121B: ALGEBRAIC TOPOLOGY

Contents

6. Poincaré Duality

These are notes from before.

Theorem 6.1 (Poincaré Duality)**.** *Suppose that M is a compact, connected, oriented n-manifold (without boundary). Then there is an isomorphism*

$$D : H^k(M;\mathbb{Z}) \xrightarrow{\approx} H_{n-k}(M;\mathbb{Z})$$

which is given by cap product with the fundamental class $[M] \in H_n(M;\mathbb{Z})$:

$$D\alpha = [M] \cap \alpha$$

To explain this theorem, I need to give definitions of:

(1) topological manifold
(2) orientation of a manifold
(3) fundamental class $[M] \in H_n(M)$ (also called "orientation class")
(4) cap product

The proof of Poincaré duality uses

(1) "good coverings" and
(2) Mayer-Vietoris sequences

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6.1. **Manifolds.**

Definition 6.2. A (topological) *n-manifold* is a Hausdorff space M which is locally homeomorphic to (n-dimensional) Euclidean space.

Hausdorff means any two points are contained in disjoint open neighborhoods. "Locally homeomorphic to Euclidean space" means: For every $x \in M$ there is a neighborhood U of x in M and a homeomorphism $\phi : U \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is homeomorphic to an open ball in $\mathbb{R}^n$ we can assume that ϕ is a homeomorphism $U \cong B_1(0)$. Sometimes we need a ball-within-a-ball. Then we take $V = \phi^{-1}(B_\epsilon(0))$ for some $0 < \epsilon < 1$.

6.1.1. *Charts and good atlases.* The pair (U, ϕ) is called a *chart* for the manifold M. A collection of charts $\{(U_\alpha, \phi_\alpha)\}$ which covers M in the sense that $\bigcup U_\alpha = M$ is called an *atlas* for M. We want to consider the case when these atlases are "good" even though good atlases may not exist (or at least may be difficult to prove that they exist).

Definition 6.3. We say that V is a *good open ball neighborhood* of x in M if there is a chart (U, ϕ) so that the closure of V lies in U,

$$V = \phi^{-1} B_\epsilon(0)$$

and

$$\overline{V} = \phi^{-1} \overline{B}_\epsilon(0)$$

We call $\overline{V}$ a *good disk nbd* of x. (We also say V is a *good open ball* and $\overline{V}$ is a *good disk*.

Proposition 6.4.
(1) *Every $x \in M$ has a good open nbh $V \subset M$*
(2) *$H_n(M, M - V) \cong H_n(M, M - \overline{V}) \cong \mathbb{Z}$ for every good open nbh V of x.*

Definition 6.5. A *good covering* of an n-manifold M is a collection of good open balls U_α in M so that

(1) Any nonempty finite intersection of the U_α is a good open ball.
(2) $M = \bigcup U_\alpha$

A *good covering* of a subset $A \subseteq M$ is the same as above except that the second condition is replaced by: $A \subseteq \bigcup U_\alpha$.

Theorem 6.6. *Every compact subset $A \subseteq \mathbb{R}^n$ has an arbitrarily small good covering.*

Arbitrarily small means inside any open neighborhood of A.

The following theorem shows that most manifolds have good coverings.

Theorem 6.7. *Any second countable C^∞-manifold has a good covering.*

Second countable means it has a countable basis. This implies (quoting a number of difficult theorems) that you can properly embed the manifold into Euclidean space so that the image is C^∞. This gives a metric on M and we can talk about *geodesic* subsets. These are subsets U which contain the shortest path in M connecting any two points of U. Every point in M has a geodesic open neighborhood (any sufficiently small ϵ-ball). The intersection of two geodesic open sets is obviously geodesic. So, this covering is good. ("Geodesic" takes the place of "convex" in the earlier proof.)

6.2. **Orientation.** I need to define orientation in several settings

(1) as generator $\alpha_x \in H_n(M|x) = H_n(M, M - \{x\}) \cong \mathbb{Z}$
(2) as generator $\alpha_V \in H_n(M|V) = H_n(M, M - V) \cong \mathbb{Z}$ for good open balls V
(3) Orientation of M as a consistent family of point orientations.
(4) Orientation of M is a compatible family of orientations on a covering of M.
(5) orientation class $[M] \in H_n(M) = H_n(M, M - M)$

6.2.1. *Generator of $H_n(M, M - \{x\})$.* I reminded you about excision:

$$H_*(X, A) \cong H_*(X - K, A - K)$$

if K is a closed subset of the interior of A. (We can "excise" or "cut out" K.) Since each $x \in M$ has a neighborhood $U \cong \mathbb{R}^n$,

$$H_n(M, M - \{x\}) \cong H_n(U, U - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \mathbb{Z}$$

Later, I started to use the shorthand notation:

$$H_n(M|x) := H_n(M, M - x) = H_n(M, M - \{x\})$$

A generator of $H_n(M|x) \cong \mathbb{Z}$ is denoted α_x and called an *orientation* for M at x. There are two orientations at x. The other one is $-\alpha_x$.

6.2.2. *Orientation of M.* A (local) *orientation* of M is defined to be an orientation α_x at each point $x \in M$ which are "consistent" or "compatible" in the following sense. For every $x \in M$ there is a good open nbh V of x in M and an element $\alpha_V \in H_n(M|V) \cong \mathbb{Z}$ so that $\alpha_V|y = \alpha_y$ for all $y \in V$.

$$\alpha_x \in H_n(M|x) \leftarrow \exists \alpha_V \in H_n(M|V) \to \alpha_y \in H_n(M|y)$$

The orientations α_x, α_y are compatible since they are the restrictions of the same orientation α_V on V. Since these maps are isomorphisms, α_V is unique.

6.2.3. *Fundamental class.* One way to get a consistent family of point orientations α_x is if there happens to be an element α_M of $H_n(M)$ which restricts to α_x for all $x \in M$. This is a consistent family by the following diagram:

$$\alpha_M \in H_n(M) = H_n(M|M) \to \alpha_V \in H_n(M|V) \to \alpha_x \in H_n(M|x)$$

The main theorem about orientations is that, if M is compact and oriented, this class exists and is unique. It is denoted $[M]$ and called the *fundamental class* of a compact oriented manifold M.

Here is the way we started to do this in class.

Theorem 6.8. *Suppose that M is a manifold with local orientation $\{\alpha_x \,|\, x \in U\}$ where U is an open neighborhood of a compact set A. Then*

(1) *There is a unique $\alpha_A \in H_n(M|A)$ so that $\alpha_A|x = \alpha_x$ for all $x \in A$ and*
(2) *$H_m(M|A) = 0$ for all $m > n$.*

If $A = U = M$ then $H_*(M|M) = H_*(M, M - M) = H_*(M)$. So,

Corollary 6.9. *Suppose that M is a compact manifold with local orientation $\{\alpha_x \,|\, x \in M\}$. Then*

(1) *There is a unique $\alpha_M \in H_n(M)$ so that $\alpha_M|x = \alpha_x$ for all $x \in M$ and*
(2) *$H_m(M) = 0$ for all $m > n$.*

Lemma 6.10. *Suppose that $A \subseteq M$ is any subset and $\alpha_x, \beta_x, x \in A$ are compatible families of orientations along A. Let*

$$B = \{x \in A \,|\, \alpha_x = \beta_x\}, \quad C = \{x \in A \,|\, \alpha_x \neq \beta_x\}.$$

Then $A = B \coprod C$ and B, C are both relatively open subsets of A.

