S^n \vee S^n \to X$, where $S^n \vee S^n$ is the space obtained from $S^n \amalg S^n$ by identifying basepoints. The homotopy H defines a map $\overline{H}\colon S^n \wedge I_+ = (S^n \times I)/(* \times I) \to Y$. We have a commutative diagram

$$\begin{array}{ccc} S^n \vee S^n & \xrightarrow{(f,g)} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ S^n \wedge I_+ & \xrightarrow{\overline{H}} & Y \end{array}$$

Since the left-hand vertical map is a relative CW complex (obtained by attaching an $n+1$ disk to $S^n \vee S^n$), we can find a lift in this diagram, giving us the desired homotopy between f and g. Thus $\pi_n(p, x)$ is injective as well, so p is a weak equivalence. □

We must now prove that every trivial fibration is in I'-inj. This statement is claimed without proof in both [**Qui67**] and [**DS95**]. This would seem to indicate that there is a simple proof; the author, however, has been unable to find one. We will therefore give a complete proof, referring to [**Spa81**] where necessary.

LEMMA 2.4.11. *Suppose $p\colon X \to Y$ is a map. Then $p \in I'$-inj if and only if the map $Q(i, p)\colon X^B \to P(i, p) = X^A \times_{Y^A} Y^B$ is surjective for all maps $i\colon A \to B$ in I'. In particular, if $Q(i, p)$ is a trivial fibration for all $i \in I'$, then $p \in I'$-inj.*

PROOF. The first part holds by an adjointness argument, using the fact that the domains and codomains of the maps of I' are locally compact Hausdorff. The second part holds because all trivial fibrations are surjective. To see this, suppose $q\colon W \to Z$ is a trivial fibration. Then $\pi_0(q)$ is surjective, so given $z \in Z$, thee is a point $w \in W$ and a path $H\colon I \to Z$ from qw to z. Since q is a fibration, we can lift this path to a path $H'\colon I \to X$ such that $H'(0) = w$. Then $qH'(1) = H(1) = z$, so q is surjective. □

We can now outline the proof.

THEOREM 2.4.12. *Every trivial fibration is in I'-inj.*

PROOF. Suppose $p\colon X \to Y$ is a trivial fibration. By Lemma 2.4.11, it suffices to show that the map $Q(i, p)$ is a trivial fibration, where $i\colon S^{n-1} \to D^n$ is the boundary inclusion. By Lemma 2.4.13, $Q(i, p)$ is a fibration. Consider the pullback square

$$\begin{array}{ccc} P(i,p) & \longrightarrow & Y^{D^n} \\ \big\downarrow & & \big\downarrow \\ X^{S^{n-1}} & \longrightarrow & Y^{S^{n-1}} \end{array}$$

By Corollary 2.4.14, the right-hand vertical map is a fibration. By Lemma 2.4.17, the bottom horizontal map is a weak equivalence. By Proposition 2.4.18, the top horizontal map is also a weak equivalence. Using Lemma 2.4.15, we find that the composite $X^{D^n} \to P(i, p) \to Y^{D^n}$ is a weak equivalence. The two out of three property then guarantees that $Q(i, p)$ is a weak equivalence, as required. □

We begin the detailed analysis by showing that the map $Q(i, p)$ is a fibration.

LEMMA 2.4.13. *Suppose $p\colon X \to Y$ is a fibration, and $i\colon S^{n-1} \to D^n$ is the boundary inclusion. Then the map $Q(i,p)$ is a fibration.*

PROOF. By the same adjointness argument used in Lemma 2.4.11, it suffices to show that the map

$$(D^m \times S^{n-1} \times I) \amalg_{D^m \times S^{n-1} \times \{0\}} (D^m \times D^n \times \{0\}) \to D^m \times D^n \times I$$

is in J-cof for all $m, n \geq 0$. The pair (D^n, S^{n-1}) is homeomorphic to the pair $(I^n, \partial I^n)$, where I^n is the n-cube $I \times I \times \cdots \times I$, and the boundary is the collection of points where at least one coordinate is 0 or 1. Therefore the map

$$f\colon (S^{n-1} \times I) \amalg_{S^{n-1} \times \{0\}} (D^n \times \{0\}) \to D^n \times I$$

is homeomorphic to the map

$$(\partial I^n \times I) \amalg_{\partial I^n \times \{0\}} (I^n \times \{0\}) \to I^n \times I$$

which is in turn homeomorphic to the map $D^n \times \{0\} \to D^n \times I$, by flattening out the sides of the box $(\partial I^n \times I) \cup I^n \times \{0\}$. Thus the map f is in J-cof, and the map $D^m \times f$ is homeomorphic to $D^{m+n} \times \{0\} \to D^{m+n} \times I$, so is also in J-cof. $\square$

COROLLARY 2.4.14. *Every topological space is fibrant. Hence the map $Y^{D^n} \to Y^{S^{n-1}}$ is a fibration for all $n \geq 0$.*

PROOF. Every map of J is the inclusion of a retract. Hence every map of the form $Y \to *$ has the right lifting property with respect to J, so is a fibration. It follows from Lemma 2.4.13 applied to the fibration $Y \to *$ that the map $Y^{D^n} \to Y^{S^{n-1}}$ is a fibration. $\square$

LEMMA 2.4.15. *If $p\colon X \to Y$ is a weak equivalence, so is $p^{D^n}\colon X^{D^n} \to Y^{D^n}$.*

PROOF. Let 0 denote the origin in D^n. The evaluation at 0 map $q\colon Z^{D^n} \to Z$ has a section $j\colon Z \to Z^{D^n}$ that takes z to the constant map at z. The composite qj is the identity, and the composite jq is homotopic to the identity by the homotopy $H\colon Z^{D^n} \times I \to Z^{D^n}$ defined by $H(f,t)(x) = f(tx)$. Therefore q is a homotopy equivalence, and hence a weak equivalence. The lemma then follows using the two out of three property. $\square$

We must still show that $p^{S^{n-1}}$ is a weak equivalence, and that weak equivalences are preserved by pullbacks through fibrations. The basic tool for both of these arguments is the long exact homotopy sequence of a fibration.

LEMMA 2.4.16. *Suppose $p\colon X \to Y$ is a fibration in* **Top**, *and $x \in X$. Let $F = p^{-1}(p(x))$, and $i\colon F \to X$ denote the inclusion. Then there is a long exact sequence*

$$\begin{aligned} \dots \to \pi_{n+1}(Y, p(x)) \xrightarrow{d_*} \pi_n(F, x) \xrightarrow{\pi_n(i,x)} \pi_n(X, x) \xrightarrow{\pi_n(p,x)} \pi_n(Y, p(x)) \\ \xrightarrow{d_*} \pi_{n-1}(F, x) \to \dots \xrightarrow{\pi_0(p,x)} \pi_0(Y, p(x)). \end{aligned}$$

which is natural with respect to commutative squares

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \longrightarrow & Y' \end{array}$$

where p and p' are fibrations. Here d_* is a group homomorphism $\pi_n(Y, p(x)) \to \pi_{n-1}(F, x)$ when $n > 1$, and exactness just means the image of one map is the kernel of the next.

This lemma is proved in [**Spa81**, Theorem 7.2.10], but the proof is better left as an exercise for the interested reader.

LEMMA 2.4.17. *Suppose $p\colon X \to Y$ is a weak equivalence. Then $p^{S^n}\colon X^{S^n} \to Y^{S^n}$ is a weak equivalence for all $n \geq -1$, where $S^{-1} = \emptyset$.*

PROOF. The proof is by induction on n. The pushout square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & S^n \end{array}$$

gives rise to a pullback square

$$\begin{array}{ccc} Z^{S^n} & \longrightarrow & Z^{D^n} \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & Z^{S^{n-1}} \end{array}$$

for any Z, where, by Lemma 2.4.13, the right-hand vertical map, and hence also the left-hand vertical map, is a fibration. The map p induces a map between this pullback square with $Z = X$ to this pullback square with $Z = Y$. The homotopy long exact sequence of these fibrations then allows us to do the induction step. To be more precise, given a point $\alpha \in X^{S^n}$, let F_X denote the fiber of $X^{S^n} \to X$ containing α. Let F_Y denote the corresponding fiber of $Y^{S^n} \to Y$ containing $p\alpha$. We first show that $F_X \to F_Y$ is a weak equivalence. Note that F_X is also the fiber of $X^{D^n} \to X^{S^{n-1}}$, and similarly for F_Y, so a five-lemma argument using the inductive hypothesis and the long exact sequence of a fibration shows that the map $F_X \to F_Y$ induces an isomorphism on positive-dimensional homotopy. The five-lemma runs into trouble on π_0, but a point of F_X just a basepoint-preserving map $(S^n, *) \to (X, \alpha(0))$, and similarly for F_Y. By definition, $\pi_0(F_X, \alpha) = \pi_n(X, \alpha(0))$, and similarly for Y. Hence, by hypothesis, the induced map $\pi_0(F_X, \alpha) \to \pi_0(F_Y, p\alpha)$ is an isomorphism, and thus $F_X \to F_Y$ is a weak equivalence.

Now we would like to use the five-lemma argument again to show that $X^{S^n} \to Y^{S^n}$ is a weak equivalence. Once again, there is no difficulty with positive-dimensional homotopy, but we run into trouble with π_0. Suppose we have two points α and β of X^{S^n} that are sent to the same path component of Y^{S^n}. Since π_0 does not depend on the choice of basepoint, we may as well choose α as our basepoint. Then the long exact sequence implies that $\alpha(*)$ and $\beta(*)$ lie in the basepoint path component of X. We can then repeat the usual five-lemma argument to conclude that $\pi_0(p^{S^n})$ is injective.

To see that $\pi_0(p^{S^n})$ is surjective, note that $\pi_0(X^{S^n}, \alpha) \cong [S^n, X]$, where $[A, Z]$ means (free) homotopy classes of maps from A to Z. Similarly, $\pi_0(Y^{S^n}, p\alpha) \cong [S^n, Y]$. Now, any element $[f]$ of $[S^n, Y]$ represents an element of $\pi_n(Y, y)$ for some $y \in Y$. Since $\pi_0(p)$ is an isomorphism, there is a point $x \in X$ such that $p(x)$ is in the same path component of Y. Choose a path ω from y to $p(x)$. Then, as in the discussion preceding Definition 2.4.3, there is an isomorphism $h_\omega\colon \pi_n(Y, y) \to$

$\pi_n(Y, p(x))$, given by conjugating with ω. Choose a preimage $[g] \in \pi_n(X, x)$ of $h_\omega[f]$. Then pg is homotopic to $h_\omega f$ by a basepoint-preserving homotopy, and it follows that pg is freely homotopic to f. See [**Spa81**, Section 7.3] for more details. □

Finally, we prove that topological spaces are *right proper*, in the following sense.

PROPOSITION 2.4.18. *Suppose we have a pullback square*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

in **Top**, *where p is a fibration and g is a weak equivalence. Then f is a weak equivalence.*

PROOF. This is another five-lemma argument. Given $w \in W$, let F denote $q^{-1}(q(w))$. Then the induced map $F \to F' = p^{-1}(pf(w))$ is a homeomorphism. Hence the induced map $\pi_n(F, w) \to \pi_n(F', f(w))$ is an isomorphism for all n. The map $\pi_n(Z, q(w)) \to \pi_n(Y, gq(w))$ is also an isomorphism for all n since g is a weak equivalence. Lemma 2.4.16 and a diagram chase then show that $\pi_n(f, w)$ is an isomorphism for $n \geq 1$. In fact, this same diagram chase also shows that $\pi_0(f)$ is an injection, using the trick of changing the basepoint that we used the proof of Lemma 2.4.17.

We still must show that $\pi_0(f)$ is surjective. Suppose $x \in X$. Then there is a point $z \in Z$ and a path $\alpha \colon D^1 \to Y$ from $p(x)$ to $g(z)$, since $\pi_0(g)$ is surjective. Because p is a fibration, we can find a lift of this path to a path $\beta \colon I \to X$ such that $\beta(0) = x$. In particular, $(z, \beta(1)) \in W$ and there is a path in X from $f(z, \beta(1)) = \beta(1)$ to x. Hence $\pi_0(f)$ is surjective, as required. □

We have now completed the proof of Theorem 2.4.12. The recognition theorem 2.1.19 then immediately implies that topological spaces form a model category.

THEOREM 2.4.19. *There is a finitely generated model structure on* **Top** *with I' as the set of generating cofibrations, J as the set of generating trivial cofibrations, and the weak equivalences as above. Every object of* **Top** *is fibrant, and the cofibrant objects are retracts of relative cell complexes.*

The corollary below then follows from Proposition 1.1.8 and Lemma 2.1.21.

COROLLARY 2.4.20. *There is a finitely generated model structure on the category* $\mathbf{Top}_*$ *of pointed topological spaces, with generating cofibrations I'_+ and generating trivial cofibrations J_+. Every object is fibrant, and a map is a cofibration, weak equivalence, or fibration if and only if its image in* **Top** *is so.*

There are several model categories associated with the model category of topological spaces that we now consider. As we have discussed above, the function space $(-)^X$ is not a right adjoint to the product $- \times X$ in general. This is a serious drawback with the category **Top**, but there are several subcategories of **Top** which do not have this drawback. We will discuss two of them.

DEFINITION 2.4.21. Let X be a topological space.

1. X is *weak Hausdorff* if, for every continuous map $f \colon K \to X$, where K is compact Hausdorff, the image $f(K)$ is closed in X.

2. A subset U of X is *compactly open* if for every continuous map $f\colon K \to X$ where K is compact Hausdorff, $f^{-1}(U)$ is open in K. Similarly, U is *compactly closed* if for every such map f, $f^{-1}(U)$ is closed in K.
3. X is a *Kelley space*, or a *k-space*, if every compactly open subset is open, or equivalently, if every compactly closed subset is closed. A k-space that is also weak Hausdorff is called a *compactly generated* space. We denote the full subcategory of **Top** consisting of k-spaces by $\mathbf{K}$, and the full subcategory of $\mathbf{K}$ consisting of compactly generated spaces by $\mathbf{T}$.
4. The *k-space topology* on X, denoted kX, is defined by letting U be open in kX if and only if U is compactly open in X.
5.

The definitive source for k-spaces and compactly generated spaces is [**Lew78**, Appendix]; see also [**Wyl73**]. The category $\mathbf{T}$ is the most commonly used category of topological spaces in algebraic topology; in particular, it is used in [**LMS86**], and hence also in [**EKMM97**], and also in [**HSS98**].

The basic facts about k-spaces and compactly generated spaces are contained in the following omnibus proposition. See [**Lew78**, Appendix] for the proof.

PROPOSITION 2.4.22.
1. *The inclusion functor $\mathbf{K} \to \mathbf{Top}$ has a right adjoint and left inverse $k\colon \mathbf{Top} \to \mathbf{K}$ that takes X to X with its k-space topology.*
2. *The inclusion functor $\mathbf{T} \to \mathbf{K}$ has a left adjoint and right inverse $w\colon \mathbf{K} \to \mathbf{T}$ that takes X to its maximal weak Hausdorff quotient.*
3. *$\mathbf{K}$ has all small limits and colimits, where colimits are taken in $\mathbf{Top}$ and limits are taken by applying k to the limit in $\mathbf{Top}$.*
4. *$\mathbf{T}$ has all small limits and colimits, where limits are taken in $\mathbf{K}$ and colimits are taken by applying w to the colimit in $\mathbf{K}$.*
5. *For $X, Y \in \mathbf{K}$, define $C(X, Y)$ to be the set of continuous maps from X to Y, given the topology generated by the subbasis $S(f, U)$. Here U is an open set in Y, $f\colon K \to X$ is a continuous map from a compact Hausdorff space K into X, and $S(f, U)$ is the set of all $g\colon X \to Y$ such that $(g \circ f)(K) \subseteq U$. Define $\operatorname{Hom}(X, Y)$ to be $kC(X, Y)$. Then we have a natural isomorphism $\mathbf{K}(k(X \times Y), Z) \to \mathbf{K}(X, \operatorname{Hom}(Y, Z))$ for all $X, Y, Z \in \mathbf{K}$.*
6. *If $X \in \mathbf{K}$ and $Y \in \mathbf{T}$, then $C(X, Y)$ is weak Hausdorff. Hence, for $X, Y, Z \in \mathbf{T}$, we have a natural isomorphism $\mathbf{T}(k(X \times Y), Z) \to \mathbf{T}(X, \operatorname{Hom}(Y, Z))$.*

The biggest drawback of $\mathbf{T}$ is that it is difficult to understand colimits. However, in practive most colimits are already weak Hausdorff, so there is no need to apply w. This is true for transfinite compositions of injections and pushouts of closed inclusions [**Lew78**, Appendix]. We also point out that an adjointness argument shows that w preserves the k-space product.

Both $\mathbf{K}$ and $\mathbf{T}$ are model categories in their own right.

THEOREM 2.4.23. *The category $\mathbf{K}$ of k-spaces admits a finitely generated model structure, where a map is a cofibration (fibration, weak equivalence) if and only if it is so in $\mathbf{Top}$. The inclusion functor $\mathbf{K} \to \mathbf{Top}$ is a Quillen equivalence.*

PROOF. We define a map to be a weak equivalence if and only if it is so in **Top**, and we use the same sets I' of generating cofibrations and J of generating trivial cofibrations. Then it is clear that a map is a fibration or trivial fibration if and only if it is so in **Top**, and hence that the trivial fibrations form the class I'-inj. If

$f \in \mathbf{K}$ is a cofibration in **Top**, then it has the left lifting property with respect to all maps in I'-inj, and hence is also a cofibration in $\mathbf{K}$. Conversely, suppose $f \in I'$-cof in $\mathbf{K}$. Then f is a retract of a transfinite composition of pushouts of I'. Since the forgetful functor $\mathbf{K} \to \mathbf{Top}$ preserves colimits, it follows that f is in I'-cof in **Top**. Therefore, a map is a cofibration if and only if it is so in **Top**. Similarly, a map is in J-cof in $\mathbf{K}$ if and only f it is a map in $\mathbf{K}$ which is in J-cof as a map in **Top**. Thus, the trivial cofibrations coincide with J-cof, and the cofibrations are closed T_1 inclusions. The recognition theorem 2.1.19 therefore applies.

To see that the forgetful functor is a Quillen equivalence, note that it certainly reflects weak equivalences between cofibrant objects. It follows from Corollary 1.3.16 that we need only show the map $kX \to X$ is a weak equivalence. But, if A is compact Hausdorff, then $\mathbf{Top}(A, X) = \mathbf{Top}(A, kX)$. It follows that $kX \to X$ is a weak equivalence as required. □

The corollary below then follows from Proposition 1.1.8 and Proposition 1.3.17.

COROLLARY 2.4.24. *There is a finitely generated model structure on the category $\mathbf{K}_*$ of pointed k-spaces, with generating cofibrations I'_+ and generating trivial cofibrations J_+. A map is a cofibration, fibration, or weak equivalence if and only if it is so in* **Top**. *The inclusion functor is a Quillen equivalence $\mathbf{K}_* \to \mathbf{Top}_*$.*

Similarly, we have the following theorem.

THEOREM 2.4.25. *The category $\mathbf{T}$ of compactly generated spaces admits a finitely generated model structure, where a map is a cofibration (fibration, weak equivalence) if and only if it is so in $\mathbf{K}$. The functor $w\colon \mathbf{T} \to \mathbf{K}$ is a Quillen equivalence.*

PROOF. We use the same argument as in Theorem 2.4.23. We use the same generating sets I' and J. The same argument shows that a map is a fibration in $\mathbf{T}$ if and only if it is a fibration in $\mathbf{K}$, and hence that trivial fibrations coincide with I'-inj. Once again, a map in $\mathbf{T}$ that is a cofibration in $\mathbf{K}$ has the left lifting property with respect to all maps in I'-inj, so is a cofibration in $\mathbf{T}$. The converse is slightly more complicated, since the forgetful functor $\mathbf{T} \to \mathbf{K}$ does not preserve all colimits. However, it does preserves pushouts of closed inclusions and transfinite compositions of injections, and this is sufficient to guarantee that a cofibration in $\mathbf{T}$ is also a cofibration in $\mathbf{K}$. The same argument implies that a map in $\mathbf{T}$ is in J-cof as a map of $\mathbf{T}$ if and only if it is J-cof as a map of $\mathbf{K}$. It follows that the trivial cofibrations are the class J-cof, and that the cofibrations are closed inclusions. (Note that every space in $\mathbf{T}$ is T_1). The recognition theorem 2.1.19 then completes the proof that $\mathbf{T}$ is a cofibrantly generated model category.

The forgetful functor $\mathbf{T} \to \mathbf{K}$ obviously preserves fibrations and trivial fibrations and reflects weak equivalences. By Corollary 1.3.16, to show that $w\colon \mathbf{K} \to \mathbf{T}$, it suffices to show that $X \to wX$ is a weak equivalence for all cofibrant X. However, cofibrant X are already weak Hausdorff, since w preserves the colimits used to form a cofibrant X from I', so in fact $wX \to X$ is an isomorphism for cofibrant X. □

We get the standard corollary for pointed compactly generated spaces.

COROLLARY 2.4.26. *There is a finitely generated model structure on $\mathbf{T}_*$ with generating cofibrations I'_+ and generating trivial cofibrations J_+. A map is a cofibration, fibration, or weak equivalence if and only if it is so in* **Top**. *The functor $w_*\colon \mathbf{K}_* \to \mathbf{T}_*$ is a Quillen equivalence.*

2.5. Chain complexes of comodules over a Hopf algebra

Suppose that k is a field and B is a commutative Hopf algebra over k. Let B-comod denote the category of left B-comodules, and let $\mathrm{Ch}(B)$ denote the category of chain complexes of left B-comodules. There is some ambiguity of notation here, since $\mathrm{Ch}(B)$ could also denote the category of chain complexes of B-modules, but this ambiguity should be easily tolerated. We will put a cofibrantly generated model category structure on $\mathrm{Ch}(B)$ so the associated homotopy category is the stable homotopy category considered in [**HPS97**, Section 9.5].

Throughout this section, the symbol $A \otimes B$ will mean $A \otimes_k B$, and the symbol $\mathrm{Hom}(A, B)$ will mean $\mathrm{Hom}_k(A, B)$.

2.5.1. The category of B-comodules. Since the category of B-comodules is considerably less familiar to most mathematicians than the category of modules over a ring, we will need to prove some basic results about this category first. These results can also be found in [**HPS97**, Section 9.5].

First we remind the reader that a commutative Hopf algebra over k is, by definition, a cogroup object in the category of commutative k-algebras. Equivalently, a commutative Hopf algebra B is a commutative k-algebra B, whose unit we always denote by $\eta\colon k \to B$ and whose multiplication we always denote by $\mu\colon B \otimes B \to B$, together with maps of algebras $\Delta\colon B \to B \otimes B$ (the comultiplication or diagonal), $\varepsilon\colon B \to k$ (the counit), and $\chi\colon B \to B$ (the conjugation or inverse), satisfying the following conditions. We require that Δ is coassociative, so that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. We require that Δ is counital, so that $(\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = 1_B$, where we have used the identification $B \otimes k \cong B \cong k \otimes B$. And we require that the inverse be an inverse, so that $\mu(1 \otimes \chi)\Delta = \eta\varepsilon = \mu(\chi \otimes 1)\Delta$. Note that, in considering Δ as a map of algebras, the multiplication on $B \otimes B$ that we use is the composite

$$B \otimes B \otimes B \otimes B \xrightarrow{1 \otimes T \otimes 1} B \otimes B \otimes B \otimes B \xrightarrow{\mu \otimes \mu} B \otimes B$$

where T is the twist map. We have an obvious notion of a map of Hopf algebras as well.

Since χ corresponds to the inverse map, the usual properties of the inverse map hold for χ. For example, we have $\chi^2 = 1$ and, corresponding to the relation $(xy)^{-1} = y^{-1}x^{-1}$, we have $\Delta\chi = T(\chi \otimes \chi)\Delta$.

A standard example of a commutative Hopf algebra is $F(G, k)$, the algebra of functions from a finite group G to k (dual to the group ring $k[G]$). Any affine group scheme over k corresponds to a Hopf algebra over k. For example $SL_2(k)$ corresponds to the Hopf algebra $k[x_{11}, x_{12}, x_{21}, x_{22}][d^{-1}]$, where d is the determinant $x_{11}x_{22} - x_{12}x_{21}$, and where the comultiplcation is dual to matrix multiplication. Thus $\Delta(x_{11}) = x_{11} \otimes x_{11} + x_{12} \otimes x_{21}$.

We can also consider graded commutative Hopf algebras. In this case, we would require B to be a graded commutative k-algebra. That is, we would define $T(x \otimes y) = (-1)^{d(x)d(y)}(y \otimes x)$, where $d(x)$ denotes the degree of x. The dual Steenrod algebra A_* is an example of a graded commutative Hopf algebra.

Given a commutative Hopf algebra B, its dual $B^* = \mathrm{Hom}(B, k)$ is always an algebra, but need not have any kind of diagonal map. Indeed, there is a natural map $B^* \otimes B^* \to (B \otimes B)^*$, but this map is only an isomorphism when B is finite-dimensional. Thus we can get a multiplication on B^* dual to Δ, but not always a comultiplication. If B is graded, B^* is defined using the graded Hom, so $B^*_n =$

$\operatorname{Hom}(B_{-n}, k)$. In this case, B^* is a graded algebra, which has a comultiplication if B is finite-dimensional in each degree.

Given a commutative Hopf algebra B, recall that a (left) B-comodule is a k-vector space M equipped with a map of vector spaces $\psi\colon M \to B \otimes M$ which is coassociative and counital. That is, we have $(\Delta \otimes 1)\psi = (1 \otimes \psi)\psi$ and $(\varepsilon \otimes 1)\psi = 1_M$ under the identification $M \cong k \otimes M$. Given comodules M and N, a comodule map $M \xrightarrow{f} N$ is a vector space map such that $\psi f = (1 \otimes f)\psi$. Of course, if B is graded, we require a B-comodule to be graded and the coaction ψ to be a graded map.

If M is a B-comodule, we can think of M as a B^*-module using the structure map

$$B^* \otimes M \xrightarrow{1 \otimes \psi} B^* \otimes B \otimes M \xrightarrow{\mathrm{ev} \otimes 1} M$$

where ev is the evaluation map $B^* \otimes B \to k$. This defines a functor from the category of B-comodules to the category of B^*-modules which is obviously full and faithful. Hence the category of B-comodules is isomorphic to a full subcategory of the category of B^*-modules. We must determine exactly which subcategory this is.

Choose a basis $\{b_i\}$ for B over k, which should be homogeneous if B is graded. We will commonly write $\psi(m) = \sum b_i \otimes m_i$ for $m \in M$, where M is a left B-comodule. Of course, all but finitely many of the m_i must be 0 in this description.

The most important fact about comodules is the following.

Lemma 2.5.1. *Suppose M is a B-comodule, and $m \in M$. Then the subcomodule generated by m is finite-dimensional.*

Proof. Write $\psi(m) = \sum b_i \otimes m_i$ as above. Let M' denote the vector space spanned by the m_i. Then M' is a subcomodule of M, as may be seen by applying $1 \otimes \psi$ and using coassociativity. Since M' contains m and is finite-dimensional, the result follows. □

Corollary 2.5.2. *Suppose B is a commutative Hopf algebra and M is a* simple *B-comodule. That is, suppose M is nonzero and has no nontrivial proper subcomodules. Then M is finite-dimensional.*

Proof. Take a nonzero element $m \in M$. Then the comodule generated by m must be M, since M is simple. Lemma 2.5.1 completes the proof. □

Corollary 2.5.3. *Suppose B is a commutative Hopf algebra over a field k. Then every nonzero comodule has a simple subcomodule.*

Proof. Certainly every comodule has a finite-dimensional subcomodule, by Lemma 2.5.1. Since every one-dimensional comodule is simple, we can prove by induction on the dimension that every finite-dimensional comodule has a simple subcomodule. □

Lemma 2.5.1 motivates the following definition.

Definition 2.5.4. Define a B^*-module M to be *tame* if, for all $m \in M$, the submodule generated by m is finite-dimensional.

If M is a B-comodule, then when we think of M as a B^*-module as above, M is tame. This leads to the following proposition.

PROPOSITION 2.5.5. *There is an isomorphism of categories which is the identity on objects between the category of left B-comodules and the category of tame left B^*-modules. Furthermore, the inclusion functor from left B-comodules to left B^*-modules has a right adjoint R.*

PROOF. We have already seen that a B-comodule M can be made into a B^*-module. In concrete terms, we define for $f \in B^*$ and $m \in M$, $fm = \sum f(b_i)m_i$, where $\psi(m) = \sum b_i \otimes m_i$ and $\{b_i\}$ is a basis for B. In particular, the sub-B^*-module generated by m is the sub-B-comodule generated by m, so M is a tame B^*-module.

Conversely, suppose M is a B^*-module, with structure map $B^* \otimes M \to M$. This structure map corresponds to a map $M \to \operatorname{Hom}(B^*, M)$. There is an inclusion $B \otimes M \xrightarrow{i} \operatorname{Hom}(B^*, M)$ defined by $i(b \otimes m)(f) = f(b)m$. The image of i is the set of $g \in \operatorname{Hom}(B^*, M)$ which factor through a finite-dimensional quotient of B^*. In particular, if M is tame, the map $M \to \operatorname{Hom}(B^*, M)$ factors through $B \otimes M$, giving us the required B-comodule structure.

Now, given an arbitrary B^*-module M, we can define RM to be the submodule consisting of all m such that B^*m is finite-dimensional. Then RM is obviously tame and corresponds to a B-comodule, which we also denote RM. This defines a functor R which is both a left inverse and a right adjoint to the inclusion functor. □

COROLLARY 2.5.6. *The category of B-comodules has all small limits and colimits.*

PROOF. Given a functor F from a small category to B-comodules, let $\operatorname{colim} F$ denote a colimit of F in the category of vector spaces. Because the tensor product preserves colimits, there is a unique B-comodule structure on $\operatorname{colim} F$, and this comodule structure makes $\operatorname{colim} F$ into a colimit in the category of B-comodules.

This is not true with limits. There may be no B-comodule structure on $\lim F$, but there is certainly a B^*-module structure on it making $\lim F$ into a limit in the category of B^*-modules. Hence $R(\lim F)$ is a B-comodule, which one can check is a limit of F in the category of B-comodules. □

Corollary 2.5.6 then implies that $\operatorname{Ch}(B)$ also has all small limits and colimits, taken dimensionwise.

COROLLARY 2.5.7. *Every B-comodule is small. Every finite-dimensional B-comodule is finite.*

PROOF. For the first part, we already know that every B^*-module is small, by Example 2.1.6. For the second part, suppose we have a limit ordinal λ and a λ-sequence $X\colon \lambda \to BB\text{-comod}$. Suppose A is finite-dimensional. The usual arguments show that the map $\operatorname{colim} BB\text{-comod}(A, X_\alpha) \to BB\text{-comod}(A, \operatorname{colim} X_\alpha)$ is injective, and that any map $A \to \operatorname{colim} X_\alpha$ factors through a map of vector space $g\colon A \to X_\alpha$ for some $\alpha < \lambda$. The map g may not be a comodule map. However, for each basis element a of A, there is a $\beta_a < \lambda$ so that g respects the diagonal of a when we go out to X_{β_a}. Since A is finite-dimensional, we can find a simgle β and a factorization of f through a map of comodules $A \to X_\beta$, as required. □

The same argument as in Lemma 2.3.2 then shows that every object on $\operatorname{Ch}(B)$ is small, and that every totally finite-dimensional complex is finite.

We now show that the category of B-comodules is a closed symmetric monoidal category. See Section 4.1 for a precise definition of a closed symmetric monoidal category. The main point is that there is a tensor product and a Hom functor. Indeed, given comodules M and N, we can put a B-comodule structure on $M \otimes N$ by the composite

$$M \otimes N \xrightarrow{\psi \otimes \psi} B \otimes M \otimes B \otimes N \xrightarrow{1 \otimes T \otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{\mu \otimes 1 \otimes 1} B \otimes M \otimes N$$

where ψ denotes the comodule structure on M and N, T is the commutativity isomorphism of the tensor product, and μ is the multiplication map of B. We require B to be commutative in order for this composite to be coassociative. We leave it to the reader to check that this tensor product is commutative, associative, and unital up to coherent isomorphism. The unit is the trivial comodule k. If B is graded, then $M \otimes N$ is defined to be the graded tensor product, so that $(M \otimes N)_m = \bigoplus M_k \otimes N_{m-k}$, and the B-comodule structure is defined as above.

We claim that this tensor product has a right adjoint (in each variable). This right adjoint will of course be related to $\operatorname{Hom}_k(M, N)$, but it is rather complicated to define it. First suppose M and N are tame B^*-modules. Dual to the multiplication and conjugation on B we have maps $\Delta^* \colon B^* \to \operatorname{Hom}_k(B, B^*)$ and $\chi^* \colon B^* \to B^*$. We are going to define a B^*-module structure on $\operatorname{Hom}_k(M, N)$ by a horrendous formula, which is necessary since we are not assuming that B is finite-dimensional. Recall we have chosen a basis $\{b_i\}$ for B; we let $\{b_i^*\}$ denote the dual basis for B^*. Given $f \in \operatorname{Hom}_k(M, N)$ and $u \in B^*$, we define uf by the formula

$$uf(x) = \sum_i b_i^* f[\chi^*((\Delta^* u)(b_i))x].$$

Since M and N are both tame, this sum is in fact finite. Indeed, B^*x is finite-dimensional, so $f(B^*x)$ is as well; thus $b_i^* f(B^*x)$ is zero for almost all i. We leave it to the reader to check that this gives a B^*-module structure on $\operatorname{Hom}_k(M, N)$. If B is graded, we get a graded B^*-module structure on the graded Hom.

Hence, given B-comodules M and N, we get a B-comodule $R\operatorname{Hom}_k(M, N)$. We leave it to the reader to verify that this functor is indeed right adjoint to the tensor product.

Now we consider injective B-comodules. First note that the forgetful functor U from B-comodules to k-vector spaces has a right adjoint. Indeed, given a vector space V, this right adjoint takes V to the comodule $B \otimes V$ with structure map $\Delta \otimes 1$. A map $M \xrightarrow{f} V$ of vector spaces induces a comodule map $(1 \otimes f)\psi$. Conversely, a comodule map $g \colon M \to B \otimes V$ induces a vector space map $\varepsilon g \colon M \to V$. A comodule of the form $B \otimes V$ for some vector space V is called a *cofree comodule.* Note that adjointness implies that cofree comodules are injective.

PROPOSITION 2.5.8. *Suppose B is a commutative Hopf algebra over a field k.*

(a) *For any comodule M, there is a natural isomorphism $B \otimes M \xrightarrow{t} B \otimes UM$, where $B \otimes UM$ denotes the cofree comodule on UM and $B \otimes M$ denotes the tensor product of comodules.*
(b) *Every comodule is isomorphic to a subcomodule of an injective comodule. In particular, there are enough injectives in the category of B-comodules and so we can define the functors $\operatorname{Ext}_B^n(M, N)$ for comodules M and N as usual.*
(c) *A comodule is injective if and only if it is a retract of a cofree comodule.*
(d) *Coproducts of injective comodules are injective.*
(e) *If I is injective and M is any comodule, then $I \otimes M$ is injective.*

(f) *A comodule I is injective if and only if, for all inclusions $M \xrightarrow{i} N$ of finite-dimensional comodules and all maps $f\colon M \to I$, there is an extension $g\colon N \to I$ such that $gi = f$.*

(g) *A comodule I is injective if and only if $Ext^1_B(M, I) = 0$ for all simple comodules M.*

Proof. For part (a), define t as the composite

$$B \otimes M \xrightarrow{1\otimes\psi} B \otimes B \otimes M \xrightarrow{\mu\otimes 1} B \otimes M$$

because μ is a map of coalgebras, t is a comodule map. The inverse of t is the composite

$$B \otimes M \xrightarrow{1\otimes\psi} B \otimes B \otimes M \xrightarrow{1\otimes\chi\otimes 1} B \otimes B \otimes M \xrightarrow{\mu\otimes 1} B \otimes M$$

We leave it to the reader to check that these maps are inverse isomorphisms of comodules.

For part (b), suppose M is a comodule. The injection $k \xrightarrow{\eta} B$ of comodules gives an injection $M \to B \otimes M \cong B \otimes UM$ of comodules. Since $B \otimes UM$ is cofree, it is injective. We then define $\operatorname{Ext}^*(M, N)$ in the usual way, by taking an injective resolution of N, applying $\operatorname{Hom}_B(M, -)$ to it, and taking homology.

For part (c), we have already seen that any cofree comodule, and hence any retract of a cofree comodule, is injective. By part (b), any injective embeds into a cofree comodule, and this embedding must split.

Part (d) follows from part (c) and the fact that direct sums of cofree comodules are cofree. Similarly, if I is injective and M is any comodule, then I is a retract of $B \otimes I$. Hence $I \otimes M$ is a retract of $B \otimes (I \otimes M) \cong B \otimes U(I \otimes M)$, which is cofree. Thus $I \otimes M$ is injective.

For part (f), we use Zorn's lemma. Suppose I satisfies the hypotheses of part (f), $i\colon M \to N$ is an arbitrary injection of comodules, and $f\colon M \to I$ is a map. Let S denote the set of all pairs (P, h), where P is a subcomodule of N containing $i(M)$ and $hi = f$. Partially order S by defining $(P, h) \leq (P', h')$ if and only if $P \subseteq P'$ and h' is an extension of h. It is easy to see that a totally ordered subset of S has an upper bound, so by Zorn's lemma S has a maximal element (N', g). We claim that $N' = N$. Indeed, suppose not, and choose an $n \in N$ but not in N'. Let L denote the subcomodule generated by n, which is finite-dimensional. Then the restriction of g defines a map $L \cap N' \to I$, which by hypothesis can be extended to a map $g'\colon L \to I$. Then g and g' define an extension $L + N' \to I$ of g, contradicting the maximality of (N', g). Hence we must have had $N' = N$, and so I is injective.

Finally, for part (g), suppose $\operatorname{Ext}^1_B(M, I) = 0$ for all simple comodules M. We use induction on the dimension to show that $\operatorname{Ext}^1_B(N, I) = 0$ for all finite-dimensional comodules N. We can certainly get started, since every one-dimensional comodule is simple. Now suppose we have proved it for all comodules of dimension $< n$, and N has dimension n. If N is simple, there is nothing to prove. If not, then N has a subcomodule N' of smaller dimension. We then have an exact sequence

$$\operatorname{Ext}^1_B(N/N', I) \to \operatorname{Ext}^1_B(N, I) \to \operatorname{Ext}^1_B(N', I)$$

so we must have $\operatorname{Ext}^1_B(N, I) = 0$. Now suppose $M \xrightarrow{i} N$ is an arbitrary inclusion of finite-dimensional comodules. Then we have an exact sequence

$$0 \to \operatorname{Hom}_B(N/M, I) \to \operatorname{Hom}_B(N, I) \to \operatorname{Hom}_B(M, I) \to \operatorname{Ext}^1_B(N/M, I) = 0$$

Hence any map $M \to I$ has an extension to N. By part (f), it follows that I is injective. $\square$

2.5.2. Weak equivalences. We now describe the weak equivalences in our model structure on $\mathrm{Ch}(B)$.

First note that the tensor product on the category of B-comodules extends to a tensor product on $\mathrm{Ch}(B)$. Indeed, given chain complexes X and Y of B-comodules, we define

$$(X \otimes Y)_n = \bigoplus_m X_m \otimes Y_{n-m}$$

where m runs through all integers, including the negative ones. We define $d(x \otimes y) = dx \otimes y + (-1)^m x \otimes dy$ on $X_m \otimes Y_{n-m}$. In a similar fashion, we also get a Hom functor using the Hom functor on B-comodules.

Now choose a specfic injective resolution Lk of the trivial comodule k. We think of Lk as a complex of injective comodules concentrated in nonpositive degrees.

Definition 2.5.9. Suppose B is a commutative Hopf algebra over a field k, M is a simple B-comodule (see Corollary 2.5.2), $X \in \mathrm{Ch}(B)$, and n is an integer. Define the *nth homotopy group of X with respect to M*, $\pi_n^M(X)$, to be the vector space $[S^n M, Lk \otimes X]$ of chain homotopy classes of chain maps from $S^n M$ to $Lk \otimes X$. Here $S^n M$ is the complex whose only nonzero comodule is M in degree n, and Lk is an injective resolution of k.

The reader who has forgotten the definition of chain homotopy can consult the paragraph preceding Lemma 2.3.8.

Note that $\pi_n^M(X)$ does not depend on the choice of injective resolution Lk of k. Indeed, any two such injective resolutions are chain homotopy equivalent, and so will still be chain homotopy equivalent after tensoring with X.

Note as well that k itself is a simple comodule. If M is an arbitrary comodule, then $Lk \otimes M$ is an injective resolution for M, by Proposition 2.5.8. Hence we have $\pi_n^k(M) \cong \mathrm{Ext}_B^n(k, M)$. There are many cases when k is the only simple comodule, as for example when $B = F(G, k)$, G is a finite p-group, and k has characteristic p. When B is a connected graded Hopf algebra (*i.e.* when $B_0 = k$ and $B_n = 0$ for $n < 0$), the only simple (graded) comodules are one-dimensional (but can be in any degree).

Finally, note that $\pi_n^M(X)$ is functorial in X.

Definition 2.5.10. Suppose B is a commutative Hopf algebra over a field k. Define a map $f\colon X \to Y$ in $\mathrm{Ch}(B)$ to be a *weak equivalence* if $\pi_n^M(f)$ is an isomorphism for all simple comodules M and integers n.

We need some basic properties of weak equivalences. First note that weak equivalences obviously form a subcategory, are closed under retracts, and obey the two out of three axiom.

Lemma 2.5.11. (a) *Suppose* $0 \to W \to X \to Y \to 0$ *is a short exact sequence in* $\mathrm{Ch}(B)$. *Then there is an induced long exact sequence*

$$\dots \to \pi_{n+1}^M Y \to \pi_n^M W \to \pi_n^M X \to \pi_n^M Y \to \pi_{n-1}^M W \to \dots$$

for all simple comodules M.

(b) *Suppose we have a pushout square in* $\mathrm{Ch}(B)$

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ D \arrow[r] & Y \end{tikzcd}$$

where f is an injective weak equivalence. Then g is an injective weak equivalence.

(c) *Suppse λ is an ordinal, and $X\colon \lambda \to \mathrm{Ch}(B)$ is a λ-sequence of weak equivalences. Then the transfinite composition $X_0 \to \operatorname{colim} X_\alpha$ of X is a weak equivalence. More generally, if X is an arbitrary λ-sequence, and λ is a limit ordinal, the map $\operatorname{colim} \pi_n^M(X_\alpha) \to \pi_n^M(\operatorname{colim} X_\alpha)$ is an isomorphism for all simple comodules M and integers n.*

Proof. For part (a), we have a short exact sequence

$$0 \to Lk \otimes W \to Lk \otimes X \to Lk \otimes Y \to 0$$

which is split since $Lk \otimes W$ is injective in each dimension by Proposition 2.5.8. A class in $\pi_n^M Y$ is represented by a map $M \to Z_n(Lk \otimes Y)$ of comodules. Applying the splitting, we get a map $M \to (Lk \otimes X)_n$. Applying d, we get a map $M \to Z_{n-1}(Lk \otimes X)$, which in fact is a map $M \to Z_{n-1}(Lk \otimes W)$. The reader can verify by standard diagram chases that this defines a map $\pi_n^M Y \to \pi_{n-1}^M W$ and that the associated long sequence is exact.

For part (b), let K denote the cokernel of f. By part (a) we have $\pi_n^M K = 0$ for all simple comodules M and integers n. Since we have a short exact sequence

$$0 \to C \xrightarrow{g} D \to K \to 0,$$

part (a) implies that g is also a weak equivalence.

For part (c), we prove the more general statement, from which the first statement follows easily. So we must show that the map $\operatorname{colim} \pi_n^M(X_\alpha) \to \pi_n^M(\operatorname{colim} X_\alpha)$ is an isomorphism. This is a consequence of the smallness argument used to prove that all modules are small in Example 2.1.6. We first show that our map is surjective. A class $[g]$ in $\pi_n^M(\operatorname{colim} X_\alpha)$ is represented by a map $g\colon S^n M \to Lk \otimes \operatorname{colim} X_\alpha$. The map g is determined by $g_n\colon M \to Z_n(\operatorname{colim} Lk \otimes X_\alpha)$ of comodules, where $Z_n Y$ is the cycles in dimension n of the chain complex Y. Both the tensor product functor and the cycles functor commute with colimits. Since M is finite-dimensional, g_n must factor through a map $h_n\colon M \to Z_n(Lk \otimes X_\beta)$ for some $\beta < \lambda$. (Recall that g_n obviously factors through a map h'_n of sets, but by going farther out we can get a map of comodules.) The map $h\colon S^n M \to Lk \otimes X_\alpha$ represents a class $[h]$ in $\pi_n^M(X_\alpha)$ which hits the class $[g]$.

We now show that the map $\operatorname{colim} \pi_n^M(X_\alpha) \to \pi_n^M(\operatorname{colim} X_\alpha)$ is injective. Let $\varphi_\beta\colon X_\beta \to \operatorname{colim} X_\alpha$ denote the structure maps of the colimit, and $\varphi_{\beta,\gamma}\colon X_\beta \to X_\gamma$ denote the structure maps of X. Suppose $[g] \in \pi_n^M(X_\beta)$ goes to 0. This means that there is a map $h\colon M \to (Lk \otimes \operatorname{colim} X_\alpha)_{n+1}$ such that $dh = \varphi_\beta g_n$. As before, this map must factor through a comodule map $h'\colon M \xrightarrow{Lk \otimes X_\gamma}{}_{n+1}$ for some $\gamma < \lambda$. By going out farther, if necessary, we can also arrange that $dh' = \varphi_{\beta,\gamma} g$. It follows that $[g]$ goes to 0 in $\operatorname{colim} \pi_n^M(X_\alpha)$, as required. □

2.5.3. The model structure. We now want to define a model structure on $\mathrm{Ch}(B)$ with the homotopy isomorphisms as the weak equivalences.

Definition 2.5.12. Let B be a commutative Hopf algebra over a field k. Define J' to be a set of maps containing a representative of each isomorphism class of inclusions $i\colon M \to N$ of finite-dimensional comodules. Then define the set J in $\mathrm{Ch}(B)$ to be the set of maps $D^n j$, where n is an integer and $j \in J'$. Define the set I to be the union of J and the maps $S^{n-1}M \to D^n M$, where n is an integer and M runs through the isomorphism classes of *simple* comodules. Define a map in $\mathrm{Ch}(B)$ to be a *cofibration* if it is in I-cof, and define a map to be a *fibration* if it is J-inj.

Proposition 2.5.13. *Every map in J-cof is a trivial cofibration in* $\mathrm{Ch}(B)$.

Proof. Since $J \subseteq I$, J-cof $\subseteq I$-cof, so every map of J-cof is a cofibration. Since every object of $\mathrm{Ch}(B)$ is small, the small object argument applies. Thus, every map in J-cof is a retract of a map in J-cell. It therefore suffices to show that transfinite compositions of pushouts of maps of J are weak equivalences. In light of Lemma 2.5.11, it suffices to show that the maps of J are weak equivalences (and injections). The complex $D^n M$ is chain homotopy equivalent to 0, so $Lk \otimes D^n M$ is also chain homotopy equivalent to 0. It follows that the maps of J are weak equivalences, as required. ☐

Proposition 2.5.14. *A map $p\colon X \to Y$ in* $\mathrm{Ch}(B)$ *is a fibration if and only if $p_n\colon X_n \to Y_n$ is a surjection with injective kernel for all n.*

Proof. Suppose first that p is a fibration. Consider an element y in Y_n. The subcomodule M generated by y is finite-dimensional, by Lemma 2.5.1. We have a commutative diagram

$$\begin{tikzcd} D^n 0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D^n M \arrow[r] & Y \end{tikzcd}$$

Since p is a fibration, there is a lift $D^n M \to X$. The image of the class y is a preimage of $y \in Y_n$. Hence each p_n is surjective.

Now let A be the kernel of p, and let $i\colon A \to X$ denote the inclusion map. We want to show that A_n is injective. We will use Proposition 2.5.8. Suppose we have a map $M \xrightarrow{f} A_n$ and an injection $M \xrightarrow{g} N$, where N is finite-dimensional. The map f corresponds to a map $D^n M \xrightarrow{f'} A$, so we get a commutative diagram

$$\begin{tikzcd} D^n M \arrow[r, "i\circ f'"] \arrow[d, "D^n g"'] & X \arrow[d, "p"] \\ D^n N \arrow[r, "0"] & Y \end{tikzcd}$$

Since p is a fibration, we get a lift $D^n N \xrightarrow{h'} X$. Since $ph' = 0$, we can think of h' as a map $D^n N \to A$, corresponding to $h\colon N \to A_n$. This map h is an extension of f, so A_n is injective.

Now suppose that p is surjective with dimensionwise injective kernel. Suppose we have a commutative diagram

$$\begin{tikzcd} D^n M \arrow[r, "f"] \arrow[d, "D^n g"'] & X \arrow[d, "p"] \\ D^n N \arrow[r, "h"] & Y \end{tikzcd}$$

where g is an inclusion of finite-dimensional comodules. This diagram is equivalent to the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f_n"] \arrow[d, "g"'] & X_n \arrow[d, "p_n"] \\ N \arrow[r, "h_n"] & Y_n \end{tikzcd}$$

and we want to find a lift $k\colon N \longrightarrow X_n$ in this diagram. Since p_n is a surjection with injective kernel, there is a splitting $q\colon Y_n \longrightarrow X_n$. Note that $p_n(f_n - qh_n g) = 0$, so $f_n - qh_n g$ defines a map $M \longrightarrow A_n$. Since A_n is injective, there is a map $r\colon N \longrightarrow A_n$ such that $rg = f_n - qh_n g$. Then one can easily check that the map $r + qh_n\colon N \longrightarrow X_n$ gives the required lift. □

Next, we characterize the trivial fibrations. We need a lemma first. This lemma is the key fact that makes this construction of a model structure on $\mathrm{Ch}(B)$ work.

LEMMA 2.5.15. *Suppose A is a complex of injective comodules in* $\mathrm{Ch}(B)$. *Then the map $A \xrightarrow{j} Lk \otimes A$ is a chain homotopy equivalence. In particular, $\pi_n^M A \cong [S^n M, A]$ for all simple comodules M and integers n.*

PROOF. The plan of the proof is as follows. We first show that j is a chain homotopy equivalence when A is a bounded above complex of injectives. We then use this to conclude that $\pi_n^M A \cong [S^n M, A]$ for all complexes of injective comodules A, simple comodules M, and integers n. We then show that this implies that j is a chain homotopy equivalence for arbitrary complexes of injectives A.

Let C denote the cokernel of j. If C is chain homotopic to 0, then j is a chain homotopy equivalence. Indeed, since A is a complex of injectives, j is a split inclusion in each dimension (so C is a complex of injectives as well). Thus, the differential on $Lk \otimes A$ must be of the form $d(a, c) = (da + \varphi c, dc)$, where the maps $\varphi_n\colon C_n \longrightarrow A_{n-1}$ can be any comodule maps such that $\varphi d = -d\varphi$. Given a contracting homotopy $D_n\colon C_n \longrightarrow C_{n+1}$ such that $dD + Dd = 1_C$, we define a chain homotopy inverse $r\colon Lk \otimes A \longrightarrow A$ to j by $r(a, c) = a - \varphi Dc$. The reader can verify that r is a chain map and that $rj = 1_A$. The map that takes (a, c) to $(0, Dc)$ is a chain homotopy between jr and $1_{Lk\otimes A}$. Hence j is a chain homotopy equivalence if C is chain homotopic to 0.

Now suppose A is bounded above. The map $k \longrightarrow Lk$ is a homology isomorphism, so j is also a homology isomorphism, since we are tensoring over a field. Thus C is a bounded above complex of injectives with no homology. We construct the contracting homotopy D by downward induction on n. For n sufficently large, C_n and C_{n+1} are 0, so we take $D_n = 0$. Now suppose we have constructed D_n such that $dD_{n+1} + D_n d = 1_C$ for all $n > m$. Define a function $E_m\colon Z_m C \longrightarrow C_{m+1}$ as follows. Given a cycle x, there is a y such that $dy = x$, since C has no homology. Define $E_m x = y - dD_{m+1}y$. This is well-defined, since if $dz = x$ as well, then $y - z$ is a cycle, so there is a w such that $dw = y - z$. It follows that $D_{m+1}(y-z) = w - dD_{m+2}w$, so $dD_{m+1}(y-z) = dw = y - z$. Define $D_m \colon C_m \longrightarrow C_{m+1}$ to be an extension of E_m to all of C_m.

We have now proved that j is a chain homotopy equivalence for all bounded above complexes of injectives A. Suppose A is an arbitrary complex of injectives. Let A^n be the cotruncation of A at dimension n, so that $(A^n)_i = 0$ if $i > n$ and $(A^n)_i = A_i$ if $i \leq n$. The differential on A^n is the same as the differential on A in degrees $\leq n$ and 0 elsewhere. The map $j^n \colon A^n \longrightarrow Lk \otimes A^n$ is a chain homotopy equivalence, so induces an isomorphism $[S^iM, A^n] \longrightarrow [S^iM, Lk \otimes A^n]$ for all integers i and simple comodules M. There are obvious chain maps $A^n \longrightarrow A^{n+1}$, and A is the colimit of the A^n. Similarly, $L(k) \otimes A$ is the colimit of the $L(k) \otimes A^n$, and so j is the colimit of the j^n. Since $[S^iM, -]$ commutes with colimits, by the argument used to prove part (c) of Lemma 2.5.11, we find that j induces an isomorphism $[S^iM, A] \longrightarrow \pi_i^M(A)$ for any complex of injectives A.

It follows that the cokernel C of j is a complex of injectives such that $[S^nM, C] = 0$ for all simple comodules M and integers n. We will show this forces C to be chain homotopic to 0. This will prove that j is a chain homotopy equivalence, as above. The short exact sequence

$$0 \longrightarrow Z_nC \longrightarrow C_n \xrightarrow{d} B_{n-1}C \longrightarrow 0$$

gives rise to an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_B(M, Z_nC) &\longrightarrow \operatorname{Hom}_B(M, C_n) \\ \longrightarrow \operatorname{Hom}_B(M, B_{n-1}C) &\longrightarrow \operatorname{Ext}^1_B(M, Z_nC) \longrightarrow 0 \end{aligned}$$

for all simple comodules M, since C_n is injective. On the other hand, a map $f \colon M \longrightarrow Z_nC$ corresponds to a chain map $S^nM \longrightarrow C$. A chain homotopy between f and 0, which must exist by hypothesis, is a map $g \colon M \longrightarrow C_{n+1}$ such that $dg = f$. Thus the map $\operatorname{Hom}_B(M, C_n) \longrightarrow \operatorname{Hom}_B(M, Z_{n-1}C)$ is surjective, and in particular the map $\operatorname{Hom}_B(M, C_n) \longrightarrow \operatorname{Hom}_B(M, B_{n-1}C)$ is surjective. Hence $\operatorname{Ext}^1_B(M, Z_nC) = 0$ for all simple comodules M, and so, by Proposition 2.5.8, Z_nC is injective.

It follows that there is a retraction $r \colon C_{n+1} \longrightarrow Z_{n+1}C$. and a section $q \colon BX_n \longrightarrow C_{n+1}$. In particular, $C_{n+1} \cong Z_{n+1}C \oplus B_nC$, so B_nC is injective as well. Hence $Z_nC \cong B_nC \oplus H_nC$. But the map $\operatorname{Hom}_B(M, B_nC) \longrightarrow \operatorname{Hom}_B(M, Z_nC)$ is an isomorphism (we saw above that it was surjective and it is obviously injective) for all simple comodules M, so $\operatorname{Hom}_B(M, H_nC) = 0$ for all simple comodules M. It follows from Corollary 2.5.3 that $H_nC = 0$, so $Z_nC = B_nC$.

We now define a chain homotopy $D \colon C_n \longrightarrow C_{n+1}$ as the composite

$$C_n \xrightarrow{r} Z_nC = B_nC \xrightarrow{q} C_{n+1}$$

We leave it to the reader to verify that $dD + Dd = 1_C$, so that C is chain homotopic to 0. □

Proposition 2.5.16. *A map $p \colon X \longrightarrow Y$ in $\operatorname{Ch}(B)$ is a trivial fibration if and only if it has the right lifting property with respect to I.*

Proof. Suppose first that p is a trivial fibration. Let A denote the kernel of p, so that A is dimensionwise injective by Proposition 2.5.14. Pick a simple comodule M. The long exact sequence in homotopy shows that $\pi_*^M A = 0$. Hence $[S^i M, A] = 0$ as well, by Lemma 2.5.15. Suppose we have a commutative diagram

$$\begin{tikzcd} S^{n-1}M \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ D^n M \arrow[r] & Y \end{tikzcd}$$

We need to show that there is a lift $D^n M \to X$. This diagram corresponds to a comodule map $f\colon M \to Z_{n-1}X$ and a comodule map $g\colon M \to Y_n$ such that $pf = dg$. A lift in the diagram corresponds to a map $r\colon M \to X_n$ such that $pr = g$ and $dr = f$. Choose a splitting $q\colon Y_n \to X_n$ such that $pq = 1$. Then $p \circ (f - dqg) = 0$, so $f - dqg$ is really a map $M \to Z_{n-1}A$. Since $[S^n M, A_{n-1}] = 0$, there is a comodule map $h\colon M \to A_n$ such that $dh = f - dqg$. The map $h + qg\colon M \to X_n$ gives the desired lift $D^n M \to X$, and so p has the right lifting property with respect to I.

Now suppose p has the right lifting property with respect to I. Then p is a fibration, since $J \subseteq I$. Let A denote the kernel of p, so that A is dimensionwise injective. We want to show that p is a weak equivalence. By Lemma 2.5.15 and the long exact sequence in homotopy, it suffices to show that $[S^n M, A] = 0$ for all simple comodules M and all integers n. But this is clear: if $f\colon S^n M \to A$ is a chain map, then there is a map $g\colon D^{n+1}M \to X$ such that $pg = 0$ and the composite $S^n M \to D^{n+1} M \xrightarrow{g} X$ is f, since p has the right lifting property with respect to I. The map g corresponds to a map $M \to A_{n+1}$ which gives a chain homotopy between f and 0. □

The following theorem then follows immediately from Theorem 2.1.19.

Theorem 2.5.17. *Suppose B is a commutative Hopf algebra over a field k. Then the category $\mathrm{Ch}(B)$ of chain complexes of B-comodules is a finitely generated model category with generating cofibrations I, generating trivial cofibrations J, and weak equivalences the homotopy isomorphisms. The fibrations are the surjections with dimensionwise injective kernel.*

The homotopy category of $\mathrm{Ch}(B)$ is the stable homotopy category considered in [**HPS97**, Section 9.5].

To complete our description of the model structure on $\mathrm{Ch}(B)$, we identify the cofibrations.

Proposition 2.5.18. *The cofibrations in $\mathrm{Ch}(B)$ are the injective maps.*

Proof. Certainly every map of I is an injection, so every cofibration is an injection. Conversely, suppose $i\colon K \to L$ is an inclusion. Given a diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ L \arrow[r, "g"] & Y \end{tikzcd}$$

where p is a trivial fibration, we must show there is a lift. Let A denote the kernel of p. Then A is a complex of injectives with no homotopy, so by the proof of Lemma 2.5.15, A is chain homotopy equivalent to 0 by a chain homotopy $D\colon A_n \to A_{n+1}$ such that $dD + Dd = 1_A$. Furthermore, the map p is a dimensionwise split surjection. Choose a splitting $X_n \cong Y_n \oplus A_n$ in each dimension. With respect to this splitting, we can write the differential on X as $d(y,a) = (dy, ky + da)$, where $dk = -kd$. The map f can then be written $f = (gi, f_2)$, where $f_2 d = kgi + df_2$. If $h\colon L_n \longrightarrow A_{n-1}$ is a map, the pair (g,h) will define a lift in our diagram if and only if $hi = f_2$ and $hd = kg + dh$.

Now, there is certainly a map $h'\colon L_n \longrightarrow A_{n-1}$ such that $h'i = f_2$, since A_{n-1} is injective. Choose such maps for all n. Then one can check that the map $\alpha = h'd - dh' - kg\colon L_n \longrightarrow A_{n-1}$ satisfies $\alpha i = 0$, so factors through a map $\beta\colon M_n \longrightarrow A_{n-1}$, where M is the cokernel of i. One can also check that $d\beta = -\beta d$. Let $j\colon L \longrightarrow M$ denote the evident map. Then one can check that $h' + D\beta j$ gives us the required map $h\colon L_n \longrightarrow A_n$. $\square$

The following lemma will be useful later.

LEMMA 2.5.19. *Suppose X is a bounded above complex in* $\mathrm{Ch}(B)$ *with no homology. Then X also has no homotopy, so becomes trivial in* $\operatorname{Ho}\mathrm{Ch}(B)$.

PROOF. Since X is bounded above, so is $Lk \otimes X$. Since tensoring over a field is exact, $Lk \otimes X$ has no homology. Then Lemma 2.3.17 and Lemma 2.3.19 imply that $Lk \otimes X$ is chain homotopic to 0, and hence has no homotopy. $\square$

Now suppose $B \xrightarrow{f} B'$ is a map of commutative Hopf algebras over k. Then, given a B-comodule M, we can make M into a B'-comodule by the structure map $M \xrightarrow{\psi} B \otimes M \xrightarrow{f\otimes 1} B' \otimes M$. Let us denote this B'-comodule by FM. Conversely, given a B'-comodule M, we can get a B-comodule UM by letting UM be the set of all m such that $\psi(m)$ is in the image of $f \otimes 1$. We leave it to the reader to check that UM is a B-comodule and that U is right adjoint to F. The functors F and U induce corresponding functors $F\colon\ \mathrm{Ch}(B) \longrightarrow \mathrm{Ch}(B')$ and $U\colon\ \mathrm{Ch}(B') \longrightarrow \mathrm{Ch}(B)$.

