PARAMETRIZED SPECTRA, A USER'S GUIDE

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Abstract. This document is intended to be a primer or "user's guide" for parametrized spectra, with a focus on how the subject is applied in fixed-point theory. It is an informal companion to the longer reference [Mal19].

Contents

1. An informal summary

1.1. **The basics.** Let B be a topological space. A **parametrized spectrum** over B is essentially a fibration over B whose fibers are spectra, rather than spaces.

There are two basic ways to make parametrized spectra: suspension spectra and Eilenberg-Maclane spectra.

Recall that you can make a suspension spectrum $\Sigma^\infty F$ any time you have a space F and a chosen basepoint $*$. In the parametrized world, you can make a parametrized suspension spectrum $\Sigma^\infty_B E$ any time you have a fibration $E \to B$ and a chosen section $B \subseteq E$. Over each point $b \in B$, this is just the suspension spectrum of the fiber $\Sigma^\infty E_b$, with basepoint coming from the chosen section.

If you don't have a convenient choice of basepoint section, you can always add a disjoint copy of B, giving the fibration $E_{+B} = E \amalg B$ over B. Its suspension spectrum is denoted $\Sigma^\infty_{+B} E$. Each fiber is $\Sigma^\infty_+ E_b$, the suspension spectrum of $(E_b)_+$.

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Next recall that every abelian group A has an Eilenberg-Maclane spectrum HA. In the parametrized world, every bundle of abelian groups $\mathcal{A} \to B$ has a parametrized Eilenberg-Maclane spectrum $H\mathcal{A}$. As above, the fiber over $b \in B$ is the Eilenberg-Maclane spectrum of the abelian group $\mathcal{A}_b$.

What if we want the fiber to be something other than a suspension or Eilenberg-Maclane spectrum? Then assume for simplicity that $B \simeq BG$ for a topological group G. To make a parametrized spectrum over B whose fiber is any spectrum we want, all we have to do is name the fiber spectrum F and give it a G-action. This is because parametrized spectra over B are equivalent to spectra with a G-action, just as for bundles.[1]

A map of parametrized spectra $X \to Y$ over B is a **stable equivalence** when it induces a stable equivalence on fiber spectra $X_b \to Y_b$ for every $b \in B$. Inverting these equivalences gives the homotopy category of spectra over B.

There are many operations we can perform on parametrized spectra: suspension, looping, cofiber sequences, fiber sequences, homotopy colimits and limits, etc. But these all commute with passing to one fiber. So:

Meta-Theorem: Any operation you can perform to ordinary spectra, you can also perform on parametrized spectra by doing it to each fiber.

There are a few more operations that don't arise this way. The first is the smash product. The above meta-theorem says we can take two parametrized spectra X and Y over B and smash their fibers together. This is the **internal smash product** $X \wedge_B Y$. While this is true, it's actually more natural to start with a spectrum X over A, a spectrum Y over B, and to make an **external smash product** $X \overline{\wedge} Y$. This is a spectrum over $A \times B$ whose fiber over (a, b) is the smash product of the fibers $X_a \wedge Y_b$. If $A = B$ we can always make this and then restrict to the diagonal of B to get $X \wedge_B Y$ back.

Just like fibrations, parametrized spectra can also be pulled back. If $f \colon A \to B$ is a map of topological spaces then every spectrum X over B has a pullback f^*X over A. As with bundles, the fiber of f^*X over $a \in A$ is just the fiber of X over $f(a) \in B$. The pullback has both a left adjoint $f_!$ and a right adjoint f_*. The left adjoint takes a spectrum X over A to the pushout of X and B along A. The right adjoint takes sections along each fiber of $A \to B$.

This description of $f_!$ doesn't actually preserve fibrations over B. We fix this by changing the definition of parametrized spectra, dropping the requirement that their levels are fibrations over B. (We'll implicitly keep everything cofibrant though.) Now we have a bigger category of parametrized spectra, and the ones we considered before are now the fibrant objects. There is a fibrant replacement functor P that brings us back to the original category.

[1] This raises an obvious question: why study parametrized spectra at all if they are equivalent to spectra with G-actions? The answer is that some applications (e.g. [KW07]) are much more natural to state and study with parametrized language.

With the correct assumptions, the three operations f^*, $f_!$, and $\overline{\wedge}$ preserve equivalences, and therefore define operations on the homotopy category. They can also be commuted past each other: for any pair of maps the product commutes with pullback and pushforward,

$$(f \times g)_!(X \overline{\wedge} Y) \cong (f_! X) \overline{\wedge} (g_! Y)$$
$$(f \times g)^*(X \overline{\wedge} Y) \cong (f^* X) \overline{\wedge} (g^* Y).$$

To "commute pullbacks with pushforwards" we consider any homotopy pullback square

(1.1.1)
$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "f"] & \\ B \arrow[dr, "p"'] & & C \arrow[dl, "q"] \\ & D. & \end{tikzcd}$$

Then there is a (Beck-Chevalley) isomorphism $g_! f^* \simeq p^* q_!$. So in total, if we take any two of the three operations f^*, $f_!$, and $\overline{\wedge}$, there is some rule for how to commute them past each other.