B relatively open means $B = U \cap A$ where U is an open set.

Proof. This follows from the definition of compatibility. □

Lemma 6.11. *The theorem is true if U is an open ball and A is a good closed disk.*

Lemma 6.12. *We may assume that A is a finite union of good closed disks. In other words, if the theorem above holds in the case when A is a finite union of good closed disks then it is true for all compact A.*

During the proof we also saw the proof of the following statement.

Lemma 6.13. *If $\gamma \in H_*(M|A)$ with A compact then there is an open nbd U of A in M and $\widetilde{\gamma} \in H_*(M|U)$ so that $\widetilde{\gamma}|A = \gamma$.*

Proof of Lemma 6.12. Suppose that the theorem holds when $A = D = \cup D_i$. Then I showed that the theorem holds for all compact A.

(1) The first step was to show that $H_m(M|A) = 0$ for $m > n$. The proof was by contradiction. Suppose that $H_m(M|A) \neq 0$. Then it has a nonzero element $\gamma \neq 0$. This is represented by a relative cocycle: $\gamma = [c]$ where $c = \sum n_i \sigma_i$ where $\sigma_i : \Delta^m \to M$. By definition of relative homology, we have $\partial c \in C_{m-1}(M - A)$. This implies that the support of ∂c is disjoint from A. If $\partial c = \sum a_j \tau_j, a_j \in \mathbb{Z}, \tau_j : \Delta^{m-1} \to M$ then the *support* of ∂c is defined to be the union of the images of these mappings:

$$K = \operatorname{supp}(\partial c) := \cup \tau_j(\Delta^{m-1}).$$

This is disjoint from A. So, if we take $V = U - K$, the same chain c is a relative cocycle in $C_m(M, M - V)$. So $\widetilde{\gamma} = [c] \in H_m(M|V)$ is an element which restricts to γ. But, we can cover A with a finite number of closed disks D_i contained in V and we get the following diagram.

$$\begin{array}{ccc} \widetilde{\gamma} \in H_m(M|V) & \longrightarrow & \overline{\gamma} \in H_m(M| \cup D_i) = 0 \\ \downarrow & \swarrow & \\ \gamma \in H_m(M|A) & & \end{array}$$

He have by assumption that $H_m(M| \cup D_i) = 0$. So, $\gamma = 0$.

I pointed out that this argument proves Lemma 6.13.

(2) The existence of α_A is clear. We just choose a covering of A with good closed disks $D_i \subseteq U$. Then there is a unique $\alpha_D \in H_n(M| \cup D_i)$ which restricts to α_x for all $x \in D = \cup D_i$. Take $\alpha_A = \alpha_D | A$.

The uniqueness of α_A follows from the uniqueness of α_D. Suppose that β_A is another element with the property that

$$\alpha_A | x = \beta_A | x = \alpha_x \quad \forall x \in A$$

Then $\gamma = \alpha_A - \beta_A$ has the property that $\gamma_x = 0$ for all $x \in A$. As before, γ lifts to $\widetilde{\gamma} \in H_n(M|V)$ and we can find disks D_i so that $A \subseteq D = \cup D_i \subseteq V$ and $\overline{\gamma} = \widetilde{\gamma}|D \in H_n(M|D)$ so that $\overline{\gamma}|A = \gamma$. However, the assumed uniqueness of α_D implies that $\overline{\gamma} = 0$. (If $\overline{\gamma} \neq 0$ then $\beta_D = \alpha_D + \overline{\gamma}$ would be another choice of α_D.)

I ran out of time so I skipped the use of Lemma 6.10. What we actually have is that $\gamma | x = 0$ for all $x \in A$ and we need $\gamma | x$ for all $x \in D$ in order for the last step to work. However, Lemma 6.10 tells us that the set B of all points $y \in D$ for which $\alpha_D | y = \beta_D | y$ (or equivalently, $\overline{\gamma} | y = 0$) is both open and closed. So, any disk D_i which meets B is contained in B. So, B is a union of disks D_i which contains A. So we may take $B = D$ and we are done. □

Lemma 6.14 (Mayer-Vietoris argument)**.** *If Theorem 6.8 hold for* $A_1 \subset U_1$*,* $A_2 \subseteq U_2$ *and* $A_1 \cap A_2 \subseteq U_1 \cap U_2$ *then it holds for* $A_1 \cup A_2 \subseteq U_1 \cup U_2$*.*

Proof. (2): We have a long exact sequence:

$$\to \underbrace{H_{m+1}(M|A_1 \cap A_2)}_{0} \to H_m(M|A_1 \cup A_2) \to \underbrace{H_m(M|A_1)}_{0} \oplus \underbrace{H_m(M|A_2)}_{0} \to H_m(M|A_1 \cap A_2) \to \cdots$$

This shows that $H_m(M|A_1 \cup A_2) = 0$ for $m > n$.

For $m = n$ we get:

$$0 \to H_n(M|A_1 \cup A_2) \to \underbrace{H_n(M|A_1)}_{\alpha_1} \oplus \underbrace{H_n(M|A_2)}_{\alpha_2} \to H_n(M|A_1 \cap A_2) \to \cdots$$

The uniqueness statement implies that $\alpha_1|A_1 \cap A_2 = \alpha_2|A_1 \cap A_2$. The exactness of the sequence implies that there is a unique element $\alpha \in H_n(M|A_1 \cup A_2)$ which restricts to α_1, α_2. But this property is equivalent to the property that $\alpha|x = \alpha_x$ for all $x \in A_1 \cup A_2$ by assumption. □

Lemma 6.15. *If* $A = \cup D_i$ *and the disks* D_i *form a good covering (i.e., any finite intersection is also ambiantly homeomorphic to a disk), e.g., if these are round disks in* $\mathbb{R}^n$*, the theorem holds.*

Ambiantly homeomorphic to a disk means there is a homeomorphism of a neighborhood of the set onto an open ball which maps the set onto a closed disk.

Proof. This follows by induction on the number of disks using the Mayer-Vietoris argument (previous lemma). □

Lemma 6.16. *The theorem holds for* $M = \mathbb{R}^n$ *for any compact* A *in any open set* U*.*

Proof. Cover A with round disks D_i each in a round ball $B_i \subseteq U$. Then

$$A \subseteq \cup D_i \subseteq \cup B_i \subseteq U$$

and we can apply the previous lemma to $\cup D_i$ and use the "enough to show for union of disks" argument to go from $\cup D_i$ to arbitrary compact $A \subseteq \mathbb{R}^n$. □

Proof of Theorem 6.8. We assume that $A = \cup D_i = D_1 \cup \cdots \cup D_k$ and $U = \cup V_i$. Suppose the theorem holds for $k-1$. Then we know it is true for $A_1 = D_1 \cup \cdots \cup D_{k-1} \subseteq U$ and $A_2 = D_k \subseteq B_k$.

