Weak Mutual Majority Criterion for Voting Rules

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Abstract

We study a novel axiom for voting rules: *the weak mutual majority criterion* (WMM). A voting rule satisfies WMM if whenever some k candidates receive top k ranks from a qualified majority that consists of more than $q = k/(k+1)$ of voters, the rule selects the winner among these k candidates. WMM lies between the two standard axioms: it is stronger than *the majority criterion* (here $k = 1$ and $q = 1/2$) and weaker than *the mutual majority criterion* (MM, here for any k the size of majority is fixed $q = 1/2$).

The widespread *plurality rule* satisfies WMM (but not MM). Moreover, for any k the bound $q = k/(k+1)$ is tight. *The plurality with runoff rule*, *the Dodgson's rule*, *the Condorcet least reversal rule*, *the Simpson's rule*, and *the Young's rule* satisfy WMM, for most of these rules we also find tight bounds on the size of the qualified majority q.

The well-known *Black's rule* and its positional version do not satisfy WMM: for $k > 1$ its tight bound $q = 1$. We propose two modifications of the Black's positional rule that satisfy WMM: *the qualified mutual majority rule* with the tight bound $q = k/(k+1)$ and *the convex median voting rule* with the tight bound $q = (3k-1)/(4k)$.

Keywords. Positional voting rules, single winner elections, plurality voting rule, plurality with runoff, instant runoff voting, qualified mutual majority rule, mutual majority criterion

1 Introduction

Plurality voting rule and its modifications are the most popular rules in political elections across the world. In the single-winner elections two versions of the plurality rule are most common: the instant-runoff voting (aka preferential vote or single transferable vote) and the plurality with runoff voting (aka the two-round system). In the **plurality with runoff** rule each voter first casts a vote, the two candidates with the highest number of votes proceed to the second round where each voter casts a vote for one of them and the winner is determined by a simple majority.[1] In the **instant-runoff** rule each voter submits a ballot with a rank-ordered list of the candidates. The candidate that gets the least number of the first positions in the ballots is eliminated and his ballots are redistributed among other candidates according to the second position, the candidates with the lowest number of the first positions keep being eliminated one by one until there is a candidate that receives more than half of votes.[2]

[1] A version of plurality with run-off is used for presidential elections in France and Russia. The US Presidential election system with primaries also resembles the plurality with runoff rule given the dominant positions of the two political parties.

[2] The instant runoff system is currently used in parliamentary elections in Australia, presidential elections in India and Ireland; according to the Center of Voting and Democracy fairvote.org [13] the instant-runoff and plurality with runoff rules have the highest prospects for adoption in the US.

One of the very basic criteria a voting rule is desired to satisfy is the **majority criterion**: if one of the candidates gets more than half of votes he must be selected by the rule. Both the instant-runoff rule and the two-round system satisfy the majority criterion as the majority winner is chosen in the first round. The crucial issue arises when no candidate has received more than half of votes.

In this case, a straightforward generalization of the majority criterion is the **mutual majority criterion**: if more than half of voters prefer a group of some k candidates over each other candidates (in any order), then the rule must select one of these k candidates. Instant-runoff rule satisfies the mutual majority criterion: if the majority prefers some group of candidates over all others, then when being eliminated the candidates from this group will transfer their votes to other candidates from the group until some candidate from the group gets a majority of votes.

In contrast, the plurality with runoff rule fails the mutual majority criterion. To see that consider the following example. Let there be five candidates with the following percentage of first positions: Bernie (17%), Donald (25%), Hillary (24%), John (17%), and Ted (17%). Assume further that Bernie, John and Ted form the group preferred by a mutual majority: 51% of voters rank them higher than both Donald and Hillary. The mutual majority criterion demands that either Bernie, John or Ted is selected, but in the plurality with runoff rule all of them are eliminated in the first round and the winner is either Donald or Hillary.

In this paper we study where exactly – between majority and mutual majority – lies the frontier of the plurality with runoff rule lie. To do that we consider a novel criterion that is stronger than the majority criterion and weaker than the mutual majority criterion. The **weak mutual majority criterion** requires that if there is a group of k candidates that get top k positions by a *qualified* majority of more than quota $q = k/(k+1)$ of voters, then the rule must select one of these k candidates.

We show that the plurality rule and the plurality with runoff rule satisfy the weak mutual majority criterion. Moreover the bound of $q = k/(k+1)$ of voters is tight for the plurality rule and $q = k/(k+2)$ for the plurality with runoff rule: for each smaller quota we find a counterexample.

We further study other voting rules, which are popular in the literature, with respect to the weak mutual majority criterion. We show that *the Dodgson's rule* (aka Lewis Carroll's rule), *the Young's rule*, *the Condorcet least-reversal rule* and *the Simpson's rule* (aka maximin rule) satisfy the weak mutual majority criterion. We prove that *the Black's rule* fails this criterion. We also find the tight bounds on the size of the qualified mutual majority for most of these rules.

We also study the weak mutual majority criterion for important type of voting rules – positional voting rules – where the outcome of the elections depends only on the aggregate data. Specifically, in a **positional voting rule** each voter casts a rank-ordered ballot, and the winner is chosen based on how many first positions, second positions, etc. each candidate receives (while the individual preferences do not matter anymore). The most popular positional rule in the literature is the **Black's positional rule**: it selects the majority winner whenever possible[3], otherwise selects the candidate with the highest average position (i.e. the candidate with the highest Borda score). We show that the Black's positional rule does not satisfy the weak mutual majority criterion. We also show that this can be fixed and we provide two modifications of the Black's positional rule that satisfy the weak mutual majority criterion: *the convex median voting rule* and *the qualified mutual majority voting rule*. For these two rules we also find the tight bounds on the size of the qualified mutual majority.

The summary of our results is presented in the table below. For each of the listed rules

[3] The Black's (non-positional) rule selects the Condorcet winner whenever possible, that is the candidate that beats any other candidate in pairwise majority comparisons.

we find a tight bound on the size q of the qualified mutual majority. If this bound equals 1/2 (as in the case of the instant-runoff rule), then the mutual majority criterion is satisfied. If this bound is not higher than $k/(k+1)$, then the weak mutual majority criterion is satisfied, otherwise (as in the case of the Black's rule with $q=1$) the criterion is not satisfied.

