ON CLOSED CATEGORIES OF FUNCTORS

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Received November 7, 1969

The purpose of the present paper is to develop in further detail the remarks, concerning the relationship of Kan functor extensions to closed structures on functor categories, made in "Enriched functor categories" [1] §9. It is assumed that the reader is familiar with the basic results of closed category theory, including the representation theorem. Apart from some minor changes mentioned below, the terminology and notation employed are those of [1], [3], and [5].

Terminology

A closed category V in the sense of Eilenberg and Kelly [3] will be called a <u>normalised</u> closed category, V: $V_0 \to Ens$ being the <u>normalisation</u>. Throughout this paper V is taken to be a <u>fixed normalised symmetric monoidal closed category</u> (V_0, $\otimes$, I, r, ℓ, a, c, V, [-,-], p) with V_0 admitting all small limits (inverse limits) and colimits (direct limits). It is further supposed that if the limit or colimit in V_0 of a functor (with possibly large domain) exists then a definite choice has been made of it. In short, we place on V those hypotheses which both allow it to replace the cartesian closed category of (small) sets Ens as a ground category and are satisfied by most "natural" closed categories.

As in [1], an <u>end in</u> B of a V-functor T: $A^{op} \otimes A \to B$ is a V-natural family α_A: K $\to$ T(AA) of morphisms in B_0 with the property that the family $B(1,\alpha_A)$: B(BK) $\to$ B(B,T(AA)) in V_0 is universally V-natural in A for each $B \in \mathcal{B}$; then an end in V turns out to be simply a family α_A: K → T(AA) of morphisms in V_o which is universally V-natural in A. The dual concept is called a coend.

From [1] we see that the choice of limits and colimits made in V_o determines a definite end and coend of each V-functor T: $A^{op} \otimes A \to V$ for which such exist. These are denoted by s_A: $\int_A T(AA) \to T(AA)$ and s^A: $T(AA) \to \int^A T(AA)$ respectively. We can now construct, for each pair A, $\mathcal{B}$ of V-categories with A small, a definite V-category $[A,\mathcal{B}]$ having V-functors S, T, ...: $A \to \mathcal{B}$ as its objects, and having $[A,\mathcal{B}](S,T) = \int_A \mathcal{B}(SA,TA)$. An element $\alpha \in V\int_A \mathcal{B}(SA,TA)$ clearly corresponds, under the projections Vs_A: $V\int_A \mathcal{B}(SA,TA) \to \mathcal{B}_o(SA,TA)$, to a V-natural family of morphisms α_A: SA → TA in the sense of [3]. It is convenient to call α, rather than the family $\{\alpha_A\}$ of its components, a V-natural transformation from S to T; for then the underlying ordinary category $[A,\mathcal{B}]_o$ is the category of V-functors and V-natural transformations.

Limits and colimits in the functor category $[A,\mathcal{B}]$ will always be computed evaluationwise, so that the choice of limits and colimits made in V fixes a choice in $[A,V]$ for each small V-category A. Included in this rule are the concepts of cotensoring and tensoring, which were seen in [5] to behave like limits and colimits respectively.

In order to replace the category of sets by the given normalised closed category V, we shall "lift" most of the usual terminology. A V-monoidal category $\overline{V}$ is a V-category A together with a V-functor $\bar{\otimes}$: $A \otimes A \to A$, an object $\bar{I} \in A$, and V-natural isomorphisms $\bar{a}$: $(A \bar{\otimes} B)\bar{\otimes} C \cong A\bar{\otimes}(B\bar{\otimes}C)$, $\bar{r}$: $A\bar{\otimes}\bar{I} \cong A$, and $\bar{l}$: $\bar{I}\bar{\otimes}A \cong A$, satisfying the usual coherence axioms for a monoidal category - namely axioms MC2 and MC3 of [3]. If, furthermore, $-\bar{\otimes}A$ and $A\bar{\otimes}-$: $A \to A$ both have (chosen) right V-adjoints for each $A \in A$, then $\bar{V}$ is called a V-<u>biclosed category</u> (see Lambek [8]). A V-symmetry for a V-monoidal category $(A, \bar{\otimes}, \bar{I}, \bar{r}, \bar{l}, \bar{a})$ is a V-natural isomorphism $\bar{c}$: $A\bar{\otimes}B \cong B\bar{\otimes}A$ satisfying the coherence axioms MC6 and MC7 of [3]. Finally we come to the concept of a V-<u>symmetric-monoidal-closed category</u> which can be described simply as a V-biclosed category with a V-symmetry; we do not insist on a "V-normalisation" as part of this structure. An obvious example of such a category is V itself, where $\bar{\otimes}$ is taken to be the V-functor Ten: $V \otimes V \to V$ defined in [3] Theorem III.6.9.

We note here that, for V-functors S: $A \to C$ and T: $B \to D$, the symbol $S \otimes T$ may have two distinct meanings. In general it is the canonical V-functor $A \otimes B \to C \otimes D$ which sends the object (ordered pair) $(A,B) \in A \otimes B$ to the object $(SA,TB) \in C \otimes D$, as defined in [3] Proposition III.3.2. When C and D are both V, however, we shall also use $S \otimes T$ to denote the composite

$$A \otimes B \xrightarrow{S \otimes T} V \otimes V \xrightarrow{\text{Ten}} V.$$

The context always clearly indicates the meaning.

<u>Henceforth we work entirely over V and suppose that the unqualified words "category", "functor", "natural transformation", etc. mean</u> "V-<u>category</u>", "V-<u>functor</u>", "V-<u>natural transformation", etc</u>.

1. Introduction

Let A be a small category and regard A^{op} as a full subcategory of $[A,V]$, identifying $A \in A^{op}$ with the left represented functor $L^A\colon A \to V$ in the usual way. For each $S \in [A,V]$ we have the canonical expansion (see [1]) $S \cong \int^A SA \otimes L^A$ which asserts the density (adequacy) of A^{op} in $[A,V]$. If $[A,V]$ has the structure of a biclosed category $\bar{V}$ then, in view of this expansion, the value $S \bar{\otimes} T$ of $\bar{\otimes}\colon [A,V] \otimes [A,V] \to [A,V]$ at (S,T) is essentially determined by the values $L^A \bar{\otimes} T$, because $-\bar{\otimes} T$ has a right adjoint. These in turn are determined by the values $L^A \bar{\otimes} L^B$, because each $L^A \bar{\otimes} -$ has a right adjoint. Writing $P(ABC)$ for $(L^A \bar{\otimes} L^B)(C)$, we see that the functor $\bar{\otimes}$ is essentially determined by the functor $P\colon A^{op} \otimes A^{op} \otimes A \to V$, in the same way that the multiplication in a linear algebra is determined by structure constants.

These considerations suggest what is called in section 3 a premonoidal structure on A. This consists of functors $P\colon A^{op} \otimes A^{op} \otimes A \to V$ and $J\colon A \to V$, together with certain natural isomorphisms corresponding to associativity, left-identity, and right-identity morphisms, which satisfy suitable axioms; a monoidal structure is a special case. Before attempting to write the axioms down, we collect in section 2 the properties of ends and coends that we shall need.

The main aim of this paper is to show that, from a premonoidal structure on a small category A, there results a canonical biclosed structure on the functor category $[A,V]$; this is done in section 3. As one would expect, biclosed structures on $[A,V]$ correspond bijectively to premonoidal structures on A to within "isomorphism". However we do not formally prove this assertion, which would require the somewhat lengthy introduction of premonoidal functors to make it clear what "isomorphism" was intended.

The concluding sections contain descriptions of some commonly occurring types of premonoidal structure on a (possibly large) category A. The case in which the premonoidal structure is actually monoidal is discussed in section 4. In section 5 we provide the data for a premonoidal structure which arises when the hom-objects of A are comonoids ($\otimes$-coalgebras) in V in a natural way. In both cases the tensor-product and internal-hom formulas given in section 3 for the biclosed structure on $[A,V]$ may be simplified to allow comparison with the corresponding formulas for some well-known examples of closed functor categories.

2. Induced Natural Transformations

Natural transformations, in both the ordinary and extraordinary senses, are treated in [2] and [3]. Our applications of the rules governing their composition with each other (and with functors) are quite straightforward and will not be analysed in detail.

The following dualisable lemmas on induced naturality are expressed in terms of coends.

Lemma 2.1. *Let* $T: A^{op}\otimes A\otimes B \to C$ *be a functor and let* $\alpha_{AB}: T(AAB) \to SB$ *be a coend over* A *for each* $B \in B$. *Then there* *exists a unique functor* $S\colon \mathcal{B} \to \mathcal{C}$ *making the family* α_{AB} *natural in* B.

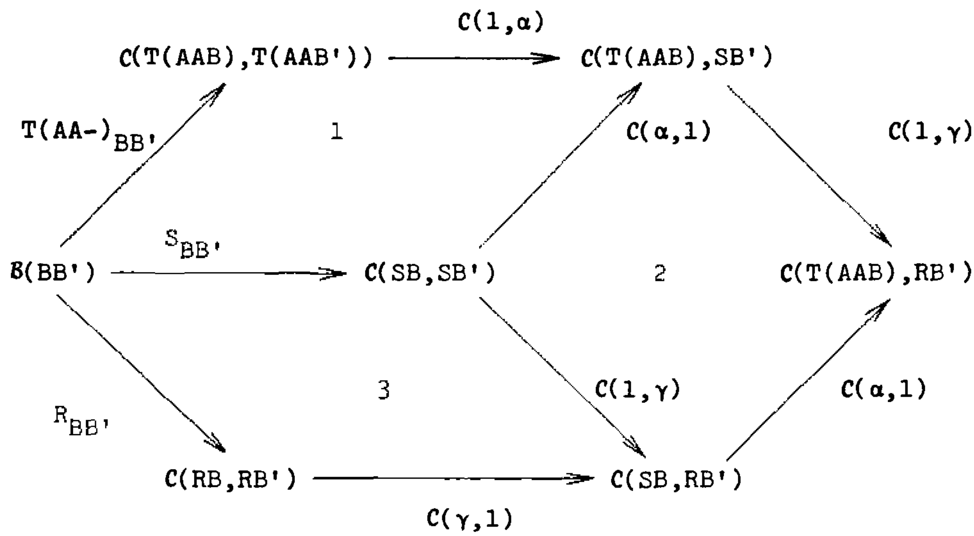
Proof. For each pair B, B' $\in \mathcal{B}$ consider the diagram

$$\begin{array}{ccc}
\mathcal{B}(BB') & \xrightarrow{\quad S_{BB'} \quad} & \mathcal{C}(SB,SB') \\
{\scriptstyle T(AA-)_{BB'}}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\alpha,1)} \\
\mathcal{C}(T(AAB),T(AAB')) & \xrightarrow[\quad \mathcal{C}(1,\alpha) \quad]{} & \mathcal{C}(T(AAB),SB')
\end{array},$$

Because $\mathcal{C}(\alpha,1)$ is an end and $\mathcal{C}(1,\alpha).T(AA-)_{BB'}$ is natural in A we can define $S_{BB'}$ to be the unique morphism making this diagram commute. The functor axioms VF1' and VF2' of [3] are easily verified for this definition of S using the fact that $\mathcal{C}(\alpha,1)$ is an end. S is then the unique functor making α_{AB} natural in B.