PROPOSITION 2.5.20. *Suppose $f\colon B \longrightarrow B'$ is a map of commutative Hopf algebras over a field k. Then the induced adjunction $(F, U, \varphi)\colon$* $\mathrm{Ch}(B) \longrightarrow \mathrm{Ch}(B')$ *is a Quillen adjunction.*

PROOF. It is obvious that F preserves injections, and hence cofibrations. It is also clear that F takes the generating trivial cofibrations of $\mathrm{Ch}(B)$ to some of the generating trivial cofibrations of $\mathrm{Ch}(B')$. The result follows from Lemma 2.1.20. $\square$

A more interesting example of a Quillen adjunction arises as follows. Let B be a finite-dimensional commutative Hopf algebra over a field k such that B^* is a Frobenius algebra over k. For example, B could be $F(G,k)$ where G is a finite group, or B could be a graded connected finite Hopf algebra. In this case, the categories B^*-mod and BB-comod are isomorphic, since every B^*-module is tame. We therefore identify BB-comod with B^*-mod.

We will construct a Quillen adjunction $F\colon\ B^*\text{-mod} \longrightarrow \mathrm{Ch}(B)$, where B^*-mod is given the model structure of Section 2.2. To do so, let Tk be a Tate resolution of the ground field k. Recall that Tk is a complex of projectives (which are also injectives, of course) with no homology, such that $Z_0 Tk = k$. The usual way to construct Tk is to splice a projective resolution $P_* \longrightarrow k$ with an injective resolution $k \longrightarrow I_*$, so that $(Tk)_n = P_{n-1}$ if $n > 0$ and $(Tk)_n = I_n$ if $n \le 0$. In particular, the cycles in degree 0 are just k. Then we define $FM = Tk \otimes M$. The right adjoint $U\colon\ \mathrm{Ch}(B) \longrightarrow B^*$-mod of F is then defined by $UX = Z_0 \operatorname{Hom}(Tk, X)$. Since we are tensoring over a field, F preserves injections, and hence cofibrations. To show that F is a Quillen functor, we have to show that F takes the generating trivial cofibration $0 \to B^*$ to a weak equivalence in $\mathrm{Ch}(B)$. Thus it suffices to show that $Tk \otimes B$ is chain homotopy equivalent to 0. But the complex which is B in degree 0 and $P_* \otimes B$ in positive degrees is a bounded below complex of projectives with no homology, so it is chain homotopy equivalent to 0. Similarly, the complex which is B in degree 1 and $I_* \otimes B$ in nonpositive degrees is a bounded above complex of injectives with no homology, so is chain homotopy equivalent to 0. By splicing these chain homotopy equivalences, we find that $Tk \otimes B$ is chain homotopy equivalent to 0, as required.

The Quillen functor F induces an embedding of $\operatorname{Ho} B^*$-mod into $\operatorname{Ho}\mathrm{Ch}(B)$ as a full subcategory, as explained in [**HPS97**, Secion 9.6].

CHAPTER 3

Simplicial sets

This chapter is devoted to the central example of simplicial sets. This example will recur throughout the book, so the reader is advised at least to skim this section. It turns out to be quite difficult to prove that simplicial sets form a model category. We follow the proof given in [**GJ97**, Chapter 1], which is similar, but not identical, to the original proof of Quillen [**Qui67**]. Standard references for simplicial sets include [**May67**], [**Qui67**], and [**BK72**].

3.1. Simplicial sets

We begin by reminding the reader of some basic definitions and properties of simplicial sets.

Recall that the simplicial category Δ is the category with objects

$$[n] = \{0, 1, \dots, n\}$$

for $n \geq 0$ and $\Delta([n], [k])$ the set of weakly order-preserving maps f from $[n]$ to $[k]$, so that $x \leq y$ implies $f(x) \leq f(y)$. Note that Δ has two obvious subcategories: the category Δ_+ of injective order-preserving maps, and the category Δ_- of surjective order-preserving maps. Furthermore, every morphism in Δ can be factored uniquely into a morphism in Δ_- followed by a morphism in Δ_+. In fact, Δ is generated by the morphisms $d^i \colon [n-1] \longrightarrow [n] \in \Delta_+$ for $n \geq 1$ and $0 \leq i \leq n$, where the image of d^i does not include i, and the morphisms $s^i \colon [n] \longrightarrow [n-1] \in \Delta_-$ for $n \geq 1$ and $0 \leq i \leq n-1$, where s^i identifies i and $i+1$. All the relations among these maps are implied by the *cosimplicial identities*:

$$\begin{array}{rcll} d^j d^i & = & d^i d^{j-1} & (i < j) \\ s^j d^i & = & d^i s^{j-1} & (i < j) \\ & = & \mathrm{id} & (i = j, j+1) \\ & = & d^{i-1} s^j & (i > j+1) \\ s^j s^i & = & s^{i-1} s^j & (i > j) \end{array}$$

If $\mathcal{C}$ is any category, the category of *cosimplicial objects in* $\mathcal{C}$ is the functor category $\mathcal{C}^{\Delta}$, and the category of *simplicial objects in* $\mathcal{C}$ is the functor category $\mathcal{C}^{\Delta^{\mathrm{op}}}$. Note that these functor categories have whatever colimits and limits exist in $\mathcal{C}$, taken objectwise. The most important example is when $\mathcal{C}$ is the category of sets, in which case we denote $\mathcal{C}^{\Delta^{\mathrm{op}}}$ by **SSet**, and refer to **SSet** as the *category of simplicial sets*.

If K is a simplicial set, we denote $K[n]$ by K_n and refer to K_n as the set of n-simplices of K. If $x \in K_n$, the integer n is referred to as the *dimension of* x. Dual to the d^i we have the *face maps* $d_i \colon K_n \longrightarrow K_{n-1}$ for $n \geq 1$ and $0 \leq i \leq n$. Dual to the s^i we have the *degeneracy maps* $s_i \colon K_{n-1} \longrightarrow K_n$ for $n \geq 1$ and $0 \leq i \leq n-1$.

These maps are subject to the *simplicial identities*

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \quad (i < j) \\
d_i s_j &= s_{j-1} d_i \quad (i < j) \\
&= \mathrm{id} \quad (i = j, j+1) \\
&= s_j d_{i-1} \quad (i > j) \\
s_i s_j &= s_j s_{i-1} \quad (i > j)
\end{aligned}$$

A simplicial set K is equivalent to a collection of sets K_n and maps d_i and s_i as above satisfying the simplicial identities. A map of simplicial sets $f\colon K \longrightarrow L$ is equivalent to a collection of maps $f_n\colon K_n \longrightarrow L_n$ commuting with the face and degeneracy maps.

LEMMA 3.1.1. *Every simplicial set is small.*

PROOF. Suppose K is a simplicial set and the cardinality of the set of simplices of K is κ. Note that κ is infinite. We claim that K is κ-small. Indeed, suppose λ is a κ-filtered ordinal and $X\colon \lambda \longrightarrow \mathbf{SSet}$ is a λ-sequence. Given a map $f\colon K \longrightarrow \operatorname{colim} X_\alpha$ of simplicial sets, there is an $\alpha_n < \lambda$ such that f_n factors through X_{α_n}, the set K_n is κ-small. Since κ is infinite, there is an $\alpha < \lambda$ such that f factors through a map of sets $g\colon K \longrightarrow X_\alpha$. The map g may not be a map of simplicial sets. However, for each pair (x, i), where x is a simplex of K and d_i is a face map applicable to x, there is a $\beta_{(x,i)}$ such that $g(d_i x)$ becomes equal to $d_i g x$ in $X_{\beta_{(x,i)}}$. There are κ such pairs (x, i), so there is a $\beta < \lambda$ and a factorization of f through X_β compatible with the face maps. A similar argument shows that we can make the factorization compatible with the degeneracy maps as well.

This shows that the map $\operatorname{colim} \mathbf{SSet}(K, X_\alpha) \longrightarrow \mathbf{SSet}(K, \operatorname{colim} X_\alpha)$ is surjective. The smallness of each K_n shows this map is injective as well. $\square$

Given a simplicial set K and a simplex x of K, any image of x under arbitrary iterations of face maps is called a *face* of x. Similarly, any image of x under arbitrary iterations of degeneracy maps is called a *degeneracy* of x. We include the case of 0 iterations, so x is both a face and degeneracy of itself. A simplex x is called *non-degenerate* if it is a degeneracy only of itself. A simplicial set is called *finite* if it has only finitely many non-degenerate simplices. Given any simplex x of K, there is a unique non-degenerate simplex y of K such that x is a degeneracy of y. Indeed, we can take y to be a simplex of smallest dimension such that x is a degeneracy of y. The simplicial identities imply that there is a unique such simplex, and that every simplex z such that x is a degeneracy of z is in fact a degeneracy of y.

LEMMA 3.1.2. *Finite simplicial sets are finite.*

PROOF. Suppose K is a finite simplicial set, λ is a limit ordinal, and $X\colon \lambda \longrightarrow \mathbf{SSet}$ is a λ-sequence. We must show that the canonical map $\operatorname{colim} \mathbf{SSet}(K, X_\alpha) \longrightarrow \mathbf{SSet}(K, \operatorname{colim} X_\alpha)$ is an isomorphism. We first show it is injective. Suppose $f, g\colon K \longrightarrow X_\alpha$ are maps that become equal in the colimit. Since K is finite, we can go out far enough in the colimit so that f and g are equal on the nondegenerate simplices of K. But then they are equal on all the simplices of K, since every simplex is a degeneracy of a nondegenerate simplex, and f and g are simplicial maps. This shows that the canonical map is injective.

Now suppose we have a map $K \longrightarrow \operatorname{colim} X_\alpha$. For each nondegenerate simplex x of K, there is an $\alpha_x < \lambda$ and a simplex $y_x \in X_{\alpha_x}$ such that $f(x) = i_{\alpha_x} y_x$, where $i_{\alpha_x}\colon X_{\alpha_x} \longrightarrow \operatorname{colim} X_\alpha$ is the structure map. Since there are only finitely many nondegenerate simplices of K, we can assume that $\alpha_x = \alpha$, independent of x. We can then define a map $g\colon K \to X_\alpha$, compatible with the degeneracy maps, such that $i_\alpha g = f$, using the fact that every simplex is a degeneracy of a unique nondegenerate simplex. The map g may not be compatible with the face maps, however. Nevertheless, for each face $d_i x$ of a nondegenerate simplex x, we can find an $\alpha_{(i,x)} < \lambda$ such that $gd_i x$ and $d_i gx$ become equal in $X_{\alpha_{(i,x)}}$. Since there are only finitely many such pairs, we can find a $\beta < \lambda$ and a map $h\colon K \to X_\beta$ such that $i_\beta h = f$ and h is compatible with all the degeneracy maps and h is compatible with the face maps when applied to nondegenerate simplices. The simplicial identities then imply that h is a simplicial map, as required. □

There is a very important functor $\Delta \to \mathbf{SSet}$, typically denoted $\Delta[-]$, defined by the functor $\Delta(-,-)\colon \Delta^{\mathrm{op}} \times \Delta \to \mathbf{Set}$. That is, $\Delta[n]$ is the functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$ which takes $[k]$ to $\Delta([k],[n])$. The simplicial set $\Delta[n]$ has $\binom{n}{k}$ nondegenerate k-simplices, corresponding to the injective order-preserving maps $[k] \to [n]$, and in particular one nondegenerate n-simplex i_n. There is a natural isomorphism $\mathbf{SSet}(\Delta[n], K) \cong K_n$ which takes f to $f(i_n)$.

Another important example of a simplicial set is $\partial\Delta[n]$, the boundary of $\Delta[n]$, whose nondegenerate k-simplices correspond to nonidentity injective order-preserving maps $[k] \to [n]$. Similarly, given an r with $0 \leq r \leq n$, the simplicial set $\Lambda^r[n]$, the r-horn of $\Delta[n]$, has nondegenerate k-simplices all injective order-preserving maps $[k] \to [n]$ except the identity and the injective order-preserving map $[n-1] \xrightarrow{d^r} [n]$ whose image does not contain r. The simplicial set $\Lambda^r[n]$ is the closed star of the vertex r in $\Delta[n]$. Geometrically, $\Lambda^r[n]$ is obtained from $\Delta[n]$ by omitting the interior of $\Delta[n]$ and the interior of the $n-1$-dimensional face opposite to r. Said another way, consider the category $\mathcal{D}$ whose objects are nonidentity injective order-preserving maps $[k] \to [n]$ whose image contains r, and whose morphisms are commutative triangles. Then $\Lambda^r[n] = \operatorname{colim}_{\mathcal{D}} \Delta[k]$.

This idea of constructing $\Lambda^r[n]$ as a colimit of copies of $\Delta[k]$ is a general one. Indeed, given a simplicial set K, let ΔK be the category whose objects are maps $\Delta[n] \to K$ of simplicial sets, for some n. A morphism from $f\colon \Delta[k] \to K$ to $g\colon \Delta[n] \to K$ is a map $[k] \to [n]$ in Δ making the obvious triangle commutative. The category ΔK is called the *category of simplices of* K in [**DHK**]. Note that a map $K \to L$ of simplicial sets induces an obvious functor $\Delta K \to \Delta L$, so that this construction defines a functor from $\mathbf{SSet}$ to the category of small categories.

The category ΔK is very important and useful. One of the reasons for this is the following simple lemma.

LEMMA 3.1.3. *Given a simplicial set K, the colimit of the functor $\Delta K \to \mathbf{SSet}$ that takes $f\colon \Delta[n] \to K$ to $\Delta[n]$ is K itself.*

PROOF. Use the isomorphism $K_n \cong \mathbf{SSet}(\Delta[n], K)$. □

The advantage of this description of the category of simplices is that it is functorial in the simplicial set K. However, if one is working with a specific simplicial set K, it is often more helpful to consider the category of nondegenerate simplices $\Delta' K$. An object of $\Delta' K$ is a map $\Delta[n] \xrightarrow{f} K$ such that fi_n is nondegenerate. A morphism is an *injective* order-preserving map $[k] \to [n]$ making the obvious triangle commute. We then have the following lemma, whose proof we leave to the reader.

LEMMA 3.1.4. *Given a simplicial set K, a colimit of the functor $\Delta' K \to \mathbf{SSet}$ that takes $f\colon \Delta[n] \to K$ to $\Delta[n]$ is K itself.*

Another important use of the category of simplices is to show that any cosimplicial object gives rise to a functor from simplicial sets.

PROPOSITION 3.1.5. *Suppose $\mathcal{C}$ is a category with all small colimits. Then the category $\mathcal{C}^{\Delta}$ is equivalent to the category of adjunctions $\mathbf{SSet} \to \mathcal{C}$. We denote the image of $A^{\bullet} \in \mathcal{C}^{\Delta}$ under this equivalence by $(A^{\bullet} \otimes -, \mathcal{C}(A^{\bullet}, -), \varphi)\colon \mathbf{SSet} \to \mathcal{C}$.*

PROOF. Suppose first that we have an adjunction $(F, U, \varphi)\colon \mathbf{SSet} \to \mathcal{C}$. Then the composite $\Delta \to \mathbf{SSet} \to \mathcal{C}$, where the first functor takes $[n]$ to $\Delta[n]$, is an object of $\mathcal{C}^{\Delta}$. This clearly defines a functor from adjunctions to $\mathcal{C}^{\Delta}$. Conversely, given a simplicial set K, there is a functor $\Delta K \to \Delta$ which takes a simplex $\Delta[n] \to K$ to $[n]$. We have a corresponding restriction functor $\mathcal{C}^{\Delta} \to \mathcal{C}^{\Delta K}$. On the other hand, we also have the colimit functor $\mathcal{C}^{\Delta K} \to \mathcal{C}$. Given $A^{\bullet} \in \mathcal{C}^{\Delta}$, we define $A^{\bullet} \otimes K$ to be the image of $A^{\bullet}$ under the composite functor $\mathcal{C}^{\Delta} \to \mathcal{C}^{\Delta K} \to \mathcal{C}$. Since a map of simplicial sets induces a functor $\Delta K \to \Delta L$, the map $A^{\bullet} \otimes -$ is really a functor. Since the identity map of $\Delta[n]$ is cofinal in the category $\Delta\Delta[n]$ we get an isomorphism $A^{\bullet} \otimes \Delta[n] \cong A^{\bullet}[n]$. Conversely, if F preserves colimits, then there is a natural isomorphism $F(\Delta[-]) \otimes K \to FK$.

The right adjoint $\mathcal{C}(A^{\bullet}, -)$ of the functor $A^{\bullet} \otimes -$. Given $Y \in \mathcal{C}$, the simplicial set $\mathcal{C}(A^{\bullet}, Y)$ is defined to have n-simplices $\mathcal{C}(A^{\bullet}[n], Y)$. The adjointness isomorphism is the composite

$$
\begin{aligned}
\mathcal{C}(A^{\bullet} \otimes K, Y) &\cong \mathcal{C}(\operatorname{colim}_{\Delta K} A^{\bullet}[n], Y) \cong \lim_{\Delta K} \mathcal{C}(A^{\bullet}[n], Y) \\
&\cong \lim_{\Delta K} \mathbf{SSet}(\Delta[n], \mathcal{C}(A^{\bullet}, Y)) \cong \mathbf{SSet}(\operatorname{colim}_{\Delta K} \Delta[n], \mathcal{C}(A^{\bullet}, Y)) \\
&\cong \mathbf{SSet}(K, \mathcal{C}(A^{\bullet}, Y))
\end{aligned}
$$

□

COROLLARY 3.1.6. *Suppose $\mathcal{C}$ is a pointed category with all small colimits. Then the category $\mathcal{C}^{\Delta}$ is equivalent to the category of adjunctions $\mathbf{SSet}_* \to \mathcal{C}$. We denote the image of $A^{\bullet} \in \mathcal{C}^{\Delta}$ under this equivalence by $(A^{\bullet} \wedge -, \mathcal{C}(A^{\bullet}, -), \varphi)\colon \mathbf{SSet} \to \mathcal{C}$. Furthermore, we have a natural isomorphism $A^{\bullet} \wedge K_+ \cong A^{\bullet} \otimes K$, where $A^{\bullet} \otimes -$ is the functor of Proposition 3.1.5.*

PROOF. Just as in the proof of Proposition 1.3.5, an adjunction $\mathbf{SSet} \to \mathcal{C}$ gives rise to an adjunction $\mathbf{SSet}_* \to \mathcal{C}_* = \mathcal{C}$, since $\mathcal{C}$ is pointed. If F is the left adjoint of the old adjunction and F_* is the left adjoint of the new adjunction, we have the pushout diagram

$$
\begin{array}{ccc}
F(*) & \xrightarrow{Fv} & FX \\
\downarrow & & \downarrow \\
* & \longrightarrow & F_*(X, v)
\end{array}
$$

We have also proved in Proposition 1.3.5 that $F_*(X_+) = (FX)_+$, which is isomorphic to FX itself when $\mathcal{C}$ is pointed. We leave it to the reader to prove that this correspondence is an equivalence of categories between adjunctions $\mathbf{SSet} \to \mathcal{C}$ and adjunctions $\mathbf{SSet}_* \to \mathcal{C}$. □

REMARK 3.1.7.
1. Proposition 3.1.5 has a simplicial analog as well, obtained by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$ in Proposition 3.1.5. That is, if $\mathcal{C}$ has all small limits, there is an equivalnce of categories between $\mathcal{C}^{\Delta^{\mathrm{op}}}$ and adjunctions $\mathbf{SSet}^{\mathrm{op}} \to \mathcal{C}$. We denote the image of a simplicial object $A_\bullet$ by $(\mathrm{Hom}(-, A_\bullet), \mathcal{C}(-, A_\bullet), \varphi)$. We might also write $\mathrm{Hom}(-, A_\bullet) = A_\bullet^-$. There is an analogous pointed version.
2. A functor $\mathcal{C} \to \mathcal{C}^\Delta$ gives rise, under the equivalence of Proposition 3.1.5, to a functor from $\mathcal{C}$ to adjunctions from $\mathbf{SSet}$ to $\mathcal{C}$. This gives rise to a bifunctor $-\otimes -\colon \mathcal{C} \times \mathbf{SSet} \to \mathcal{C}$. The functor $A \otimes -\colon \mathbf{SSet} \to \mathcal{C}$ will have a right adjoint, but the functor $- \times K \to \mathcal{C} \to \mathcal{C}$ need not have a right adjoint in general. A similar remark holds in the simplicial case, and in the pointed case.

We now give some important examples of this construction. We have an obvious functor $\mathbf{SSet} \to \mathbf{SSet}^\Delta$ that takes a simplicial set K to the cosimplicial simplicial set $K \times \Delta[-]$. The associated bifunctor $\mathbf{SSet} \times \mathbf{SSet} \to \mathbf{SSet}$ is just the product functor $(K, L) \mapsto K \times L$, since the product obviously commutes with colimits. Its adjoint is the function complex functor $(K, L) \mapsto \mathrm{Map}(K, L)$, where an n-simplex of $\mathrm{Map}(K, L)$ is a map of simplicial sets $K \times \Delta[n] \to L$. In the terminology of the next chapter, this makes $\mathbf{SSet}$ into a closed symmetric monoidal category.

As another example, in the category $\mathbf{Top}$ of topological spaces and continuous maps, let $|\Delta[n]| \subseteq \mathbf{R}^n$ denote the convex hull of the points $e_0, e_1, \dots, e_n$, where $e_0 = (0, \dots, 0)$ and e_i has ith coordinate 1 and all other coordinates 0. That is, $|\Delta[n]|$ consists of all points $(t_1, \dots, t_n) \in \mathbf{R}^n$ such that $t_i \geq 0$ for all i and $\sum t_1 \leq 1$. We refer to $|\Delta[n]|$ as the standard topological n-simplex. An order preserving map $[m] \xrightarrow{f} [n]$ obviously induces a linear, hence continuous, map $|\Delta[m]| \to |\Delta[n]|$. Indeed, just send e_i to $e_{f(i)}$.

Hence $|\Delta[-]|$ is a cosimplicial topological space. By Proposition 3.1.5, we get an induced adjunction $(|\ |, \mathrm{Sing}, \varphi)\colon \mathbf{SSet} \to \mathbf{Top}$. The left adjoint $|\ |$ is called the *geometric realization*. The right adjoint Sing is called the *singular functor*. Note that $|\Delta[n]|$ is a compact Hausdorff space, so in particular is in $\mathbf{K}$, the category of k-spaces. Since $\mathbf{K}$ is closed under colimits in $\mathbf{Top}$, it follows that the adjunction $(|\ |, \mathrm{Sing}, \varphi)$ can be thought as of an adjunction $\mathbf{SSet} \to \mathbf{K}$ as well (without changing the definitions of the functors). In fact, $|K|$ is Hausdorff, though we do not need this fact. It will follow from Proposition 3.2.2 that $|K|$ is a cell complex, and hence weak Hausdorff.

The following lemma is of crucial importance.

LEMMA 3.1.8. *As a functor* $\mathbf{SSet} \to \mathbf{K}$ *the geometric realization preserves finite products.*

PROOF. Since the product preserves colimits in each variable in both $\mathbf{SSet}$ and $\mathbf{K}$, it suffices to verify that the natural map $|\Delta[m] \times \Delta[n]| \to |\Delta[m]| \times |\Delta[n]|$ is a homeomorphism. Since both the source and target of this continuous map are compact Hausdorff spaces (we will see this below for the source), it suffices to show the map is a bijection. Proving this is somewhat combinatorially intricate. The reader is encouraged to have a specific example, say $m = 3$ and $n = 2$, in mind while reading this proof.

We must first understand the nondegenerate simplices of $\Delta[m] \times \Delta[n]$. A p-simplex of $\Delta[m] \times \Delta[n]$ is the same thing as an order-preserving map $[p] \to [m] \times [n]$, where $(a,b) \leq (a',b')$ in $[m]\times[n]$ if and only if $a \leq a'$ and $b \leq b'$. It is convenient to visualize $[m]\times[n]$ as the integer lattice between $(0,0)$ and (m,n). A non-degenerate p-simplex is an injective order-preserving map $[p] \longrightarrow [m]\times[n]$, or, equivalently, a chain in $[m]\times[n]$. Any such chain can be expanded to a maximal chain $[m+n] \longrightarrow [m]\times[n]$, and therefore any nondegenerate simplex of $\Delta[m]\times\Delta[n]$ is a face of a nondegenerate $m+n$-simplex. Such a maximal chain is a path along the integer lattice from $(0,0)$ to (m,n) which always goes right or up. It is convenient to label the vertices of such a path, giving $(0,0)$ the label 0, the next vertex the label 1, and so on, until (m,n) has the label $m+n$. Then such a path is completely determined by the labels on the ends of the horizontal segments. For example, there are two such paths from $(0,0)$ to $(1,1)$. The one which goes right first has 1 as the label on the end of its horizontal segment, and the one which goes up first has 2 as the label on the end of its horizontal segment. This constructs a one-to-one correspondence between maximal chains of $[m]\times[n]$ and m-subsets of $\{1,2,\dots,m+n\}$, of which there are $\binom{m+n}{m}$.

Now, let $c(i)$ for $1 \leq i \leq \binom{m+n}{m}$ be the complete list of maximal chains of $[m]\times[n]$. Given any chain c, let n_c denote the number of edges in c. The considerations above show that $\Delta[m]\times\Delta[n]$ is the coequalizer in **SSet** of the two maps

$$f,g\colon \coprod_{1\leq i<j\leq\binom{m+n}{m}} \Delta[n_{c(i)\cap c(j)}] \longrightarrow \coprod_{1\leq i\leq\binom{m+n}{m}} \Delta[n_{c(i)}]$$

where f is induced by the inclusion $c(i)\cap c(j) \longrightarrow c(i)$, and g is induced by the inclusion $c(i)\cap c(j) \longrightarrow c(j)$. For example, $\Delta[1]\times\Delta[1]$ is the union of two copies of $\Delta[2]$, corresponding to the chains $\{(0,0),(0,1),(1,1)\}$ and $\{(0,0),(1,0),(1,1)\}$, attached along the 1-simplex corresponding to the chain $\{(0,0),(1,1)\}$.

Since the geometric realization is a left adjoint, it preserves coequalizers. This shows in particular that $|\Delta[m]\times\Delta[n]|$ is compact Hausdorff. We now describe the maps $h_i\colon |\Delta[m+n]| \longrightarrow |\Delta[m]|\times\Delta[n]$ defined by the composite

$$|\Delta[n_{c(i)}]| \longrightarrow |\Delta[m]\times\Delta[n]| \longrightarrow |\Delta[m]|\times|\Delta[n]|$$

Let us denote a point of $|\Delta[m+n]|$ by $z=(z_1,\dots,z_{m+n})$, where $z_i \geq 0$ for all i and $\sum z_i \leq 1$. Similarly, denote a point of $|\Delta[m]|\times|\Delta[n]|$ as $(u,v) = (u_1,\dots,u_m,v_1,\dots,v_n)$. Suppose $c(i)$ corresponds to the m-subset $\{a_1<\dots<a_m\}$ of $\{1,\dots,m+n\}$ whose complement is $\{b_1<\dots<b_n\}$. Write $a_{m+1}=m+n+1=b_{n+1}$. Then $h_i z=(u,v)$, where $u_j=\sum_{k=a_j}^{a_{j+1}-1} z_k$ and $v_j=\sum_{k=b_j}^{b_{j+1}-1} z_k$. We leave it to the reader to verify that h_i is injective.

Given a point (u,v) of $|\Delta[m]|\times|\Delta[n]|$, we must find a chain $c(i)$ and a point in $|\Delta[n_{c(i)}]|$ hitting (u,v) under h_i. We must also show that different choices for $c(i)$ are related by the coequalizer diagram describing $\Delta[m]\times\Delta[n]$.

To find $c(i)$, we let $w_j=u_j+\dots+u_m$, and $x_j=v_j+\dots+v_n$. We then write the set of x_j and w_j in descending order $y_1\geq y_2\geq\dots\geq y_{m+n}$. There may be more than one way to do this, of course. Each w_j must be some y_{k_j}. The set of the k_j is an m-subset of $m+n$, so corresponds to a maximal chain $c(i)$. Now let $z_j=y_j-y_{j+1}$, where $y_{m+n+1}=0$. Then $h_i(z_1,\dots,z_{m+n})=(u,v)$ as required. We leave it to the reader to verify that the ambiguity in the choice of $c(i)$ corresponds exactly to points in $\coprod|\Delta[n_{c(i)\cap c_j}]|$. Thus the map $|\Delta[m]\times\Delta[n]| \longrightarrow |\Delta[m]|\times|\Delta[n]|$ is bijective, and so a homeomorphism, as required. $\square$

This lemma is also proved in [**GZ67**, Section III.3], but using a different definition of $|\Delta[n]|$. Our proof is based on their proof, however.

This proof does not work in **Top**, because the product does not preserve colimits unless one of the factors is locally compact Hausdorff. The geometric realization of any simplicial set is Hausdorff, but is not always locally compact.

We will see later that the geometric realization preserves other kinds of finite limits as well as products.

3.2. The model structure on simplicial sets

We now want to put a model structure on **SSet**, using Theorem 2.1.19 as always. In this section, we will define the model structure, but we will not be able to complete the proof that **SSet** is a model category.

DEFINITION 3.2.1. Define the set I to consist of the canonical inclusions $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$. Define the set J to consist of the canonical inclusions $\Lambda^r[n] \to \Delta[n]$ for $n > 0$ and $0 \leq r \leq n$. A map $f \in$ **SSet** is a *cofibration* if and only if it is in I-cof. A map $f \in$ **SSet** is a *fibration* (sometimes called a *Kan fibration*) if and only if it is in J-inj. A map $f \in$ **SSet** is a *weak equivalence* if and only if $|f|$ is a weak equivalence in **Top**. The maps in J-cof are called *anodyne extensions*.

Given a fibration $p\colon X \to Y$ and a vertex $v\colon \Delta[0] \to Y$, we will often refer to the pullback $\Delta[0] \times_Y X$ as the *fiber of* p *over* v.

The cofibrations in **SSet** are particularly simple.

PROPOSITION 3.2.2. *A map $f\colon K \to L$ in* **SSet** *is a cofibration if and only if it is injective. In particular, every simplicial set is cofibrant. Furthermore, every cofibration is a relative I-cell complex.*

PROOF. Certainly the maps of I are injective. Since injections are closed under pushouts, transfinite compositions, and retracts, every element of I-cof is an injection as well. Conversely, suppose $K \xrightarrow{f} L$ is injective. We write f as a countable composition of pushouts of coproducts of maps of I, thereby showing that $f \in I$-cell. Define $X_0 = K$. Having defined X_n and an injection $X_n \to L$ which is an isomorphism on simplices of dimension less than n, let S_n denote the set of n-simplices of L not in the image of X_n. Each such simplex s is necessarily non-degenerate, and corresponds to a map $\Delta[n] \to L$. The restriction of s to $\partial\Delta[n]$ factors uniquely through X_n. Define X_{n+1} as the pushout in the diagram

$$\begin{tikzcd} \coprod_S \partial\Delta[n] \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod_S \Delta[n] \arrow[r] & X_{n+1} \end{tikzcd}$$

Then the inclusion $X_n \to L$ extends to a map $X_{n+1} \to L$. This extension is surjective on simplices of dimension $\leq n$, by construction. It is also injective, since we are only adding non-degenerate simplices. The map $f\colon K \to L$ is a composition of the sequence X_n, so f is a relative I-cell complex. □

Since the maps of J are injective, $J \subseteq I$-cof, and so J-cof $\subseteq I$-cof.

Note that $|\Delta[n]|$ is homeomorphic to D^n, and this homeomorphism takes $|\partial\Delta[n]|$ to S^{n-1}. Of course, D^n is also homeomorphic to $D^{n-1} \times [0,1]$, and one can choose this homeomorphism to take $|\Lambda^r[n]|$ to D^{n-1}. By Lemma 2.1.8, it follows that $|I\text{-cof}|$ consists of cofibrations of k-spaces, and that $|J\text{-cof}|$ consists of trivial cofibrations of k-spaces. Furthermore, Lemma 2.1.8 also implies that the singular functor takes fibrations of k-spaces to Kan fibrations and trivial fibrations of k-spaces to maps of I-inj.

The following proposition is then immediate.

PROPOSITION 3.2.3. *Every anodyne extension is a trivial cofibration of simplicial sets.*

These comments also allow us to prove the following fact about geometric realizations.

LEMMA 3.2.4. *The geometric realization functor* $|\ |\colon \mathbf{SSet} \to \mathbf{K}$ *preserves all finite limits, and in particular, preserves pullbacks.*

PROOF. We have already seen that the geometric realization functor preserves finite products in Lemma 3.1.8. It therefore suffices to prove that the geometric realization preserves equalizers. Suppose K is the equalizer in $\mathbf{SSet}$ of two maps $f, g\colon L \to M$. Let Z be the equalizer in $\mathbf{Top}$ of $|f|$ and $|g|$. The map $\emptyset \to M$ is an injection, and hence is in I-cell by Proposition 3.2.2. Thus $|M|$ is a cell complex. It is well-known that every cell complex is Hausdorff; one can prove it by transfinite induction, using the fact that cells themselves are normal and that the inclusion of the boundary of a cell is a neighborhood deformation retract. It follows that Z is a closed subspace of $|L|$. In particular, Z is a k-space, so is also the equalizer in $\mathbf{K}$. Now, $|K|$ is also (homeomorphic to) a closed subspace of $|L|$. Indeed, $K \to L$ is an injection, and so is in I-cell by Proposition 3.2.2. Thus $|K| \to |L|$ is a relative cell complex in $\mathbf{K}$, and any such is a closed inclusion by Lemma 2.4.5. Since the image of $|K|$ in $|L|$ is obviously contained in Z, it suffices to show that every point of Z is in the image of $|K|$. So take a $z \in Z$. The point z must be in the interior of a $|x|$ for a unique non-degenerate simplex x of L. By definition of the geometric realization, the only way for $|f|(z)$ to equal $|g|(z)$ is if $fx = gx$. Hence x is a (necessarily non-degenerate) simplex of K, and so z is in the image of $|K|$ as required. □

To complete the proof that $\mathbf{SSet}$ forms a model category, we must show that a map $f\colon K \to L$ is a trivial fibration if and only if it is in I-inj. We can prove part of this now, after the following lemma.

LEMMA 3.2.5. *Suppose* $f\colon K \to L$ *is in* I*-inj. Then* $|f|$ *is a fibration.*

PROOF. Since f has the right lifting property with respect to I, f ahas the right lifting property with respect to all inclusions of simplicial sets, by Proposition 3.2.2. In particular, we can find a lift in the following commutative square.

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d, "{(id,f)}"'] & K \arrow[d, "f"] \\
K \times L \arrow[r, "p_2"] & L
\end{tikzcd}$$

where p_2 is the projection. This lift makes f into a retract of p_2. Hence $|f|$ is a retract of $|p_2|$, which is a fibration since the geometric realization preserves products by Lemma 3.1.8. Thus $|f|$ is a fibration. □

PROPOSITION 3.2.6. *Suppose* $f\colon K \to L$ *is in* I*-inj. Then* f *is a trivial fibration.*

Proof. Since $J \subseteq I$-cell, it is clear that f is a fibration. We must show that $|f|$ is a weak equivalence. Let $F = f^{-1}(v)$ be the fiber of f over some vertex $v \in L$, so that we have a pullback diagram

$$\begin{array}{ccc} F & \longrightarrow & K \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \Delta[0] & \xrightarrow{v} & L \end{array}$$

Then by Lemma 3.2.5 and Lemma 3.2.4, $|f|$ is a fibration with fiber $|F|$.

Now, note that the map $F \to \Delta[0]$ has the right lifting property with respect to I, and hence with respect to all inclusions by Proposition 3.2.2. In particular, F is nonempty, so we can find a 0-simplex w in F. We denote the resulting map $F \to \Delta[0] \xrightarrow{w} F$ by w as well. We can then find a lift $H\colon F \times \Delta[1] \to F$ in the commutative square

$$\begin{array}{ccc} F \times \partial\Delta[1] & \xrightarrow{(id,w)} & F \\ \big\downarrow & & \big\downarrow \\ F \times \Delta[1] & \longrightarrow & \Delta[0] \end{array}$$

Since the geometric realization preserves products, $|H|$ is a homotopy between the identity map of F and a constant map, so $|F|$ is contractible.

By the long exact homotopy sequence of the fibration $|f|$, we are then reduced to showing that $|f|$ is surjective on path components. But one can easily see that any point in $|L|$ is in the same path component as the realization of some vertex x of L. Since f has the right lifting property with respect to all inclusions, f is surjective, and in particular surjective on vertices. Thus there is a vertex y of K such that $f(y) = x$, and so the path component containing y goes to the path component containing x. □

To prove the converse of Proposition 3.2.6, we must develop a considerable amount of homotopy theory in **SSet**, which we begin to do in the next section.

3.3. Anodyne extensions

The goal of this section is to prove the following theorem.

Theorem 3.3.1. *Suppose $i\colon K \to L$ is an inclusion of simplicial sets, and $p\colon X \to Y$ is a fibration of simplicial sets. Then the induced map*

$$\operatorname{Map}_{\square}(i,p)\colon\ \operatorname{Map}(L,X) \to \operatorname{Map}(K,X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L,Y)$$

is a fibration.

Recall that the analogue of this theorem in **Top**, Lemma 2.4.13, was essential to the proof that **Top** is a model category.

In particular, if X is a fibrant simplicial set and $K \to L$ is an inclusion, Theorem 3.3.1 implies that the induced map $\operatorname{Map}(L,X) \to \operatorname{Map}(K,X)$ is a fibration.

At first glance this theorem may seem to have little to do with the title of the section. However, they are actually very closely related. Indeed, Theorem 3.3.1 is equivalent to the following theorem.

THEOREM 3.3.2. *For every anodyne extension $f\colon A \to B$ and inclusion $i\colon K \to L$ of simplicial sets, the induced map*

$$i \,\square\, f\colon P(i,f) = (K \times B) \amalg_{K\times A} (L \times A) \to L \times B$$

is an anodyne extension.

The proof that Theorem 3.3.1 is equivalent to Theorem 3.3.2 is an exercise in adjointness, using the fact that fibrations from the class J-inj = (J-cof)-inj. We leave the details to the reader.

In order to prove Theorem 3.3.2 we will need to construct some anodyne extensions.

LEMMA 3.3.3. *Let $i\colon \partial\Delta[n] \to \Delta[n]$ denote the boundary inclusion for $n \geq 0$, and let $f\colon \Lambda^\varepsilon[1] \to \Delta[1]$ denote the obvious inclusion, for $\varepsilon = 0$ or 1. Then the map $i \,\square\, f\colon P(i,f) = (\partial\Delta[n] \times \Delta[1]) \amalg_{\partial\Delta[n]\times\Lambda^\varepsilon[1]} (\Delta[n] \times \Lambda^\varepsilon[1]) \to \Delta[n] \times \Delta[1]$ is an anodyne extension.*

PROOF. Recall from the proof of Lemma 3.1.8 that a (non-degenerate) k-simplex of $\Delta[n] \times \Delta[1]$ is just an (injective) order-preserving map $[k] \to [n] \times [1]$. There are thus $n+1$ non-degenerate $n+1$-simplices x_j of $\Delta[n] \times \Delta[1]$. The $n+1$-simplex x_j, for $0 \leq j \leq n$, is the maximal chain

$$x_j = ((0,0), \ldots, (j,0), (j,1), \ldots, (n,1)).$$

Every simplex of $\Delta[n]\times\Delta[1]$ is a degeneracy of a face of an x_j. All the compatibility between the x_j's is implied by the relation

$$d_{j+1}x_j = d_{j+1}x_{j+1} = ((0,0), \ldots, (j,0), (j+1,1), \ldots, (n,1))$$

for $0 \leq j < n$. Furthermore, we have $d_i x_j \in \partial\Delta[n] \times \Delta[1]$ unless $i = j$ or $j+1$, and $d_0 x_0 \in \Delta[n] \times \Lambda^1[1]$ and $d_{n+1}x_n \in \Delta[n] \times \Lambda^0[1]$. Hence, to get from $(\partial\Delta[n] \times \Delta[1]) \amalg_{\partial\Delta[n]\times\Lambda^1[1]} (\Delta[n] \times \Lambda^1[1])$ to $\Delta[n] \times \Delta[1]$, we first attach x_0 along $\Lambda^1[n+1]$, since all the faces except $d_1 x_0$ are already there. We then attach x_1 along $\Lambda^2[n+1]$, since $d_1 x_1 = d_1 x_0$ is already there. Continuing in this fashion, we find that our map is the composite of $n+1$ pushouts of maps of J, and hence is an anodyne extension. When $\varepsilon = 1$ instead, we start by attaching x_n and work our way downwards in a similar fashion. □

From Lemma 3.3.3 we can construct many more anodyne extensions.

PROPOSITION 3.3.4. *Let $i\colon K \to L$ be an inclusion of simplicial sets, and let $f\colon \Lambda^\varepsilon[1] \to \Delta[1]$ be the usual inclusion, where $\varepsilon = 0$ or 1. Then the map $i \,\square\, f\colon P(i,f) \to L \times \Delta[1]$ is an anodyne extension.*

PROOF. Lemma 3.3.3 says that the set $I \,\square\, f$ consists of anodyne extensions. This means that the maps of $I \,\square\, f$ have the left lifting property with respect to J-inj. By adjointness, we find that the maps of I have the left lifting property with respect to $\mathrm{Map}_\square(f, J\text{-inj})$. It follows that any map i in I-cof has the left lifting property with respect to $\mathrm{Map}_\square(f, J\text{-inj})$. Using adjointness again, we find that $i \,\square\, f$ has the left lifting property with respect to J-inj. Therefore, $i \,\square\, f$ is an anodyne extension, as required. □

We can then give an alternative characterization of anodyne extensions. Let J' denote the set of maps $J \,\square\, f$, where f is one of the maps $\Lambda^\varepsilon[1] \to \Delta[1]$.

Proposition 3.3.5. *A map $g\colon K \to L$ of simplicial sets is an anodyne exension if and only if it is in J'-cof.*

Proof. Proposition 3.3.4 implies that every map of J' is an anodyne extension, and hence that J'-cof $\subseteq J$-cof. To prove the converse, we will show that the maps of J are retracts of maps of J'. So suppose $k < n$. We will construct a commutative diagram

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d] & (\Lambda^k[n] \times \Delta[1]) \amalg_{\Lambda^k[n]\times\{0\}} (\Delta[n] \times \{0\}) \arrow[r] \arrow[d] & \Lambda^k[n] \arrow[d] \\
\Delta[n] \arrow[r, "g"] & \Delta[n] \times \Delta[1] \arrow[r, "r_k"] & \Delta[n]
\end{tikzcd}$$

displaying a map of J as a retract of a map of J'. Here g is induced by the inclusion of ordered sets $[n] \to [n] \times [1]$ that takes j to $(j, 1)$. The map r_k is induced by the map of ordered sets $[n] \times [1] \to [n]$ that takes $(j, 1)$ to j and $(j, 0)$ to j if $j \leq k$ and to k if $j > k$. It is then clear that $r_k g$ is the identity, and we leave it to to the reader to check that r_k does indeed send $\Lambda^k[n] \times \Delta[1]$ and $\Delta[n] \times \{0\}$ to $\Lambda^k[n]$.

This particular retraction will not work when $k = n$. However, in this case we can still construct a similar retraction, by letting $g'\colon \Delta[n] \to \Delta[n] \times \Delta[1]$ be induced by the map $j \mapsto (j, 0)$ of ordered sets, and letting $r'\colon \Delta[n] \times \Delta[1] \to \Delta[n]$ be induced by the map of ordered sets $(j, 0) \mapsto j$, $(j, 1) \mapsto n$. $\square$

We can now prove Theorem 3.3.2, and hence Theorem 3.3.1.

Proof of Theorem 3.3.2. We must show that $i \,\square\, f$ is an anodyne extension for all inclusions i and anodyne extensions f. We first verify that $i \,\square\, J'$ consists of anodyne extensions. Indeed, we have $i \,\square\, J' = i \,\square\, (J \,\square\, g)$, where g is one of the maps $\Lambda^\varepsilon[1] \to \Delta[1]$ for $\varepsilon = 0$ or 1. A beautiful property of the box product is that it is associative (up to isomorphism), so we have $i \,\square\, (J \,\square\, g) \cong (i \,\square\, J) \,\square\, g$. This associativity is tedious, but elementary, to verify. We will return to it in the next chapter. In any case, one can easily check that $i \,\square\, J$ consists of inclusions. Proposition 3.3.4 then implies that $(i \,\square\, J) \,\square\, g$ consists of anodyne extensions, and hence that $i \,\square\, J'$ consists of anodyne extensions.

We now show that this implies that $i \,\square\, f$ is an anodyne extension for all anodyne extensions f, by a similar argument to Proposition 3.3.4. Indeed, we have just seen that the maps of $i \,\square\, J'$ have the left lifting property with respect to J-inj. Adjointness implies that the maps of J' have the left lifting property with respect to $\mathrm{Map}_\square(i, J\text{-inj})$. But then the maps of J'-cof must also have the left lifting property with respect to $\mathrm{Map}_\square(i, J\text{-inj})$. Applying adjointness again, we find that $i \,\square\, f$ has the left lifting property with respect to J-inj, and is therefore an anodyne extension, for all $f \in J'$-cof. Since every anodyne extension is in J'-cof by Proposition 3.3.5, the proof is complete. $\square$

3.4. Homotopy groups

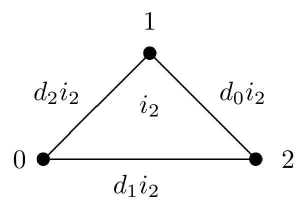
In this section, we use the results of the previous section to construct the homotopy groups of a fibrant simplicial set. The main result of this section is that, if X is a fibrant simplicial set with no nontrivial homotopy groups, then the map $X \to \Delta[0]$ has the right lifting property with respect to I. This is the prototype for the goal result that a trivial fibration has the right lifting property with respect to I. We also show that a fibration gives rise to a long exact sequence of homotopy groups. This result will be needed later, when we compare the homotopy groups of a fibrant simplicial set and the homotopy groups of its geometric realization.

We begin by defining $\pi_0 K$ for fibrant simplicial sets X.

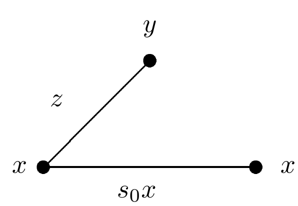
Definition 3.4.1. Suppose X is a fibrant simplicial set, and $x, y \in X_0$ are 0-simplices. Define x to be *homotopic* to y, written $x \sim y$, if and only if there is a 1-simplex $z \in X_1$ such that $d_1 z = x$ and $d_0 z = y$.

Lemma 3.4.2. *Suppose X is a fibrant simplicial set. Then homotopy of vertices is an equivalence relation. We denote the set of equivalence classes by $\pi_0 X$.*

Proof. Homotopy of vertices is obviously reflexive, since if $x \in X_0$, we have $d_1 s_0 x = d_0 s_0 x = x$. Now suppose $x \sim y$, so we have a 1-simplex z such that $d_1 z = x$ and $d_0 z = y$. Then we get a map $f \colon \Lambda^0[2] \to X$ which is $s_0 x$ on $d_1 i_2$ and z on $d_2 i_2$. It is easiest to see this pictorially. We think of $\Delta[2]$ as the following picture.

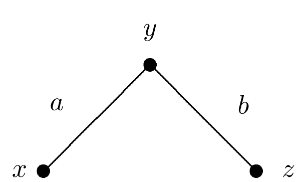


Then we think of the map f as the following picture.



Because X is fibrant, there is an extension of f to a 2-simplex w of X_2. Then $d_0 w$ is the required homotopy from y to x, as is clear from the picture.

Now suppose $x \sim y$ and $y \sim z$, so that we have 1-simplices a and b such that $d_1 a = x$, $d_0 a = d_1 b = y$, and $d_0 b = z$. Then a and b define a map $f \colon \Lambda^1[2] \to X$, as in the following picture.



Since X is fibrant, there is an extension of f to 2-simplex c of X_2. Then $d_1 c$ is the required homotopy from x to z. □

The justification for calling the equivalence classes $\pi_0 X$ is provided by the following lemma.

LEMMA 3.4.3. *Suppose X is a fibrant simplicial set. Then there is a natural isomorphism $\pi_0 X \cong \pi_0|X|$.*

PROOF. The natural map $\pi_0 X \longrightarrow \pi_0|X|$ takes a vertex v to the path component of $|X|$ containing $|v|$. Since $|\Delta[n]|$ is path connected for $n > 0$, this map is surjective, as every point of $|X|$ is in the path component of a vertex. To prove the converse, define, for $\alpha \in \pi_0 X$, the sub-simplicial set X_α of X to consist of all simplices x of X with a vertex in α (where a vertex is a 0-dimensional face). One can easily see that X_α is indeed a sub-simplicial set of X, and that $X = \coprod_{\alpha \in \pi_0 X} X_\alpha$. Since the geometric realization preserves coproducts, which are disjoint unions in **Top**, the proof is complete. □

In light of this lemma, we refer to elements of $\pi_0 X$ as *path components* of X. Note that π_0 is a functor from fibrant simplicial sets to sets. If v is a vertex of a fibrant simplicial set X, $\pi_0(X, v)$ is the pointed set $\pi_0 X$ with basepoint the equivalence class $[v]$ of v.

Naturally we would like to extend this definition of homotopy of vertices to homotopy of n-simplices.

DEFINITION 3.4.4. Suppose X is a fibrant simplicial set, and $v \in X_0$ is a vertex. For any Y, let us denote the map $Y \longrightarrow \Delta[0] \xrightarrow{v} X$ by v as well, and refer to it as the constant map at v. Let F denote the fiber over v of the fibration $\mathrm{Map}(\Delta[n], X) \longrightarrow \mathrm{Map}(\partial\Delta[n], X)$. This map is a fibration by Theorem 3.3.1. Then we define the *nth homotopy group* $\pi_n(X, v)$ of X at v to be the pointed set $\pi_0(F, v)$.

Note that $\pi_n(X, v)$ is the set of equivalence classes $[\alpha]$ of n-simplices $\alpha \colon \Delta[n] \longrightarrow X$ that send $\partial\Delta[n]$ to v, under the equivalence relation defined by $\alpha \sim \beta$ if there is a homotopy $H \colon \Delta[n] \times \Delta[1] \longrightarrow X$ such that H is α on $\Delta[n] \times \{0\}$, β on $\Delta[n] \times \{1\}$, and is the constant map v on $\partial\Delta[n] \times \Delta[1]$. This is just a translation of the definition. However, if we defined the homotopy groups this way, it would not be obvious that the homotopy relation is in fact an equivalence relation.

It is not obvious at this point that the homotopy groups are in fact groups. We do not need this for the proof that **SSet** forms a model category, so we will not prove it directly. However, we will prove in Proposition 3.6.3 that $\pi_n(X, v) \cong \pi_n(|X|, |v|)$ for a fibrant simplicial set X. Thus $\pi_n(X, v)$ is a group for $n \geq 1$ which is abelian for $n \geq 2$.

Given a map $f \colon X \longrightarrow Y$ and a vertex v of X, there is an induced map $f_* \colon \pi_n(X, v) \longrightarrow \pi_n(Y, f(v))$, making the homotopy groups functorial.

We have the following expected lemma giving an alternative characterization of when an n-simplex is homotopic to the constant map.

LEMMA 3.4.5. *Suppose X is a fibrant simplicial set, v is a vertex of X, and $\alpha \colon \Delta[n] \longrightarrow X$ is an n-simplex of X such that $d_i \alpha = v$ for all i. Then $[\alpha] = [v] \in \pi_n(X, v)$ if and only there is an $n+1$-simplex x of X such that $d_{n+1} x = \alpha$ and $d_i x = v$ for $i \leq n$.*

PROOF. Suppose first that $[\alpha] = [v]$. Then there is a homotopy $H\colon \Delta[n] \times \Delta[1] \to X$ from α to v which is v on $\partial\Delta[n] \times \Delta[1]$. We can then define a map $G\colon \partial\Delta[n+1] \times \Delta[1] \to X$ by $G \circ (d^i \times 1) = v$ for $i < n+1$ and $G \circ (d^{n+1} \times 1) = H$. Then G is just v on $\partial\Delta[n+1] \times \{1\}$, so we have a commutative diagram

$$\begin{array}{ccc} (\partial\Delta[n+1] \times \Delta[1]) \amalg_{\partial\Delta[n+1]\times\{1\}} (\Delta[n+1] \times \{1\}) & \xrightarrow{G \amalg v} & X \\ \downarrow & & \downarrow \\ \Delta[n+1] \times \Delta[1] & \longrightarrow & \Delta[0] \end{array}$$

Since X is fibrant, there is a lift $F\colon \Delta[n+1] \times \Delta[1] \to X$ in this diagram. The $n+1$-simplex $F(\Delta[n+1] \times \{0\})$ is the desired x such that $d_{n+1}x = \alpha$ and $d_i x = v$ for $i \le n$.

Conversely, suppose we have an $n+1$-simplex x such that $d_{n+1}x = \alpha$ and $d_i x = v$ for $i \le n$. We define a map

$$G\colon (\Lambda^{n+1}[n+1] \times \Delta[1]) \amalg_{\Lambda^{n+1}[n+1]\times\partial\Delta[1]} (\Delta[n+1] \times \partial\Delta[1]) \to X$$

by defining it to be v on $\Lambda^{n+1}[n+1] \times \Delta[1]$ and $\Delta[n+1] \times \{1\}$, and defining it to be x on $\Delta[n+1] \times \{0\}$. Then, since X is fibrant, there is an extension of G to a map $F\colon \Delta[n+1] \times \Delta[1] \to X$. Let $H = F \circ (d^{n+1} \times 1)$. Then H is the desired homotopy between α and v. □

More generally, if $f, g\colon K \to X$ are any maps of simplicial sets, we refer to a map $H\colon K \times \Delta[1] \to X$ such that H is f on $K \times \{0\}$ and g on $K \times \{1\}$ as a *homotopy* from f to g. The resulting homotopy relation is not always an equivalence relation, but it will be when X is fibrant. Indeed, in that case, f and g are homotopic if and only if they are homotopic as vertices of the fibrant simplicial sets $\mathrm{Map}(K, X)$.

An important example of a homotopy is provided by the following lemma.

LEMMA 3.4.6. *The vertex n is a deformation retract of $\Delta[n]$, in the sense that there is a homotopy $H\colon \Delta[n] \times \Delta[1] \to \Delta[n]$ from the identity map to the constant map at n which sends $n \times \Delta[1]$ to n. Furthermore, this homotopy restricts to a deformation retraction of $\Lambda^n[n]$ onto its vertex n.*

PROOF. As we saw in the proof of Lemma 3.1.8, a simplex of $\Delta[n] \times \Delta[1]$ is a chain of the ordered set $[n] \times [1]$. Hence a homotopy $\Delta[n] \times \Delta[1] \to \Delta[n]$ is equivalent to a map of ordered sets $[n] \times [1] \to [n]$. An obvious such map is the map which takes $(k, 0)$ to k and $(k, 1)$ to n. The homotopy corresponding to this map is the desired deformation retraction. □

Though we have just constructed a homotopy from the identity map of $\Delta[n]$ to the constant map at n, there is no homotopy going the other direction. Indeed, such a homotopy would have to be induced by a map of ordered sets that takes $(k, 0)$ to n and $(k, 1)$ to k, and there is no such map. Thus, homotopy is not an equivalence relation on self-maps of $\Delta[n]$, proving that $\Delta[n]$ is not fibrant.

We can now prove the main result of this section.

PROPOSITION 3.4.7. *Suppose X is a non-empty fibrant simplicial set with no non-trivial homotopy groups. Then the map $X \to \Delta[0]$ is in I-inj.*

PROOF. We must show that any map $f\colon \partial\Delta[n] \to X$ has an extension to $\Delta[n]$. We can assume $n > 0$ since X is non-empty. We first point out that if f and g are homotopic, and g has an extension $g'\colon \Delta[n] \to X$, then f also extends to $\Delta[n]$.

Indeed, g' together with a homotopy $H\colon \partial\Delta[n]\times\Delta[1]\to X$ from f to g define a map

$$(\partial\Delta[n]\times\Delta[1])\amalg_{\partial\Delta[n]\times\{1\}}(\Delta[n]\times\{1\})\to X$$

Since X is fibrant, there is an extension of this map to a homotopy $G\colon \Delta[n]\times\Delta[1]\to X$. Then $G(\Delta[n]\times\{0\})$ is the desired extension of f.

Consider the composition $H'\colon \Lambda^n[n]\times\Delta[1]\xrightarrow{H}\Lambda^n[n]\xrightarrow{f}X$, where H is the deformation retraction of $\Lambda^n[n]$ onto n of Lemma 3.4.6 and f is really the restriction of f. Then H' and f define a map

$$(\Lambda^n[n]\times\Delta[1])\amalg_{\Lambda^n[n]\times\{0\}}(\partial\Delta[n]\times\{0\})\to X$$

Since X is fibrant, there is an extension $G\colon \partial\Delta[n]\times\Delta[1]\to X$. The map G is a homotopy from f to a map g such that $g\circ d^i=f(n)$ for $i<n$. In particular, $g\circ d^n$ represents a class in $\pi_{n-1}(X,f(n))$. By assumption, then, $[g\circ d^n]=[f(n)]$. By Lemma 3.4.5, there is an n-simplex g' such that $d_ig'=f(n)$ for $i<n$ and $d_ng'=g\circ d^n$. Thus g' is an extension of g. Hence f also has an extension, as required. $\square$

For later use, we now construct the long exact sequence in homotopy of a fibration. So suppose $p\colon X\to Y$ is a fibration of fibrant simplicial sets, and v is a vertex of X. Let F denote the fiber of p over $p(v)$. We will construct a map $\partial\colon \pi_n(Y,p(v))\to\pi_{n-1}(F,v)$ as follows. Given a class $[\alpha]\in\pi_n(Y,p(v))$, define $\partial[\alpha]=[d_n\gamma]$, where γ is a lift in the diagram

$$\begin{array}{ccc}\Lambda^n[n] & \xrightarrow{v} & X\\ \Big\downarrow & & \Big\downarrow{\scriptstyle p}\\ \Delta[n] & \xrightarrow{\alpha} & Y\end{array}$$

Since p is a fibration, such a lift γ exists. The commutativity of the diagram implies that $d_n\gamma$ lies in F, and it is easy to see that $d_id_n\gamma=v$ for all i, so that $[d_n\gamma]\in\pi_{n-1}(F,v)$.

LEMMA 3.4.8. *The map ∂ is well-defined.*

PROOF. Suppose we have a possibly different representative $\beta\colon \Delta[n]\to Y$ for $[\alpha]$, and a lift $\delta\colon \Delta[n]\to X$ in the diagram

$$\begin{array}{ccc}\Lambda^n[n] & \xrightarrow{v} & X\\ \Big\downarrow & & \Big\downarrow{\scriptstyle p}\\ \Delta[n] & \xrightarrow{\beta} & Y\end{array}$$

We must show that $d_n\gamma$ and $d_n\delta$ represent the same homotopy class. Since α and β represent the same homotopy class of Y, there is a homotopy $H\colon \Delta[n]\times\Delta[1]\to Y$ from α to β which is the constant map pv on $\partial\Delta[n]\times\Delta[1]$. Hence we have a commutative diagram

$$\begin{array}{ccc}(\Lambda^n[n]\times\Delta[1])\amalg_{\Lambda^n[n]\times\partial\Delta[1]}(\Delta[n]\times\partial\Delta[1]) & \xrightarrow{f} & X\\ \Big\downarrow & & \Big\downarrow{\scriptstyle p}\\ \Delta[n]\times\Delta[1] & \xrightarrow{H} & Y\end{array}$$

where f is the constant map v on $\Lambda^n[n] \times \Delta[1]$, f is the map γ on $\Delta[n] \times \{0\}$, and f is the map δ on $\Delta[n] \times \{1\}$. Since the left vertical map is an anodyne extension by Theorem 3.3.2, there is a lift $G\colon \Delta[n] \times \Delta[1] \longrightarrow X$ in this diagram. But then the composite

$$\Delta[n-1] \times \Delta[1] \xrightarrow{d^n \times 1} \Delta[n] \times \Delta[1] \xrightarrow{G} X$$

actually lands in the fiber F over pv and is the desired homotopy between $d_n\gamma$ and $d_n\delta$. $\square$

It follows from Lemma 3.4.8 that the boundary map is also natural for maps of fibrations. We leave the verification of this to the reader.

LEMMA 3.4.9. *Suppose $p\colon X \longrightarrow Y$ is a fibration of fibrant simplicial sets, and that v is a vertex of X. Let $i\colon F \longrightarrow X$ denote the inclusion of the fiber of p over $p(v)$. Then the sequence of pointed sets*

$$\dots \xrightarrow{\partial} \pi_n(F, v) \xrightarrow{i_*} \pi_n(X, v) \xrightarrow{p_*} \pi_n(Y, p(v))$$
$$\xrightarrow{\partial} \pi_{n-1}(F, v) \xrightarrow{i_*} \dots \xrightarrow{i_*} \pi_0(X, v) \xrightarrow{p_*} \pi_0(Y, p(v))$$

is exact, in the sense that the kernel (defined as the preimage of the basepoint) is equal to the image at each spot.