Summary of results

	Voting rule	$q(k), k>1$	$\sup_k q(k)$	reference
1	Instant-runoff rule	1/2	1/2	[38]
2	Condorcet least reversal rule	$(5k-2)/(8k)$	5/8	Theorem 3.6
3	Convex median rule	$(3k-1)/(4k)$	3/4	Theorem 4.3
4	Plurality with runoff rule	$k/(k+2)$	1	Theorem 3.3
5-6	Simpson's rule	$(k-1)/k$	1	Theorem 3.4
5-6	Young's rule	$(k-1)/k$	1	Theorem 3.5
7-8	Plurality rule	$k/(k+1)$	1	Theorem 3.1
7-8	Qualified mutual majority rule	$k/(k+1)$	1	Theorem 4.2
9	Black's rule	1	1	Theorem 4.1

Notes. The voting rules are ordered according to the minimal size of the qualified majority $q(k)$ for $k>4$, even k for Condorcet least reversal rule, and high enough number of candidates. These rules satisfy the majority criterion and therefore $q(1)=1/2$. The instant-runoff rule satisfies mutual majority criterion and therefore $q(k)=1/2$.

The paper proceeds as follows. Section 2 presents the model and the necessary definitions. Section 3 presents voting rules that do not satisfy the mutual majority criterion but satisfy the weak mutual majority criterion. The final section 4 presents results on positional voting rules.

2 The Model

2.1 Voting problem

Consider a **voting problem** where $n \geq 1$ voters $I = \{1,\ldots,n\}$ select one winner among $m \geq 1$ candidates (alternatives) $A = \{a_1,\ldots,a_m\}$. Let $L(A)$ be the set of linear orders (complete, transitive and antisymmetric binary relations) on the set of candidates A.

Each voter $i \in I$ is endowed with a **preference relation** $\succ_i \in L(A)$. Preference relation $\succ_i$ corresponds to a unique ranking bijection $R_i : A \to \{1,\ldots,m\}$, where R_i^a is the relative rank that voter i gives to candidate a,

$$R_i^a = |\{b \in A : b \succ_i a\}| + 1, \quad a \in A, \quad i \in \{1,\ldots,n\}.$$

The collection of the individual preferences $\succ = (\succ_1,\ldots,\succ_n) \in L(A)^n$ as well as corresponding ranks $(R_1,\ldots,R_n)$ are referred to as a **preference profile**. (There exist $m!$ different linear orders and $(m!)^n$ different profiles.)

Table 1 provides an example of preference profile for $n=7$ voters over $m=5$ candidates. Here voters are assumed to be anonymous which allows to group voters with the same individual preferences. Each column represents some group of voters, the number of voters in the group is in the top row; the candidates are listed below according to the preference of the group starting from the most preferred.

Table 1. Preference profile

2	2	2	1
a	b	c	c
b	a	d	d
c	c	a	b
d	d	b	a

Given a preference profile we determine function $h(a, b)$ as the number of voters that prefer candidate a over candidate b,

$$h(a, b) = |\{i : a \succ_i b, \quad 1 \leq i \leq n\}|, \quad a, b \in A, \quad a \neq b.$$

Matrix h with elements $h(a, b)$ is called **the tournament matrix**. (Note that $h(a, b) = n - h(b, a)$ for each $a \neq b$.)

Table 2 provides the tournament matrix for the preference profile from Table 1.

Table 2. Tournament matrix

	a	b	c	d
a		**4**	**4**	**4**
b	3		**4**	**4**
c	3	3		**7**
d	3	3	0	

We say that candidate a **(weakly) dominates** candidate b, or **(weakly) wins in pairwise majority comparison**, if $h(a, b) > n/2$ ($h(a, b) \geq n/2$). For arbitrary disjoint subsets of candidates $A_1, A_2 \in A$ we say that A_1 dominates A_2, if for each element $a \in A_1$ and each element $b \in A_2$ a dominates b.

For some subset $B \subseteq A$, a candidate is called **the (weak) Condorcet winner** [9],[4] and is denoted as $CW(B)$, if (weakly) dominates any other candidate in this subset,

$$CW(B) = \{b \in B : h(b, a) > n/2 \quad \text{for all} \quad a \in B \setminus b\}, \quad B \subseteq A.$$

Similarly, **Condorcet loser** in some subset $B \subseteq A$ is a candidate that loses in pairwise comparisons to each candidate in this subset.

$$CL(B) = \{b \in B : h(b, a) < n/2 \quad \text{for all} \quad a \in B \setminus b\}, \quad B \subseteq A.$$

It is easy to see that the set of Condorcet winners CW is either a singleton or empty.

Let the **positional vector** be the vector $n(a) = (n_1(a), \ldots, n_m(a))$, where $n_l(a)$ is the number of voters for whom candidate a has rank l in individual preferences,

$$n_l(a) = |\{i : R_i^a = l, \quad 1 \leq i \leq n\}|, \quad a \in A, \quad l \in \{1, \ldots, m\}.$$

The definition implies that the positional vector has nonnegative elements, $n_l(a) \geq 0$ for each l, and the sum of elements is equal to the number of voters $\sum\limits_{l=1}^{m} n_l(a) = n$.

Let the **cumulative standings** be the vector $N(a) = (N_1(a), \ldots, N_m(a))$, where $N_k(a)$ is number of voters for whom candidate a is not below rank k in individual preference, i.e. $N_k(a) = n_1(a) + \ldots + n_k(a)$.

Candidate a is called **majority winner**, if $n_1(a) > n/2$.

The collection of positional vectors for all candidates is called **positional matrix** $(n(a_1), \ldots, n(a_m)) = n(\succ)$.

Table 3 provides positional matrix for preference profile in Table 1.

Table 3. Positional matrix

Rank	a	b	c	d
1	2	2	3	0
2	2	2	0	3
3	2	1	4	0
4	1	2	0	4

[4] The collection [27] contains English translations of original works by Borda, Condorcet, Nanson, Dodgson and other early researches.