Lemma 2.2. *Let* $T\colon A^{op}\otimes A\otimes \mathcal{B} \to \mathcal{C}$ *and* $S,R\colon \mathcal{B} \to \mathcal{C}$ *be functors, let* $\alpha_{AB}\colon T(AAB) \to SB$ *be a coend over* A, *natural in* B, *and let* $\beta_{AB}\colon T(AAB) \to RB$ *be natural in* A *and* B. *Then the induced family* $\gamma_B\colon SB \to RB$ *is natural in* B.

Proof. For each pair B, B' $\in \mathcal{B}$ consider the diagram



The commutativity of region 1 and that of the exterior express the naturality in B of α and β respectively. Region 2 clearly commutes hence, because $C(\alpha,1)$ is an end, region 3 commutes for each pair $B,B' \in \mathcal{B}$, as required.

By similar arguments we obtain

Lemma 2.3. *Let* $T\colon A^{op}\otimes A\otimes \mathcal{B}^{op}\otimes\mathcal{B} \to C$ *and* $S\colon \mathcal{B}^{op}\otimes\mathcal{B} \to C$ *be functors, let* $\alpha_{ABB'}\colon T(AABB') \to S(BB')$ *be a coend over* A, *natural in* B *and* B', *and let* $\beta_{AB}\colon T(AABB) \to C$ *be natural in* A *and* B. *Then the induced family* $\gamma_B\colon S(BB) \to C$ *is natural in* B.

Lemma 2.4. *Let* $T\colon A^{op}\otimes A \to C$ *and* $R\colon \mathcal{B}^{op}\otimes\mathcal{B} \to C$ *be functors, let* $\alpha_A\colon T(AA) \to C$ *be a coend over* A, *and let* $\beta_{AB}\colon T(AA) \to R(BB)$ *be natural in* A *and* B. *Then the induced family* $\gamma_B\colon C \to R(BB)$ *is natural in* B.

Let $\mathcal{A}$ be a category and let T(AA-) be a functor into $\mathcal{V}$ whose coend s^A: T(AA-) $\to \int^A$T(AA-) over A $\in \mathcal{A}$ exists for all values of the extra variables "-". Then, by Lemma 2.1, $\int^A$T(AA-) is canonically functorial in these extra variables. In the special case where T(AA-) = S(A-)$\otimes$T(A-) for functors S and R into $\mathcal{V}$ (with different variances in A) we will frequently abbreviate this notation to s^A: S(A-)$\otimes$R(A-) $\to$ S(A-)$\underline{\otimes}$R(A-), leaving the repeated dummy variable A in the expression S(A-)$\underline{\otimes}$R(A-) to indicate the domain of $\int^\cdot$.

In order to handle expressions formed entirely by the repeated use of $\underline{\otimes}$, it is convenient to introduce the following considerations which we do not formalise completely. To each expression $\underline{N}$ which is formed by one or more uses of $\underline{\otimes}$, there corresponds an expression N in which each $\underline{\otimes}$ is replaced by $\otimes$, the dummy variables in $\underline{N}$ becoming repeated variables in N; for example, if $\underline{N}$ is (RA$\underline{\otimes}$S(AB))$\underline{\otimes}$T(BC) for functors R: $\mathcal{A} \to \mathcal{V}$, S: $\mathcal{A}^{op}\otimes\mathcal{B} \to \mathcal{V}$, and T: $\mathcal{B}^{op}\otimes\mathcal{C} \to \mathcal{V}$, then N is (RA$\otimes$S(AB))$\otimes$T(BC). Moreover, there is a canonical natural transformation $q = q_N$: N $\to \underline{N}$ defined, as follows, by induction on the number of occurrences of $\underline{\otimes}$ in $\underline{N}$. If $\underline{N}$ contains no occurrence of $\underline{\otimes}$ then N = $\underline{N}$ and q_N = 1; otherwise $\underline{N} = \underline{N}'\underline{\otimes}\underline{N}''$ and q_N is the composite

$$N'\otimes N'' \xrightarrow{q'\otimes q''} \underline{N}'\otimes\underline{N}'' \xrightarrow{s} \underline{N}'\underline{\otimes}\underline{N}''.$$

In the above example, q is the composite

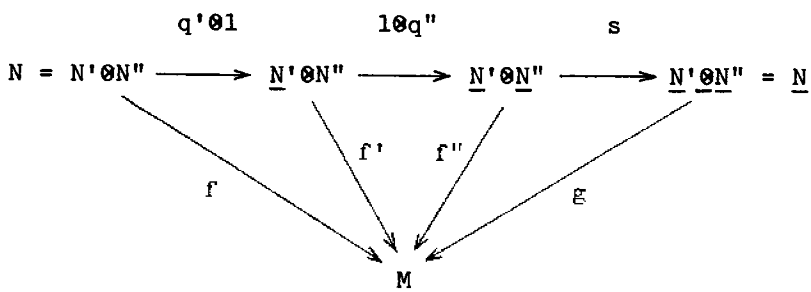
$$(RA\otimes S(AB))\otimes T(BC) \xrightarrow{s\otimes 1} (RA\underline{\otimes}S(AB))\otimes T(BC) \xrightarrow{s} (RA\underline{\otimes}S(AB))\underline{\otimes}T(BC)$$

and this is natural in A, B, and C; we say that the variables A and B are "summed out" by q.

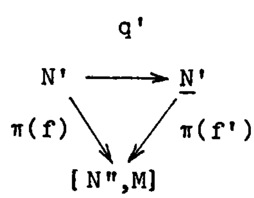
The path $q_N: N \to \underline{N}$ is in fact a multiple coend over all those variables in N which are summed out by q_N:

Lemma 2.5. Let M be a functor into V and let $f: N \to M$ be a natural transformation which is, in particular, natural in all the repeated variables in N which are summed out by $q_N: N \to \underline{N}$. Then f factors as $g.q_N$ for a unique natural transformation $g: \underline{N} \to M$.

Proof. This is by induction on the number of occurrences of $\underline{\otimes}$ in $\underline{N}$. If $\underline{N}$ contains no occurrence of $\underline{\otimes}$ the result is trivial; otherwise $\underline{N} = \underline{N}'\underline{\otimes}\underline{N}''$ and we can factor f in three steps:



First consider the transform $\pi(f): N' \to [N'',M]$ of f under the tensor-hom adjunction isomorphism $\pi = Vp$ of V. By the induction hypothesis and routine naturality considerations, the diagram



commutes for a unique morphism $\pi(f'): \underline{N}' \to [N'',M]$ where f': $\underline{N}'\otimes N'' \to M$ is natural in all the variables not summed out by q'. Similarly f' factors as $f''.(1\otimes q'')$ for a unique morphism f": $\underline{N}'\otimes\underline{N}'' \to M$ which is natural in all the variables not summed out by either of q' or q". Finally, because s is a coend, f" factors as g.s for a unique g: $\underline{N} \to M$ which is natural in all the remaining variables in $\underline{N}$ and M by Lemmas 2.2, 2.3, and 2.4. Combining these steps, we have that f factors through $q_N = s(q'\otimes q'')$ in the required manner.

When the transformation f in Lemma 2.5 is of the form q'.n for a path q': $N' \to \underline{N}'$, the induced transformation g: $\underline{N} \to \underline{N}'$ is denoted by $\underline{n}$. Such induced transformations are a necessary part of the concept of a premonoidal category and we consider three relevant special cases below.

First, if n: $N \to N'$ is a natural isomorphism constructed entirely from the coherent data isomorphisms a, r, ℓ, c of V then $\underline{n}$: $\underline{N} \to \underline{N}'$ is a natural isomorphism and is called an <u>induced coherence isomorphism</u>. In view of the uniqueness assertion of Lemma 2.5, and the original coherence of a, r, ℓ, c, it is clear that induced coherence isomorphisms are coherent. In other words, the induced coherence isomorphism $\underline{n}$: $\underline{N} \to \underline{N}'$ is completely determined by the arrangement of $\underline{\otimes}$ in $\underline{N}$ and $\underline{N}'$; consequently we shall not label such isomorphisms.

Secondly, when $n = h\otimes k$: $S(A-)\otimes R(A-) \to S'(A-)\otimes R'(A-)$ for natural transformations h: $S \to S'$ and k: $R \to R'$, let us write

$h\underline{\otimes}k$ for $\underline{h\otimes k}$. This not only makes the symbol $\underline{\otimes}$ $\mathcal{Ens}$-functorial whenever it is defined on objects, but also makes the coend s^A: $S(A-)\otimes R(A-) \to S(A-)\underline{\otimes}R(A-)$ $\mathcal{Ens}$-natural in S and R. Under reasonable conditions the same observations can be made at the $\mathcal{V}$-level.

If we restrict our attention to functors into $\mathcal{V}$ with small domains then the functors themselves may be regarded as extra variables. For example, let T: $\mathcal{A}^{op}\otimes\mathcal{A}\otimes\mathcal{B} \to \mathcal{V}$ be a functor with $\mathcal{A}$ and $\mathcal{B}$ small. Then $\int^A T(AAB)$ is canonically functorial in T and B for we can write T(AAB) = F(AATB) where F is the composite

$$\mathcal{A}^{op}\otimes\mathcal{A}\otimes([\mathcal{A}^{op}\otimes\mathcal{A}\otimes\mathcal{B},\mathcal{V}]\otimes\mathcal{B}) \cong [\mathcal{A}^{op}\otimes\mathcal{A}\otimes\mathcal{B},\mathcal{V}]\otimes(\mathcal{A}^{op}\otimes\mathcal{A}\otimes\mathcal{B}) \xrightarrow[E]{} \mathcal{V},$$

and where E is the evaluation functor defined in [1] §4. Similarly, if S(A-) and R(A-) are functors into $\mathcal{V}$ with small domains (and different variances in A) then $S(A-)\underline{\otimes}R(A-)$ is functorial in S and R in a unique way that makes s^A: $S(A-)\otimes R(A-) \to S(A-)\underline{\otimes}R(A-)$ natural in S and R.