When G is a topological group, the equivalence between spectra over BG and G-spectra respects all three of these operations. The pullback f^* corresponds to restricting a G-action to an H-action along a homomorphism $H \to G$, and their left and right adjoints agree, too. The external smash product $\overline{\wedge}$ corresponds to taking a G-spectrum X and an H-spectrum Y and making $X \wedge Y$ into a $G \times H$-spectrum in the obvious way.

Ordinary spectra represent cohomology theories. Parametrized spectra over B represent cohomology theories on the category of spaces over B. Given a space $X \to B$ and a parametrized spectrum $\mathcal{E} \to B$, we get two maps

$$* \xleftarrow{r} X \xrightarrow{p} B.$$

We then define homology and cohomology of X with twisted $\mathcal{E}$-coefficients as

$$H_n(X; \mathcal{E}) = \pi_n(r_! p^* \mathcal{E}), \qquad H^n(X; \mathcal{E}) = \pi_{-n}(r_* p^* \mathcal{E}).$$

In other words, we pull back the coefficient system $\mathcal{E}$ to X, then quotient the X to a point to get homology, or take sections over X to get cohomology. When the coefficient system is trivial, $\mathcal{E} = B \times E$, these recover the usual definitions, because the first operation becomes the smash product $X_+ \wedge E$ and the second becomes the function spectrum $F(X_+, E)$. In particular, twisted K-theory and Thom spectra of various kinds naturally arise as parametrized spectra. The Poincaré duality theorem also extends to a version that uses twisted coefficients.

1.2. **Bicategories and traces.** Now for duality theory. Recall that in a symmetric monoidal category, if an object X is dualizable with dual DX, we can take the **trace** of any map $f\colon X \to X$ by

$$I \xrightarrow{\text{coevaluation}} DX \otimes X \xrightarrow{\cong} X \otimes DX \xrightarrow{f \otimes 1} X \otimes DX \xrightarrow{\text{evaluation}} I.$$

In vector spaces, this is the usual trace of a matrix. If X is a finite CW complex and $f\colon X \to X$, the trace of $\Sigma^\infty_+ f$ in the stable homotopy category is the element of $\pi_0(\mathbb{S}) = \mathbb{Z}$ given by the Lefschetz number $L(f) \in \mathbb{Z}$.

If we fix one base space B, the homotopy category of spectra over B has a symmetric monoidal structure coming from $\wedge_B$. A spectrum is **fiberwise dualizable** if it's dualizable in this structure. This is equivalent to asking that each fiber is dualizable, i.e. a finite spectrum. So if we have a fibration $E \to B$ with finite CW fiber and a fiberwise map $f\colon E \to E$, we can trace $\Sigma^\infty_{+B} f$ in the category of spectra over B. This gives the **fiberwise Lefschetz number**. It is a map of sphere spectra over B, or equivalently a map of spaces

$$B \longrightarrow \Omega^\infty \mathbb{S}$$

that on each $b \in B$ hits the component of $\pi_0(\Omega^\infty \mathbb{S}) = \mathbb{Z}$ corresponding to the Lefschetz number of the fiber map $f_b\colon E_b \to E_b$. So by taking a trace in parametrized spectra, we've essentially topologized the Lefschetz number, turning it from a number into a map of spaces.

There is a stronger fixed-point invariant called the Reidemeister trace $R(f)$, and it also arises by a trace, but it's a "non-commutative" or "bicategorical" trace as defined in [Pon10]. To understand what this means, consider non-commutative rings A, B, C, ... and bimodules over these. We can tensor bimodules ${}_AM_B$ and ${}_BN_C$ over B to get an (A, C)-bimodule $M \odot N := M \otimes_B C$. This makes bimodules into something like a monoidal category, except that the rings change. To be precise, they form a **bicategory**. We can still talk about dualizability on the left or on the right. An (A, B)-bimodule ${}_AM_B$ is dualizable on the right (over B) iff it is finitely generated projective as a B-module. It is dualizable on the left (over A) iff it is finitely generated projective as an A-module.

Furthermore we can take a "circular product" $\langle\langle M \odot N \rangle\rangle$ of the bimodules ${}_AM_B$ and ${}_BN_A$ by tensoring them and then dividing out both the A and B actions. We can do this for any circular list of bimodules, or even just one bimodule. For a single bimodule ${}_BM_B$, the circular product is called the **shadow** of M, or $\langle\langle M \rangle\rangle$.