A mapping $C(B, \succ)$ that to each nonempty subset $B \subseteq A$ and each preference profile gives a choice set is called a **social choice rule**,

$$C : 2^A \setminus \emptyset \times L(A)^n \to 2^A,$$

where $C(B, \succ) \subseteq B$ for any B; $C(B, \succ) = B$, if $|B| = 1$; and $C(B, \succ) = C(B, \succ')$, if preference profiles $\succ, \succ'$ coincide on B.

A rule C is **positional**, if for each two preference profiles $\succ$ and $\succ'$ with the same positional matrices on some subset $B \subseteq A$, the choice sets also coincide, $C(B, \succ) = C(B, \succ')$.

One important class of social choice rules is the class of **positional scoring rules**, where each candidate among m candidates gets $s_1, \ldots, s_m$ points for corresponding rank in individual preference (scoring weight) and then these points are summed across all voters. If $s_1 \geq s_2 \geq \ldots \geq s_m$, and at least one inequality is strict, then the rule is called **monotonic scoring rule**. The rule is called **strictly monotonic scoring rule** if $s_1 > s_2 > \ldots > s_m$. (Note that the scoring weights can be defined separately for each subset $B \subseteq A$.)

2.2 Criteria for voting rules

This subsection defines the criteria that are critical for the results of the paper and the social choice rules considered below. Note that the general criteria of universality, non imposition, anonymity, neutrality, unanimity, and homogeneity are satisfied by all social choice rules considered in this paper. These criteria are usually desired to be satisfied for any preference profile.

Condorcet (C) criterion. For each preference profile, if some candidate is a Condorcet winner, then the choice set is a singleton and coincides with this candidate.

Majority (Maj) criterion. For each preference profile, if some candidate is top-ranked by more than half of voters, then the choice set is a singleton and coincides with this candidate.

Mutual majority (MM) criterion [38]. For each preference profile, if more than half of voters give to some k candidates ($B = \{b_1, \ldots, b_k\}$ $1 \leq k < m$) top k ranks in an arbitrary order, then the choice set is included in B.

Weak mutual majority (WMM) criterion [24]. For each preference profile, if more than $k/(k+1)$ of voters give to some k candidates ($B = \{b_1, \ldots, b_k\}$ $1 \leq k < m$) top k ranks in an arbitrary order, then the choice set is included in B.

It is apparent from the definitions that *weak mutual majority criterion* follows from *mutual majority criterion*, and at the same time *weak mutual majority criterion* implies *majority criterion*.

For some preference profile, if for each strictly monotonic scoring rule candidate a gets strictly higher score than candidate b, then we say that a **positionally dominates** b.

As shown in [24, 36], positional dominance is equivalent to the following definition based on stochastic dominance. Candidate a **positionally dominates** candidate b if and only if the cumulative standings for a are not less than for b, that is for each $k = 1, \ldots, m-1$ we have $N_k(a) \geq N_k(b)$ and at least one of the inequalities is strict.

Positional dominance criterion (PD) [16, 17, 24, 36]. For each preference profile $\succ \in L(A)^n$, if candidate a positionally dominates candidate b, then candidate b is not in the choice set $b \notin C(A, \succ)$. Moreover if candidate a positionally dominates each other candidate, then the choice set is a singleton and coincides with it $C(A, \succ) = \{a\}$.

Strict monotonicity (SM) [18, 30, 35]. For each preference profile $\succ \in L(A)^n$, each subset $B \subseteq A$ and each candidate from the choice set $a \in C(B, \succ)$, if some voter increases the position of candidate a by one without changing positions of other candidates, then for the new preference profile $\succ'$ this candidate remains in the choice set and also the choice set becomes weakly smaller: $a \in C(B, \succ') \subseteq C(B, \succ)$.

Positive responsiveness (PR)[5] [18, 26, 38]. For each preference profile $\succ \in L(A)^n$, each subset $B \subseteq A$ and each candidate from the choice set $a \in C(B, \succ)$, if some voter increases the position of candidate a by one without changing positions of other candidates, then for the new preference profile $\succ'$ candidate a becomes the sole winner $C(B, \succ') = \{a\}$.

3 Voting rules satisfying the majority criterion

For completeness of results, we should mention well-studied voting rules that satisfy the mutual majority criterion: Nanson's [27, 28], Baldwin's [2], single transferable vote [21, 22], Coombs [10], sequential pairwise majority, maximal likelihood [23], ranked pairs [39], beat paths [31], median voting rule [3], Bucklin's, majoritarian compromise [33], q-approval fallback bargaining [6], and any refinement of the top cycle [19, 32]. For interested readers we advise [7, 15, 37, 38, 41, 44] for details.

This section considers the classic social choice rules that satisfy the *majority criterion* but do not satisfy the *mutual majority criterion*. In case of only two candidates each rule satisfying the *majority criterion* coincides with the **simple majority rule** where the winner is the candidate that gets at least half of votes.[6] In what follows we consider the case of $m > 2$ candidates.

In the **plurality (Pl)** voting rule each voter casts a vote for her most preferred candidate, and the candidate that receives the highest number of votes is declared to be a winner. This rule is a monotonic scoring rule where the top candidate gets 1 point and other candidates get 0 points,

$$\mathrm{Pl}(A, \succ) = \{a \in A : n_1(a) \geq n_1(b) \quad \text{for all} \quad b \in A \setminus a\}.$$

The **lexicographic** voting rule is a generalized scoring rule where the winner is determined by the number of top positions. In case of a tie, among the candidates with the same number of top positions wins the candidate with the highest number of second positions, and so forth.

Theorem 3.1. Plurality voting rule satisfies the weak mutual majority criterion; for each $m > k \geq 1$, $q = k/(k+1)$ is the tight bound.

Proof.

Let $m \geq 3$, and let more than $nk/(k+1)$ of voters support some subset $B \subsetneq A$ with k candidates, $m > |B| = k \geq 1$. Then all together candidates in B receive strictly more than $nk/(k+1)$ of top positions, while candidates from $A \setminus B$ all together receive strictly less than $n/(k+1)$ top positions. Therefore, at least one of the candidates in B receives strictly more than $n/(k+1)$ of top positions, and each candidate from $A \setminus B$ receives strictly less than $n/(k+1)$ of top positions. Therefore, the plurality voting rule can only select a candidate from set B.