Lastly, let S(A-) be a functor into $\mathcal{V}$ which is covariant in $A \in \mathcal{A}$. As part of the data for S, we have a family of morphisms S_{AB}: $\mathcal{A}(AB) \to [S(A-),S(B-)]$ which is natural in A and B and also in the extra variables in S. Transforming this family by the tensor-hom adjunction of $\mathcal{V}$, we get a transformation $\pi^{-1}(S_{AB})$: $\mathcal{A}(AB)\otimes S(A-) \to S(B-)$ which is natural in A and B and the extra variables in S. As a result of the generalised "higher" representation theorem (see [1], §3 and §5), this induces the Yoneda isomorphism

$$y_{S,B}\colon \mathcal{A}(AB)\underline{\otimes}S(A-) \to S(B-).$$

By Lemma 2.2, we then have

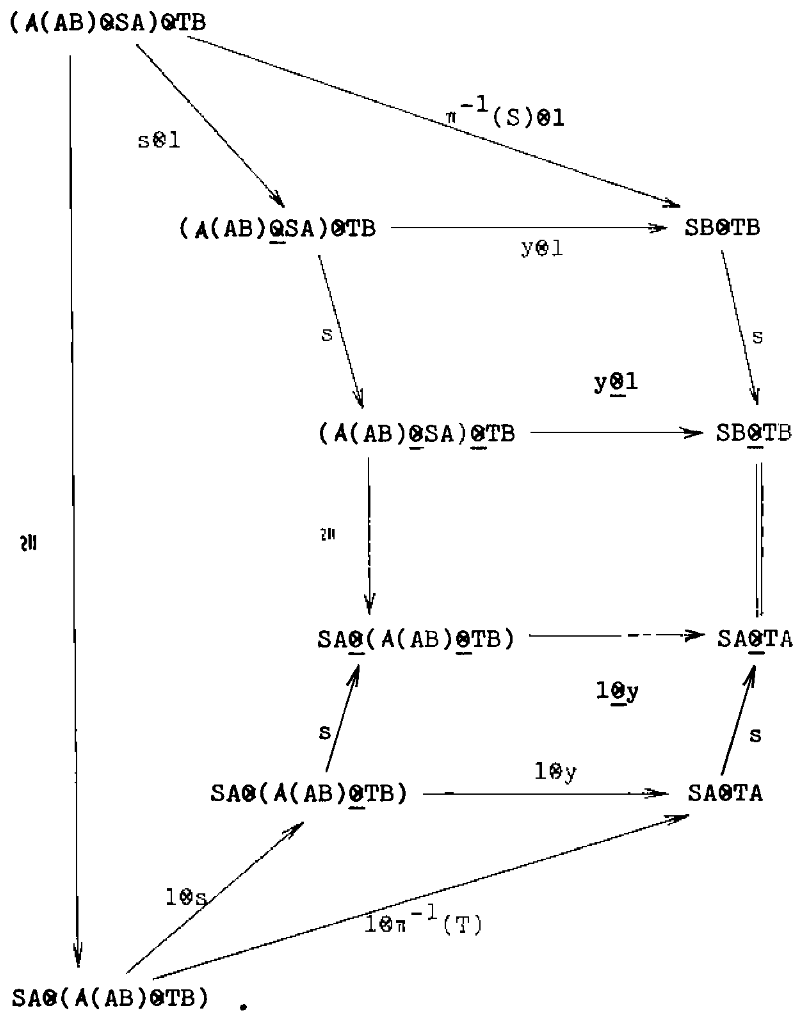
Lemma 2.6. *The Yoneda isomorphism* $y_{S,A}$ *is natural in* A *and in the extra variables in* S; *if the domain of* S *is small then it is natural in* S.

The following diagram lemmas for the Yoneda isomorphism y are all proved using [3] Proposition II.7.4 which we shall refer to as the representation theorem. These lemmas are presented here in their most convenient forms for application in sections 3 and 4.

Lemma 2.7. *Given functors* $S\colon \mathcal{A} \to \mathcal{V}$ *and* $T\colon \mathcal{A}^{op} \to \mathcal{V}$ *for which* $SA\underline{\otimes}TA$ *exists, the following diagram commutes*:

$$\begin{array}{ccc}
(\mathcal{A}(AB)\underline{\otimes}SA)\underline{\otimes}TB & \xrightarrow{\;y\underline{\otimes}1\;} & SB\underline{\otimes}TB \\
\big\downarrow \wr & & \big\| \\
SA\underline{\otimes}(\mathcal{A}(AB)\underline{\otimes}TB) & \xrightarrow[\;1\underline{\otimes}y\;]{} & SA\underline{\otimes}TA \;.
\end{array}$$

Proof. Replacing $\underline{\otimes}$ by $\otimes$, y by its definition, etc., we obtain a new diagram:



By Lemma 2.5, s(s⊗1) is a coend over A and B hence it suffices to prove that the exterior of this new diagram commutes for all A, B ∈ A. This is easily seen to be so on applying the representation theorem; put B = A and compose both exterior legs with

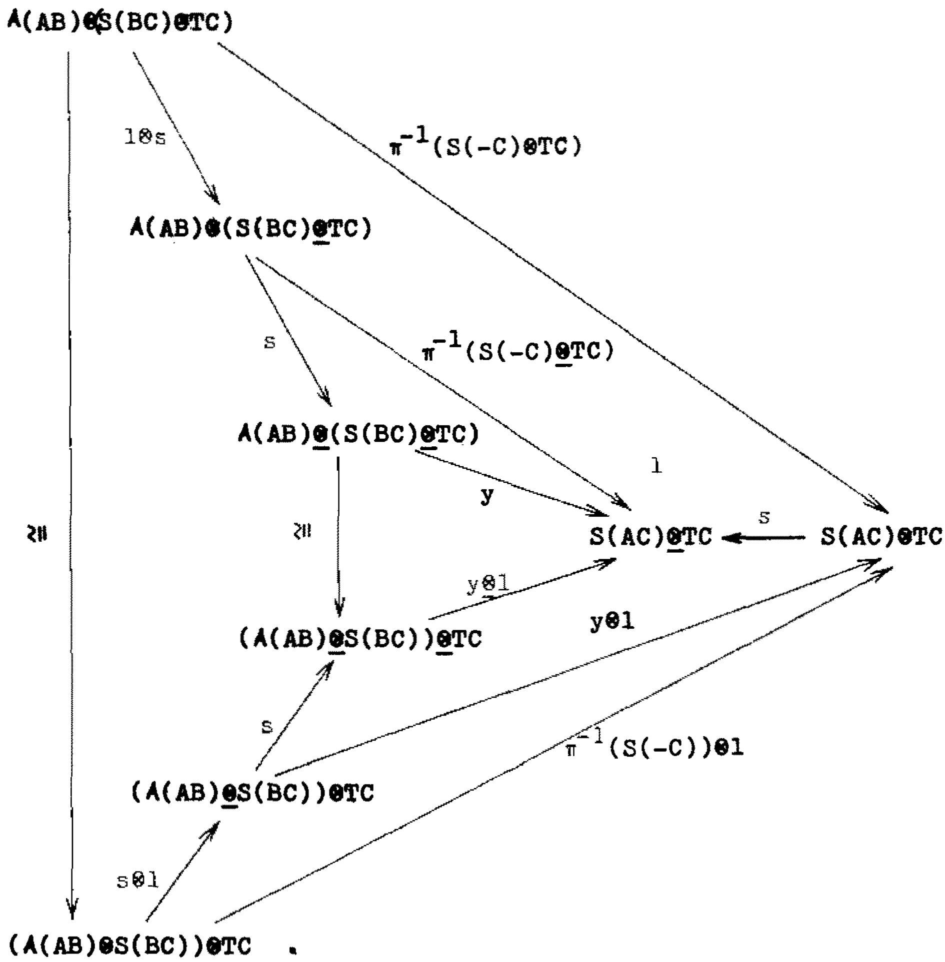
$$(I\otimes SA)\otimes TA \xrightarrow[(j_A\otimes 1)\otimes 1]{} (A(AA)\otimes SA)\otimes TA;$$

the resulting diagram commutes, hence the original one does.

Lemma 2.8. *Given functors* S: $\mathcal{A}^{op}\otimes\mathcal{B} \to \mathcal{V}$ *and* T: $\mathcal{B}^{op} \to \mathcal{V}$ *for which* $S(AC)\underline{\otimes}TC$ *exists for each* $A \in \mathcal{A}$, *the following diagram commutes for each* $A \in \mathcal{A}$:

$$\begin{array}{ccc} \mathcal{A}(AB)\underline{\otimes}(S(BC)\underline{\otimes}TC) & & \\ \downarrow\cong & \searrow^{y} & \\ & & S(AC)\underline{\otimes}TC \\ & \nearrow_{y\underline{\otimes}1} & \\ (\mathcal{A}(AB)\underline{\otimes}S(BC))\underline{\otimes}TC\ . & & \end{array}$$

Proof. Again replacing $\underline{\otimes}$ by $\otimes$, y by its definition, etc., we obtain a new diagram:



In this diagram the region labelled 1 commutes because it is the transform of the diagram

$$\begin{array}{ccc} A(AB) & \xrightarrow{S(-C)\otimes TC} & [S(BC)\otimes TC, S(AC)\otimes TC] \\ \Big\downarrow{\scriptstyle S(-C)\underline{\otimes} TC} & & \Big\downarrow{\scriptstyle [1,s]} \\ [S(BC)\underline{\otimes} TC, S(AC)\underline{\otimes} TC] & \xrightarrow[{[s,1]}]{} & [S(BC)\otimes TC, S(AC)\underline{\otimes} TC] \end{array}$$

which expresses the naturality of $s = s^C\colon S(AC)\otimes TC \to S(AC)\underline{\otimes}TC$ in A. Hence, because $s(1\otimes s)$ is a coend over B and C by Lemma 2.5, it suffices to prove that the exterior of the new diagram commutes for all A, B $\in \mathcal{A}$ and C $\in \mathcal{B}$. Again this is a simple consequence of the representation theorem.

The remaining lemmas are obtained by the same type of argument.