Using this circular product, we define the **trace** of a morphism $f\colon M \to M$. So long as ${}_AM_B$ is dualizable over B, it has a dual ${}_BM^*_A$, and the trace is the composite

$$\langle\langle {}_AA_A \rangle\rangle \xrightarrow{\text{coev}} \langle\langle {}_AM_B \odot {}_BM^*_A \rangle\rangle \xrightarrow{\cong} \langle\langle {}_BM^*_A \odot {}_AM_B \rangle\rangle \xrightarrow{1\odot f} \langle\langle {}_BM^*_A \odot {}_AM_B \rangle\rangle \xrightarrow{\text{ev}} \langle\langle {}_BB_B \rangle\rangle.$$

If A is commutative and B is an A-algebra, this becomes an element of $B/[B, B]$ that takes the trace of the matrix of f and then mods out by commutators in the ring B so that the trace is independent of the choice of basis.

The Reidemeister trace is a trace in the bicategory $\mathcal{E}x$ of parametrized spectra. The "rings" are now topological spaces A, B, and C, and the "(A, B)-bimodules" are parametrized spectra over the product $A \times B$. These are multiplied the way we compose spans or correspondences: if ${}_AX_B$ is parametrized over $A \times B$ and ${}_BY_C$ is parametrized over $B \times C$ then their product $X \odot Y$ is the external smash product $X \overline{\wedge} Y$, pulled back and pushed forward along the maps

$$A \times B \times B \times C \xleftarrow{1 \times \Delta_B \times 1} A \times B \times C \xrightarrow{1 \times \pi_B \times 1} A \times C.$$

If X and Y happen to be suspension spectra of fibrations E and E', this is just the suspension spectrum of the fiber product $E \times_B E'$. When these operations are derived, this becomes a homotopy pullback $E \times^h_B E'$, in other words pairs of points whose images in B are joined along some path.

Similarly, the circular product takes two suspension spectra over $A \times B$ and $B \times A$ and returns the suspension spectrum of their fiber product $E \times^h_{A \times B} E'$. Circular products for longer lists are similar. The shadow takes a space $E \to B \times B$ to the space of points $e \in E$ and paths in B connecting its two images together.

If X is a finite CW complex then $X \to * \times X$ has a suspension spectrum that is dualizable over X. Taking the trace of a map $f \colon X \to X$ in the non-commutative sense then gives the Reidemeister trace $R(f)$. It is a map of ordinary, non-parametrized spectra $\mathbb{S} \to \Sigma^\infty_+ \Lambda^f X$, where $\Lambda^f X$ is the shadow of the space $(f, \mathrm{id}_X) \colon X \to X \times X$.

The space $\Lambda^f X$ consists of points $x \in X$ and paths from $f(x)$ back to x. Notice that the "constant paths" in this loop space are exactly the fixed points of f. Informally, $R(f)$ is counting these fixed points with their indices, but remembering where they are located in the space $\Lambda^f X$, which has the effect of sorting the fixed points into bins that tell us which ones could possibly be combined.

This bicategory of parametrized spectra is equivalent to a similar bicategory whose rings are suspension ring spectra $\Sigma^\infty_+ G$ and whose modules are bimodule spectra. The space B corresponds to the ring $\Sigma^\infty_+ \Omega B$ and the homotopy fiber products $E \times^h_B E'$ correspond to bar constructions over $\Sigma^\infty_+ \Omega B$. This gives us a conceptual way of seeing why

$$\mathrm{THH}(\Sigma^\infty_+ \Omega B) \simeq \Sigma^\infty_+ \Lambda B.$$

It is because $\mathrm{THH}(\Sigma^\infty_+ \Omega B)$ is the tensor of this ring $\Sigma^\infty_+ \Omega B$ with itself, while the free loop space ΛB consists of points of B joined by a path to themselves. The equivalence of bicategories takes one to the other.

We can also use this correspondence to interpret $R(f)$ algebraically using ring spectra, or geometrically using free loop spaces and Pontryagin-Thom collapses.

2. A technical summary

The next part of the guide is for the user who needs a firmer technical grip. We will state all of the relevant definitions and results as concisely as possible.

2.1. **Over a single base space.** A **retractive space** is a pair of maps $B \xrightarrow{i} X \xrightarrow{p} B$ composing to the identity of B. We call X the total space, $i(B)$ the basepoint section, and

$X_b = p^{-1}(b)$ the fiber over $b \in B$. Note that X_b is a based space with basepoint $i(b)$, so a retractive space is really a family of based spaces indexed by B.