For any smaller quota $q < k/(k+1)$ we can always find the following counterexample. Let the total number of voters be $n = k+1$ and let k voters give candidates from set B top k positions such that each of these candidates gets the top position exactly once. Let some voter give the top position to some other candidate $a \notin B$. Then the plurality voting rule selects all candidates from the set $B \cup a$. ■

Corollary 3.2. The iterative[7] plurality rule, lexicographic rule, and the iterative lexicographic rule satisfy the weak mutual majority criterion.

[5] PR is one of the strongest forms of monotonicity criterion

[6] In case of only $m = 2$ candidates the simple majority rule is the most natural as it satisfies a number of other important axioms according to May's Theorem [26].

[7] If some rule is used repeatedly while the choice set is decreasing, this rule is called iterative.

The **plurality with runoff (RV)** voting rule proceeds in two rounds: first the two candidates with the highest number of votes are determined, then the winner is chosen between the two using simple majority.

Theorem 3.3. Plurality with runoff satisfies the weak mutual majority criterion; for each $m-1=k\geq 1$, $q=1/2$ is the tight bound; for each $m-1>k>1$, $q=k/(k+2)$ is the tight bound.

Proof.

In case $m=3$ the mutual majority criterion holds ($q=1/2$).

In case $k=m-1$, and $q=1/2$, in the second round there is at least one candidate from the supported k candidates, and this candidate wins.

Let $m>3$, and let more than $nk/(k+2)$ of voters support some subset $B\subsetneq A$ with k candidates, $m-1>|B|=k>1$. Then all together candidates in B receive strictly more than $nk/(k+2)$ of top positions, while candidates from $A\setminus B$ all together receive strictly less than $2n/(k+2)$ top positions. Therefore, at least one of the candidates in B and at most one of the candidates in $A\setminus B$ receive strictly more than $n/(k+2)$ of top positions. Thus, in the second round there is at least one candidate from set B. Even if the second candidate is from $A\setminus B$, this second candidate loses to the candidate from B by simple majority. Hence, the winner is from B.

For any smaller quota $q<k/(k+2)$ we can always find the following counterexample. Let the total number of voters be $n=(k+2)n_1+2$ and let kn_1 voters give k candidates from set B top k positions such that each candidate in B gets the top position exactly n_1 times. Consider the other $2*(n_1+1)$ voters and two other candidates $a_1,a_2\notin B$. Let n_1+1 voters top-rank candidate a_1 and the other n_1+1 voters top-rank candidate a_2. Then candidates a_1 and a_2 make it to the second round.

If we set $n_1>2q/(k-kq-2q)$ then set B is supported by more than qn voters. ■

According to **the Simpson's rule** (also known as maximin voting rule) [34, 43] each candidate receives points equal to the minimal number of votes that this candidate gets compared to any other candidate,

$$Si(a)=\min_{b\in A\setminus\{a\}} h(a,b).$$

The winner is the candidate with the highest number of points.

Theorem 3.4. Simpson's rule satisfies the weak mutual majority criterion; for each $m>k>1$, $q=(k-1)/k$ is the tight bound.

Proof.

In case $m=3$ the mutual majority criterion holds ($q=1/2$).

Let $m>3$, and more than $n(k-1)/k$ voters top-rank $k\geq 2$ candidates, denote this subset of candidates as $B=\{b_1,\ldots,b_k\}$. It is easy to see that each candidate in $A\setminus B$ gets less than n/k of Simpson's scores (a candidate from $A\setminus B$ gets the highest score when it is top-ranked by all voters that do not top-rank B).

Denote the number of the first positions of some candidate $b\in B$ among all other candidates in B as $n_1(b,B)$:

$$n_1(b,B)=|\{i: b\succ_i b' \quad \text{for each} \quad b'\in B\setminus b\}|. \tag{1}$$

Since the total number of first positions is fixed $n_1(b_1,B)+\ldots+n_1(b_k,B)=n$, there is a candidate $b\in B$ with the number of top positions weakly higher than the average $n_1(b,B)\geq n/k$.

Hence, there is a candidate that receives not less than n/k of scores, and each candidate from $A\setminus B$ gets less than n/k and cannot be the winner.

To see that the bound $q = (k-1)/k$ is tight consider the following counterexample in Table 4: each candidate $b \in B$ receives exactly qn/k first positions, qn/k second positions and so on from the qualified majority of qn voters, while all voters outside of the qualified majority top-rank some other candidate a_1 and also prefer all candidates in $A \setminus B$ over candidates in B.

Table 4. Preference profile

$\frac{qn}{k}$	...	$\frac{qn}{k}$	$\frac{(1-q)n}{k}$	...	$\frac{(1-q)n}{k}$
b_1	...	b_k	a_1	...	a_1
b_2	...	b_1	...	...	...
...	...	...	a_{m-k}	...	a_{m-k}
b_k	...	b_{k-1}	b_1	...	b_k
a_1	...	a_1	...	...	...
...	...	...	b_{k-1}	...	b_{k-2}
a_{m-k}	...	a_{m-k}	b_k	...	b_{k-1}

Notes. The qualified majority of qn voters give exactly qn/k first, second and so on positions to each candidate $b_i \in B$, all preferences over remaining alternatives $A \setminus B$ are the same. The other $(1-q)n$ voters prefer each candidate in $A \setminus B$ over each candidate in B, and have identical relative ordering of candidates within these two sets. This type of cyclical preferences over B is known as a Condorcet k-tuple.

For each $k > 1$ we can set $n = k^2$ and $q = (k-1)/k$. Then set B is supported by $n(k-1)/k$ voters, while each candidate from the set $B \cup a_1$ gets the same Simpson's score. ■

Note that this tight bound $q = (k-1)/k$ for the Simpson's rule is closely related with the tight bound of q-majority equilibrium [20, 25], and with the minimal quota, that guarantees acyclicity of preferences [11, 14, 40].

By the **Young's rule** [8, 43] the winner is the candidate that needs the least number of voters to be removed for this candidate to become the (weak) Condorcet winner.

Theorem 3.5. Young's rule satisfies the weak mutual majority criterion; for each $m > k > 1$, $q = (k-1)/k$ is the tight bound.

Proof.