Lemma 2.9. *Given functors* $S\colon \mathcal{A}^{op}\otimes\mathcal{B} \to \mathcal{V}$ *and* $T\colon \mathcal{B}^{op} \to \mathcal{V}$ *for which* $TC\underline{\otimes}S(AC)$ *exists for each* $A \in \mathcal{A}$, *the following diagram commutes for each* $A \in \mathcal{A}$:

$$\begin{array}{ccc} \mathcal{A}(AB)\underline{\otimes}(TC\underline{\otimes}S(BC)) & & \\ & \searrow^{y} & \\ \downarrow\wr\| & & TC\underline{\otimes}S(AC) \\ & \nearrow_{1\underline{\otimes}y} & \\ TC\underline{\otimes}(\mathcal{A}(AB)\underline{\otimes}S(BC)) & & \end{array}$$

.

Lemma 2.10. *For any functors* $S\colon \mathcal{A} \to \mathcal{B}$ *and* $T\colon \mathcal{B}^{op} \to \mathcal{V}$ *the following diagram commutes for each* $A \in \mathcal{A}$:

$$\begin{array}{ccc} \mathcal{A}(AB)\underline{\otimes}(\mathcal{B}(SB,C)\underline{\otimes}TC) & \xrightarrow{1\underline{\otimes}y} & \mathcal{A}(AB)\underline{\otimes}TSB \\ \downarrow y & & \downarrow y \\ \mathcal{B}(SA,C)\underline{\otimes}TC & \xrightarrow[y]{} & TSA \end{array}$$

,

<u>Lemma 2.11</u> <u>For any functor</u> T: $\mathcal{A}\otimes\mathcal{B} \to \mathcal{V}$ <u>the following diagram commutes for all</u> B, D $\in \mathcal{A}$:

$$\begin{array}{ccc}
\mathcal{A}(AB)\underline{\otimes}(\mathcal{B}(CD)\underline{\otimes}T(AC)) & \xrightarrow{1\underline{\otimes}y} & \mathcal{A}(AB)\underline{\otimes}T(AD) \\
\Big\downarrow y & & \Big\downarrow y \\
\mathcal{B}(CD)\underline{\otimes}T(BC) & \xrightarrow[y]{} & T(BD)
\end{array},$$

3. Premonoidal Categories

We emphasise again that, unless otherwise indicated, all concepts are relative to the given normalised symmetric monoidal closed category $\mathcal{V}$.

<u>Definition 3.1</u> A premonoidal category $\mathcal{P} = (\mathcal{A},P,J,\lambda,\rho,\alpha)$ over $\mathcal{V}$ consists of

a category $\mathcal{A}$,

a functor P: $\mathcal{A}^{op}\otimes\mathcal{A}^{op}\otimes\mathcal{A} \to \mathcal{V}$,

a functor J: $\mathcal{A} \to \mathcal{V}$,

and natural isomorphisms

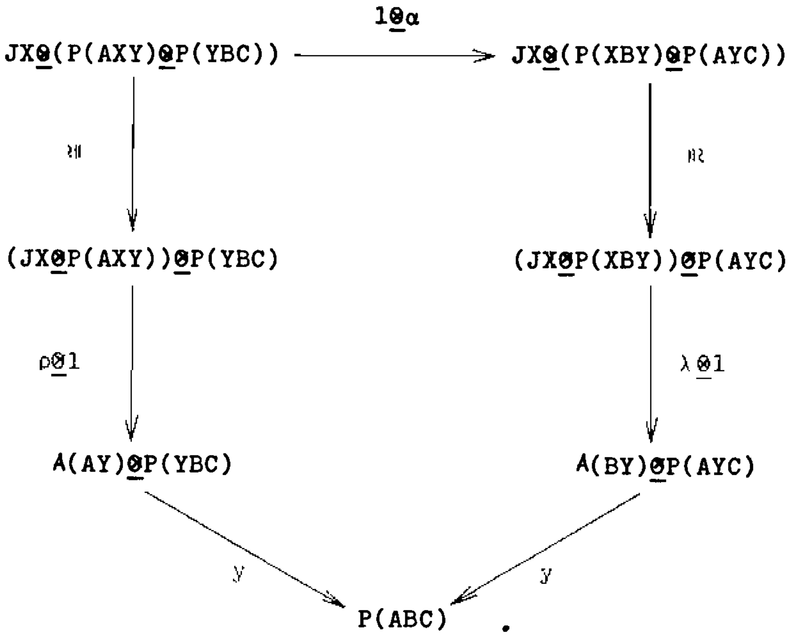
$\lambda = \lambda_{AB}$: $JX\underline{\otimes}P(XAB) \to \mathcal{A}(AB)$,

$\rho = \rho_{AB}$: $JX\underline{\otimes}P(AXB) \to \mathcal{A}(AB)$,

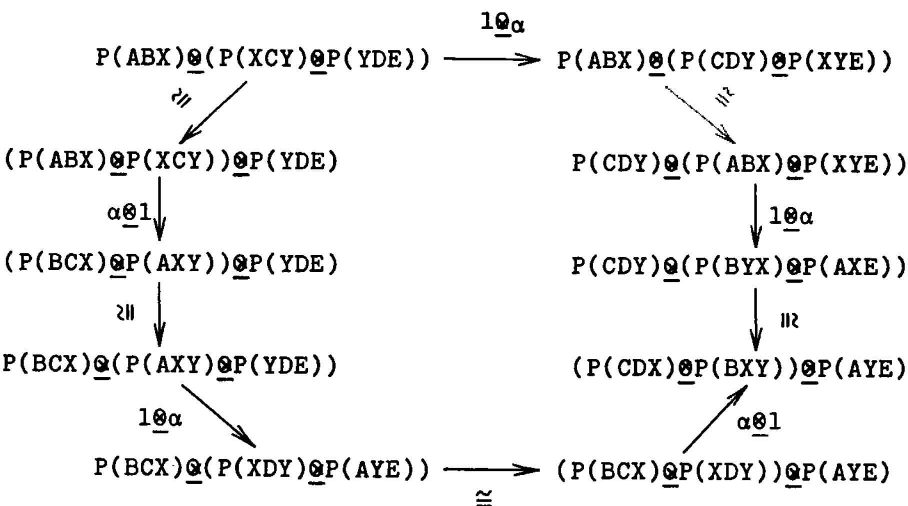
$\alpha = \alpha_{ABCD}$: $P(ABX)\underline{\otimes}P(XCD) \to P(BCX)\underline{\otimes}P(AXD)$,

satisfying the following two axioms:

PC1. For all A,B,C ∈ A, the following diagram commutes:



PC2. For all A,B,C,D,E ∈ A, the following diagram commutes:



<u>Remark 3.2</u> It is assumed in the definition that the requisite $\underline{\otimes}$'s exist for the given $\mathcal{A}$, P, and J. They do so, by hypothesis on $\mathcal{V}$, when $\mathcal{A}$ is small. They also exist whenever $P(AB-): \mathcal{A} \to \mathcal{V}$ and $J: \mathcal{A} \to \mathcal{V}$ are representable for all $A,B \in \mathcal{A}$.

In the remainder of this section we will suppose that $\mathcal{A}$ is <u>small</u> and show that each premonoidal structure $\mathcal{P}$ on $\mathcal{A}$ "extends" to a biclosed structure $[\mathcal{P},\mathcal{V}]$ on the functor category $[\mathcal{A},\mathcal{V}]$. For the monoidal part define a tensor-product $*: [\mathcal{A},\mathcal{V}]\otimes[\mathcal{A},\mathcal{V}] \to [\mathcal{A},\mathcal{V}]$ by

$$S*T = \int^A SA\otimes\int^B TB\otimes P(AB-) = SA\underline{\otimes}(TB\underline{\otimes}P(AB-)) \tag{3.1}$$

for all $S,T \in [\mathcal{A},\mathcal{V}]$; this expression is canonically functorial in S and T by the considerations of section 2. Next, let $J \in [\mathcal{A},\mathcal{V}]$ be the identity-object of $*$, and define natural isomorphisms $\ell^* = \ell^*_T: J*T \to T$ and $r^* = r^*_T: T*J \to J$ as the respective composites

$$J*T = JX\underline{\otimes}(TA\underline{\otimes}P(XA-)) \cong (JX\underline{\otimes}P(XA-))\underline{\otimes}TA \xrightarrow[\lambda\underline{\otimes}1]{} \mathcal{A}(A-)\underline{\otimes}TA \xrightarrow[y]{} T$$

and

$$T*J = TA\underline{\otimes}(JX\underline{\otimes}P(AX-)) \cong (JX\underline{\otimes}P(AX-))\underline{\otimes}TA \xrightarrow[\rho\otimes1]{} \mathcal{A}(A-)\underline{\otimes}TA \xrightarrow[y]{} T.$$

Lastly, define a natural isomorphism $a^* = a^*_{RST}: (R*S)*T \to R*(S*T)$ as the composite

$$
\begin{aligned}
(R*S)*T &= (RA\underline{\otimes}(SB\underline{\otimes}P(ABX)))\underline{\otimes}(TC\underline{\otimes}P(XC-)) \\
&\cong RA\underline{\otimes}(SB\underline{\otimes}(TC\underline{\otimes}(P(ABX)\underline{\otimes}P(XC-)))) \\
&\xrightarrow[1\underline{\otimes}(1\underline{\otimes}(1\underline{\otimes}\alpha))]{} RA\underline{\otimes}(SB\underline{\otimes}(TC\underline{\otimes}(P(BCX)\underline{\otimes}P(AX-)))) \\
&\cong RA\underline{\otimes}((SB\underline{\otimes}(TC\underline{\otimes}P(BCX)))\underline{\otimes}P(AX-)) \\
&= R*(S*T).
\end{aligned}
$$