The external smash product of retractive spaces $A \to X \to A$ and $B \to Y \to B$ is a retractive space $X \,\overline{\wedge}\, Y$ over $A \times B$, defined by the pushout

$$\begin{tikzcd} (X \times B) \cup_{A \times B} (A \times Y) \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ A \times B \arrow[r] & X \,\overline{\wedge}\, Y. \end{tikzcd}$$

Over each point (a, b) its fiber is just the smash product of the fibers $X_a \wedge Y_b$. So in the special case where the first retractive space is $* \to X \to *$, the external smash product $X \,\overline{\wedge}\, Y$ is just Y with every fiber smashed with X. We define the **fiberwise suspension** of a retractive space Y to be the external smash product

$$\Sigma_B Y := S^1 \,\overline{\wedge}\, Y.$$

By the above discussion, this has the effect of suspending every fiber of Y. Σ_B has a right adjoint is called fiberwise based loops Ω_B, which takes based loops of each fiber of Y. More generally, the external smash product has a right adjoint in each variable.

A parametrized sequential spectrum over B is a sequence of retractive spaces X_n over B, and bonding maps

$$\sigma \colon \Sigma_B X = S^1 \,\overline{\wedge}\, X_n \to X_{1+n}.$$

A **parametrized orthogonal spectrum** over B also has a continuous fiberwise action of the orthogonal group $O(n)$ on X_n, preserving the basepoint section, such that the composite

$$\sigma^p \colon S^p \wedge X_q \to \ldots \to X_{p+q}$$

is $O(p) \times O(q)$-equivariant. A map of spectra $X \to Y$ consists of maps of retractive spaces $X_n \to Y_n$ commuting with the above structure. The **fiber spectrum** X_b is the evident spectrum formed from the based spaces $(X_n)_b$, $n \geq 0$; this is just an ordinary sequential or orthogonal spectrum.

As in the classical case [MMSS01], these are equivalent to diagrams of retractive spaces indexed by categories $\mathscr{N}$ and $\mathscr{J}$, respectively. A **free spectrum** $F_n X$ is a free diagram on one of the objects of this category:

$$(F_n(X))_m = \mathscr{J}(n, m) \,\overline{\wedge}\, A \qquad \text{or} \qquad (F_n(X))_m = S^{n-m} \,\overline{\wedge}\, A.$$

A **level equivalence** of parametrized spectra is a map $X \to Y$ such that each $X_n \to Y_n$ is a weak equivalence of topological spaces. A **level fibration** is similarly a map in which each $X_n \to Y_n$ is a fibration. This can mean either a Serre fibration (shorthand q-fibration), or a Hurewicz fibration (shorthand h-fibration). Every spectrum is level equivalent to one that is level fibrant.

A **stable equivalence** of spectra is a map that, after making source and target level fibrant, induces an isomorphism on the stable homotopy groups of each fiber spectrum X_n.

Let $\mathcal{OS}(B)$ denote the category of orthogonal spectra over B, and $\operatorname{Ho}\mathcal{OS}(B)$ the homotopy category we get by inverting the stable equivalences. The rest of this document is only concerned with orthogonal spectra, but any statement that does not involve smash products also holds for sequential spectra.

To work effectively up to stable equivalence, we also need cofibrations, and it turns out that we need a few different kinds. A map of retractive spaces $X \to Y$ over B is a **q-cofibration** if it is a retract of a relative cell complex, an **h-cofibration** if it just has the homotopy extension property, and an **f-cofibration** if it has the version of the homotopy extension property where everything is done fiberwise over B. We have the implications

$$f\text{-cofibration} \Longrightarrow h\text{-cofibration} \Longleftarrow q\text{-cofibration}$$

For spectra, the class of **free f-cofibrations** is generated by applying the free spectrum functor to the class of all f-cofibrations of f-cofibrant spaces, then closing under pushouts, transfinite compositions, and retracts. The free q- and h-cofibrations are defined similarly.

There is a **level model structure** on spectra over B with the free q-cofibrations, level equivalences, and level q-fibrations. There is also a **stable model structure** with the free q-cofibrations, and stable equivalences. The fibrations in the stable model structure are maps that are level q-fibrations, and in addition each of the squares

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & \Omega_B^j X_{i+j} \arrow[d] \\ Y_i \arrow[r] & \Omega_B^j Y_{i+j} \end{tikzcd} \tag{2.1.1}$$

is a homotopy pullback square. (We can either apply Ω_B strictly or right-derive it, the condition turns out to be the same). Therefore a parametrized spectrum X is **stably fibrant** when each $X_n \to B$ is a Serre fibration and each of the maps $X_n \to \Omega_B X_{n+1}$ is a weak equivalence.

If G is a topological group whose underlying space is q-cofibrant, and BG is its classifying space, then the stable model structure on spectra over BG is Quillen equivalent to the usual Quillen model category of $\mathbb{S}[G]$-modules, or spectra with a G-action. The left adjoint takes each spectrum X with a G-action to the spectrum that is $EG \times_G X_n \to BG$ at each spectrum level, and the right adjoint takes each parametrized spectrum to something that is equivalent to the fiber spectrum over one point.