By the previous lemma, we know that the theorem holds for $A_1 \cap A_2 \subseteq U \cap B_k = B_k$ since this is some compact set in an open ball. Therefore, the MV argument shows that the theorem holds for $A_1 \cup A_2 = A$ in $U \cup B_k = U$. □

6.2.4. *Triangulated manifolds.* We just showed that every compact oriented n-manifold M has an orientation class $[M] = \alpha_M \in H_n(M)$. This is the unique homology class which restricts to α_x at every $x \in M$. If M is triangulated then I showed that $[M]$ is represented by a simplician n-cycle given by the sum of the n-simplices with some sign on each n-simplex. As an example I gave the octahedral triangulation of S^2 into 8 triangles.

Definition 6.17. A *triangulation* of any topological space X is a simplicial complex K and a homeomorphism $h : |K| \to X$ where $|K|$ is the geometric realization of K.

First, I explained the difference between K and $|K|$. I pointed out that K is a set which is finite if and only if $|K|$ is compact. The simplicial complex K is the set whose elements are the simplices of K. This set is partially ordered by inclusion.

Recall that the *geometric realization* of K is defined to be the union of the affine simplices corresponding to the elements of K:

$$|K| = \bigcup_{\sigma \in K_n} \Delta^n(\sigma) \subseteq \mathbb{R}^V$$

Here $\Delta^n(\sigma)$ is the set of all convex linear combinations of the vertices of σ. The vertices of K should be placed in "general position" in some Euclidean space and the standard way to do that is to make the vertices the unit coordinate vectors in $\mathbb{R}^V$ where $V = K_0$ is the set of vertices of K. So,

$$\Delta^n = \{(t_0, t_1, \cdots, t_n) \in \mathbb{R}^{n+1} \,|\, t_i \geq 0 \text{ and } \sum t_i = 1\}$$

Theorem 6.18. *Suppose that M is a triangulated n-manifold, i.e., we have a homeomorphism $h : M \cong |K|$ where K is an n-dimensional simplicial complex. Suppose that M is compact and oriented and that $[M] = \alpha_M \in H_n(M)$ is the fundamental class. Then, in the simplicial chain complex,*

$$[M] = \left[\sum \pm\sigma_i\right]$$

In other words, the (unique) n-chain representing the homology class $[M]$ is a linear combination of the n-simplices where the coefficient of each n-simplex is ± 1.

This follows from the following lemma. Let M be any n-manifold and let $x \in M$. Choose any disk neighborhood D of x in M. Thus, we have a homeomorphism $\phi : D \to D^n$, the unit disk in $\mathbb{R}^n$, so that $\phi(x) = 0$. This gives a mapping $\overline{\phi} : D \to S^n = D^n/\partial D^n$ given by pinching the boundary ∂D^n to a point which we call the South pole

(sp). We call $\overline{\phi}(x) = 0 \in D^n/\partial = S^n$ the North pole (np). Consider the continuous mapping $f : M \to S^n = D^n/\partial D^n$ given as follows.

$$f(y) = \begin{cases} \overline{\phi}(y) & \text{if } y \in D \\ sp & \text{otherwise} \end{cases}$$

Lemma 6.19. *If M is compact and oriented then the mapping $f_* : H_n(M) \to H_n(S^n) \cong \mathbb{Z}$ sends $[M]$ to a fundamental class $[S^n]$ which is also a generator of $H_n(S^n)$.*

Proof. The mapping $f : M \to S^n$ sends x to the North pole and the complement of x to the complement of np. So we get the following commuting diagram.

$$\begin{array}{ccc} H_n(M) & \xrightarrow{f_*} & H_n(S^n) \\ \downarrow & & \downarrow \cong \\ H_n(M|x) & \xrightarrow{\cong} & H_n(S^n|np) \end{array}$$

Since the fundamental class $[M]$ maps to the generator $\alpha_x \in H_n(M|x)$ by definition, $f_*[M]$ must be a generator of S^n. This generator is a fundamental class for S^n since it maps to a generator of $H_n(S^n|z)$ for any $z \in S^n$. □

Proof of Theorem 6.18. Suppose that $[M] = [\sum n_i\sigma_i]$. Then each σ_i is an n-simplex and therefore homoemorphic to a disk $\phi : \sigma_i \cong D^n$. The mapping $f : M \to S^n$ in the lemma sends every other simplex to the South pole. So,

$$f_*[M] = [S^n] = f_*(\sum n_i\sigma_i) = f_*(n_i\sigma_i) = n_i f_*(\sigma_i).$$

Since this is a generator of $H_n(S^n)$ we must have $n_i = \pm 1$ for every i. □

6.3. **Orientation sheaf.** The next thing we did was to construct the orientation sheaf $\widetilde{M}$ for any n-manifold M:

(1) As a set, $\widetilde{M}$ is the set of all pairs (x, α_x) where $x \in M$ and $\alpha_x \in H_n(M|x)$ is an orientation of M at x.
(2) The topology on $\widetilde{M}$ is given by the basic open sets $\langle U, \alpha_U \rangle$ which is given by letting U be an open subset of M and $\alpha_U \in H_n(M|U)$ be an orientation along U, i.e., so that $\alpha_U|x \in H_n(M|x)$ is a generator for every $x \in U$.
(3) The mapping $p : \widetilde{M} \to M$ is given by $p(x, \alpha_x) = x$.

One obvious thing that you can say is that $p : \widetilde{M} \to M$ is a 2-1 mapping since every point x has exactly two possible orientations α_x. Another easy point is that, if M is oriented then $\langle M, [M] \rangle$ is a basic open set.

Lemma 6.20. *The subsets $\langle U, \alpha_U \rangle \subseteq \widetilde{M}$ form a basis for a topology on $\widetilde{M}$.*

First I recalled the definition. A collection of "basic open sets" forms a *basis* for a topology if for any two basic open sets U, V and any $x \in U \cap V$ there exists a third basic open set W so that $x \in W \subseteq U \cap V$. This condition implies that the intersection of any two basic open sets is a union of basic open sets. Thus we can define a set to be *open* if it is the union of basic open sets and the axioms for open sets will be satisfied:

(1) Any union of open sets is open.
(2) Any finite intersection of open sets is open.
(3) The empty set and the whole space are open.