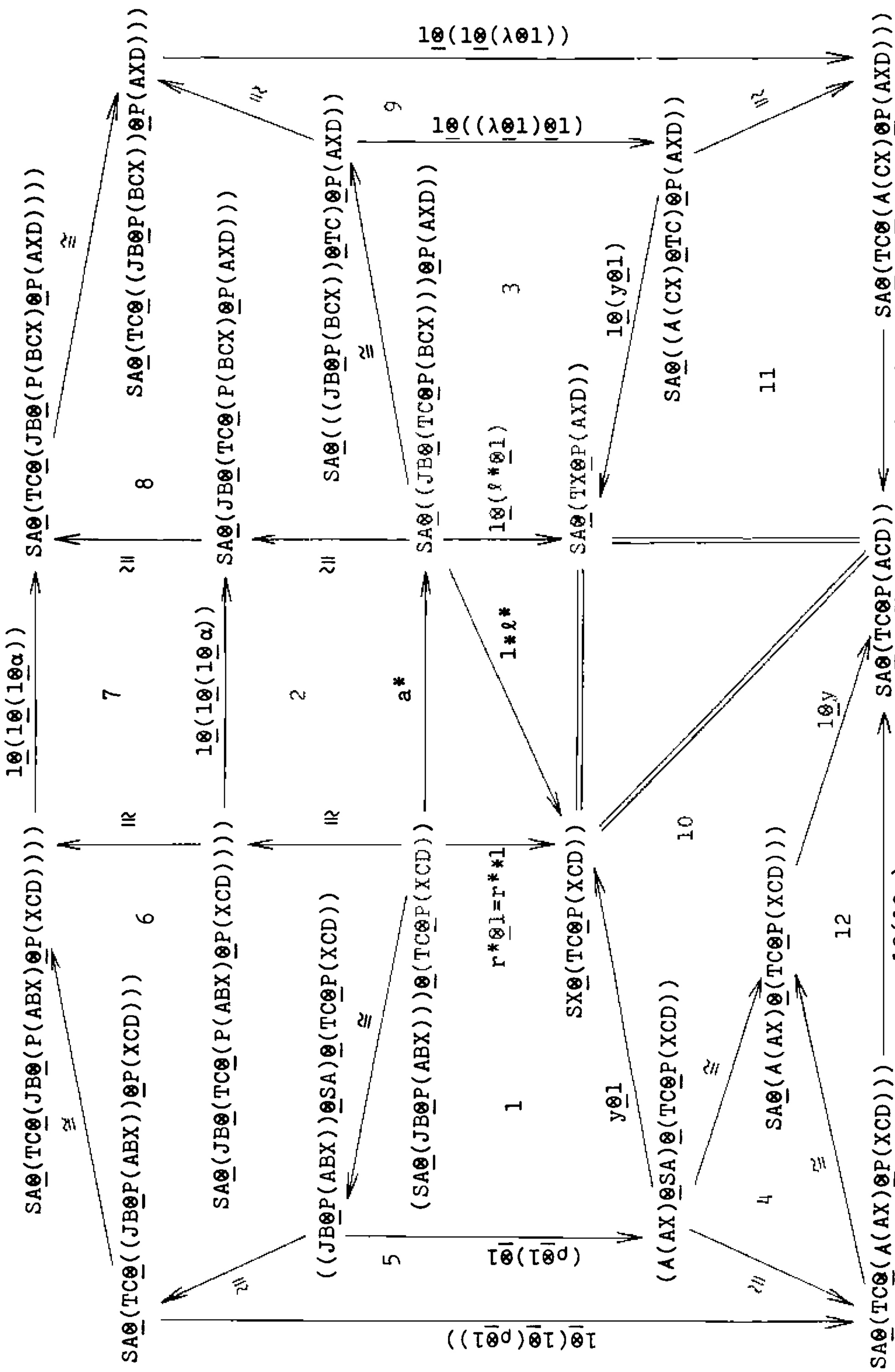
Then ℓ^*, r^*, and a^* are natural by Lemmas 2.5 and 2.6.

<u>Theorem 3.3</u> $[P,V] = ([A,V],*,J,\ell^*,r^*,a^*)$ <u>is a monoidal category admitting a biclosed structure</u>.

<u>Proof</u> First, to show that $[P,V]$ is a monoidal category, we need to prove PC1 ⇒ MC2 and PC2 ⇒ MC3. The first of these is obtained by considering diagram (3.2) in which the exterior commutes by PC1; 1, 2, and 3 commute by the definitions of $*$, r^*, a^*, and ℓ^*; 4, 5, 6, 7, 8, and 9 commute by the naturality and coherence of the induced coherence isomorphisms (Lemma 2.5 and the succeeding remarks); 10 and 11 commute by Lemma 2.7; and 12 commutes by Lemma 2.9. The proof of PC2 ⇒ MC3 requires a diagram that is too large for the space available but, apart from the definitions of $*$ and a^*, uses only the naturality and coherence of the induced coherence isomorphisms involved.

To complete the structure on $[A,V]$ to that of a biclosed category, consider the composite isomorphism:

Diagram (3.2)


1⊗(1⊗(λ⊗1))
1⊗((λ⊗1)⊗1)
SA⊗(TC⊗(JB⊗(P(BCX)⊗P(AXD))))
SA⊗(TC⊗((JB⊗P(BCX))⊗P(AXD)))
SA⊗(((JB⊗P(BCX))⊗TC)⊗P(AXD))
SA⊗((A(CX)⊗TC)⊗P(AXD))
SA⊗(TC⊗(A(CX)⊗P(AXD)))
SA⊗(JB⊗(TC⊗(P(BCX)⊗P(AXD))))
SA⊗((JB⊗(TC⊗P(BCX)))⊗P(AXD))
1⊗(y⊗1)
1⊗(ℓ*⊗1)
SA⊗(TX⊗P(AXD))
1⊗(1⊗y)
SA⊗(TC⊗P(ACD))
1⊗(1⊗(1⊗α))
a*
1*ℓ*
1⊗y
SA⊗(TC⊗(JB⊗(P(ABX)⊗P(XCD))))
SA⊗(JB⊗(TC⊗(P(ABX)⊗P(XCD))))
(SA⊗(JB⊗P(ABX)))⊗(TC⊗P(XCD))
r*⊗1=r**1
SX⊗(TC⊗P(XCD))
SA⊗(A(AX)⊗(TC⊗P(XCD)))
SA⊗(TC⊗((JB⊗P(ABX))⊗P(XCD)))
((JB⊗P(ABX))⊗SA)⊗(TC⊗P(XCD))
y⊗1
(A(AX)⊗SA)⊗(TC⊗P(XCD))
SA⊗(TC⊗(A(AX)⊗P(XCD)))
(ρ⊗1)⊗1
1⊗(1⊗(ρ⊗1))
1
2
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7
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10
11
12

$$
\begin{aligned}
[\mathcal{A},\mathcal{V}](R*S,T) &= \int_C[(R*S)C,TC] \\
&= \int_C[\int^A RA\otimes\int^B SB\otimes P(ABC),TC] \\
&\cong \int_C\int_A[RA\otimes\int^B SB\otimes P(ABC),TC] \\
&\xrightarrow[\int\int p]{} \int_C\int_A[RA,[\int^B SB\otimes P(ABC),TC]] \\
&\cong \int_A\int_C[RA,[\int^B SB\otimes P(ABC),TC]] \\
&\cong \int_A[RA,\int_C[\int^B SB\otimes P(ABC),TC]] \\
&= \int_A[RA,(T/S)A] \text{ say,} \\
&= [\mathcal{A},\mathcal{V}](R,T/S),
\end{aligned}
$$

where the unlabelled isomorphisms are the canonical ones which assert that limit-preserving functors preserve ends and that repeated ends commute (see [1] §3). Assuming that each of the ends involved is made functorial in its extra variables using the dual form of Lemma 2.1, we see that each isomorphism is natural in R, S, and T, by the dual form of Lemma 2.2. Consequently $-*S$ has a right adjoint $-/S$, given by the formula

(3.3) $$T/S = \int_C[\int^B SB\otimes P(-BC),TC]$$

for all $S,T \in [\mathcal{A},\mathcal{V}]$. Similarly we have the natural composite

$$
\begin{aligned}
[\mathcal{A},\mathcal{V}](S*R,T) &= \int_C[\int^A SA\otimes\int^B RB\otimes P(ABC),TC] \\
&\cong \int_C[\int^B RB\otimes\int^A SA\otimes P(ABC),TC] \\
&\cong \int_B[RB,\int_C[\int^A SA\otimes P(ABC),TC]] \\
&= \int_B[RB,(S\backslash T)B] \text{ say,} \\
&= [\mathcal{A},\mathcal{V}](R,S\backslash T).
\end{aligned}
$$

Thus $S*-$ has a right adjoint $S\backslash -$, given by the formula

(3.4) $$S\backslash T = \int_C[\int^A SA\otimes P(A-C),TC]$$

for all $S,T \in [A,V]$. This completes the proof.

<u>Definition 3.4</u> A <u>symmetry</u> for the premonoidal category P is a natural isomorphism

$$\sigma = \sigma_{ABC}: P(ABC) \to P(BAC)$$

satisfying the following two axioms:

PC3. $\sigma^2 = 1$

PC4. For all $A,B,C,D \in A$, the following diagram commutes:

$$\begin{array}{ccc}
P(ABX)\underline{\otimes}P(XCD) & \xrightarrow{\alpha} & P(BCX)\underline{\otimes}P(AXD) \\
\downarrow{\scriptstyle \sigma\underline{\otimes}1} & & \downarrow{\scriptstyle 1\underline{\otimes}\sigma} \\
P(BAX)\underline{\otimes}P(XCD) & & P(BCX)\underline{\otimes}P(XAD) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
P(ACX)\underline{\otimes}P(BXD) & \xrightarrow[\sigma\underline{\otimes}1]{} & P(CAX)\underline{\otimes}P(BXD)\ .
\end{array}$$

<u>Remark 3.5</u> This definition does not, of course, require A to be small.

It remains to be shown that $[P,V]$ admits a symmetric monoidal closed structure whenever P has a symmetry. For this, define a natural isomorphism $c^* = c^*_{ST}: S*T \to T*S$ as the composite

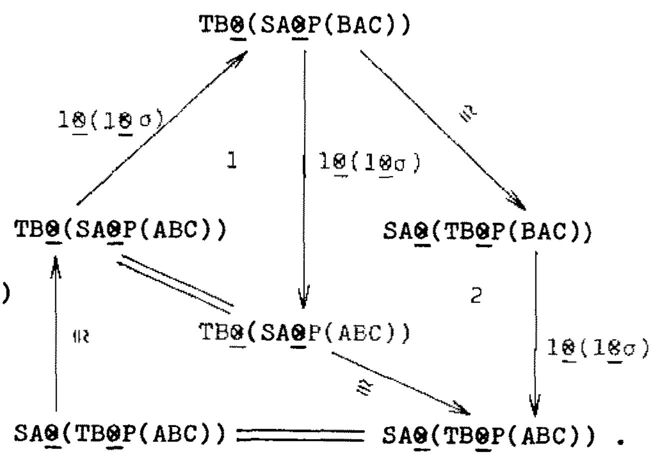
$$S*T = SA\underline{\otimes}(TB\underline{\otimes}P(AB-)) \cong TB\underline{\otimes}(SA\underline{\otimes}P(AB-)) \xrightarrow[1\underline{\otimes}(1\underline{\otimes}\sigma)]{} TB\underline{\otimes}(SA\underline{\otimes}P(BA-)) = T*S.$$

Again, the naturality of c* is a consequence of Lemma 2.5.