The external smash product is a left Quillen bifunctor with respect to the level and stable model structures – this means that it preserves cofibrant objects and equivalences between them. However, taking smash products tend to destroy fibrancy. The external smash product of fibrant objects is not even level fibrant. This is one reason why we also want to use h-cofibrations and h-fibrations.

Since q-cofibrant implies h-cofibrant, the above model structures allow us to replace any spectrum X by a freely h-cofibrant spectrum $QX \xrightarrow{\sim} X$. After this, any freely h-cofibrant spectrum Y can be replaced by a spectrum $Y \xrightarrow{\sim} PY$, where PY is freely f-cofibrant and level h-fibrant. In total, X can therefore be replaced by PQX, which is freely f-cofibrant and level h-fibrant. In many examples this is not even necessary – for instance suspension spectra or free spectra on f-cofibrant, h-fibrant spaces are already freely f-cofibrant and level h-fibrant.

These are convenient conditions to have because they interact well with the smash product:

- $\overline{\wedge}$ preserves freely h- or f-cofibrant spectra.
- $\overline{\wedge}$ preserves level and stable equivalences of freely h-cofibrant spectra.
- $\overline{\wedge}$ preserves the condition of being both freely f-cofibrant and level h-fibrant.

Therefore, if we stick with spectra that are cofibrant and fibrant in this sense, $\overline{\wedge}$ always preserves equivalences and does not destroy fibrancy!

The last bullet point above is one of the most important technical results in [Mal19]. It turns out to be a key property for making parametrized spectra into a bicategory and a symmetric monoidal bifibration.

2.2. **Over all base spaces.** Instead of defining a category $\mathcal{R}(B)$ of retractive spaces over B, we could define a larger category $\mathcal{R}$ of all retractive spaces. An object is a pair (A, X) of a base space A and a retractive space X over A. A map is a commuting diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

The external smash product $\overline{\wedge}$ defines a functor on this larger category $\mathcal{R}$, sending (A, X) and (B, Y) to $(A \times B, X \,\overline{\wedge}\, Y)$.

We similarly define a larger category $\mathcal{OS}$ of orthogonal spectra over all base spaces. An object is a pair (A, X) where A is a space and X is a spectrum over A. A map consists of a map of base spaces $A \to B$, and maps of retractive spaces as above for each spectrum level, commuting with suspension and the $O(n)$-action. This category could also be interpreted as a category of enriched diagrams in $\mathcal{R}$ indexed by $\mathscr{J}$, that are constant on the base space.[2] There is a projection functor $\mathcal{OS} \to \mathbf{Top}$, and each fiber is the category $\mathcal{OS}(B)$ of parametrized spectra over one base space.

The external smash product of spectra extends to this larger category as well, sending (A, X) and (B, Y) to $(A \times B, X \,\overline{\wedge}\, Y)$. The cofibrant and fibrant replacements Q and P from above

[2]See [HSS19] for a generalization that allows you to vary the base space as well.

also extend to functors on the category of all parametrized spectra, that always preserve the base space. This allows us to make all spectra cofibrant and fibrant in a uniform way.

For each map of base spaces $f\colon A \to B$, there are two operations we can perform on retractive spaces, summarized by the diagram below.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ f^*Y \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ X \arrow[r] \arrow[d] & f_!X \arrow[d] \\ A \arrow[r, "f"] & B. \end{tikzcd}$$

Here f^*Y is the pullback $A \times_B Y$, and $f_!X$ is the pushout $X \cup_A B$. These define functors $f^*\colon \mathcal{R}(B) \to \mathcal{R}(A)$ and $f_!\colon \mathcal{R}(A) \to \mathcal{R}(B)$, respectively, and these functors are adjoints. The same operations can be performed on parametrized spectra by doing the above constructions at each spectrum level. They form a Quillen adjunction for the level and stable model structures, and they are a Quillen equivalence if f is a weak homotopy equivalence.

The arrows of the form $f^*Y \to Y$ are called **cartesian arrows**. They satisfy a universal property in the larger category of all parametrized spaces or spectra, so any two cartesian arrows over the same f with the same endpoint Y are canonically isomorphic. This is convenient because if we want to prove that $\overline{\wedge}$ commutes with pullback, we don't have to pick isomorphisms and then keep track of them. We can just say that $\overline{\wedge}$ preserves cartesian arrows. Similarly, an arrow of the form $X \to f_!X$ is called a **co-cartesian arrow**, and $\overline{\wedge}$ preserves co-cartesian arrows.

All together, $\mathcal{OS}$ is a **symmetric monoidal bifibration** over **Top**. This means that cartesian and cocartesian arrows always exist, and that $\overline{\wedge}$ is a functor of spectra lying over the functor $\times$ of spaces that preserves the cartesian and co-cartesian arrows. In addition, we get Beck-Chevalley isomorphisms $g_!f^* \simeq p^*q_!$ for each pullback square of base spaces. This gives us a convenient calculus in which to swap these operations with each other.