In case $m = 3$ the mutual majority criterion holds ($q = 1/2$).

Let $m > 3$, and let more than $n(k-1)/k$ voters top-rank $k \geq 2$ candidates, denote this subset of candidates as $B = \{b_1, \ldots, b_k\}$. For each candidate from $A \setminus B$ to make him the Condorcet winner, we need to remove more than $n(k-2)/k$ voters (i.e. at least the entire qualified majority).

Consider some candidate $b \in B$ with a higher than average number of top positions $n_1(b, B) \geq n/k$ (as defined in equation (1)). For b to win, at most $n(k-2)/k$ of voters have to be removed.

The example from Table 4 shows that the bound $(k-1)/k$ is tight. ■

According to the **Condorcet least-reversal rule** (the simplified Dodgson's rule) [38] the winner is, informally, the candidate $a \in A$ that needs the least number of reversals in pairwise comparisons in order to become the Condorcet winner. Formally, the winner d minimizes the following sum of losing margins compared to each other candidate c:

$$p_d^{CLR} = \sum_{c \in A \setminus d} \max\left\{\frac{n}{2} - h(d,c), 0\right\}.$$

Theorem 3.6. Condorcet least-reversal rule satisfies the weak mutual majority criterion; for each $m > k \geq 2$ and for each even k, $q = (5k-2)/(8k)$ is the tight bound; for each $m > k \geq 1$ and for each odd k, $q = (5k^2 - 2k + 1)/(8k^2)$ is the tight bound.

Proof.

Again we use the preference profile in Table 4. Let's first show that it is the worst possible profile for each candidate $b \in B$ to win by the Condorcet least-reversal rule, i.e. it has the maximum minimum score p_b^{CLR} among all candidates $b \in B$. To maximize the minimum score p^{CLR} for candidates in B we can maximize the scores for the subset B separately: $\sum\limits_{c \in B \setminus b}$. This is true, because the another part $\sum\limits_{c \in A \setminus B}$ is zero whenever $q \geq 1/2$.

According to Proposition 5 in [29], each tournament matrix with k candidates has unique representation as the sum of its transitive matrix and its Condorcet k-tuple matrix. Thus, the maximal element of the transitive matrix gets not more total scores p^{CLR} than in the k-tuple matrix only. Hence, the profile in Table 4 qualifies as the worst case.

Next we find the bound for the profile in Table 4. Each candidate $a \in A \setminus B$ gets at least $p_a^{CLR} \geq nk(2q-1)/2$ points.

Candidate b_1 gets $n/k, 2n/k, \ldots, (k-1)n/k$ pairwise majority wins against candidates $b_k, \ldots, b_2$ correspondingly. For even k the score for each $b \in B$ is $p_b^{CLR} = n(k-2)/8$, for odd k the score is $p_b^{CLR} = n(k-1)^2/(8k)$. Setting these scores equal to the score $p_{a_1}^{CLR} = nk(2q-1)/2$ received by a_1 we get the tight bounds. ■

The classic **Dodgson's** [8, 12, 27] winner is determined as the candidate that needs the least upgrades by one position in individual preferences that makes him the Condorcet winner. To satisfy homogeneity property such upgrades are allowed to perform for non integer amount of voters in order to make a weak Condorcet winner.

Theorem 3.7. The Dodgson's rule satisfies the weak mutual majority criterion; the tight bound for the quota is not lower than $(5k-2)/(8k)$ in case of even $k \geq 2$, and $(5k^2-2k+1)/(8k^2)$ in case of odd $k \geq 1$.

Proof.

Let the qualified majority (more than $k/(k+1)$) of voters support some subset $B \subsetneq A$ with k candidates. Then each candidate $a \in A \setminus B$ gets less than $n/(k+1)$ votes in pairwise comparison against each candidate in set B. Upgrading candidate a by one position in the preference profile adds not more than one vote in a pairwise comparison against each candidate in set B. Therefore candidate a needs more than $k(\frac{n}{2}-\frac{n}{k+1})$ upgrades to become a Condorcet winner. A candidate in B that gets more than $n/(k+1)$ votes (i.e. top positions) needs not more than $(k-1)(\frac{n}{2}-\frac{n}{k+1})$ upgrades in the preference profile in order to become a Condorcet winner. Since $(k-1)(\frac{n}{2}-\frac{n}{k+1}) < k(\frac{n}{2}-\frac{n}{k+1})$, the Dodgson rule selects from set B and thus satisfies the weak mutual majority criterion.

The second statement of the theorem follows from the calculations for profile in Table 4. ■

4 Black's rule and its modifications

In this section we introduce the Black's rule and the Black's positional rule, show that they do not satisfy the weak mutual majority criterion. Then we modify the Black's positional rule so that it satisfies the criterion.

4.1 Black's rule

The **Borda rule** [5, 27] is a convex scoring rule where the first best candidate in an individual preference gets $m-1$ points, the second best candidate gets $m-2$, …, the last gets 0 points.

The following facts will be helpful to establish the main result of this section.

Borda score can be calculated using the positional vector $n(a)$ as follows:

$$Bo(a) = \sum_{i=1}^{m} n_i(a)(m-i), \quad a \in A. \tag{2}$$

The candidate with the highest total score wins. The score can also be calculated using the tournament matrix:

$$Bo(a) = \sum_{b \in A \setminus \{a\}} h(a,b), \quad a \in A. \tag{3}$$

The previous equation readily shows that the Borda score of a Condorcet loser is always lower than the average of all candidates, $Bo(CL(A))$ ¡ $n(m-1)/2$. Similarly, the Borda score of a Condorcet winner is always higher than the average of all candidates, $Bo(CW(A))$ ¿ $n(m-1)/2$.

The **Black's rule** [4] selects the Condorcet winner. If the Condorcet winner does not exist, then the candidate with the highest Borda score (3) is selected.

The **Black's positional rule** selects the majority winner. If the majority winner does not exist, then the candidate with the highest Borda score (2) is selected.

Theorem 4.1. Black's rule satisfies the weak mutual majority criterion in case of $m \leq 4$ candidates. For each $m \geq 5$, Black's rule fails the weak mutual majority criterion; there is a counterexample if and only if $m > (k+1)^2/2$; for each $m > k > 1$, $q = (2m-k-1)/(2m)$ is the tight bound.