<u>Theorem 3.6</u> <u>If</u> σ <u>is a symmetry for</u> P <u>then</u> c* <u>is a symmetry for</u> $[P, V]$.

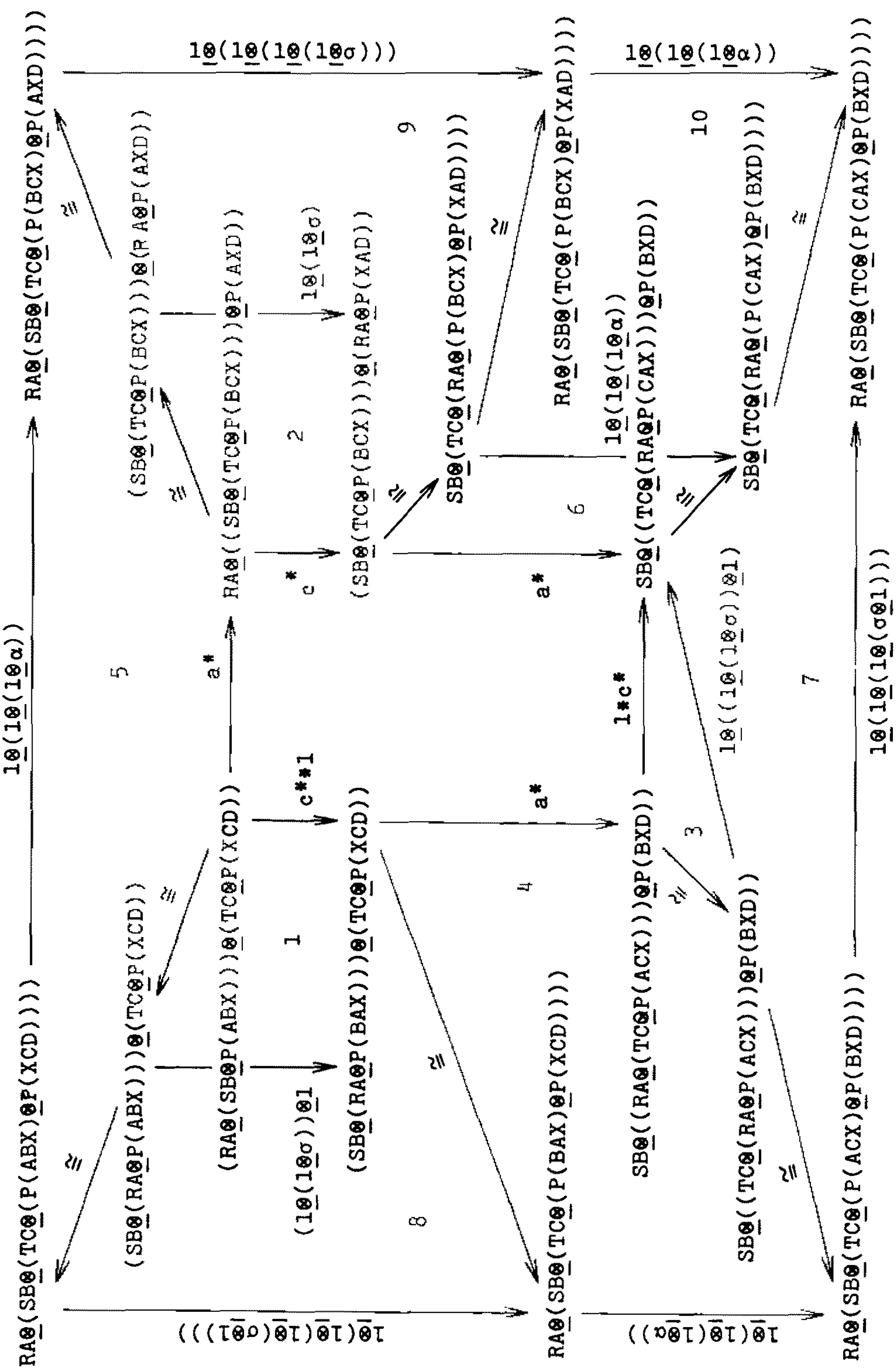
<u>Proof</u> To prove PC3 ⇒ MC6 consider diagram (3.5):

(3.5)



Region 1 commutes by PC3, and region 2 commutes by the naturality of the induced coherence isomorphism involved; hence the exterior commutes and so, by definition of c*, MC6 is satisfied. To prove PC4 ⇒ MC7 consider diagram (3.6), in which the exterior commutes by PC4; 1, 2, and 3 commute by the definitions of * and c*; 4, 5, and 6 commute by the definition of a*; and 7, 8, 9, and 10 commute by the naturality and coherence of induced coherence isomorphisms.

Diagram (3.6)


RA⊗(SB⊗(TC⊗(P(ABX)⊗P(XCD))))
RA⊗(SB⊗(TC⊗(P(BCX)⊗P(AXD))))
RA⊗(SB⊗(TC⊗(P(BCX)⊗P(XAD))))
RA⊗(SB⊗(TC⊗(P(CAX)⊗P(BXD))))
RA⊗(SB⊗(TC⊗(P(BAX)⊗P(XCD))))
RA⊗(SB⊗(TC⊗(P(ACX)⊗P(BXD))))
1⊗(1⊗(1⊗(1⊗σ)))
1⊗(1⊗(1⊗α))
1⊗(1⊗(1⊗α))
1⊗(1⊗(1⊗(σ⊗1)))
1⊗(1⊗(1⊗(σ⊗1)))
1⊗(1⊗(1⊗α))
a*
c*
c*⊗1
1⊗c*
1⊗(1⊗σ)
1⊗(1⊗α)
(1⊗(1⊗σ))⊗1
1⊗((1⊗(1⊗σ))⊗1)
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4. Monoidal Categories

A monoidal category is a particular instance of a premonoidal category. Let $(A,\overline{\otimes},\overline{I},\overline{l},\overline{r},\overline{a})$ be a monoidal structure on A; we write $\otimes$ for $\overline{\otimes}$, etc. when the meaning is clear. The data for the corresponding premonoidal category P are obtained by taking $P(ABC)$ to be $A(A\otimes B,C)$ and JA to be $A(I,A)$, and by defining λ, ρ, and α by the commutativity of the diagrams

$$\begin{array}{ccc} JX\underline{\otimes}P(XAB) & \xrightarrow{\lambda} & A(AB) \\ \| & & \downarrow A(l,1) \\ A(IX)\underline{\otimes}A(X\otimes A,B) & \xrightarrow[y]{} & A(I\otimes A,B) \end{array} \quad ,$$

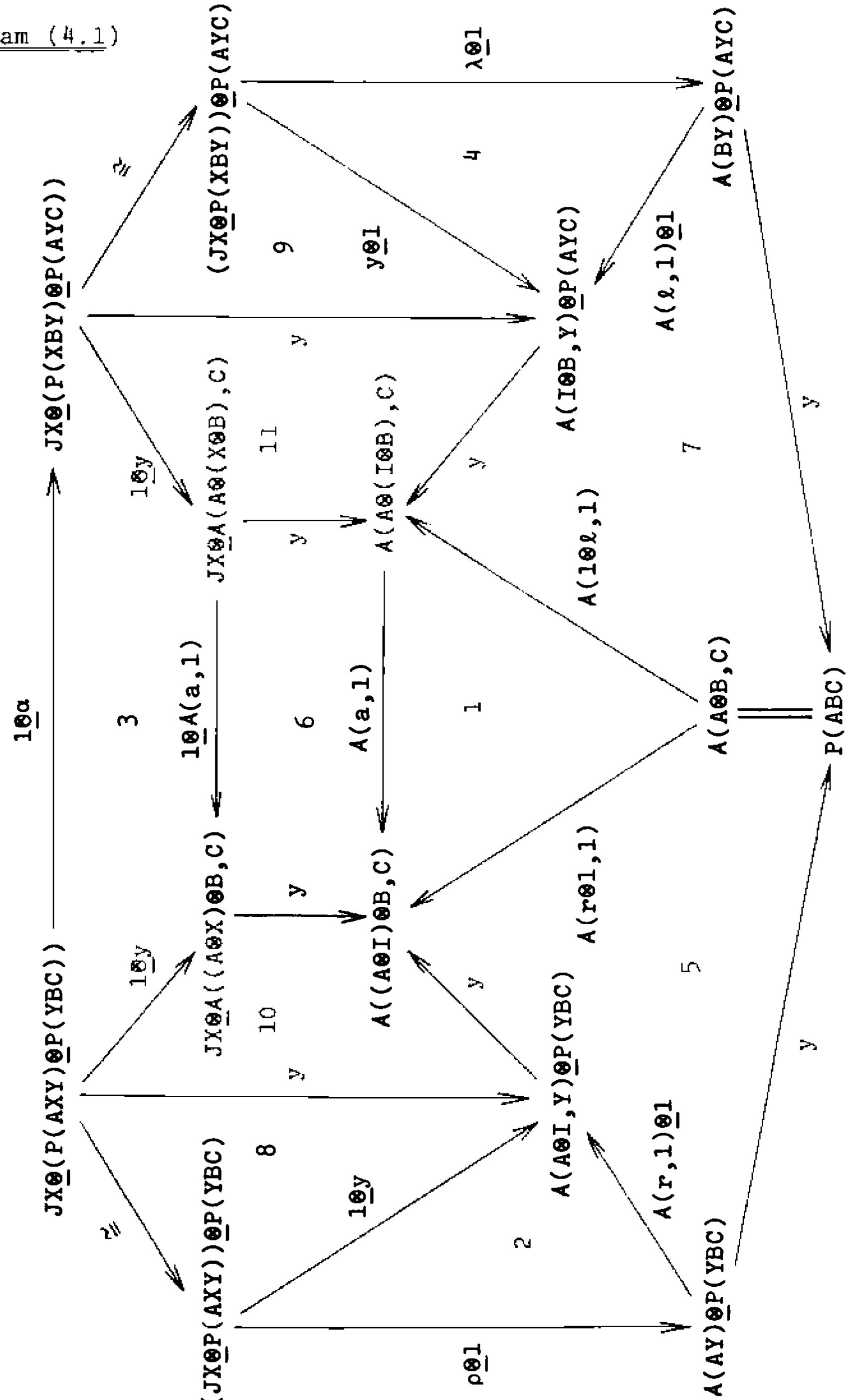
$$\begin{array}{ccc} JX\underline{\otimes}P(AXB) & \xrightarrow{\rho} & A(AB) \\ \| & & \downarrow A(r,1) \\ A(IX)\underline{\otimes}A(A\otimes X,B) & \xrightarrow[y]{} & A(A\otimes I,B) \end{array} \quad ,$$

and

$$\begin{array}{ccc} P(ABX)\underline{\otimes}P(XCD) = A(A\otimes B,X)\underline{\otimes}A(X\otimes C,D) & \xrightarrow{y} & A((A\otimes B)\otimes C,D) \\ \alpha \downarrow & & \uparrow A(a,1) \\ P(BCX)\underline{\otimes}P(AXD) = A(B\otimes C,X)\underline{\otimes}A(A\otimes X,D) & \xrightarrow[y]{} & A(A\otimes(B\otimes C),D) \end{array} \quad .$$

<u>Theorem 4.1</u> $P = (A,P,J,\lambda,\rho,\alpha)$ <u>is a premonoidal category</u>.