Another convenient fact is rigidity. Under mild assumptions, any functor isomorphic to a composite of smash products, pullbacks, and pushforwards is in fact uniquely isomorphic, and has no automorphisms. As a result, when we write down a natural isomorphism that commutes these functors past each other, not only is the natural isomorphism canonical, it is actually unique.

2.3. **Inverting stable equivalences.** Now we turn back to the stable equivalences. The pushforward $f_!$ preserves all notions of cofibration, and weak equivalences between freely h-cofibrant spectra. The pullback f^* preserves all notions of fibration, and weak equivalences between level q-fibrant spectra. In total, this means that if we use freely f-cofibrant and level h-fibrant spectra, both f^* and $\overline{\wedge}$ preserve these spectra and all equivalences between them. $f_!$ destroys fibrancy (unless f itself is a fibration of spaces), but this can be corrected by applying P again. This gives a convenient framework to compose and re-arrange these operations in different orders, without losing their homotopy meaning.

If we want to invert the stable equivalences, but retain the fact that spectra form a bifibration over spaces, we should invert the maps $(A, X) \to (B, Y)$ that are homeomorphisms $A \cong B$ followed by stable equivalences $X \xrightarrow{\sim} Y$. This gives a new category $\mathrm{Ho}\,\mathcal{OS}$ that still projects to **Top**, and each fiber category is the homotopy category $\mathrm{Ho}\,\mathcal{OS}(B)$ of spectra over a single base space. The larger category $\mathrm{Ho}\,\mathcal{OS}$ still has cartesian and cocartesian arrows, but now they are of the form $f^*Y \to Y$ where f^*Y is the pullback *and* Y is fibrant, equivalently f^*Y is the right-derived pullback $\mathbb{R}f^*Y$. Similarly, the co-cartesian arrows are now $X \to f_!X$ where X is cofibrant, or $f_!X$ is the left-derived pushforward $\mathbb{L}f_!X$. The Beck-Chevalley maps are now maps of composites of derived functors, $\mathbb{L}g_!\mathbb{R}f^* \to \mathbb{R}p^*\mathbb{L}q_!$, and they are isomorphisms if (f, g, p, q) form a *homotopy* pullback square.

The left-derived external smash product $\overline{\wedge}^{\mathbb{L}}$ turns this into a symmetric monoidal category, and still preserves the cartesian and co-cartesian arrows. (The proof relies critically on freely f-cofibrant, level h-fibrant replacement.) This allows us to swap smash products, pullbacks, and pushforwards in the homotopy category just as easily as we did on the point-set level. In fact, the isomorphisms we get on the homotopy category using universal properties, agree with the unique isomorphisms we had on the point-set level when the inputs are freely f-cofibrant and level h-fibrant.

We can now define the **internal smash product** $\wedge_B$ by taking spectra X and Y over B, taking the external smash product over $B \times B$, then pulling back along the diagonal $B \to B \times B$. Using the above commutations of $\overline{\wedge}$ with pullback, this is associative and commutative up to canonical isomorphism, making the category $\mathcal{OS}(B)$ of spectra over B into a symmetric monoidal category. Doing the same construction with derived smash product and pullback gives the derived internal smash product $\wedge^{\mathbb{M}} = \mathbb{R}\Delta_B^* \overline{\wedge}^{\mathbb{L}}$ on $\mathrm{Ho}\,\mathcal{OS}(B)$. It is a composite of a left-derived and a right-derived functor, but it still agrees with the point-set internal smash product when X and Y are freely f-cofibrant and level h-fibrant.

We can then explicitly prove that a bundle of compact ENRs E over a compact ENR base B is dualizable in $\mathrm{Ho}\,\mathcal{OS}(B)$ with respect to the internal smash product $\wedge_B$. Writing out the coevaluation and evaluation maps, we get an explicit formula for the trace of a fiberwise map $f\colon E \to E$. This is the **fiberwise Lefschetz number** of f.