PROOF. The proof of this lemma is mostly straightforward using Lemma 3.4.5 and Lemma 3.4.8. We leave most of it to the reader. We will prove that the kernel of ∂ is contained in the image of p_*, however. Suppose $\partial[\alpha] = [v]$, where $\alpha\colon \Delta[n] \longrightarrow Y$ has $d_i\alpha = pv$ for all i. Let $\gamma\colon \Delta[n] \longrightarrow X$ be a lift in the commutative diagram

$$\begin{tikzcd} \Lambda^n[n] \arrow[r, "v"] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r, "\alpha"] & Y \end{tikzcd}$$

Then there is a homotopy $H\colon \Delta[n-1] \times \Delta[1] \longrightarrow F$ from $d_n\gamma$ to v. We use H to define a commutative diagram

$$\begin{tikzcd} (\partial\Delta[n] \times \Delta[1]) \amalg_{\partial\Delta[n]\times\{0\}} (\Delta[n] \times \{0\}) \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \times \Delta[1] \arrow[r, "\alpha\pi_1"] & Y \end{tikzcd}$$

where π_1 is the projection onto the first factor. Indeed, we define f to be γ on $\Delta[n] \times \{0\}$, to be iH on the face $d_n i_n \times 1$ of $\partial\Delta[n] \times \Delta[1]$, and to be v on the other faces $d_i i_n \times 1$ for $i < n$. The left vertical map is an anodyne extension, so there is a lift $G\colon \Delta[n] \times \Delta[1] \longrightarrow X$. The n-simplex $\beta = G(\Delta[n] \times \{1\})$ defines a class in $\pi_n(X, v)$ such that $p_*[\beta] = [\alpha]$, as required. $\square$

3.5. Minimal fibrations

We have seen in the last section that the map $F \longrightarrow \Delta[0]$ is in I-inj when F is a fibrant simplicial set with no homotopy. In this section we first point out that this implies a lifting result for some locally trivial fibrations. We then point out that every fibration is locally fiberwise homotopy equivalent to a locally trivial fibration. We thus try to determine what we need to know about a fibration to guarantee that this local fiberwise homotopy equivalence is actually an isomorphism. This leads us to the notion of a minimal fibration. We show that minimal fibrations are locally trivial, and that every fibration is closely approximated by a minimal fibration.

Reasoning by analogy with topological spaces, we would expect many fibrations of simplicial sets to be locally trivial, in the following sense.

DEFINITION 3.5.1. Suppose $p\colon X \to Y$ is a fibration of simplicial sets. We say that p is *locally trivial* if, for every simplex $\Delta[n] \xrightarrow{y} Y$ of Y, the pullback fibration $y^*X = \Delta[n] \times_Y X \xrightarrow{y^*p} \Delta[n]$ is isomorphic over $\Delta[n]$ to a product fibration $\Delta[n] \times F_v \xrightarrow{\pi_1} \Delta[n]$.

If p is locally trivial, then the simplicial set F_v in Definition 3.5.1 is of course isomorphic to the fiber of p over a vertex v of the simplex y.

We then have the following corollary to Proposition 3.4.7.

COROLLARY 3.5.2. *Suppose $p\colon X \to Y$ is a locally trivial fibration of simplicial sets such that every fiber of p is non-empty and has no non-trivial homotopy groups. Then p is in I-inj.*

PROOF. Suppose we have a commutative square

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta[n] \arrow[r, "v"] & Y \end{tikzcd}$$

A lift in this square is equivalent to a lift in the square

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & v^*X \arrow[d, "v^*p"] \\ \Delta[n] \arrow[r, equal] & \Delta[n] \end{tikzcd}$$

Since p is locally trivial, this is equivalent to a lift in a square of the form

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "f"] \arrow[d] & \Delta[n] \times F \arrow[d, "\pi_1"] \\ \Delta[n] \arrow[r, equal] & \Delta[n] \end{tikzcd}$$

A lift in this square is equivalent to an extension of the map $\pi_2 f\colon \partial\Delta[n] \to F$ to $\Delta[n]$. But since F is isomorphic to a fiber of p, F is a non-empty fibrant simplicial set which has no nontrivial homotopy groups. Thus Proposition 3.4.7 completes the proof. □

It would be unreasonable to expect that every fibration of simplicial sets is locally trivial. However, since $\Delta[n]$ is simplicially contractible onto its vertex n (Lemma 3.4.6), we would expect any fibration over $\Delta[n]$ to be at least homotopy equivalent to a product fibration. This is in fact the case, as we now prove.

PROPOSITION 3.5.3. *Suppose $p\colon X \to Y$ is a fibration of simplicial sets, and suppose $f, g\colon K \to Y$ are maps such that there exists a homotopy from f to g. Then the pullback fibrations $f^*p\colon f^*X \to K$ and $g^*p\colon g^*X \to K$ are fiber homotopy equivalent. That is, there are maps $\theta_*\colon f^*X \to g^*X$ and $\omega_*\colon g^*X \to f^*X$ such that $g^*p \circ \theta_* = f^*p$ and $f^*p \circ \omega_* = g^*p$ and there are homotopies from $\theta_*\omega_*$ to* *the identity of g^*X and from $\omega_*\theta_*$ to the identity of f^*X that cover the constant homotopy of K.*

Proof. Let $h\colon K \times \Delta[1] \to Y$ be a homotopy from f to g. Let us denote the inclusion $Z = Z \times \{0\} \to Z \times \Delta[1]$ by i_0 (although it is actually $1 \times d^1$), and the denote the other inclusion $Z = Z \times \{1\} \to Z \times \Delta[1]$ by i_1. Thus $hi_0 = f$ and $hi_1 = g$.

Then we have a pullback square

$$\begin{array}{ccc} f^*X & \xrightarrow{r_f} & h^*X \\ {\scriptstyle f^*p}\downarrow & & {\scriptstyle h^*p}\downarrow \\ K & \xrightarrow{i_0} & K \times \Delta[1] \end{array}$$

and another pullback square

$$\begin{array}{ccc} g^*X & \xrightarrow{r_g} & h^*X \\ {\scriptstyle g^*p}\downarrow & & {\scriptstyle h^*p}\downarrow \\ K & \xrightarrow{i_1} & K \times \Delta[1] \end{array}$$

Hence we have a commutative square

$$\begin{array}{ccc} f^*X & \xrightarrow{r_f} & h^*X \\ {\scriptstyle i_0}\downarrow & & {\scriptstyle h^*p}\downarrow \\ f^*X \times \Delta[1] & \xrightarrow{f^*p \times 1} & K \times \Delta[1] \end{array}$$

Since h^*p is a fibration, there is a lift to a homotopy $\theta\colon f^*X \times \Delta[1] \to h^*X$. Then we have $h^*p \circ \theta i_1 = (f^*p \times 1) \circ i_1 = i_1 \circ f^*p$. Hence the pair $\theta \circ i_1$ and f^*p define a map $\theta_*\colon f^*X \to g^*X$ such that $r_g\theta_* = \theta \circ i_1$ and $g^*p\theta_* = f^*p$.

Similarly, we find a homotopy $\omega\colon g^*X \times \Delta[1] \to h^*X$ such that $h^*p \circ \omega = g^*p \times 1$ and $\omega i_1 = r_g$. This induces a map $\omega_*\colon g^*X \to f^*X$ such that $f^*p \circ \omega_* = g^*p$ and $r_f\omega_* = \omega \circ i_0$.

We would like to find a homotopy from $\omega_*\theta_*$ to the identity of f^*X. Such a homotopy would induce a homotopy from the map $r_f\omega_*\theta_* = \omega(\theta_* \times 1)i_0$ to the map $r_f\colon f^*X \to h^*X$. We can construct such a homotopy by lifting in the diagram

$$\begin{array}{ccc} f^*X \times \Lambda^2[2] & \xrightarrow{H} & h^*X \\ \downarrow & & {\scriptstyle h^*p}\downarrow \\ f^*X \times \Delta[2] & \xrightarrow{f^*p \times s^0} & K \times \Delta[1] \end{array}$$

Here H is the map θ on $f^*X \times d_0\Delta[2]$ and is the map $\omega(\theta_* \times 1)$ on $f^*X \times d_2\Delta[2]$. Since $\theta i_1 = \omega(\theta_* \times 1)i_1$ the map H makes sense. Since $h^*p \circ \theta = f^*p \times 1 = h^*p \circ \omega(\theta_* \times 1)$, this diagram commutes. Hence there is a lift G in this diagram. Let γ denote G on $f^*X \times d_2\Delta[2]$, so that γ is a homotopy from $r_f\omega_*\theta_*$ to r_f such that $h^*p \circ \gamma = f^*p \times s^0d^2 = i_0 \circ f^*p \circ \pi_1$. It follows that γ induces a map $\gamma_*\colon f^*X \times \Delta[1] \to f^*X$ such that $r_f\gamma_* = \gamma$ and $f^*p \circ \gamma = f^*p \circ i_1$. Thus γ_* is the required fiberwise homotopy from $\omega_*\theta_*$ to the identity map.

In a similar fashion we can construct a homotopy from $\theta_*\omega_*$ to the identity of g^*X which covers the constant homotopy, as required. □

COROLLARY 3.5.4. *Suppose $p\colon X \to Y$ is a fibration of simplicial sets, and let $\Delta[n] \xrightarrow{y} Y$ be a simplex of Y. Then the pullback $y^*X \to \Delta[n]$ is fiber homotopy equivalent to the product fibration $\Delta[n] \times F_n \xrightarrow{\pi_1} \Delta[n]$, where F_n is the fiber of p over the vertex $y(n)$.*

PROOF. The identity map of $\Delta[n]$ is homotopic to the constant map n, by Lemma 3.4.6. Hence Proposition 3.5.3 completes the proof. $\square$

Now we want to put restrictions on the fibration p so that the fiber homotopy equivalence of Corollary 3.5.4 must actually be an isomorphism. It suffices to consider the following situation. Suppose we have two fibrations $p\colon X \to Y$ and $q\colon Z \to Y$ over the same base, and two maps $f, g\colon X \to Z$ covering the identity map of Y such that g is an isomorphism. Suppose as well that they are fiber homotopic, so that there is a homotopy from f to g which convers the constant homotopy. Given a fiber homotopy equivalence, we would take g to be the identity map and f to be the composite of one of the maps with a homotopy inverse. We would like to conclude that f is also an isomorphism.

Let us just try to prove that f is an isomorphism on vertices. Suppose z is a vertex of Z. Then there is a vertex x of X such that $gx = z$ since g is an isomorphism. The homotopy gives us a path from fx to gx which covers the constant path of $qgx = qfx$. Hence fx and gx are vertices of a fiber of q which are in the same path component of that fiber. If we knew that every path component of every fiber of q has only one vertex, we could conclude that $fx = z$. Similarly, suppose $fx = fy$. Then the homotopy gives us a path from fx to gx, and from fy to gy, in the fiber of q over qfx. Since the fiber is fibrant, this gives a path from gx to gy in that fiber of q. Once again, if we knew that every path component of every fiber of q had only one vertex, we could conclude that $gx = gy$, and so $x = y$.

If we wanted to extend this approach to n-simplices for positive n, we would want to assume we had already proven that f is an isomorphism on lower dimensional simplices. These considerations lead to the following definition.

DEFINITION 3.5.5. A fibration $p\colon X \to Y$ in **SSet** is called a *minimal fibration* if and only if for every $n \geq 0$, every path component of every fiber of the fibration $\mathrm{Map}_\square(i, p)\colon \mathrm{Map}(\Delta[n], X) \to \mathrm{Map}(\partial\Delta[n], X) \times_{\mathrm{Map}(\partial\Delta[n], Y)} \mathrm{Map}(\Delta[n], Y)$ has only one vertex. More generally, we define two n-simplices x and y of X to be *p-related* if they represent vertices in the same path component of the same fiber of $\mathrm{Map}_\square(i, p)$. We write $x \sim_p y$ if x and y are p-related. Note that this relation is an equivalence relation, by Lemma 3.4.2, and that p is a minimal fibration if and only if $x \sim_p y$ implies $x = y$. Also note that $x \sim_p y$ if and only if $p(x) = p(y)$, $d_i x = d_i y$ for all i such that $0 \leq i \leq n$, and there is a homotopy $\Delta[n] \times \Delta[1] \xrightarrow{H} X$ from X to Y such that pH is the constant homotopy and H is constant on $\partial\Delta[n]$.

As expected, we then have the following lemma.

LEMMA 3.5.6. *Suppose $p\colon X \to Y$ and $q\colon Z \to Y$ are fibrations of simplicial sets, and that q is a minimal fibration. Suppose $f, g\colon X \to Z$ are two maps such that $qf = qg = p$. Suppose $H\colon X \times \Delta[1] \to Z$ is a homotopy from f to g such that $qH = p\pi_1$. Then if g is an isomorphism, so is f.*

PROOF. Naturally, we prove that f is an isomorphism on n-simplices by induction on n. So suppose f is an isomorphism on k-simplices for all $k < n$. We first show that f is surjective on n-simplices. Let z be an n-simplex of Z. For each i with $0 \leq i \leq n$, there is a unique simplex x_i of X such that $fx_i = d_i z$. The x_i define a map $x'\colon \partial\Delta[n] \to X$ such that $fx' = zi$, where i is the inclusion $\partial\Delta[n] \to \Delta[n]$. We then have a commutative diagram

$$\begin{tikzcd} (\partial\Delta[n] \times \Delta[1]) \amalg_{\partial\Delta[n]\times\{0\}} (\Delta[n] \times \{0\}) \arrow[r, "(H\circ(x'\times 1))\amalg z"] \arrow[d] & Z \arrow[d, "q"] \\ \Delta[n] \times \Delta[1] \arrow[r, "qz\pi_1"] & Y \end{tikzcd}$$

Hence there is a lift $G\colon \Delta[n] \times \Delta[1] \to Z$ in this diagram. Since g is surjective, there is an n-simplex x of X such that $gx = Gi_1$, the end of the homotopy G. Then $gd_i x = gx_i$, so since g is an isomorphism, the restriction of x to $\partial\Delta[n]$ is x'.

The map G is a homotopy from z to gx. We want to find a homotopy from fx to z. But H defines a homotopy from fx to gx, via the composite $H \circ (x \times 1)$. Hence we get a map $K\colon \Delta[n] \times \Lambda^2[2]$ which is G on $\Delta[n] \times d_0 i_2$ and $H \circ (x \times 1)$ on $\Delta[n] \times d_1 i_2$. We then get the following commutative diagram.

$$\begin{tikzcd} (\Delta[n] \times \Lambda^2[2]) \amalg_{\partial\Delta[n]\times\Lambda^2[2]} (\partial\Delta[n] \times \Delta[2]) \arrow[r, "K\amalg(H\circ(x'\times s^0))"] \arrow[d] & Z \arrow[d, "q"] \\ \Delta[n] \times \Delta[2] \arrow[r, "qz\pi_1"] & Y \end{tikzcd}$$

A lift $G'\colon \Delta[n] \times \Delta[2] \to Z$ in this diagram exists. Then $G'(\Delta[n] \times d_2 i_2)$ is a fiberwise homotopy from fx to z which fixes the boundary. Since q is minimal, we have $fx = z$ and so f is surjective.

We now show that f is injective. Suppose x and y are n-simplices of X such that $fx = fy$. By induction, we have $d_i x = d_i y$ for all i. Let us denote the restriction of x (or y) to $\partial\Delta[n]$ by $x'\colon \partial\Delta[n] \to X$. Now $H \circ (x \times 1)$ is a homotopy from fx to gx, and $H \circ (y \times 1)$ is a homotopy from $fy = fx$ to gy. Hence we get a map $K\colon \Delta[n] \times \Lambda^0[2] \to Z$ which is $H \circ (x \times 1)$ on $\Delta[n] \times d_2 i_2$ and is $H \circ (y \times 1)$ on $\Delta[n] \times d_1 i_2$. Thus we get a commutative diagram

$$\begin{tikzcd} (\Delta[n] \times \Lambda^0[2]) \amalg_{\partial\Delta[n]\times\Lambda^0[2]} (\partial\Delta[n] \times \Delta[2]) \arrow[r, "K\amalg(H\circ(x'\times s^1))"] \arrow[d] & Z \arrow[d, "q"] \\ \Delta[n] \times \Delta[2] \arrow[r, "px\pi_1"] & Y \end{tikzcd}$$

using the fact that $px = py$. Let $G\colon \Delta[n] \times \Delta[2] \to Z$ be a lift in this diagram. Then $G(\Delta[n] \times d_0 i_2)$ is a fiberwise homotopy from gx to gy that fixes the boundary. Since q is minimal, we must have $gx = gy$, and so $x = y$ as required. $\square$

COROLLARY 3.5.7. *Suppose $p\colon X \to Y$ is a minimal fibration of simplicial sets. Then p is locally trivial.*

PROOF. First note that the pullback of a minimal fibration is again a minimal fibration. Indeed, the reader can check that a pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

induces a pullback square

$$\begin{array}{ccc} \operatorname{Map}(\Delta[n], X') & \longrightarrow & \operatorname{Map}(\Delta[n], X) \\ {\scriptstyle \operatorname{Map}_{\square}(i,p')}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Map}_{\square}(i,p)} \\ P' & \longrightarrow & P \end{array}$$

where

$$P' = \operatorname{Map}(\Delta[n], Y') \times_{\operatorname{Map}(\partial\Delta[n], Y')} \operatorname{Map}(\partial\Delta[n], X')$$

and

$$P = \operatorname{Map}(\Delta[n], Y) \times_{\operatorname{Map}(\partial\Delta[n], Y)} \operatorname{Map}(\partial\Delta[n], X)$$

Hence every fiber of $\operatorname{Map}(i, p')$ is isomorphic to a fiber of $\operatorname{Map}(i, p)$. It follows easily that p' is minimal if p is. Corollary 3.5.4 and Lemma 3.5.6 then complete the proof. $\square$

We now need to show that every fibration is close to a minimal fibration in some sense. We begin with the following lemma, which says that every fibration looks minimal on the degenerate simplices.

LEMMA 3.5.8. *Suppose $p\colon X \to Y$ is a fibration of simplicial sets, and suppose x and y are degenerate n-simplices of X such that $x \sim_p y$. Then $x = y$.*

PROOF. We will actually prove that if $d_i x = d_i y$ for all i such that $0 \leq i \leq n$, then $x = y$. Since x and y are degenerate, we have $x = s_i d_i x$ and $y = s_j d_j y$ for some i and j. We claim that we can assume that $i = j$, where the conclusion is obvious. Indeed, if $i \neq j$, we can assume $i < j$. Then we have

$$x = s_i d_i y = s_i d_i s_j d_j y = s_i s_{j-1} d_i d_j y = s_j s_i d_i d_j y.$$

Hence we have

$$s_j d_j x = s_j d_j s_j s_i d_i d_j y = s_j s_i d_i d_j y = x$$

and so we can assume $i = j$. Thus $x = y$. $\square$

We now show that every fibration is close to a minimal one.

THEOREM 3.5.9. *Suppose $p\colon X \to Y$ is an arbitrary fibration of simplicial sets. Then we can factor p as $X \xrightarrow{r} X' \xrightarrow{p'} Y$, where p' is a minimal fibration and r is a retraction onto a subsimplicial set X' of X such that $r \in I$-inj.*

PROOF. Let T be a set of simplices of X containing one simplex from each p-equivalence class. By Lemma 3.5.8, every degenerate simplex is in T. Let S denote the set of all subsimplicial sets of X all of whose simplices lie in T. Partially order S by inclusion. Then Zorn's lemma obviously applies, so we can chose a maximal element X' of S. Note that if $x \in T$ is an n-simplex such that $d_i x \in X'$ for all i, then $x \in X'$. Indeed, otherwise the subsimplicial set generated by X' and x, all of whose simplices are either in X', equal to x, or degenerate, contradicts the maximality of X'.

If the restriction $p'\colon X' \to Y$ is a fibration, it is obviously minimal. We will show that p' is a retract of p, hence a fibration. To do so, we again use Zorn's lemma, this time applied to pairs (Z, H), where Z is a sub-simplicial set of X containing X', and $H\colon Z \times \Delta[1] \to X$ is a homotopy such that H is the inclusion on $Z \times \{0\}$, maps $Z \times \{1\}$ into X', is constant on $X' \times \Delta[1]$, and such that pH is the constant homotopy of p restricted to Z. Let (Z, H) be a maximal such pair. We must show that $Z = X$. If not, consider a simplex $x\colon \Delta[n] \to X$ of minimal dimension which does not belong to Z. Then we have a pushout square

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & Z \\ \Big\downarrow & & \Big\downarrow \\ \Delta[n] & \xrightarrow{x} & Z' \end{array}$$

where Z' is the sub-simplicial set of X generated by Z and x. We want to extend H to Z'. Such an extension is equivalent to a map $\Delta[n] \times \Delta[1] \xrightarrow{H'} X$ such that H' is x on $\Delta[n] \times \{0\}$, pH' is the constant homotopy at px, H' extends H on $\partial\Delta[n] \times \Delta[1]$, and H' of $\Delta[n] \times \{1\}$ is a simplex of X'. Equivalently, we are looking for a 1-simplex w of the fiber of $\mathrm{Map}_{\square}(i, p)$ over the point defined by px and H, such that $d_1 w$ is x and $d_0 w$ is in X'. Certainly x is in some path component of this fiber, so we can find such a 1-simplex w with $d_0 w$ in T. But then $d_i d_0 w$ is in X' for all i, so w must itself be in X', as pointed out above. Hence we can extend H to Z', contradicting the maximality of (Z, H). We must therefore have had $Z = X$. It follows that p' is a retract of p, and hence a minimal fibration.

Let $H\colon X \times \Delta[1] \to X$ denote the homotopy we have just constructed. Let j denote the inclusion $X' \to X$. Let r be the composite $X \cong X \times \{1\} \to X \times \Delta[1] \xrightarrow{H} X'$, so that r is a retraction of X onto X', and H is a homotopy between the identity and jr which is constant on X' and such that pH is constant. We must still show that r has the right lifting property with respect to I. Suppose we have a commutative diagram

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{u} & X \\ {\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle r} \\ \Delta[n] & \xrightarrow{v} & X' \end{array}$$

Then we have a commutative diagram

$$\begin{array}{ccc} (\partial\Delta[n] \times \Delta[1]) \amalg_{\partial\Delta[n]\times\{1\}} (\Delta[n] \times \{1\}) & \xrightarrow{(H\circ(u\times 1))\amalg(j\circ v)} & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ \Delta[n] \times \Delta[1] & \xrightarrow{p'\circ v\circ \pi_1} & Y \end{array}$$

as we leave to the reader to check. Let $G\colon \Delta[n] \times \Delta[1] \to X$ be a lift in this diagram, and let v_1 be the n-simplex of X defined by G on $\Delta[n] \times \{0\}$. Then $v_1 i = u$.

We must show that $rv_1 = v$. Now G is a homotopy from v_1 to jv, so rG is a homotopy from rv_1 to $rjv = v$. This homotopy may not leave the boundary fixed, however. But $rH(v_1 \times 1)$ is a homotopy from rv_1 to itself which is just what we need to make a fiberwise homotopy from rv_1 to v. Indeed, these two homotopies together define a map $\Delta[n] \times \Lambda^0[2] \xrightarrow{K} X'$ so that $K(\Delta[n] \times d_1 i_2) = rG$ and

$K(\Delta[n] \times d_2 i_2) = rH(v_1 \times 1)$. We then get a commutative diagram

$$\begin{tikzcd} (\partial\Delta[n] \times \Delta[2]) \amalg_{\partial\Delta[n]\times\Lambda^0[2]} (\Delta[n] \times \Lambda^0[2]) \arrow[r, "(r\circ H\circ(u\times s^1))\amalg K"] \arrow[d] & X' \arrow[d, "p'"] \\ \Delta[n] \times \Delta[2] \arrow[r, "p'v\pi_1"] & Y \end{tikzcd}$$

There is a lift $G' \colon \Delta[n] \times \Delta[2] \longrightarrow X'$ in this diagram, and the map $G'(\Delta[n] \times d_0 i_2)$ is a fiberwise homotopy from rv_1 to v that fixes the boundary. Since p' is minimal, it follows that $rv_1 = v$, as required. □

Corollary 3.5.10. *Suppose p is a fibration of simplicial sets such that every fiber of p is non-empty and has no non-trivial homotopy groups. Then p is in I-inj.*

Proof. We write $p = p'r$ as in Theorem 3.5.9, where r has the right lifting property with respect to I and p' is minimal. Since p' is a retract of p, the fibers of p' are retracts of the fibers of p. Hence every fiber of p' is non-empty and has no non-trivial homotopy groups. Since p' is minimal, it is locally trivial by Corollary 3.5.7. Hence Corollary 3.5.2 shows that p' also has the right lifting property with respect to I. Thus p does as well. □

Note that Corollary 3.5.10 does not imply that every trivial fibration has the right lifting property with respect to I, because we do not at the moment know anything about the relationship between homotopy of fibrant simplicial sets and weak equivalences. We will remedy this deficiency in the next section.

3.6. Fibrations and geometric realization

In this section we complete the proof that simplicial sets form a model category. We show that the geometric realization preserves fibrations and use this to show that the homotopy groups of a fibrant simplicial set are isomorphic to the homotopy groups of its geometric realization. We also show that the geometric realization is part of a Quillen equivalence from simplicial sets to topological spaces.

We begin by showing that the geometric realization of a locally trivial fibration is a fibration.

Proposition 3.6.1. *Suppose $p \colon X \longrightarrow Y$ is a locally trivial fibration of simplicial sets. Then $|p|$ is a fibration.*

Proof. We must show that we have lifting in any commutative diagram of the form

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & {|X|} \arrow[d, "|p|"] \\ D^n \times I \arrow[r, "f"] & {|Y|} \end{tikzcd}$$

Write $\emptyset \longrightarrow Y$ as a transfinite composition of pushouts of maps of I. Taking the geometric realization and applying Lemma 2.4.7, we find that the image of f intersects the interior of only finitely many simplices, so that the image of f is contained in $|Y'|$ for some finite sub-simplicial set Y' of Y. A lifting in the diagram above is equivalent to a lifting in the diagram

$$\begin{tikzcd} D^n \arrow[r] \arrow[d] & {|Y'| \times_{|Y|} |X| \cong |Y' \times_Y X|} \arrow[d] \\ D^n \times I \arrow[r, "f"] & {|Y'|} \end{tikzcd}$$

We can therefore assume that Y' is a finite simplicial set, since the fibration $Y' \times_Y X \to Y'$ is locally trivial. This allows us to use [**Spa81**, Theorem 2.7.13], which says that any locally trivial map over a paracompact Hausdorff space is a Hurewicz fibration, and hence a Serre fibration.

We are thus reduced to showing that the geometric realization of a locally trivial fibration over a finite base is a locally trivial map. By induction on the non-degenerate simplices, we are reduced to the following situation. Suppose we have a locally trivial fibration $p\colon X \to Y$ and a pushout square

$$\begin{tikzcd} \partial\Delta[n] \arrow[r, "f"] \arrow[d] & Z \arrow[d, "g"'] \\ \Delta[n] \arrow[r, "h"] & Y \end{tikzcd}$$

such that $|Z \times_Y X \to Z|$ is locally trivial. We must show that $|p|$ is locally trivial. The idea of the proof is simple: we have a trivialization over $|\Delta[n]|$ of $|p|$ since p is locally trivial. By induction, we have trivializations over sufficiently small neighborhoods in Z. To get trivializations over suffficently small neighborhoods in Y, we must make these two trivializations compatible, which we can do because $|\partial\Delta[n]|$ is a deformation retract of $|\Delta[n]|$.

In more detail, since p is locally trivial, we have an isomorphism $\Delta[n] \times F \to \Delta[n] \times_Y X$ over $\Delta[n]$. Since the geometric realization preserves finite limits, this induces a homeomorphism $|\Delta[n]| \times |F| \xrightarrow{\psi} |\Delta[n]| \times_{|Y|} |X|$ over $|\Delta[n]|$. In particular, $|p|$ is locally trivial over points in the image of the interior of $|\Delta[n]|$. Any other point z in $|Y|$ is also in $|Z|$, so, by induction, there is a neighborhood U in $|Z|$ and a homeomorphism $U \times F' \xrightarrow{\varphi} U \times_{|Y|} |X|$ over U. Let $U' = |f|^{-1}U$. By pulling back, we get a homeomorphism $U' \times F' \xrightarrow{\varphi'} U' \times_{|Y|} |X|$ over U'. If U' is empty, then U is also open in $|Y|$, so we are done. Otherwise, thicken U' to get an open set V in $|\Delta[n]|$ containing U' such that U' is a deformation retract of V. Then $V' = V \amalg_{U'} U$ is a neighborhood of z in Y, so we must exhibit a trivialization of $|p|$ over V'. We do so by modifying the trivialization ψ over V so it agrees with the trivialization φ on U'. First note that F and F' are homeomorphic, since they are both homeomorphic to the fiber of p over a point (in the image) of U'. Hence we can assume ψ is a homeomorphism $V \times F' \xrightarrow{\psi} V \times_{|Y|} |X|$ over V. Now $\psi^{-1}\varphi'$ is a homeomorphism $U' \times F' \to U' \times F'$ over U', so can be written $\psi^{-1}\varphi'(u, f) = (u, \alpha(u, f))$. Similarly, $(\varphi')^{-1}\psi(u, f) = (u, \beta(u, f))$ on $U' \times F'$, so we have $\alpha(u, \beta(u, f)) = f = \beta(u, \alpha(u, f))$. Let $r\colon V \to U'$ denote the retraction. Then we have a map $r^*(\psi^{-1}\varphi')\colon V \times F' \to V \times F'$ over V defined by $r^*(\psi^{-1}\varphi')(v, f) = (v, \alpha(rv, f))$. Then $r^*(\psi^{-1}\varphi')$ is an extension of $\psi^{-1}\varphi'$ and a homeomorphism whose inverse takes (v, f) to $(v, \beta(rv, f))$. Thus we get a trivialization $V \times F' \xrightarrow{\rho = \psi \circ r^*(\psi^{-1}\varphi')} V \times_{|Y|} |X|$ over V whose restriction to U' is

φ'. Hence ρ and φ patch together to define a trivialization of $|p|$ over $V \amalg_{U'} U$, as required. $\square$

We can now prove Quillen's result [**Qui68**] that the geometric realization preserves fibrations.

COROLLARY 3.6.2. *Suppose p is a fibration of simplicial sets. Then $|p|$ is a (Serre) fibration of compactly generated topological spaces.*

PROOF. Write $p = p'r$ as in Theorem 3.5.9. Then p' is minimal, and hence locally trivial, so $|p'|$ is a fibration by Proposition 3.6.1. On the other hand, r has the right lifting property with respect to I, so is a fibration by Lemma 3.2.5. Thus p is a fibration as well. $\square$

We now prove that the definition of homotopy groups we gave for fibrant simplicial sets does match the homotopy of the geometric realization.

PROPOSITION 3.6.3. *Suppose X is a fibrant simplicial set, and v is a vertex of X. Then there is a natural isomorphism $\pi_n(X, v) \cong \pi_n(|X|, |v|)$.*

PROOF. The idea of the proof is to use induction on n, as we have already seen that π_0 works well in Lemma 3.4.3. For this, we need a way to relate the $(n+1)$st homotopy group to the nth homotopy group. This is provided by the path space fibration.

Given a fibrant simplicial set X and a vertex v of X, consider the following commutative diagram.

$$\begin{tikzcd}
PX \arrow[r] \arrow[d, "q"'] & \mathrm{Map}(\Delta[1], X) \arrow[d, "\mathrm{Map}(i,X)"] \\
X \arrow[r, "{(v,1)}"] \arrow[d] & X \times X \cong \mathrm{Map}(\partial\Delta[1], X) \arrow[d, "\pi_1"] \\
\Delta[0] \arrow[r] & X
\end{tikzcd}$$

Here the top square is a pullback square, so this defines the path space PX, and the bottom square is also a pullback square. Hence $q\colon PX \to X$ is a fibration. Note as well that the outer square must be a pullback square. But the vertical composite $\mathrm{Map}(\Delta[1], X) \to X$ is isomorphic to $\mathrm{Map}(d^1, X)$. Since $d^1\colon \Delta[0] \to \Delta[1]$ is an anodyne extension, it follows from adjointness and Theorem 3.3.2 that $\mathrm{Map}(d^1, X)$ has the right lifting property with respect to I. Hence $PX \to \Delta[0]$ also has the right lifting property with respect to I.

It follows easily from this that $\pi_n(PX, v)$ is trivial, where v is the constant path at v. Indeed, given an $\alpha\colon \Delta[n] \to PX$ such that $d_i\alpha = v$ for all i, we can find an $n+1$-simplex x such that $d_{n+1}x = \alpha$ and $d_i x = v$ for $i < n+1$, since $PX \to \Delta[0]$ has the right lifting property with respect to I. Lemma 3.4.5 then implies that $\alpha \sim v$. Let us denote the fiber of $q\colon PX \to X$ over v by ΩX, the loop space. Applying the long exact sequence of the fibration q, we find an isomorphism $\pi_{n+1}(X, v) \xrightarrow[\cong]{\partial} \pi_n(\Omega X, v)$. This isomorphism is natural in the pair (X, v), since the boundary map is and the path fibration is.

Applying the geometric realization, we get a fibration $|PX| \to |X|$ with fiber $|\Omega X|$ by Corollary 3.6.2. The map $PX \to \Delta[0]$ is a weak equivalence by Lemma 3.2.6.

Hence $|PX| \to |\Delta[0]|$ is a weak equivalence, so $|PX|$ has no non-trivial homotopy groups. Thus we get a natural isomorphism $\pi_{n+1}(|X|, |v|) \cong \pi_n(|\Omega X|, |v|)$. Since we have a natural isomorphism $\pi_n(\Omega X, v) \cong \pi_n(|\Omega X|, |v|)$ by induction, this completes the proof. $\square$

It is useful to give an explicit construction of the isomorphism $\pi_n(X, v) \to \pi_n(|X|, |v|)$ for fibrant simplicial sets X. A class of $\pi_n(X, v)$ is represented by a map $\Delta[n] \to X$ that sends the boundary to v. Applying the geometric realization and some homeomorphisms gives a map $D^n \to |X|$ that sends S^{n-1} to $|v|$. This map then defines a map $S^n \cong D^n/S^{n-1} \to |X|$ which defines an element of $\pi_n(|X|, |v|)$. We leave it to the reader to check that this map is well-defined and compatible with the isomorphism $\pi_{n+1}(X, v) \cong \pi_n(\Omega X, v)$.

We will also need to know later that the group structure on $\pi_1(X, v)$ given by the isomorphism $\pi_1(X, v) \cong \pi_1(|X|, |v|)$ can be defined simplicially, for a fibrant simplicial set X. Indeed, an element of $\pi_1(X, v)$ is represented by a 1-simplex α of X such that $d_0\alpha = d_1\alpha = v$. Given another such 1-simplex β, we get a map $\Lambda^1[2] \to X$ which is α on $d_2 i_2$ and β on $d_0 i_2$. Since X is fibrant, there is an extension to a 2-simplex γ, and we define $\alpha * \beta = d_1\gamma$. We could verify explicitly that the homotopy class of $\alpha * \beta$ depends only on the homotopy classes of α and β, and that this defines a group structure on $\pi_1(X, v)$. However, it is clear that $|\alpha * \beta|$ represents the same element of $\pi_1(|X|, |v|)$ as $|\alpha| * |\beta|$, where we use the usual group structure in $\pi_1(|X|, |v|)$. Thus this definition must induce the group structure on $\pi_1(X, v)$.

We can now complete the proof that simplicial sets form a model category.

THEOREM 3.6.4. *Suppose p is a trivial fibration of simplicial sets. Then p has the right lifting property with respect to I.*

PROOF. It suffices to show that the fibers of p are non-empty and have no non-trivial homotopy groups, by Corollary 3.5.10. Let F be a fiber of p over a vertex v. Then $|F|$ is the fiber of the fibration $|p|$ over $|v|$ by Corollary 3.6.2. Since $|p|$ is a weak equivalence, $|F|$ has no non-trivial homotopy groups and is non-empty. It follows from Proposition 3.6.3 that F has no non-trivial homotopy groups (and is non-empty), as required. $\square$

Theorem 3.6.4 and the results of Section 3.2 are what we need to apply the recognition theorem 2.1.19.

THEOREM 3.6.5. *The category of simplicial sets is a finitely generated model category with generating cofibrations I, generating trivial cofibrations J, and weak equivalences the maps whose geometric realization is a weak equivalence.*

Proposition 1.1.8 and Lemma 2.1.21 imply the following corollary.

COROLLARY 3.6.6. *The category* $\mathbf{SSet}_*$ *of pointed simplicial sets is a finitely generated model category, where a map is a cofibration, fibration, or weak equivalence if and only if it is so in* $\mathbf{SSet}$.

THEOREM 3.6.7. *The geometric realization and singular complex define a Quillen equivalence* $\mathbf{SSet} \to \mathbf{K}$*, and a Quillen equivalence* $\mathbf{SSet}_* \to \mathbf{K}_*$.

PROOF. The second statement follows from the first and Proposition 1.3.17. For the first statement, it is clear that the geometric realization preserves cofibrations and trivial cofibrations, since it preserves the generating cofibrations and trivial cofibrations. Also, the geometric realization reflects weak equivalences by definition. Thus, by Corollary 1.3.16, it suffices to show that the map $|\operatorname{Sing} X| \to X$ is a weak equivalence for all k-spaces X. For this, it suffices to show that the map $\pi_i(|\operatorname{Sing} X|, v) \to \pi_i(X, v)$ is an isomorphism for every point v of X, since such points are in one-to-one correspondence with vertices of $\operatorname{Sing} X$, and every point of $|\operatorname{Sing} X|$ is in the same path component as a vertex. Since $\operatorname{Sing} X$ is fibrant, we have an isomorphism $\pi_i(\operatorname{Sing} X, v) \xrightarrow{\cong} \pi_i(|\operatorname{Sing} X|, v)$. The composite map $\pi_i(\operatorname{Sing} X, v) \to \pi_i(X, v)$ is the map induced by the adjunction; an element of $\pi_i(\operatorname{Sing} X, v)$ is represented by a map $\Delta[i] \to \operatorname{Sing} X$ sending $\partial\Delta[i]$ to v. This map is adjoint to a map $D^i \cong |\Delta[i]| \to X$ sending S^{i-1} to v, and so to a map $S^i \to X$ sending the basepoint to v. This represents an element of $\pi_i(X, v)$. We can also run this adjunction backwards, and we can apply it to homotopies as well. Thus the map $\pi_i(\operatorname{Sing} X, v) \to \pi_i(X, v)$ is an isomorphism, as required. □

The model category **SSet** is extremely important, and we will use it often during the rest of this book. One very useful property of **SSet** is the following.

Proposition 3.6.8. *Suppose $\mathcal{C}$ is a model category, and $F\colon$ **SSet** $\to \mathcal{C}$ is a functor which preserves colimits and cofibrations. Then F preserves trivial cofibrations (and hence weak equivalences) if and only if $F(\Delta[n]) \to F(\Delta[0])$ is a weak equivalence for all $n \geq 0$.*

Proof. The only if part is straightforward. For the if part, note that the hypotheses implies that $F(\Delta[k]) \to F(\Delta[n])$ is a weak equivalence for any map $\Delta[k] \to \Delta[n]$. It suffices to prove that $F(\Lambda^r[n]) \to F(\Delta[n])$ is a weak equivalence for all $n > 0$ and $0 \leq r \leq n$. We will actually prove that, if L is a subcomplex of $\Lambda^r[n]$ generated by a nontrivial collection of the $(n-1)$-dimensional faces, then the map $F(\Delta[n-1]) \to F(L)$ induced by the inclusion of one of the faces into L is a weak equivalence. This implies the desired result, since in particular $F(\Delta[n-1]) \to F(\Lambda^r[n])$ is a weak equivalence, and so $F(\Lambda^r[n]) \to F(\Delta[n])$ must also be a weak equivalence by the two-out-of-three axiom.

The base case $n = 1$ is simple, since $\Lambda^r[1] = \Delta[0]$. So suppose $n > 1$ and our induction hypothesis holds for $n-1$. Suppose L is a subcomplex of $\Lambda^r[n]$ generated by a nontrivial collection of top-dimensional faces. Attaching these faces one at a time we get a sequence

$$L_0 = \Delta[n-1] \to L_1 \to \dots \to L_k = L$$

Each of the maps $L_i \to L_{i+1}$ fits into a pushout square

$$\begin{tikzcd} K_i \arrow[r] \arrow[d] & \Delta[n-1] \arrow[d] \\ L_i \arrow[r] & L_{i+1} \end{tikzcd}$$

where K_i is the intersection of the new face with the faces already added. Then K_i is a subcomplex of $\Lambda^s[n-1]$ for some s (there may be some reindexing necessary that prevents us from taking $s = r$). Since K_i is generated by a nontrivial collection of top-dimensional faces, the induction hypothesis guarantees that $F(K_i) \to F(\Delta[n-1])$ is a weak equivalence. Since all the maps involved are cofibrations and F preserves cofibrations and pushouts, we find that $F(L_i) \to F(L_{i+1})$ is a trivial cofibration. Thus $F(\Delta[n-1]) \to F(L)$ is a trivial cofibration as well, completing the induction step. □

Corollary 3.6.9. *Suppose* $\mathcal{C}$ *is a model category, and* $F\colon \mathbf{SSet}_* \to \mathcal{C}$ *is a functor which preserves colimits and cofibrations. Then* F *preserves trivial cofibrations if and only if* $F(\Delta[n]_+) \to F(\Delta[0]_+)$ *is a weak equivalence for all* $n \geq 0$.

Proof. The only if part is straightforward. For the if part, let F' denote the composite of F with the operation of attaching a disjoint basepoint. Then Proposition 3.6.8 implies that F' preserves trivial cofibrations. In particular, $F(\Lambda^r[n]_+) \to F(\Delta[n]_+)$ is a trivial cofibration for all $n > 0$ and $0 \leq r \leq n$. Since these are the generating trivial cofibrations for $\mathbf{SSet}_*$ and F preserves colimits, F preserves all trivial cofibrations. $\square$

CHAPTER 4

Monoidal model categories

This chapter is devoted to the analogues of rings and modules in the theory of model categories. The analogue of a ring is called a *monoidal model category*; the analogue of a module over a monoidal model category $\mathcal{C}$ is called a $\mathcal{C}$-*model category*. Most of the examples we have considered so far are monoidal model categories. Simplicial sets, pointed simplicial sets, chain complexes of modules over a commutative ring, and chain complexes of comodules over a commutative Hopf algebra all form monoidal model categories. Topological spaces do not; however k-spaces and compactly generated topological spaces do form monoidal model categories.

Given a homotopy category $\mathcal{C}$, the major reason one would like to find a monoidal model category $\mathcal{D}$ and an equivalence $\operatorname{Ho}\mathcal{D} \cong \mathcal{C}$ is so that one can consider model categories of monoids and modules over them. In stable homotopy theory, the monoids in question are often known as A_∞-ring spectra. We do not consider model categories of monoids and modules in this book; however, the reader should be well equipped to read the papers [**SS97**] and [**Hov98a**].

Before we can talk about monoidal model categories and modules over them, we need to discuss rings and modules in the 2-category of categories. These are called monoidal categories and categories with an action of a monoidal category, respectively. However, the 2-category of model categories is based on the 2-category of categories and adjunctions, rather than the 2-category of categories and functors. We therefore need to discuss closed monoidal categories and categories with a closed action of a closed monoidal category. We do this in Section 4.1.

In Section 4.2 we define monoidal model categories and modules over them. The model category **SSet** of simplicial sets is a monoidal model category, and an **SSet**-model category is just a simplicial model category, first introduced by Quillen [**Qui67**]. The definition of monoidal model category given here is original, as far as the author is aware. The definition one would expect, based on Quillen's SM7 axiom, is not quite sufficient in case the unit is not cofibrant.

The last section of the chapter, Section 4.3, is devoted to proving that the homotopy pseudo-2-functor is compatible with these definitions. So, for example, the homotopy category of a monoidal model category is naturally a closed monoidal category.

The material in this chapter does not seem to have appeared in the literature before, though none of it will be a surprise to model category theorists.

4.1. Closed monoidal categories and closed modules

In this section, we remind the reader of the definitions of closed monoidal categories and modules over them, and define the associated 2-categories. This material is probably standard, but the author knows of no other source for all of it. This section is of necessity extremely abstract, though also simple. It is organized as follows. First we define all of the structures we need in the 2-category of categories, then extend them to the 2-category of categories and adjunctions. For each structure, we first define the structure then define the 2-category of such structures. If we were working in algebra, our progression would be rings, commutative rings, modules over a ring, algebras over a ring, and finally central and commutative algebras over a commutative ring. In the world of categories, these are called, respectively, monoidal categories, symmetric monoidal categories, modules over a monoidal category, algebras over a monoidal category, and finally central and symmetric algebras over a symmetric monoidal category. In the world of categories and adjunctions we simply add the word "closed" to each of these terms.

We begin with monoidal categories.

Definition 4.1.1. A *monoidal* structure on a category $\mathcal{C}$ is a tensor product bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, a unit object $S \in \mathcal{C}$, a natural associativity isomorphism $a\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, a natural left unit isomorphism $\ell\colon S \otimes X \to X$, and a natural right unit isomorphism $r\colon X \otimes S \to X$ such that three coherence diagrams commute. These coherence diagrams can be found in any reference on category theory, such as [**ML71**]. There is one for four-fold associativity, one equating the two different ways to get from $(X \otimes S) \otimes Y$ to $X \otimes Y$ using the associativity and unit isomorphisms, and one saying that ℓ and r agree on $S \otimes S$. A *monoidal category* is a category together with a monoidal structure on it.

The simplest example of a monoidal category is the category of sets under the Cartesian product. Other examples include the category of topological spaces under the product, the category of simplicial set under the product, the category of modules over a commutative ring under the tensor product, and the category of comodules over a Hopf algebra under the tensor product. A slightly more complicated example is the category of bimodules over a ring, where the monoidal structure is given by the tensor product of bimodules.

In order to define a 2-category of monoidal categories, we need to know what a functor of monoidal categories is.

Definition 4.1.2. Given monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, m, α) satisfying certain properties, where F is a functor from $\mathcal{C}$ to $\mathcal{D}$, m is a natural isomorphism $m\colon FX \otimes FY \to F(X \otimes Y)$, and $\alpha\colon FS \to S$ is an isomorphism. In order for (F, m, α) to be a monoidal functor, three coherence diagrams must commute. One of these equates the two obvious ways to get from $(FX \otimes FY) \otimes FZ$ to $F(X \otimes (Y \otimes Z))$, one equates the two obvious ways to get from $FS \otimes FX$ to FX, and one equates the two obvious ways to get from $FX \otimes FS$ to FX.

We leave it to the reader to define composition of monoidal functors and verify that it is associative and unital. We often abuse notation and refer to a monoidal functor F, leaving the isomorphisms m and α implicit.

A typical example of a monoidal functor is the free R-module functor that takes a set X to the free R-module $R\{X\}$, where R is a commutative ring. Another example is the geometric realization functor $|\ |\colon \mathbf{SSet} \to \mathbf{K}$. We will discuss this example further below.

Finally, we define a monoidal natural transformation.

DEFINITION 4.1.3. Given two monoidal functors $F, F' \colon \mathcal{C} \to \mathcal{D}$ of monoidal categories, a *monoidal natural transformation* from F to F' is a natural transformation $\tau \colon F \to F'$ which is compatible with m and α. That is, $\alpha' \circ \tau_S = \alpha \colon FS \to S$, and the diagram

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{m} & F(X \otimes Y) \\ {\scriptstyle \tau\otimes\tau}\big\downarrow & & {\scriptstyle \tau}\big\downarrow \\ F'X \otimes F'Y & \xrightarrow{m'} & F'(X \otimes Y) \end{array}$$

commutes.

We leave it to the reader to verify that vertical and horizontal compositions of monoidal natural transformations are again monoidal natural transformations. It is then easy to check that we get a 2-category of monoidal categories, monoidal functors, and monoidal natural transformations.

We now move on to commutative rings.

DEFINITION 4.1.4. A *symmetric monoidal structure* on a category $\mathcal{C}$ is a monoidal structure and a commutativity isomorphism $T \colon X \otimes Y \to Y \otimes X$ satisfying four additional coherence diagrams. One of these says that T is the identity on $S \otimes S$, one that $T^2 = 1$, one that $r = T\ell$, and one equates the two different ways of getting from $(X \otimes Y) \otimes Z$ to $Y \otimes (Z \otimes X)$ using the associativity and commutativity isomorphism. A category with a symmetric monoidal structure is a *symmetric monoidal category.*

Note that the right unit isomorphism in a symmetric monoidal category is redundant, and so we usually drop it from the structure. All of the monoidal categories mentioned above are symmetric monoidal, with the exception of the category of bimodules over a ring.

DEFINITION 4.1.5. Given symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a *symmetric monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a monoidal functor (F, m, α) such that the diagram

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{m} & F(X \otimes Y) \\ {\scriptstyle T}\big\downarrow & & {\scriptstyle F(T)}\big\downarrow \\ FY \otimes FX & \xrightarrow{m} & F(Y \otimes X) \end{array}$$

is commutative.

The free R-module functor discussed above is a symmetric monoidal functor from the category of sets to the category of R-modules, when R is commutative. For an example of a monoidal functor that is not symmetric monoidal, consider the category of $\mathbf{Z}$-graded vector spaces over a field of characteristic different from 2. We give this category the usual monoidal structure using the graded tensor product. There are two different symmetric monoidal structures we can put on this monoidal category. We can define $T(x \otimes y) = y \otimes x$ for homogeneous elements x and y, or we can define $T(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$. Then the identity functor of this monoidal category, thought of as a functor from one symmetric monoidal structure to the other, is a monoidal functor that is not symmetric monoidal.

The composition of symmetric monoidal functors is the same as the composition of monoidal functors. We leave it to the reader to check that we get a 2-category of symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations. The forgetful 2-functor from symmetric monoidal categories to monoidal categories is faithful on 1-morphisms and full and faithful on 2-morphisms.

We now discuss modules.

DEFINITION 4.1.6. Suppose $\mathcal{C}$ is a monoidal category. A *right* $\mathcal{C}$*-module structure* on a category $\mathcal{D}$ is a triple $(\otimes, a, r)$, where $\otimes\colon\ \mathcal{D}\times\mathcal{C}\to\mathcal{D}$ is a functor, a is a natural isomorphism $(X\otimes K)\otimes L\to X\otimes(K\otimes L)$, and r is a natural isomorphism $X\otimes S\to X$ making three coherence diagrams commute. One of these is four-fold associativity, one is the unit diagram equating the two ways to get from $X\otimes(S\otimes K)$ to $X\otimes K$, and one is a compatibility diagram between the unit isomorphisms, relating the two ways to get from $X\otimes(K\otimes S)$ to $X\otimes K$. A *right* $\mathcal{C}$*-module* is a category equipped with a right $\mathcal{C}$-module structure.

One could also define left modules, of course. We will often drop the word "right" and just refer to $\mathcal{C}$-modules. Every category with all coproducts is a module over the monoidal category of sets, where $A\otimes X$ is just a colimit of the functor $X\to\mathcal{C}$ which takes every element of x to A; *i.e.* $A\otimes X$ is the coproduct of A with itself $|\,X\,|$ times.

DEFINITION 4.1.7. Suppose $\mathcal{C}$ is a monoidal category, and $\mathcal{D}$ and $\mathcal{E}$ are $\mathcal{C}$-modules. A $\mathcal{C}$*-module functor* from $\mathcal{D}$ to $\mathcal{E}$ is a functor $F\colon\ \mathcal{D}\to\mathcal{E}$ and a natural isomorphism $m\colon\ FX\otimes K\to F(X\otimes K)$ such that two coherence diagrams commute. One of these equates the two ways of getting from $(FX\otimes K)\otimes L$ to $F(X\otimes(K\otimes L))$, and the other equates the two ways to get from $FX\otimes S$ to FX. As usual, we often refer to a $\mathcal{C}$-module functor F, abusing notation. Given two $\mathcal{C}$-module functors F and F' from $\mathcal{D}$ to $\mathcal{E}$, $\mathcal{C}$*-module natural transformation* from F to F' is a natural transformation $\tau\colon\ F\to F'$ such that the diagram

$$\begin{array}{ccc} FX\otimes K & \xrightarrow{m} & F(X\otimes K) \\ {\scriptstyle\tau\otimes 1}\big\downarrow & & {\scriptstyle\tau}\big\downarrow \\ F'X\otimes K & \xrightarrow{m} & F'(X\otimes K) \end{array}$$

commutes.

As above, a category $\mathcal{C}$ with all coproducts can be made into a **Set**-module. Given two such categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F\colon\ \mathcal{C}\to\mathcal{D}$ is a **Set**-module functor if and only F preserves coproducts. Any natural transformation is a **Set**-module natural transformation.

As usual, we leave it to the reader to check that we get a 2-category of $\mathcal{C}$-modules, $\mathcal{C}$-module functors, and $\mathcal{C}$-module natural transformations.

From modules, we go to algebras.

DEFINITION 4.1.8. Given a monoidal category $\mathcal{C}$, a $\mathcal{C}$*-algebra structure* on a category $\mathcal{D}$ is a monoidal structure on $\mathcal{D}$ together with a monoidal functor $i\colon\ \mathcal{C}\to\mathcal{D}$. A $\mathcal{C}$*-algebra* is a category equipped with $\mathcal{C}$-algebra structure. A $\mathcal{C}$*-algebra functor* is a monoidal functor $F\colon\ \mathcal{D}\to\mathcal{E}$ together with a monoidal natural isomorphism $\rho\colon\ F\circ i_{\mathcal{D}}\to i_{\mathcal{E}}$. A $\mathcal{C}$*-algebra natural transformation* from F to F' is a monoidal natural transformation $\tau\colon F \to F'$ such that the diagram

$$\begin{array}{ccc} F(i(K)) & \xrightarrow{\rho} & i(K) \\ {\scriptstyle \tau\circ i}\Big\downarrow & & \Big\| \\ F'(i(K)) & \xrightarrow{\rho'} & i(K) \end{array}$$

commutes.

For example, let $\mathcal{C}$ be the category of left R-modules for a commutative ring R, and suppose we have a map of R-algebras $S \to T$. Then we get a $\mathcal{C}$-algebra functor F from the category of left S-modules to the category of left T-modules, which takes the S-module M to $T \otimes_S M$. Note that in this case, F does not preserve the map i on the nose, since $Fi(M) = T \otimes_S (S \otimes_R M)$, which is canonically isomorphic, but not equal, to $T \otimes_R M$.

As usual, we leave it to the reader to check that we get a 2-category of $\mathcal{C}$-algebras, $\mathcal{C}$-algebra functors, and $\mathcal{C}$-algebra natural transformations. Note that there is a forgetful 2-functor from $\mathcal{C}$-algebras to $\mathcal{C}$-modules. Indeed, given a $\mathcal{C}$-algebra $\mathcal{D}$, we put a $\mathcal{C}$-module structure on it by defining $X \otimes K = X \otimes iK$, where $K \in \mathcal{C}$. We leave it to the reader to show that this idea can be extended to define the forgetful 2-functor.

Now we consider the case where the underlying monoidal category $\mathcal{C}$ is symmetric monoidal.

Definition 4.1.9. Suppose $\mathcal{C}$ is a symmetric monoidal category. Then a *symmetric $\mathcal{C}$-algebra structure* on a category $\mathcal{D}$ is a symmetric monoidal structure on $\mathcal{D}$ together with a symmetric monoidal functor $i\colon \mathcal{C} \to \mathcal{D}$. A *symmetric $\mathcal{C}$-algebra* is a category equipped with a symmetric $\mathcal{C}$-algebra structure. A *symmetric $\mathcal{C}$-algebra functor* of symmetric $\mathcal{C}$-algebras is a symmetric monoidal functor which is also a $\mathcal{C}$-algebra functor.

As usual, we get a 2-category of symmetric $\mathcal{C}$-algebras, symmetric $\mathcal{C}$-algebra functors, and $\mathcal{C}$-algebra natural transformations. There is of course a forgetful 2-functor from the 2-category of symmetric $\mathcal{C}$-algebras to the 2-category of $\mathcal{C}$-algebras.

It is more interesting to consider central $\mathcal{C}$-algebras.

Definition 4.1.10. Suppose $\mathcal{C}$ is a symmetric monoidal category. Then a *central $\mathcal{C}$-algebra structure* on a category $\mathcal{D}$ is a $\mathcal{C}$-algebra structure together with a natural transformation $t\colon iX \otimes Y \to Y \otimes iX$ satisfying four coherence diagrams. The first such coherence diagram says that t^2 is the identity, the second equates the two obvious ways of getting from $iX \otimes iY$ to $i(Y \otimes X)$ using the commutativity isomorphism T of $\mathcal{C}$ on one path and t on the other, the third equates the two obvious ways of getting from $iS \otimes X$ to X, using t and r on one path and ℓ on the other, and the fourth equates the two obvious ways of getting from $(iX \otimes Y) \otimes Z$ to $Y \otimes (Z \otimes iX)$ using t and a. A *central $\mathcal{C}$-algebra* is a category equipped with a central $\mathcal{C}$-algebra structure.

A typical example of a central $\mathcal{C}$-algebra is the category of modules over a commutative Hopf algebra (over a field) as an algebra over the category of vector spaces. This will always be central, but will only be symmetric when the Hopf algebra is cocommutative.

DEFINITION 4.1.11. Suppose $\mathcal{C}$ is a symmetric monoidal category. Given two central $\mathcal{C}$-algebras $\mathcal{D}$ and $\mathcal{E}$, a *central* $\mathcal{C}$*-algebra functor* is a $\mathcal{C}$-algebra functor F such that the diagram

$$\begin{array}{ccccc} F(iX)\otimes FY & \xrightarrow{m} & F(iX\otimes Y) & \xrightarrow{Ft} & F(Y\otimes iX) \\ {\scriptstyle \rho\otimes 1}\downarrow & & & & \uparrow{\scriptstyle m} \\ iX\otimes FY & \xrightarrow{t} & FY\otimes iX & \xrightarrow{1\otimes\rho^{-1}} & FY\otimes F(iX) \end{array}$$

commutes.

We then get a 2-category of central $\mathcal{C}$-algebras, central $\mathcal{C}$-algebra functors, and $\mathcal{C}$-algebra natural transformations. There is a forgetful 2-functor from symmetric $\mathcal{C}$-algebras to central $\mathcal{C}$-algebras defined by letting t be the restriction of the commutativity isomorphism of $\mathcal{D}$.

This completes our tour of the algebra of categories. We now indicate how these definitions must be changed to work in the 2-category of categories and adjunctions.

DEFINITION 4.1.12. Suppose $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are categories. An *adjunction of two variables* from $\mathcal{C}\times\mathcal{D}$ to $\mathcal{E}$ is a quintuple $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$, where $\otimes\colon \mathcal{C}\times\mathcal{D}\longrightarrow \mathcal{E}$, $\mathrm{Hom}_r\colon \mathcal{D}^{\mathrm{op}}\times\mathcal{E}\longrightarrow\mathcal{C}$, and $\mathrm{Hom}_\ell\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{E}\longrightarrow\mathcal{D}$ are functors, and φ_r and φ_ℓ are natural isomorphisms

$$\mathcal{C}(C, \mathrm{Hom}_r(D,E)) \xrightarrow[\cong]{\varphi_r^{-1}} \mathcal{E}(C\otimes D, E) \xrightarrow[\cong]{\varphi_\ell} \mathcal{D}(D, \mathrm{Hom}_\ell(C,E)).$$

We often abuse notation by referring to $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell)$, or even $\otimes$ alone, as an adjunction of two variables, leaving the adjointness isomorphisms implicit.

Now we simply change all the definitions above so that every bifunctor in sight is an adjunction of two variables, and every functor in sight is an adjunction. The first such definition is the following.

DEFINITION 4.1.13. A *closed monoidal structure* on a category $\mathcal{C}$ is an octuple

$$(\otimes, a, \ell, r, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$$

where $(\otimes, a, \ell, r)$ is a monoidal structure on $\mathcal{C}$ and $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)\colon \mathcal{C}\times\mathcal{C}\longrightarrow\mathcal{C}$ is an adjunction of two variables. A *closed monoidal category* is a category equipped with a closed structure.

One could think of the definition of a closed monoidal category as the pullback of the definition of a monoidal category and the definition of an adjunction of two variables over the bifunctor $\otimes$. Virtually every standard example of a monoidal category is in a fact a closed monoidal category. Sets, for example, form a closed monoidal category, where $\mathrm{Hom}_r(X,Y) = \mathrm{Hom}_\ell(X,Y) = Y^X$. Similarly, modules over a commutative ring R form a closed monoidal category, where $\mathrm{Hom}_r = \mathrm{Hom}_\ell = \mathrm{Hom}_R$. Even bimodules over a ring form a closed monoidal category, though in this case Hom_r and Hom_ℓ are different. It requires some care to get the definitions correct in this case. However, the category of topological spaces is a symmetric monoidal category which is not closed. The categories of k-spaces and compactly generated spaces are closed symmetric monoidal categories (see Definition 2.4.21).

DEFINITION 4.1.14. A *closed monoidal functor* between closed monoidal categories is a quintuple $(F, m, \alpha, U, \varphi)$, where (F, m, α) is a monoidal functor and (F, U, φ) is an adjunction.

Most monoidal functors of closed monoidal categories are in fact closed monoidal functors. For example, the monoidal functor from R-modules to S-modules induced by a map of rings $R \to S$ is actually a closed monoidal functor, where the adjoint is given by the forgetful functor from S-modules to R-modules.

We then get a 2-category of closed monoidal categories, closed monoidal functors, and monoidal natural transformations. In a similar fashion, we get 2-categories of closed symmetric monoidal categories, closed modules and closed algebras over a given closed monoidal category, and symmetric and central closed algebras over a given closed symmetric monoidal category. In the case of a closed module or closed algebra $\mathcal{D}$ over $\mathcal{C}$, we usually write $X^K \in \mathcal{D}$ instead of $\mathrm{Hom}_r(K, X)$ for $K \in \mathcal{C}$ and $X \in \mathcal{D}$. We then often write $\mathrm{Hom}(X, Y)$ or $\mathrm{Map}(X, Y) \in \mathcal{C}$ instead of $\mathrm{Hom}_\ell(X, Y)$ for $X, Y \in \mathcal{D}$. Only two other new things happen, so we discuss those and leave the rest of the details to the reader.

Firstly, in the case of closed symmetric monoidal categories, the commutativity isomorphism defines a natural isomorphism between $\mathrm{Hom}_\ell(X, Y)$ and $\mathrm{Hom}_r(X, Y)$, so it is usual to drop the subscript.