Diagram (4.1)


JX⊗P(P(XBY)⊗P(AYC))
(JX⊗P(XBY))⊗P(AYC)
A(BY)⊗P(AYC)
A(I⊗B,Y)⊗P(AYC)
JX⊗A(A⊗(X⊗B),C)
A(A⊗(I⊗B),C)
A(A⊗B,C)
P(ABC)
JX⊗P(P(AXY)⊗P(YBC))
JX⊗A((A⊗X)⊗B,C)
A((A⊗I)⊗B,C)
A(A⊗I,Y)⊗P(YBC)
(JX⊗P(AXY))⊗P(YBC)
A(AY)⊗P(YBC)
1⊗α
λ⊗1
y⊗1
1⊗y
y
A(ℓ,1)⊗1
A(1⊗ℓ,1)
1⊗A(a,1)
A(a,1)
A(r⊗1,1)
A(r,1)⊗1
ρ⊗1
1
2
3
4
5
6
7
8
9
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11

<u>Proof</u> Naturality of the data λ, ρ, and α follows from the naturality of the Yoneda isomorphism y (Lemma 2.6) and of $\bar{\ell}$, $\bar{r}$, and $\bar{a}$. It remains to establish axioms PC1 and PC2; we shall only provide the diagram (4.1) for MC2 $\Rightarrow$ PC1. In this diagram region 1 commutes by axiom MC2; 2, 3, and 4 commute by the definitions of ρ, α, and λ; 5, 6, and 7 commute by the evident $\mathcal{E}ns$-naturality of y; 8 and 9 commute by Lemma 2.8; and 10 and 11 commute by Lemma 2.10. Hence the exterior commutes, as required. To prove MC3 $\Rightarrow$ PC2 one requires a similar (but larger) diagram; the only additional results needed are Lemmas 2.9 and 2.11.

Monoidal structures on $\mathcal{A}$ are in fact characterised among premonoidal ones by the representability of J and of $P(AB-): \mathcal{A} \to \mathcal{V}$ for all $A,B \in \mathcal{A}$. It is also straightforward to verify that a symmetry $\bar{c}$ for $\bar{\otimes}$ provides a symmetry σ for P, defined by:

$$\begin{array}{ccc} P(ABC) & \xrightarrow{\ \sigma\ } & P(BAC) \\ \Vert & & \Vert \\ \mathcal{A}(A\otimes B, C) & \xleftarrow[\mathcal{A}(c,1)]{} & \mathcal{A}(B\otimes A, C) \end{array}$$

commutes.

Now suppose that $(\mathcal{A},\bar{\otimes},\bar{I},\bar{\ell},\bar{r},\bar{a})$ is a small monoidal category.

Applying the generalised representation theorem to the internal-hom formulas (3.3) and (3.4) for $[P,V]$, we get:

$$
\begin{aligned}
(T/S)A &= \int_C [\int^B SB \otimes P(ABC), TC] \\
&= \int_C [\int^B SB \otimes A(A \otimes B, C), TC] \\
&\cong \int_B [SB, \int_C [A(A \otimes B, C), TC]] \\
&\cong \int_B [SB, T(A \otimes B)],
\end{aligned}
$$

and

$$
\begin{aligned}
(S\backslash T)A &= \int_C [\int^B SB \otimes P(BAC), TC] \\
&\cong \int_B [SB, T(B \otimes A)],
\end{aligned}
$$

for all $S,T \in [A,V]$ and $A \in A$. If, in addition, the functor $P(A-C)\colon A^{op} \to V$ admits a representation $A(-,A\backslash C)\colon A^{op} \to V$ for all $A,C \in A$ then, on applying the generalised representation theorem to the tensor-product formula (3.1) for $[P,V]$, we get a convolution formula:

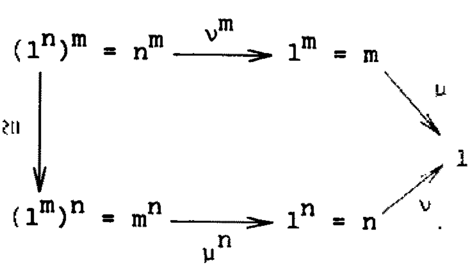
$$
\begin{aligned}
(S*T)C &= \int^A SA \otimes \int^B TB \otimes P(ABC) \\
&= \int^A SA \otimes \int^B TB \otimes A(B, A\backslash C) \\
&\cong \int^A SA \otimes \int^B A(B, A\backslash C) \otimes TB \\
&\cong \int^A SA \otimes T(A\backslash C)
\end{aligned}
$$

for all $S,T \in [A,V]$ and $C \in A$.

Here are three examples of closed functor categories that arise in this way. For certain choices of the ground category V (e.g. $V = Ens$ and $V = Ab$) these examples are quite well known.

<u>Example 4.2</u> If A is a category with only one object $\bar{I}$ whose endomorphism - monoid (i.e. endomorphism - algebra) $\{M = A(\bar{I},\bar{I})$, $\mu: M \otimes M \to M$, $\eta: I \to M\}$ is <u>commutative</u> then we can, by [3] Proposition III.4.2, define a functor $\bar{\otimes}: A \otimes A \to A$ with the data $\bar{I} \bar{\otimes} \bar{I} = \bar{I}$ and $\mu: M \otimes M \to M$. Taking each of $\bar{\ell}$, $\bar{r}$, $\bar{a}$, and $\bar{c}$ to be the identity transformation of the identity functor on A, we see that $(A, \bar{\otimes}, \bar{I}, \bar{\ell}, \bar{r}, \bar{a}, \bar{c})$ is a symmetric monoidal category. In this example we may also take $\bar{I} \backslash \bar{I} = \bar{I} = \bar{I} \bar{\otimes} \bar{I}$ because $A^{op} = A$; it is then easy to check that $[P,V]$ is the category of M-modules with the usual tensor-product and internal-hom.

<u>Example 4.3</u> Let $V = Ens$ and let A be a (finitary) <u>commutative theory</u> in the sense of Linton [9]. Recall that commutativity of A means that, for each m-ary operation $\mu \in A(m,1)$ and n-ary operation $\nu \in A(n,1)$, the following diagram commutes:



This condition is sufficient for the existence of a functor $\bar{\otimes}: A \times A \to A$ defined by $m \bar{\otimes} n = n^m$ and $\mu \bar{\otimes} \nu = \mu . \nu^m$. Let $\bar{I} = 1$, let $\bar{\ell}$, $\bar{r}$, and $\bar{a}$ be the appropriate identity isomorphisms, and let $\bar{c}$ be the canonical "switching" isomorphism shown in the diagram. These data provide A with the structure of a symmetric monoidal category which, in turn, yields a symmetric monoidal closed structure $[P, \mathcal{E}ns]$ on the category $[A, \mathcal{E}ns]$ of A-prealgebras. When this structure is restricted to the full subcategory of $[A, \mathcal{E}ns]$ determined by the A-algebras, we obtain the usual symmetric monoidal closed category of algebras over the commutative theory A.

In fact the above assertions remain valid if we replace $\mathcal{E}ns$ by any cartesian closed V (having small limits and colimits); a finitary V-theory is a V-category A having for objects the non-negative integers 0, 1, ... , n, ... and having the property that $n = 1^n$ in A_0 and $A(m,n) \cong A(m,1)^n$ in V_0 for all $m,n \in A$; the definition of commutativity is the V-analogue of the above and can easily be deduced from [3] III Proposition 4.2. For still further generalisation see Kock [6] and [7].

Example 4.4 First let us note that the monoidal closed normalisation functor $V: V \to \mathcal{E}ns$ has a monoidal closed left adjoint $F: \mathcal{E}ns \to V$ which sends a set X to the copower $\sum_X I$ in V_0 of X copies of I (see [5] §5 for related generalities). The induced monoidal functor $F_\#: \mathcal{E}ns_\# \to V_\#$ (of [3] Proposition III.3.6) sends the $\mathcal{E}ns$-category C to the category $F_\# C$ whose objects are those of C and whose hom-objects are given by $(F_\# C)(AB) = \sum_{C(AB)} I$ in V.

Now let C be the discrete $\mathcal{E}ns$-category whose object-set is the abelian group of integers $\mathbf{Z}$. Putting $A = F_\# C$, defining

$m\bar{\otimes}n = m + n$, $n\backslash m = m - n$, $\bar{I} = 0 \in \mathbb{Z}$, and taking $\bar{\ell}$, $\bar{r}$, $\bar{a}$, and $\bar{c}$ to be the appropriate identity transformations, we obtain a symmetric monoidal closed structure on A. Because C is a discrete $\mathcal{E}ns$-category, $\int^{\cdot}$ reduces to $\sum$ and $\int_{\cdot}$ to Π in V_0 so that the resulting symmetric monoidal closed structure on the category $[A,V]$ of $\mathbb{Z}$-graded objects in V is given by

$$(X*Y)_m = \int^n X_n \otimes Y_{n\backslash m} = \sum_{n\in\mathbb{Z}} X_n \otimes Y_{m-n}$$

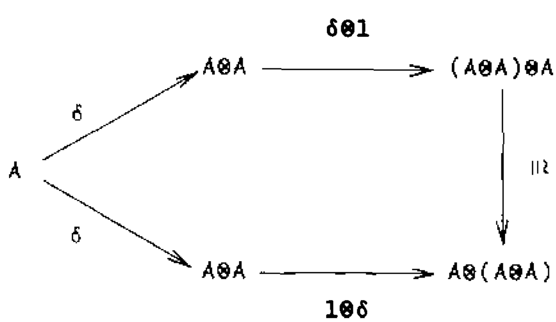
and

$$(Y/X)_m = \int_n [X_n, Y_{m\otimes n}] = \prod_{n\in\mathbb{Z}} [X_n, Y_{m+n}]$$

for all $X,Y \in [A,V]$ and $m \in \mathbb{Z}$.