2.4. **The bicategory $\mathcal{E}x$.** We make a bicategory $\mathcal{E}x$ of parametrized spectra by taking the "rings" to be base spaces and the "(A, B) bimodules" to be parametrized spectra over $A \times B$. The "maps of bimodules" are then maps in the homotopy category $\mathrm{Ho}\,\mathcal{OS}(A \times B)$. We define composition products as follows, where all the functors are derived.

$$\odot\colon \mathrm{Ho}\,\mathcal{OS}(A \times B) \times \mathrm{Ho}\,\mathcal{OS}(B \times C) \to \mathrm{Ho}\,\mathcal{OS}(A \times C)$$

$$X \odot Y := (r_B)_!(\Delta_B)^*(X \overline{\wedge} Y),$$

$$\begin{tikzcd}
X \odot Y \arrow[d] & \Delta_B^*(X \barwedge Y) \arrow[l] \arrow[r] \arrow[d] & X \barwedge Y \arrow[d] \\
A \times C & A \times B \times C \arrow[l, "r_B"'] \arrow[r, "\Delta_B"] & A \times B \times B \times C
\end{tikzcd}$$

This plays the role of tensoring two bimodules over a ring. On suspension spectra

$$\Sigma^\infty_{+(A\times B)} X \odot \Sigma^\infty_{+(B\times C)} Y,$$

this operation just becomes the fiber product $\Sigma^\infty_{+(A\times C)} X \times_B Y$. The unit is $U_B = \Sigma^\infty_{+(B\times B)} B$ and the shadow or self-product of a spectrum X over $B \times B$ is

$$\langle\!\langle X \rangle\!\rangle := (r_B)_!(\Delta_B)^* X,$$

$$\begin{tikzcd}
\langle\!\langle X \rangle\!\rangle \arrow[d] & \Delta_B^* X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\
* & B \arrow[l, "r_B"'] \arrow[r, "\Delta_B"] & B \times B.
\end{tikzcd}$$

This entire procedure is very general, any symmetric monoidal bifibration can be turned into a shadowed bicategory in this way. So although we applied it to the homotopy category of parametrized spectra and used derived smash products, pullbacks, and pushforwards, we could just as well have done this with the point-set category of parametrized spectra and used strict smash products, pullbacks, and pushforwards.

In fact, the two are related – if we make a point-set bicategory first, then invert equivalences and derive the operations in a canonical way, we get the same bicategory as if we had made the bicategory from the homotopy category of parametrized spectra. Therefore $\mathcal{E}x$ can refer to either one of these constructions.

This is good because it gives us the ability to stay at the point-set level longer, where we can use rigidity to more easily check that things commute. To avoid getting lost, all you have to remember is that on freely f-cofibrant, level h-fibrant spectra, the derived bicategory operations are canonically equivalent to the strict bicategory operations. You can even drop the fibrancy condition and just apply P whenever you need it.

For each map of base spaces $f\colon A \to B$, we form the **base-change** spectrum of f by taking the space A over $A \times B$, with projection map (id_A, f), adding a disjoint copy of $A \times B$, and taking the suspension spectrum:

$$[A \to B] := \Sigma^\infty_{+(A\times B)} A_{(\mathrm{id},f)}.$$

The composition product $- \odot [A \to B]$ is just the pushforward $f_!$, and the composition product $[A \to B] \odot -$ is just the pullback f^*. The base change functor can also be defined so it points the other way, $[B \leftarrow A]$.

These base-change spectra are freely f-cofibrant, and applying P makes them level h-fibrant. Explicitly, applying P replaces the space $A \to A \times B$ with the path space $A^I \times_A B^I \to A \times B$, which is now fibrant. As a result, the shadow of a unit $\langle\!\langle U_B \rangle\!\rangle$ in the homotopy category is the suspension spectrum of the free loop space $\Sigma^\infty_+ \Lambda B$, while the shadow of $\left[B \xrightarrow{f} B\right]$ is the suspension spectrum of the twisted free loop space $\Sigma^\infty_+ \Lambda^f B$, where $\Lambda^f B$ consists of paths in B from any point x to its image $f(x)$.

When we have composable morphisms $A \to B \to C$ we get canonical isomorphisms of base-change objects

$$[A \to B] \odot [B \to C] \cong [A \to C]\,.$$

This is true both on the point-set level and on the homotopy category, where we apply P before taking the product on the left.

Now we can do duality theory in $\mathcal{E}x$. A spectrum over $A \times B$ turns out to be dualizable over A when, if we pull it back to $A \times *$, we get a retract of a finite cell spectrum. So if X is a finite CW complex, the suspension spectrum $[X \to *] = \Sigma^\infty_{+(X \times *)} X$ is dualizable over X.

Given a map $f\colon X \to X$, the trivial observation that $X \to X \to *$ equals $X \to *$ gives an isomorphism of base-change objects

$$f\colon\ [X \to *] \xrightarrow{\cong} [X \to X] \odot [X \to *]$$

This is not quite a self-map of $[X \to *]$, but it's a self-map with coefficients $[X \to X]$, so when X is a finite CW complex, we can still take its trace as follows.