Secondly, there is a duality 2-functor on the 2-category of closed $\mathcal{C}$-modules, where $\mathcal{C}$ is a closed *symmetric* monoidal category. Indeed, given a closed module $\mathcal{D}$, we define $D\mathcal{D}$ to be $\mathcal{D}^{\mathrm{op}}$, where the $\mathcal{C}$-action is given by $(X, K) \mapsto \mathrm{Hom}_r(K, X)$, and the closed structure is given by the functors $(X, K) \mapsto X \otimes K$ and $(X, Y) \mapsto \mathrm{Hom}_\ell(Y, X)$. The adjointness isomorphisms are easily defined. The associativity isomorphism on $D\mathcal{D}$ corresponds to a map $\mathrm{Hom}(L, \mathrm{Hom}_r(K, X)) \to \mathrm{Hom}_r(K \otimes L, X)$ in $\mathcal{D}$ which can be defined by adjointness, although it requires the commutativity isomorphism of $\mathcal{C}$. The unit isomorphism is similar. The dual of a morphism (F, U, φ, m) from $\mathcal{D}$ to $\mathcal{E}$ is the morphism $(U, F, \varphi^{-1}, (Dm)^{-1})$. Here Dm is the natural isomorphism $Dm\colon U(\mathrm{Hom}_r(K, X)) \to \mathrm{Hom}_r(K, UX)$ in $\mathcal{D}$ which is dual to m. The dual of a 2-morphism is defined in the same way as it is in $\mathbf{Cat}_{ad}$. We leave it to the reader to verify that D is a contravariant 2-functor whose square is the identity, as usual.

4.2. Monoidal model categories and modules over them

In the last section, we defined the 2-categories of closed categories, symmetric closed categories, closed modules and closed algebras over a given closed category, and symmetric and central closed algebras over a symmetric closed category. We now want to construct analogous 2-categories of model categories. These notions have been around implicitly for a long time, but so far as the author knows, have never been written down before.

All we had to do to get from monoidal categories to closed categories was to define what an adjunction of two variables is. Similarly, the crucial step (but not the only step) needed to define monoidal model categories is to define a Quillen adjunction of two variables. The following definition is based on Quillen's SM7 axiom [**Qui67**], and is also found in [**DHK**]. See also Theorem 3.3.2. The author may have first heard it from Jeff Smith.

Definition 4.2.1. Given model categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, an adjunction of two variables $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is called a *Quillen adjunction of two variables*, if, given a cofibration $f\colon U \to V$ in $\mathcal{C}$ and a cofibration $g\colon W \to X$ in $\mathcal{D}$, the induced map

$$f \mathbin{\square} g\colon P(f, g) = (V \otimes W) \amalg_{U \otimes W} (U \otimes X) \to V \otimes X$$

is a cofibration in $\mathcal{E}$ which is trivial if either f or g is. We refer to the left adjoint F of a Quillen adjunction of two variables as a *Quillen bifunctor*, and often abuse notation by using the term "Quillen bifunctor $\otimes$" when we really mean "Quillen adjunction of two variables $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$."

The map $f \,\square\, g$ occuring in Definition 4.2.1 is sometimes called the *pushout product* of f and g.

The following lemma is then an exercise in adjointness, using the fact that cofibrations, trivial cofibrations, fibrations, and trivial fibrations are all characterized by lifting properties. We have seen it before for simplicial sets in 3.3.2.

LEMMA 4.2.2. *Suppose $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are model categories and*

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)$$

is an adjunction of two variables $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$. Then the following are equivalent:

1. *$\otimes$ is a Quillen bifunctor.*
2. *Given a cofibration $g\colon W \to X$ in $\mathcal{D}$ and a fibration $p\colon Y \to Z$ in $\mathcal{E}$, the induced map*
$$\mathrm{Hom}_{r,\square}(g,p)\colon\ \mathrm{Hom}_r(X,Y) \to \mathrm{Hom}_r(X,Z) \times_{\mathrm{Hom}_r(W,Z)} \mathrm{Hom}_r(W,Y)$$
is a fibration in $\mathcal{C}$ which is trivial if either g or p is.
3. *Given a cofibration $f\colon U \to V$ in $\mathcal{C}$ and a fibration $p\colon Y \to Z$ in $\mathcal{E}$, the induced map*
$$\mathrm{Hom}_{\ell,\square}(f,g)\colon\ \mathrm{Hom}_\ell(V,Y) \to \mathrm{Hom}_\ell(V,Z) \times_{\mathrm{Hom}_\ell(U,Z)} \mathrm{Hom}_\ell(U,Y)$$
is a fibration in $\mathcal{D}$ which is trivial if either f or p is.

REMARK 4.2.3. Suppose $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a Quillen bifunctor. Then, if C is cofibrant, the functor $C \otimes -\colon \mathcal{D} \to \mathcal{E}$ is a Quillen functor with right adjoint $\mathrm{Hom}_\ell(C,-)$. Similarly, if D is cofibrant, the functor $-\otimes D$ is a Quillen functor with right adjoint $\mathrm{Hom}_r(D,-)$. Also, if E is fibrant, the functor $\mathrm{Hom}_r(-,E)\colon \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$ is a Quillen functor. Its right adjoint is the functor $\mathrm{Hom}_\ell(-,E)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Here we are giving $\mathcal{C}^{\mathrm{op}}$ the opposite model category structure, as usual.

Just as was the case for Quillen functors (Lemma 2.1.20), it is easier to test whether a given adjunction of two variables is a Quillen bifunctor when the domain model categories are cofibrantly generated.

LEMMA 4.2.4. *Suppose $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is an adjunction of two variables, I is a set of maps in $\mathcal{C}$, I' is a set of maps in $\mathcal{D}$, and K is a set of maps in $\mathcal{E}$. Suppose as well that $I \,\square\, I' \subseteq K$. Then $(I\text{-}cof) \,\square\, (I'\text{-}cof) \subseteq K\text{-}cof$.*

PROOF. We first show that $(I\text{-cof}) \,\square\, I' \subseteq K\text{-cof}$. Indeed, since $I \,\square\, I' \subseteq K \subseteq K\text{-cof}$, adjointness implies that the maps of I have the left lifting property with respect to $\mathrm{Hom}_{r,\square}(I', K\text{-inj})$. It follows that every map of I-cof has the left lifting property with respect to $\mathrm{Hom}_{r,\square}(I', K\text{-inj})$. Applying adjointness again, we find that the maps of $(I\text{-cof}) \,\square\, I'$ have the left lifting property with respect to K-inj. Hence $(I\text{-cof}) \,\square\, I' \subseteq K\text{-cof}$. A similar argument using Hom_ℓ shows that $(I\text{-cof}) \,\square\, (I'\text{-cof}) \subseteq K\text{-cof}$. $\square$

COROLLARY 4.2.5. *Suppose $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is an adjunction of two variables between model categories. Suppose as well that $\mathcal{C}$ and $\mathcal{D}$ are cofibrantly generated, with generating cofibrations I and I' respectively, and generating trivial cofibrations*

J and J' respectively. Then $\otimes$ is a Quillen bifunctor if and only if $I \square I'$ consists of cofibrations and both $I \square J'$ and $J \square I'$ consist of trivial cofibrations.

We now define our notion of a monoidal model category.

Definition 4.2.6. A *monoidal model category* is a closed category $\mathcal{C}$ with a model structure making $\mathcal{C}$ into a model category, such that the following conditions hold.

1. The monoidal structure $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a Quillen bifunctor.
2. Let $QS \xrightarrow{q} S$ be the cofibrant replacement for the unit S, obtained by using the functorial factorizations to factor $0 \to S$ into a cofibration followed by a trivial fibration. Then the natural map $QS \otimes X \xrightarrow{q\otimes 1} S \otimes X$ is a weak equivalence for all cofibrant X. Similarly, the natural map $X \otimes QS \xrightarrow{1\otimes q} X \otimes S$ is a weak equivalence for all cofibrant X.

Note that this second condition is automatic if S is cofibrant.

We have a similar definition of a *symmetric monoidal model category*. In this case, we only need one side of the second condition.

As Jeff Smith pointed out to the author, when $\mathcal{C}$ is a monoidal category, the pushout product defines a monoidal structure on the category $\operatorname{Map}\mathcal{C}$ of arrows of $\mathcal{C}$: the identity is the identity map of the unit object S, and the associativity isomorphism is constructed from the associativity isomorphism of $\mathcal{C}$ by commuting $\otimes$ with pushouts. When $\mathcal{C}$ is closed monoidal, so is $\operatorname{Map}\mathcal{C}$: the adjoints are given by $\operatorname{Hom}_{r,\square}$ and $\operatorname{Hom}_{\ell,\square}$.

The second condition in Definition 4.2.6 is easy to forget, but is essential when the unit is not cofibrant. The following lemma, suggested to the athor by Stefan Schwede, gives an alternative characterization of this condition.

Lemma 4.2.7. *Suppose $\mathcal{C}$ is a closed monoidal category that is also a model category. Then the following are equivalent.*

(a) *The map $QS \otimes X \to X$ is a weak equivalence for all cofibrant X.*
(b) *The map $X \to \operatorname{Hom}_r(QS, X)$ is a weak equivalence for all fibrant X.*

Similarly, the following are equivalent.

(a') *The map $X \otimes QS \to X$ is a weak equivalence for all cofibrant X.*
(b') *The map $X \to \operatorname{Hom}_\ell(QS, X)$ is a weak equivalence for all fibrant X.*

Proof. The map $q\colon QS \to S$ induces a natural transformation between the Quillen functor $QS \otimes -$ and the identity functor. The result then follows from Corollary 1.4.4, part (b). $\square$

We now give some examples of monoidal model categories.

Proposition 4.2.8. *The model category* **SSet** *of simplicial sets forms a symmetric monoidal model category.*

Proof. The symmetric monoidal structure on **SSet** is of course the product. The adjoint is given by the function complex $\operatorname{Map}(X, Y)$, defined in Section 3.1. It is clear that **SSet** is a closed symmetric monoidal category.

The cofibrations in **SSet** are just the monomorphisms. The unit $*$ is thus cofibrant, so it suffices to verify that the product is a Quillen bifunctor. The pushout product of any two monomorphisms is easily seen to be a monomorphism. The trivial cofibrations in **SSet** are the anodyne extensions, studied in Section 3.3.

Thus Theorem 3.3.2 says that the pushout product of a trivial cofibration with a cofibration is a trivial cofibration, as required. □

It will follow from this theorem that pointed simplicial sets also form a symmetric monoidal model category. In fact, something more general is true.

PROPOSITION 4.2.9. *Suppose $\mathcal{C}$ is a monoidal model category whose unit is the terminal object $*$, and that $*$ is cofibrant. Then $\mathcal{C}_*$ is also a monoidal model category, which is symmetric if $\mathcal{C}$ is.*

See Proposition 1.1.8 for a discussion of the model structure on $\mathcal{C}_*$.

PROOF. We define $(X, v) \wedge (Y, w)$ to be the pushout in the diagram

$$\begin{tikzcd} X \amalg Y \arrow[r, "(X\otimes w)\amalg(v\otimes Y)"] \arrow[d] & X \otimes Y \arrow[d] \\ * \arrow[r] & X \wedge Y \end{tikzcd}$$

The reader can readily verify that this is a monoidal structure, with unit $(*)_+ = * \amalg *$. To construct the associativity isomorphism, use the fact that $X \otimes -$ commutes with colimits to write $X \wedge (Y \wedge Z)$ as the quotient of $X \otimes (Y \otimes Z)$ by the coproduct of $Y \otimes Z$, $X \otimes Z$ and $X \otimes Y$, and similarly for $(X \wedge Y) \wedge Z$. This monoidal structure is symmetric if and only if $\otimes$ is so.

We define the adjoint $\mathrm{Hom}_{r,*}(X, Y)$ as the pullback in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{r,*}(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_r(X,Y) \arrow[d, "\mathrm{Hom}_r(v,Y)"] \\ * \arrow[r, "\mathrm{Hom}_r(*,w)"'] & \mathrm{Hom}_r(*,Y) \end{tikzcd}$$

The basepoint of $\mathrm{Hom}_{r,*}(X, Y)$ is the zero map $X \longrightarrow * \xrightarrow{w} Y$. We define the other adjoint $\mathrm{Hom}_{\ell,*}(X, Y)$ in similar fashion. We leave it to the reader to verify the required adjunctions.

Since $*$ is cofibrant in $\mathcal{C}$, the unit $*_+$ is cofibrant in $\mathcal{C}_*$. Hence to complete the proof we need only verify that $\wedge$ is a Quillen bifunctor. Note that $X_+ \wedge Y_+$ is naturally isomorphic to $(X \otimes Y)_+$, so that the disjoint basepoint functor $\mathcal{C} \longrightarrow \mathcal{C}_*$ is a closed monoidal functor. This implies that $f_+ \,\square\, g_+ \cong (f \,\square\, g)_+$. Let I denote the cofibrations in $\mathcal{C}$ and let I' denote the cofibrations in $\mathcal{C}_*$. Then, because $\mathcal{C}$ is monoidal, $I_+ \square I_+ \subseteq I_+ \subseteq I'$. It follows from Lemma 4.2.4 that $(I_+\text{-cof}) \square (I_+\text{-cof}) \subseteq I'$. But we claim that $I_+\text{-cof} = I'$. Indeed, one can easily verify using adjointness that I_+-inj is the class of trivial fibrations in $\mathcal{C}_*$. It follows that $I_+\text{-cof} = I'$, so that $I' \,\square\, I' \subseteq I'$. A similar argument shows that $f \,\square\, g$ is a trivial cofibration if both f and g are cofibrations in $\mathcal{C}_*$ and one of them is trivial. □

It is essential that $\mathcal{C}$ be a closed monoidal category in order to ensure that $\mathcal{C}_*$ is a monoidal category. Indeed, the smash product on $\mathbf{Top}_*$ fails to be associative.

COROLLARY 4.2.10. *The model category $\mathbf{SSet}_*$ of pointed simplicial sets is a symmetric monoidal model category.*

The model category **Top** of topological spaces is not a monoidal model category, because it is not a closed monoidal category

PROPOSITION 4.2.11. *The model categories* $\mathbf{K}$ *and* $\mathbf{T}$ *of k-spaces and compactly generated spaces are symmetric monoidal model categories.*

PROOF. We leave to the reader the easy proof that $\mathbf{K}$ and $\mathbf{T}$ are symmetric monoidal categories under the k-space product. It follows from part 5 of Proposition 2.4.22 that $\mathbf{K}$ and $\mathbf{T}$ are in fact closed symmetric monoidal categories. The unit $*$ of the product is cofibrant, so it suffices to show that the product is a Quillen bifunctor. One can verify this directly, using the generators and Lemma 4.2.4. However, it also follows from the facts that $\mathbf{SSet}$ is a monoidal model category, and that the geometric realization is a monoidal functor. Indeed, let I denote the generating cofibrations of $\mathbf{SSet}$. Then $|I| \mathbin{\square} |I| \cong |I \mathbin{\square} I|$ because the geometric realization is monoidal. Since $I \mathbin{\square} I \subseteq I$-cof since $\mathbf{SSet}$ is a monoidal model category, we have $|I| \mathbin{\square} |I| \subseteq |I|$-cof. But the set $|I|$ is homeomorphic to the set of generating cofibrations of $\mathbf{K}$ (or $\mathbf{T}$). Lemma 4.2.4 completes the proof in this case, and a similar argument works when one of the cofibrations is trivial. □

COROLLARY 4.2.12. *The model categories* $\mathbf{K}_*$ *and* $\mathbf{T}_*$ *are symmetric monoidal model categories.*

PROPOSITION 4.2.13. *Let R be a commutative ring. Then* $\mathrm{Ch}(R)$*, the category of unbounded chain complexes of R-modules, given the model structure of Definition 2.3.3, is a symmetric monoidal model category.*

PROOF. First we recall that $\mathrm{Ch}(R)$ is indeed a closed symmetric monoidal category. Given chain complexes X and Y, we define

$$(X \otimes Y)_n = \bigoplus_k X_k \otimes_R Y_{n-k}$$

where $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$. The unit is the complex S^0 consisting of R in degree 0. The commutativity isomorphism is defined by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for homogeneous elements x and y. We leave it to the reader to construct the required natural associativity and unit isomorphisms, and to verify that the coherence diagrams commute, making $\mathrm{Ch}(R)$ into a symmetric monoidal category. To see that $\mathrm{Ch}(R)$ is in fact a closed symmetric monoidal category, we define

$$\mathrm{Hom}(X, Y)_n = \prod_k \mathrm{Hom}_R(X_k, Y_{n+k})$$

with $(df)(x) = df(x) + (-1)^{n+1} f(dx)$ for $f \in \mathrm{Hom}_R(X_k, Y_{n+k})$. It is easy to make a mistake with the signs above. One way to check that the signs are right is to verify that $\{f_k\} \in \prod_k \mathrm{Hom}_R(X_k, Y_k)$ is a cycle if and only if it is a chain map, which is true with our sign convention. Another way is to check the required adjointness, which we leave to the reader.

As the unit S^0 is cofibrant, it suffices to verify that the tensor product is a Quillen bifunctor. Recall that the generating cofibrations are the maps $S^{n-1} \to D^n$, and the generating trivial cofibrations are the maps $0 \to D^n$. The pushout product of two generating cofibrations is an injection with bounded below dimensionwise projective cokernel. Hence by Lemma 2.3.6, the cokernel is cofibrant. Proposition 2.3.9 then implies that the pushout product of two generating cofibrations is a cofibration. Lemma 4.2.4 implies that the pushout product of any two cofibrations is a cofibration.

To complete the proof, we must verify that the pushout product of a generating cofibration with a generating trivial cofibration is a weak equivalence. The pushout product of $S^{n-1} \to D^n$ and $0 \to D^m$ is the map $D^{m+n-1} \to D^m \otimes D^n$, which is a weak equivalence as required. $\square$

Note that the injective model structure (Definition 2.3.12) does not make $\mathrm{Ch}(R)$ into a monoidal model category, in general. Indeed, let $R = \mathbf{Z}$, and consider the pushout product of the injective cofibration $\mathbf{Z} \to \mathbf{Q}$ (concentrated in degree 0) with the injective cofibration $0 \to \mathbf{Z}/2\mathbf{Z}$. This pushout product is the map $\mathbf{Z}/2\mathbf{Z} \to 0$, which is certainly not an injective cofibration.

Even if R is not commutative, there is a tensor product pairing $\mathrm{Ch}(R^{\mathrm{op}}) \times \mathrm{Ch}(R) \to \mathrm{Ch}(\mathbf{Z})$. This is always a Quillen bifunctor, by the same argument used to prove Proposition 4.2.13.

Recall the model category $\mathrm{Ch}(B)$ of chain complexes of comodules over a commutative Hopf algebra B over a field k from Section 2.5.

PROPOSITION 4.2.14. *The model category* $\mathrm{Ch}(B)$ *is a symmetric monoidal model category.*

PROOF. Recall from Section 2.5 that the category of B-comodules is a closed symmetric monodal category. The monoidal structure is given by the tensor product $M \otimes_k N$ over the ground field k, using the coalgebra structure of B. The adjoint is given by the largest comodule contained in $\mathrm{Hom}_k(M, N)$. As remarked at the beginnning of Section 2.5.2, the tensor product extends to the category $\mathrm{Ch}(B)$ of chain complexes of B-comodules, just as in the proof of Proposition 4.2.13. The commutativity isomorphism again takes $x \otimes y$ to $(-1)^{|x||y|} y \otimes x$, and the unit is the trivial comodule k concentrated in degree 0. Similarly, the Hom functor also extends to $\mathrm{Ch}(B)$, just as in the proof of Proposition 4.2.13. We leave it to the reader to check that with these definitions $\mathrm{Ch}(B)$ is a closed symmetric monoidal category.

Recall that the cofibrations in $\mathrm{Ch}(B)$ the monomorphisms. Thus, the unit k is cofibrant, and, since tensoring over k is exact, the pushout product of two cofibrations is a cofibration. The generating trivial cofibrations are the maps $D^n i : D^n M \to D^n N$, where i is an inclusion of finite-dimensional comodules. The generating cofibrations are these plus the maps $S^{n-1}M \to D^n M$ for simple comodules M. In either case, one can check easily that the pushout product $f \square g$ of a generating cofibration f with a generating trivial cofibration g is a monomorphism with bounded acyclic cokernel C. Applying Lemma 2.5.19, we find that C has no homotopy. The long exact sequence in homotopy (Lemma 2.5.11) then implies that $f \square g$ is a homotopy isomorphism, so a weak equivalence as required. $\square$

Finally, we consider the stable category of modules over a Frobenius ring. In this case, we will have to assume R is a Hopf algebra over a field k in order to obtain a monoidal structure.

PROPOSITION 4.2.15. *Suppose R is a Frobenius ring which is also a finite-dimensional Hopf algebra over a field k. Then the category R-mod is a monoidal model category when given the model structure of Definition 2.2.5. It is symmetric if and only if R is cocommutative.*

PROOF. The monoidal structure on R-mod is given by $M \otimes_k N$, where R acts via its diagonal $R \to R \otimes_k R$. That is, if we write $\Delta r = \sum r' \otimes r''$, then $r(m \otimes n) = \sum r'm \otimes r''n$. The adjoints $\mathrm{Hom}_r(M, N)$ and $\mathrm{Hom}_\ell(M, N)$ are both isomorphic to $\mathrm{Hom}_k(M, N)$ as vector spaces. The R-action on $\mathrm{Hom}_r(M, N)$ is defined by $rg(m) = \sum r'g(\chi(r'')n)$, and the R-action on $\mathrm{Hom}_\ell(M,N)$ is defined by $rg(m) = \sum \chi(r')g(r''n)$. We leave it to the reader to verify that this makes R-mod into a closed monoidal category, which is symmetric if and only if R is cocommutative.

Since everything is cofibrant in R-mod, to verify that R-mod is a monoidal model category, we need only check that $\otimes$ is a Quillen bifunctor. Recall that the cofibrations are simply the injections. Since we are tensoring over a field, if f and g are injections, so is $f \square g$. If f is one of the generating cofibrations $\mathfrak{a} \to R$ and g is the generating trivial cofibration $0 \to R$, then $f \square g$ is the injection $\mathfrak{a} \otimes_k R \to R \otimes_k R$. To show that this is a weak equivalence, we must show that the cokernel $R/\mathfrak{a} \otimes_k R$ is a projective R-module. To do this, we must check that, given a surjection $f\colon M \to N$, any map $g\colon R/\mathfrak{a} \otimes_k R \to N$ lifts to M. But adjointness implies that is suffices to lift the adjoint $R \to \mathrm{Hom}_\ell(R/\mathfrak{a}, N)$ of g to $\mathrm{Hom}_\ell(R/\mathfrak{a}, M)$. Since Hom_k is exact and R is projective, we can find such a lift. A similar proof shows that $g \square f$ is a weak equivalence. $\square$

In all of these examples, the unit is cofibrant. The reader may therefore wonder if the unit condition in Definition 4.2.6 is really necessary. We will see below that the role this condition plays is to make sure that the unit isomorphism descends to the homotopy category. In practice, the unit is usually cofibrant, but the category of S-modules introduced in [**EKMM97**] is an example of a monoidal model category where the unit is not cofibrant.

We now define the 2-category of monoidal model categories.

DEFINITION 4.2.16. Given monoidal model categories $\mathcal{C}$ and $\mathcal{D}$, a *monoidal Quillen adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a Quillen adjunction (F, U, φ) such that F is a monoidal functor and such that the map $Fq\colon F(QS) \to FS$ is a weak equivalence. This last condition is redundant if S is cofibrant, but is necessary in general to make sure the unit isomorphism α passes to the homotopy category. We usually refer to a monoidal Quillen adjunction by its left adjoint F, whch we refer to as a *monoidal Quillen functor.*

We claim that monoidal model categories, monoidal Quillen functors, and monoidal natural transformations form a 2-category. As usual, we leave most of this to the reader. One must check, among other things, that if G and F are monoidal Quillen functors, the map $GFq\colon GFQS \to GFS$ is still a weak equivalence. To see this consider the diagram

$$\begin{tikzcd}
GQFQS \arrow[r, "GQFq_S"] \arrow[d, "Gq_{FQS}"'] & GQFS \arrow[r, "GQ\alpha"] \arrow[d, "Gq_{FS}"'] & GQS \arrow[d, "Gq_S"'] \\
GFQS \arrow[r, "GFq_S"] & GFS \arrow[r, "G\alpha"] & GS
\end{tikzcd}$$

where α is the unit isomorphism of the monoidal functor F. This diagram commutes because q is natural. The map Gq_S is a weak equivalence since G is a monoidal Quillen functor, and the maps $GQ\alpha$ and $G\alpha$ are isomorphisms. Hence the map Gq_{FS} is a weak equivalence. The map $GQFq_S$ is a weak equivalence since F is a monoidal Quillen functor and GQ preserves weak equivalences. The map Gq_{FQS} is a weak equivalence since G preserves weak equivalences between cofibrant objects. Hence the only other map in the diagram, GFq_S, must also be a weak equivalence.

There is an analogous 2-category of symmetric monoidal model categories, where the morphisms are symmetric monoidal Quillen adjunctions.

The simplest example of a symmetric monoidal Quillen functor is probably the disjoint basepoint functor $\mathbf{SSet} \to \mathbf{SSet}_*$, whose right adjoint is the forgetful functor. Similarly, the disjoint basepoint functors $\mathbf{K} \to \mathbf{K}_*$ and $\mathbf{T} \to \mathbf{T}_*$ are symmetric monoidal Quillen functors. Also, if $f\colon R \to S$ is a homomorphism of commutative rings, the induced Quillen adjunction $\mathrm{Ch}(R) \to \mathrm{Ch}(S)$ (whose left adjoint tensors with S and whose right adjoint is restriction of scalars) is a symmetric monoidal Quillen functor.

Another example is the geometric realization.

Proposition 4.2.17. *The geometric realization is a symmetric monoidal Quillen functor* $|\ |\colon \mathbf{SSet} \to \mathbf{K}$*, and extends to a symmetric monoidal Quillen functor* $|\ |\colon \mathbf{SSet}_* \to \mathbf{K}_*$.

Proof. We have already seen in Theorem 3.6.7 that the geometric realization is a Quillen equivalence. In Lemma 3.1.8 we saw that the geometric realization preserves products as well. One then just has to check that the coherence diagrams commute, which we leave to the reader. □

The reader might be tempted to think that we can construct a monoidal Quillen functor $\mathbf{SSet} \to \mathrm{Ch}(R)$ by tensoring with R to get a simplicial R-module, and then normalizing to get a chain complex by, for example, taking the alternating sum of the d_i. This is what the Eilenberg-Zilber theorem is about: so far as the author knows there is no monoidal Quillen functor $\mathbf{SSet} \to \mathrm{Ch}(R)$, though there is a Quillen functor which preserves products up to weak equivalence.

We also have the notion of a module over a monoidal model category.

Definition 4.2.18. Given a monoidal model category $\mathcal{C}$, a $\mathcal{C}$-*model category* is a $\mathcal{C}$-module $\mathcal{D}$ with a model structure making $\mathcal{D}$ into a model category such that the following conditions hold.

1. The action map $\otimes\colon \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ is a Quillen bifunctor.
2. If $QS \xrightarrow{q} S$ is the cofibrant replacement for S in $\mathcal{C}$, then the map $X \otimes QS \xrightarrow{1\otimes q} X \otimes S$ is a weak equivalence for all cofibrant $X \in \mathcal{D}$.

Again, this second condition is automatic when S is cofibrant in $\mathcal{C}$. An $\mathbf{SSet}$-model category is called a *simplicial model category*. A $\mathcal{C}$-*Quillen functor* from the $\mathcal{C}$-model category $\mathcal{D}$ to the $\mathcal{C}$-model category $\mathcal{E}$ is a Quillen functor which is also a $\mathcal{C}$-module functor.

Simplicial model categories were introduced by Quillen in [**Qui67**] and have been studied by many other authors since. Our definition is slightly different from Quillen's, as he only required an action by finite simplicial sets. This means that under our definition **Top** is not a simplicial model category, though it is under Quillen's. The categories $\mathbf{SSet}$, $\mathbf{SSet}_*$, $\mathbf{K}$, $\mathbf{K}_*$, $\mathbf{T}$ and $\mathbf{T}_*$ are all simplicial model categories, using the monoidal Quillen functors discussed above to define the action, but $\mathrm{Ch}(R)$ and $\mathrm{Ch}(B)$ for a ring R and a Hopf algebra B are not. On the other hand, for any ring R, $\mathrm{Ch}(R)$ is a $\mathrm{Ch}(\mathbf{Z})$-model category.

We leave it to the reader to verify that, given a monoidal model category $\mathcal{C}$, we get a 2-category of $\mathcal{C}$-model categories, $\mathcal{C}$-Quillen functors, and $\mathcal{C}$-module natural transformations. Furthermore, a monoidal Quillen functor $\mathcal{C} \to \mathcal{D}$ induces a forgetful 2-functor from $\mathcal{D}$-model categories to $\mathcal{C}$-model categories.

Note that, if $\mathcal{C}$ is pointed, then every $\mathcal{C}$-model category $\mathcal{D}$ is pointed as well. Indeed, the map $S \longrightarrow *$ in $\mathcal{C}$ induces a map $X \cong X \otimes S \longrightarrow X \otimes *$ in $\mathcal{D}$ for any $X \in \mathcal{D}$. But the functor $X \otimes -$ is a left adjoint, so we must have $X \otimes * = 0$, the initial object of $\mathcal{D}$. Taking X to be the terminal object 1 of $\mathcal{D}$, we get a map $1 \longrightarrow 0$, which must be an isomorphism.

Proposition 4.2.19. *Suppose $\mathcal{C}$ is a monoidal model category whose unit is the terminal object $*$, and suppose $*$ is cofibrant. If $\mathcal{D}$ is a $\mathcal{C}$-model category, then $\mathcal{D}_*$ is naturally a $\mathcal{C}_*$-model category. There is an equivalence of categories between pointed $\mathcal{C}$-model categories and $\mathcal{C}_*$-model categories.*

Proof. We have seen in Proposition 4.2.9 that $\mathcal{C}_*$ is a monoidal model category, and that the disjoint basepoint functor $\mathcal{C} \longrightarrow \mathcal{C}_*$ is a monoidal Quillen functor. Any $\mathcal{C}_*$-model category is automatically pointed, as we have seen above, and therefore we get a forgetful functor from $\mathcal{C}_*$-model categories to pointed $\mathcal{C}$-model categories. The same argument used in Proposition 4.2.9 shows that $\mathcal{D}_*$ is a $\mathcal{C}_*$-model category in a natural way. If $\mathcal{D}$ is already pointed, then $\mathcal{D}_*$ is naturally isomorphic to $\mathcal{D}$, giving the desired equivalence between pointed $\mathcal{C}$-model categories and $\mathcal{C}_*$-model categories. $\square$

So, for example, a pointed simplicial model category is the same thing as a $\mathbf{SSet}_*$-model category.

There is a duality 2-functor on the 2-category of $\mathcal{C}$-model categories, which is defined just as it is in the category $\mathcal{C}$-modules except that we also reverse the model structure. We leave the details to the reader.

Given a monoidal model category $\mathcal{C}$, we can form the 2-category of algebras over it as well.

Definition 4.2.20. Suppose $\mathcal{C}$ is a monoidal model category. A *monoidal $\mathcal{C}$-model category* is a monoidal model category $\mathcal{D}$ together with a monoidal Quillen functor $\mathcal{C} \longrightarrow \mathcal{D}$. A *monoidal $\mathcal{C}$-Quillen functor* is a monoidal Quillen functor which respects $\mathcal{C}$, so is a $\mathcal{C}$-algebra functor.

We leave it to the reader to verify that we get a 2-category of monoidal $\mathcal{C}$-model categories, monoidal $\mathcal{C}$-Quillen functors,, and $\mathcal{C}$-algebra natural transformations.

We have analogous 2-categories of symmetric and central monoidal $\mathcal{C}$-model categories. We leave the details to the reader. The examples above of symmetric monoidal Quillen functors also furnish examples of symmetric monoidal $\mathcal{C}$-model categories. For example, $\mathbf{K}$ is a symmetric monoidal $\mathbf{SSet}$-model category.

4.3. The homotopy category of a monoidal model category

In this section, we prove the expected result that the homotopy category of a monoidal model category is a closed monoidal category. Of course, we actually prove that the homotopy pseudo-2-functor $\mathbf{Mod} \to \mathbf{Cat}_{ad}$ extends to a pseudo-2-functor from monoidal model categories to closed monoidal categories. We have similar results for the five other sorts of categories we have considered.

We first show that a Quillen bifunctor induces a bifunctor on the homotopy category.

Proposition 4.3.1. *Suppose $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ are model categories, and*

$$(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_\ell, \varphi_r, \varphi_\ell)\colon\ \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$$

is a Quillen adjunction of two variables. Then the total derived functors define an adjunction of two variables $(\otimes^L, R\operatorname{Hom}_r, R\operatorname{Hom}_\ell, R\varphi_r, R\varphi_\ell)\colon \operatorname{Ho}\mathcal{C} \times \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{E}$.

PROOF. We use Remark 4.2.3 throughout this proof, so the reader might wish to review it. First note that $0 \cong C \otimes 0 \cong 0 \otimes D$, where 0 denotes the initial object, since $\otimes$ is a left adjoint in each variable. Similarly, $\operatorname{Hom}_r(0, E) \cong 1 \cong \operatorname{Hom}_r(D, 1)$, where 1 denotes the terminal object, and $\operatorname{Hom}_\ell(0, E) \cong 1 \cong \operatorname{Hom}_\ell(C, 1)$. It follows that $\otimes$ preserves cofibrant objects, since the map $0 \to A \otimes B$ is isomorphic to the map $A \otimes 0 \to A \otimes B$. Similarly, Hom_r and Hom_ℓ preserve fibrant objects, giving the first variable the opposite model category structure as usual.

We show that the total left derived functor $\otimes^L$ exists by showing that $\otimes$ preserves trivial cofibrations between cofibrant objects. Indeed, if $f\colon C \to C'$ and $g\colon D \to D'$ are trivial cofibrations of cofibrant objects, then, by Remark 4.2.3, both $C \otimes C' \xrightarrow{C \otimes g} C \otimes D'$ and $C \otimes D' \xrightarrow{f \otimes D'} C' \otimes D'$ are trivial cofibrations. Hence their composite $f \otimes g$ is also a trivial cofibration. Similarly, Hom_r and Hom_ℓ preserve trivial fibrations between fibrant objects, so have total right derived functors.

To define $R\varphi_r$, note first that φ_r defines a natural isomorphism

$$\tilde{\varphi}_r\colon\ [C \otimes D, E] \xrightarrow{\cong} [C, \operatorname{Hom}_r(D, E)]$$

of functors from $(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \times \mathcal{E})_f$ to sets, where the superscript denotes the full subcategory of fibrant objects. To see this, we must show that φ_r is compatible with the homotopy relation. To do so, we use the Quillen bifunctor property to show that, if $C \times I$ is a cylinder object on a cofibrant object C, and D is cofibrant, then $(C \times I) \otimes D$ is a cylinder object on the cofibrant object $C \otimes D$. There is a similar statement in the second variable. There are also similar statements for Hom_r (and Hom_ℓ), which we summarize by saying that Hom_r preserves path objects in either variable when thought of as a functor $(\mathcal{D}^{\mathrm{op}} \times \mathcal{E})_f \to \mathcal{C}_f$. It follows easily from this that $\tilde{\varphi}_r$ exists and is a natural isomorphism.

Since $\tilde{\varphi}_r$ is a natural isomorphism between functors which preserve weak equivalences, it induces a natural isomorphism

$$\operatorname{Ho}\tilde{\varphi}_r\colon\ [C \otimes D, E] \xrightarrow{\cong} [C, \operatorname{Hom}_r(D, E)]$$

of functors from $\operatorname{Ho}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \times \mathcal{E})_f$ to sets. We then define $R\varphi_r$ as the composite

$$\begin{aligned}[C \otimes^L D, E] = [QC \otimes QD, E] \xrightarrow{\cong} [QC \otimes QD, RE] &\xrightarrow[\cong]{\operatorname{Ho}\tilde{\varphi}_r} [QC, \operatorname{Hom}_r(QD, RE)] \\ &\xrightarrow{\cong} [C, \operatorname{Hom}_r(QD, RE)] = [C, (R\operatorname{Hom}_r)(D, E)]\end{aligned}$$

where the first and third arrows are induced by the isomorphisms (in the homotopy category) $E \to RE$ and $QC \to C$.

We have a similar construction for $R\varphi_\ell$. $\square$

It follows easily from Proposition 4.3.1 that a natural transformation of Quillen bifunctors induces a functorial derived natural transformation between the total left derived functors of the Quillen bifunctors.

THEOREM 4.3.2. *Suppose* $\mathcal{C}$ *is a (symmetric) monoidal model category. Then* $\operatorname{Ho}\mathcal{C}$ *can be given the structure of a closed (symmetric) monoidal category. The adjunction of two variables* $(\otimes^L, R\operatorname{Hom}_r, R\operatorname{Hom}_\ell)$ *which is part of the closed structure on* $\operatorname{Ho}\mathcal{C}$ *is the total derived adjunction of* $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_\ell)$. *The associativity and* *unit isomorphisms (and the commutativity isomorphism in case $\mathcal{C}$ is symmetric) on* $\operatorname{Ho}\mathcal{C}$ *are derived from the corresponding isomorphisms of* $\mathcal{C}$.

PROOF. Proposition 4.3.1 implies that the adjunction $(\otimes^L, R\operatorname{Hom}_r, R\operatorname{Hom}_\ell)$ exists. We must construct the associativity, unit, and commutativity isomorphisms (if applicable) and show the required coherence diagrams commute.

Consider first the associativity isomorphism

$$a\colon (X \otimes^L Y) \otimes^L Z \to X \otimes^L (Y \otimes^L Z).$$

Recall that, by definition,

$$(X \otimes^L Y) \otimes^L Z = Q(QX \otimes QY) \otimes QZ$$

and

$$X \otimes^L (Y \otimes^L Z) = QX \otimes Q(QY \otimes QZ).$$

We can then define a to be the composite

$$Q(QX \otimes QY) \otimes QZ \xrightarrow[\cong]{q\otimes 1} (QX \otimes QY) \otimes QZ$$
$$\xrightarrow[\cong]{a} QX \otimes (QY \otimes QZ) \xrightarrow[\cong]{1\otimes q^{-1}} QX \otimes Q(QY \otimes QZ)$$

Here we are using the fact that $a\colon (QX \otimes QY) \otimes QZ \to QX \otimes (QY \otimes QZ)$ is actually natural in the homotopy category, not just the actual category. To see this, use the argument in the proof of Proposition 4.3.1 that $\operatorname{Ho}\tilde{\varphi}_r$ exists.

One can also define this more formally by noting that the derived natural transformation of $a\colon \otimes\circ(\otimes\times 1) \to \otimes\circ(1\times\otimes)$ is a natural isomorphism $L(\otimes\circ(\otimes\times 1)) \to L(\otimes\circ(1\times\otimes))$, and constructing natural isomorphisms $\otimes^L\circ(\otimes^L\times 1) \to L(\otimes\circ(\otimes\times 1))$ and $\otimes^L\circ(1\times\otimes^L) \to L(\otimes\circ(1\times\otimes))$.

The commutativity isomorphism is easier, as we define $T\colon X\otimes^L Y \to Y \otimes^L X$ as the derived natural transformation of $T\colon X\otimes Y \to Y\otimes X$. That is, T is just $T\colon QX\otimes QY \to QY \otimes QX$. This assumes $\mathcal{C}$ is symmetric, of course.

The left unit isomorphism is the composite $QS\otimes QX \xrightarrow{q\otimes 1} S\otimes QX \xrightarrow{\ell} QX$. It is an isomorphism since $q\otimes 1$ is a weak equivalence when X is cofibrant. This is the reason for this condition in the definition of a monoidal model category. The construction of the right unit isomorphism is similar.

We leave the reader to check that the necessary coherence diagrams commute, which is straightforward since we have explicit descriptions of the maps involved. One can use the functoriality of the derived natural transformation to do some of the work. □

Naturally we want this correspondence between monoidal model categories and closed monoidal categories to be functorial.

THEOREM 4.3.3. *The homotopy pseudo-2-functor of Theorem 1.4.3 lifts to a pseudo-2-functor from monoidal model categories to closed monoidal categories, and further lifts to a pseudo-2-functor from symmetric monoidal model categories to closed symmetric monoidal categories.*

PROOF. Suppose $(F, U, \varphi, m, \alpha)$ is a monoidal Quillen adjunction from $\mathcal{C}$ to $\mathcal{D}$. We then get an adjunction $(LF, RU, R\varphi)\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$. We must construct natural isomorphisms $\alpha_{LF}\colon (LF)S \to S$ and $m_{LF}\colon (LF)X\otimes^L(LF)Y \to (LF)(X\otimes^L Y)$ and show that the required coherence diagrams commute.

We define α_{LF} to be the composite

$$(LF)S = F(QS) \xrightarrow{Fq} FS \xrightarrow{\alpha_F} S$$

which is an isomorphism, since by hypothesis Fq is a weak equivalence in $\mathcal{C}$. This is in fact the reason for making this assumption on F.

There are two different methods one could use to define m. What we are looking for is a natural isomorphism between composites of total derived functors, so we can use the derived natural isomorphism between the total derived functors of the composites and the isomorphism commuting "composites" and "total derived" past one another. Or we can just define m concretely, as the composite

$$(LF)X \otimes^L (LF)Y = QFQX \otimes QFQY \xrightarrow{q_{FQX} \otimes q_{FQY}} FQX \otimes FQY$$

$$\xrightarrow{m_F} F(QX \otimes QY) \xrightarrow{Fq^{-1}_{QX \otimes QY}} FQ(QX \otimes QY) = (LF)(X \otimes^L Y)$$

Here m is an isomorphism because both the tensor product and F preserve cofibrant objects and weak equivalences between cofibrant objects. The same reasoning shows that m_F is actually natural on the homotopy category level, not just the model category level.

We will leave it to the reader to verify that with these definitions, the appropriate coherence diagrams commute, making $(LF, RU, R\varphi, m, \alpha)$ into a closed monoidal functor, which is a closed symmetric monoidal functor if $(F, U, \varphi, m, \alpha)$ is so.

If τ is a monoidal natural transformation of monoidal Quillen functors, then one can easily check that $L\tau$ is compatible with the multiplication and unit isomorphisms just defined, so is a monoidal natural transformation. We leave it to the reader to check that compositions behave correctly, so that we do get a pseudo-2-functor as required. $\square$

The following theorem is proved in the same way.

THEOREM 4.3.4. *Suppose $\mathcal{C}$ is a monoidal model category. Then the homotopy pseudo-2-functor of Theorem 1.4.3 lifts to define*:

1. *A pseudo-2-functor from $\mathcal{C}$-model categories to closed* $\operatorname{Ho}\mathcal{C}$*-modules which is compatible with duality*;
2. *A pseudo-2-functor from monoidal $\mathcal{C}$-model categories to closed* $\operatorname{Ho}\mathcal{C}$*-algebras*;
3. *If $\mathcal{C}$ is a symmetric monoidal model category, a pseudo-2-functor from symmetric (resp. central) monoidal $\mathcal{C}$-model categories to closed symmetric (resp. central)* $\operatorname{Ho}\mathcal{C}$*-algebras.*

In particular, the geometric realization defines an equivalence of closed symmetric monoidal categories $\operatorname{Ho}\mathbf{SSet} \to \operatorname{Ho}\mathbf{K}$ and $\operatorname{Ho}\mathbf{SSet}_* \to \operatorname{Ho}\mathbf{K}_*$.

CHAPTER 5

Framings

In the last chapter we saw, among other things, that the homotopy category of a simplicial model category is naturally a closed $\mathrm{Ho}\,\mathbf{SSet}$-module. The main goal of this chapter is to show that in fact the homotopy category of any model category is naturally a closed $\mathrm{Ho}\,\mathbf{SSet}$-module. This seems to be saying that simplicial sets play almost the same role in model category theory as the integers do in ring theory, and explains why there are so many results about simplicial model categories in the literature. Almost all of those results will in fact hold for general model categories using the techniques in this section.

The author is very pleased with this result, and so must take great pains to point out that its essentials are not due to him, but rather to Dwyer and Kan [**DK80**]. We have used the formulation of the results of Dwyer and Kan that appears in an early draft of [**DHK**]. In this chapter, most of the results that do not contain one of the phrases "homotopy category", "2-category", or "pseudo-2-functor" are taken from [**DHK**].

The outline of this chapter is as follows. In order to construct the closed $\mathrm{Ho}\,\mathbf{SSet}$-module structure on the homotopy category of a model category, we need simplicial and cosimplicial resolutions of objects in a model category $\mathcal{C}$. A cosimplicial resolution will be an object in the functor category $\mathcal{C}^{\Delta}$, where Δ is, as usual, the category of finite totally ordered sets. To be able to work with such functors, we will need a model structure on $\mathcal{C}^{\Delta}$. We begin in Section 5.1 by putting a model structure on $\mathcal{C}^{\mathcal{B}}$, where $\mathcal{B}$ is a direct or inverse category. The category Δ is nether a direct nor an inverse category, but is instead what is known as a *Reedy category.* We examine diagrams over Reedy categories in Section 5.2. We also define framings and construct the framing associated to a model category there. A framing induces bifunctors analogous to the functors that define a simplicial model category. Before we investigate these functors, we take a brief detour in Section 5.3 to prove a lemma about bisimplicial sets. In Section 5.4, we study how the functors induced by a framing interact with the model structure. In particular, we show that the framing on a model category $\mathcal{C}$ gives rise to an adjunction of two variables $\mathrm{Ho}\,\mathcal{C} \times \mathrm{Ho}\,\mathbf{SSet} \to \mathrm{Ho}\,\mathcal{C}$. In Section 5.5, we show that this adjunction is actually part of a closed $\mathrm{Ho}\,\mathbf{SSet}$-module structure. Finally, in Section 5.6, we show that we get the promised pseudo-2-functor. We also show here that the homotopy category of a monoidal model category is naturally a closed $\mathrm{Ho}\,\mathbf{SSet}$-algebra. We would like to say that the homotopy category of a monoidal model category is naturally a *central* closed $\mathrm{Ho}\,\mathbf{SSet}$ algebra, and that the homotopy category of a symmetric monoidal model category is naturally a symmetric closed $\mathrm{Ho}\,\mathbf{SSet}$-algebra, but we are unable to prove this.

5.1. Diagram categories

Before we can introduce framings, we need to consider diagrams in a model category, and show that we sometimes get a model category of diagrams. The results in this section are mostly taken from [**DHK**].

Recall that an ordinal is defined inductively to be the totally ordered set of all smaller ordinals. If λ is an ordinal, we often think of λ as a category where there is one map from α to β if and only if $\alpha \leq \beta$.

Definition 5.1.1. Suppose $\mathcal{B}$ is a small category and λ is an ordinal.

1. A functor $f\colon \mathcal{B} \longrightarrow \lambda$ is called a *linear extension* if the image of a nonidentity map is a nonidentity map. We then refer to $f(i)$ as the *degree* of i. Note that all nonidentity maps raise the degree.
2. $\mathcal{B}$ is a *direct category* if there is a linear extension $\mathcal{B} \longrightarrow \lambda$ for some ordinal λ.
3. Dually, $\mathcal{B}$ is an *inverse category* if there is a linear extension $\mathcal{B}^{\mathrm{op}} \longrightarrow \lambda$ for some ordinal λ.

Note that the dual of a direct category is an inverse category, and vice versa.

In a direct category or inverse category, there is a kind of induction procedure, controlled by the latching or matching space functors that we now define.

Definition 5.1.2. Suppose $\mathcal{C}$ is a category with all small colimits, $\mathcal{B}$ is a direct category, and i is an object of $\mathcal{B}$. We define the *latching space* functor $L_i\colon \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}$ as follows. Let $\mathcal{B}_i$ be the category of all non-identity maps with codomain i in $\mathcal{B}$, and define L_i to be the composite

$$L_i\colon \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\mathcal{B}_i} \xrightarrow{\operatorname{colim}} \mathcal{C}$$

where the first arrow is restriction. Note that we have a natural transformation $L_iX \longrightarrow X_i$. Similarly, if $\mathcal{B}$ is an inverse category and $\mathcal{C}$ has all small limits, we define the *matching space* functor to be the composite

$$M_i\colon \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\mathcal{B}^i} \xrightarrow{\lim} \mathcal{C}$$

where $\mathcal{B}^i$ is the category of all non-identity maps with domain i in $\mathcal{B}$, and the first arrow is restriction. We have a natural transformation $X_i \longrightarrow M_iX$.

We can use the latching space functors to define a model category structure on $\mathcal{C}^{\mathcal{B}}$ for a direct category $\mathcal{B}$ and a model category $\mathcal{C}$.

Theorem 5.1.3. *Given a model category $\mathcal{C}$ and a direct category $\mathcal{B}$, there is a model structure on $\mathcal{C}^{\mathcal{B}}$, where a map $\tau\colon X \longrightarrow Y$ is a weak equivalence or a fibration if and only if the map $\tau_i\colon X_i \longrightarrow Y_i$ is so for all i. Furthermore, $\tau\colon X \longrightarrow Y$ is a (trivial) cofibration if and only if the induced map $X_i \amalg_{L_iX} L_iY \longrightarrow Y_i$ is a (trivial) cofibration for all i. Dually, if $\mathcal{B}$ is an inverse category, then we have a model structure on $\mathcal{C}^{\mathcal{B}}$ where the weak equivalences and cofibrations are the objectwise ones, and a map $\tau\colon X \longrightarrow Y$ is a (trivial) fibration if and only if the induced map $X_i \longrightarrow Y_i \times_{M_iY} M_iX$ is a (trivial) fibration for all i.*

To prove Theorem 5.1.3, we first prove that the lifting axiom holds. We concentrate on the direct category case, as the inverse category case is dual.

PROPOSITION 5.1.4. *Suppose $\mathcal{B}$ is a direct category, and $\mathcal{C}$ is a model category. Suppose we have a commutative square in $\mathcal{C}^{\mathcal{B}}$*

$$\begin{CD} A @>>> X \\ @V{f}VV @VV{p}V \\ B @>>> Y \end{CD}$$

where p is an objectwise fibration and where the map $g_i\colon A_i \amalg_{L_i A} L_i B \to B_i$ is a cofibration for all $i \in \mathcal{B}$. Then, if either p_i is a trivial fibration for all i or g_i is a trivial cofibration for all i, there is a lift $B \to X$.

PROOF. We will only prove the case when g_i is a trivial cofibration, as the other case is similar. We will show the required lift exists using transfinite induction. There is a linear extension $d\colon \mathcal{B} \to \lambda$ for some ordinal λ, and for $\beta \leq \lambda$, we define $\mathcal{B}_{<\beta}$ to be the full subcategory of $\mathcal{B}$ consisting of all i such that $d(i) < \beta$. Similarly, for $Z \in \mathcal{C}^{\mathcal{B}}$, we let $Z_{<\beta}$ be the restriction of Z to $\mathcal{B}_{<\beta}$. We will construct by transfinite induction on β, a lift $h_{<\beta}$ in the diagram

$$\begin{CD} A_{<\beta} @>>> X_{<\beta} \\ @VVV @VVV \\ B_{<\beta} @>>> Y_{<\beta} \end{CD}$$

such that, for all $\alpha < \beta$, the restriction of $h_{<\beta}$ to $B_{<\alpha}$ is $h_{<\alpha}$. The case $\beta = 0$ is trivial. If β is a limit ordinal and we have constructed $h_{<\alpha}$ for all $\alpha < \beta$, then we define $h_{<\beta}$ on $B_{<\beta}$ as the map induced by the $h_{<\alpha}$ for $\alpha < \beta$. That is, given an $i \in \mathcal{B}$ with $di < \beta$, there is an $\alpha < \beta$ such that $di < \alpha$, so we define $h_{<\beta}$ on X_i to be $h_{<\alpha}$ on X_i.

For the successor ordinal case, suppose we have defined $h_{<\beta}$. Then, for each element i of degree β, we have a commutative square

$$\begin{CD} A_i \amalg_{L_i A} L_i B @>>> X_i \\ @V{g_i}VV @VV{p_i}V \\ B_i @>>> Y_i \end{CD}$$

where the map $L_i B \to X_i$ is defined using $h_{<\beta}$. Since g_i is a trivial cofibration, we can find a lift in this diagram. Putting these together for the different i of degree β defines an extension $h_{<\beta+1}$ of $h_{<\beta}$, as required. □

COROLLARY 5.1.5. *Suppose $\mathcal{B}$ is a direct category and $\mathcal{C}$ is a model category. If $f\colon A \to B$ is a map in $\mathcal{C}^{\mathcal{B}}$ such that the map $A_i \amalg_{L_i A} L_i B \xrightarrow{g_i} B_i$ is a (trivial) cofibration for all i, then the map* $\operatorname{colim} f\colon \operatorname{colim} A \to \operatorname{colim} B$ *is a (trivial) cofibration.*

Given Theorem 5.1.3, Corollary 5.1.5 is just saying that the colimit is a left Quillen functor.

The dual of this corollary holds when $\mathcal{B}$ is an inverse category, as usual.

PROOF. Again, we concentrate on the case where g_i is a trivial cofibration for all i, as the other case is similar. Given a fibration $p\colon X \to Y$ in $\mathcal{C}$, we must show that we can find a lift in any commutative square

$$\begin{array}{ccc} \operatorname{colim} A & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ \operatorname{colim} B & \longrightarrow & Y \end{array}$$

But finding a lift in this square is equivalent to finding a lift in the commutative square

$$\begin{array}{ccc} A & \longrightarrow & c^*X \\ \Big\downarrow & & \Big\downarrow \\ B & \longrightarrow & c^*Y \end{array}$$

where c^*Z denotes the constant diagram on Z. Now Proposition 5.1.4 implies that we can find such a lift. $\square$

We can now prove Theorem 5.1.3.

Proof of Theorem 5.1.3. It suffices to prove the case when $\mathcal{B}$ is direct, since the isomorphism $\mathcal{C}^{\mathcal{B}^{\mathrm{op}}} \cong (\mathcal{C}^{\mathrm{op}})^{\mathcal{B}}$ converts the latching space to the matching space. The category $\mathcal{C}^{\mathcal{B}}$ has all small colimits and limits, taken objectwise. The two-out-of-three axiom is clear.

For the moment, let us refer to a map $A \longrightarrow B$ in $\mathcal{C}^{\mathcal{B}}$ which has the property that the map $A_i \amalg_{L_i A} L_i B \longrightarrow B_i$ is a trivial cofibration for all i as a *good trivial cofibration*. A good trivial cofibration is certainly a cofibration, and we claim that it is also a weak equivalence. Indeed, by Corollary 5.1.5, the map $L_i A \longrightarrow L_i B$ is a trivial cofibration for all i. It follows that the map $A_i \longrightarrow A_i \amalg_{L_i A} L_i B$ is also a trivial cofibration. Hence the map $A_i \longrightarrow B_i$ is a composition of two trivial cofibrations, hence is also a trivial cofibration. Thus every good trivial cofibration is a trivial cofibration. Later in the proof, we will show that the converse is also true.

Now, we leave it to the reader to check that weak equivalences, fibrations, cofibrations, and good trivial cofibrations are all closed under retracts. Proposition 5.1.4 shows that cofibrations have the left lifting property with respect to trivial fibrations, and that good trivial cofibrations have the left lifting property with respect to fibrations.

Now we construct the functorial factorizations of maps $A \longrightarrow B$. For concreteness, we will do the factorization into a good trivial cofibration followed by a fibration. The construction of the other factorization is similar. Recall that we have a degree function $d\colon \mathcal{B} \longrightarrow \lambda$. We construct compatible functorial factorizations on $\mathcal{C}^{\mathcal{B}_{<\beta}}$ by transfinite induction on $\beta \leq \lambda$, where $\mathcal{B}_{<\beta}$ is the full subcategory of all i such that $d(i) < \beta$. The base case of the induction is $\beta = 1$. Here we use the functorial factorization in $\mathcal{C}$ to factor $A_i \longrightarrow B_i$ for all i of degree 0. Now suppose we have constructed a functorial factorization on $\mathcal{C}^{\mathcal{B}_{<\beta}}$. We extend this to a functorial factorization on $\mathcal{C}^{\mathcal{B}_{<\beta+1}}$ as follows. Given a map $A \longrightarrow B$ of diagrams, we have the functorial factorization $A_{<\beta} \longrightarrow Z_{<\beta} \longrightarrow B_{<\beta}$. Given an i of degree β, we then have a map $A_i \amalg_{L_i A} L_i Z \longrightarrow B_i$. We use the functorial factorization in $\mathcal{C}$ to factor this into a trivial cofibration $A_i \amalg_{L_i A} L_i Z \longrightarrow Z_i$ followed by a fibration $Z_i \longrightarrow B_i$. Combining these for the different i of degree β, we get the required functorial factorization on $\mathcal{C}^{\mathcal{B}_{<\beta+1}}$. To complete the induction, we need to consider limit ordinals β. Suppose we have defined compatible functorial factorizations on $\mathcal{C}^{\mathcal{B}_{<\gamma}}$ for all $\gamma < \beta$. Then they clearly combine to define a functorial factorization on $\mathcal{C}^{\mathcal{B}_{<\beta}}$, as required.

To complete the proof, we must show that every trivial cofibration is a good trivial cofibration. So suppose $f\colon X \to Y$ is any trivial cofibration. Then we can factor it as $X \xrightarrow{g} Z \xrightarrow{p} Y$, where g is a good trivial cofibration and p is a (necessarily trivial) fibration. By lifting in the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Y & = & Y \end{array}$$

we see that f is a retract of g. This implies that f is also a good trivial cofibration. □

The following corollary is the immediate from Theorem 5.1.3 and Corollary 5.1.5.

Corollary 5.1.6. *Suppose $\mathcal{C}$ is a model category and $\mathcal{B}$ is a direct category. Then the colimit functor* colim: $\mathcal{C}^{\mathcal{B}} \to \mathcal{C}$ *is a left Quillen functor, left adjoint to the functor c that takes an object to the constant diagram at that object. Dually, if $\mathcal{B}$ is an inverse category, the limit functor* lim: $\mathcal{C}^{\mathcal{B}} \to \mathcal{C}$ *is a right Quillen functor, right adjoint to c.*

Remark 5.1.7. Suppose $\mathcal{B}$ is a direct category and $\mathcal{C}$ is a model category. Then a cofibration in $\mathcal{C}^{\mathcal{B}}$ is in particular an objectwise cofibration. Indeed, it follows from Corollary 5.1.5 that, if $f\colon X \to Y$ is a cofibration, then the map $L_iX \to L_iY$ is a cofibration for all i. Hence the map $X_i \to X_i \amalg_{L_iX} L_iY$ is also a cofibration for all i. The map $X_i \to Y_i$ is then the composition of two cofibrations, so is a cofibration. Similarly, if $\mathcal{B}$ is an inverse category, a fibration in $\mathcal{C}^{\mathcal{B}}$ is in particular an objectwise fibration.

Remark 5.1.8. In case $\mathcal{C}$ is cofibrantly generated and $\mathcal{B}$ is a direct category, the model structure of Theorem 5.1.3 on $\mathcal{C}^{\mathcal{B}}$ is cofibrantly generated. To see this, one first constructs, for all $i \in \mathcal{B}$, a left adjoint F_i to the evaluation functor $\mathrm{Ev}_i\colon \mathcal{C}^{\mathcal{B}} \to \mathcal{C}$. Then, if I is the set of generating cofibrations of $\mathcal{C}$, $FI = \bigcup_{i\in\mathcal{B}} F_iI$ is a set of generating cofibrations for $\mathcal{C}^{\mathcal{B}}$, and similarly for the generating trivial cofibrations. The domains of the maps of FI are small relative to the cofibrations of $\mathcal{C}^{\mathcal{B}}$ by adjointness, using the fact that the cofibrations on $\mathcal{C}^{\mathcal{B}}$ are in particular objectwise cofibrations.

We do not know if the model structure on $\mathcal{C}^{\mathcal{B}}$ is cofibrantly generated when $\mathcal{C}$ is so and $\mathcal{B}$ is an inverse category.

5.2. Diagrams over Reedy categories and framings

In this section we define the notion of a left and right framing on a model category and prove that such framings always exist. A framing is a functorial choice of simplicial and cosimplicial resolutions for each object A in a model category $\mathcal{C}$. Hence to define and study framings, we need to consider diagrams over the simplicial category Δ and analogous categories known as Reedy categories. The material in this section is taken from [**DHK**].

Recall from Section 3.1 that the simplicial category Δ has two obvious subcategories: the category Δ_+ of injective order-preserving maps, and the category Δ_-

of surjective order-preserving maps. The subcategory Δ_+ is a direct category, and the subcategory Δ_- is an inverse category. Furthermore, every morphism in Δ can be factored uniquely into a morphism in Δ_- followed by a morphism in Δ_+. It is this property of Δ which we abstract, following [**DHK**], to define the notion of a Reedy category.

DEFINITION 5.2.1. A *Reedy category* is a triple $(\mathcal{B}, \mathcal{B}_+, \mathcal{B}_-)$ consisting of a small category $\mathcal{B}$ and two subcategories $\mathcal{B}_+$, and $\mathcal{B}_-$, such that there exists a functor $d\colon \mathcal{B} \longrightarrow \lambda$, called a *degree function*, for some ordinal λ, such that every nonidentity map in $\mathcal{B}_+$ raises the degree, every nonidentity map in $\mathcal{B}_-$ lowers the degree, and every map $f \in \mathcal{B}$ can be factored uniquely as $f = gh$, where $h \in \mathcal{B}_-$ and $g \in \mathcal{B}_+$. In particular, $\mathcal{B}_+$ is a direct category and $\mathcal{B}_-$ is an inverse category. By abuse of notation, we often say $\mathcal{B}$ is a Reedy category, leaving the subcategories implicit.