5. Other Examples

Another type of premonoidal structure arises when a category A has the structure of a comonoid in the monoidal category $V_{\#}$. Such a comonoid consists of a comultiplication functor $\delta\colon A \to A\otimes A$ and a counit functor $\varepsilon\colon A \to I$ satisfying the following coassociative and left and right counit laws (in which the unlabelled isomorphisms are the data isomorphisms of $V_{\#}$):



commutes, and

$$\begin{array}{ccccc} I\otimes\mathcal{A} & \xleftarrow{\varepsilon\otimes 1} & \mathcal{A}\otimes\mathcal{A} & \xrightarrow{1\otimes\varepsilon} & \mathcal{A}\otimes I \\ \wr\Big\downarrow & & \Big\uparrow\delta & & \Big\downarrow\wr \\ \mathcal{A} & = & \mathcal{A} & = & \mathcal{A} \end{array}$$

commute. These laws imply that δ sends an object $A \in \mathcal{A}$ to $(A,A) \in \mathcal{A}\otimes\mathcal{A}$ and that the morphisms $\delta_{AB}\colon \mathcal{A}(AB) \to \mathcal{A}(AB)\otimes\mathcal{A}(AB)$ and $\varepsilon_{AB}\colon \mathcal{A}(AB) \to I$ provide a comonoid structure on the hom-object $\mathcal{A}(AB)$ for each pair of objects $A,B \in \mathcal{A}$.

A premonoidal structure $\mathcal{P}$ is now defined on $\mathcal{A}$ with the following data. Take P and J to be the respective composites

$$(\mathcal{A}^{op}\otimes\mathcal{A}^{op})\otimes\mathcal{A} \xrightarrow{1\otimes\delta} (\mathcal{A}^{op}\otimes\mathcal{A}^{op})\otimes(\mathcal{A}\otimes\mathcal{A}) \xrightarrow{\mathrm{Hom}(\mathcal{A}\otimes\mathcal{A})} \mathcal{V}$$

and

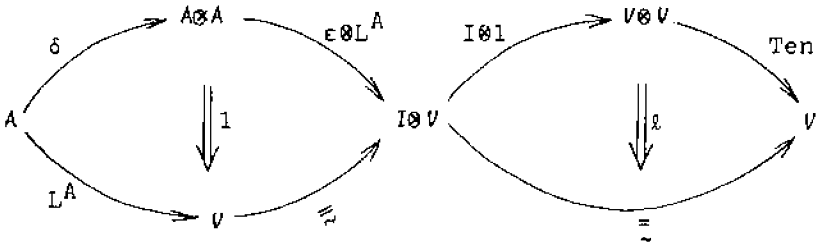
$$\mathcal{A} \xrightarrow{\varepsilon} \mathcal{I} \xrightarrow{I} \mathcal{V},$$

so that, on objects, $P(ABC) = \mathcal{A}(AC)\otimes\mathcal{A}(BC)$ and $JA = I$. Define $\lambda = \lambda_{AB}$ as the composite

$$JX\underline{\otimes}P(XAB) = JX\underline{\otimes}(\mathcal{A}(XB)\otimes\mathcal{A}(AB)) \cong (\mathcal{A}(XB)\underline{\otimes}JX)\otimes\mathcal{A}(AB)$$

$$\xrightarrow[y\otimes 1]{} JB\otimes\mathcal{A}(AB) \xrightarrow[\ell]{} \mathcal{A}(AB),$$

noting that, for each $A \in \mathcal{A}$, the last arrow is actually the B-component of the horizontal composite



of natural transformations. Similarly, define $\rho = \rho_{AB}$ as the composite

$$JX\underline{\otimes}P(AXB) = JX\underline{\otimes}(A(AB)\otimes A(XB)) \cong A(AB)\otimes(A(XB)\underline{\otimes}JX)$$

$$\xrightarrow[1\otimes y]{} A(AB)\otimes JB \xrightarrow[r]{} A(AB),$$

and $\alpha = \alpha_{ABCD}$ as the composite

$$P(ABX)\underline{\otimes}P(XCD) = P(ABX)\underline{\otimes}(A(XD)\otimes A(CD)) \cong (A(XD)\underline{\otimes}P(ABX))\otimes A(CD)$$

$$\xrightarrow[y\otimes 1]{} P(ABD)\otimes A(CD) = (A(AD)\otimes A(BD))\otimes A(CD)$$

$$\xrightarrow[a]{} A(AD)\otimes(A(BD)\otimes A(CD)) = A(AD)\otimes P(BCD)$$

$$\xrightarrow[1\otimes y^{-1}]{} A(AD)\otimes(A(XD)\underline{\otimes}P(BCX)) \cong P(BCX)\underline{\otimes}(A(AD)\otimes A(XD))$$

$$= P(BCX)\underline{\otimes}P(AXD).$$

Furthermore, if the comultiplication δ is commutative we can define a symmetry $\sigma = \sigma_{ABC}$ for P as

$$P(ABC) = A(AC)\otimes A(BC) \xrightarrow[c]{} A(BC)\otimes A(AC) = P(BAC).$$

We now suppose that $\mathcal{A}$ is small and, as in section 4, use the generalised representation theorem to reduce the tensor-product and internal-hom formulas (3.1), (3.3), and (3.4) for $[\mathcal{P},\mathcal{V}]$:

$$\begin{aligned}(S*T)C &= \int^A SA\otimes\int^B TB\otimes P(ABC)\\ &= \int^A SA\otimes\int^B TB\otimes(\mathcal{A}(AC)\otimes\mathcal{A}(BC))\\ &\cong (\int^A SA\otimes\mathcal{A}(AC))\otimes(\int^B TB\otimes\mathcal{A}(BC))\\ &\cong (\int^A \mathcal{A}(AC)\otimes SA)\otimes(\int^B \mathcal{A}(BC)\otimes TB)\\ &\cong SC\otimes TC\end{aligned}$$

for all $S,T \in [\mathcal{A},\mathcal{V}]$ and $C \in \mathcal{A}$, and

$$\begin{aligned}(T/S)A &= \int_C[\int^B SB\otimes P(ABC),TC]\\ &\cong \int_C[\int^B P(ABC)\otimes SB,TC]\\ &= \int_C[\int^B(\mathcal{A}(AC)\otimes\mathcal{A}(BC))\otimes SB,TC]\\ &\cong \int_C[\mathcal{A}(AC)\otimes\int^B\mathcal{A}(BC)\otimes SB,TC]\\ &\cong \int_C[\mathcal{A}(AC)\otimes SC,TC],\end{aligned}$$

$$\begin{aligned}(S\backslash T)A &= \int_C[\int^B SB\otimes P(BAC),TC]\\ &\cong \int_C[SC\otimes\mathcal{A}(AC),TC]\end{aligned}$$

for all $S,T \in [\mathcal{A},\mathcal{V}]$ and $A \in \mathcal{A}$.

It is easy to find instances of this biclosed structure and several commonly-occurring examples are given below.

Example 5.1 If A is a comonoid in $V_\#$ with only one object then its hom-object is a Hopf monoid in V, and $[P,V]$ is the usual biclosed category of modules over this monoid (cf. [3] IV §5).

Example 5.2 If V is cartesian closed then $V_\#$ is a cartesian monoidal category, hence every V-category A admits a unique (commutative) comonoid structure in $V_\#$ with the diagonal functor $A \to A \times A$ as comultiplication and the unique functor $A \to I$ as counit. Taking A small, the reduced tensor-product formula given above shows $[P,V]$ to be cartesian closed.

Example 5.3 Let F: $\mathcal{E}ns \to V$ be the monoidal closed functor described in Example 4.4, and let C be any $\mathcal{E}ns$-category. $\mathcal{E}ns$ is cartesian closed so C is a comonoid in $\mathcal{E}ns_\#$ and this induces an evident (commutative) comonoid structure on $F_\# C$. Hence, when C is small, the category $[F_\# C, V]$, whose underlying $\mathcal{E}ns$-category $[F_\# C, V]_0$ consists of the ordinary $\mathcal{E}ns$-functors from C to V_0 and the $\mathcal{E}ns$-natural transformations between them, automatically admits a symmetric monoidal closed structure over V. For $V = Ab$, this fact was pointed out by P. Freyd in [4].

The types of premonoidal category noted here, and in section 4, are far from being exhaustive. We have not, for instance, considered the premonoidal category which yields the following canonical biclosed structure on the category $[A^{op} \otimes A, V]$ of "bimodules" over an arbitrary small category A:

$$(S*T)(AB) = \int^C S(AC)\otimes T(CB) = S(AC)\underline{\otimes}T(CB),$$

$$(T/S)(AB) = \int_C [S(BC),T(AC)],$$

$$(S\backslash T)(AB) = \int_C [S(CA),T(CB)],$$

$$J(AB) = A(AB)$$

$$\ell^* = A(AC)\underline{\otimes}T(CB) \xrightarrow[y]{} T(AB),$$

$$r^* = T(AC)\underline{\otimes}A(CB) \xrightarrow[\underline{c}]{} A(CB)\underline{\otimes}T(AC) \xrightarrow[y]{} T(AB),$$

$$a^* = (R(AC)\underline{\otimes}S(CD))\underline{\otimes}T(DB) \xrightarrow[\underline{a}]{} R(AC)\underline{\otimes}(S(CD)\underline{\otimes}T(DB)),$$

where $R,S,T \in [A^{op}\otimes A, V]$ and $(A,B) \in A^{op}\otimes A$ (axioms MC2 and MC3 for this definition of * are easily verified). In this case it is decidedly easier to describe the biclosed functor category than to give an explicit premonoidal structure on $A^{op}\otimes A$.

In conclusion I wish to express my gratitude to Professor Max Kelly for his helpful advice and discussions during the preparation of this article.

Remark The editor informs me that J. Bénabou, in lectures in 1967, had proposed the consideration of premonoidal categories, in the case $V = \mathcal{E}ns$, defining the functor P by considerations of section 4 above.

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