Proof. Suppose that $(x, \alpha_x) \in \langle U, \alpha_U \rangle \cap \langle V, \beta_V \rangle$. This means that $x \in U \cap V$ and that $\alpha_U|x = \alpha_x = \beta_V|x$. We need to find a basic open nbh $\langle W, \alpha_W \rangle$ of (x, α_x) so that $\langle W, \alpha_W \rangle \subseteq \langle U, \alpha_U \rangle \cap \langle V, \beta_V \rangle$. This is easy: Just take W to be any good open ball neighborhood of x in $U \cap V$. Then $H_n(M|W) \cong H_n(M|x)$. So, there is a unique element $\alpha_W \in H_n(M|W)$ which restricts to α_x. We looked at the following diagram which we had before:

$$\begin{array}{ccccc} H_n(M|U) & \longrightarrow & H_n(M|W) & \longleftarrow & H_n(M|V) \\ & \searrow & \downarrow \cong & \swarrow & \\ & & H_n(M|x) & & \end{array}$$

Since $\alpha_U|x = \beta_V|x$, we must have $\alpha_U|W = \beta_V|W$. Therefore, $(x, \alpha_x) \in \langle W, \alpha_W \rangle \subseteq \langle U, \alpha_U \rangle \cap \langle V, \beta_V \rangle$ as required. $\square$

Lemma 6.21. *$p : \widetilde{M} \to M$ is an open mapping (i.e. it sends open sets to open sets).*

Proof. This is obvious since $p\langle U, \alpha_U\rangle = U$ is open. $\square$

Lemma 6.22. *$p : \widetilde{M} \to M$ is continuous.*

Proof. Take any open set U in M and let (x, α_x) be a point in the inverse image (i.e., $x \in U$). Then we need to find a basic open nbh $\langle V, \alpha_V\rangle$ of (x, α_x) so that $V \subseteq U$. But this is easy. Just take V to be any good open nbh of x in U. Then α_x extends uniquely to an orientation α_V for V. $\square$

Theorem 6.23. *$p : \widetilde{M} \to M$ is a two fold covering map.*

Proof. We needed to show that M is covered by open sets U so that $p^{-1}U = V_1 \coprod V_2$ where each V_i maps homeomorphically onto U.

Finding U, V_1, V_2 was easy: We took the covering of M by good open balls U. Each open ball was constructed with a center point x and $H_n(M|U) \cong H_n(M|x) \cong \mathbb{Z}$. Take the two generators $\alpha_U, \beta_U \in H_n(M|U)$. Then we let $V_1 = \langle U, \alpha_U\rangle, V_2 = \langle U, \beta_U\rangle$. Since p is open and continuous, p maps V_1 and V_2 homeomorphically onto U. The only thing we needed to prove was that V_1 and V_2 are disjoint.

This was again a repetition of a previous argument. (If this proof was organized better, we could probably avoid all this repetition, but I hope this made it easier to understand.) We let $B = \{y \in U \mid \alpha_y = \beta_y\}$ and $C = \{y \in U \mid \alpha_y \neq \beta_y\}$ where I used the notation $\alpha_y = \alpha_U|y$ and $\beta_y = \beta_U|y$. Then, as before, we have that B, C are both open and $U = B \coprod C$. Since U is an open ball it is connected. So, it cannot be a union of two disjoint open sets. So, either B is empty or C is empty. Since $x \in C$, we have that B is empty. But this implies that V_1, V_2 are disjoint since $B = p(V_1 \cap V_2)$. $\square$

Corollary 6.24. (1) *Any simply connected manifold is orientable.*
(2) *If M is a connected manifold and $\pi_1 M$ has no subgroups of index 2 then M is orientable.*

Proof. I pointed out that (1) is a special case of (2) and (2) follows from covering space theory. The nontrivial 2-fold covering spaces of M are in 1-1 correspondence with index 2 subgroups of $\pi_1 M$. So, if $\pi_1 M$ has no index 2 subgroups, it means that $\widetilde{M}$ must be the trivial 2-fold covering, i.e., $\widetilde{M} = M \coprod M$, the disjoint union of two copies of M. Each of these gives a section of M which gives a compatible family of orientations at all points in M. $\square$

6.4. **Cap product.** The cap product will be a mapping

$$\cap : H_n(X,A) \otimes H^p(X) \to H_{n-p}(X,A)$$

which makes $H_* = \bigoplus H_n(X,A)$ into a graded module over the graded ring $H^*(X) = \bigoplus H^p(X)$.

6.4.1. *definition.*

Definition 6.25. If $\sigma : \Delta^n \to X$ is any singular simplex in X and $\varphi \in C^p(X,\mathbb{Z})$ is any integral cochain then

$$\sigma \cap \varphi = \varphi(f_p\sigma)b_q\sigma \quad \in C_q(X)$$

where $p + q = n$.

Since $\varphi(f_p\sigma)$ is a number, we have:

$$\partial(\sigma \cap \varphi) = \varphi(f_p\sigma)\partial b_q\sigma = (\varphi f_p \otimes \partial b_q)\sigma$$

I needed another formula for the cap product to get a better formula for the boundary. If $c = \sum a_i\sigma_i \in C_n(X)$ is an n-chain then:

$$c \cap \varphi = (\varphi \otimes 1_q)\Delta_{pq}c$$

Since $\Delta_{pq} = f_p \otimes b_q$, this is the same formula as the one above. The formula for the boundary is now:

$$\partial(c \cap \varphi) = (\varphi \otimes \partial)\Delta_{pq}c$$

The reason this is a better notation is that we can write down the fact that Δ is a chain map: $\Delta\partial = \partial^\otimes\Delta$ applied to c we get:

$$\Delta\partial c = \partial^\otimes\Delta c = \sum_{p+q=n}(\partial_p \otimes 1_q + (-1)^p 1_p \otimes \partial_q)\Delta c$$

So,

$$(\partial c) \cap \varphi = (\varphi \otimes 1)\Delta\partial c = (\varphi\partial \otimes 1 + (-1)^p\varphi \otimes \partial)\,\Delta c$$

$$(\partial c) \cap \varphi = c \cap \delta\varphi + (-1)^p\partial(c \cap \varphi)$$

Solving for $\partial(c \cap \varphi)$ we get:

$$\boxed{\partial(c \cap \varphi) = (-1)^p(\partial c) \cap \varphi + (-1)^{p+1}c \cap \delta\varphi}$$

Looking at this formula we see immediately that, if c is a cycle and φ is a cocycle then $c \cap \varphi$ is a cycle. So, we get an induced map

$$H_n(X) \otimes H^p(X) \xrightarrow{\cap} H_q(X)$$

Next, suppose that $c \in C_n(X,A)$ is a relative cycle and $\delta\varphi = 0$. Then $\partial c \in C_{n-1}(A)$. So,

$$\partial(c \cap \varphi) = (-1)^p(\partial c) \cap \varphi \quad \in C_q(A)$$

making $c \cap \varphi$ a relative cycle. Thus we get an induced map:

$$H_n(X, A) \otimes H^p(X) \xrightarrow{\cap} H_q(X, A)$$

Finally, and this is the most surprising one, if $c \in C_n(X, A)$ is a relative cycle (so that $\partial c \in C_{n-1}(A)$) and $\varphi \in C^p(X, A)$ is a relative cocycle (so that $\delta\varphi = 0$ and $\varphi(C_p(A)) = 0$) then

$$\partial(c \cap \varphi) = (-1)^p(\partial c) \cap \varphi = 0$$

which means we get an induced map

$$H_n(X, A) \otimes H^p(X, A) \xrightarrow{\cap} H_q(X)$$

Now change the notation: $A = X - K$. Then $H_n(X, A) = H_n(X|K)$ and $H^p(X, A) = H^p(X|K)$. So, cap product is:

$$H_n(X|K) \otimes H^p(X|K) \xrightarrow{\cap} H_q(X)$$

Special case: Suppose that $n = p$ and X is connected. Then we have:

$$H_n(X, A) \otimes H^n(X, A) \xrightarrow{\cap} H_0(X) \xrightarrow{\approx} \mathbb{Z}.$$