Hence Δ is a Reedy category, as is Δ^{op}. Indeed, given any Reedy category $\mathcal{B}$, the category $\mathcal{B}^{\mathrm{op}}$ is also a Reedy category, where $(\mathcal{B}^{\mathrm{op}})_- = (\mathcal{B}_+)^{\mathrm{op}}$ and $(\mathcal{B}^{\mathrm{op}})_+ = (\mathcal{B}_-)^{\mathrm{op}}$. Also, if $\mathcal{B}$ and $\mathcal{B}'$ are both Reedy categories, so is their product, in the obvious way. Another example of a Reedy category is the category of simplices ΔK of a simplicial set K (see Section 3.1).

In any Reedy category, we can define both latching and matching space functors.

DEFINITION 5.2.2. Suppose $\mathcal{C}$ is a category with all small colimits and limits, and $\mathcal{B}$ is a Reedy category. For each object i of $\mathcal{B}$, we define the *latching space* functor L_i as the composite $\mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\mathcal{B}_+} \xrightarrow{L_i} \mathcal{C}$, where the latter functor is the latching space functor defined for direct categories in Definition 5.1.2. Similarly, we define the *matching space* functor M_i as the composite $\mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\mathcal{B}_-} \xrightarrow{M_i} \mathcal{C}$, where the latter functor is the matching space functor defined for inverse categories in Definition 5.1.2. Note that we have natural transformations $L_iA \longrightarrow A_i \longrightarrow M_iA$ defined for $A \in \mathcal{C}^{\mathcal{B}}$.

For example, if $\mathcal{B}$ is the simplicial category Δ, then $L_1A = A_0 \amalg A_0$ and $M_1A = A_0$. Of course L_0A is the initial object and M_0A is the terminal object. Dually, if $\mathcal{B}$ is Δ^{op}, then $L_1A = A_0$ and $M_1A = A_0 \times A_0$.

The beauty of the latching and matching space functors is that they allow us to define diagrams and maps of diagrams inductively.

REMARK 5.2.3. Let $\mathcal{C}$ be a category with all small colimits and limits. Suppose $\mathcal{B}$ is a Reedy category, with degree function $d\colon \mathcal{B} \longrightarrow \lambda$. Define $\mathcal{B}_{<\beta}$, for an ordinal $\beta \leq \lambda$, to be the full subcategory consisting of all i with $d(i) < \beta$. Suppose we have a functor $X\colon \mathcal{B}_{<\beta} \longrightarrow \mathcal{C}$. For any i with $d(i) = \beta$, we then have a map $L_iX \longrightarrow M_iX$. Then an extension of X to a functor $X'\colon \mathcal{B}_{<\beta+1} \longrightarrow \mathcal{C}$ is equivalent to factorizations $L_iX \longrightarrow X'_i \longrightarrow M_iX$ for all i such that $d(i) = \beta$. Indeed, given a nonidentity map $i \longrightarrow j$, where $d(i)$ and $d(j)$ are both $\leq \beta$, there is a unique factorization $i \xrightarrow{r} k \xrightarrow{s} j$, where $r \in \mathcal{B}_+$ and $s \in \mathcal{B}_-$. It is then clear how to define the map $X'_i \longrightarrow X'_j$, as the composite

$$X'_i \longrightarrow M_iX \longrightarrow X_k \longrightarrow L_jX \longrightarrow X'_j$$

Similarly, an extension of a natural transformation $\tau\colon X \to Y\colon \mathcal{B}_{<\beta} \to \mathcal{C}$ is equivalent to maps $X_i' \to Y_i'$ for $d(i) = \beta$ such that the diagrams

$$\begin{tikzcd} L_iX \arrow[r] \arrow[d, "L_i\tau"'] & X_i' \arrow[r] \arrow[d] & M_iX \arrow[d, "M_i\tau"'] \\ L_iY \arrow[r] & Y_i' \arrow[r] & M_iY \end{tikzcd}$$

are commutative. The situation is even simpler with regard to limit ordinals. If β is a limit ordinal, a functor $X\colon \mathcal{B}_{<\beta} \to \mathcal{C}$ is equivalent to a collection of compatible functors $X_\gamma\colon \mathcal{B}_{<\gamma} \to \mathcal{C}$ for all $\gamma < \beta$, and a natural transformation $X \to Y$ is equivalent to a collection of compatible natural transformations $X_\gamma \to Y_\gamma$ for all $\gamma < \beta$.

EXAMPLE 5.2.4. As an example of the procedure discussed in Remark 5.2.3, let $\mathcal{B} = \Delta$. We typically write an object $X \in \mathcal{C}^\Delta$ as $X^\bullet$, with nth term $X^\bullet[n]$. Suppose we start with an object $A \in \mathcal{C}$, and think of this as $X^\bullet[0]$. There are two obvious inductive choices of $X^\bullet[n]$: we can take $X^\bullet[n]$ to be either $L_nX^\bullet$ or $M_nX^\bullet$. In the first case, we are led to the cosimplicial object $\ell^\bullet A$ whose nth space is the $n+1$-fold coproduct of A. In the second case, we are led to the cosimplicial object $r^\bullet A$, whose nth space is A itself. We leave it to the reader to verify that the functor $\ell^\bullet\colon \mathcal{C} \to \mathcal{C}^\Delta$ is a left adjoint to the functor $\mathrm{Ev}_0\colon \mathcal{C}^\Delta \to \mathcal{C}$ which takes $X^\bullet$ to $X^\bullet[0]$, and that $r^\bullet$ is a right adjoint to Ev_0. Similarly, in the simplicial case, where we use the obvious dual notation, there is a left adjoint $\ell_\bullet$ and a right adjoint $r_\bullet$ to $\mathrm{Ev}_0\colon \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}$. We have $\ell_\bullet A[n] = A$ and $r_\bullet A[n]$ equal to the $n+1$-fold product of A.

We can use these latching and matching space functors to define a model category structure on Reedy diagrams in a model category.

THEOREM 5.2.5. *Suppose $\mathcal{C}$ is a model category and $\mathcal{B}$ is a Reedy category. Then there is a model structure on $\mathcal{C}^\mathcal{B}$ defined as follows. A map $f\colon X \to Y$ is a weak equivalence if and only if f_i is a weak equivalence for all $i \in \mathcal{B}$. The map f is a (trivial) cofibration if and only if the map $X_i \amalg_{L_iX} L_iY \to Y_i$ is a (trivial) cofibration for all $i \in \mathcal{B}$. The map f is a (trivial) fibration if and only if the map $X_i \to Y_i \times_{M_iY} M_iX$ is a (trivial) fibration for all $i \in \mathcal{B}$.*

PROOF. Certainly the category $\mathcal{C}^\mathcal{B}$ has all small limits and colimits, taken objectwise. By definition, a map is a cofibration or weak equivalence if and only if it is so in the model category $\mathcal{C}^{\mathcal{B}_+}$ of Theorem 5.1.3. The two-out-of-three axiom and the retract axiom for cofibrations and weak equivalences follow immediately, as does the characterization of trivial cofibrations. Similarly, a map is a fibration or weak equivalence if and only if it is so in $\mathcal{C}^{\mathcal{B}_-}$. The retract axiom for fibrations and the characterization of trivial fibrations follow immediately.

Now suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"'] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

where i is a cofibration, p is a fibration, and one of them is trivial. We must construct a lift. We do this by transfinite induction, combining the proof of Proposition 5.1.4 with Remark 5.2.3. We leave most of the details to the reader. The key point is that an extension of a partial lift defined on $\mathcal{B}_{<\beta}$ (in the terminology of Remark 5.2.3) to $\mathcal{B}_{<\beta+1}$ is equivalent to a lift in the diagram

$$\begin{array}{ccc} A_i \amalg_{L_iA} L_iB & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & Y_i \times_{M_iY} M_iX \end{array}$$

for each i of degree β. We can always find such a lift since the left vertical map is a cofibration, the right vertical map is a fibration, and one of them is a weak equivalence.

The proof of the functorial factorization axiom is similar to the direct category case, proved in Theorem 5.1.3. We use transfinite induction. Given a map $X \to Y$, we first use the functorial factorization in $\mathcal{C}$ to define $X_i \to Z_i \to Y_i$ for all i of degree 0. The limit ordinal case of the induction is easy, as pointed out in Remark 5.2.3. For the successor ordinal case, suppose we have defined a partial functorial factorization $X_i \to Z_i \to Y_i$ for all i of degree $< \beta$. An extension of this is equivalent to a functorial factorization of the map

$$X_i \amalg_{L_iX} L_iZ \to Y_i \times_{M_iY} M_iZ$$

for all i of degree β, which we construct using the functorial factorization in $\mathcal{C}$. $\square$

The model structure on $\mathcal{C}^{\mathcal{B}}$ of Theorem 5.2.5 is called the *Reedy model structure* in [**DHK**]. Note that, if $\mathcal{C}$ is a model category and $\mathcal{B}$ is a Reedy category, the Reedy model structure on $\mathcal{C}^{\mathcal{B}^{\mathrm{op}}}$ is the same, under the obvious isomorphism, as the Reedy model structure on $(\mathcal{C}^{\mathrm{op}})^{\mathcal{B}}$. Similarly, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are both Reedy categories, the Reedy model structure on $\mathcal{C}^{\mathcal{B}_1 \times \mathcal{B}_2}$ is the same, under the obvious isomorphisms, as the Reedy model structure on $(\mathcal{C}^{\mathcal{B}_1})^{\mathcal{B}_2}$ and on $(\mathcal{C}^{\mathcal{B}_2})^{\mathcal{B}_1}$. This can be seen by commuting the latching space colimits with each other and the matching space limits with each other. We leave the proof to the reader.

One might expect the colimit to be a left Quillen functor when $\mathcal{B}$ is a Reedy category. This is false in general, but there are important examples where it is true, such as in the following useful lemma. We learned this lemma from [**DHK**].

LEMMA 5.2.6 (The cube lemma). *Suppose $\mathcal{C}$ is a model category, and we have pushout squares X_i*

$$\begin{array}{ccc} P_i & \xrightarrow{f_i} & Q_i \\ \downarrow & & \downarrow \\ R_i & \longrightarrow & S_i \end{array}$$

for $i = 0, 1$ such that f_0 and f_1 are cofibrations and all objects are cofibrant. Suppose we have a map $X_0 \to X_1$ of pushout squares such that each of the maps $P_0 \to P_1$, $Q_0 \to Q_1$, and $R_0 \to R_1$ is a weak equivalence. Then the induced map $S_0 \to S_1$ is a weak equivalence.

PROOF. Let $\mathcal{B}$ be the category with three objects a, b, and c and two non-identity morphisms $a \to b$ and $a \to c$. We make $\mathcal{B}$ into a Reedy category in a non-standard way, by letting the map $a \to b$ raise degree, but letting the map $a \to c$ lower the degree. Then a cofibrant object of $\mathcal{C}^{\mathcal{B}}$ with the Reedy model structure is precisely a diagram $C \leftarrow A \xrightarrow{f} B$ of cofibrant objects where f is a cofibration. So we just need to prove that the colimit functor from $\mathcal{C}^{\mathcal{B}}$ to $\mathcal{C}$ is a left Quillen functor. To do so, we show that the constant functor $\mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{B}}$ preserves fibrations (it obviously preserves weak equivalences). But a map from the diagram $C \longleftarrow A \longrightarrow B$ into the diagram $C' \longleftarrow A' \longrightarrow B'$ is a fibration if and only if the maps $B \longrightarrow B'$, $C \longrightarrow C'$, and $A \longrightarrow A' \times_{C'} C$ are fibrations. It follows easily from this characterization that the constant functor preserves fibrations, as required. □

The Reedy model structure now allows us to define framings. Recall from Example 5.2.4 the two functors $\ell^\bullet, r^\bullet \colon \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ which are left and right adjoints respectively to Ev_0. Note that there is a natural transformation $\ell^\bullet \longrightarrow r^\bullet$ which is the identity in degree 0 (and the fold map in higher degrees).

Definition 5.2.7. Suppose $\mathcal{C}$ is a model category, and A is an object of $\mathcal{C}$.

1. A *cosimplicial frame* on A is a factorization $\ell^\bullet A \longrightarrow A^* \longrightarrow r^\bullet A$ of the canonical map $\ell^\bullet A \longrightarrow r^\bullet A$ into a cofibration in $\mathcal{C}^{\Delta}$ followed by a weak equivalence, which is an isomorphism in degree 0. Given two cosimplicial frames A^* and $\overline{A^*}$ on A, a *map of cosimplicial frames over* A is a map $A^* \longrightarrow \overline{A^*}$ in $\mathcal{C}^{\Delta}$ making the evident diagram commute. We also refer to a map $A^* \longrightarrow B^*$ in $\mathcal{C}^{\Delta}$ as a *map of cosimplicial frames* if A^* is a cosimplicial frame for A and B^* is a cosimplicial frame for B.
2. A *left framing* on $\mathcal{C}$ is a functor $\mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$, written $A \mapsto A^*$, together with a natural isomorphism $A \cong A^*[0]$, such that A^* is a cosimplicial frame on A when A is cofibrant.
3. Dually, a *simplicial frame* on A is a factorization $\ell_\bullet A \longrightarrow A_* \longrightarrow r_\bullet A$ of the canonical map $\ell_\bullet A \longrightarrow r_\bullet A$ into a weak equivalence followed by a fibration, which is an isomorphism in degree 0. A *map of simplicial frames over* A is a map of simplicial objects making the evident diagram commute.
4. A *right framing* on $\mathcal{C}$ is a functor $\mathcal{C} \longrightarrow \mathcal{C}^{\Delta^{\mathrm{op}}}$, written $A \mapsto A_*$, together with a natural isomorphism $A \cong A_*[0]$, such that A_* is a simplicial frame on A when A is fibrant.
5. A *framing* on $\mathcal{C}$ is a left framing together with a right framing.

Note that a map $\ell^\bullet A \xrightarrow{f} A^*$ is equivalent to a map $A \xrightarrow{f_0} A^*[0]$, and f is a cofibration if and only f_0 is a cofibration and the map $L_n A^* \longrightarrow A^*[n]$ is a cofibration for all positive n. Similarly, a map $A^* \xrightarrow{g} r^\bullet A$ is equivalent to a map $g_0 \colon A^*[0] \longrightarrow A$. Hence a cosimplicial frame on A is a cosimplicial object A^* together with an isomorphism $A \cong A^*[0]$ such that the induced map $A^*[n] \longrightarrow A$ is a weak equivalence for all n and the map $L_n A^* \longrightarrow A^*[n]$ is a cofibration for all positive n. In particular, $A^*[1]$ is a cylinder object on A for any cosimplicial frame A^* on A. A map of cosimplicial frames over A is just a map of cosimplicial objects which is compatible with the isomorphism in degree 0. The dual remarks hold for simplicial frames.

Note that a right framing on $\mathcal{C}$ is equivalent to a left framing on $D\mathcal{C} = \mathcal{C}^{\mathrm{op}}$. Hence to prove a result about framings, it typically suffices to prove only the left half of it. Also note that Ev_0 is both a left and right Quillen functor, thought of as a functor from either $\mathcal{C}^{\Delta}$ or $\mathcal{C}^{\Delta^{\mathrm{op}}}$ to $\mathcal{C}$. Hence the functors $\ell^\bullet$ and $\ell_\bullet$ are left Quillen functors, and the functors $r^\bullet$ and $r_\bullet$ are right Quillen functors. In particular, a cosimplicial frame A^* on a cofibrant object A is cofibrant in $\mathcal{C}^{\Delta}$, and a simplicial frame A_* on a fibrant object A is fibrant in $\mathcal{C}^{\Delta^{\mathrm{op}}}$.

We now show that framings always exist.

THEOREM 5.2.8. *If $\mathcal{C}$ is a model category, then the functorial factorization on $\mathcal{C}$ induces a framing on $\mathcal{C}$. We reserve the notation A° and $A_\circ$ for the images of A under the left and right framings so constructed. Then A° is a cosimplicial frame on A for all A (not just cofibrant A), and $A_\circ$ is a simplicial frame on A for all A (not just fibrant A). Furthermore, the maps $A^\circ \to r^\bullet A$ are trivial fibrations in $\mathcal{C}^\Delta$, and the maps $\ell_\bullet A \to A_\circ$ are trivial cofibrations in $\mathcal{C}^{\Delta^{\mathrm{op}}}$.*

PROOF. It suffices to construct the left framing, by duality. We apply the method used to construct the functorial factorization in $\mathcal{C}^\Delta$ in Theorem 5.2.5. We cannot apply the functorial factorization directly, since the result will not be an isomorphism in degree 0. So instead we let $A^\circ[0] = A$, and then proceed by induction, using the functorial factorization in $\mathcal{C}$ to factor the map

$$L_n A^\circ \amalg_{L_n \ell^\bullet A} \ell^\bullet A[n] = L_n A^\circ \to M_n A^\circ = M_n A^\circ \times_{M_n r^\bullet A} r^\bullet A[n]$$

into a cofibration $L_n A^\circ \to A^\circ[n]$ followed by a trivial fibration $A^\circ[n] \to M_n A^\circ$. $\square$

Note that the framing of Theorem 5.2.8 is canonically attached to the model category $\mathcal{C}$, in the sense that no choices are involved, as the functorial factorization is part of the structure of a model category. However, we still need to consider other simplicial and cosimplicial frames, because Quillen functors will not preserve the canonical frames in general, just as they do not preserve the functorial factorizations.

REMARK 5.2.9. Proposition 3.1.5 implies that a cosimplicial frame A^* in a model category $\mathcal{C}$ induces adjoint functors $A^* \otimes -\colon \mathbf{SSet} \to \mathcal{C}$ and $\mathcal{C}(A^*, -)\colon \mathcal{C} \to \mathbf{SSet}$. Dually, a simplicial frame Y_* induces functors $\mathrm{Hom}(-, Y_*)\colon \mathbf{SSet}^{\mathrm{op}} \to \mathcal{C}$ and $\mathcal{C}(-, Y_*)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$. Remark 3.1.7 implies that the framing of Theorem 5.2.8 induces adjoint bifunctors $\mathcal{C} \times \mathbf{SSet} \to \mathcal{C}$, which we denote by $(A, K) \mapsto A \otimes K$, and $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{SSet}$, which we denote by $(A, Y) \mapsto \mathrm{Map}_\ell(A, Y)$ and refer to as the *left function complex.* Dually, the framing also induces adjoint bifunctors $\mathbf{SSet}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, which we denote by $(K, Y) \mapsto \mathrm{Hom}(K, Y)$ or Y^K, and $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{SSet}$, which we denote $(A, Y) \mapsto \mathrm{Map}_r(A, Y)$ and refer to as the *right function complex.* Note that the functor $A \otimes -$ is a left adjoint, but the functor $- \otimes K$ need not be.

REMARK 5.2.10. If $\mathcal{C}$ is a simplicial model category (see Definition 4.2.18), then the functor $A \mapsto A \otimes \Delta[-]$ defines a left framing on $\mathcal{C}$. Indeed, the map $L_n(A \otimes \Delta[-]) \to A \otimes \Delta[n]$ is the map $A \otimes \partial\Delta[n] \to A \otimes \Delta[n]$, which is a cofibration when A is cofibrant. Note that $A \otimes \Delta[-]$ need not be a cosimplicial frame on A unless A is cofibrant. Similarly, the functor $A \mapsto A^{\Delta[-]}$ defines a right framing on $\mathcal{C}$, and $A^{\Delta[-]}$ need not be a simplicial frame on A unless A is fibrant. In particular, this framing is not the same as the canonical framing constructed in Theorem 5.2.8.

5.3. A lemma about bisimplicial sets

Our next goal is to investigate the homotopy properties of the left and right function complexes induced by the framing of Theorem 5.2.8. Before we can do so, though, we need a technical lemma about bisimplicial sets. Essentially, this lemma says that the diagonal is a Quillen functor. This short section is devoted to proving this lemma.

We have an obvious diagonal functor $\Delta \to \Delta \times \Delta$ that takes $[n]$ to $([n], [n])$. This functor induces a functor $\mathrm{diag}\colon \mathbf{SSet}^{\Delta^{\mathrm{op}}} \to \mathbf{SSet}$ from the category of bisimplicial sets $\mathbf{SSet}^{\Delta^{\mathrm{op}}}$ to the category of simplicial sets $\mathbf{SSet}$ by restriction. An n-simplex of $\operatorname{diag} X_\bullet$ is an n-simplex of $X_\bullet[n]$.

The diagonal functor is left adjoint to the functor which takes a simplicial set K to the bisimplicial set $\operatorname{Map}(\Delta[-], K)$. Here we are thinking of $\Delta[-]$ as a functor $\Delta \longrightarrow \mathbf{SSet}$, or a cosimplicial simplicial set. To see this adjunction is rather tricky, so we provide some details. First suppose $\mathcal{C}$ is a category with all small colimits and limits, and consider the functor $\mathrm{Ev}_n \colon \mathcal{C}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{C}$ which takes X to $X[n]$. This functor has a left adjoint F_n, where $F_n K = K \times \Delta[n]$. That is, $(F_n K)_m = \coprod_{\Delta[n]_m} K$, and the simplicial structure is given by the simplicial structure of $\Delta[n]$. Furthermore any simplicial object X is the coequalizer in a diagram of the form

$$\coprod_{[k]\longrightarrow[m]} F_k X_m \rightrightarrows \coprod_n F_n X_n \longrightarrow X$$

Here the top map takes $F_k X_m$ to $F_k X_k$ and is induced by the structure map $X_m \longrightarrow X_k$ of X. The bottom map takes $F_k X_m = X_m \times \Delta[k]$ to $F_m X_m = X_m \times \Delta[m]$ and is induced by the map $\Delta[k] \longrightarrow \Delta[m]$. Therefore, a functor that commutes with colimits, such as the diagonal functor, is completely determined by its effect on the $F_n K$. One can easily check that $\operatorname{diag} F_n K = K \times \Delta[n]$, where the product is now a product of simplicial sets. This implies that diag is left adjoint to the functor $K \mapsto \operatorname{Map}(\Delta[-], K)$, as required.

Lemma 5.3.1. *Suppose $X_\bullet \in \mathbf{SSet}^{\Delta^{\mathrm{op}}}$ is a bisimplicial set such that, for all maps $[k] \longrightarrow [n]$ in Δ, the induced map $X_\bullet[n] \longrightarrow X_\bullet[k]$ is a weak equivalence of simplicial sets. Then the map $X_\bullet[0] \longrightarrow \operatorname{diag} X_\bullet$ is also a weak equivalence.*

Proof. Note that the hypothesis immediately implies that the map $\ell_\bullet X_\bullet[0] \longrightarrow X_\bullet$ is a weak equivalence in the Reedy model structure, where $\ell_\bullet K$ is the constant bisimplicial set on the simplicial set K. Since $\operatorname{diag} \ell_\bullet K = K$, it suffices to show that the diagonal functor preserves weak equivalences.

We claim that the diagonal functor is a left Quillen functor. Indeed, to prove this it suffices to show that the functor $K \mapsto \operatorname{Map}(\Delta[-], K)$ preserves fibrations and trivial fibrations. But we have $M_n \operatorname{Map}(\Delta[-], K) = \operatorname{Map}(\partial\Delta[n], K)$, as one can easily verify using the description of $\partial\Delta[n]$ as the colimit of its nondegenerate simplices. Hence, given a (trivial) fibration $K \longrightarrow L$, we must show that the induced map

$$\operatorname{Map}(\Delta[n], K) \longrightarrow \operatorname{Map}(\Delta[n], L) \times_{\operatorname{Map}(\partial\Delta[n], L)} \operatorname{Map}(\partial\Delta[n], K)$$

is a (trivial) fibration. But this follows immediately from the fact that simplicial sets form a monoidal model category.

Hence the diagonal functor preserves weak equivalences between Reedy cofibrant bisimplicial sets. However, every bisimplicial set is Reedy cofibrant, because the simplicial identities force the map $L_n X_\bullet \longrightarrow X_\bullet[n]$ to be injective. This completes the proof. $\square$

5.4. Function complexes

We have seen that the framing on a model category gives rise to left and right function complexes, as well as functors corresponding to tensoring with a simplicial set and mapping out of a simplicial set. In this section, we examine the homotopy properties of these functors. They are not Quillen bifunctors in general, but they preserve enough of the model structure to have total derived functors. Furthermore, the total right derived functors of the left and right function complexes coincide, giving us an adjunction of two variables $\operatorname{Ho}\mathcal{C} \times \operatorname{Ho}\mathbf{SSet} \to \operatorname{Ho}\mathcal{C}$. We show in the next section that this is part of a closed $\operatorname{Ho}\mathbf{SSet}$-module structure on $\operatorname{Ho}\mathcal{C}$.

Proposition 5.4.1. *Let $\mathcal{C}$ be a model category. Suppose $f\colon A^\bullet \to B^\bullet$ is a cofibration in $\mathcal{C}^\Delta$ with respect to the Reedy model structure, and $g\colon K \to L$ is a cofibration of simplicial sets. Then the induced map $f \mathbin{\square} g\colon (A^\bullet \otimes L) \amalg_{A^\bullet \otimes K} (B^\bullet \otimes K) \to B^\bullet \otimes L$ is a cofibration in $\mathcal{C}$, which is trivial if f is. Dually, if $p\colon Y_\bullet \to Z_\bullet$ is a fibration in $\mathcal{C}^{\Delta^{\mathrm{op}}}$, the map $\operatorname{Hom}_\square(g,p)\colon \operatorname{Hom}(L, Y_\bullet) \to \operatorname{Hom}(K, Y_\bullet) \times_{\operatorname{Hom}(K,Z_\bullet)} \operatorname{Hom}(L, Z_\bullet)$ is a fibration which is trivial if p is.*

Proof. In the cofibration case, we can assume that g is one of the generating cofibrations $\partial\Delta[n] \to \Delta[n]$, using the method of Lemma 4.2.4 and the fact that $A^\bullet \otimes -$ has a right adjoint. We claim that the map $A^\bullet \otimes \partial\Delta[n] \to A^\bullet \otimes \Delta[n]$ is isomorphic to the map $L_n A^\bullet \to A^\bullet[n]$. Indeed, recall from Lemma 3.1.4 that $\partial\Delta[n]$ is the colimit of the functor $X\colon \mathcal{B} \to \mathbf{SSet}$, where $\mathcal{B}$ is the direct category whose objects are all nonidentity injective order-preserving maps $[k] \to [n]$ and whose morphisms are injective order-preserving maps $[k] \to [m]$ making the obvious triangle commute. The functor X just takes $[k] \to [n]$ to $\Delta[k]$. Since the functor $A^\bullet \otimes -$ commutes with colimits, it follows that $A^\bullet \otimes \partial\Delta[k] = \operatorname{colim} X'$, where $X'\colon \mathcal{B} \to \mathcal{C}$ takes $[k] \to [n]$ to $A^\bullet \otimes \Delta[k] = A^\bullet[k]$. But this colimit is the definition of the latching space $L_n A^\bullet$, and our claim follows. Under this isomorphism, the map $f \mathbin{\square} g$ corresponds to the map $A^\bullet[n] \amalg_{L_n A^\bullet} L_n B^\bullet \to B^\bullet[n]$, which is a cofibration since f is.

Now, if f is a trivial cofibration, the same argument shows that $f \mathbin{\square} g$ is a trivial cofibration when g is the map $\partial\Delta[n] \to \Delta[n]$. Hence $f \mathbin{\square} g$ will be a trivial cofibration for any cofibration g. The statements about simplicial frames follow by duality. ∎

Corollary 5.4.2. *Suppose $\mathcal{C}$ is a model category and K is a simplicial set. The functor $\mathcal{C}^\Delta \to \mathcal{C}$ that takes $A^\bullet$ to $A^\bullet \otimes K$ preserves cofibrations and trivial cofibrations.*

Notice that we do not prove that $f \mathbin{\square} g$ is a trivial cofibration if it is only assumed that g is a trivial cofibration. We think this is not true in general, though we do have the following result.

Proposition 5.4.3. *Suppose $\mathcal{C}$ is a model category, and $f\colon A^* \to B^*$ is a cofibration of cosimplicial frames of A and B respectively. Suppose in addition A, and hence B, are cofibrant. Then, if $g\colon K \to L$ is a trivial cofibration of simplicial sets, the induced map $f \mathbin{\square} g\colon Q = (A^* \otimes L) \amalg_{A^* \otimes K} (B^* \otimes K) \to B^* \otimes L$ is a trivial cofibration. Dually, if $p\colon Y_* \to Z_*$ is a fibration of simplicial frames on fibrant objects, then $\operatorname{Hom}_\square(g,p)\colon \operatorname{Hom}(L, Y_*) \to \operatorname{Hom}(K, Y_*) \times_{\operatorname{Hom}(K,Z_*)} \operatorname{Hom}(L, Z_*)$ is a trivial fibration.*

Proof. Since A is cofibrant, so is A^*. Hence the functor $A^* \otimes -\colon \mathbf{SSet} \to \mathcal{C}$ preserves cofibrations, by Proposition 5.4.1. This functor also preserves colimits, and of course the map $A^* \otimes \Delta[n] \to A^* \otimes \Delta[0] = A$ is a weak equivalence. Hence Proposition 3.6.8 applies to show that $A^* \otimes -$ preserves trivial cofibrations. In particular, the map $A^* \otimes K \to A^* \otimes L$ is a trivial cofibration. It follows that the map $B^* \otimes K \to Q$ is also a trivial cofibration. Since B^* is also cofibrant, the map

$B^* \otimes K \longrightarrow B^* \otimes L$ is a trivial cofibration. The two-out-of-three axiom then implies that $f \square g$ is a weak equivalence, as required. The statement about simplicial frames follows by duality. $\square$

We then have the following corollary.

COROLLARY 5.4.4. *Suppose $\mathcal{C}$ is a model category.*

1. *Suppose A is a cofibrant object of $\mathcal{C}$, and A^* is a cosimplicial frame on A. Then the functor $A^* \otimes -\colon$ **SSet** $\longrightarrow \mathcal{C}$ preserves cofibrations and trivial cofibrations and its right adjoint $\mathcal{C}(A^*, -)\colon \mathcal{C} \longrightarrow$ **SSet** preserves fibrations and trivial fibrations. In particular, the adjunction $(A \otimes -, \mathrm{Map}_\ell(A, -), \varphi)$ induced by the framing of Theorem 5.2.8 is a Quillen adjunction.*
2. *Suppose Y is a fibrant object of $\mathcal{C}$, and Y_* is a simplicial frame on Y. Then the functor $\mathrm{Hom}(-, Y_*)\colon$ **SSet** $\longrightarrow \mathcal{C}^{\mathrm{op}}$ preserves cofibrations and trivial cofibrations and its right adjoint $\mathcal{C}(-, Y_*)\colon \mathcal{C}^{\mathrm{op}} \longrightarrow$ **SSet** preserves fibrations and trivial fibrations, where we use the dual model structure on $\mathcal{C}^{\mathrm{op}}$. In particular, the adjunction $(\mathrm{Hom}(-, Y), \mathrm{Map}_r(-, Y), \varphi)$ induced by the framing of Theorem 5.2.8 is a Quillen adjunction.*

Corollary 5.4.4 implies that, if A is cofibrant, the functor $A \otimes -$ preserves weak equivalences between cofibrant objects. We also need the functor $- \otimes K$ to preserve weak equivalences between cofibrant objects, for a simplicial set K. To se this, note that Corollary 5.4.2 implies the following proposition.

PROPOSITION 5.4.5. *Suppose $\mathcal{C}$ is a model category, A and B are cofibrant objects, A^* is a cosimplicial frame on A, and B^* is a cosimplicial frame on B. If $f\colon A^* \longrightarrow B^*$ is a map of cosimplicial objects which is a weak equivalence in degree 0, then f induces a natural weak equivalence $A^* \otimes K \longrightarrow B^* \otimes K$ for all simplicial sets K. Dually, if X_* and Y_* are simplicial frames on fibrant objects X and Y, and $g\colon X_* \longrightarrow Y_*$ is a simplicial map which is a weak equivalence in degree 0, then g induces a natural weak equivalence $\mathrm{Hom}(K, X_*) \longrightarrow \mathrm{Hom}(K, Y_*)$ for all simplicial sets K.*

PROOF. As usual, it suffices to prove the cosimplicial case. The map $A^* \longrightarrow B^*$ is a weak equivalence of cofibrant objects of $\mathcal{C}^{\Delta}$, so the proposition follows immediately from Corollary 5.4.2 and Ken Brown's lemma 1.1.12. $\square$

COROLLARY 5.4.6. *Suppose $\mathcal{C}$ is a model category, given the framing of Theorem 5.2.8, and K is a simplicial set. Then the functor $- \otimes K\colon \mathcal{C} \longrightarrow \mathcal{C}$ preserves weak equivalences between cofibrant objects. Dually, the functor $\mathrm{Hom}(K, -)$ preserves weak equivalences between fibrant objects.*

We would like to conclude that $\mathrm{Map}_\ell(-, Y)$ preserves weak equivalences between cofibrant objects when Y is fibrant, and that $\mathrm{Map}_r(A, -)$ preserves weak equivalences between fibrant objects when A is cofibrant, but we do not have the adjointness necessary to conclude this. However, if $\mathcal{C}$ is a simplicial model category, the two mapping spaces $\mathrm{Map}_\ell(A, Y)$ and $\mathrm{Map}_r(A, Y)$ are equal, so it is not unreasonable to expect them to be weakly equivalent in general. This is in fact the case, at least when A is cofibrant and Y is fibrant.

PROPOSITION 5.4.7. *Suppose $\mathcal{C}$ is a model category, A^* is a cosimplicial frame on a cofibrant object A, and Y_* is a simplicial frame on a fibrant object Y. Then* *there are weak equivalences*

$$\mathcal{C}(A^*, Y) \longrightarrow \operatorname{diag} \mathcal{C}(A^*, Y_*) \longleftarrow \mathcal{C}(A, Y_*).$$

PROOF. First consider $\mathcal{C}(A^*, Y_*)$ as a bisimplicial set $X_\bullet$, where

$$X_\bullet[n] = \mathcal{C}(A^*, Y_*[n])$$

Each of the structure maps $Y_*[k] \longrightarrow Y_*[n]$ is a weak equivalence of fibrant objects, since Y is fibrant. Thus Corollary 5.4.4 and Ken Brown's lemma 1.1.12 imply that each of the structure maps of $X_\bullet$ is a weak equivalence. Lemma 5.3.1 then implies that the induced map $\mathcal{C}(A^*, Y) \longrightarrow \operatorname{diag} \mathcal{C}(A^*, Y_*)$ is a weak equivalence. The other case is proved similarly, using the other order of indexing. □

The following corollary then follows from Proposition 5.4.7 and Corollary 5.4.4.

COROLLARY 5.4.8. *Suppose $\mathcal{C}$ is a model category, given the framing of Theorem 5.2.8. If Y is a fibrant object of $\mathcal{C}$, the functor $\operatorname{Map}_\ell(-.Y)$ preserves weak equivalences between cofibrant objects of $\mathcal{C}$. If A is a cofibrant object of $\mathcal{C}$, then the functor $\operatorname{Map}_r(A, -)$ preserves weak equivalences between fibrant objects of $\mathcal{C}$.*

We summarize the results of this section in the following theorem.

THEOREM 5.4.9. *Suppose $\mathcal{C}$ is a model category, given the framing of Theorem 5.2.8. Then the total left derived functors of $- \otimes -\colon \mathcal{C} \times \mathbf{SSet} \longrightarrow \mathcal{C}$ and $\operatorname{Hom}(-,-)\colon \mathbf{SSet} \times \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$ exist. We denote them by $(X, K) \mapsto X \otimes^L K$ and $(K, X) \mapsto R\operatorname{Hom}(K, X)$ respectively. The total right derived functors of $\operatorname{Map}_\ell(-,-)$ and $\operatorname{Map}_r(-,-)$ exist and are naturally isomorphic. We denote them by $R\operatorname{Map}_\ell(-,-)$ and $R\operatorname{Map}_r(-,-)$ respectively. There are natural isomophisms*

$$[X \otimes^L K, Y] \xrightarrow{\cong} [K, R\operatorname{Map}_\ell(X, Y)] \xrightarrow{\cong} [K, R\operatorname{Map}_r(X, Y)] \xrightarrow{\cong} [X, R\operatorname{Hom}(K, Y)]$$

so we have an adjunction of two variables $\operatorname{Ho}\mathcal{C} \times \operatorname{Ho}\mathbf{SSet} \longrightarrow \operatorname{Ho}\mathcal{C}$. There is also a natural isomorphism $X \otimes^L \Delta[0] \cong X$.

PROOF. Corollaries 5.4.4 and 5.4.6 imply that the functors $-\otimes-$ and $\operatorname{Hom}(-,-)$ preserve weak equivalences between cofibrant objects, where we think of the latter as a functor $\operatorname{Hom}(-,-)\colon \mathbf{SSet} \times \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$ and use the dual model structure on $\mathcal{C}^{\mathrm{op}}$. Hence their total left derived functors exist. Corollaries 5.4.4 and 5.4.8 imply that the functors $\operatorname{Map}_\ell(-,-)$ and $\operatorname{Map}_r(-,-)$, thought of as functors from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathbf{SSet}$, preserve weak equivalences between fibrant objects, and hence their total right derived functors exist. Proposition 5.4.7 implies that $R\operatorname{Map}_\ell(X, Y)$ is naturally isomorphic to $R\operatorname{Map}_r(X, Y)$. Indeed, we have

$$\begin{aligned} R\operatorname{Map}_\ell(X, Y) = \mathcal{C}((QX)^\circ, RY) &\xrightarrow{\cong} \operatorname{diag} \mathcal{C}((QX)^\circ, (RY)_\circ) \\ &\xrightarrow{\cong} \mathcal{C}(QX, (RY)_\circ) = R\operatorname{Map}_r(X, Y) \end{aligned}$$

where the equalities are true by definition, and the arrows are natural isomorphisms in $\operatorname{Ho}\mathbf{SSet}$.

Now, since QX is cofibrant, the functor $QX \otimes -$ is a left Quillen functor, adjoint to $\operatorname{Map}_\ell(QX, -)$, by Corollary 5.4.4. Hence we get an isomorphism, natural in K and Y,

$$[X \otimes^L K, Y] = [QX \otimes QK, Y] \cong [K, \operatorname{Map}_\ell(QX, RY)] = [K, R\operatorname{Map}_\ell(X, Y)]$$

where the equalities are by definition. The isomorphism is natural in X as well, since a map $X \longrightarrow X'$ induces a natural transformation $QX \otimes - \longrightarrow QX' \otimes -$. The dual argument constructs the other natural isomorphism we need to complete the proof. □

5.5. Associativity

In this section, we show that the adjunction of two variables $\operatorname{Ho}\mathcal{C}\times\operatorname{Ho}\mathbf{SSet}\to\operatorname{Ho}\mathbf{SSet}$ of Theorem 5.4.9 is part of a closed $\operatorname{Ho}\mathbf{SSet}$-module structure on $\operatorname{Ho}\mathcal{C}$. The only real difficulty is associativity.

We begin by studying the uniqueness of cosimplicial frames. The first thing to notice is that any two cosimplicial frames on the same object are weakly equivalent.

LEMMA 5.5.1. *Suppose $\mathcal{C}$ is a model category and $A \xrightarrow{f} B$ is a map of $\mathcal{C}$. For any cosimplicial frame A^* of A, there is a map $A^*\to B^\circ$ of cosimplicial frames, which is the map f in degree 0. Here B° is the cosimplicial frame on B constructed in Theorem 5.2.8. Dually, for any simplicial frame B_* of B, there is a map $A_\circ\to B_*$ of simplicial frames which is f in degree 0.*

PROOF. Consider the diagram

$$\begin{array}{ccc}
\ell^\bullet A & \longrightarrow & B^\circ \\
\Big\downarrow & & \Big\downarrow \\
A^* & \longrightarrow & r^\bullet B
\end{array}$$

The top horizontal map is the composite $\ell^\bullet A\to\ell^\bullet B\to B^\circ$, and the bottom horizontal map is the composite $A^*\to r^\bullet A\to r^\bullet B$. In particular, the square is commutative. The left vertical map is a cofibration in the Reedy model structure on $\mathcal{C}^\Delta$ by the definition of a cosimplicial frame, and the right vertical map is a trivial fibration. Hence we can find a lift $A^*\to B^\circ$ as required. □

A map of cosimplicial frames $A^*\to B^*$ induces a natural transformation $A^*\otimes - \to B^*\otimes -$, and if A and B are cofibrant, a derived natural transformation $A^*\otimes^L - \to B^*\otimes^L -$. We now show that this derived natural transformation depends only on the map $A\to B$.

LEMMA 5.5.2. *Suppose $\mathcal{C}$ is a model category, and A^* (resp. B^*) is a cosimplicial frame on a cofibrant object A (resp. B). Suppose $f,g\colon A^*\to B^*$ are maps of cosimplicial frames which agree in degree 0. Then the derived natural transformations $\tau_f,\tau_g\colon A^*\otimes^L K\to B^*\otimes^L K$ are equal.*

PROOF. Let γ denote the functor from a model category to its homotopy category. Then we have $\gamma f=\gamma g$. Indeed, because f and g agree in degree 0, they become equal upon composing with the weak equivalence $B^*\to r^\bullet B$. For now, let $F\colon \mathcal{C}^\Delta\to\mathcal{C}$ denote the functor that takes $X^\bullet$ to $X^\bullet\otimes K$. Then F preserves cofibrations and trivial cofibrations, by Corollary 5.4.2, and hence has a total left derived functor LF. Of course $(LF)\gamma f=(LF)\gamma g$, since $\gamma f=\gamma g$. By definition, this means that $\gamma F(Qf)=\gamma F(Qg)$, where Q denotes the cofibrant replacement functor in $\mathcal{C}^\Delta$. Since A^* and B^* are already cofibrant and F preserves weak equivalences between cofibrant objects, it follows that $\gamma Ff=\gamma Fg$, as we wanted to prove. □

THEOREM 5.5.3. *Suppose $\mathcal{C}$ is a model category. Then the framing of Theorem 5.2.8 makes $\operatorname{Ho}\mathcal{C}$ into a closed $\operatorname{Ho}\mathbf{SSet}$-module.*

PROOF. The functors, adjointness isomorphisms, and unit isomorphism that make up most of the closed action were constructed in Theorem 5.4.9. We need to construct the associativity isomorphism $(X \otimes^L K) \otimes^L L \xrightarrow{a} X \otimes^L (K \times L)$. Here we have used the symbol $K \times L$ to denote both the product in **SSet** and the derived product in Ho **SSet**, though they are not equal. Since every object of **SSet** is cofibrant, however, they are isomorphic, so this should cause no confusion. It turns out to be slightly more convenient to construct the inverse of the associativity isomorphism, so we do so, without changing notation.

To construct the associativity isomorphism, suppose A is cofibrant, and consider the cosimplicial object $A \otimes (K \times \Delta[-])$. Since $A \otimes -$ is a left Quillen functor, the map

$$A \otimes (K \times \Delta[n]) \to A \otimes (K \times \Delta[0]) \cong A \otimes K$$

is a weak equivalence. Furthermore, since $K \times \partial\Delta[n] \to K \times \Delta[n]$ is a cofibration of simplicial sets, the map

$$L_n(A \otimes (K \times \Delta[-])) = A \otimes (K \times \partial\Delta[n]) \to A \otimes (K \times \Delta[n])$$

is a cofibration. Thus $A \otimes (K \times \Delta[-])$ is a cosimplicial frame on $A \otimes K$.

Hence there is a map of cosimplicial frames $A \otimes (K \times \Delta[-]) \to (A \otimes K)^\circ$, by Lemma 5.5.1. This map induces a natural (in L) weak equivalence

$$A \otimes (K \times \Delta[-]) \otimes L \to (A \otimes K) \otimes L$$

by Proposition 5.4.5. But since $A \otimes (K \times -)$ commutes with colimits, we have a natural isomorphism

$$A \otimes (K \times \Delta[-]) \otimes L \cong A \otimes (K \times L).$$

Altogether then, we have a weak equivalence, natural in L,

$$a\colon\ A \otimes (K \times L) \to (A \otimes K) \otimes L.$$

We then define the associativity isomorphism $a\colon\ A \otimes^L (K \times L) \to (A \otimes^L K) \otimes^L L$ to be the composite

$$\begin{aligned} QA \otimes Q(QK \times QL) &\xrightarrow{QA \otimes q} QA \otimes (QK \times QL) \xrightarrow{a} (QA \otimes QK) \otimes QL \\ &\xrightarrow{(q \otimes QL)^{-1}} Q(QA \otimes QK) \otimes QL \end{aligned}$$

A priori, this is natural only in L. We must show that a is natural in both A and K as well, and that the appropriate coherence diagrams commute.

We first show that a is natural in K. Given a map $K \to K'$ and a cofibrant A, we have a (non-commutative) square of cosimplicial frames

$$\begin{array}{ccc} A \otimes (K \times \Delta[-]) & \longrightarrow & (A \otimes K)^\circ \\ \big\downarrow & & \big\downarrow \\ A \otimes (K' \times \Delta[-]) & \longrightarrow & (A \otimes K')^\circ \end{array}$$

This square is non-commutative, but it does commute in degree 0. It follows from Lemma 5.5.2 that the square

$$\begin{array}{ccc} A \otimes (K \times L) & \xrightarrow{a} & (A \otimes K) \otimes L \\ \big\downarrow & & \big\downarrow \\ A \otimes (K' \times L) & \xrightarrow{a} & (A \otimes K') \otimes L \end{array}$$

is commutative in the homotopy category, so a is natural in K.

A similar argument shows that a is natural in A. The coherence diagrams are all proved similarly. The trickiest one is the four-fold associativity diagram, so we prove that it commutes and leave the other two coherence diagrams to the reader. Recall that the four-fold associativity diagram looks like this:

$$\begin{tikzcd}
A \otimes^L (K \times (L \times M)) \arrow[r, "a"] \arrow[d, "1\otimes a"'] & (A \otimes^L K) \otimes^L (L \times M) \arrow[d, "a"] \\
A \otimes^L ((K \times L) \otimes^L M) \arrow[d, "a"'] & ((A \otimes^L K) \otimes^L L) \otimes^L M \arrow[d, equal] \\
(A \otimes^L (K \times L)) \otimes^L M \arrow[r, "a\otimes 1"] & ((A \otimes^L K) \otimes^L L) \otimes^L M
\end{tikzcd}$$

Now consider the cosimplicial object $A \otimes (K \times (L \times \Delta[-]))$, for A a cofibrant object of $\mathcal{C}$. One can check that this is a cosimplicial frame on $A \otimes (K \times L)$. We will show that both composites in the four-fold associativity diagram are induced by maps of cosimplicial frames $A \otimes (K \times (L \times \Delta[-])) \to ((A \otimes K) \otimes L)^\circ$ which are the associativity weak equivalence in degree 0. It will follow that their derived natural transformations are equal by Lemma 5.5.2, so the four-fold associativity diagram commutes.

The counterclockwise composite is induced by a map of cosimplicial frames which is the composite

$$\begin{aligned} A \otimes (K \times (L \times \Delta[-])) \cong A \otimes ((K \times L) \times \Delta[-]) &\to (A \otimes (K \times L))^\circ \\ &\xrightarrow{a^\circ} ((A \otimes K) \otimes L)^\circ \end{aligned}$$

Here the second map is any map of cosimplicial frames over $A \otimes (K \times L)$.

We now show that the first map in the clockwise composite in the four-fold associativity diagram is induced by a map of cosimplicial frames covering the associativity weak equivalence. The cosimplicial object $(A \otimes K) \otimes (L \times \Delta[-])$ is a cosimplicial frame on $(A \otimes K) \otimes L$, and so there is a map

$$A \otimes (K \times (L \times \Delta[-])) \to (A \otimes K) \otimes (L \times \Delta[-])$$

which is the associativity isomorphism in degree 0. This map induces a weak equivalence

$$A \otimes (K \times (L \times M)) \to (A \otimes K) \otimes (L \times M)$$

and hence an isomorphism of the total derived functors, which we claim is the isomorphism obtained from the associativity weak equivalence

$$A \otimes (K \times (L \times M)) \xrightarrow{a_{A,K,L\times M}} (A \otimes K) \otimes (L \times M).$$

To see this, note that the associativity weak equivalence is natural in the last variable, and therefore commutes with colimits. Hence $a_{A,K,L\times M}$ is the colimit of the maps

$$A \otimes (K \times (L \times \Delta[n])) \to (A \otimes K) \otimes (L \times \Delta[n])$$

for $\Delta[n] \to M$ running though the simplices of M. It follows that $a_{A,K,L\times M}$ is induced by some map of cosimplicial frames

$$A \otimes (K \times (L \times \Delta[-])) \to (A \otimes K) \otimes (L \times \Delta[-])$$

covering the associativity isomorphism, and then Lemma 5.5.2 implies that it doesn't matter which one we pick.

Then the map $(A \otimes^L K) \otimes^L (L \times M) \to ((A \otimes^L K) \otimes^L L) \otimes^L M$ is induced by a map of cosimplicial frames $(A \otimes K) \otimes (L \times \Delta[-]) \to ((A \otimes K) \otimes L)^{\circ}$ which is the identity in degree 0. Hence the clockwise composite in the four-fold associativity diagram is induced by a map of cosimplicial frames covering the associativity weak equivalence, as claimed. $\square$

If $\mathcal{C}$ is actually a simplicial model category, then we already have an action of $\operatorname{Ho}\mathbf{SSet}$ on $\operatorname{Ho}\mathcal{C}$ induced by the simplicial structure. We will see in the next section that these two actions are naturally isomorphic. The point is that, if $A \in \mathcal{C}$ is cofibrant, then $A \otimes \Delta[-]$ is a cosimplicial frame on A and, if A is fibrant, $\operatorname{Hom}_r(\Delta[-], A)$ is a simplicial frame on A.

5.6. Naturality

In this section, we show that the closed action of $\operatorname{Ho}\mathbf{SSet}$ defined in the last few sections on $\operatorname{Ho}\mathcal{C}$ for any model category $\mathcal{C}$ is in fact preserved by Quillen adjunctions. This means that the homotopy pseudo-2-functor can be lifted to a pseudo-2-functor from the 2-category of model categories to the 2-category of closed $\operatorname{Ho}\mathbf{SSet}$-modules. We also show that there is a similar pseudo-2 functor from monoidal model categories to closed $\operatorname{Ho}\mathbf{SSet}$-algebras. We would like to assert that the homotopy category of a monoidal model category is in fact a *central* closed $\operatorname{Ho}\mathbf{SSet}$-algebra, but we have been unable to prove that in general. This does hold in every example we know of, however. This section is rather technical, especially near the end.

LEMMA 5.6.1. *Suppose $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction of model categories, A is a cofibrant object of $\mathcal{C}$, A^* is a cosimplicial frame on A, Y is a fibrant object of $\mathcal{D}$, and Y_* is a simplicial frame on Y. Then FA^* is a cosimplicial frame on FA and UY_* is a simplicial frame on UY.*

PROOF. Since F commutes with colimits, we have $L_n(FA^*) \cong F(L_n A^*)$. Since F preserves cofibrations, it follows that the map $L_n(FA^*) \to FA^*[n]$ is a cofibration for positive n. It is also a cofibration for $n = 0$ since A, and hence FA, is cofibrant. The map $FA^*[n] \to FA$ is a weak equivalence since F preserves weak equivalences between cofibrant objects, by Ken Brown's lemma 1.1.12. The simplicial case is dual. $\square$

THEOREM 5.6.2. *The homotopy pseudo-2-functor of Theorem 1.4.3 can be lifted to a pseudo-2-functor* $\mathbf{Ho}\colon \mathbf{Mod} \to \operatorname{Ho}\mathbf{SSet}$*-Mod which commutes with the duality 2-functor. The resulting pseudo-2-functor from simplicial model categories to closed* $\operatorname{Ho}\mathbf{SSet}$*-modules is naturally isomorphic to the pseudo-2-functor of Theorem 4.3.4.*

Implicit in the statement of Theorem 5.6.2 is a notion of natural isomorphism of pseudo-2-functors. A natural isomorphism of pseudo-2-functors is the same thing as a natural isomorphism of functors, except it must also preserve 2-morphisms. That is, given pseudo-2-functors F and G, we need a natural isomorphism $FX \xrightarrow{\tau_X} GX$ such that, given a 2-morphism $\alpha\colon f \to g$ of morphisms from X to Y, we have $1_{\tau_Y} * F\alpha = G\alpha * \tau_X$. Here $*$ denotes the horizontal composition of 2-morphisms, as in Section 1.4.

PROOF. Suppose $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction of model categories. We first show that $(LF, RU, R\varphi)$ is a morphism of closed $\operatorname{Ho}\mathbf{SSet}$-modules. To do this, we need to construct a coherent natural isomorphism $m\colon (LF)(A \otimes^L K) \xrightarrow{\cong} (LF)A \otimes^L K$.

Suppose A is a cofibrant object of $\mathcal{C}$. Then $F(A^\circ)$ is a cosimplicial frame on FA, so by Lemma 5.5.1, there is a map of cosimplicial frames $F(A^\circ) \to (FA)^\circ$ over FA. This map induces a weak equivalence, natural in K,

$$F(A \otimes K) \underset{\simeq}{\xrightarrow{m}} F(A^\circ) \otimes K \to FA \otimes K.$$

Here we are using the fact that F commutes with colimits. We therefore get an isomorphism, natural in K, $(LF)(A \otimes^L K) \to (LF)A \otimes^L K$ in the homotopy category. To be precise, this isomorphism is the composite

$$FQ(QA \otimes QK) \underset{\cong}{\xrightarrow{Fq}} F(QA \otimes QK) \underset{\cong}{\xrightarrow{m}} FQA \otimes QK \underset{\cong}{\xrightarrow{q^{-1}\otimes 1}} QFQA \otimes QK$$

We must show that this isomorphism in natural in A and makes the necessary coherence diagrams commute. Suppose we have a map $A \to B$. Then we get a possibly non-commutative square

$$\begin{tikzcd}
F(A^\circ) \arrow[r] \arrow[d] & (FA)^\circ \arrow[d] \\
F(B^\circ) \arrow[r] & (FB)^\circ
\end{tikzcd}$$

This square does commute in degree 0, however. It follows from Lemma 5.5.2 that m is natural in A as well as K.

The unit coherence diagram follows from the fact that the map $F(A^\circ) \to (FA)^\circ$ is the identity in degree 0. The associativity coherence diagram is the following.

$$\begin{tikzcd}
(LF)(A \otimes^L (K \times L)) \arrow[r, "m"] \arrow[d, "Fa"'] & (LF)A \otimes^L (K \times L) \arrow[r, "a"] & ((LF)A \otimes^L K) \otimes^L L \arrow[d, equal] \\
(LF)((A \otimes^L K) \otimes^L L) \arrow[r, "m"] & (LF)(A \otimes^L K) \otimes^L L \arrow[r, "m\otimes 1"] & ((LF)A \otimes^L K) \otimes^L L
\end{tikzcd}$$

Similarly to the proof of Theorem 5.5.3, we claim that each of these composites is induced by a map of cosimplicial frames $F(A \otimes (K \times \Delta[-])) \to (FA \otimes K)^\circ$ over the weak equivalence $m\colon F(A \otimes K) \to FA \otimes K$ (for cofibrant A, of course). It will then follow from Lemma 5.5.2 that the associativity coherence diagram commutes.

The counterclockwise composite is induced by the composite

$$F(A \otimes (K \times \Delta[-])) \to F((A \otimes K)^\circ) \to (F(A \otimes K))^\circ \xrightarrow{m^\circ} (FA \otimes K)^\circ$$

which is a map of cosimplicial frames covering m. The clockwise composite is induced by the composite

$$F(A \otimes (K \times \Delta[-])) \xrightarrow{m} FA \otimes (K \times \Delta[-]) \to (FA \otimes K)^\circ$$

which is another map of cosimplicial frames covering m, as required.

We have now shown that $(LF, RU, R\varphi)$ is a morphism of closed $\operatorname{Ho}\mathbf{SSet}$-modules. We must still show that this multiplication isomorphism m is pseudo-functorial. That is, we must show that the following diagram commutes.

$$\begin{tikzcd}
LG(LF(A\otimes^L K)) \arrow[r, "LG(m_F)"] \arrow[d, "\cong"'] & LG(LFA\otimes^L K) \arrow[r, "m_G"] & LG(LFA)\otimes^L K \arrow[d, "\cong"'] \\
L(GF)(A\otimes^L K) \arrow[r, "m_{GF}"] & L(GF)A\otimes^L K \arrow[r, equal] & L(GF)A\otimes^L K
\end{tikzcd}$$

where G and F are two composable left Quillen functors. The counterclockwise composite in this diagram is induced by a map of cosimplicial frames $GF(A^\circ)\to(GFA)^\circ$ over GFA. The clockwise composite is induced by $GF(A^\circ)\to G((FA)^\circ)\to(GFA)^\circ$, which is also a map of cosimplicial frames over GFA. It follows from Lemma 5.5.2 that this diagram commutes.

We must now show that if $\tau\colon F\to F'$ is a natural transformation of Quillen adjunctions (F,U,φ) and (F',U',φ') from $\mathcal{C}$ to $\mathcal{D}$, then $L\tau$ preserves the isomorphism m. That is, we must show the diagram

$$\begin{tikzcd}
LF(A\otimes^L K) \arrow[r, "m"] \arrow[d, "\tau"'] & LFA\otimes^L K \arrow[d, "\tau\otimes 1"'] \\
LF'(A\otimes^L K) \arrow[r, "m"] & LF'A\otimes^L K
\end{tikzcd}$$

commutes. But, as usual, each of these composites is induced by maps of cosimplicial frames $F(A^\circ)\to(F'A)^\circ$ covering τ, so the diagram commutes by Lemma 5.5.2. The claim about duality just follows from the fact that a cosimplicial frame on an object of $\mathcal{C}$ is the same thing as a simplicial frame of that object of $\mathcal{C}^{\mathrm{op}}$.

Finally, suppose $\mathcal{C}$ is already a simplicial model category. We claim that the identity map of $\operatorname{Ho}\mathcal{C}$ is a natural isomorphism between the pseudo-2-functor of Theorem 4.3.4 and the pseudo-2-functor just constructed. We must then define a natural isomorphism $A\otimes^L K\to A^\circ\otimes^L K$, where $A^\circ\otimes^L K$ denotes the action coming from the framing. We may as well assume that A is cofibrant. By Lemma 5.5.1, there is a map of cosimplicial frames $A\times\Delta[-]\to A^\circ$ covering the identity. This map induces our required isomorphism, and Lemma 5.5.2 guarantees that this makes the identity functor into a closed $\operatorname{Ho}\mathbf{SSet}$-module isomorphism. We must still show that it is natural for simplicial Quillen adjunctions (F,U,φ). Certainly the underlying functor LF is the same whether we think of $\mathcal{C}$ as a simplicial model category or not, but we must show that the diagram

$$\begin{tikzcd}
FA\otimes^L K \arrow[r] \arrow[d, "m"'] & (FA)^\circ\otimes^L K \arrow[d, "m"'] \\
F(A\otimes^L K) \arrow[r] & F(A^\circ\otimes^L K)
\end{tikzcd}$$

is commutative. Now, we have a diagram of cosimplicial frames

$$\begin{tikzcd}
F(A\otimes\Delta[-]) \arrow[r] \arrow[d] & F(A^\circ) \arrow[d] \\
FA\otimes\Delta[-] \arrow[r] & (FA)^\circ
\end{tikzcd}$$

using Lemma 5.5.1. This diagram does not commute, but it does commute in degree 0. Hence Lemma 5.5.2 guarantees that the first diagram commutes. Thus the identity functor is a natural isomorphism of closed Ho **SSet**-modules. It remains to show that this natural isomorphism is compatible with 2-morphisms. But this is automatic, since a 2-morphism of Ho **SSet**-modules is just a natural transformation of functors which happens to preserve the structure. Since the natural isomorphism in question is the identity, the underlying 2-morphisms will be the same. □

If $\mathcal{C}$ is a model category, we have just seen that $\operatorname{Ho}\mathcal{C}$ is naturally a closed Ho **SSet**-module. This means that we have functors $L_X\colon \operatorname{Ho}\mathbf{SSet} \to \operatorname{Ho}\mathcal{C}$ and $R_K\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$ for any object X of $\mathcal{C}$ and any simplicial set K. These functors realize left and right "multiplication". However, we do not understand these functors equally well. Indeed, L_X is the total derived functor of the Quillen functor $QX \otimes -$, so must be a Ho **SSet**-module functor. The associated isomorphism $L_X(K) \otimes^L L \to L_X(K \times L)$ is nothing more than the associativity isomorphism. On the other hand, R_K is not the total derived functor of a Quillen functor in general, so *a priori* we do not know if R_K is a Ho **SSet**-module functor.

LEMMA 5.6.3. *Suppose $\mathcal{C}$ is a model category and K is a simplicial set. Let $R_K\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$. denote the functor $R_K(X) = X \otimes^L K$. Define a natural isomorphism $m\colon R_K X \otimes^L L \to R_K(X \otimes^L L)$ as the composite*

$$(X \otimes^L K) \otimes^L L \xrightarrow{a} X \otimes^L (K \times L) \xrightarrow{1 \otimes T} X \otimes^L (L \times K) \xrightarrow{a^{-1}} (X \otimes^L L) \otimes^L K$$

Then R_K is a Ho **SSet***-module functor with this structure.*

PROOF. It suffices to check that the appropriate coherence diagrams commute. This is a long diagram chase which we leave to the reader. It involves the naturality of a, the four-fold associativity diagram, and the coherence of commutativity and associativity in Ho **SSet**. □

REMARK 5.6.4. Suppose $\mathcal{C}$ is a simplicial model category, and K is a simplicial set. Then any map of cosimplicial frames $A \otimes \Delta[-] \to A^\circ$ covering the identity gives a natural isomorphism $A \otimes^L K \to A^\circ \otimes^L K = R_K(A)$. Thus R_K is the total left derived functor of a Quillen functor, so it is a Ho **SSet**-module functor with a possibly different structure map m'. We claim that m' and m are equal. The proof of this claim is another argument with cosimplicial frames. Indeed, m' is induced by any map of cosimplicial frames $(A \otimes \Delta[-]) \otimes K \to (A \otimes K)^\circ$ covering the identity. Associating and twisting as in the definition of m above each give maps of cosimplicial frames covering the identity, so m and m' are equal.