In the theorem above, we actually find the tight bound of quota for the Borda rule. In particular case $k = 1$, this quota equals $q = (m-1)/m$, and also was calculated in [1].

4.2 Qualified mutual majority rule

In this section we generalize the definition of majority winner using the idea of the support by a qualified majority.

We say that a (nonempty) subset of candidates $B \subseteq A$ **is positionally supported by a qualified majority**, if

$$\frac{1}{s}\sum_{i=1}^{s}\sum_{a \in B} n_i(a) > \frac{nk}{k+1} \quad \text{for each} \quad s = 1, \ldots, |B|, \tag{4}$$

where $k = |B| \geq 1$.

If some subset of candidates is supported by a qualified majority, then condition (4) is satisfied, therefore this subset is also positionally supported by a qualified majority.

Let's define **the qualified (mutual majority) set** as the intersection of all subsets positionally supported by a qualified majority,

$$QMM(\succ) = \bigcap_{B \subseteq A} B : \quad B \quad \text{satisfies} \quad (4).$$

Theorem 4.2. For any number of voters n and candidates m the following statements are true:

1) For each preference profile $\succ$ the set $QMM(\succ)$ is nonempty;
2) $QMM(\succ)$ is a strictly monotonic choice rule ;
3) $QMM(\succ)$ satisfies the weak mutual majority criterion; for each $m > k \geq 1$, $q = k/(k+1)$ is the tight bound;
4) $QMM(\succ)$ with tie-breaking based on Borda score satisfies positive response, positional

dominance, fails Condorcet criterion for $m \geq 3$, and fails the mutual majority criterion for $m \geq 4$.

Remark 1. If we choose a smaller quota in condition (4) $q < k/(k+1)$, then for each $m > k$ we can find a profile such that the qualified set is empty $QMM = \emptyset$. For instance, consider a profile with $n = k+1$ voters such that for the first voter $a_1 \succ_1 a_2 \succ_1 \ldots \succ_1 a_{k+1}$, for the second voter $a_2 \succ_2 a_3 \succ_2 \ldots \succ_2 a_1$, ..., for the last $(k+1)$'th voter $a_{k+1} \succ_{k+1} a_1 \succ_{k+1} \ldots \succ_{k+1} a_k$.

Remark 2. According to the incompatibility theorem in [24] in case of $m \geq 3$ candidates the qualified mutual majority set (with tie-breaking based on Borda score) does not satisfy the continuity criterion [35, 42], independence of Pareto dominated candidates and independence of clones [39].

4.3 Convex median voting rule

Based on truncated Borda score [24] defines the **convex median voting rule (CM)** in the following way.

First for some positional vector $n(a)$ and some real number $t \in (0, +\infty)$ define the truncated Borda score [16] as

$$B_t(a) = t \cdot n_1(a) + (t-1)n_2(a) + \ldots + (t - \lfloor t \rfloor)n_{\lfloor t \rfloor + 1}(a), \quad t \in (0, +\infty),$$

where formally put $n_i(a) = 0$ for $i > m$. The definition implies that $B_{m-1}(a) = Bo(a)$, $B_k(a) = N_1(a) + \ldots + N_k(a)$ – cumulative standings for $k = 1, \ldots, m-1$.

For each candidate a define the score of convex median using the following formula:

$$\mathrm{CM}(a) = \begin{cases} m - 1 - \frac{1}{2}\max\left\{t \in [1, 2(m-1)] : \frac{B_t(a)}{t} \leq \frac{n}{2}\right\}, & n_1(a) \leq \frac{n}{2}, \\ m - 2 + \frac{n_1(a)}{n}, & n_1(a) > \frac{n}{2}, \end{cases}$$

The winner is the candidate with the highest value of the convex median, ties are broken using Borda scores (2).

In [24] it is proved that the convex median satisfies positional dominance, positive responsiveness and weak mutual majority, but fails Condorcet and mutual majority criteria. In this paper we find the tight bounds of the quota.

Theorem 4.3. The convex median voting rule satisfies the weak mutual majority criterion; for each $m > 2k$, $q = (3k-1)/(4k)$ is the tight bound; for each $m = k+1$, $q = 1/2$ is the tight bound; for each $2k \geq m > k+1$, the tight bound q satisfies to inequality $\frac{1}{2} < q < \frac{3k-1}{4k}$ and to equation

$$4k(m-k-1)q^2 + (5k^2 + 5k - 2mk - m^2 + m)q + m(m-1-2k) = 0.$$

5 Conclusions

We studied the novel axiom called the weak mutual majority criterion with respect to the most popular voting rules. Our first focus was to study plurality rules and to find how well they respect the preferences of the qualified majority. The instant-runoff rule respects the qualified majority extremely well as it satisfies the mutual majority criterion. We show that plurality with runoff does it slightly better than the simple plurality rule as it has a smaller tight bound. According to this criterion, all other considered rules are located between the instant-runoff rule and the plurality rule.

Our second focus was on positional rules. We show that the standard Black's positional rule does not satisfy the weak mutual majority criterion. We also show that positionallity by itself does not preclude the weak mutual majority criterion and propose two positional rules that satisfy it.

One specific open question arises from the incomplete result regarding the Dodgson's rule: in contrast to other results, Theorem 3.7 does not specify the tight bound on the size of the qualified majority. The value of the tight bound seems to be a hard question, as the Dodgson's rule is known to be not very operational. It is not easy to check whether the profile in Table 4 gives the worst case for each candidate in group of mutually supported candidates B and at the same time the best case for some other candidate in $A \setminus B$.

A more general open question is the analysis of the aggregative properties of voting rules in practically-relevant scenarios. In this paper the main results are based on the worst-case analysis. Future research can make use of more realistic scenarios inspired by theories of individual decision-making, empirical results and experiments on voting.

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APPENDIX

Theorem 4.1. Black's rule satisfies the weak mutual majority criterion in case of $m \leq 4$ candidates. For each $m \geq 5$, Black's rule fails the weak mutual majority criterion; there is a counterexample if and only if $m > (k+1)^2/2$; for each $m > k > 1$, $q = (2m-k-1)/(2m)$ is the tight bound.