This mapping is the *evaluation map* since, for $c = \sum a_i\sigma_i$, we get

$$c \cap \varphi = \sum a_i\sigma_i \cap \varphi = \sum a_i\varphi(f_n\sigma_i) \otimes b_0\sigma_i \mapsto \sum a_i\phi(\sigma_i) = \phi(c)$$

Theorem 6.26. *$C_*(X, A)$ is a graded right module over the graded associative ring $C^*(X)$ and $H_*(X, A)$ is a graded right module over the graded commutative and associative ring $H^*(X)$.*

Proof. This was the following calculation. If $\sigma : \Delta^n \to X$ and φ, ψ or cochains of degree p, q resp and $n = p + q + r$ then

$$(\sigma^n \cap \varphi_p) \cap \psi_q = a\psi_q(f_q\tau)b_r\tau$$

where

$$\sigma^n \cap \varphi_p = a\tau^{q+r} = \underbrace{\varphi(f_p\sigma)}_{a}\underbrace{b_{q+r}\sigma}_{\tau}$$

But $f_q(\tau) = m_q\sigma$, the middle q-face of σ, and $b_r\tau = \beta_r\sigma$. So,

$$(\sigma^n \cap \varphi) \cap \psi = \varphi(f_p\sigma)\psi(m_q\sigma)b_r\sigma$$

$$= (\varphi \cup \psi)(f_{p+q}\sigma)b_r\sigma = \sigma \cap (\varphi \cup \psi)$$

Here the superscripts and subscripts are only to keep track of dimensions and degrees. ∎

6.4.2. *Statement of Poincaré Duality.*

Theorem 6.27. *If M is a compact oriented n-manifold then we have an isomorphism:*

$$[M]\cap : H^p(M) \xrightarrow{\approx} H_{n-p}(M)$$

Example 6.28. Let $M = \mathbb{R}P^n = S^n/\pm 1$. This is n-dimensional projective space. It is orientable if n is odd. The homology of $M = \mathbb{R}P^3$ is given by

coef :	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Q}$
$H_0(M) =$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Q}$
$H_1(M) =$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$H_2(M) =$	0	$\mathbb{Z}/2$	0
$H_3(M) =$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Q}$

Poincaré duality for $p = 0$ tells up that $H_3(M) \cong H^0(M) = \mathbb{Z}$. Next, we compared what we get from the UCT and PD:

$$H_2(M) \cong_{PD} H^1(M;\mathbb{Z}) \cong_{UCT} \mathrm{Hom}(\underbrace{H_1(M)}_{\mathbb{Z}/2},\mathbb{Z})\otimes \mathrm{Ext}(\underbrace{H_0(M)}_{\mathbb{Z}},\mathbb{Z}) = 0\oplus 0 = 0$$

$$H_1(M) \cong_{PD} H^2(M;\mathbb{Z}) \cong_{UCT} \mathrm{Hom}(\underbrace{H_2(M)}_{0},\mathbb{Z})\otimes \mathrm{Ext}(\underbrace{H_1(M)}_{\mathbb{Z}/2},\mathbb{Z}) = 0\oplus \mathbb{Z}/2 = \mathbb{Z}/2$$

$$H_0(M) \cong_{PD} H^3(M;\mathbb{Z}) \cong_{UCT} \mathrm{Hom}(\underbrace{H_3(M)}_{\mathbb{Z}},\mathbb{Z})\otimes \mathrm{Ext}(\underbrace{H_2(M)}_{0},\mathbb{Z}) = \mathbb{Z}\oplus 0 = \mathbb{Z}$$

If we take coefficients in a field then we get the following.

Corollary 6.29. *If F is a field then*

$$H_{n-p}(M;F) \cong_{PD} H^p(M;F) \cong_{UCT} H_p(M;F)$$

So, in the chart above, the last two columns are vertically symmetrical.

6.4.3. *geometric interpretation of Poincaré duality.* In response to questions I explained that Poincaré duality reflects that fact that a triangulated manifold $M \cong |K|$ has a "dual triangulation" by "dual cells" which are actually not simplices but a CW-decomposition of the manifold.

Geometrically, each p simplex σ^p has a dual $n-p$ cell $D(\sigma)^{n-p}$ which goes to the center of every n simplex containing σ. This gives a CW complex $X \cong M$ and the cellular chain complex of X is the "upside down" version of the simplicial chain complex of K. In other words, τ^{p-1} is in the boundary of σ^p if and only if the dual $n-p$ cell $D(\sigma)$ is in the boundary of the $n-p+1$ cell $D(\tau)$. Algebraically, the matrix that gives the boundary map $C_{n-p+1}(X) \to C_{n-p}(X)$ is the transpose of the boundary map $C_p(K) \to C_{p-1}(K)$. This duality between these two chain complexes is given by cap product with the fundamental class which is the sum of the n-simplices.

6.5. **Proof for good coverings.** The proof of Poincar'e duality that I presented was a very long version of what is in Hatcher's book. The basic idea is to use a Mayer-Vietoris argument. So, we need a local version of the theorem which holds for open balls and then I need to paste them together.

The first step was to look at the special case of a disk inside of an open ball. Suppose that K is a disk contained in an open ball U and the pair (U, K) is homeomorphic to the pair $(B_1(0), \overline{B}_{1/2}(0))$ where these balls are in $\mathbb{R}^n$. Call these *good disk-ball pairs* Then

$$H_n(U|K) = H_n(U, U - K) \cong Z$$

and $H_k(U|K) = 0$ for $k \neq n$. Any generator $\alpha_K \in H_n(U|K)$ is an orientation. The first case of Poincaré duality was the following.

Lemma 6.30. *Cap product with α_K gives an isomorphism*

$$\alpha_K \cap : H^p(U|K) \xrightarrow{\approx} H_{n-p}(U)$$

Proof. Since U is a ball, it is contractible and has only homology in degree 0: $H_0(U) \cong \mathbb{Z}$. But, in degree $p = n$, cap product is the evaluation map. So, cap product with α_K is evaluation at α_K:

$$\mathbb{Z} \cong H^n(U|K) \xrightarrow{eval} H_0(U) = \mathbb{Z}$$
$$\varphi \mapsto \varphi(\alpha_K).$$

This is surjective and thus an isomorphism since the identity map in $H_n(U|K) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ maps to $\pm 1 \in \mathbb{Z}$. □

Starting with this one example, we can get everything using the Mayer-Vietoris sequence if we have a good covering.