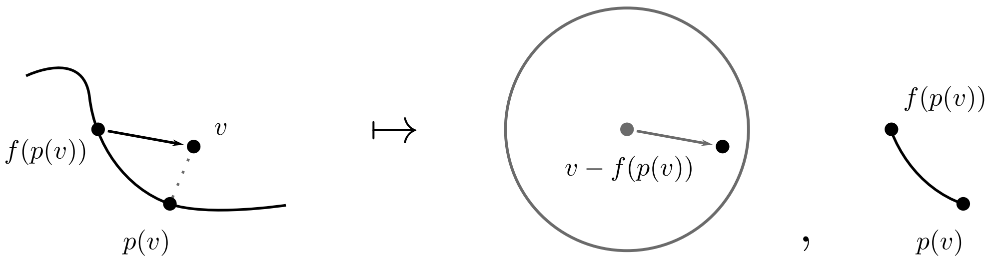
$$\begin{tikzcd}
\langle\langle U_* \rangle\rangle \arrow[r, "c"] & \langle\langle [* \leftarrow X] \odot [X \to *] \rangle\rangle \arrow[r, "f", "\cong"'] & \langle\langle [* \leftarrow X] \odot [X \to X] \odot [X \to *] \rangle\rangle \arrow[d, leftrightarrow, "\cong"'] \\
\langle\langle [X \to X] \rangle\rangle & \langle\langle U_X \odot [X \to X] \rangle\rangle \arrow[l, leftrightarrow, "\cong"'] & \langle\langle [X \to *] \odot [* \leftarrow X] \odot [X \to X] \rangle\rangle \arrow[l, "e"']
\end{tikzcd}$$

This gives a map of spectra from $\langle\langle U_* \rangle\rangle = \mathbb{S}$ to $\langle\langle [X \to X] \rangle\rangle = \Sigma^\infty_+ \Lambda^f X$, the **Reidemeister trace** of X.

The Costenoble-Waner duality theorem gives explicit descriptions of each map in this trace. In particular, the coevaluation map c is a Pontryagin-Thom collapse to a neighborhood of X in $\mathbb{R}^n$, and the evaluation map e is a scanning map that creates a short vector and a short path in X. More explicitly, if $p\colon N \to X$ is the projection of a mapping cylinder neighborhood of X back to X and B_ϵ is a ball in $\mathbb{R}^n$ of radius ϵ, then the maps above have the formula

$$\begin{array}{lrl}
 & v & \in \mathbb{R}^n \\
\mapsto & (v, p(v)) & \in N \times_X X \\
\mapsto & (v, p(v), f(p(v)) & \in N \times_X X_{(\mathrm{id},f)} \times_X X \\
\mapsto & (f(p(v)), v, p(v)) & \in (X \times N) \times_{X \times X} X_{(f,\mathrm{id})} \\
\mapsto & (v - f(p(v)), \gamma_{f(p(v)),v}, p(v)) & \in B_\epsilon \times X^I \times_{X \times X} X_{(f,\mathrm{id})} \\
\mapsto & (v - f(p(v)), \gamma_{f(p(v)),v}) & \in B_\epsilon \times \Lambda^f X
\end{array}$$

when $\|v - f(p(v))\| < \epsilon$, and everything else is sent to the basepoint. The path $\gamma_{f(p(v)),v}$ is a very short path in X from $f(p(v))$ to $p(v)$, which comes about because $v - f(p(v))$ is small and therefore X has a contractible neighborhood near $p(v)$. The simplest way to make this precise is to take a straight line in $\mathbb{R}^n$ from $f(p(v))$ to v and then use p to project it to make a path in X. See the illustration below.



The above formulas define maps of spaces, in particular their composite is a map $S^n \to B_\epsilon/\partial B_\epsilon \wedge (\Lambda^f X)_+$. The corresponding maps of spectra are the n-fold desuspensions of these.

The equivalence between G-spectra and spectra over BG extends to their bicategories and shadows. There is an equivalence between $\mathcal{E}x$ and a bicategory whose "rings" are topological groups G and "bimodules" from G to H are spectra with $G \times H^{\mathrm{op}}$-actions. Equivalently, the rings are the ring spectra $\Sigma^\infty_+ G$ and the bimodules are bimodule spectra over these. The composition product is left-derived smash product over G, in other words the bar construction $B(-, \Sigma^\infty_+ G, -)$. The shadow divides out the diagonal G-action on a $G \times G^{\mathrm{op}}$-spectrum X in a homotopically correct way. In other words, the shadow is topological Hochschild homology with coefficients in X, $\mathrm{THH}(\Sigma^\infty_+ G; X)$. The equivalence of bicategories gives a conceptual proof that

$$\mathrm{THH}(\Sigma^\infty_+ G) \simeq \Sigma^\infty_+ \Lambda BG, \qquad \text{or} \qquad \mathrm{THH}(\Sigma^\infty_+ \Omega B) \simeq \Sigma^\infty_+ \Lambda B,$$

and also proves that the "geometric" Reidemeister trace performed in $\mathcal{E}x$ agrees with the "algebraic" Reidemeister trace performed with ring spectra.

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