Proof.

In case $m = 2, 3$ the mutual majority criterion holds ($q = 1/2$).

Let's show that the weak mutual majority criterion holds in case $m = 4$. If three candidates are supported by a qualified mutual majority, then the fourth candidate is a Condorcet loser and cannot get the highest Borda score. If two candidates are supported by a qualified majority (in this case more than 2/3 of voters), then these two candidates get a Borda score higher than $11n/3$, and at least one of them gets more than $11n/6$. Each of the other two candidates gets at most $5n/3$ and therefore cannot win.

Let's show that the weak mutual majority criterion does not hold in case $m = 5$. Consider the preference profile of $n = 20$ voters (Table 5), where two candidates $\{b_1, b_2\}$ are supported by a qualified majority (more than 2/3 of voters). There is no Condorcet winner in this example, therefore the Black's rule selects three candidates $\{b_1, b_2, a_1\}$ which Borda score equals 52 points. (Note that candidates b_1 and b_2 are weak Condorcet winners.)

Table 5. Preference profile

7	7	3	3
b_1	b_2	a_1	a_1
b_2	b_1	a_2	a_2
a_1	a_1	a_3	a_3
a_2	a_2	b_1	b_2
a_3	a_3	b_2	b_1

For $m > 5$ one can construct an analogous example by adding new candidates directly below a_3 in each voter's preference relation.

Moreover for $m \geq 9$ one can provide an example where even weak Condorcet winner does not exist. Consider the preference profile of $n = 13$ voters (Table 6), where three candidates $\{b_1, b_2, b_3\}$ are supported by a qualified majority (more than 3/4 of voters). For candidates b_1, b_2, b_3 the Borda score is 73, while for candidate a_1 it is 74.

Table 6. Preference profile

4	3	3	1	1	1
b_1	b_3	b_2	a_1	a_1	a_1
b_2	b_1	b_3	a_2	a_2	a_2
b_3	b_2	b_1	a_3	a_3	a_3
a_1	a_1	a_1	a_4	a_4	a_4
a_2	a_2	a_2	a_5	a_5	a_5
a_3	a_3	a_3	a_6	a_6	a_6
a_4	a_4	a_4	b_3	b_1	b_2
a_5	a_5	a_5	b_2	b_3	b_3
a_6	a_6	a_6	b_1	b_2	b_1

For each $m > k > 1$, one can find the tight bound for the quota $q = q(k, m)$ using the following equation:

$$(1 - q)(m - 1) + q(m - k - 1) = q\frac{m - 1 + m - k}{2} + (1 - q)\frac{k - 1}{2},$$

where the left part is the maximal Borda score for any $a \notin B$, and the right part is the minimal maximal Borda score for any $b \in B$.

Thus, a counterexample exists if and only if

$$q(k, m) = 1 - \frac{k + 1}{2m} > \frac{k}{k + 1},$$

and it is equivalent to $m > (k + 1)^2/2$.

■

Theorem 4.2. For any number of voters n and candidates m the following statements are true:
1) For each preference profile $\succ$ the set $QMM(\succ)$ is nonempty;
2) $QMM(\succ)$ is a strictly monotonic choice rule ;
3) $QMM(\succ)$ satisfies the weak mutual majority criterion; for each $m > k \geq 1$, $q = k/(k+1)$ is the tight bound;
4) $QMM(\succ)$ with tie-breaking based on Borda score satisfies positive response, positional dominance, fails Condorcet criterion for $m \geq 3$, and fails the mutual majority criterion for $m \geq 4$.

Proof.

1) Let's define the following rule

$$M(\succ) = \bigcap_{B \subseteq A} B : \sum_{a \in B} n_1(a) > \frac{n|B|}{|B|+1}.$$

Subsets $B \subseteq A$ in the definition of rule $M(\succ)$, are determined by the following condition:

$$n_1(B) = \sum_{a \in B} n_1(a) > \frac{n|B|}{|B|+1}. \tag{5}$$

From the definition of M it readily follows that for each profile $\succ$ we have $M(\succ) \subseteq QMM(\succ)$. Let's show that $M(\succ)$ is always nonempty.

Assume that among the subsets satisfying condition (5) the smallest power is $k = 1$. In this case there is a majority winner, some candidate $a : n_1(a) > n/2$. Each subset not containing a has fewer than $n/2$ top positions and does not satisfy condition (5). Therefore, $QMM(\succ) = \{a\}$.

Assume now that among the subsets satisfying condition (5) the smallest power is $k = 2$. **In step** $l = 1$ consider set B_1 that satisfies condition 5 and such that $B_1 = 2$. Denote $\cap_1 = B_1$, where $|\cap_1| = 2$. Condition (5) implies $n_1(B_1) > 2n/3$, thus $n_1(A \setminus \cap_1) < n/3$, and each $B \subseteq A \setminus \cap_1$ has $n_1(B) < n/3$. Thus, either each subset satisfying condition (5) contains $\cap_1$, and $QMM(\succ) = \cap_1$, or **in step** $l = 2$ there exists some set B_2 satisfying condition (5) such that $\cap_2 = B_1 \cap B_2$, and $|\cap_2| = 1$. Condition (5) implies $n_1(B_2) > 2n/3$, therefore $n_1(B_1)+n_1(B_2)+n_1(A\setminus\cap_2) \leq 2n$. Therefore, $n_1(A\setminus\cap_2) < 2n/3$, and each subset $B \subseteq A \setminus \cap_2$ contains $n_1(B) < 2n/3$. Thus, each subset satisfying condition (5) contains $\cap_2$, and $QMM(\succ) = \cap_2$.

Assume now that among the subsets satisfying condition (5) the minimum power is k, such that $3 \leq k \leq m$. **In step** $l = 1$ consider subset B_1, such that B_1 satisfies condition (5), and $|B_1| = k$. Denote $\cap_1 = B_1$, where $|\cap_1| = k$. Condition (5) implies $n_1(B_1) > kn/(k+1)$, therefore $n_1(A\setminus\cap_1) < n/(k+1)$, and each subset $B \subseteq A\setminus\cap_1$ has $n_1(B) < n/(k+1)$. Thus, either each subset satisfying condition (5) contain $\cap_1$, and $QMM(\succ) = \cap_1$, or we proceed to the **next step** $l = 2$.