Lemma 6.31. *Suppose that K, L are compact subsets of U, V and $\alpha \in H_n(M|K \cup L)$ is an orientation along $K \cup L$. Let $\alpha_K \in H_n(M|K) = H_n(U|K), \alpha_L \in H_n(M|L) = H_n(V|L)$ be the restrictions of α to K and L. Suppose that*

$$\alpha_K \cap : H^p(M|K) \cong H_{n-p}(U)$$

and similarly for (V, L) and $(U \cap V, K \cap L)$. Then the corresponding statement holds for $(U \cup V, K \cup L)$, i.e.,

$$\alpha \cap : H^p(M|K) \cong H_{n-p}(U \cup V)$$

Proof. We get a map from one MV sequence to another:

$$\begin{array}{ccccc} H^p(M|K \cap L) & \longrightarrow & H^p(M|K) \oplus H^p(M|L) & \longrightarrow & H^p(M|K \cup L) \\ \cong \downarrow \alpha_{K\cap L}\cap & & \cong \downarrow \alpha_K \cap \oplus \alpha_L \cap & & \downarrow \alpha_{K \cup L} \cap \\ H_{n-p}(U \cap V) & \longrightarrow & H_{n-p}(U) \oplus H_{n-p}(V) & \longrightarrow & H_{n-p}(U \cup V) \end{array}$$

So, the lemma follows from the 5-lemma. There is one technical point which I glossed over in class which was the commutativity of the diagram. Not the part drawn above but the boundary map. I will explain that below (but not in class). □

Lemma 6.32. *Suppose that (V_i, D_i) is a finite collection of good disk ball pairs in M so that*

(1) *Any finite intersection of the disks $K = D_{j_1} \cap D_{j_2} \cap \cdots \cap D_{j_k}$ is a disk if it is nonempty and the intersection V of the corresponding balls V_{j_i} is a ball and the pair (V, K) is a good disk-ball pair.*
(2) *If any finite intersection of the disks $D_{j_1} \cap D_{j_2} \cap \cdots \cap D_{j_k}$ is empty then the intersection of the corresponding balls U_{j_i} is also empty.*

Suppose also that $\alpha_D \in H_n(M|D) = H_n(U|D)$ is an orientation along $D = \cup V_i$. Then, cap product with α_D gives an isomorphism

$$\alpha_D \cap : H^p(U|K) \cong H_{n-p}(U)$$

where $U = \cup V_i$.

Proof. This just follows from the previous two lemmas by induction on the number of disks. Suppose the lemma holds for $m-1$. Then it holds for $K = D_1 \cup \cdots D_{m-1}$ inside of $U = V_1 \cup \cdots \cup V_{m-1}$. Also

$$K \cap D_m = (D_1 \cap D_m) \cup (D_2 \cap D_m) \cup \cdots \cup (D_{m-1} \cap D_m)$$

is a union of $m-1$ disks which have good intersections and it lies in $U \cap V_m = \bigcup (V_i \cap V_m)$ which is a union of balls with good intersections. So, the condition holds for this pair and we can use MV to get it on the union. □

But what if we don't have a good system of disks and balls? The situation in $\mathbb{R}^n$ is as follows. Suppose that K is a compact subset of an open set U. Then, we can cover K with a finite number of small round disks D_i which are contained in open balls V_i which intersect in a good way. (The formula I gave in class was to let ϵ be the smallest distance between any finite intersection of the disks with a disk which does not meet that intersection. Then let each ball B_i be the ball of radius $\epsilon/3$ greater than the radius of the disk that it surrounds.) This implies the following.

Lemma 6.33. *Let K be any compact subset of $\mathbb{R}^n$ and let U be an open neighborhood of K. Then there exists another compact set D and* *another open set V with $K \subseteq D \subseteq V \subseteq U$ so that the theorem holds for (U, D), i.e.,*

$$\alpha_D \cap : H^p(\mathbb{R}^n | D) \xrightarrow{\approx} H_{n-p}(V)$$

where α_D is any orientation class for D.

What we now want to do is to take the direct limit of both sides.

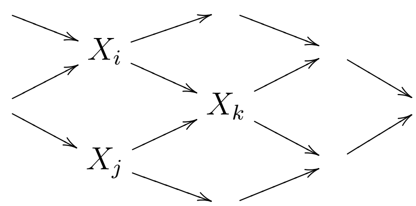
$$\{\alpha_D \cap\} : \varinjlim H^p(\mathbb{R}^n | D) \xrightarrow{\approx} \varinjlim H_{n-p}(V)$$

The result will be an isomorphism between compactly generated cohomology on the left and homology of U on the right since homology commutes with direct limit:

$$H_c^p(\mathbb{R}^n | U) \xrightarrow{\approx} H_{n-p}(U)$$

6.6. **Direct limit.** I went over the definition and basic properties of direct limits.

Definition 6.34. A *directed system* is a diagram:



consisting of

(a) X_i which are abelian groups or topological spaces of objects in some other category indexed by $i \in I$ where I is a partially ordered set (*poset*), i.e. a set with a transitive, antireflexive relation ($i < j, j < k \Rightarrow i < k$ and $i < j \Rightarrow i \neq j$).
(b) $f_{ji} : X_i \to X_j$ either a homomorphism of groups or a continuous mapping of spaces for all $i < j$.

And these should satisfy two conditions. (I decided that we can't call it a "directed system" if we don't use the usual definition.)

(1) The first condition is that, for all $i, j \in I$ there is a $k \in I$ so that $i, j < k$, as indicated in the diagram above.
(2) The second condition is that

$$f_{kj} f_{ji} = f_{ki}$$

for all $i < j < k$. I.e., the entire big diagram commutes.

6.6.1. *direct limit.* The *direct limit* of a directed system is denoted $\lim_{\rightarrow} X_i$ and can be defined in two ways: by a universal property and by a formula.

The *universal property* of the direct limit $X_\infty = \lim_{\rightarrow} X_i$ is the following.

(1) For all $i \in I$ we have a map $h_i : X_i \to X_\infty$.
(2) For all $i < j \in I$ we have a commuting diagram: $h_j \circ f_{ji} = h_i$.
(3) For any system $(Z, g_i : X_i \to Z)$ consisting of an object Z and arrows $g_i : X_i \to Z$ so that $g_j \circ f_{ji} = g_i$ for all $i < j$, there is a unique arrow $g_\infty : X_\infty \to Z$ so that $g_\infty \circ h_i = g_i$ for all i. I.e., the giant diagram including all of the above mentioned objects and arrows commutes.

It is clear that, if the direct limit exists, then it is unique. For abelian groups the existence is given by the formula:

$$X_\infty = \lim_{\to} X_i = \frac{\bigoplus X_i}{x_i \in X_i \sim f_{ji}(x_i) \in X_j}$$

Example 6.35. The first example was a really dumb one which actually need later. If the diagram consists of the same group repeated: $X_i = G$ for all i and $f_{ji} = id_G$ for all $i < j$ then $\lim_\to X_i = G$.

The second more interesting example was the limit of the diagram where $I = \{1, 2, 3, \cdots\}$, $X_i = \mathbb{Z}$ for all i but $X_i = \mathbb{Z} \to X_{i+1} = \mathbb{Z}$ was multiplication by $i + 1$:

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{3} \mathbb{Z} \xrightarrow{4} \cdots$$

In this case, $X_\infty = \mathbb{Q}$ with $h_i = \frac{1}{i!} : X_i = \mathbb{Z} \to \mathbb{Q}$ given by dividing by $i!$

As a third more general example suppose that $I = \{1, 2, 3, \cdots\}$, the groups X_i are all subgroups of some group $X = \cup X_i$ and suppose that $f_{ji} : X_j \to X_i$ is the inclusion map for all $i < j$:

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow X_4 \hookrightarrow \cdots$$

Then $X_\infty = \lim_\to X_i = X$.