Now, we can think of the associativity isomorphism as a natural transformation $L_{X \otimes K} \to L_X \circ L_K$. The four-fold associativity coherence diagram merely states that this natural transformation is a Ho **SSet**-module natural transformation of Ho **SSet**-module functors.

One can prove by long diagram chases that the associativity isomorphism is also a Ho **SSet**-module natural transformation when thought of as a natural transformation $R_L \circ L_X(K) \to L_X \circ R_L(K)$ or when thought of as a natural transformation $R_L \circ R_K(X) \to R_{K \times L}(X)$. We leave these diagram chases to the reader. These diagram chases and Lemma 5.6.3 go a long way towards putting L_X and R_K on an equal footing, but they do not go far enough, as we will soon see.

Now suppose $\mathcal{C}$ is a monoidal model category. Then $\operatorname{Ho}\mathcal{C}$ is a closed monoidal category, and also a closed module over Ho **SSet**. Clearly these operations must be compatible in an appropriate sense. We have the following theorem.

THEOREM 5.6.5. *The restriction of the homotopy pseudo-2-functor to the 2-category of monoidal model categories lifts to a pseudo-2-functor to the 2-category of closed* $\operatorname{Ho}\mathbf{SSet}$*-algebras. The resulting pseudo-2-functor from monoidal* $\mathbf{SSet}$*-model categories to closed* $\operatorname{Ho}\mathbf{SSet}$*-algebras is naturally isomorphic to the pseudo-2-functor of Theorem 4.3.4.*

PROOF. This theorem is a purely formal consequence of the results we have already proven. Indeed, given a monoidal model category $\mathcal{C}$, we know already that $\operatorname{Ho}\mathcal{C}$ is a closed monoidal category and a closed $\operatorname{Ho}\mathbf{SSet}$-module. There is then a functor $i\colon \operatorname{Ho}\mathbf{SSet} \to \operatorname{Ho}\mathcal{C}$ defined by $i(K) = S \otimes^L K$. It is clear that i is a left adjoint, since its right adjoint is the functor $R\operatorname{Map}_\ell(S, -)$. We must show that i is monoidal. The map $\alpha\colon i(S) \to S$ is the map r_S, where r is the unit isomorphism of the $\operatorname{Ho}\mathbf{SSet}$-module structure.

To construct the multiplicativity isomorphism $\mu\colon iK \otimes^L iL \to i(K \times L)$, we must do a little work. The functor $L'_X\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$ defined by $L'_X(Y) = X \otimes^L Y$ is the total left derived functor of a left Quillen functor. Hence it respects the $\operatorname{Ho}\mathbf{SSet}$-module structure. That is, there is a coherent natural isomorphism

$$m^\ell_{X,Y,K}\colon (X \otimes^L Y) \otimes^L K \to X \otimes^L (Y \otimes^L K).$$

Here coherence means a four-fold associativity diagram involving X, Y, K, and L, and the associativity isomorphism a of the $\operatorname{Ho}\mathbf{SSet}$-module structure, commutes. It also means a simpler diagram involving the unit isomorphism r commutes. Similarly, there is a coherent natural isomorphism

$$m^r_{X,Y,K}\colon (X \otimes^L Y) \otimes^L K \to (X \otimes^L K) \otimes^L Y$$

using the functor R'_Y.

We now define $\mu\colon iK \otimes^L iL \to i(K \times L)$ as the composite

$$\begin{aligned}(S \otimes^L K) \otimes^L (S \otimes^L L) &\xrightarrow{(m^\ell_{S\otimes K,S,L})^{-1}} ((S \otimes^L K) \otimes^L S) \otimes^L L \\ &\xrightarrow{r'_{S\otimes K}\otimes 1} (S \otimes^L K) \otimes^L L \xrightarrow{a} S \otimes^L (K \times L)\end{aligned}$$

Here r' is the right unit isomorphism of the closed category $\operatorname{Ho}\mathcal{C}$. We must now check that the required coherence diagrams commute, making i into a monoidal functor.

The left unit isomorphism ℓ of the closed category $\operatorname{Ho}\mathcal{C}$ is a $\operatorname{Ho}\mathbf{SSet}$-module natural transformation $L'_S(X) \to X$, since it is the derived natural transformation of a natural transformation of left Quillen functors. This gives a coherence diagram that says $\ell_{X\otimes K} \circ m^\ell_{S,X,K} = \ell_X$. Similarly, we have $r'_{X\otimes K} \circ m^r_{X,S,K} = r'_X$, where r' is the right unit isomorphism of $\operatorname{Ho}\mathcal{C}$.

The associativity isomorphism A of the closed category $\operatorname{Ho}\mathcal{C}$ can be thought of as a natural transformation $L'_{X\otimes Y}(Z) \to L'_X L'_Y(Z)$, as a natural transformation $R'_Z L'_X(Y) \to L'_X R'_Z(Y)$, or as a natural transformation $R'_Z R'_Y(X) \to R'_{Y\otimes Z}(X)$. In any of these cases, it is the total derived natural transformation of a natural transformation of Quillen functors. It is therefore a $\operatorname{Ho}\mathbf{SSet}$-module natural transformation. This gives us three large associativity coherence diagrams, which we leave to the reader to write down.

Using these coherence diagrams, it is not difficult, though it is long, to check that i is a monoidal functor. We leave this check to the reader. We also leave to the reader the check that the closed $\operatorname{Ho}\mathbf{SSet}$-module structure on $\operatorname{Ho}\mathcal{C}$ induced by i is naturally isomorphic to the one we started with.

Now, suppose $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is a monoidal Quillen functor. We must show that $(LF, RU, R\varphi)$ is a closed $\operatorname{Ho}\mathbf{SSet}$-algebra map. We know already that LF is a closed monoidal functor and a $\operatorname{Ho}\mathbf{SSet}$-module functor. It suffices to construct a natural isomorphism $\rho\colon (LF)i_{\mathcal{C}}K \to i_{\mathcal{D}}K$ of monoidal functors. The natural transformation ρ is the composite

$$(LF)(S \otimes^L K) \xrightarrow{m^{-1}} (LF)(S) \otimes^L K \xrightarrow{\alpha\otimes 1} S \otimes^L K$$

where m is the isomorphism realizing LF as a $\operatorname{Ho}\mathbf{SSet}$-module functor, and α is the unit isomorphism of the closed functor LF. We leave it to the reader to verify that ρ is compatible with the multiplicativity isomorphisms.

Now suppose we have a natural transformation $\tau\colon F \to F'$ between monoidal Quillen functors. Then $L\tau$ is a natural transformation of monoidal functors and a $\operatorname{Ho}\mathbf{SSet}$-module natural transformation. Hence $L\tau$ is compatible with both m and α, so will be compatible with ρ as well. Thus $L\tau$ is a $\operatorname{Ho}\mathbf{SSet}$-algebra natural transformation, as required.

As usual we leave it to the reader to check that we do get a pseudo-2-functor with these definitions. We also leave to the reader the check that, if $\mathcal{C}$ is a monoidal $\mathbf{SSet}$-model category, the identity functor gives a natural isomorphism from the pseudo-2-functor of Theorem 4.3.4 to the one just constructed. □

We now come back to the functor R_K considered in Lemma 5.6.3. When $\mathcal{C}$ is a monoidal model category, there is a natural isomorphism $R_{iK} \to R_K$, given by the composite

$$X \otimes^L (S \otimes^L K) \xrightarrow{(m^{\ell}_{X,S,K})^{-1}} (X \otimes^L S) \otimes^L K \xrightarrow{r'_X\otimes 1} X \otimes^L K$$

This composite is induced by a map of cosimplicial frames $QX \otimes (QS)^{\circ} \to (QX)^{\circ}$ covering the weak equivalence $QX \otimes QS \xrightarrow{1\otimes q} QX \otimes S \xrightarrow{r} QX$.

The functor R_{iK} is the total derived functor of a Quillen functor, so is a $\operatorname{Ho}\mathbf{SSet}$-module functor. The functor R_K is also a $\operatorname{Ho}\mathbf{SSet}$-module functor, as we have seen in Lemma 5.6.3. We make the following conjecture.

Conjecture 5.6.6. *Suppose $\mathcal{C}$ is a monoidal model category and K is a simplicial set. The natural isomorphism $R_{iK} \to R_K$ defined above is an isomorphism of* $\operatorname{Ho}\mathbf{SSet}$*-module functors.*

This conjecture is certainly technical, but it is also important, as we will see below.

Proposition 5.6.7. *Suppose $\mathcal{C}$ is a monoidal model category, T is a cofibrant replacement for the unit S, and T^* is a cosimplicial frame on T equipped with a natural transformation $T^*[m] \otimes T^*[n] \to T^* \otimes (\Delta[m] \times \Delta[n])$ extending the obvious isomorphism when m or n is 0. Then $\mathcal{C}$ satisfies Conjecture 5.6.6.*

Proof. Define a left framing on $\mathcal{C}$ by $X \mapsto X^* = X \otimes T^*$. Note that X^* is a cosimplicial frame on X when X is cofibrant, but may not be in general. Note also that this framing commutes with colimits. Define $X * K = (X \otimes T^*) \otimes K$. Then, as a functor of X, $X * K$ commutes with colimits and preserves cofibrations and trivial cofibrations. Indeed, to see that $- * K$ commutes with colimits, simply commute colimits past one another. To see that $- * K$ preserves cofibrations and trivial cofibrations, note that if $X \to Y$ is a (trivial) cofibration, then $X \otimes T^* \to Y \otimes T^*$ is a (trivial) Reedy cofibration. Now use Proposition 5.4.1. It follows that $- * K$ has a total derived functor $- *^L K$, and that this functor is a $\operatorname{Ho}\mathbf{SSet}$-module functor.

In fact, $- *^L K$ is naturally isomorphic to $- \otimes^L (S \otimes^L K)$ by a $\operatorname{Ho}\mathbf{SSet}$-module natural isomorphism. The corresponding natural isomorphism $X *^L K \to X \otimes^L K$ is induced by a map of cosimplicial frames $QX \otimes T^* \to (QX)^\circ$ covering the weak equivalence $QX \otimes T \to QX$. We must check that this natural isomorphism is a $\operatorname{Ho}\mathbf{SSet}$-module natural isomorphism. That is, we must check that the diagram

$$\begin{tikzcd}
(X \otimes^L L) *^L K \arrow[r] \arrow[d] & (X *^L K) \otimes^L L \arrow[r] & (X \otimes^L K) \otimes^L L \\
(X \otimes^L L) \otimes^L K & X \otimes^L (K \times L) \arrow[l] \arrow[r] & X \otimes^L (L \times K) \arrow[u]
\end{tikzcd}$$

is commutative. We have drawn the isomorphisms in this diagram in the direction corresponding to the maps of cosimplicial frames which induce them. It is the fact that these directions do not match up well that prevents us from proving this diagram commutes in general. However, it clearly suffices to prove this diagram commutes when we replace the upper left corner by $(X *^L L) *^L K$. By taking colimits, the map $T^*[m] \otimes T^*[n] \to T^*(\Delta[m] \times \Delta[n])$ induces a map of cosimplicial frames $(X \otimes T^*) * K \to X *^L (K \times \Delta[-])$, and so a natural isomorphism $(X *^L) *^L K \to X *^L (K \times L)$. Using this isomorphism, we find that we only need check that the following diagram commutes.

$$\begin{tikzcd}
(X * L) * K \arrow[r] \arrow[d] & (X \otimes L) * K \arrow[r] & (X * K) \otimes L \arrow[r, equal] & (X * K) \otimes L \arrow[d] \\
X * (K \times L) \arrow[r] & X \otimes (K \times L) \arrow[r] & X \otimes (L \times K) \arrow[r] & (X \otimes K) \otimes L
\end{tikzcd}$$

We have removed the superscript "L" from this diagram for reasons of space. Both composites in this diagram are induced by maps of cosimplicial frames covering the weak equivalence $QX * K \to QX \otimes K$, and so they are equal. □

Corollary 5.6.8. *Suppose $\mathcal{C}$ is a monoidal* **SSet***-model category. Then $\mathcal{C}$ satisfies Conjecture 5.6.6.*

Proof. The cosimplicial frame $S \otimes \Delta[-]$ satisfies the hypothesis of Proposition 5.6.7. (The unit S is automatically cofibrant). □

Remark 5.6.9. Notice that, if T^* is a cosimplicial frame in $\mathcal{C}$ satisfying the hypothesis of Proposition 5.6.7, and if $F\colon \mathcal{C} \to \mathcal{D}$ is a monoidal Quillen functor, then FT^* is also a cosimplicial frame satisfying the hypothesis of Proposition 5.6.7. Therefore any monoidal $\mathcal{C}$-model category will satisfy Conjecture 5.6.6.

Corollary 5.6.10. *Every monoidal* $\operatorname{Ch}(\mathbf{Z})$*-model category satisfies Conjecture 5.6.6.*

Proof. In view of Remark 5.6.9, it suffices to construct a cosimplicial frame S^* on the unit S in $\operatorname{Ch}(\mathbf{Z})$ with a natural map $S^*[m] \otimes S^*[n] \to S^* \otimes (\Delta[m] \times \Delta[n])$. We define $S^*[m]_k$ to be the free abelian group on the nondegenerate k-simplices of $\Delta[m]$, with the boundary map defined as the alternating sum of the faces. For example, $S^*[1]$ is $\mathbf{Z}$ in degree 1 and $\mathbf{Z} \oplus \mathbf{Z}$ in degree 0, with the boundary map taking 1 to $(1, -1)$. One can easily check that this defines a cosimplicial frame on S. The map $S^*[m] \otimes S^*[n] \to S^* \otimes (\Delta[m] \times \Delta[n])$ is the Eilenberg-Zilber map. □

Note that all of the monoidal model categories we have considered in this book are either monoidal **SSet**-model categories or monoidal Ch(**Z**)-model categories.

Conjecture 5.6.6 is equivalent to asserting that the diagram below commutes, where we have dropped the superscript on the tensor product.

$$\begin{array}{ccccc}
(X\otimes(S\otimes K))\otimes L & \xrightarrow{m^r} & (X\otimes L)\otimes(S\otimes K) & \xrightarrow{(m^\ell)^{-1}} & ((X\otimes L)\otimes S)\otimes K \\
{\scriptstyle (m^\ell)^{-1}\otimes 1}\downarrow & & & & \downarrow{\scriptstyle a} \\
((X\otimes S)\otimes K)\otimes L & \xrightarrow{a} & (X\otimes S)\otimes(K\times L) & \xrightarrow{1\otimes T_{K,L}} & (X\otimes S)\otimes(L\times K)
\end{array}$$

The importance of Conjecture 5.6.6 is made clear by the following theorem.

THEOREM 5.6.11. *The homotopy pseudo-2-functor of Theorem 5.6.5 lifts to a homotopy pseudo-2-functor from monoidal model categories satisfying Conjecture 5.6.6 to* central *closed* Ho **SSet**-*algebras.*

PROOF. Suppose $\mathcal{C}$ is a monoidal model category. We use the notation of Theorem 5.6.5. We define $t\colon\ iK\otimes^L X\to X\otimes^L iK$ as the composite

$$(S\otimes^L K)\otimes^L X \xrightarrow{(m^r_{S,X,K})^{-1}} (S\otimes^L X)\otimes^L K \xrightarrow{\ell_X\otimes 1} X\otimes^L K$$
$$\xrightarrow{(r'_X)^{-1}\otimes 1} (X\otimes^L S)\otimes^L K \xrightarrow{m^\ell_{X,S,K}} X\otimes^L(S\otimes^L K)$$

All of the coherence diagrams except the diagram

$$\begin{array}{ccc}
iK\otimes^L iL & \xrightarrow{t} & iL\otimes^L iK \\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\
i(K\times L) & \xrightarrow{iT} & i(L\times K)
\end{array}$$

commute using the coherence isomorphisms discussed in the proof of Theorem 5.6.5. Those coherence diagrams can be used to reduce this last diagram to the commutative diagram of Conjecture 5.6.6. Thus Ho $\mathcal{C}$ is a central Ho **SSet**-algebra with this structure if and only if Conjecture 5.6.6 holds for $\mathcal{C}$. It is then an extremely long diagram chase to check that a monoidal Quillen functor F induces a central Ho **SSet**-algebra functor. This diagram chase does not require Conjecture 5.6.6, but it does require realizing that the natural isomorphism $FX\otimes^L FY\to F(X\otimes^L Y)$ is a Ho **SSet**-module natural transformation. It also requires a great deal of patience, so we will leave it to the interested reader. □

Since Conjecture 5.6.6 is true for simplicial monoidal model categories, we get the following corollary.

COROLLARY 5.6.12. *The homotopy pseudo-2-functor can be lifted to a pseudo-2-functor from (not necessarily central) monoidal* **SSet**-*model categories to central closed* Ho **SSet**-*algebras. Similarly, the homotopy pseudo-2-functor can be lifted to a pseudo-2-functor from monoidal* Ch(**Z**)-*model categories to central closed* Ho **SSet**-*algebras.*

Another immediate corollary is the following.

COROLLARY 5.6.13. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from symmetric monoidal model categories satisfying Conjecture 5.6.6 to symmetric closed* Ho **SSet**-*algebras.*

Proof. Let T denote the commutativity isomorphism of $\operatorname{Ho}\mathcal{C}$, and t the centrality isomorphism. Let i denote the functor $\operatorname{Ho}\mathbf{SSet} \to \operatorname{Ho}\mathcal{C}$. We must show that $T_{iK,X} = t_{iK,X}$. This is yet another diagram chase. We use the fact that T is the derived natural transformation of a natural transformation of Quillen functors, so is a $\operatorname{Ho}\mathbf{SSet}$-module natural transformation. This means $m^{\ell}_{Y,X,K} \circ (T_{X,Y} \otimes 1) = T_{X\otimes K,Y} \circ m^{r}_{X,Y,K}$. Applying this to the definition of t, and using the fact that T changes the left unit to the right unit, we get the required result. $\square$

5.7. Framings on pointed model categories

In this section, we show that if $\mathcal{C}$ is a pointed model category, then $\operatorname{Ho}\mathcal{C}$ is a closed $\operatorname{Ho}\mathbf{SSet}_*$-module, where $\mathbf{SSet}_*$ denotes the category of pointed simplicial sets.

It follows from Corollary 3.1.6 that a cosimplicial frame A^* on an object A of a pointed model category $\mathcal{C}$ induces an adjunction $(A^* \wedge -, \mathcal{C}(A^*, -), \varphi)$ from $\mathbf{SSet}_*$ to $\mathcal{C}$, whose restriction to $\mathbf{SSet}$ is the adjunction considered in Section 5.4. That is, we have $A^* \wedge K_+ \cong A^* \otimes K$. Similarly, a simplicial frame Y_* on an object Y induces an adjunction $(\operatorname{Hom}_*(-, Y_*), \mathcal{C}(-, Y_*), \varphi)$ from $\mathbf{SSet}^{\mathrm{op}}$ to $\mathcal{C}$. Again, we have $\operatorname{Hom}_*(K_+, Y_*) \cong \operatorname{Hom}(K, Y_*)$.

By Remark 3.1.7, the framing of Theorem 5.2.8 induces a bifunctor $\mathcal{C}\times\mathbf{SSet}_* \to \mathcal{C}$, denoted $(A, K) \mapsto A\wedge K$, and an adjoint $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{SSet}_*$, denoted by $(A, Y) \mapsto \operatorname{Map}_{*\ell}(A, Y)$. Furthermore, we have $A \wedge K_+ \cong A \otimes K$ for an unpointed simplicial set K. Dually, we also get a bifunctor $\mathbf{SSet}_*^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, denoted by $(K, Y) \mapsto \operatorname{Hom}_*(K, Y)$, and an adjoint $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{SSet}_*$ denoted by $(A, Y) \mapsto \operatorname{Map}_{*r}(A, Y)$.

The results of Section 5.4 go through almost without change in the pointed case.

Proposition 5.7.1. *Let $\mathcal{C}$ be a pointed model category. Suppose $f\colon A^\bullet \to B^\bullet$ is a cofibration in $\mathcal{C}^\Delta$ with respect to the Reedy model structure, and $g\colon K \to L$ is a cofibration of pointed simplicial sets. Then the induced map $f \,\square\, g\colon (A^\bullet \wedge L) \amalg_{A^\bullet\wedge K} (B^\bullet \wedge K) \to B^\bullet \wedge L$ is a cofibration in $\mathcal{C}$, which is trivial if f is. Dually, if $p\colon Y_\bullet \to Z_\bullet$ is a fibration in $\mathcal{C}^{\Delta^{\mathrm{op}}}$, the map $\operatorname{Hom}_\square(g, p)\colon \operatorname{Hom}_*(L, Y_\bullet) \to \operatorname{Hom}_*(K, Y_\bullet) \times_{\operatorname{Hom}_*(K,Z_\bullet)} \operatorname{Hom}_*(L, Z_\bullet)$ is a fibration which is trivial if p is.*

Proof. We can assume g is the map $\partial\Delta[n]_+ \to \partial\Delta[n]$, in which case the proposition follows from Proposition 5.4.1. $\square$

Proposition 5.7.2. *Suppose $\mathcal{C}$ is a pointed model category, and $f\colon A^* \to B^*$ is a cofibration of cosimplicial frames of A and B respectively. Suppose in addition A, and hence B, are cofibrant. Then, if $g\colon K \to L$ is a trivial cofibration of pointed simplicial sets, the induced map $f\square g\colon (A^*\wedge L)\amalg_{A^*\wedge K}(B^*\wedge K) \to B^*\wedge L$ is a trivial cofibration. Dually, if $p\colon Y_* \to Z_*$ is a fibration of simplicial frames on fibrant objects, then $\operatorname{Hom}_\square(g, p)\colon \operatorname{Hom}_*(L, Y_*) \to \operatorname{Hom}_*(K, Y_*) \times_{\operatorname{Hom}_*(K,Z_*)} \operatorname{Hom}_*(L, Z_*)$ is a trivial fibration.*

Proof. We can assume g is one of the maps $\Lambda^r[n]_+ \to \Delta[n]_+$, in which case the proposition follows from Proposition 5.4.3. $\square$

We then get pointed analogs of the rest of the results of Section 5.4 without difficulty, though one must check that the zig-zag of weak equivalences from $\mathcal{C}(A^*, Y)$ to $\mathcal{C}(A, Y_*)$ preserves the basepoint. The results of Section 5.5 and Section 5.6 extend to the pointed case with no difficulty.

We then get the following theorems, whose proofs are the same as the proofs of Theorem 5.6.2 and Theorem 5.6.5 respectively. Note that the 2-category of pointed model categories is just the full sub-2-category whose objects are pointed model categories, and similarly for the 2-category of pointed monoidal model categories.

THEOREM 5.7.3. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from pointed model categories to closed* $\operatorname{Ho}\mathbf{SSet}_*$*-modules which commutes with the duality 2-functor.*

THEOREM 5.7.4. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from pointed monoidal model categories to closed* $\operatorname{Ho}\mathbf{SSet}_*$*-algebras.*

We make the analogous conjecture to Conjecture 5.6.6 as well.

CONJECTURE 5.7.5. *Suppose* $\mathcal{C}$ *is a pointed monoidal model category and* K *is a pointed simplicial set. The natural isomorphism* $R_{iK} \to R_K$ *defined as in Conjecture 5.6.6 is an isomorphism of* $\operatorname{Ho}\mathbf{SSet}_*$*-module functors.*

Then the analog of Proposition 5.6.7 goes through without difficulty, and so all monoidal $\mathbf{SSet}_*$-model categories and all monoidal $\operatorname{Ch}(\mathbf{Z})$-model categories satisfy Conjecture 5.7.5.

THEOREM 5.7.6. *The homotopy pseudo-2-functor of Theorem 5.7.4 lifts to a homotopy pseudo-2-functor from pointed monoidal model categories satisfying Conjecture 5.7.5 to* central $\operatorname{Ho}\mathbf{SSet}_*$*-algebras.*

COROLLARY 5.7.7. *The homotopy pseudo-2-functor can be lifted to a pseudo-2-functor from (not necessarily central) monoidal* $\mathbf{SSet}_*$*-model categories to central closed* $\operatorname{Ho}\mathbf{SSet}_*$*-algebras. It can also be lifted to a functor from monoidal* $\operatorname{Ch}(\mathbf{Z})$*-model categories to central closed* $\operatorname{Ho}\mathbf{SSet}_*$*-algebras.*

COROLLARY 5.7.8. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from pointed symmetric monoidal model categories satisfying Conjecture 5.7.5 to symmetric closed* $\operatorname{Ho}\mathbf{SSet}_*$*-algebras.*

CHAPTER 6

Pointed model categories

We have just seen that the homotopy category of a model category is naturally a closed $\operatorname{Ho}\mathbf{SSet}$-module, and that the homotopy category of a pointed model category is naturally a closed $\operatorname{Ho}\mathbf{SSet}_*$-module. The homotopy category of a pointed model category has additional structure as well, as was pointed out by Quillen in [**Qui67**, Sections I.2 and I.3]. The purpose of this chapter is to study this additional structure.

We begin in Section 6.1 with the suspension and loop functors. These exist in any closed $\operatorname{Ho}\mathbf{SSet}_*$-module, but there are a number of results specific to the homotopy category of a pointed model category that we will need later. The results in this section are all proved in [**Qui67**, Section I.2]. In Section 6.2 we define the cofiber and fiber sequences in the homotopy category of a pointed model category, and in Section 6.3 we discuss some of their properties. These sections are both based on [**Qui67**, Section I.3], but they have some new features. In particular, we prove that Verdier's octahedral axiom holds, and we give a different version of the compatibility between cofiber and fiber sequences. In Section 6.4 we study the naturality of cofiber sequences. The main new feature in this section is that we show that the closed $\operatorname{Ho}\mathbf{SSet}_*$-module structure respects the cofiber and fiber sequences.

In Section 6.5 we pull together the properties of cofiber and fiber sequences to define a pre-triangulated category. A pre-triangulated category is, then, a closed $\operatorname{Ho}\mathbf{SSet}_*$-module together with some cofiber and fiber sequences which satisfy the same properties as those in the homotopy category of a pointed model category. The main point of these, for us, is to provide a 2-category in which the homotopy pseudo-2-functor can land. Finally, in Section 6.6, we define closed monoidal pre-triangulated categories and show that the homotopy category of a pointed monoidal model category is naturally such a thing.

6.1. The suspension and loop functors

This section is devoted to the study of the suspension and loop functors that exist in the homotopy category of a pointed model category. These functors were introduced in [**Qui67**]. We adopt a slightly different approach, using the framing constructed in Chapter 5.

Before giving the definition of the suspension and loop functors, we recall that in a pointed category with colimits and limits, we define the *cokernel*, or *cofiber*, of a map $f\colon X \to Y$ to be the coequalizer $g\colon Y \to Z$ of f and the zero map. In practice, we usually think of g as the pushout in the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

though most coequalizers are not pushouts. We usually abuse notation and just refer to the cokernel or cofiber Z. Similarly, the *kernel* or *fiber* of f is the equalizer of f and the zero map, or equivalently, the pullback of f through the zero map.

DEFINITION 6.1.1. Suppose $\mathcal{C}$ is a pointed model category. The *suspension functor* $\Sigma\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is the functor $X \mapsto X \wedge^L S^1$ defined by the closed action of $\operatorname{Ho}\mathbf{SSet}_*$ on $\operatorname{Ho}\mathcal{C}$ given in Section 5.7. Dually, the *loop functor* $\Omega\colon \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is the functor $X \mapsto R\operatorname{Hom}_*(S^1, X)$.

The suspension functor is of course left adjoint to the loop functor. Note that, by definition, $\Sigma X = QX \wedge S^1$. Recall that the pointed simplicial set S^1 is the cokernel of the map $\partial\Delta[1]_+ \to \Delta[1]_+$. Furthermore, $X \wedge \partial\Delta[1]_+ = X \otimes \partial\Delta[1] = X \vee X$. Also, $X \wedge \Delta[1]_+ = X \otimes \Delta[1] = X \times I$, the functorial cylinder object obtained from the functorial factorization. Hence ΣX is the cokernel in $\mathcal{C}$ of the map $QX \vee QX \to QX \times I$ including the two ends of the functorial cylinder object on QX. If X is cofibrant, ΣX is naturally isomorphic (in the homotopy category, of course) to the cokernel in $\mathcal{C}$ of the map $X \vee X \to X \times I$, using the pointed analog of Corollary 5.4.6. This is the original definition of the suspension given in [**Qui67**].

Dually, by definition, we have $\Omega X = \operatorname{Hom}_*(S^1, RX)$. Writing S^1 as a cokernel as above, we find that ΩX is the kernel in $\mathcal{C}$ of the map $(RX)^I \to RX \times RX$ projecting the canonical path object of RX onto its two ends. If X is fibrant, we do not have to apply R first. This is the original definition of the loop functor given in [**Qui67**].

The following lemma is extremely useful.

LEMMA 6.1.2. *Suppose $\mathcal{C}$ is a pointed model category, A is a cofibrant object, and Y is a fibrant object. Then we have natural isomorphisms*

$$\pi_t \operatorname{Map}_{*\ell}(A, Y) \cong \pi_t \operatorname{Map}_{*r}(A, Y) \cong [\Sigma^t A, Y] \cong [A, \Omega^t Y]$$

for all nonnegative integers t.

PROOF. Using associativity of the $\operatorname{Ho}\mathbf{SSet}_*$-action, we have a natural isomorphism $\Sigma^t X \cong X \wedge^L S^t$ in $\operatorname{Ho}\mathcal{C}$. Hence we have

$$[\Sigma^t X, Z] \cong [X \wedge^L S^t, Z] \cong [S^t, R\operatorname{Map}_{*\ell}(X, Z)] \cong \pi_t R\operatorname{Map}_{*\ell}(X, Z).$$

Since A is cofibrant and Y is fibrant, we then get $[\Sigma^t A, Y] \cong \pi_t \operatorname{Map}_{*\ell}(A, Y)$ as required. □

REMARK 6.1.3. Suppose A is cofibrant and X is fibrant in a pointed model category $\mathcal{C}$. We can use Lemma 6.1.2 to describe $[\Sigma A, X]$. An element of $\pi_1 \operatorname{Map}_{*r}(A, X)$ is represented by an unpointed map $\Delta[1] \xrightarrow{\widetilde{h}} \operatorname{Map}_{*r}(A, X)$ whose restrictions to $\Delta[0]$ are both 0. This corresponds, via adding a disjoint basepoint and adjointness, to a map $h : A \to X^{\Delta[1]} = X^I$ such that $p_0 h = p_1 h = 0$. Two such one-simplices $\widetilde{h}$ and $\widetilde{h}'$ give the same element of $\pi_1 \operatorname{Map}_{*r}(A, X)$ if and only if there is a map $\Delta[1] \times \Delta[1] \xrightarrow{\widetilde{H}} \operatorname{Map}_{*r}(A, X)$ such that $\widetilde{H}$ is $\widetilde{h}$ on $\Delta[1] \times \{0\}$, $\widetilde{h}'$ on $\Delta[1] \times \{1\}$, and

0 on $\partial\Delta[1] \times \Delta[1]$. Again using adjointness, we find that h and h' represent the same element of $[\Sigma A, X]$ if and only if there is a map $H\colon A \to X^{\Delta[1]\times\Delta[1]}$ such that $p_0^0 H = h$, $p_0^1 H = h'$, and $p_1^0 H = p_1^1 H = 0$. Here $p_0^0\colon X^{\Delta[1]\times\Delta[1]} \to X^{\Delta[1]}$ is the trivial fibration dual to the inclusion $\Delta[1] \times \{0\} \to \Delta[1] \times \Delta[1]$, and similarly for p_0^1, p_1^0, and p_1^1. Such a map H is like a right homotopy between the right homotopies h and h', though the definition we have just given is different from that of [**Qui67**, Section I.2].

In the next section we will need to know that a right homotopy between right homotopies induces a corresponding left homotopy between right homotopies. This lemma is our version of [**Qui67**, Lemma I.2.1].

LEMMA 6.1.4. *Suppose $\mathcal{C}$ is a pointed model category, A is cofibrant, and X is fibrant. Suppose $h.h'\colon A \to X^I$ satisfy $p_0 h = p_0 h' = p_1 h = p_1 h' = 0$. Then h and h' represent the same element of $[\Sigma A, X]$ if and only if there is a map $H\colon A \times I \to X^I$ such that $Hi_0 = h$, $Hi_1 = h'$, and $p_0 H = p_1 H = 0$.*

PROOF. By Remark 6.1.3, h and h' represent the same element of $[\Sigma A, X]$ if and only if there is a map $\widetilde{H}\colon A \to X^{\Delta[1]\times\Delta[1]}$ such that $p_0^0 H = h$, $p_0^1 H = h'$, and $p_1^0 H = p_1^1 H = 0$. Let

$$P = (\Delta[1] \times \{1\}) \amalg_{\partial\Delta[1]\times\{1\}} (\partial\Delta[1] \times \Delta[1])$$

Then the inclusion $P \to \Delta[1] \times \Delta[1]$ is a trivial cofibration of simplicial sets. Hence the dual map $X^{\Delta[1]\times\Delta[1]} \to X^P$ is a trivial fibration. Furthermore, X^P is a pullback of $X^{\Delta[1]\times\{1\}}$ and $X^{\partial\Delta[1]\times\Delta[1]}$. We then get a commutative diagram

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(\widetilde{H}, i_1^1 h')} & X^{\Delta[1]\times\Delta[1]} \\ {\scriptstyle (i_0,i_1)}\big\downarrow & & \big\downarrow \\ A \times I & \xrightarrow{(h's, 0)} & X^P \end{array}$$

Here i_0, i_1, and s are the structure maps of the functorial cylinder object $A \times I$, and i_1^1 is the map induced by the surjection $\Delta[1] \times \Delta[1] \to \Delta[1] \times \{1\}$. Hence there is a lift $G\colon A \times I \to X^{\Delta[1]\times\Delta[1]}$. Let H denote the map $p_0^0 G\colon A \times I \to X^{\Delta[1]}$. Then $Hi_0 = h$, $Hi_1 = h'$, and $p_0 H = p_1 H = 0$, as required.

Conversely, suppose we have such an H. Let Q denote the boundary of $\Delta[1] \times \Delta[1]$, which is the pushout of $\partial\Delta[1] \times \Delta[1]$ and $\Delta[1] \times \partial\Delta[1]$ over $\partial\Delta[1] \times \partial\Delta[1]$. Then we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{i_1^0 h} & X^{\Delta[1]\times\Delta[1]} \\ {\scriptstyle i_0}\big\downarrow & & \big\downarrow \\ A \times I & \xrightarrow{H'} & X^Q \end{array}$$

where H' is the map whose projection to $X^{\Delta[1]\times\{0\}}$ is hs, whose projection to $X^{\Delta[1]\times\{1\}}$ is H, and whose projection to $X^{\partial\Delta[1]\times\Delta[1]}$ is 0. As the left vertical map is a trivial cofibration and the right vertical map is a fibration, there is a lift G in this diagram. Then Gi_1 is the required map $\widetilde{H}\colon A \to X^{\Delta[1]\times\Delta[1]}$. □

Of course, $[\Sigma A, X]$ is also isomorphic to $\pi_1 \operatorname{Map}_{*\ell}(A, X)$. Hence an element of $[\Sigma A, X]$ also has a representative of the form $h\colon A \times I \to X$ where $hi_0 = hi_1 = 0$. There is a dual lemma to Lemma 6.1.4 as well. It will be important later to be able to tell when a map $h\colon A \times I \to X$ represents the same homotopy class as a map $k\colon A \to X^I$.

LEMMA 6.1.5. *Suppose $\mathcal{C}$ is a pointed model category, A is cofibrant, and X is fibrant. Suppose $h\colon A \times I \to X$ satisfies $hi_0 = hi_1 = 0$, and $k\colon A \to X^I$ satisfies $p_0 k = p_1 k = 0$. Then h and k represent the same element of $[\Sigma A, X]$ under the isomorphism $\pi_1 \operatorname{Map}_{*\ell}(A, X) \cong \pi_1 \operatorname{Map}_{*r}(A, X)$ if and only if there is a map $H\colon A \times I \to X^I$ such that $Hi_0 = k$, $Hi_1 = 0$, $p_0 H = h$, and $p_1 H = 0$. Such a map H is called a* correspondence *between h and k.*

PROOF. Recall that the isomorphism $\pi_1 \operatorname{Map}_{*\ell} \cong \pi_1 \operatorname{Map}_{*r}(A, X)$ is induced by the weak equivalences $\mathcal{C}(A^\circ, X) \to \operatorname{diag} \mathcal{C}(A^\circ, X_\circ) \leftarrow \mathcal{C}(A, Y_\circ)$. Let $r\colon X \to X^I$ and $s\colon A \times I \to A$ be structure maps of the functorial path and cylinder objects. Then if there is a homotopy in $\operatorname{diag} \mathcal{C}(A^\circ, Y_\circ)$ between rh and ks, then h and k represent the same homotopy class. We cannot assert the converse since $\operatorname{diag} \mathcal{C}(A^\circ, Y_\circ)$ need not be fibrant. Such a homotopy corresponds to a map $\Delta[1] \times \Delta[1] \to \operatorname{diag} \mathcal{C}(A^\circ, Y_\circ)$ with certain properties. This is equivalent to two 2-simplices H_0 and H_1 of $\operatorname{diag} \mathcal{C}(A^\circ, Y_\circ)$ with $d_0 H_0 = ks$, $d_1 H_0 = d_1 H_1$, $d_2 H_0 = d_0 H_1 = 0$, and $d_2 H_1 = rh$. These 2-simplices are actually maps $A \times \Delta[2] \to X^{\Delta[2]}$, and, for example, $d_0 H_0$ is really the map $Y^{d^0} \circ H_0 \circ (A \times d^0)$.

Let us suppose first that there is a correspondence $H\colon A \times I \to X^I$ such that $Hi_0 = k$, $Hi_1 = 0$, $p_0 H = h$, and $p_1 H = 0$. Define H_0 to be the composite $Y^{s^1} \circ H \circ (A \times s^0)$ and H_1 to be the composite $Y^{s^0} \circ H \circ (A \times s^1)$. It is an exercise in the simplicial identities to verify that H_0 and H_1 satisfy the required properties, so that h and k represent the same homotopy class.

Conversely, suppose h and k represent the same homotopy class. We first construct a correspondence between h and some map k'. To do so, let H' be a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "i_1"'] & X^I \arrow[d, "{(p_0, p_1)}"] \\ A \times I \arrow[r, "{(h,0)}"] & X \times X \end{tikzcd}$$

Then G is a correspondence between h and $H' i_0$, which we denote k'. Hence h and k' represent the same homotopy class, so k and k' also represent the same homotopy class. Lemma 6.1.4 then gives us a map $H''\colon A \times I \to X^I$ such that $p_0 H'' = p_1 H'' = 0$, $H'' i_0 = k$, and $H'' i_1 = k'$. We then have a commutative diagram

$$\begin{tikzcd} A \times \Lambda^1[2] \arrow[r, "G"] \arrow[d] & X^I \arrow[d, "{(p_0, p_1)}"] \\ A \times \Delta[2] \arrow[r, "{(h \circ (A \times s^1), 0)}"] & X \times X \end{tikzcd}$$

where G is the map which is H'' on $A \times d_2 i_2$ and H' on $A \times d_0 i_2$. Let F be a lift in this diagram. Then $F \circ (A \times d^1)$ is the required correspondence between h and k. $\square$

Now, recall that a *cogroup structure* on an object X of a (pointed) category $\mathcal{C}$ is a lift of the functor $\mathcal{C}(X, -)$ from $\mathcal{C}$ to (pointed) sets to a functor to groups.

When $\mathcal{C}$ has coproducts, a cogroup structure is equivalent to a counit map $X \longrightarrow 0$, a comultiplication map $X \longrightarrow X \amalg X$, and a co-inverse map $X \longrightarrow X$ which make a coassociativity diagram, a left and right counit diagram, and a left and right co-inverse diagram all commute. Dually, a *group structure* on an object X is a lift of the functor $\mathcal{C}(-, X)$ to groups. When $\mathcal{C}$ has products, a group structure on X is equivalent to a unit map $1 \longrightarrow X$, a multiplication map $X \times X \longrightarrow X$, and an inverse map $X \longrightarrow X$ making an associativity diagram, a left and right unit diagram and a left and right inverse diagram all commute. An object equipped with a cogroup structure is called a *cogroup object*, or just a cogroup, and an object equipped with a group structure is called a *group object*, or just a group. We have evident notions of homomorphisms of groups and cogroups as well.

COROLLARY 6.1.6. *Suppose $\mathcal{C}$ is a pointed model category. Then the iterated suspension functor Σ^t lifts to a functor to the category of cogroups in* $\operatorname{Ho}\mathcal{C}$ *and homomorphisms for $t \geq 1$. If $t \geq 2$, Σ^t lifts to a functor to the category of abelian cogroups in* $\operatorname{Ho}\mathcal{C}$ *and homomorphisms. Dually, the iterated loop functor Ω^t lifts to a functor to the category of groups in* $\operatorname{Ho}\mathcal{C}$ *and homomorphisms for $t \geq 1$. If $t \geq 2$, Ω^t lifts to a functor the category of abelian groups in* $\operatorname{Ho}\mathcal{C}$ *and homomorphisms.*

PROOF. This follows from Lemma 6.1.2, the Quillen equivalence between $\mathbf{Top}_*$ and $\mathbf{SSet}_*$, and the well-known fact that, for topological spaces X, $\pi_t(X, x)$ is naturally a group for $t \geq 1$ and an abelian group for $t \geq 2$. $\square$

REMARK 6.1.7. It is useful to have an explicit construction for the product in $[\Sigma A, X]$ for A cofibrant and X fibrant. We can get such an explicit construction by translating the definition of the group structure in π_1 of a simplicial set (see the remarks following Proposition 3.6.3). We find that if we have two maps $h, h' : A \longrightarrow X^I$ representing elements $[h], [h'] \in [\Sigma A, X]$, their product $[h][h']$ is represented by the map $h * h'$ defined as follows. Note that $X^{\Lambda^1[2]}$ is the pullback of two copies of X^I over the maps p_1 and p_0. Thus, the maps h, h' define a map $A \longrightarrow X^{\Lambda^1[2]}$, since $p_1 h = p_0 h'$. Since A is cofibrant and the map $X^{\Delta[2]} \longrightarrow X^{\Lambda^1[2]}$ is a trivial fibration, there is a lift to a map $A \xrightarrow{H} X^{\Delta[2]}$. Define $h * h' = X^{d^1} H$. One can prove directly that $[h * h']$ is independent of the lift H and choice of representatives h and h', but this is unnecessary since we already know that $[\Sigma A, X]$ is a group and this is the group structure for it. Note that we can define $h * h'$, though not uniquely, as long as $p_1 h = p_0 h'$, and then we will have $p_0(h * h') = p_0 h$ and $p_1(h * h') = p_1 h'$. It is still true that $h * h'$ is well-defined up to an appropriate notion of homotopy, but we do not prove this nor do we need it.

The unit in $[\Sigma A, X]$ is $[0]$. The inverse in $[\Sigma A, X]$ is given by a similar construction as the product. A map $h\colon A \longrightarrow X^I$ representing an element of $[\Sigma A, X]$, together with the 0 map, defines a map $A \longrightarrow X^{\Lambda^0[2]}$. We choose a lift to a map $H\colon A \longrightarrow X^{\Delta[2]}$. Then $X^{d^0} H$ represents the inverse of $[h]$.

6.2. Cofiber and fiber sequences

This section is devoted to proving that there is a natural coaction in $\operatorname{Ho}\mathcal{C}$ of the cogroup ΣA on the cofiber of a cofibration of cofibrant objects $A \longrightarrow B$ in a pointed model category $\mathcal{C}$. This allows to define cofiber sequences, and, by duality, fiber sequences. In the next two sections we study some properties of cofiber and fiber sequences. We prove approximately the same results in this section as in the first four pages of [**Qui67**, Section I.3], but we use a somewhat different method.

Our construction of a coaction of ΣA on the cofiber C of a cofibration $A \xrightarrow{f} B$ of cofibrant objects in a pointed model category $\mathcal{C}$ will use the results and notations about homotopies of Section 1.2. To construct such a coaction, it is necessary and sufficent to construct a natural right action of the group $[\Sigma A, X]$ on $[C, X]$ for all X. In fact, by using the natural isomorphism (in $\operatorname{Ho}\mathcal{C}$) $X \to RX$, we need only construct such a natural action for fibrant X.

We construct such a right action as follows. Denote the map $B \to C$ by g. Given a map $h\colon A \to X^I$ representing an element $[h]$ of $[\Sigma A, X]$ and a map $u\colon C \to X$ representing an element $[u]$ of $[C, X]$, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & X^I \arrow[d, "p_0"] \\ B \arrow[r, "ug"] & X \end{tikzcd}$$

Since p_0 is a trivial fibration and f is a cofibration, there is a lift $\alpha\colon B \to X^I$. Since $p_1 h_1 f = p_1 h = 0$, there is a unique map $w\colon C \to X$ such that $wg = p_1\alpha$. We then define $[u] \odot [h] = [w]$.

Dually, suppose $p\colon E \to B$ is a fibration of fibrant objects with fiber $i\colon F \to E$. If $h\colon A \times I \to B$ represents an element of $[A, \Omega B]$ and $u\colon A \to F$ represents an element of $[A, F]$, let $\alpha\colon A \times I \to E$ be a lift in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "iu"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "h"] & B \end{tikzcd}$$

Then we define $[u] \odot [h] = [w]$, where $w\colon A \to F$ is the unique map such that $if = \alpha i_1$.

It is enlightening to take $\mathcal{C} = \mathbf{Top}_*$ and $A = S^0$. Then, given a loop h in B and a point u in F, $[u] \odot [h]$ is defined by taking a lift of h to a path α which starts at u, and taking its other endpoint w.

THEOREM 6.2.1. *Suppose $f\colon A \to B$ is a cofibration of cofibrant objects with cofiber $g\colon B \to C$ in a pointed model category $\mathcal{C}$ and X is fibrant. Then the pairing $([u], [h]) \mapsto [u] \odot [h]$ constructed above defines a natural right action of the group $[\Sigma A, X]$ on $[C, X]$, so defines a right coaction of ΣA on C. Dually, suppose $p\colon E \to B$ is a fibration of fibrant objects with fiber $i\colon F \to E$ in $\mathcal{C}$ and A is cofibrant. Then the pairing $([u], [h]) \mapsto [u] \odot [h]$ constructed above defines a natural right action of the group $[A, \Omega B]$ on $[A, F]$, so defines a right action of the group object ΩB on the fiber F.*

The fibration half of Theorem 6.2.1 follows immediately from the cofibration half and duality. We will theorefore concentrate on the cofibration half. We will prove Theorem 6.2.1 in a series of lemmas. We need a preliminary lemma about cylinder objects.

LEMMA 6.2.2. *Suppose $f\colon A \to B$ is a cofibration of cofibrant objects in a pointed model category $\mathcal{C}$, with cofiber $g\colon B \to C$. Then there are cylinder objects B' for B and C' for C and maps $A \times I \xrightarrow{f'} B' \xrightarrow{g'} C'$ such that g' is the cofiber of* *the cofibration f' and the following diagram is commutative.*

$$\begin{tikzcd}
A \amalg A \arrow[r, "(i_0,i_1)"] \arrow[d, "f\amalg f"'] & A \times I \arrow[r, "s"] \arrow[d, "f'"'] & A \arrow[d, "f"'] \\
B \amalg B \arrow[r, "(i_0,i_1)"] \arrow[d, "g\amalg g"'] & B' \arrow[r, "s"] \arrow[d, "g'"'] & B \arrow[d, "g"'] \\
C \amalg C \arrow[r, "(i_0,i_1)"] & C' \arrow[r, "s"] & C
\end{tikzcd}$$

PROOF. Let Q denote the pushout in the diagram

$$\begin{tikzcd}
A \amalg A \arrow[r, "(i_0,i_1)"] \arrow[d, "f\amalg f"'] & A \times I \arrow[d, "e"'] \\
B \amalg B \arrow[r, "j"] & Q
\end{tikzcd}$$

so that j and e are cofibrations. The fold map $B \amalg B \longrightarrow B$ together with the composite $A \times I \xrightarrow{s} A \xrightarrow{f} B$ define a map $Q \longrightarrow B$. If we factor this into a cofibration $Q \xrightarrow{k} B'$ followed by a trivial fibration $B' \xrightarrow{s} B$, we find that B' is a cylinder object for B, where $(i_0, i_1) = kj$. It follows that the first two rows of our diagram are commutative, where $f' = ke$.

Now we define $g'\colon B' \longrightarrow C'$ as the cofiber of the cofibration f'. Then there are induced maps $C \amalg C \xrightarrow{(i_0,i_1)} C' \xrightarrow{s} C$ factoring the fold map of C and making our diagram commutative. We must show that (i_0, i_1) is a cofibration and s is a weak equivalence. We use the result and method of the cube lemma 5.2.6. By applying the result of the cube lemma 5.2.6 to the pushout squares defining C' and C, we find that s is a weak equivalence. By applying the method of the cube lemma 5.2.6 to the pushout square defining $C \amalg C$ as the cofiber of $f \amalg f$ and the pushout square defining C', we get a cofibration in the Reedy model structure on $\mathcal{C}^{\mathcal{B}}$, where $\mathcal{B}$ is the category with three objects used in the cube lemma 5.2.6. The only thing to check here is that the map $Q \longrightarrow B'$ is a cofibration, which of course it is. Since the colimit functor is a left Quillen functor, as in the proof of the cube lemma 5.2.6, we find that the map $C \amalg C \longrightarrow C'$ is a cofibration. □

With this lemma in hand, we can prove that our pairing is well-defined.

LEMMA 6.2.3. *Suppose $f\colon A \longrightarrow B$ is a cofibration of cofibrant objects with cofiber C in a pointed model category $\mathcal{C}$, and X is fibrant. Then the pairing $([u],[h]) \mapsto [u] \odot [h]$ defines a map $[C,X] \times [\Sigma A, X] \longrightarrow [C,X]$.*

PROOF. Define the maps u, h, and α as in the definition of $[u] \odot [h]$. Suppose $h'\colon A \longrightarrow X^I$ is a (possibly) different representative for $[h]$, $v\colon C \longrightarrow X$ is a (possibly) different representative of $[u]$, and β is a lift in the diagram

$$\begin{tikzcd}
A \arrow[r, "h'"] \arrow[d, "f"'] & X^I \arrow[d, "p_0"'] \\
B \arrow[r, "vg"] & X
\end{tikzcd}$$

Let w be the unique map such that $wg = p_1\alpha$, and let w' be the unique map such that $w'g = p_1\beta$. We must show that $[w] = [w']$. By Lemma 6.1.4, there is a map $H\colon A \times I \to X^I$ such that $Hi_0 = h$, $Hi_1 = h'$, and $p_0 H = p_1 H = 0$. We use the notation and cylinder objects of Lemma 6.2.2 and its proof. By Corollary 1.2.6, there is a homotopy $K\colon C' \to X$ from u to v. We then get a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{(H,\alpha\amalg\beta)} & X^I \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ B' & \xrightarrow{Kg'} & X \end{array}$$

Let $\widetilde{G}\colon B' \to X^I$ be a lift in this diagram. Then $p_1\widetilde{G} \circ ke = p_1 H = 0$, so there is a unique map $G\colon C' \to X$ such that $Gq = p_1\widetilde{G}$. It follows that G is a homotopy from w to w', as required. $\square$

LEMMA 6.2.4. *Suppose $f\colon A \to B$ is a cofibration of cofibrant objects with cofiber C in a pointed model category $\mathcal{C}$. Then the map $[C, X] \times [\Sigma A, X] \to [C, X]$ constructed above is natural for maps of fibrant objects X.*

PROOF. Suppose $q\colon X \to Y$ is a map of fibrant objects. The induced map $[\Sigma A, X] \to [\Sigma A, Y]$ takes the class $[h]$ represented by $h\colon A \to X^I$ to $[q^I h]$. Similarly, the induced map $[C, X] \to [C, Y]$ takes $[u]$ to $[qu]$. Now let α be a lift in the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X^I \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ B & \xrightarrow{ug} & X \end{array}$$

Then $[u] \odot [h] = [w]$, where w is the unique map $C \to X$ such that $wg = p_1\alpha$. But $q^I\alpha$ is a lift in the diagram

$$\begin{array}{ccc} A & \xrightarrow{q^I h} & Y^I \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ B & \xrightarrow{qug} & Y \end{array}$$

Thus, by Lemma 6.2.3, $[qu] \odot [q^I h] = [v]$, where v is the unique map $C \to Y$ such that $vg = p_1 q^I \alpha$. But $p_1 q^I \alpha = q p_1 \alpha = qwg$. Thus $v = qw$, so $[qu] \odot [q^I h] = q([u] \odot [h])$, as required. $\square$

We can now finish the proof of Theorem 6.2.1.

PROOF OF THEOREM 6.2.1. It remains to show that the product $\odot$ is associative and unital. The unit of $[\Sigma A, X]$ is the zero map. In this case, we can choose our lift α in the definition of $[u] \odot [h]$ to be rug, where $r\colon X \to X^I$ is one of the structure maps of the path object X^I. Hence $p_1\alpha = ug$, so $[u] \odot [0] = [u]$, as required. To check associativity, we use the description of the product on $[\Sigma A, X]$ given in Remark 6.1.7. So suppose $h, h'\colon A \to X^I$ represent elements of $[\Sigma A, X]$, and $u\colon C \to X$. Given a lift α used to define $u \odot h$, we have $([u] \odot [h]) \odot [h'] = [k]$, where k is the unique map such that $kg = p_1\beta$, and where β is a lift in the square

$$\begin{array}{ccc} A & \xrightarrow{h'} & X^I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow{p_1\alpha} & X \end{array}$$

Thus we have $p_0\beta = p_1\alpha$, so we can define $\alpha * \beta$ as in Remark 6.1.7. We find that $\alpha * \beta$ is a lift in the square

$$\begin{array}{ccc} A & \xrightarrow{h*h'} & X^I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow{ug} & X \end{array}$$

for a particular choice of $h * h'$. Hence $[u] \odot ([h][h']) = [q]$ for the unique map q such that $qg = p_1(\alpha * \beta)$. But $p_1(\alpha * \beta) = p_1\beta = kg$. Hence $q = k$, so $([u] \odot [h]) \odot [h'] = [u] \odot ([h][h'])$, as required. □

The coaction of Theorem 6.2.1 is natural for maps of cofibrations as well.

Proposition 6.2.5. *Suppose $\mathcal{C}$ is a pointed model category and we have a commutative square of cofibrant objects*

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ A & \xrightarrow{f} & B \end{array}$$

where f' and f are cofibrations, with cofibers $g' \colon B' \to C'$ and $g \colon B \to C$ respectively. Then the induced map $q_3 \colon C' \to C$ is equivariant in $\operatorname{Ho}\mathcal{C}$ with respect to the cogroup homomorphism Σq_1.

The corresponding statement for fibrations holds by duality.

Proof. Suppose X is fibrant, $h \colon A \to X^I$ represents an element of $[\Sigma A, X]$, and $u \colon C \to X$ represents an element of $[C, X]$. We must show that $([u] \odot [h])q_3 = [uq_3] \odot [hq_1]$. To see this, let α be a lift in the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X^I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow{ug} & X \end{array}$$

so that $[u] \odot [h] = [w]$, where w is the unique map such that $wg = p_1\alpha$. Then αq_2 is a lift in the diagram

$$\begin{array}{ccc} A' & \xrightarrow{hq_1} & X^I \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle p_0} \\ B' & \xrightarrow{uq_3g'} & X \end{array}$$

Thus $[uq_3] \odot [hq_1] = [r]$ for the unique map r such that $rg' = p_1\alpha q_2$. But we can take $r = wq_3$, as the reader can check. □

With Theorem 6.2.1 in hand, we make the following definition.

Definition 6.2.6. Suppose $\mathcal{C}$ is a pointed model category. A *cofiber sequence* in $\operatorname{Ho}\mathcal{C}$ is a diagram $X \to Y \to Z$ in $\operatorname{Ho}\mathcal{C}$ together with a right coaction of ΣX on Z which is isomorphic in $\operatorname{Ho}\mathcal{C}$ to a diagram of the form $A \xrightarrow{f} B \xrightarrow{g} C$ where f is a cofibration of cofibrant objects in $\mathcal{C}$ with cofiber g and where C has the right ΣA-coaction given by Theorem 6.2.1. Dually, a *fiber sequence* is a diagram $X \to Y \to Z$ together with a right action of ΩZ on X which is isomorphic to a diagram $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration of fibrant objects with fiber i and where F has the right ΩB-action given by Theorem 6.2.1.

Note that what this isomorphism means precisely in the cofiber sequence case is that there are isomorphisms $\alpha\colon X \to A$, $\beta\colon Y \to B$, and $\gamma\colon Z \to C$ making the evident diagrams commute and such that γ is equivariant with respect to the cogroup isomorphism $\Sigma\alpha$.

Note also that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho}\mathcal{C}$, in particular we have $gf = 0$ in $\operatorname{Ho}\mathcal{C}$.

A cofiber sequence has associated to it a boundary map, which we now define.

Definition 6.2.7. Suppose $\mathcal{C}$ is a pointed model category, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho}\mathcal{C}$. The *boundary map* is the map $\partial\colon Z \to \Sigma X$ in $\operatorname{Ho}\mathcal{C}$ which is the composite

$$Z \to Z \amalg \Sigma X \xrightarrow{0\times 1} \Sigma X$$

where the first map is the coaction. Dually, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence, the *boundary map* is the map $\partial\colon \Omega Z \to X$ which is the composite

$$\Omega Z \xrightarrow{(0,1)} X \times \Omega Z \to Z$$

Note that, if $\theta \in [\Sigma A, X]$, then $\theta\partial = [0] \odot \theta$. Similarly, if $\theta \in [A, \Omega Z]$, then $\partial\theta = [0] \odot \theta$.

6.3. Properties of cofiber and fiber sequences

In this section we study some of the properties of the cofiber and fiber sequences defined in the previous section. We concentrate on cofiber sequences, as the corresponding properties of fiber sequences follow by duality. The results of this section are mostly proved by Quillen in [**Qui67**, Section I.3].

We begin with some simple properties.

Lemma 6.3.1. *The collection of cofiber sequences is replete in the homotopy category of a pointed model category $\mathcal{C}$. That is, any diagram isomorphic to a cofiber sequence is a cofiber sequence. Dually, the collection of fiber sequences is replete as well.*

Lemma 6.3.1 follows immediately from the definition of cofiber sequences. One must be careful to note that $X' \to Y' \to Z'$ is isomorphic to a cofiber sequence $X \to Y \to Z$ if and only if there is a commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "q_1"'] & Y' \arrow[r] \arrow[d, "q_2"'] & Z' \arrow[d, "q_3"'] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

where the q_i are isomorphisms and q_3 is Σq_1-equivariant.

LEMMA 6.3.2. *For any X in a pointed model category $\mathcal{C}$, the diagram $* \longrightarrow X \xrightarrow{1} X$ together with the trivial coaction of $\Sigma * = *$ on X is a cofiber sequence in* $\operatorname{Ho}\mathcal{C}$. *Dually, the diagram $X \xrightarrow{1} X \longrightarrow *$ with the trivial action of $\Omega * = *$ on X is a fiber sequence.*

PROOF. The cofibration $* \longrightarrow QX$ has cofiber $QX \xrightarrow{1} QX$. $\square$

The following lemma is part (i) of [**Qui67**, Propopsition I.3.5].

LEMMA 6.3.3. *Suppose $f\colon X \longrightarrow Y$ is an arbitrary map in* $\operatorname{Ho}\mathcal{C}$, *where $\mathcal{C}$ is a pointed model category. Then there is a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ for some g. Dually, there is a fiber sequence $W \xrightarrow{h} X \xrightarrow{f} Y$ for some h.*

PROOF. The composite $QX \xrightarrow{q_X} X \xrightarrow{f} Y \xrightarrow{r_Y} RY$ is a map in $\operatorname{Ho}\mathcal{C}$ from a cofibrant object to a fibrant object. It is therefore represented by a map $f'\colon QX \longrightarrow RY$ of $\mathcal{C}$. Factor f' into a cofibration $i\colon QX \longrightarrow Y'$ followed by a trivial fibration $p\colon Y' \longrightarrow RY$. Let $g'\colon Y' \longrightarrow Z$ denote the cofiber of i. Give Z the coaction of ΣX given by the composite

$$Z \longrightarrow Z \amalg \Sigma QX \xrightarrow{1 \amalg \Sigma q_X} Z \amalg \Sigma X$$

where the first map is the coaction of ΣQX on Z. Then we have a commutative diagram in $\operatorname{Ho}\mathcal{C}$

$$\begin{array}{ccccc}
QX & \xrightarrow{i} & Y' & \xrightarrow{g'} & Z \\
{\scriptstyle q_X}\downarrow & & {\scriptstyle r_Y^{-1}p}\downarrow & & \| \\
X & \xrightarrow{f} & Y & \xrightarrow{g'p^{-1}r_Y} & Z
\end{array}$$

The vertical maps are isomorphisms and the identity map of Z is Σq_X-equivariant. Since the top row is a cofiber sequence, so is the bottom row, as required. $\square$

We now move on to some less trivial properties of cofiber sequences. The following is [**Qui67**, Proposition I.3.3].

PROPOSITION 6.3.4. *Suppose $\mathcal{C}$ is a pointed model category, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in* $\operatorname{Ho}\mathcal{C}$. *Then the sequence $Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$, where ∂ is the boundary map of Definition 6.2.7, becomes a cofiber sequence when ΣX is given the ΣY-coaction*

$$\Sigma X \longrightarrow \Sigma X \amalg \Sigma X \xrightarrow{1 \amalg \Sigma f} \Sigma X \amalg \Sigma Y \xrightarrow{1 \amalg i} \Sigma X \amalg \Sigma Y$$

where the first map is the cogroup structure map and i is the cogroup inverse map of ΣY.

Proposition 6.3.4 and duality imply the corresponding result for fiber sequences as well, whose exact formulation we leave to the reader. Note that Proposition 6.3.4 implies in particular that $\partial g = 0$ in a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. Also note that one can apply Proposition 6.3.4 any number of times, to generate a "long exact sequence" often called the Puppe sequence.