...

In step $l \in \{2, \ldots, k-1\}$ there exists a subset B_1 satisfying condition (5) such that $\cap_l = B_1 \cap \ldots \cap B_l$, and $1 \leq |\cap_l| \leq k - l + 1$. Condition (5) implies $n_1(B_l) > kn/(k+1)$, therefore $n_1(B_1) + \ldots + n_1(B_l) + n_1(A \setminus \cap_l) \leq l \cdot n$. Thus, $n_1(A \setminus \cap_l) < l \cdot n/(k+1)$, and each subset $B \subseteq A \setminus \cap_l$ has $n_1(B) < l \cdot n/(k+1) \leq kn/(k+1)$. Thus, either each subset satisfying condition (5) contains $\cap_l$, and $QMM(\succ) = \cap_l$, or **in step** $l+1$ ... **in step** $l = k$ there is a subset B_k satisfying condition (5) such that $\cap_k = B_1 \cap \ldots \cap B_k$, $|\cap_k| = 1$, $n_1(A \setminus \cap_k) < kn/(k+1)$, and $QMM(\succ) = \cap_k$.

2) Consider a profile $\succ$ and an arbitrary candidate $a \in QMM(\succ)$. Consider now another profile $\succ'$ derived from $\succ$ by upgrading a by one position in the preference of some voter. Each set satisfying condition (4) for the profile $\succ$ contains candidate a and thus this subset also satisfies condition (4) for the profile $\succ'$. Therefore set $QMM(\succ')$ is contained in the original set $QMM(\succ') \subseteq QMM(\succ)$.

For a contradiction, assume that $a \notin QMM(\succ')$. Then there is some set B' satisfying condition (4) for profile $\succ'$ such that $a \notin B'$. But B' also satisfies condition(4), which contradicts that $a \in QMM(\succ)$.

3) If the qualified majority supports some subset of candidates then this subset is contained in the intersection from the definition of the qualified majority (4). For each smaller coalition $q < k/(k+1)$ the counterexample is the same as in the proof of Theorem 3.1 for the plurality rule.

4) Since the Borda rule satisfies positive response and the qualified set is strictly monotonic, then the hybrid rule also satisfied positive response.

Each candidate a that positionally dominates some candidate b from the qualified set also belongs to the qualified set. Candidate a gets a higher Borda score than candidate b does, therefore the hybrid rule satisfies the positional dominance criterion. Then due to the incompatibility Theorem [24] the Condorcet criterion is not satisfied for $m \geq 3$.

Consider the preference profile for $n = 7$ voters and $m = 4$ candidates (Table 1) where candidates $\{a, b\}$ are mutually supported by a simple majority. According to condition (4) the qualified set consists of three candidates $\{a, b, c\}$, which get 12, 11 and 13 Borda points respectively. The example can be generalized to the case of $m > 4$ candidates where the new candidates are added to the individual preferences below candidates a, b, c, d. Thus the mutual majority criterion is not satisfied for $m \geq 4$.

Let $m = 3$ and let candidates $\{a, b\}$ be supported by a simple majority. Then candidate c is a majority loser and gets the lowest Borda score $Bo(c) < n < \max\{Bo(a), Bo(b)\}$.

If either of the subsets $\{a\}$, $\{b\}$ or $\{a, b\}$ satisfies condition (4), then either a or b wins. If subset $\{a, c\}$ satisfies condition (4), then, adding up the two inequalities in condition (4) we get $Bo(a) + Bo(c) > 2n$. The latter inequality implies $Bo(b) < n < Bo(a)$ and thus subset $\{b, c\}$ does not satisfy condition (4). Therefore the qualified set in this case is either $\{a, c\}$ or just $\{a\}$ and candidate a is the winner. If the qualified set contains all three candidates, then based on the Borda score we select either a or b. Thus for $m = 3$ the mutual majority criterion is satisfied. ∎

Theorem 4.3. The convex median voting rule satisfies the weak mutual majority criterion; for each $m > 2k$, $q = (3k - 1)/(4k)$ is the tight bound; for each $m = k + 1$, $q = 1/2$ is the tight bound; for each $2k \geq m > k+1$, the tight bound q satisfies to inequality $\frac{1}{2} < q < \frac{3k-1}{4k}$ and to equation

$$4k(m - k - 1)q^2 + (5k^2 + 5k - 2mk - m^2 + m)q + m(m - 1 - 2k) = 0.$$

Proof.

Let qn voters ($q > 0$) mutually support some subset $B = \{b_1, \ldots, b_k\}$. Then each candidate $a \notin B$ gets the following truncated Borda score:

$$\frac{B_{2kq}(a)}{2kq} \leq (1 - q)n + \frac{(2kq - k)qn}{2kq} = \frac{n}{2}.$$

Let $m > 2k$ and $q > (3k - 1)/(4k)$. It is sufficient to show that for some $b \in B$ its truncated Borda score is higher: $B_{2kq}(b)/(2kq) > n/2$. For a contradiction assume the opposite:

$$\frac{B_{2kq}(b)}{2kq} \leq \frac{n}{2} \quad \text{for each} \quad b \in B.$$

Then

$$\frac{(2kq)n_1(b) + \ldots + (2kq - k + 1)n_k(b)}{2kq} \leq \frac{n}{2} \quad \text{for each} \quad b \in B,$$

whence, after summing up k inequalities, we get:

$$\frac{qnk(4kq - k + 1)}{4kq} < \frac{nk}{2}.$$

The latter inequality contradicts the assumption $q > (3k - 1)/(4k)$.

To show that the bound is tight we again use the preference profile from Table 4.

Similarly we find a tight bound for the case $2k \geq m \geq k+1$:

$$\min_{\succ} \max_{b \in B} \frac{B_{2kq}(b)}{2kq} = \frac{\frac{qn}{k}\frac{(4kq-k+1)}{2}k}{2kq} + \frac{\frac{(1-q)n}{k}\frac{(4kq-m)}{2}(2k-m+1)}{2kq} = \frac{n}{2},$$

which leads to equation 5 and also to a special case $m = k+1, q = 1/2$.

$\blacksquare$