For topological spaces the formula for the direct limit is:

$$X_\infty = \lim_{\to} X_i = \frac{\coprod X_i}{x_i \in X_i \sim f_{ji}(x_i) \in X_j}$$

with the quotient topology, i.e., a subset of X_∞ is open if and only if its inverse image in each X_i is open. The following is pretty clear.

Lemma 6.36. *Suppose that*

$$U_1 \subseteq U_2 \subseteq U_3 \cdots$$

is an increasing sequence of open subsets of some space X and $X = \cup U_i$. Then $X = \lim_\to U_i$.

6.6.2. *cofinal subsystems.*

Definition 6.37. A *cofinal* subsequence or subsystem of a directed system consists of the objects $\{X_n \,|\, n \in J\}$ where J is a subset of I so that for all $i \in I$ there is an $n \in J$ so that $i \le n$.

Lemma 6.38. *A cofinal subsystem of a directed system is directed.*

Proof. If $i, j \in J$ then there exists some $k \in I$ so that $i, j < k$. But then there is an $n \ge k$ in J so $i, j < k \le n \in J$. □

Theorem 6.39.

$$\varinjlim_{n\in J} X_n = \varinjlim_{i\in I} X_i$$

We also needed one other property of direct limits of groups:

Lemma 6.40. $\lim_\rightarrow$ *is an exact functor. In other words, any directed system of short exact sequences*

$$0 \to A_i \to B_i \to C_i \to 0$$

gives a short exact sequence of direct limits:

$$0 \to \lim_{\rightarrow} A_i \to \lim_{\rightarrow} B_i \to \lim_{\rightarrow} C_i \to 0$$

Since $C_i = B_i/A_i$, this implies that

$$\lim_{\rightarrow} \frac{B_i}{A_i} = \frac{\lim_\rightarrow B_i}{\lim_\rightarrow A_i}$$

Proof. Right exactness follows from the fact that $\lim_\rightarrow$ is a left adjoint functor by the universal property:

$$\operatorname{Hom}(\lim_{\rightarrow} X_i, Z) \cong \operatorname{Hom}(\{X_i\}, \{Z\})$$

The right adjoint is $C(Z) =$ the constant system $Z_i = Z$.

Left exactness (the fact that $\lim_\rightarrow A_i \subset \lim_\rightarrow B_i$) follows from the fact that any element of the limit comes from some $a \in A_i$ and if it goes to zero in $\lim_\rightarrow B_i$ then it goes to zero in some B_j and therefore in A_j. □

6.6.3. *compactly generated cohomology.* I defined this and used two different notations for the same thing.

Definition 6.41. If X is any topological space then the *compactly generated cohomology* groups of X are defined to be the direct limit:

$$H^p_c(X) := \lim_{\rightarrow} H^p(X|K)$$

where the limit is taken over all compact subsets $K \subseteq X$.

I also used the notation:

$$H^p_c(M|U) = \lim_{\rightarrow} H^p(M|K)$$

where the direct limit is over all compact K in U. But when U is open this is the same thing by excision: $H^p_c(M|U) = H^p_c(U)$.

Lemma 6.42. *If U is an open subset of $\mathbb{R}^n$ then $H^p_c(U)$ is the direct limit of the groups $H^p(\mathbb{R}^n|D)$ where D runs over all subsets of U which are finite unions of good disks.*

Proof. This follows from the fact that these unions of good disk D form a cofinal subsystem of the directed system of all compact subsets of U. That is the gist of the statement that for any compact $K \subseteq U$ we have a good pair (V, D) so that $K \subseteq D \subseteq V \subseteq U$. □

Theorem 6.43. *If U is any open subset of $\mathbb{R}^n$ and α_U is any orientation of U then*

$$\alpha_U \cap : H^p(U) \xrightarrow{\approx} H_{n-p}(U)$$

Proof. We take the cofinal system of good disk-ball unions:

$$H^p(M|D) \cong H_{n-p}(V)$$

We just need to know that $\lim_{\rightarrow} H_*(V) = H_*(U)$. □

6.6.4. *homology commutes with direct limit.* The general theorem is that for any directed system of topological space,

$$H_k\left(\lim_{\rightarrow} X_i\right) \cong \lim_{\rightarrow} H_k(X_i).$$

However, I only proved it in the case that we need when the directed system is a collection of open subsets U_i of $X = \cup V_i$.

Lemma 6.44. *Suppose that $\{U_i\}$ is a collection of open subsets of X so that*

1. *The system is directed, i.e., for all i, j there is a k so that $U_i \cup U_j \subseteq U_k$.*
2. *$\cup U_i = X$.*

Then $X = \lim_{\rightarrow} U_i$.

Proof. I will show that X satisfies the universal property of direct limit.

Suppose that Z is any topological space and $g_i : U_i \to Z$ are continuous mappings so that $g_j \circ f_{ji} = g_i$, i.e., $g_j|U_i = g_i$ for all $U_i \subseteq U_j$. Then, we need to show that, for all i,j, we have

$$g_i|U_i \cap U_j = g_j|U_i \cap U_j$$

The proof of this is simple: there is a k so that $U_i \cup U_j \subseteq U_k$. And we are assuming that $g_i = g_k|U_i$ and $g_j = g_k|U_j$. It follows that $g_i = g_j$ on $U_i \cap U_j$. Since the family of mappings $g_i : U_i \to Z$ agree on all overlaps, they define a mapping $g : X = \cup U_i \to Z$. And g is continuous since, for any open set $W \subseteq Z$, $g^{-1}(W) = \cup g_i^{-1}(W)$ is a union of open set and thus open. □

Lemma 6.45. *For any compact $K \subseteq U$, there is a U_i is the directed system of open subsets of X which contains K.*

Proof. This is easy. Each point in K is contained in some U_i. Take a finite subcovering. Then, since the system is directed, there exists some U_k which contains all of the U_i in this finite covering. Then $K \subseteq U_k$. □

Theorem 6.46. *Suppose that $\{U_i\}$ is a directed system of open subsets of a space X and $X = \cup U_i$. Then $H_k(\lim_{\rightarrow} X_i) \cong \lim_{\rightarrow} H_k(X_i)$.*

Proof. Since $\lim_{\rightarrow}$ is exact, we have:

$$\lim_{\rightarrow} H_k(U_i) = \lim_{\rightarrow} \frac{Z_k(U_i)}{B_k(U_i)} = \frac{\lim_{\rightarrow} Z_k(U_i)}{\lim_{\rightarrow} B_k(U_i)} =_? \frac{Z_k(X)}{B_k(X)} = H_k(X)$$

All we need is to show that $\lim_{\rightarrow} Z_k(U_i) = Z_k(X)$ and similarly for $B_k(X)$. Since $Z_k(U_i)$ is a subgroup of $Z_k(X)$ we just need to show that every element of $Z_k(X)$ lies in some $Z_k(U_i)$. Let $c = \sum a_i \sigma_i \in Z_k(X)$. Then, recall that the support of c is the union of the images of the mappings $\sigma_i : \Delta^k \to X$. This is compact. By the lemma above, there is some U_i which contains the support of c. But then $c \in Z_k(U_i)$ and we are done. □

This finally completes the proof of Theorem 6.43 which we need to finish this really long proof of Poincaré duality.