PROOF. We can assume f is actually a cofibration $A \longrightarrow B$ in $\mathcal{C}$ with cofiber $g\colon B \longrightarrow C$. Define the mapping cone C' of f via the pushout diagram

$$\begin{tikzcd} A \amalg A \arrow[r, "(i_0,i_1)"] \arrow[d, "(f,0)"'] & A \times I \arrow[d, "a"] \\ B \arrow[r, "g'"] & C' \end{tikzcd}$$

By manipulating pushouts, one can check that the cofiber of g' is the map $h'\colon C' \longrightarrow A \wedge S^1$ induced by the zero map on B and the canonical map $A \times I \longrightarrow A \wedge S^1$. We therefore have a cofiber sequence $B \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A$, which we will show is isomorphic to $B \xrightarrow{g} C \xrightarrow{\partial} \Sigma A$ in $\operatorname{Ho}\mathcal{C}$.

Note that there is a map $b\colon C' \longrightarrow C$ such that $bg' = g$ induced by the identity map on B, the map $A \amalg A \xrightarrow{1 \amalg 0} A$, and the unique map $A \times I \longrightarrow *$. It is not clear from this description that b is a weak equivalence. To see this we must manipulate pushouts. Let $(I, 1)$ denote be the pointed simplicial set $\Delta[1]$ with basepoint 1. Then $A \wedge (I, 1)$ is the cone on A, and there is a map $i_0'\colon A \longrightarrow A \wedge (I, 1)$ induced by i_0. We claim that there is a pushout square

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "f"'] & A \wedge (I,1) \arrow[d] \\ B \arrow[r, "g'"] & C' \end{tikzcd}$$

The proof of this involves examining $\mathcal{C}(C', -)$, and we leave the details to the reader. The map $b\colon C' \longrightarrow C$ is induced by the identity on B and A and the map $A \wedge (I, 1) \longrightarrow *$. Since the latter map is a weak equivalence (left to the reader), the cube lemma 5.2.6 implies that b is a weak equivalence as well.

We now show that $\partial b = h'\colon C' \longrightarrow \Sigma A$ in $\operatorname{Ho}\mathcal{C}$. To see this, suppose X is fibrant, and we have an element θ of $[\Sigma A, X]$ represented both by $j\colon A \times I \longrightarrow X$ and $k\colon A \longrightarrow X^I$. Then $\theta h'$ is represented by the map $u\colon C' \longrightarrow X$ which is j on $A \times I$ and 0 on B. To calculate $\theta\partial$, we choose a lift H in the diagram

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & X^I \arrow[d, "p_0"] \\ B \arrow[r, "0"] & X \end{tikzcd}$$

Then $\theta\partial$ is represented by the map $c\colon C \longrightarrow X$ such that $cg = p_1 H$. It follows that $\theta\partial b$ is represented by the map $u'\colon C' \longrightarrow X$ which is 0 on $A \times I$ and $p_1 H$ on B. We must show that u and u' are homotopic. To see this, choose a correspondence H' between j and k using Lemma 6.1.5. Define $H''\colon C' \longrightarrow X^I$ to be H' on $A \times I$ and H on B. Then H'' is the required homotopy between u and u'.

We therefore have a commutative diagram in $\operatorname{Ho}\mathcal{C}$

$$\begin{tikzcd} B \arrow[r, "g'"] \arrow[d, equal] & C' \arrow[r, "h'"] \arrow[d, "b"] & \Sigma A \arrow[d, equal] \\ B \arrow[r, "g"] & C \arrow[r, "\partial"] & \Sigma A \end{tikzcd}$$

Since the vertical maps are isomorphisms and the top sequence is a cofiber sequence, so is the bottom one. It remains to show that the coaction of ΣB on ΣA induced by g' has the stated form.

In order to compute this coaction, let X be fibrant as before, and let $\theta \in [\Sigma A, X]$ be represented by $j\colon A \times I \to X$. Let $h\colon B \to X^I$ represent an element in $[\Sigma B, X]$. Let H be a lift in the diagram

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "g'"'] & X^I \arrow[d, "p_0"] \\ C' \arrow[r, "u"] & X \end{tikzcd}$$

where u is the map which is 0 on B and is j on $A \times I$. Then H is h on B and some map K on $A \times I$. We have $Ki_0 = hf$, $Ki_1 = 0$, and $p_0 K = j$. The action of $[h]$ on θ is given by $\theta \odot [h] = [p_1 K]$.

Now, $[p_1 K] = [k]$ for some map $k\colon A \to X^I$. Let $G\colon A \times I \to X^I$ be a correspondence between $p_1 K$ and k. Then we have $p_0 G = p_1 K$, $p_1 G = 0$, $Gi_0 = k$, and $Gi_1 = 0$. We then have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "i_1"'] & X^{\Delta[2]} \arrow[d] \\ A \times I \arrow[r, "{(K,G)}"] & X^{\Lambda^1[2]} \end{tikzcd}$$

analogous to the diagram in Remark 6.1.7. Let F be a lift in this diagram, and let $K * G = X^{d^1} F$. Then we have $p_0(K * G) = p_0 K = j$, $p_1(K * G) = p_1 G = 0$, and $(K * G)i_1 = 0$. Furthermore, $(K * G)i_0$ is a possible choice for $Ki_0 * Gi_0 = (hf) * k$. Thus $K * G$ is a correspondence between j and a possible choice for $hf * k$. Hence $\theta = [hf](\theta \odot [h])$, where the multiplication is in the group $[\Sigma A, X]$. Thus $\theta \odot [h] = \theta [hf]^{-1}$, as required. □

The following is part (ii) of [**Qui67**, Proposition I.3.5]. Our proof is somewhat simpler.

PROPOSITION 6.3.5. *Suppose $\mathcal{C}$ is a pointed model category and we have a commutative diagram in* $\operatorname{Ho} \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \end{tikzcd}$$

where the rows are cofiber sequences. Then there is a (nonunique) map $\gamma\colon Z \to Z'$ which is $\Sigma\alpha$-equivariant and satisfies $\gamma g = g'\beta$. The dual statement for fiber sequences holds as well.

PROOF. We can assume that our cofiber sequences are actually sequences $A \xrightarrow{f} B \xrightarrow{g} C$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C'$ of maps of $\mathcal{C}$, where f and f' are cofibrations, g is the cofiber of f, g' is the cofiber of f', and all objects are cofibrant.

We claim that we can also assume that A' and B' are fibrant. Indeed, let $B'' = R(RA' \amalg_{A'} B')$. Then we have a cofibration $f''\colon RA' \to B''$, and we let

$g''\colon B'' \to C''$ be its cofiber. We then have a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "r_{A'}"'] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] \\ RA' \arrow[r, "f''"] & B'' \arrow[r, "g''"] & C'' \end{tikzcd}$$

The first two vertical maps are weak equivalences, so the cubes lemma 5.2.6 guarantees that the last vertical map is also a weak equivalence. Proposition 6.2.5 guarantees that it is $\Sigma r_{A'}$-equivariant in $\operatorname{Ho}\mathcal{C}$. Hence we may as well replace the top cofiber sequence by the bottom cofiber sequence, so we can assume A' and B' are fibrant.

Now, since A' and B' are fibrant, α is represented by some map $p\colon A \to A'$, and β is represented by some map $q\colon B \to B'$. We have $[qf] = [f'p]$. Let $h\colon A \to (B')^I$ be a homotopy from qf to $f'p$. Let H be a lift in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & (B')^I \arrow[d, "p_0"] \\ B \arrow[r, "q"] & B' \end{tikzcd}$$

Let $q' = p_1 H$. Then $[q'] = [q] = \beta$, and $q'f = p_1 H = f'p$. Thus p and q' induce a map $r\colon C \to C'$. The class $\gamma = [r]$ makes the required diagram commute and is $\Sigma\alpha$-equivariant by Proposition 6.2.5. □

We now prove that a version of Verdier's octahedral axiom holds for the cofiber sequences in the homotopy category of a pointed model category. Quillen states that this axiom holds in [**Qui67**, Section I.3], but says that it is not worth the effort to write it down. Readers of [**HPS97**] will realize that, on the contrary, Verdier's octahedral axiom is very important, at least in the stable situation. It is the only tool we have for getting at the cofiber of a composite. Incidentally, the axiom as stated below bears no resemblance to an octahedron: it is called the octahedral axiom by analogy to the triangulated case, discussed in the next chapter, where it can be written as an octahedron.

PROPOSITION 6.3.6. *Let $\mathcal{C}$ be a pointed model category. Suppose we have maps $X \xrightarrow{v} Y \xrightarrow{u} Z$ in $\operatorname{Ho}\mathcal{C}$. Then there exist cofiber sequences*

$$X \xrightarrow{v} Y \xrightarrow{d} U$$

$$X \xrightarrow{uv} Z \xrightarrow{a} V$$

$$Y \xrightarrow{u} Z \xrightarrow{f} W$$

and

$$U \xrightarrow{r} V \xrightarrow{s} W$$

in $\operatorname{Ho}\mathcal{C}$ such that $au = rd$, $sa = f$, r is ΣX-equivariant, s is Σv-equivariant, and the ΣU-coaction on W is the composite

$$W \to W \amalg \Sigma Y \xrightarrow{1 \amalg \Sigma d} W \amalg \Sigma U.$$

There is of course a dual statement for fiber sequences in $\operatorname{Ho}\mathcal{C}$, which follows by duality.

PROOF. We may as well assume that X, Y, and Z are cofibrant and fibrant, and that the maps v and u are cofibrations in $\mathcal{C}$. We can then take d to be the cofiber of v, a to be the cofiber of uv, and f to be the cofiber of u. The first three cofiber sequences are then immediate.

The map $r : U \longrightarrow V$ is then the pushout of the map u and the identity map on $*$ over the identity map on X. Then $au = rd$ by construction, and r is ΣX-equivariant in $\operatorname{Ho}\mathcal{C}$ by Proposition 6.2.5. It also follows that r is a cofibration. The easiest way to see this is to check that r has the left lifting property with respect to trivial fibrations, but it also follows from the methods of the cube lemma 5.2.6.

Similarly, the map $s : V \longrightarrow W$ is the pushout of the identity maps on Z and $*$ over the map v. Then $sa = f$ and Proposition 6.2.5 implies that s is Σv-equivariant in $\operatorname{Ho}\mathcal{C}$. Furthermore, by commuting colimits we find that s is the cofiber of r. Hence we do get a cofiber sequence $U \xrightarrow{r} V \xrightarrow{s} W$ in $\operatorname{Ho}\mathcal{C}$.

We must still check that the two coactions of ΣU on W agree. To see this, let B be a fibrant object of $\mathcal{C}$, let $g\colon W \longrightarrow B$ be a map, and let $h\colon U \longrightarrow B^I$ represent a class in $[\Sigma U, B]$. We must show that $[g] \odot [h] = [g] \odot [hd]$, where the first product is the action of $[\Sigma U, B]$ on $[W, B]$ and the second product is the action of $[\Sigma Y, B]$ on $[W, B]$. Let H be a lift in the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & B^I \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ V & \xrightarrow{gs} & B \end{array}$$

Then $[g] \odot [h] = [k]$, where k is the unique map such that $ks = p_1 H$. One can then readily verify that Ha is a lift in the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{hd} & B^I \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ Z & \xrightarrow{gf} & B \end{array}$$

Since $p_1 Ha = ksa = kf$, we conclude that $[g] \odot [hd] = [k]$ as well. □

We close this section by investigating how the cofiber and fiber sequences in $\operatorname{Ho}\mathcal{C}$ interact. The following proposition is closely related to [**Qui67**, Proposition 6], but is somewhat stronger and easier to use.

PROPOSITION 6.3.7. *Let $\mathcal{C}$ be a pointed model category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho}\mathcal{C}$ and $X' \xrightarrow{i} Y' \xrightarrow{p} Z'$ is a fiber sequence in $\operatorname{Ho}\mathcal{C}$. Suppose in addition we have a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{\partial} & \Sigma X \\ {\scriptstyle \alpha}\big\downarrow & & {\scriptstyle \beta}\big\downarrow & & & & \big\downarrow{\scriptstyle \widetilde{\alpha}^{-1}} \\ \Omega Z' & \xrightarrow{\partial} & X' & \xrightarrow{i} & Y' & \xrightarrow{p} & Z' \end{array}$$

where $\widetilde{\alpha}^{-1}$ is the inverse of the adjoint of α with respect to the group structure on $[\Sigma X, Z']$. Then there is a fill-in map $\gamma\colon Z \longrightarrow Y'$ making the diagram commute.

Similarly, if we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\widetilde{\delta}^{-1}"'] & Y \arrow[r, "g"] & Z \arrow[r, "\partial"] \arrow[d, "\gamma"'] & \Sigma X \arrow[d, "\delta"'] \\ \Omega Z' \arrow[r, "\partial"] & X' \arrow[r, "i"] & Y' \arrow[r, "p"] & Z' \end{tikzcd}$$

then there is a fill-in map $\beta\colon Y \longrightarrow X'$ making the diagram commute.

Proof. Note that the two assertions of Proposition 6.3.7 are dual, so it suffices to prove the second statement. As usual, we can assume our cofiber sequence is of the form $A \xrightarrow{f} B \xrightarrow{g} C$ where f is a cofibration in $\mathcal{C}$ with cofiber g and A, B, and C are cofibrant. In fact, we can replace C by the mapping cone C', as in the proof of Proposition 6.3.4. Recall that C' is the pushout in the diagram

$$\begin{tikzcd} A \amalg A \arrow[r, "(i_0,i_1)"] \arrow[d, "(f,0)"'] & A \times I \arrow[d] \\ B \arrow[r, "g'"] & C' \end{tikzcd}$$

and that the cofiber of g' is $\partial'\colon C' \longrightarrow A \wedge S^1$. Furthermore, the sequence $A \xrightarrow{f} B \xrightarrow{g'} C' \xrightarrow{\partial'} \Sigma A$ is isomorphic in $\operatorname{Ho}\mathcal{C}$ to the sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\partial} \Sigma A$, as was proved in the course of proving Proposition 6.3.4. Dually, we can assume our fiber sequence is of the form $F \xrightarrow{i} E \xrightarrow{p} D$ where p is a fibration with fiber i and F, E, and D are fibrant. Altogether then, we have a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\widetilde{\delta}^{-1}"'] & B \arrow[r, "g'"] & C' \arrow[r, "\partial'"] \arrow[d, "\gamma"'] & \Sigma A \arrow[d, "\delta"'] \\ \Omega D \arrow[r, "\partial"] & F \arrow[r, "i"] & E \arrow[r, "p"] & D \end{tikzcd}$$

in $\operatorname{Ho}\mathcal{C}$.

Now, choose a representative $c\colon C' \longrightarrow E$ of γ and a representative $h\colon A \wedge S^1 \longrightarrow D$ of δ. We will also denote by h the map $A \times I \longrightarrow D$ induced by h. By hypothesis, we have $[pc] = [h\partial']$. We claim that we can assume that $ph = h\partial'$. Indeed, let $H\colon C' \times I \longrightarrow D$ be a homotopy from ph to $h\partial'$. Then we have a commutative diagram

$$\begin{tikzcd} C' \arrow[r, "c"] \arrow[d, "i_0"'] & E \arrow[d, "p"'] \\ C' \times I \arrow[r, "H"] & D \end{tikzcd}$$

Let $G\colon C' \times I \longrightarrow E$ be a lift in this diagram. Then G is a homotopy from c to Gi_1, so we can replace c by Gi_1. Since $pGi_1 = Hi_1 = h\partial'$, this means we can assume $pc = h\partial'$, as claimed.

Since $pc = h\partial'$, we have $pcg' = h\partial' g' = 0$ in $\mathcal{C}$, so there is a unique map $b\colon B \longrightarrow F$ such that $ib = cg'$. The map c is then ib on B and some map $H\colon A \times I \longrightarrow E$ on $A \times I$. Since $pc = h\partial'$, we have $pH = h$. Since c is ib on B, we have $Hi_0 = ibf$ and $Hi_1 = 0$. Let $\beta = [b]$. Then to complete the proof we must show that $[bf] = \partial(\widetilde{\delta}^{-1})$. But by definition $\partial(\widetilde{\delta}^{-1}) = 0 \odot \widetilde{\delta}^{-1}$, where the product is the action of the group

$[A, \Omega D]$ on $[A, F]$. Hence we must show that $[bf] \odot [h] = 0$. To calculate $[bf] \odot [h]$, we first consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{ibf} & E \\ {\scriptstyle i_0}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ A \times I & \xrightarrow{h} & D \end{array}$$

Given a lift K in this diagram, $[bf] \odot [h] = [k]$ for the unique map k such that $ik = Ki_1$. However, the map H is a lift in this diagram, and it satisfies $Hi_1 = 0$. Thus $k = 0$, so $[bf] \odot [h] = 0$, as required. □

6.4. Naturality of cofiber sequences

In this section, we show that the cofiber (resp. fiber) sequences in the homotopy category of a pointed model category $\mathcal{C}$ are preserved by left (resp. right) Quillen functors. We also show that cofiber and fiber sequences are preserved by the closed $\operatorname{Ho} \mathbf{SSet}_*$-module structure on $\operatorname{Ho} \mathcal{C}$ induced by the framing.

Proposition 6.4.1. *Suppose $(F, U, \varphi)\colon \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction of pointed model categories. Then $LF\colon \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$ preserves cofiber sequences. That is, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho} \mathcal{C}$ with coaction $\psi\colon Z \to Z \amalg \Sigma X$, then $(LF)X \xrightarrow{(LF)f} (LF)Y \xrightarrow{(LF)g} (LF)Z$ is a cofiber sequence in $\operatorname{Ho} \mathcal{D}$, where the coaction on $(LF)Z$ is given by the composite*

$$(LF)Z \xrightarrow{(LF)\psi} (LF)(Z \amalg \Sigma X) \cong (LF)Z \amalg (LF)\Sigma X \xrightarrow[\cong]{1 \amalg m} (LF)Z \amalg \Sigma (LF)X$$

Here m is the isomorphism $(LF)(A \wedge^L S^1) \xrightarrow{\cong} (LF)A \wedge^L S^1$ constructed in the unpointed case in Theorem 5.6.2. Dually, RU preserves fiber sequences.

Proof. We may as well assume that our cofiber sequence is of the form $A \xrightarrow{f} B \xrightarrow{g} C$ where f is a cofibration with cofiber g and A, B, and C are cofibrant. Since F is a left Quillen functor, it preserves cofibrations and colimits, so we have a cofiber sequence $FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$. Since, for cofibrant D, $(LF)D$ is naturally isomorphic to FD in $\operatorname{Ho} \mathcal{D}$ via the natural transformation q, it suffices to check the statement about the coaction.

So suppose W is fibrant in $\mathcal{D}$, $\alpha \in [FC, W]$, and $\beta \in [\Sigma FA, W]$. Let $\alpha \bullet \beta$ be the product defined by the coaction described in the statement of the proposition. We must show that $\alpha \odot \beta = \alpha \bullet \beta$. We will actually show that $\varphi(\alpha \odot \beta) = \varphi(\alpha \bullet \beta)$, where we have used φ instead of $R\varphi$ for the isomorphism $[FC, D] \cong [C, UD]$ when C is cofibrant and D is fibrant. This is simpler since we have $\varphi(\alpha \bullet \beta) = (\varphi\alpha) \odot (\varphi(\beta m))$. This equality just follows from the naturality of φ.

Recall from Lemma 5.5.1 and Lemma 5.6.1 that there is a weak equivalence of simplicial frames $(UW)_\circ \to U(W_\circ)$. In particular, there is a map $(UW)^I \xrightarrow{j} U(W^I)$ such that $Up_i \circ j = p_i$ for $i = 0, 1$. This weak equivalence of simplicial frames defines the map m, by adjointness. That is, choose a representative $h\colon A \to (UW)^I$ for $\varphi(\beta m)$. Then $\varphi^{-1}(jh)\colon FA \to W^I$ is a representative for β. Similarly, choose a representative $u\colon C \to UW$ for $\varphi\alpha$. Then $\varphi^{-1}u\colon FC \to W$ is a representative for α.

Now, let H be a lift in the commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & (UW)^I \arrow[d, "p_0"] \\ B \arrow[r, "ug"] & UW \end{tikzcd}$$

Then $(\varphi\alpha) \odot (\varphi(\beta m)) = [v]$, where v is the unique map such that $vg = p_1 H$. Note that jH satisfies $Up_0 \circ jH = ug$ and $jHf = jh$. Thus $\varphi^{-1}(jH)$ is a lift in the commutative diagram

$$\begin{tikzcd} FA \arrow[r, "\varphi^{-1}(jh)"] \arrow[d, "Ff"'] & W^I \arrow[d, "p_0"] \\ FB \arrow[r, "(\varphi^{-1}u)\circ Fg"] & W \end{tikzcd}$$

Hence $\alpha \odot \beta = [w]$, where w is the unique map such that $w \circ Fg = p_1\varphi^{-1}(jH)$. Hence $\varphi(\alpha \odot \beta) = [\varphi w]$, and φw satisfies $\varphi w \circ g = Up_1 \circ jH = p_1 H = vg$. Thus $\varphi w = v$, so $\alpha \odot \beta = (\varphi\alpha) \odot (\varphi(\beta m))$, as required. □

Proposition 6.4.1 implies that some of the functors giving the closed $\operatorname{Ho}\mathbf{SSet}_*$-module structure on $\operatorname{Ho}\mathcal{C}$ respect cofiber or fiber sequences.

Corollary 6.4.2. *Suppose $\mathcal{C}$ is a pointed model category.*

(a) *The functor $- \wedge^L -\colon \operatorname{Ho}\mathcal{C} \times \operatorname{Ho}\mathbf{SSet}_* \to \operatorname{Ho}\mathcal{C}$ preserves cofiber sequences in the second variable. That is, suppose $A \in \operatorname{Ho}\mathcal{C}$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho}\mathbf{SSet}_*$. Then $A \wedge^L X \xrightarrow{1\wedge^L f} A \wedge^L Y \xrightarrow{1\wedge^L g} A \wedge^L Z$ is a cofiber sequence in $\operatorname{Ho}\mathcal{C}$, where the coaction on $A \wedge^L Z$ is the composite*

$$\begin{aligned} A \wedge^L Z \to A \wedge^L (Z \vee \Sigma X) &\cong (A \wedge^L Z) \amalg (A \wedge^L \Sigma X) \\ &\xrightarrow[\cong]{1\wedge^L a} (A \wedge^L Z) \amalg \Sigma(A \wedge^L X). \end{aligned}$$

Here a is the (pointed) associativity isomorphism $A \wedge^L (X \wedge^L S^1) \cong (A \wedge^L X) \wedge^L S^1$ whose unpointed version was constructed in Section 5.5.

(b) *The functor $R\operatorname{Hom}_*(-,-)$ converts cofiber sequences in the first variable into fiber sequences. That is, suppose $A \in \operatorname{Ho}\mathcal{C}$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho}\mathbf{SSet}_*$. Then the sequence $R\operatorname{Hom}_*(Z, A) \to R\operatorname{Hom}_*(Y, A) \to R\operatorname{Hom}_*(X, A)$ is a fiber sequence in $\operatorname{Ho}\mathcal{C}$ where the action is given by the composite*

$$\begin{aligned} \Omega R\operatorname{Hom}_*(X, A) \times R\operatorname{Hom}_*(Z, A) &\xrightarrow{\cong} R\operatorname{Hom}_*(\Sigma X, A) \times R\operatorname{Hom}_*(Z, A) \\ &\cong R\operatorname{Hom}_*(Z \vee \Sigma X, A) \to R\operatorname{Hom}_*(Z, A) \end{aligned}$$

Here the first map is adjoint to the associativity isomorphism $\Sigma(- \wedge^L X) \cong - \wedge^L \Sigma X$.

(c) *The functor $R\operatorname{Map}_{*\ell}(-,-) \cong R\operatorname{Map}_{*r}(-,-)$ preserves fiber sequences in the second variable and converts cofiber sequences to fiber sequences in the first variable. That is, suppose $A \in \operatorname{Ho}\mathcal{C}$ and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber* *sequence in* $\operatorname{Ho}\mathcal{C}$*. Then the sequence* $R\operatorname{Map}_{*r}(A,X) \to R\operatorname{Map}_{*r}(A,Y) \to R\operatorname{Map}_{*r}(A,Z)$ *is a fiber sequence in* $\operatorname{Ho}\mathbf{SSet}_*$*, with action*

$$\begin{aligned}\Omega R\operatorname{Map}_{*r}(A,Z)\times R\operatorname{Map}_{*r}(A,X) &\cong R\operatorname{Map}_{*r}(A,\Omega Z)\times R\operatorname{Map}_{*r}(A,X)\\ &\to R\operatorname{Map}_{*r}(A,X)\end{aligned}$$

Here the first map is the adjoint to the inverse of the associativity isomorphism $A\wedge^L \Sigma- \cong \Sigma(A\wedge^L -)$*. Dually, if* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is a cofiber sequence in* $\operatorname{Ho}\mathcal{C}$*, then* $R\operatorname{Map}_{*r}(Z,A) \to R\operatorname{Map}_{*r}(Y,A) \to R\operatorname{Map}_{*}(X,A)$ *is a fiber sequence in* $\operatorname{Ho}\mathbf{SSet}_*$*, with action*

$$\begin{aligned}\Omega R\operatorname{Map}_{*r}(X,A)\times R\operatorname{Map}_{*r}(Z,A) &\cong R\operatorname{Map}_{*r}(\Sigma X,A)\times R\operatorname{Map}_{*r}(Z,A)\\ &\to R\operatorname{Map}_{*}(Z,A)\end{aligned}$$

Here the first map is adjoint to the isomorphism

$$\Omega R\operatorname{Hom}_*(-,A)\cong R\operatorname{Hom}_*(\Sigma-,A)$$

of part (b).

PROOF. Apply Proposition 6.4.1 to the Quillen functor $QA\wedge -$ and its adjoint $\operatorname{Map}_{*\ell}(QA,-)$, and also to the Quillen functor $\operatorname{Hom}_*(-,RA)$ and its adjoint $\operatorname{Map}_{*r}(-,RA)$. The fact that the coaction in part (a) is as claimed follows from coherence. That the other actions are as claimed follows by adjointness. □

We are still left with checking that the other functors giving the closed $\operatorname{Ho}\mathbf{SSet}_*$-module structure on $\operatorname{Ho}\mathcal{C}$ respect cofiber or fiber sequences. The proof of this is fairly complicated, and is similar in flavor to the proofs in Section 5.5 and Section 5.6. We first need an alternative definition of the coaction. This definition is dual to Quillen's first definition of the action [**Qui67**, Section I.3] associated to a fibration.

Let $f\colon A\to B$ be a cofibration of cofibrant objects in a pointed model category $\mathcal{C}$ with cofiber $g\colon B\to C$. Choose a cylinder object A' for A. Denote by $r\colon A'\to A'\wedge S^1$ the cofiber of $(i_0,i_1)\colon A\amalg A\to A'$. Let $\widetilde{B}$ denote the double mapping cylinder of f. That is, $\widetilde{B}$ is the pushout in the diagram

$$\begin{array}{ccc} A\amalg A & \xrightarrow{(i_0,i_1)} & A' \\ {\scriptstyle f\amalg f}\downarrow & & \downarrow \\ B\amalg B & \longrightarrow & \widetilde{B}\end{array}$$

Then the fold map $B\amalg B\to B$ and the map $fs\colon A'\to B$ induce a map $\widetilde{B}\to B$. Factor this map into a cofibration $\widetilde{B}\to B'$ followed by a trivial cofibration $s\colon B'\to B$. The B' is a cylinder object for B, and there is an induced map $f'\colon A'\to B'$, which is in fact a cofibration, such that $f'i_j = i_jf$ for $j=0,1$ and $sf'=fs$.

Now let B_f denote the single mapping cylinder on f, so that we have a pushout square

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A' \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow{j} & B_f\end{array}$$

Since i_0 is a trivial cofibration, so is j. By manipulating colimits, the reader can check that there is a pushout square

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "jf"'] & A' \arrow[d] \\ B_f \arrow[r] & \widetilde{B} \end{tikzcd}$$

We therefore get an induced trivial cofibration $B_f \xrightarrow{(i_0, f')} B'$, where the notation indicates that the restriction to B is i_0 and the restriction to A' is f'.

Consider the pushout square

$$\begin{tikzcd} B_f \arrow[r, "(i_0, f')"] \arrow[d, "(g,r)"'] & B' \arrow[d, "k"] \\ C \amalg (A' \wedge S^1) \arrow[r, "\pi"] & \widetilde{C} \end{tikzcd}$$

We then find that π is a weak equivalence. One can then verify that the map $ki_1 \colon B \to \widetilde{C}$ satisfies $ki_1 f = 0$, so induces a map $i_1 \colon C \to \widetilde{C}$. Let $t \colon A' \to A \times I$ denote a map of cylinder objects, as in the remarks preceding Proposition 1.2.5. Note that t induces a weak equivalence $A' \wedge S^1 \xrightarrow{t} A \wedge S^1$ by the cube lemma 5.2.6. Let ψ denote the composite

$$C \xrightarrow{i_1} \widetilde{C} \xrightarrow{\pi^{-1}} C \amalg (A' \wedge S^1) \underset{\cong}{\xrightarrow{1 \amalg t}} C \amalg (A \wedge S^1) \cong C \amalg^L \Sigma A$$

in $\operatorname{Ho} \mathcal{C}$. Here the last isomorphism is the inverse of the evident weak equivalence $QC \amalg Q(QA \wedge S^1) \to C \amalg (A \wedge S^1)$.

LEMMA 6.4.3. *Let $\mathcal{C}$ is a pointed model category. Suppose $f \colon A \to B$ is a cofibration of cofibrant objects with cofiber $g \colon B \to C$. Then the map $\psi \colon C \to C \amalg \Sigma A$ in $\operatorname{Ho} \mathcal{C}$ defined above is the same as the coaction of Theorem 6.2.1.*

In particular, this lemma is saying that ψ does not depend on any of the choices made in defining it.

PROOF. Let X be fibrant, and suppose $u \colon C \to X$ and and $h \colon A \to X^I$ are maps representing elements of $[C, X]$ and $[\Sigma A, X]$ respectively. We must show that $[u] \odot [h] = ([u], [h]) \circ \psi$. Let h' be a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & X^I \arrow[d, "p_0"] \\ B \arrow[r, "ug"] & X \end{tikzcd}$$

Then $p_1 h' = vg$ for a unique map v, and we have $[u] \odot [h] = [v]$.

To evaluate $([u], [h]) \circ \psi$, we will use the notation used in the definition of ψ. Let $H' \colon A \times I \to X^I$ be a correspondence of h with some map $k' \colon A \times I \to X$. Let H denote the composition of H' with the weak equivalence of cylinder objects $A' \to A \times I$ used in the definition of ψ, and similarly for $k \colon A' \to X$. Then we have $p_1 H = Hi_1 = 0$, $p_0 H = k$, and $Hi_0 = h$. Also, k and h represent the same element of $[\Sigma A, X]$ under the evident weak equivalences, since k' and h do.

Now consider the commutative diagram

$$\begin{tikzcd} B_f \arrow[r, "{(h',H)}"] \arrow[d, "{(i_0,f')}"'] & X^I \arrow[d, "p_1"] \\ B' \arrow[r, "p_1h's"] & X \end{tikzcd}$$

where B_f is the mapping cylinder on f and B' is the cylinder object for B used in the definition of ψ. Let $K\colon B' \to X^I$ be a lift in this diagram. We leave it to the reader to check that there is a map $m = ((u,k), p_0K)\colon \widetilde{C} \to X$, where we have used the same letter k for the map $A' \wedge S^1 \to X$ induced by k.

It follows that $([u],[h]) \circ \psi = [mi_1]$. The reader can check that $mi_1g = p_0Ki_1$. Now Ki_1 is a homotopy between $p_0Ki_1 = mi_1g$ and $p_1Ki_1 = p_1h' = vg$. Furthermore, $Ki_1f = 0$. Therefore, Ki_1 extends to a homotopy between mi_1 and v, as required. □

Our plan is to lift the definition of ψ by replacing each object by an appropriate cosimplicial frame on it. For this, we need a generalization of Lemma 6.2.2 to cosimplicial frames.

Lemma 6.4.4. *Suppose $f\colon A \to B$ is a cofibration of cofibrant objects in a pointed model category $\mathcal{C}$, with cofiber $g\colon B \to C$. Then there are cosimplicial frames B^* for B and C^* for C and maps $A^\circ \xrightarrow{f^*} B^* \xrightarrow{g^*} C^*$ of cosimplicial frames covering f and g respectively, such that f^* is a cofibration in the Reedy model structure with cofiber g^*.*

Proof. Recall the functors $\ell^\bullet, r^\bullet\colon \mathcal{C} \to \mathcal{C}^\Delta$ of Definition 5.2.7. A cosimplicial frame on X is a factorization of $\ell^\bullet X \to r^\bullet X$ into a cofibration $i\colon \ell^\bullet X \to X^*$ which is an isomorphism in degree 0, followed by a weak equivalence s. Let Q denote the pushout in the diagram

$$\begin{tikzcd} \ell^\bullet A \arrow[r, "i"] \arrow[d, "\ell^\bullet f"'] & A^\circ \arrow[d, "e"] \\ \ell^\bullet B \arrow[r, "j"] & Q \end{tikzcd}$$

Since i is a cofibration by the definition of the cosimplicial frame A°, so is j. Since $\ell^\bullet$ is a left Quillen functor (see Section 5.2), $\ell^\bullet f$ is a cofibration, so e is as well. Furthermore, there is an induced isomorphism $B \cong Q[0]$, and with respect to this isomorphism, e is the map f in degree 0. We have a weak equivalence $s\colon A^\circ \to r^\bullet A$ so that si is the canonical map $\ell^\bullet A \xrightarrow{\tau_A} r^\bullet A$. Then $(r^\bullet f)s$ and τ_B define a map $Q \to r^\bullet B$. We factor this map into a cofibration $Q \xrightarrow{k} B^*$ followed by a trivial fibration $B^* \xrightarrow{s} r^\bullet B$, where q and s are both the identity in degree 0. We can do this using the method of Theorem 5.2.8 since Q is isomorphic to B in degree 0. We then find that B^* is a cosimplicial frame on B with structure map $i = kj\colon \ell^\bullet B \to B^*$. We define $f^* = ke$, so that f^* is a cofibration which is isomorphic to the map f in degree 0.

Now we define $g^*\colon B^* \to C^*$ as the cofiber of the cofibration f^*. Then there are induced maps $\ell^\bullet C \xrightarrow{i} C^* \xrightarrow{s} r^\bullet C$ factoring the canonical map. Furthermore, i is an isomorphism in degree 0 and with respect to this isomorphism g^* is g in degree 0. We must show that i is a cofibration and s is a weak equivalence. The proof that

i is a cofibration is just like the proof of the corresponding fact in Lemma 6.2.2. To prove that s is a weak equivalence, note that we have a pushout square

$$\begin{tikzcd} A^\circ[n] \arrow[r, "f^*[n]"] \arrow[d] & B^*[n] \arrow[d, "g^*[n]"] \\ * \arrow[r] & C^*[n] \end{tikzcd}$$

since colimits are taken objectwise. Furthermore, $f^*[n]$ is a cofibration, as pointed out in Remark 5.1.7. Comparing this to the pushout square that defines g, we find from the cube lemma 5.2.6 that the map $C^*[n] \xrightarrow{s[n]} C$ is a weak equivalence, and hence that s is a weak equivalence. $\square$

We can now prove that the other functors associated to the framing of Theorem 5.2.8 preserve cofiber and fiber sequences.

Proposition 6.4.5. *Let $\mathcal{C}$ be a pointed model category.*

(a) *The functor $-\wedge^L -\colon\ \operatorname{Ho}\mathcal{C}\times\operatorname{Ho}\mathbf{SSet}_*$ preserves cofiber sequences in the first variable. That is, suppose K is a pointed simplicial set and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\operatorname{Ho}\mathcal{C}$. Then $X \wedge^L K \xrightarrow{f\wedge^L 1} Y \wedge^L K \xrightarrow{g\wedge^L 1} Z \wedge^L K$ is a cofiber sequence in $\operatorname{Ho}\mathcal{C}$. The coaction on $Z \wedge^L K$ is the composite*
$$\begin{aligned} Z \wedge^L K \to (Z \amalg \Sigma A) \wedge^L K &\cong (Z \wedge^L K) \amalg (\Sigma A \wedge^L K) \\ &\xrightarrow{1\amalg m} (Z \wedge^L K) \amalg \Sigma(A \wedge^L K) \end{aligned}$$
where m is the isomorphism
$$\begin{aligned} (A \wedge^L S^1) \wedge^L K \xrightarrow{a} A \wedge^L (S^1 \wedge^L K)] &\xrightarrow{1\wedge^L T} A \wedge^L (K \wedge^L S^1) \\ &\xrightarrow{a} (A \wedge^L K) \wedge^L S^1 \end{aligned}$$

(b) *The functor $R\operatorname{Hom}_*(-,-)$ preserves fiber sequences in the second variable. That is, suppose K is a pointed simplicial set and $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence in $\operatorname{Ho}\mathcal{C}$. Then $R\operatorname{Hom}_*(K,X) \to R\operatorname{Hom}_*(K,Y) \to R\operatorname{Hom}_*(K,Z)$ is a fiber sequence in $\operatorname{Ho}\mathcal{C}$. The action on $R\operatorname{Hom}_*(K,X)$ is the dual of the coaction in part (a).*

Proof. Part (b) follows from duality and part (a), so we only prove part (a). As usual, we can assume our cofiber sequence is of the form $A \xrightarrow{f} B \xrightarrow{g} C$ where f is a cofibration in $\mathcal{C}$ with cofiber g and A, B and C are all cofibrant. We choose maps of cosimplicial frames $A^\circ \xrightarrow{f^*} B^* \xrightarrow{g^*} C^*$ as in Lemma 6.4.4, so that f^* is a cofibration with cofiber g^*. Then Proposition 5.7.1 implies that $A \wedge K \to B^* \wedge K$ is a cofibration with cofiber $B^* \wedge K \to C^* \wedge K$.

In order to calculate the coaction, we choose a cofibration of cosimplicial frames $A^\circ \amalg A^\circ \xrightarrow{(i_0^*, i_1^*)} (A \times I)^*$ using Lemma 6.4.4, whose cofiber is a map of cosimplicial frames $(A \times I)^* \xrightarrow{r^*} (A \wedge S^1)^*$. Define B_f^* as the pushout of $(A \times I)^*$ and B^*, as B_f is defined in the definition of ψ. Then B_f^* is a cosimplicial frame on B_f. Similarly, define $\widetilde{B}^*$ as the pushout of $(A \amalg I)^*$ and $B^* \amalg B^*$, so that $\widetilde{B}^*$ is a cosimplicial frame on $\widetilde{B}$. We can then factor the map $\widetilde{B}^* \to B^*$ into a cofibration $\widetilde{B}^* \to (B')^*$ followed by a weak equivalence, such that the factorization in degree

0 is the one used in the definition of ψ. The proof of this is similar to the proof of Theorem 5.2.8. Then $(B')^*$ is a cosimplicial frame on B'. Finally, define $\widetilde{C}^*$ as the pushout of $C^* \amalg (A \wedge S^1)^*$ and $(B')^*$, just as we defined $\widetilde{C}$. Then $\widetilde{C}^*$ is a cosimplicial frame on $\widetilde{C}$, and we have a weak equivalence $\pi^*\colon\ C^* \amalg (A \wedge S^1)^* \longrightarrow \widetilde{C}^*$.

The functor that takes a cosimplicial frame X^* to $X^* \wedge K$ preserves cofibrations, weak equivalences between cofibrant objects, and colimits, by Proposition 5.4.1. It follows that the coaction on $C^* \wedge K$ is the composite

$$\begin{aligned} C^* \wedge K \xrightarrow{i_1^*} \widetilde{C}^* \wedge K \xrightarrow{(\pi^* \wedge 1)^{-1}} & (C^* \wedge K) \amalg ((A \wedge S^1)^* \wedge K) \\ \xrightarrow{1 \amalg t} & (C^* \wedge K) \amalg ((A \wedge K) \wedge S^1). \end{aligned}$$

where t is induced by any map of cylinder objects $(A \times I)^* \longrightarrow A^\circ \times I$. For any cosimplicial frame X^* on X, there is a map of cosimplicial frames $X^* \longrightarrow X^\circ$ by Lemma 5.5.1. Applying this idea and Lemma 5.5.2, we find that the coaction on $C \wedge K$ is the composition

$$C \wedge K \xrightarrow{\psi \wedge 1} (C \wedge K) \amalg ((A \wedge S^1) \wedge K) \xrightarrow{1 \amalg t} (C \wedge K) \amalg ((A \wedge K) \wedge S^1)$$

where t is induced by any map of cylinder objects. The map $a(1 \wedge^L T)a\colon\ (A \wedge I_+) \wedge K \longrightarrow (A \wedge K) \wedge I_+$ comes from a map of cylinder objects, so we can use it for t. □

6.5. Pre-triangulated categories

In this section, we abstract the properties of cofiber and fiber sequences in the homotopy category of a pointed model category to define a pre-triangulated category.

Let $\mathcal{S}$ be a (right) closed $\operatorname{Ho}\mathbf{SSet}_*$-module. We will denote the associated functors by $A \wedge K$, $\operatorname{Hom}(K, A)$, and $\operatorname{Map}(A, B)$. We assume that $\mathcal{S}$ is non-trivial, so has at least one object. Then, by adjointness, $A \wedge *$ is an initial object 0 of $\mathcal{S}$ for any $A \in \mathcal{S}$. Dually, $\operatorname{Hom}(*, A)$ is a terminal object 1 of $\mathcal{S}$. Furthermore, for any $A \in \mathcal{S}$, there is a map $A \wedge S^0 \longrightarrow A \wedge * = 0$. In particular, there is a map $1 \longrightarrow 0$, so $\mathcal{S}$ is pointed. We denote the initial and terminal object by $*$ rather than 0 or 1.

We have suspension and loop functors in $\mathcal{S}$, defined by $\Sigma A = A \wedge S^1$ and $\Omega A = \operatorname{Hom}(S^1, A)$. The same argument as used in Corollary 6.1.6 shows that ΣA is naturally a cogroup object and $\Sigma^t A$ is naturally an abelian cogroup object for $t \geq 2$. Dually, ΩA is naturally a group object and $\Omega^t A$ is naturally an abelian group object for $t \geq 2$.

Definition 6.5.1. Suppose $\mathcal{S}$ is a nontrivial (right) closed $\operatorname{Ho}\mathbf{SSet}_*$-module. A *pre-triangulation* on $\mathcal{S}$ is a collection of *cofiber sequences*, or *left triangles*, and *fiber sequences*, or *right triangles*, satisfying the following conditions.

(a) A cofiber sequence is in particular a diagram of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ together with a right coaction of the cogroup ΣX on Z. A fiber sequence is in particular a diagram of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ together with a right action of the group ΩZ on X.
(b) Every diagram isomorphic to a cofiber sequence is a cofiber sequence. Similarly, every diagram isomorphic to a fiber sequence is a fiber sequence. Note that the isomorphisms must take into account the action, as in Lemma 6.3.1.

(c) For any X, the diagram $* \to X \xrightarrow{1} X$ is a cofiber sequence (with the only possible coaction). The diagram $X \xrightarrow{1} X \to *$ is a fiber sequence (with the only possible action).
(d) Every map is part of a cofiber and fiber sequence, as in Lemma 6.3.3.
(e) Cofiber sequences can be shifted to the right, and fiber sequences can be shifted to the left, as in Proposition 6.3.4.
(f) Fill-in maps exist, as in Proposition 6.3.5.
(g) Verdier's octahedral axiom and its dual both hold, as in Proposition 6.3.6.
(h) Cofiber and fiber sequences are compatible, as in Proposition 6.3.7.
(i) The smash product preserves cofiber sequences in each variable. The functor $\mathrm{Hom}(-,-)$ preserves fiber sequences in the second variable and converts cofiber sequences in the first variable into fiber sequences. The functor $\mathrm{Map}(-,-)$ preserves fiber sequences in the second variable and converts cofiber sequences in the first variable into fiber sequences. See Corollary 6.4.2 and Proposition 6.4.5 for exact statements.

Having defined a pre-triangulation, we also need to define a pre-triangulated category.

DEFINITION 6.5.2. A *pre-triangulated category* is a nontrivial closed $\mathrm{Ho}\,\mathbf{SSet}_*$-module $\mathcal{S}$ with all small coproducts and products, together with a pre-triangulation on $\mathcal{S}$.

We have seen in the last four sections that the homotopy category of a pointed model category is a pre-triangulated category. Pre-triangulated categories are the unstable analog of triangulated categories, studied by many people. See, for example, [**BBD82**],[**HPS97**], and [**Nee92**]. We will discuss the relationship between triangulated and pre-triangulated categories in the next chapter.

We point out however that pre-triangulated categories, while convenient for our purposes, do not capture all the good properties of the homotopy category of a pointed model category. For example, in the homotopy category of a pointed model category, coproducts of cofiber sequences are again cofiber sequences, by Proposition 6.4.1. We have not been able to prove this in an arbitrary pre-triangulated category. One could add this as an axiom, of course.

We now discuss a few properties of pre-triangulated categories. Note first of all that the axioms for a pre-triangulated category are self-dual. That is, if $\mathcal{S}$ is a pre-triangulated category, so is $D\mathcal{S}$ with the dual $\mathrm{Ho}\,\mathbf{SSet}_*$ action. The cofiber sequences in $D\mathcal{S}$ correspond to the fiber sequences in $\mathcal{S}$, and similarly for fiber sequences. This means that properties of cofiber sequences and properties of fiber sequences are dual.

We will denote the morphisms in a pre-triangulated category by $[X,Y]$. Given a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in a pre-triangulated category, the coaction induces a map $Z \xrightarrow{\partial} \Sigma X$ as the composite $Z \to Z \amalg \Sigma X \xrightarrow{(0,1)} \Sigma X$. Note that ∂ preserves the coaction. Dually, given a fiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is an induced map $\partial \colon \Omega Z \to X$.

PROPOSITION 6.5.3. *Suppose $\mathcal{S}$ is a pre-triangulated category.*

(a) *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\mathcal{S}$, and W is an object of $\mathcal{S}$. Then we have a long exact sequence of pointed sets*

$$\dots \xrightarrow{(\Sigma\partial)^*} [\Sigma Z, W] \xrightarrow{(\Sigma g)^*} [\Sigma Y, W] \xrightarrow{(\Sigma f)^*} [\Sigma X, W]$$
$$\xrightarrow{\partial^*} [Z, W] \xrightarrow{g^*} [Y, W] \xrightarrow{f^*} [X, W]$$

This long exact sequence satisfies the following additional properties.

(i) *We have $g^*a = g^*b$ if and only if there is an $x \in [\Sigma X, W]$ such that $a \odot x = b$ under the action of the group $[\Sigma X, W]$ on $[Z, W]$.*

(ii) *Similarly, $\partial^* c = \partial^* d$ if and only if there is a $y \in [\Sigma Y, W]$ such that $c = d(\Sigma f)^* y$ under the product in the group $[\Sigma X, W]$.*

(b) *Suppose we have a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[d, "c"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z'
\end{tikzcd}$$

where the rows are cofiber sequences and where c is Σa-equivariant. Then if a and b are isomorphisms, so is c.

Of course, there is a dual proposition for fiber sequences.

Proof. For part (a), note that we have a fiber sequence

$$\mathrm{Map}(Z, W) \longrightarrow \mathrm{Map}(Y, W) \longrightarrow \mathrm{Map}(X, W)$$

in $\mathrm{Ho}\,\mathbf{SSet}_*$. Also, we have $\pi_i \mathrm{Map}(Z, W) \cong [\Sigma^i Z, W]$ by adjointness. It suffices therefore to prove that we have a long exact sequence in homotopy given a fiber sequence of pointed simplicial sets. Such a fiber sequence can be realized by a fibration of fibrant objects $p\colon E \longrightarrow B$ with fiber F. Applying the geometric realization, we get a fibration $|p|\colon |E| \longrightarrow |B|$ with fiber $|F|$, by Lemma 3.2.4 and Corollary 3.6.2. This reduces us to studying the homotopy sequence of a fibration of pointed topological spaces. Here Lemma 2.4.16 applies to give us the required long exact sequence. One could also construct this long exact sequence by hand, of course, as is done in [**Qui67**, Proposition I.3.4].

It remains to verify the improved exactness properties (i) and (ii) of part (a). Translating the problem to $\mathbf{Top}_*$ as above, we have a fibration $p\colon E \longrightarrow B$ with fiber $i\colon F \longrightarrow E$. Suppose we have $u, v \in \pi_0 F$ such that $i_* u = i_* v$. We must show there is an $\omega \in \pi_1 B$ such that $u \odot \omega = v$. Of course, u and v just correspond to (path components of) points of F. To say that $i_* u = i_* v$ just means that there is a path $\widetilde{\omega}$ from u to v in E. Then $p\widetilde{\omega}$ is a loop in B, so represents an element $\omega \in \pi_1 B$. We compute $u \odot \omega$ by finding a lift of ω to E which starts at u. The path $\widetilde{\omega}$ is such a lift, and so $u \odot \omega$ is the end of $\widetilde{\omega}$, namely v. The converse is straightforward.

Similarly, suppose $c, d \in \pi_1 B$ satisfy $\partial_* c = \partial_* d$, so that $0 \odot c = 0 \odot d$. Then there are lifts ω and φ of (representatives of) c and d to paths in E beginning at the basepoint, such that the endpoints of ω and φ lie in the same path component of F. We can therefore choose a path between these endpoints lying in F. Putting these paths together, we get a loop ρ in E whose projection down to B is homotopic to $d^{-1}c$. Thus $c = d(\Omega p)_* \rho$, as required. The converse is straightforward.

Part (b) will follow from part (a) by a complicated version of the five-lemma. To see this, suppose we have a commutative diagram as in part (b). Using the inverses of a and b and the existence of fill-in maps, we find that we can assume a and b are the identity maps. Then, for any object W, we have a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & {[\Sigma X, W]} \arrow[r, "\partial^*"] \arrow[d, equal] & {[Z, W]} \arrow[r, "g^*"] \arrow[d, "c^*"'] & {[Y, W]} \arrow[r, "f^*"] \arrow[d, equal] & {[X, W]} \arrow[d, equal] \\
\dots \arrow[r] & {[\Sigma X, W]} \arrow[r, "\partial^*"] & {[Z, W]} \arrow[r, "g^*"] & {[Y, W]} \arrow[r, "f^*"] & {[X, W]}
\end{tikzcd}$$

and we must show that c^* is an isomorphism. Note that, for $x \in [Z, W]$, we have $g^* c^*(x) = g^*(x)$. Thus, by the improved exactness of part (a), we have $c^* x = x\alpha$ for some $\alpha \in [\Sigma X, W]$. Then $c^*(x\alpha^{-1}) = (c^* x)\alpha^{-1} = x$, so c^* is surjective. Now suppose that $c^*(x) = c^*(y)$. Then the same argument shows that $y = x\beta$ for some β. Furthermore, we must have $c^*(x)\beta = c^*(x\beta) = c^*(x)$, so β is in the stabilizer $\operatorname{Stab}(c^*(x))$. It is clear that $\operatorname{Stab}(x) \subseteq \operatorname{Stab}(c^*(x))$ by the equivariance of c^*. We claim that this is in fact an equality. Indeed, if $h \in \operatorname{Stab}(c^*(x)) = \operatorname{Stab}(x\alpha)$, then $\alpha h \alpha^{-1}$ is in $\operatorname{Stab}(x)$. Since $\operatorname{Stab}(x) \subseteq \operatorname{Stab}(c^*(x))$, this means that α and α^{-1} are both in the normalizer of $\operatorname{Stab}(x)$. But conjugation by α^{-1} gives an isomorphism from $\operatorname{Stab}(x)$ to $\operatorname{Stab}(c^*(x))$. Thus $\operatorname{Stab}(c^*(x)) = \operatorname{Stab}(x)$, and so $y = x\beta = x$, as required. □

Having defined pre-triangulated categories, we now define the 2-category of pre-triangulated categories.

Definition 6.5.4. Suppose $\mathcal{S}$ and $\mathcal{T}$ are pre-triangulated categories. An *exact adjunction* from $\mathcal{S}$ to $\mathcal{T}$ is an adjunction of closed $\operatorname{Ho}\mathbf{SSet}_*$-modules (F, U, φ, m) where φ is the adjointness isomorphism and m is the natural isomorphism $FX \wedge K \to F(X \wedge K)$, such that F preserves cofiber sequences and U preserves fiber sequences. That is, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence in $\mathcal{S}$, then $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ$ is a cofiber sequence in $\mathcal{T}$, where the coaction is induced by the coaction on Z and the isomorphism $F\Sigma X \cong \Sigma FX$ induced by m. We say that F is *left exact*. There is a similar statement for fiber sequences and U, and we say that U is *right exact*.

We leave it to the reader to check that we get a 2-category of pre-triangulated categories, exact adjunctions, and natural transformations of $\operatorname{Ho}\mathbf{SSet}_*$-module functors. Such natural transformations are automatically stable. in the sense that they commute with the suspension (or smashing with any other pointed simplicial set).

The reader who is familiar with triangulated categories may suspect that the condition that U preserve fiber sequences should be redundant, as it is for triangulated categories. However, we have been unable to prove that. We can show that, if F is left exact with right adjoint U, and $p : E \to B$ is a map with fiber F, then the fiber of Up is UF. This gives us two actions of $U\Omega B$ on UF, and we do not know how to see that they are the same.

The duality we have already discussed for pre-triangulated categories gives a duality 2-functor D. We then have the following theorem, which sums up the results of this chapter so far.

THEOREM 6.5.5. *The homotopy pseudo-2-functor of Theorem 5.7.3 lifts to a pseudo-2-functor from pointed model categories to pre-triangulated categories which commutes with the duality 2-functor.*

6.6. Pointed monoidal model categories

In this brief section, we define the notion of a closed monoidal pre-triangulated category and show that the homotopy category of a pointed monoidal model category is naturally one.

We begin with the definition of a closed monoidal pre-triangulated category.

DEFINITION 6.6.1. Define a *closed monoidal pre-triangulated category* to be a closed $\operatorname{Ho}\mathbf{SSet}_*$-algebra $\mathcal{S}$ with all coproducts and products, together with a pre-triangulation on the underlying closed $\operatorname{Ho}\mathbf{SSet}_*$-module, such that the following properties hold.

(a) The functor $-\wedge-\colon \mathcal{S}\times\mathcal{S}\longrightarrow\mathcal{S}$ preserves cofiber sequences in each variable.
(b) The functors $\operatorname{Hom}_\ell(-,-), \operatorname{Hom}_r(-,-)\colon \mathcal{S}^{\mathrm{op}}\times\mathcal{S}\longrightarrow\mathcal{S}$ preserve fiber sequences in the second variable and convert cofiber sequences in the first variable into fiber sequences.

Of course, when we say that $-\wedge-$ preserves cofiber sequences in each variable, we mean that the coaction is also determined, as in Definition 6.5.4, and similarly for the right adjoints. We have analogous definitions of a *closed central monoidal pre-triangulated category* and a *closed symmetric monoidal pre-triangulated category*. In the latter case, we require that the monoidal functor $i\colon \operatorname{Ho}\mathbf{SSet}_*\longrightarrow\mathcal{S}$ be symmetric monoidal.

In a closed monoidal pre-triangulated category, we often write S^n for $\Sigma^n S = i(S^n)$, where $n\geq 0$. We then have the following lemma.

LEMMA 6.6.2. *Suppose $\mathcal{S}$ is a closed symmetric monoidal pre-triangulated category. Then the following diagram commutes.*

$$\begin{array}{ccc} S^m\wedge S^n & \xrightarrow{\mu} & S^{m+n} \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle (-1)^{mn}} \\ S^n\wedge S^m & \xrightarrow{\mu} & S^{m+n} \end{array}$$

Here μ denotes the multiplicativity isomorphism of the monoidal functor i, T denotes the commutativity isomorphism of $\mathcal{S}$, and -1 denotes the inverse of the identity with respect to the abelian group structure on S^{m+n}, unless m or n is 0, where the question does not arise.

PROOF. By using the various commutativity and associativity coherence diagrams, one can see that the lemma is automatic in case m or n is 0, and follows in general from the case $m=n=1$. It suffices to prove the analogous diagram for $m=n=1$ commutes in $\operatorname{Ho}\mathbf{SSet}_*$, and hence it suffices to prove it in $\operatorname{Ho}\mathbf{K}_*$, or in $\operatorname{Ho}\mathbf{Top}_*$. In this case, we think of $S^1\times S^1$ as the usual quotient of a square where the opposite sides are identified. The twist map is reflection about one of the diagonals. If we then identify all of the boundary of the square to get S^2, we find that the twist map is homotopic to the map that takes (x,y,z) to $(x,y,-z)$. This map is well-known to have degree -1. See [**Mun84**, Theorem 21.3], for example. $\square$

We then have an obvious notion of a closed pre-triangulated module over a closed monoidal pre-triangulated category as well. A pre-triangulated category is the same thing as a closed pre-triangulated module over the closed symmetric monoidal pre-triangulated category $\operatorname{Ho}\mathbf{SSet}_*$.

We define a morphism of closed monoidal pre-triangulated categories to be an adjunction of closed $\operatorname{Ho}\mathbf{SSet}_*$-algebras which is an exact adjunction of the underlying pre-triangulated categories. In this way we get a 2-category of closed monoidal pre-triangulated categories, where the 2-morphisms are 2-morphisms of the underlying $\operatorname{Ho}\mathbf{SSet}_*$-algebra functors.

We then get the following theorems.

THEOREM 6.6.3. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from pointed monoidal model categories to closed monoidal pre-triangulated categories.*

PROOF. Suppose $\mathcal{C}$ is a pointed monoidal model category. We have already seen that $\operatorname{Ho}\mathcal{C}$ is a closed $\operatorname{Ho}\mathbf{SSet}_*$-algebra and a pre-triangulated category. It thus suffices to show that $- \wedge^L -$ preserves cofiber sequences in each variable, and similarly for the adjoints of $- \wedge^L -$. But this follows from Proposition 6.4.1, since for any cofibrant A, $A \wedge -$ and $- \wedge A$ are left Quillen functors. We have already seen that morphisms and 2-morphisms behave correctly. □

The following theorem then follows from Theorem 5.7.6 and Corollary 5.7.8.

THEOREM 6.6.4. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from pointed monoidal model categories satisfying Conjecture 5.7.5 to closed central monoidal pre-triangulated categories, and to a pseudo-2-functor from pointed symmetric monoidal model categories satisfying Conjecture 5.7.5 to closed symmetric monoidal pre-triangulated categories.*

Of course, the homotopy category of a symmetric monoidal model category is naturally both a closed symmetric monoidal category and a closed monoidal pre-triangulated category, even if Conjecture 5.7.5 does not hold. The point is that the functor $i\colon \operatorname{Ho}\mathbf{SSet}_* \to \operatorname{Ho}\mathcal{C}$ may not be symmetric monoidal in this case.

CHAPTER 7

Stable model categories and triangulated categories

We have just seen that the homotopy category of a pointed model category $\mathcal{C}$ is naturally a pre-triangulated category. In this chapter, we examine what happens when the suspension functor is an equivalence on $\operatorname{Ho}\mathcal{C}$. We refer to a pre-triangulated category where the suspension functor is an equivalence as a *triangulated category*, and we refer to a pointed model category whose homotopy category is triangulated as a *stable model category*. Of course, there is already a well-known definition of a triangulated category, and the definition we give does not coincide with the classical definition. We justify this in Section 7.1 by showing that every triangulated category is a classical triangulated category, and that we can recover most of the structure of a triangulated category from a classical triangulated category. Our position is that every classical triangulated category that arises in nature is the homotopy category of a stable model category, so is triangulated in our sense.

For the rest of the chapter, we examine generators in the homotopy category of a stable model category. These generators are very important in [**HPS97**], and we try to uncover their precursors in the model category world. In Section 7.2, we remind the reader of the definition of an algebraic stable homotopy category, the only kind of stable homotopy category we treat in this book. This section provides some of the motivation for the next two sections. In Section 7.3, we construct weak generators in the homotopy category of a pointed cofibrantly generated model category. In Section 7.4 we discuss finitely generated model categories and show that, in this case, the weak generators of Section 7.3 are small in an appropriate sense.

The material in this chapter is all new, so far as the author knows. We do demand a little more of the reader than in previous chapters as well. In particular, we use the theory of homotopy limits of diagrams of simplicial sets from [**BK72**].

7.1. Triangulated categories

In this section we define triangulated categories and study some of their properties. Triangulated categories were first introduced by Verdier in [**Ver77**], and have been very useful since then. A good introduction to triangulated categories can be found in [**Mar83**, Appendix 2]. The definition we give is new, and is stronger than the usual one. Perhaps we should call our triangulated categories *simplicially triangulated*, but we do not, since every triangulated category with the standard definition that we know of is also a triangulated category with our stronger definition.

DEFINITION 7.1.1. A *triangulated category* is a pre-triangulated category in which the suspension functor Σ is an equivalence of categories. A pointed model category is *stable* if its homotopy category is triangulated.

We then have an obvious 2-category of triangulated categories, namely the full sub-2-category of the 2-category of pre-triangulated categories whose objects consist of triangulated categories. This full sub-2-category is closed under the duality 2-functor, since Σ is an equivalence if and only if its adjoint Ω is an equivalence.

Similarly, we have a 2-category of stable model categories. If R is a ring, the model category $\mathrm{Ch}(R)$ is stable , with any of the model structures in Section 2.3. Similarly, if B is a commutative Hopf algebra over a field, then $\mathrm{Ch}(B)$ is a stable model category. On the other hand $\mathbf{K}_*$ and $\mathbf{SSet}_*$ are definitely not stable model categories. The model categories of [**EKMM97**] and [**HSS98**] are stable model categories whose homotopy categories are equivalent to the standard stable homotopy category of spectra. In the model categories $\mathrm{Ch}(R)$ and $\mathrm{Ch}(B)$, the suspension functor is already an equivalence before passing to the homotopy category. The reader may think it preferable to require this of any stable model category. This is not reasonable, however, because changing the functorial factorization changes the definition of the suspension. The suspension may be an equivalence before passing to the homotopy category with one functorial factorization and not with another. Furthermore, the suspension is not an equivalence in the model category of symmetric spectra studied in [**HSS98**], though it is an equivalence in the homotopy category.