6.7. **Proof of Poincaré duality.** The proof is as follows. Choose a finite covering of M by open balls V_i. Let $U_m = V_1 \cup V_2 \cup \cdots \cup V_{m-1}$. We want to show by induction on m that cap product with $[M]$ induces an isomorphism

$$H^p_c(M|U_m) \cong H_{n-p}(U_m)$$

For this we use the map of Mayer-Vietoris sequences

$$\begin{array}{ccccc} H^p_c(M|U_m \cap V_m) & \longrightarrow & H^p_c(M|U_m) \oplus H^p_c(M|V_m) & \longrightarrow & H^p_c(M|U_m \cup V_m) \\ \cong \downarrow & & \cong \downarrow & & \downarrow \\ H_{n-p}(U_m \cap V_m) & \longrightarrow & H_{n-p}(U_m) \oplus H_{n-p}(V_m) & \longrightarrow & H_{n-p}(U_m \cup V_m) \end{array}$$

The fact that the top row is exact comes from the fact that it is the direct limit of the exact sequences

$$\to H^p(M|K \cap L) \to H^p(M|K) \oplus H^p(M|L) \to H^p(M|K \cup L) \to H^{p+1}(M|K \cap L) \to$$

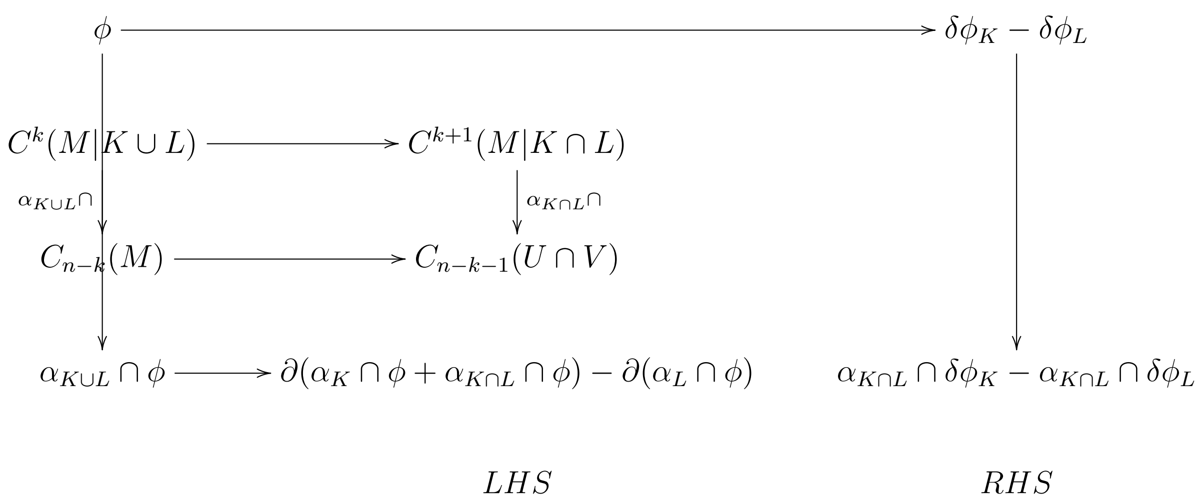
and direct limit takes exact sequences to exact sequences. The theorem vertical map on the left is an isomorphism by Theorem 6.43 since $U_m \cap V_m$ is contained in the open ball V_m. The middle vertical map is an isomorphism by induction for U_m and by Theorem 6.43 for V_m. So, we get the result we want by the 5-lemma. There is only the technical point about the commutativity of the diagram.

I also gave a very short geometric proof of Poincaré duality for triangulated manifolds. I will write both of these up later.

6.7.1. *Commutativity of the diagram.* The only thing we need to do is to show that the diagram commutes. This is very tricky.

- $M = U \cup V$
- $K \subset U, L \subset V$ are compact subsets.
- $\phi \in C^k(M|K \cup L)$ means $\phi : C_k(M) \to R$ and $\phi = 0$ on $C_k(M - (K \cup L))$.
- $\phi = \phi_K + \phi_L$ where
- $\phi_K = 0$ on $M - K$
- $\phi_L = 0$ on $M - L$
- $\alpha_{K\cup L} \in C_n(M|K \cup L)$ is the orientation class.
- $\alpha_{K\cup L} = \alpha_K + \alpha_{K\cap L} + \alpha_L$ where
- $\alpha_K \in C_n(M - L)$. So, $\phi_L(\alpha_K) = 0$.
- $\alpha_L \in C_n(M - K)$. So, $\phi_K(\alpha_L) = 0$.
- $\alpha_{K\cap L} \in C_n(U \cap V)$ is the orientation class for $K \cap L$. So, $\partial\alpha_{K\cap L}$ is disjoint from $K \cap L$.
- $\alpha_K + \alpha_{K\cap L} = \alpha_K$
- $\alpha_L + \alpha_{K\cap L} = \alpha_L$

We need to check that the following diagram commutes.



The first thing to notice is that $\partial(\alpha_{K\cap L} \cap \phi) = 0$ in homology since $\alpha_{K\cap L}$ lies in $U \cap V$. The other two terms $\partial(\alpha_K \cap \phi)$ and $-\partial(\alpha_L \cap \phi)$ are not zero since α_K, α_L do not lie inside $U \cap V$. However, $\partial\phi = 0$. So, the formula:

$$\partial(\mu \cap \phi) = (-1)^k \partial\mu \cap \phi - (-1)^k \mu \cap \delta\phi$$

implies that $\partial(\alpha_K \cap \phi) = (-1)^k \partial\alpha_K \cap \phi$ and $\partial(\alpha_L \cap \phi) = (-1)^k \partial\alpha_L \cap \phi$. But $\phi = \phi_K + \phi_L$ and α_K lies in the place where $\phi_L = 0$ and similarly for α_L. So,

$$\partial(\alpha_K \cap \phi) = (-1)^k \partial\alpha_K \cap \phi_K$$
$$\partial(\alpha_L \cap \phi) = (-1)^k \partial\alpha_L \cap \phi_L$$

So,

$$LHS \sim (-1)^k \partial\alpha_K \cap \phi_K - (-1)^k \partial\alpha_L \cap \phi_L$$

The right hand side is $\alpha_{K\cap L} \cap \delta\phi_K - \alpha_{K\cap L} \cap \delta\phi_L$ which differs by a boundary from

$$\partial\alpha_{K\cap L} \cap \phi_K - \partial\alpha_{K\cap L} \cap \phi_L$$

Now we use the fact that $\alpha_K + \alpha_{K\cap L} = \alpha_K$ with boundary $\partial\alpha_K$ disjoint from K. So, $\partial\alpha_K \cap \phi_K = 0$. Similarly for ϕ_L. So,

$$\partial\alpha_{K\cap L} \cap \phi_K = -\alpha_K \cap \phi_K$$
$$-\partial\alpha_{K\cap L} \cap \phi_L = \alpha_L \cap \phi_L$$

So,

$$RHS \sim -\alpha_K \cap \phi_K + \alpha_L \cap \phi_L \sim (-1)^{k+1} LHS$$