We have analogous definitions of a closed monoidal triangulated category and of a closed (pre-)triangulated module over a closed monoidal triangulated category. Such a closed module is in fact automatically triangulated, as the reader can easily check. The homotopy category of a stable monoidal model category is a closed monoidal triangulated category, and will be a closed central monoidal triangulated category if Conjecture 5.7.5 holds for the model category. Similarly, the homotopy category of a stable symmetric monoidal model category is both a closed symmetric monoidal category and a closed monoidal triangulated category, but we do not know it is a closed symmetric monoidal triangulated category unless Conjecture 5.7.5 holds for the model category.

We must of course relate our definition of a triangulated category to the standard one. We begin this process with the following lemma.

LEMMA 7.1.2. *Triangulated categories are additive.*

PROOF. Suppose $\mathcal{S}$ is a triangulated category. Since Σ is an equivalence, so is Σ^2. Thus we have a natural isomorphism $\Sigma^2\Omega^2 X \to X$. Since $\Sigma^2 Z$ is an abelian cogroup object for any Z, and $\Sigma^2 f$ is an abelian cogroup map for any f, this proves that every object of $\mathcal{S}$ is an abelian cogroup object and that every map is an abelian cogroup map. To complete the proof that $\mathcal{S}$ is additive, we only have to show that the canonical map $X \amalg Y \to X \times Y$ is an isomorphism. But since both coproducts and products exist in $\mathcal{S}$, this is purely formal, and we leave it to the reader. $\square$

REMARK 7.1.3. Because of this lemma, a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in a triangulated category is completely determined by f, g, and the map $Z \xrightarrow{\partial} \Sigma X$. Indeed, the coaction of ΣX on Z is a map $Z \to Z \amalg \Sigma X \cong Z \times \Sigma X$. The unit axiom forces the first component of this coaction to be 1_Z, and the second component is ∂.

For this reason, in a triangulated category $\mathcal{S}$, we will refer to a cofiber sequence, or *triangle*, as a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. Note that if we have a commutative diagram

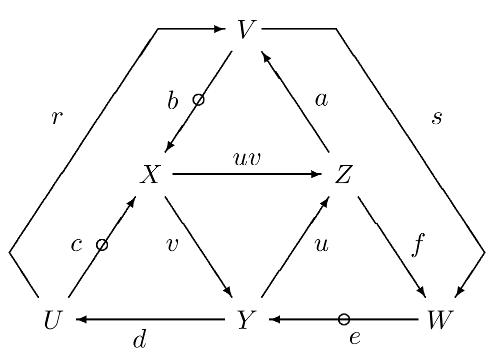
$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & & & {\scriptstyle \Sigma a}\downarrow \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'
\end{array}$$

a fill-in map $c\colon Z \longrightarrow Z'$ is Σa-equivariant if and only if $\Sigma a \circ h = h' \circ c$, so our notion of a map of cofiber sequence also translates correctly to the triangulated situation. Dually, we will refer to a fiber sequence in a triangulated category as a diagram $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$.

We now show that a triangulated category in the sense of Definition 7.1.1 is also a triangulated category in the classical sense. First we recall the triangulated version of Verdier's octahedral axiom.

Definition 7.1.4. Suppose $\mathcal{S}$ is an additive category equipped with an additive endofunctor $\Sigma\colon \mathcal{S} \longrightarrow \mathcal{S}$ and a collection of diagrams of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, called *triangles*. We abbreviate such a triangle by (X, Y, Z). We say that *Verdier's octahedral axiom holds* if, for every pair of maps $X \xrightarrow{v} Y \xrightarrow{u} Z$, and triangles (X, Y, U), (X, Z, V) and (Y, Z, W) as shown in the diagram (where a circled arrow $U \multimap X$ means a map $U \longrightarrow \Sigma X$), there are maps r and s as shown, making (U, V, W) into a triangle, such that the following commutativities hold:

$$au = rd \qquad es = (\Sigma v)b \qquad sa = f \qquad br = c$$



This is the form of the octahedral axiom given in [**HPS97**], and is equivalent to the original definition given by Verdier in the presence of the other axioms for a classical triangulated category. The reader should compare this to Proposition 6.3.6.

We can now give the classical definition of a triangulated category.

Definition 7.1.5. Suppose $\mathcal{S}$ is an additive category. A *classical triangulation* on $\mathcal{S}$ is an additive self-equivalence $\Sigma\colon \mathcal{S} \longrightarrow \mathcal{S}$ together with a collection of diagrams of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, called *triangles*, satisfying the following properties.

(a) Triangles are replete. That is, any diagram isomorphic to a triangle is a triangle.
(b) For any X, the diagram $* \longrightarrow X \xrightarrow{1} X \longrightarrow \Sigma * = *$ is a triangle.
(c) Given any map $f\colon X \longrightarrow Y$, there is a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.
(d) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle, so is $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.
(e) Given a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma a"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

whose rows are triangles and the left square is commutative, there is a map $c\colon Z \longrightarrow Z'$ making the entire diagram commute.
(f) Verdier's octahedral axiom holds.

A *classical triangulated category* is an additive category together with a classical triangulation on it.

We then have the following proposition, whose proof is just a matter of rewriting the pre-triangulated axioms in the additive case.

PROPOSITION 7.1.6. *Suppose $\mathcal{S}$ is a triangulated category. Then the suspension functor and cofiber sequences in $\mathcal{S}$ make $\mathcal{S}$ into a classical triangulated category.*

The converse to Proposition 7.1.6 is extremely unlikely to be true, though we do not know of a counterexample. A classical triangulated category is not a closed $\operatorname{Ho}\mathbf{SSet}_*$-module, and there doesn't seem to be any reason it should be. It also does not have fiber sequences, only cofiber sequences. However, that problem turns out not be a problem at all. Indeed, we will show that, in a triangulated category, the fiber sequences are completely determined by the cofiber sequences.

Before doing this, we show that, in a triangulated category, a cofiber sequence can be shifted to the left as well as to the right. The following lemma is [**Mar83**, Lemma A2.8].

LEMMA 7.1.7. *Suppose $\mathcal{S}$ is a triangulated category, and suppose*

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \xrightarrow{\Sigma h} \Sigma^2 X$$

is a cofiber sequence. Then so is $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma X$.

Note that the converse to this lemma is immediate from the axioms.

PROOF. There is some cofiber sequence $X \xrightarrow{-f} Y \xrightarrow{-g'} Z' \xrightarrow{-h'} \Sigma X$. We then get a commutative diagram

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] \arrow[d, equal] & \Sigma Y \arrow[r, "\Sigma g"] \arrow[d, equal] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X \arrow[d, equal] \\ \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g'"] & \Sigma Z' \arrow[r, "\Sigma h'"] & \Sigma^2 X \end{tikzcd}$$

where the rows are cofiber sequences. There is then a fill-in map $\Sigma Z \to \Sigma Z'$, which is an isomorphism by part (b) of Proposition 6.5.3. Since Σ is an equivalence of categories, we can write this map as Σk for some map $k\colon Z \longrightarrow Z'$. We then get an isomorphism of sequences from the desired sequence to the cofiber sequence $X \xrightarrow{-f} Y \xrightarrow{-g'} Z' \xrightarrow{-h'} \Sigma X$. □

This lemma allows us to shift cofiber sequences to the left, as we prove in the following proposition. We need some notation to do so. Let $\varepsilon_X \colon \Sigma\Omega X \longrightarrow X$ and $\eta_X \colon X \longrightarrow \Omega\Sigma X$ denote the counit and unit of the adjunction between Σ and Ω in a closed $\operatorname{Ho}\mathbf{SSet}_*$-module. Then we have $(\Omega\varepsilon_X) \circ \eta_{\Omega X} = 1$ and $\varepsilon_{\Sigma X} \circ (\Sigma\eta_X) = 1$.

PROPOSITION 7.1.8. *Suppose $\mathcal{S}$ is a triangulated category. Then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a cofiber sequence if and only if $\Omega Z \xrightarrow{-\eta_X^{-1} \circ (\Omega h)} X \xrightarrow{f} Y \xrightarrow{\varepsilon_Z^{-1} \circ g} \Sigma\Omega Z$ is a cofiber sequence.*

PROOF. Suppose first that $(\Omega Z, X, Y)$ is a cofiber sequence. Then we find by shifting to the right that the top row in the following diagram is a cofiber sequence.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\varepsilon_Z^{-1} \circ g"] \arrow[d, equal] & \Sigma\Omega Z \arrow[r, "(\Sigma\eta_X)^{-1} \circ (\Sigma\Omega h)"] \arrow[d, "\varepsilon_Z"'] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The right-most square of this diagram commutes because $(\Sigma\eta_X)^{-1} = \varepsilon_{\Sigma X}$ and because ε is natural. Thus the bottom row must also be a cofiber sequence.

Conversely, suppose (X, Y, Z) is a cofiber sequence. The commutative diagram

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\varepsilon_X"'] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] \arrow[d, "\varepsilon_Y"'] & \Sigma\Omega Z \arrow[r, "(\Sigma\varepsilon_X^{-1}) \circ (\Sigma\eta_X)^{-1} \circ (\Sigma\Omega h)"] \arrow[d, "\varepsilon_Z"'] & \Sigma^2\Omega X \arrow[d, "\Sigma\varepsilon_X"'] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

shows that the top row is also a cofiber sequence. Lemma 7.1.7 then shows that the sequence $\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{-\Omega g} \Omega Z \xrightarrow{-\varepsilon_X^{-1} \circ \eta_X^{-1} \circ \Omega h} \Sigma\Omega X$ is a cofiber sequence. Shifting this cofiber sequence to the right two places, we find that the top row of the following commutative diagram is a cofiber sequence.

$$\begin{tikzcd}
\Omega Z \arrow[r, "-\varepsilon_X^{-1} \circ \eta_X^{-1} \circ \Omega h"] \arrow[d, equal] & \Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\varepsilon_X"'] & \Sigma\Omega Y \arrow[r, "\Sigma\Omega g"] \arrow[d, "\varepsilon_Y"'] & \Sigma\Omega Z \arrow[d, "\varepsilon_Z"'] \\
\Omega Z \arrow[r, "-\eta_X^{-1} \circ \Omega h"] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

is a cofiber sequence. Hence the bottom row is as well, completing the proof. □

REMARK 7.1.9. The dual of Proposition 7.1.8 says that we can shift fiber sequences in a triangulated category to the right. That is, $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ is a fiber sequence if and only if $\Omega\Sigma X \xrightarrow{-g \circ \eta_X^{-1}} Y \xrightarrow{h} Z \xrightarrow{\Sigma f \circ \varepsilon_Z^{-1}} \Sigma X$ is a fiber sequence.

We also need to know that mapping into a cofiber sequence in a triangulated category gives an exact sequence, just as mapping out of one does.

LEMMA 7.1.10. *Suppose $\mathcal{S}$ is a classical triangulated category, and suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle in $\mathcal{S}$. Then, for any $W \in \mathcal{S}$, the sequence*

$$[W, X] \xrightarrow{f_*} [W, Y] \xrightarrow{g_*} [W, Z] \xrightarrow{h_*} [W, \Sigma X]$$

is exact.

Of course, the dual statement also holds, and tells us that mapping out of a fiber sequence in a triangulated category gives an exact sequence.

PROOF. We first show that gf and hg are both 0. Indeed, consider the commutative diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d, "0"'] & Y \arrow[r, "="] \arrow[d, equal] & Y \arrow[r] & * \arrow[d, "0"'] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

where the rows are triangles (by axiom (b) of Definition 7.1.5). There is a fill-in map $c\colon Y \to Z$ making the diagram commute. It follows that we must have $c = g$, and therefore that $hg = 0$. Similarly, by applying axiom (d) to axiom (b), we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, equal] & X \arrow[r, "0"] \arrow[d, "f"'] & * \arrow[r, "0"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \end{tikzcd}$$

Thus there is a fill-in map $* \to Z$ making the diagram commutative. This fill-in map must of course be the zero map, and so we have $gf = 0$.

Now suppose we have a map $j\colon W \to Y$ such that $gj = 0$. Then we get a commutative diagram

$$\begin{tikzcd} W \arrow[r, "0"] \arrow[d, "j"'] & * \arrow[r, "0"] \arrow[d, "0"'] & \Sigma W \arrow[r, "-\Sigma 1"] & \Sigma W \arrow[d, "\Sigma j"'] \\ Y \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \end{tikzcd}$$

Thus there is a fill-map $\Sigma W \to \Sigma X$ making the diagram commute. Since Σ is an equivalence of categories, we can write this map as Σk for some map $k\colon X \to W$. Then $\Sigma(f \circ k) = \Sigma j$, so $f \circ k = j$, as required.

Similarly, suppose we have a map $j\colon W \to Z$ such that $hj = 0$. Then the same argument, shifted over to the right one spot, yields a map k such that $j = gk$. □

We can now show that the fiber sequences in a triangulated category are completely determined by the cofiber sequences, as promised.

THEOREM 7.1.11. *Suppose $\mathcal{S}$ is a triangulated category. Then the sequence $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ is a fiber sequence if and only if the sequence $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{-\varepsilon_Z^{-1} \circ h} \Sigma\Omega Z$ is a cofiber sequence.*

PROOF. Suppose $\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{-\varepsilon_Z^{-1}\circ h} \Sigma\Omega Z$ is a cofiber sequence. There is some fiber sequence $\Omega Z \xrightarrow{f'} X' \xrightarrow{g'} Y \xrightarrow{h} Z$. Consider the commutative diagram

$$\begin{tikzcd} \Omega Z \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] & Y \arrow[r, "-\varepsilon_Z^{-1}\circ h"] \arrow[d, equal] & \Sigma\Omega Z \arrow[d, "-\varepsilon_Z"'] \\ \Omega Z \arrow[r, "f'"] & X' \arrow[r, "g'"] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

Since the top row is a cofiber sequence and the bottom row is a fiber sequence, the compatibility between cofiber and fiber sequences guarantees that there is a map $k\colon X \to X'$ making the diagram commute. We claim that k is an isomorphism. To see this, we use the five-lemma. Suppose W is an arbitrary object of $\mathcal{S}$. Then Proposition 7.1.8 and Lemma 7.1.10 imply that we have a commutative diagram where the rows are exact sequences:

$$\begin{tikzcd} {[W,\Omega Y]} \arrow[r, "(\Omega h)_*"] \arrow[d, equal] & {[W,\Omega Z]} \arrow[r, "f_*"] \arrow[d, equal] & {[W,X]} \arrow[r, "g_*"] \arrow[d, "k_*"'] & {[W,Y]} \arrow[r, "(-\varepsilon_Z^{-1}\circ h)_*"] \arrow[d, equal] & {[W,\Sigma\Omega Z]} \arrow[d, "-(\varepsilon_Z)_*"'] \\ {[W,\Omega Y]} \arrow[r, "(\Omega h)_*"] & {[W,\Omega Z]} \arrow[r, "f_*"] & {[W,X]} \arrow[r, "g_*"] & {[W,Y]} \arrow[r, "h_*"] & {[W,Z]} \end{tikzcd}$$

The five-lemma then implies that k_* is an isomorphism, so, since W was arbitrary, k is an isomorphism. We then have a commutative diagram

$$\begin{tikzcd} \Omega Z \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d, "k"'] & Y \arrow[r, "h"] \arrow[d, equal] & Z \arrow[d, equal] \\ \Omega Z \arrow[r, "f'"] & X' \arrow[r, "g'"] & Y' \arrow[r, "h"] & Z \end{tikzcd}$$

Since the bottom row is a fiber sequence, so is the top row. The proof of the converse is dual. □

Theorem 7.1.11 implies that a classical triangulated category is not so far away from a triangulated category. Indeed, given a classical triangulated category, we can recover the loop functor Ω up to natural isomorphism by taking the right (and also left) adjoint of Σ. Such an adjoint always exists for any equivalence of categories. We can then define fiber sequences as in Theorem 7.1.11. The interested reader can check that these fiber sequences satisfy all the properties of fiber sequences in a pre-triangulated category, except of course the compatibility with the (non-existent) closed $\operatorname{Ho}\mathbf{SSet}_*$-module structure. It is most instructive to check the compatibility between the cofiber and fiber sequences.

Since the fiber sequences in a triangulated category are determined by the cofiber sequences, we would expect morphisms of triangulated categories also to depend only on the cofiber sequences. The following proposition is based on [**Mar83**, Proposition A2.11].

PROPOSITION 7.1.12. *Suppose $\mathcal{S}$ and $\mathcal{T}$ are triangulated categories. Suppose $(F, U, \varphi)\colon \mathcal{S} \to \mathcal{T}$ is an adjunction of closed $\operatorname{Ho}\mathbf{SSet}_*$-modules. Then (F, U, φ) is an exact adjunction if and only if F preserves cofiber sequences.*

PROOF. If (F, U, φ) is an exact adjunction, then by definition F preserves cofiber sequences and U preserves fiber sequences. Conversely, suppose F preserves cofiber sequences. We must show that U preserves fiber sequences. Suppose

$\Omega Z \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ is a fiber sequence. We must show that the sequence $\Omega UZ \xrightarrow{Uf \circ Dm} UX \xrightarrow{Ug} UY \xrightarrow{Uh} UZ$ is a fiber sequence. Here m is the natural isomorphism $\Sigma FW \to F\Sigma W$ and $Dm\colon \Omega UZ \to U\Omega Z$ is its dual, as in Section 1.4. Note first that adjointness implies that mapping into this latter sequence produces an exact sequence. There is some fiber sequence $\Omega UZ \xrightarrow{f'} X' \xrightarrow{g'} UY \xrightarrow{Uh} UZ$, and mapping into it also produces an exact sequence. The five-lemma then implies that it suffices to construct a map $X' \to UX$ making the diagram

$$\begin{tikzcd}
\Omega UZ \arrow[r, "f'"] \arrow[d, equal] & X' \arrow[r, "g'"] \arrow[d] & UY \arrow[r, "Uh"] \arrow[d, equal] & UZ \arrow[d, equal] \\
\Omega UZ \arrow[r, "Uf \circ Dm"] & UX \arrow[r, "Ug"] & UY \arrow[r, "Uh"] & UZ
\end{tikzcd}$$

commute. We will construct this map by constructing its adjoint $FX' \to X$. Let ε' and η' denote the counit and unit of the adjunction (F, U, φ). Let $j\colon F\Omega UZ \to \Omega Z$ denote the composite $\varepsilon'_{\Omega Z} \circ F(Dm)$. Consider the commutative diagram below:

$$\begin{tikzcd}
F\Omega UZ \arrow[r, "Ff'"] \arrow[d, "j"'] & FX' \arrow[r, "Fg'"] & FUY \arrow[r, "m\circ(-F\varepsilon_{UZ}^{-1})\circ FUh"] \arrow[d, "\varepsilon'_Y"'] & \Sigma F\Omega UZ \arrow[d, "\Sigma j"'] \\
\Omega Z \arrow[r, "f"] & X \arrow[r, "g"] & Y \arrow[r, "-\varepsilon_Z^{-1}\circ h"] & \Sigma\Omega Z
\end{tikzcd}$$

Here the bottom row is a cofiber sequence by Theorem 7.1.11, and the top row is a cofiber sequence by Theorem 7.1.11 and the fact that F preserves cofiber sequences. It takes some work to verify that this diagram commutes, but it does. Since we can shift cofiber sequences over to the left in a triangulated category, there is a fill-in map $FX' \to X$. Its adjoint is the desired map $X' \to UX$. □

Another useful fact about triangulated categories is the following.

LEMMA 7.1.13. *Suppose $\mathcal{S}$ is a closed symmetric monoidal triangulated category. Let $S^{-n} = \Omega^n S$ for $n > 0$. The the following diagram is commutative for arbitrary integers m and n.*

$$\begin{tikzcd}
S^m \wedge S^n \arrow[r, "a"] \arrow[d, "T"'] & S^{m+n} \arrow[d, "(-1)^{mn}"'] \\
S^n \wedge S^m \arrow[r, "a"] & S^{m+n}
\end{tikzcd}$$

Here a is the associativity isomorphism, combined if necessary with the unit and counit of the adjoint equivalence $(\Sigma, \Omega, \varphi)$.

PROOF. The proof of this lemma is a long diagram chase. We outline the argument but leave the details to the reader. We know the lemma already for nonnegative m and n, by Lemma 6.6.2. Suppose that one of m and n is negative. Without loss of generality, let us suppose n is negative. Then, since T is a Ho $\mathbf{SSet}_*$-module natural transformation, we have a commutative diagram

$$\begin{tikzcd}
(S^m \wedge S^n) \wedge S^{-n} \arrow[r, "m^\ell"] \arrow[d, "T\wedge 1"'] & S^m \wedge (S^n \wedge S^{-n}) \arrow[d, "T"'] \\
(S^n \wedge S^m) \wedge S^{-n} \arrow[r, "m^r"] & (S^n \wedge S^{-n}) \wedge S^m
\end{tikzcd}$$

Here we have used the same notation as in Theorem 5.6.5. It follows from the coherence diagrams that m^r is determined by m^ℓ and T, and we know how T behaves on $S^m \wedge S^{-n}$. Since we also know how T behaves on $S^0 \wedge S^m$, a long diagram chase tells us that T must behave as claimed on $S^m \wedge S^n$. A similar argument allows us to go from one negative integer to two negative integers, completing the proof. $\square$

7.2. Stable homotopy categories

A stable homotopy category, as defined in [**HPS97**], is a certain kind of closed symmetric monoidal triangulated category. The goal of the rest of this chapter will be to determine what conditions we need to put on a model category so its homotopy category is a stable homotopy category. We do not entirely succeed in this goal, but we come reasonably close.

In this section, we will recall the definition of an algebraic stable homotopy category and describe the theorems we will prove in the rest of this chapter.

We begin with some definitions.

Definition 7.2.1. Suppose $\mathcal{S}$ is a pre-triangulated category, and $\mathcal{G}$ is a set of objects of $\mathcal{S}$. We say that $\mathcal{G}$ is a set of *weak generators* for $\mathcal{S}$ if $[\Sigma^n G, X] = 0$ for all $G \in \mathcal{G}$ and all $n \geq 0$ implies that $X \cong *$. If $\mathcal{S}$ is triangulated, we usually allow $\Sigma^n G = \Omega^{-n} G$ for $n < 0$ as well, without changing notation.

So, for example, S^0 is a weak generator of $\operatorname{Ho} \mathbf{SSet}_*$, and R is a weak generator of the triangulated category $\operatorname{Ho} \operatorname{Ch}(R)$, though we would have to include $\Sigma^{-n} R$ for all $n \geq 0$ if we were thinking of $\operatorname{Ho} \operatorname{Ch}(R)$ as only a pre-triangulated category.

The goal of the next section is to construct a set of weak generators for any pointed cofibrantly generated model category. The weak generators are simply the cofibers of the generating cofibrations.

However, a set of weak generators by itself is not tremendously useful. Just as in the definition of a cofibrantly generated model category, one also needs an appropriate definition of smallness. The one we adopt is the following.

Definition 7.2.2. Suppose $\mathcal{S}$ is a pre-triangulated category. An object $X \in \mathcal{S}$ is called *small* if, for every set $Y_\alpha, \alpha \in K$ of objects of $\mathcal{S}$, the induced map

$$\operatorname{colim}_{S \subseteq K, S \text{ finite}} [X, \coprod_{\alpha \in S} Y_\alpha] \to [X, \coprod_{\alpha \in K} Y_\alpha]$$

is an isomorphism.

Note that X is small if every map into a coproduct factors through a finite subcoproduct. If $\mathcal{S}$ is triangulated, then $X \in \mathcal{S}$ is small if and only if for every set $Y_\alpha, \alpha \in K$ of objects of $\mathcal{S}$, the induced map

$$\bigoplus_{\alpha \in K} [X, Y_\alpha] \to [X, \coprod_{\alpha \in K} Y_\alpha]$$

is an isomorphism. This is the definition of smallness given in [**HPS97**]. Note also that Definition 7.2.2 is the logical definition of finiteness in any category where coproducts are the only colimits one can expect to have, such as the homotopy category of a (not necessarily pointed) model category. By analogy with the definitions of small and finite given in Section 2.1, it would be more natural to use the word "finite" for these objects, and have a more general notion of smallness. At present, this does not appear to be useful, however, so there is no reason to change the standard nomenclature.

We will give sufficient conditions for an object in a pointed model category to be small in the homotopy category in Section 7.4.

Another useful property of an object in any closed symmetric monoidal category is the following.

DEFINITION 7.2.3. Suppose $\mathcal{S}$ is a closed symmetric monoidal category, and $X \in \mathcal{S}$. We say that X is *strongly dualizable* if the natural map $\operatorname{Hom}(X, S) \otimes Y \longrightarrow \operatorname{Hom}(X, Y)$ is an isomorphism for all Y.

We can now define an algebraic stable homotopy category.

DEFINITION 7.2.4. An *algebraic stable homotopy category* is a closed symmetric monoidal triangulated category $\mathcal{S}$ together with a set $\mathcal{G}$ of small strongly dualizable weak generators of $\mathcal{S}$.

These algebraic stable homotopy categories are the principal object of study in [**HPS97**]. The definition given in [**HPS97**, Definition 1.1.4] is not the same as the one given above, as it involves localizing subcategories and representability of cohomology functors, but it is proven in [**HPS97**, Theorem 2.3.2] that the definition above is equivalent to that one. Perhaps one should say almost equivalent, since we are certainly using a stronger definition of triangulated category than was used in [**HPS97**]. Also, we assumed in [**HPS97**] that the commutativity isomorphism behaved correctly on spheres, as in Lemma 7.1.13, and now we are assuming it also behaves correctly on $S \wedge K$ for any simplicial set K.

Another point is that, if the author were writing [**HPS97**] today, he would not insist that the generators be strongly dualizable. Peter May suggested this at the time, but the authors of [**HPS97**] were convinced by the importance of strong dualizability in the examples. However, there are too many examples, such as the G-equivariant stable homotopy category based on the trivial G-universe of [**EKMM97**], and the homotopy category of sheaves of spectra of [**BL96**], where the generators are not strongly dualizable. Furthermore, this condition is not amenable to understanding from the model category point of view, as far as the author can tell. We will therefore define an *algebraic stable homotopy category without duality* to be an closed symmetric monoidal triangulated category together with a set of small weak generators.

We then get a 2-category of algebraic stable homotopy categories as the evident full sub-2-category of closed symmetric monoidal triangulated categories. One could also make a requirement that the morphisms preserve the generators in an appropriate sense: see [**HPS97**, Section 3.4]. We do not do this, though.

Combining the results of the next two sections with the results already proven in this book, we get the following theorem. This theorem is close to the author's original goal when he began thinking about the material in this book.

THEOREM 7.2.5. *The homotopy pseudo-2-functor lifts to a pseudo-2-functor from finitely generated stable symmetric monoidal model categories satisfying Conjecture 5.7.5 to algebraic stable homotopy categories without duality.*

Recall that, if $\mathcal{C}$ is either a monoidal **SSet**-model category or a monoidal $\mathrm{Ch}(\mathbf{Z})$-model category, then Conjecture 5.7.5 does hold for $\mathcal{C}$.

7.3. Weak generators

The goal of this section is to construct weak generators in the homotopy category of a cofibrantly generated pointed model category. We will prove the following theorem.

THEOREM 7.3.1. *Suppose $\mathcal{C}$ is a cofibrantly generated pointed model category, with generating cofibrations I. Let $\mathcal{G}$ be the set of cofibers of maps of I. Then $\mathcal{G}$ is a set of weak generators for* $\operatorname{Ho}\mathcal{C}$.

The proof of Theorem 7.3.1 requires the notion of homotopy limits of diagrams of simplicial sets, for which we rely on [**BK72**]. The definitive treatment of homotopy colimits and homotopy limits for any model category will be in [**DHK**]; see also [**Hir97**].

We begin by studying homotopy classes of maps out of a colimit.

PROPOSITION 7.3.2. *Suppose we have a sequence of cofibrations*

$$* \to X_0 \xrightarrow{f_0} X_1 \to \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \dots$$

in a pointed model category $\mathcal{C}$, with colimit X. Suppose also that Y is fibrant. Then we have an exact sequence of pointed sets

$$* \to \lim{}^1[\Sigma X_n, Y] \to [X, Y] \to \lim[X_n, Y] \to *$$

When $\mathcal{C}$ is the category of pointed simplicial sets, this is proved in [**BK72**, Corollary IX.3.3].

PROOF. Recall that the functor $\operatorname{Map}_{r*}(-, Y)$ of Section 5.2 preserves limits, as a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{SSet}_*$. Thus $\operatorname{Map}_{r*}(X, Y) \cong \lim \operatorname{Map}_{r*}(X_n, Y)$. Furthermore, since each map $X_n \to X_{n+1}$ is a cofibration of cofibrant objects, each map $\operatorname{Map}_{r*}(X_{n+1}, Y) \to \operatorname{Map}_{r*}(X_n, Y)$ is a fibration of fibrant pointed simplicial sets, by Corollary 5.4.4. By [**BK72**, Theorem IX.3.1] we have a short exact sequence

$$* \to \lim{}^1 \pi_1 \operatorname{Map}_{r*}(X_n, Y) \to \pi_0 \operatorname{Map}_{r*}(X, Y) \to \lim \operatorname{Map}_{r*}(X_n, Y) \to *$$

But from Lemma 6.1.2, we have $\pi_0 \operatorname{Map}_{r*}(X, Y) \cong [X, Y]$, $\pi_0 \operatorname{Map}_{r*}(X_n, Y) \cong [X_n, Y]$, and $\pi_1 \operatorname{Map}_{r*}(X_n, Y) \cong [\Sigma X_n, Y]$, so we get the required short exact sequence. $\square$

Note that the colimit X in the sequence above is the coequalizer of the identity map of $\coprod X_n$ and the map $g = \coprod f_n$. In general, there is no way to take this coequalizer in $\operatorname{Ho}\mathcal{C}$ instead of in $\mathcal{C}$. However, if $\mathcal{C}$ is stable, we can find a cofiber sequence

$$\coprod X_n \xrightarrow{1-g} \coprod X_n \to X' \to \Sigma \coprod X_n$$

in $\operatorname{Ho}\mathcal{C}$. Then X' is called the *sequential colimit*, as in [**HPS97**, Section 2.2]. We can actually form X' in the homotopy category of any pointed model category, as long as each X_n is a suspension. Then we have an exact sequence of pointed sets

$$* \to \lim{}^1[\Sigma X_n, Y] \to [X', Y] \to \lim[X_n, Y] \to *$$

just as we do for X. This gives us maps $X' \to X$ and $X \to X'$ in $\operatorname{Ho}\mathcal{C}$, but we are not able to prove that these maps are isomorphisms in general.

COROLLARY 7.3.3. *Suppose $\mathcal{C}$ is a pointed model category,*

$$0 \to X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \dots$$

is a sequence of cofibrations with colimit X, and Y is a fibrant object. If $[X_n, Y]_* = 0$ for all n, then $[X, Y]_* = 0$.

PROOF. Apply Proposition 7.3.2 to $\Omega^k Y = \mathrm{Hom}_*(S^k, Y)$ for all k. $\square$

We also need a transfinite version of Corollary 7.3.3.

PROPOSITION 7.3.4. *Suppose $\mathcal{C}$ is a pointed model category, λ is an ordinal, $X\colon \lambda \to \mathcal{C}$ is a λ-sequence of cofibrations of cofibrant objects with colimit also denoted by X, and Y is a fibrant object. If $[X_\beta, Y]_* = 0$ for all $\beta < \lambda$, then $[X, Y]_* = 0$.*

PROOF. If we apply the functor $\mathrm{Map}_{r*}(-, Y)$ to our λ-sequence, we get what might be called an inverse λ-sequence of fibrations of pointed fibrant simplicial sets $Z_\beta = \mathrm{Map}_{r*}(X_\beta, Y)$. That is, the map $Z_\beta \to \lim_{\gamma<\beta} Z_\gamma$ is an isomorphism for all limit ordinals β. Furthermore, $\mathrm{Map}_{r*}(X, Y)$ is the inverse limit of this sequence.

The diagram Z_β defines a functor Z from the inverse category λ^{op} to pointed simplicial sets $\mathbf{SSet}_*$. Recall from Corollary 5.1.5 that the inverse limit functor on an inverse category is a right Quillen functor, right adjoint to the diagonal functor. Furthermore, we claim that an inverse λ-sequence W of fibrations, such as Z, is fibrant in the model structure given by Theorem 5.1.3. Indeed, given a successor ordinal β, the map $W_\beta \to M_\beta W$ is simply the map $W_\beta \to W_{\beta-1}$, which is a fibration by hypothesis. Given a limit ordinal β, the map $W_\beta \to M_\beta W$ is the map $W_\beta \to \lim_{\gamma<\beta} W_\gamma$, which is an isomorphism, and hence a fibration, for an inverse λ-sequence. Hence we have an isomorphism $\mathrm{Map}_{r*}(X, Y) \cong (R\lim)Z$ in the homotopy category $\mathrm{Ho}\,\mathbf{SSet}^{\lambda^{\mathrm{op}}}$, where $R\lim$ denotes the total right derived functor of the inverse limit.

However, there is another approach to this right derived functor, called the homotopy limit. Homotopy limits are developed for diagrams of simplicial sets such as Z in [**BK72**, Chapter XI]. The homotopy limit $\mathrm{holim}\colon \mathbf{SSet}^{\lambda^{\mathrm{op}}} \to \mathbf{SSet}$ is also a right Quillen functor, but with respect to a different model structure on diagrams. The weak equivalences are still defined objectwise, but now the fibrations are also defined objectwise. Since the weak equivalences are the same in the two model structures, they have the same homotopy categories. Furthermore, it is shown in [**BK72**, Section XI.8] that the total right derived functor $R\,\mathrm{holim}$ is right adjoint to the diagonal functor. Since $R\lim$ is also right adjoint to the diagonal functor, we have an isomorphism

$$\mathrm{Map}_{r*}(X, Y) \cong (R\lim)Z \cong (R\,\mathrm{holim})Z \cong \mathrm{holim}\,Z$$

in the homotopy category of pointed simplicial sets. The last isomorphism comes from the fact that Z is obviously fibrant in the model structure on which holim is a right Quillen functor.

The advantage of this is that we can calculate $\mathrm{holim}\,Z$. Indeed, it is proved in [**BK72**, Section XI.7] that there is a spectral sequence associated to the homotopy inverse limit of any diagram W of fibrant simplicial sets. The E_2 term is $E_2^{s,t} = \lim^s \pi_t W$, where $\lim^s$ indicates the sth derived functor of the inverse limit. In our case, $\pi_t Z_\beta = [\Sigma^t X_\beta, Y] = 0$, using Lemma 6.1.2. Hence the E_2 term is identically 0. As pointed out in [**BK72**], the only obstructions to the convergence of this spectral sequence arise from terms of the form $\lim^1_r E_r^{s,t}$, which are certainly all 0 in our case. We conclude that $\operatorname{holim} Z$ has no homotopy, and hence that $\operatorname{Map}_{r*}(X,Y)$ has no homotopy. Another application of Lemma 6.1.2 then shows that $[\Sigma^t X, Y] = 0$ for all t, as required. □

We can now prove Theorem 7.3.1.

PROOF OF THEOREM 7.3.1. We must show that if $[G,Y]_* = 0$ for all $G \in \mathcal{G}$, then $Y \cong *$ in $\operatorname{Ho}\mathcal{C}$. We can use the small object argument to factor $* \to Y$ into a cofibration $* \to Q'Y$ followed by a fibration $Q'Y \to Y$. We use Q' instead of Q since this may be a different factorization from the one canonically associated to $\mathcal{C}$. It suffices to show that $Q'Y \cong *$ in $\operatorname{Ho}\mathcal{C}$. To do so, we show that the weak equivalence $Q'Y \to RQ'Y$ is trivial, where R is the fibrant replacement functor canonically associated to $\mathcal{C}$. Note that $[G, RQ'Y]_* \cong [G,Y]_* = 0$ for all $G \in \mathcal{G}$.

By construction, $Q'Y$ is the colimit of a λ-sequence $X\colon \lambda \to \mathcal{C}$, where each map $X_\beta \to X_{\beta+1}$ fits into a pushout square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X_\beta \arrow[d] \\ B \arrow[r] & X_{\beta+1} \end{tikzcd}$$

where f is a map in I with cofiber C. Furthermore, $X_0 = 0$. Thus each X_β is cofibrant and each map $X_\beta \to X_{\beta+1}$ is a cofibration.

We show by transfinite induction that $[X_\beta, RQ'Y]_* = 0$ for all $\beta \leq \lambda$, where $X_\lambda = Q'Y$. Since $X_0 = 0$, we can certainly get started. We have a cofiber sequence $X_\beta \to X_{\beta+1} \to C$, and so also a cofiber sequence $\Sigma^n X_\beta \to \Sigma^n X_{\beta+1} \to \Sigma^n C$. Thus, since $C \in \mathcal{G}$, if $[X_\beta, RQ'Y]_* = 0$, then $[X_{\beta+1}, RQ'Y]_* = 0$. Now suppose β is a limit ordinal, and $[X_\alpha, RQ'Y]_* = 0$ for all $\alpha < \beta$. Then Proposition 7.3.4 shows that $[X_\beta, RQ'Y]_* = 0$, as required. □

7.4. Finitely generated model categories

The main objective of this seciton is to show that the weak generators constructed in the previous section are in fact small if the model category in question is finitely generated. Along the way, we prove some useful properties of finitely generated model categories.

The reader should recall that an object A of a category $\mathcal{C}$ is called *finite* relative to a subcategory $\mathcal{D}$ if, for all limit ordinals λ and λ-sequences $X\colon \lambda \to \mathcal{C}$ such that each map $X_\alpha \to X_{\alpha+1}$ is in $\mathcal{D}$, the natural map

$$\operatorname{colim} \mathcal{C}(A, X_\alpha) \to \mathcal{C}(A, \operatorname{colim} X_\alpha)$$

is an isomorphism. A cofibrantly generated model category is *finitely generated* if the generating cofibrations I and the generating cofibrations J can be chosen so that their domains and codomains are finite relative to the cofibrations.

LEMMA 7.4.1. *Suppose $\mathcal{C}$ is a finitely generated model category, λ is an ordinal, $X, Y\colon \lambda \to \mathcal{C}$ are λ-sequences of cofibrations, and $p\colon X \to Y$ is a natural transformation such that $p_\alpha\colon X_\alpha \to Y_\alpha$ is a (trivial) fibration for all $\alpha < \lambda$. Then $\operatorname{colim} p_\alpha\colon \operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ is a (trivial) fibration.*

PROOF. We prove the fibration case; the trivial fibration case is analogous. If λ is a successor ordinal or 0 there is nothing to prove, so we assume λ is a limit ordinal. It suffices to show that $\operatorname{colim} p_\alpha$ has the right lifting property with respect to J. So suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & \operatorname{colim} X_\alpha \arrow[d, "\operatorname{colim} p_\alpha"] \\ B \arrow[r, "g"'] & \operatorname{colim} Y_\alpha \end{tikzcd}$$

where i is a map of J. The map f factors through a map $f'\colon A \to X_\alpha$ for some α. The map g factors analogously through a map $g'\colon B \to Y_\alpha$. We can assume the index α is the same in both cases by simply taking the larger of the two. Now $p_\alpha f'$ may not be equal to $g'i$, but they become equal in the colimit. They must therefore be equal at some stage β. We can lift at the β stage, since p_β is a fibration, and this lift gives a lift in the original diagram. $\square$

In practice, the domains and codomains of I and J tend to be finite relative to a larger class of maps than the cofibrations. For example, in simplicial sets and chain complexes, they are finite relative to the whole category. Even in topological spaces, they are finite relative to closed T_1 inclusions. In this case, Lemma 7.4.1 will work, with the same proof, for λ-sequences X and Y in the larger subcategory. In this situation, the following corollary is useful.

COROLLARY 7.4.2. *Suppose $\mathcal{C}$ is a finitely generated model category, and suppose in addition that the domains and codomains of the generating cofibrations I are finite relative to a subcategory $\mathcal{D}$. Then, if λ is an ordinal and $X\colon \lambda \to \mathcal{C}$ is a λ-sequence of weak equivalences in $\mathcal{D}$, the map $X_0 \to \operatorname{colim} X_\alpha$ is a weak equivalence. In particular, if the domains and codomains of I are finite relative to the whole category, transfinite compositions of weak equivalences are weak equivalences.*

PROOF. Define a new λ-sequence Y and a natural trivial fibration $p\colon Y \to X$ by transfinite induction. Let $Y_0 = X_0$ and p_0 be the identity. Having defined Y_α and p_α, define $Y_\alpha \xrightarrow{i_\alpha} Y_{\alpha+1} \xrightarrow{p_{\alpha+1}} X_{\alpha+1}$ to be the functorial factorization of the composite $Y_\alpha \xrightarrow{p_\alpha} X_\alpha \xrightarrow{j_\alpha} X_{\alpha+1}$ into a cofibration followed by a trivial fibration. Since j_α is a weak equivalence, so is i_α. Having defined Y_α and p_α for all α less than a limit ordinal β, define $Y_\beta = \operatorname{colim} Y_\alpha$ and $p_\beta = \operatorname{colim} p_\alpha$. Then p_β is a trivial fibration, by the slight generalization of Lemma 7.4.1 referred to above.

Since each map $Y_\alpha \to Y_{\alpha+1}$ is a trivial cofibration, the map $X_0 \to \operatorname{colim} Y_\alpha$ is a trivial cofibration. The map $\operatorname{colim} p_\alpha\colon\ \operatorname{colim} Y_\alpha \to \operatorname{colim} X_\alpha$ is a trivial fibration by the argument of Lemma 7.4.1. Thus the map $X_0 \to \operatorname{colim} X_\alpha$ is a weak equivalence, as required. $\square$

In all the examples we hve discussed except topological spaces, the domains and codomains of the generating cofibrations and trivial cofibrations are finite relative to the whole category. In this situation, not only are weak equivalences closed under transfinite compositions, but so are fibrations and trivial fibrations, by a straightforward argument we leave to the reader.

THEOREM 7.4.3. *Suppose $\mathcal{C}$ is a pointed finitely generated model category. Suppose A is cofibrant and finite relative to the cofibrations. Then A is small in* $\operatorname{Ho}\mathcal{C}$.

We point out that the pointed hypothesis is probably not necessary. Certainly our proof below will work for unpointed simplicial sets and topological spaces, for example.

Proof. Let λ be an ordinal, and let S_λ be the set of all finite subsets of λ. We will show by transfinite induction on λ that the canonical map

$$\operatorname{colim}_{T \in S_\lambda}[A, \coprod_{\alpha \in T} X_\alpha] \to [A, \coprod_{\alpha < \lambda} X_\alpha]$$

is an isomorphism for all sets $\{X_\alpha \mid \alpha < \lambda\}$ of objects of $\operatorname{Ho}\mathcal{C}$. Since we are assuming that $\mathcal{C}$ is pointed, the inclusion of each finite subcoproduct into $\coprod X_\alpha$ is a split monomorphism. It follows easily that the canonical map above is always injective, so we only have to show, by transfinite induction, that it is surjective.

This is certainly true for finite ordinals λ, so there is no difficulty getting started. Suppose it is true for an ordinal λ, and $\{X_\alpha \mid \alpha \leq \lambda\}$ is a set of objects of $\operatorname{Ho}\mathcal{C}$. Define $Y_0 = X_0 \amalg X_\lambda$, and for $0 < \alpha < \lambda$, let $Y_\alpha = X_\alpha$. Then the induction hypothesis implies that the canonical map

$$\operatorname{colim}_{T \in S_\lambda}[A, \coprod_{\alpha \in T} Y_\alpha] \to [A, \coprod_{\alpha < \lambda} Y_\alpha] = [A, \coprod_{\alpha \leq \lambda} X_\alpha]$$

is an isomorphism. Since the set of finite subsets of $\lambda + 1$ containing λ is cofinal in the set of all finite subsets of $\lambda + 1$, it follows that the canonical map

$$\operatorname{colim}_{T \in S_{\lambda+1}}[A, \coprod_{\alpha \in T} X_\alpha] \to [A, \coprod_{\alpha \leq \lambda} X_\alpha]$$

is an isomorphism, as required.

We are left with the limit ordinal case of the induction. So suppose λ is a limit ordinal, the induction hypothesis holds for all $\beta < \lambda$, and we have a set $\{X_\alpha \mid \alpha < \lambda\}$ in $\operatorname{Ho}\mathcal{C}$. There is no loss of generality in supposing that the X_α are cofibrant. For $\beta < \lambda$, let $Y_\beta = \coprod_{\alpha < \beta} X_\alpha$. Then we have a λ-sequence of cofibrations $Y\colon \lambda \to \mathcal{C}$. of cofibrant objects, whose colimit is $\coprod X_\alpha$. The λ-sequence Y is cofibrant in the model structure on $\mathcal{C}^\lambda$ of Theorem 5.1.3. In this model structure, let $Z' = RY$, so that we have a trivial cofibration $Y \to Z'$ and a fibration $Z' \to *$. The functor $Z'\colon \lambda \to \mathcal{C}$ may not be a λ-sequence, so let Z be the associated λ-sequence, where $Z_\alpha = Z'_\alpha$ for successor ordinals α and also 0, and $Z_\beta = \operatorname{colim}_{\alpha < \beta} Z'_\alpha$ for limit ordinals β. The map $Z_\beta \to Z_{\beta+1}$ for limit ordinals β is the composite $\operatorname{colim}_{\alpha < \beta} Z'_\alpha \to Z'_\beta \to Z'_{\beta+1}$, which is a cofibration, since Z' is cofibrant. Hence Z is a λ-sequence of cofibrations. Furthermore, since each Z'_α is fibrant, so is each Z_α, using Lemma 7.4.1. Since the map $Y \to Z$ is a trivial cofibration, and the colimit is a left Quillen functor, each map $Y_\alpha \to Z_\alpha$ is a trivial cofibration. (This is obvious for successor ordinals, of course). Again using the fact that the colimit is a Quillen functor, e see that the map $\coprod X_\alpha = \operatorname{colim} Y_\alpha \to \operatorname{colim} Z_\alpha$ is a weak equivalence. Lemma 7.4.1 implies that $\operatorname{colim} Z_\alpha$ is fibrant.

Now, suppose we have a map $A \xrightarrow{f} \coprod X_\alpha$ in $\operatorname{Ho}\mathcal{C}$. Then, since A is cofibrant and $\operatorname{colim} Z_\alpha$ is fibrant, f must be represented by some map $g\colon A \to \operatorname{colim} Z_\alpha$ in $\mathcal{C}$. Since A is finite relative to the cofibrations, this map must factor through some Z_β. Hence, $\operatorname{Ho}\mathcal{C}$, f factors through $\coprod_{\alpha < \beta} X_\alpha$. By the induction hypothesis, this means that f factors through a finite subcoproduct of $\coprod X_\alpha$, as required. $\square$

COROLLARY 7.4.4. *Suppose* $\mathcal{C}$ *is a pointed finitely generated model category. Let* $\mathcal{G}$ *be the set of cofibers of the generating cofibrations* I*. Then* $\mathcal{G}$ *is a set of small weak generators for the pre-triangulated category* $\operatorname{Ho}\mathcal{C}$.

PROOF. We have already seen in Theorem 7.3.1 that $\mathcal{G}$ is a set of weak generators for $\operatorname{Ho}\mathcal{C}$. Using Theorem 7.4.3, we need to check that the cofibers of the maps of I, which are obviously cofibrant, are also finite relative to the cofibrations. This follows by commuting colimits, using the fact that the domains and codomains of the maps of I are finite relative to the cofibrations. $\square$

We point out that the definition of a finitely generated model category involves the trivial cofibrations as well as the cofibrations. This is why smallness is lost under the Bousfield localization of [**Hir97**] or [**Bou79**]. Indeed, the Bousfield localization of a model category $\mathcal{C}$ is a different model structure on the same underlying category, and the cofibrations are the same. Therefore if A was cofibrant and finite relative to the cofibrations before localization, it still is after localization. However, the trivial cofibrations change dramatically after localizing, so it is often the case that the localized model category is no longer finitely generated, and that A is no longer small in $\operatorname{Ho}\mathcal{C}$.

Also, if A is small relative to the cofibrations but not cofibrant, in a finitely generated model category, then A need not be small in $\operatorname{Ho}\mathcal{C}$. Indeed, consider the trivial module k in the derived category of $E(x)$, the exterior algebra on x over a field k. Then k is certainly finite relative to all of $\operatorname{Ch}(E(x))$. But $\operatorname{Tor}(k,k)$ is infinite, and one can easily check, using the methods of axiomatic stable homotopy theory [**HPS97**], that this is impossible for a small object in the derived category of a ring.

CHAPTER 8

Vistas

In this brief final chapter, we discuss some questions we have left unresolved in this book. This chapter has a less formal tone than the others, and concerns material the author does not know all that much about. I apologize in advance for incorrect claims or references, and for references that should be here and are not.

Consider first the 2-category of model categories. We have seen in Section 1.3 that Quillen equivalences behave like weak equivalences in this 2-category, as do natural weak equivalences between Quillen functors. As mentioned in Section 1.3, this suggests the following problem.

PROBLEM 8.1. Define a model 2-category and show that the 2-category of model categories is one. A Quillen adjunction should be a weak equivalence if and only if it is a Quillen equivalence, and a natural transformation should be a weak equivalence if and only if it is a natural weak equivalence when restricted to cofibrant objects.

The author does not really expect this problem to be solved, as he can see no reasonable definition of a fibration or cofibration. This problem is trying to get at the "homotopy theory of homotopy theories", which has also been studied by Charles Rezk. Rezk's work is unpublished, but the basic idea is to widen one's notion of a category. Instead of demanding that composition be associative, one should only demand that it be associative up to infinite higher homotopy. Any model category yields an object in Rezk's category, but the author is not certain of the situation with Quillen adjunctions and natural transformations.

PROBLEM 8.2. Understand the relationship between the 2-category of model categories and Rezk's homotopy theory of homotopy theories.

The author expects this problem to be straightforward; he just doesn't know enough about Rezk's work to solve it.

Moving on to examples, the author's experience with model categories has led him to believe that it is very helpful, in general, to have more than one model structure on the same category, with the same weak equivalences. We have seen one example of this with the two different model structure on $\mathrm{Ch}(R)$ discussed in Section 2.3, where in one model structure every object is fibrant, and in the other every object is cofibrant.

PROBLEM 8.3. Find a model structure on topological spaces, with the same weak equivalences as usual, in which every object is cofibrant, or else prove that this is impossible.

The reader's first reaction to this is probably that it must be impossible, or someone would already have done it. However, there is an obvious candidate for the cofibrations; the Hurewicz cofibrations, defined to be the maps with the left lifting property with respect to $Y^I \to Y$ for all spaces Y. These are the cofibrations in the model category considered by Strom [**Str72**], where the weak equivalences are the homotopy equivalences. The fibrations in the Strom model structure are the Hurewicz fibrations, which are maps with the right lifting property with respect to $X \to X \times I$ for all spaces X. If the Hurewicz cofibrations and the weak equivalences defined a model structure on **Top**, the fibrations would have to be Hurewicz fibrations with extra structure. The author can prove that any fibrant object in this model structure would have to be connected, so in particular S^0 is not fibrant. This may seem like a contradiction, since after all every object must be weakly equivalent to a fibrant object. But a connected space can have many path components, so it is not a contradiction.

Such a model structure on **Top** would perhaps be interesting only for its surprise value; however, it would make the study of topological symmetric spectra [**HSS98**] much simpler.

One could also ask whether there is a model structure on simplicial sets where every object is fibrant. Since simplicial sets are easy to work with anyway, this would be of less importance.

Of course, there are many examples and possible examples of model categories that we have not discussed. The general theory is that anytime there is a cohomology theory, there ought to be a model category. So, for example, in the theory of C^*-algebras there is K-theory.

Problem 8.4. Define a useful model structure on a suitable category of C^*-algebras.

Since the author knows nothing about C^*-algebras, he has no idea of such a thing is feasible. However, if it could be done, then presumably also a suitable stable model category of C^*-algebras could be defined, and K-theory would correspond to an object in this stable model category. But there would be other objects too, corresponding to other cohomology theories currently unknown. The author first discussed this idea with Jim McClure, who has done some work on the subject [**DM97**] that might be a good place to start.

In the same way, the work of Voevodsky [**Voe97**] on the cohomology of schemes seems certain to involve constructing a model category of suitable sheaves. It would be extremely useful to understand Voevodsky's work from the point of view of this book, assuming that Voevodsky has in fact not already done so.

Recall in Chapter 4 we discussed the theory of monoidal model categories. We did not discuss when one gets model categories of monoids and of modules over a monoid in a monoidal model category. This issue has been dealt with in [**SS97**] and [**Hov98a**]. However, neither of these sources addresses the following problem.

Problem 8.5. Find conditions on a symmetric monoidal model category $\mathcal{C}$ under which the category of commutative monoids in $\mathcal{C}$ and homomorphisms is again a model category, where the weak equivalences are the underlying ones.

This problem is subtle, as the following example will show. One would expect the free commutative monoid functor to be part of a Quillen adjunction from a symmetric monoidal model category $\mathcal{C}$ to the category of commutative monoids in $\mathcal{C}$. But in $\mathrm{Ch}(\mathbf{Z})$, for example, the free commutative monoid functor does not preserve (underlying) weak equivalences between cofibrant objects. Up until very recently, the author knew of no good model structure on commutative differential graded algebras over **Z**. However, Stanley [**Sta98**] has recently constructed such a model structure.

On the other hand, there is a general theory that may imply that commutative monoids are not the right thing to consider, as readers of [**KM96**] will be familiar with. One considers the free commutative algebra triple and replaces it by a weakly equivalent triple which comes from an operad and is cofibrant in some model category structure on operads. Here we are getting into waters currently too deep for the author to stand in, but the situation as I understand it is the following. If (F, G, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$, then GF is a triple (or monad) on $\mathcal{C}$ and GF is a cotriple (or comonad) on $\mathcal{D}$. That is, GF is a monoid in the monoidal category of endofunctors of $\mathcal{C}$, and GF is a comonoid in the category of endofunctors of $\mathcal{D}$. See [**ML71**] for some information about triples. One can then consider the category of algebras over a triple. For example, given a ring R, an algebra over the (left) free R-module triple on abelian groups is a (left) R-module. We then have the following generalization of Problem 8.5.

PROBLEM 8.6. Suppose T is a triple on a model category $\mathcal{C}$. Find conditions on $\mathcal{C}$ and T under which the category of T-algebras becomes a model category with the underlying weak equivalences.

The only result I know in this direction is an unpublished theorem of Hopkins [**Hop**], which requires that every object of $\mathcal{C}$ be fibrant. I believe this theorem will be published in joint work of Goerss and Hopkins.

Now, there will be times when there is no good model structure on the category of T-algebras. In this case, rather than giving up, one tries to replace T by a weakly equivalent triple T' for which there is a model structure on T'-algebras. This presumes one has a good notion of weak equivalence of triples, of course.

PROBLEM 8.7. Given a model category $\mathcal{C}$, find a model structure on the category of triples on $\mathcal{C}$.

Triples may be too general for such a model structure to exist. However, one can consider operads instead. Operads were introduced by Peter May in [**May72**]. See [**KM96**] for a good discussion of operads. Every operad gives rise to a triple, but the converse is false.

PROBLEM 8.8. Suppose $\mathcal{C}$ is a model category. Find a model structure on operads over $\mathcal{C}$. Find conditions on an operad T and $\mathcal{C}$ so that there is a model structure on T-algebras. Show that weakly equivalent operads give rise to Quillen equivalent categories of T-algebras. Develop spectral sequences for calculating homotopy classes of maps of T-algebras.

I think this problem has a much better chance of being solved than the previous ones, though my opinion may not be worth much! So far as I know, the closest approach to this problem is in [**KM96**], which does not ever mention model categories but nonetheless is about them, and the thesis of Charles Rezk [**Rez96**] .

Then, if one wants to consider commutative monoids in a symmetric monoidal model category, one would first try to construct a model structure on them. If that failed, as it will sometimes, one would replace the free commutative algebra triple by a weakly equivalent cofibrant operad, which is usually called an E_∞-operad. Then there should be a model structure on algebras over this operad. Such algebras would then be called E_∞-rings. This is the approach carried out in [**KM96**] for $\mathrm{Ch}(\mathbf{Z})$.

We now leave the abstruse world of operads and enter a slightly lower orbit. One of the main themes of this book is that one cannot tell whether a model category is simplicial by examining its homotopy category.

PROBLEM 8.9. Show that every model category is Quillen equivalent to a simplicial model category, or at least that there is a chain of Quillen equivalences from any model category to a simplicial model category. Similarly, show that every Quillen adjunction is Quillen equivalent, in an appropriate sense, to a simplicial Quillen adjunction, and that every natural transformation of Quillen adjunctions is Quillen equivalent to a simplicial natural transformation. That is, in the conjectural language of Problem 8.1, show that the model 2-category of model categories is Quillen 2-equivalent to the model 2-category of simplicial model categories.

The full statement of this problem is probably out of reach. But there may be some construction one can make that will embed a model category into a simplicial model category that might allow one to get started on this problem.

One can also consider specific examples. It has been proven by Schwede (personal communication) that the model category $\mathrm{Ch}(R)$ is Quillen equivalent to the category of HR-modules, where HR is an Eilenberg-MacLane spectrum in the category of symmetric spectra [**HSS98**]. This is a simplicial model category. A similar result is certainly true if HR is the Eilenberg-MacLane spectrum in the category of S-modules [**EKMM97**], though the author does not know a precise reference.

In fact, the only interesting model category the author knows of that is not known to be Quillen equivalent to a simplicial model category is $\mathrm{Ch}(B)$, the model category of chain complexes of comodules over a Hopf algebra B over a field k. No doubt there should be some kind of Hopf algebra structure on the Eilenberg-MacLane spectrum HB, and there should be a resulting model category of comodules over HB, but the author does not know how to carry out the details.

PROBLEM 8.10. Find a simplicial model category Quillen equivalent to $\mathrm{Ch}(B)$.

Another variation on the theme that the homotopy category of a general model category is indistinguishable from the homotopy category of a simplicial model category is of course Conjecture 5.6.6. The author's failure to prove this conjecture is his biggest disappointment in this book.

The most obvious question to ask about stable model categories is whether one can stabilize a general pointed model category. The author has given two partial answers to this question in [**Hov98b**], one based on Bousfield-Friedlander spectra [**BF78**], and one based on symmetric spectra.

Our definition of a triangulated category depends on the pre-triangulated category $\mathrm{Ho}\,\mathbf{SSet}_*$. This seems somewhat unnecessary. Dan Kan has suggested (a variation of) the following problem.

PROBLEM 8.11. Let $\mathcal{S}$ denote the homotopy category of the category of symmetric spectra of [**HSS98**], also known as the (ordinary) stable homotopy category. Then the homotopy category of a stable model category is naturally a closed triangulated $\mathcal{S}$-module.

This would be a stable analog of Theorem 5.6.2, and presumably any proof of it would be based on some kind of stable framing. The statement of this problem assumes that this stable framing will come from the model category of symmetric spectra, but that may be an unjustified assumption.

One might also hope that the results of Chapter 7 could be extended to cover the more general stable homotopy categories considered in [**HPS97**], and their unstable analogues.

Problem 8.12. Find a good definition of a localizing subcategory of a pre-triangulated category, agreeing with the definition in [**HPS97**] in the triangulated case. Show that the localizing subcategory generated by the cofibers of the generating cofibrations in a cofibrantly generated pointed model category is the whole homotopy category.

The author thinks that the definition of a pre-triangulated category will have to be strengthened to solve this problem. The homotopy category of a model category has much more structure than we have considered in this book. Indeed, suppose $\mathcal{C}$ is a model category. Then for any Reedy category $\mathcal{I}$, there is a model structure on $\mathcal{C}^{\mathcal{I}}$. Furthermore, there is the colimit adjunction from $\mathcal{C}^{\mathcal{I}}$ to $\mathcal{C}$, and the limit adjunction from $\mathcal{C}$ to $\mathcal{C}^{\mathcal{I}}$. Although the colimit and limit functors are not Quillen functors in general, they still have derived functors. This is the theory of homotopy colimits and homotopy limits; see [**DHK**]. More generally, a map of Reedy categories gives rise to relative homotopy colimits and relative homotopy limits.

Problem 8.13. Develop a 2-category of "Reedy schemes", where a Reedy scheme is a 2-functor from Reedy categories to categories. Show that the homotopy pseudo-2-functor lifts to a pseudo-2-functor from model categories to Reedy schemes. Show that the closed action of $\operatorname{Ho}\mathbf{SSet}$ on $\operatorname{Ho}\mathcal{C}$ can be recovered from the Reedy scheme of $\mathcal{C}$, as can the pre-triangulation when $\mathcal{C}$ is pointed.

This problem is so crazy that one might think that no one has ever considered it. This is actually not quite true. There is a paper of Franke [**Fra96**] which seems to consider something like this.

For the present, let us consider a (possibly transfinite) sequence in $\operatorname{Ho}\mathcal{C}$, where $\mathcal{C}$ is a model category. Such a sequence can be lifted to a sequence of cofibrations of cofibrant objects in $\mathcal{C}$. The colimit in $\mathcal{C}$ will then be well-defined up to isomorphism in $\operatorname{Ho}\mathcal{C}$. We can therefore add these (weak) colimits of sequences to the definition of a pre-triangulated category.

Problem 8.14. Define a notion of a pre-triangulated category with sequential colimits. Show that every triangulated category gives rise to such a thing, and show that the homotopy category of a pointed model category is naturally such a thing. Define a notion of cohomology functor in such a pre-triangulated category as in [**Bro62**], and show that all cohomology functors in the homotopy category of a pointed cofibrantly generated model category are representable.

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