Determination of Weights of Interacting Criteria from a Reference Set

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Abstract

In this paper, we present a model allowing to determine the weights related to interacting criteria. This is done on the basis of the knowledge of a partial ranking over a reference set of alternatives (prototypes), a partial ranking over the set of criteria, and a partial ranking over the set of interactions between pairs of criteria.

Keywords: multicriteria decision making, interacting criteria, Choquet integral.

1 Introduction

Let us consider a set of *alternatives* $A = \{a, b, c, \ldots\}$ and a set of *criteria* $N = \{1, \ldots, n\}$ in a multicriteria decision making problem. Each alternative $a \in A$ is associated with a *profile* $x^a = (x^a_1, \ldots, x^a_n) \in \mathbb{R}^n$ where x^a_i represents the utility of a related to the criterion i, with $x^a_i \in X_i \subseteq \mathbb{R}$, $i = 1, \ldots, n$. We assume that all the utilities x^a_i are defined according to the same interval scale ($X_i = X\ \forall i$).

Suppose that the preferences over A of the decision maker are known and expressed by a binary relation $\succeq_A$. In the classical multiattribute utility (MAUT) model [12], the problem consists in constructing a *utility function* $U : X^n \to \mathbb{R}$ representing the preference of the decision maker, that is such that

$$a \succ_A b \quad \Leftrightarrow \quad U(x^a) > U(x^b), \quad \forall a, b \in A.$$

The subset $S \subseteq N$ of criteria is said to be *preferentially independent* [23] of $N \setminus S$ if, for all $x_S, y_S \in \times_{i\in S} X_i$ and all $x_{N\setminus S}, z_{N\setminus S} \in \times_{i\in N\setminus S} X_i$, we have

$$(x_S, x_{N\setminus S}) \succeq (y_S, x_{N\setminus S}) \quad \Leftrightarrow \quad (x_S, z_{N\setminus S}) \succeq (y_S, z_{N\setminus S}).$$

The whole set of criteria N is said to be *mutually preferentially independent* if S is preferentially independent of $N \setminus S$ for every $S \subseteq N$.

Roughly speaking, the preference of $(x_S, x_{N\setminus S})$ over $(y_S, x_{N\setminus S})$ is not influenced by the values $x_{N\setminus S}$. For some problems this principle might be violated as it can be seen in the following example.

Example 1.1 Let us consider a decision problem involving 4 cars, evaluated on 3 criteria: price, consumption and comfort.

	price	consumption	comfort
car 1	10.000 Euro	10 $\ell/100$ km	very good
car 2	10.000 Euro	9 $\ell/100$ km	good
car 3	30.000 Euro	10 $\ell/100$ km	very good
car 4	30.000 Euro	9 $\ell/100$ km	good

Suppose the consumer (decision maker) has the following preferences:

$$\text{car 2} \succeq \text{car 1} \qquad \text{and} \qquad \text{car 3} \succeq \text{car 4}.$$

The reason may be that, as price increases, so does the importance of comfort. In this case, consumption and comfort are not preferentially independent of price.

It is known [5, 19] that the mutual preferential independence among the criteria is a necessary condition (but not sufficient) for a utility function to be additive, i.e., it can be assumed that there exists a weight vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ fulfilling $\sum_i \omega_i = 1$ such that

$$U(x^a) = \sum_{i=1}^{n} \omega_i\, x_i^a, \qquad a \in A. \tag{1}$$

In case of dependent criteria, the weighted arithmetic mean (1) can be extended to a Choquet integral, which was introduced in capacity theory:

$$U(x^a) = \sum_{i=1}^{n} x_{(i)}^a\, [\mu(A_{(i)}) - \mu(A_{(i+1)})], \tag{2}$$

where $(\cdot)$ indicates a permutation such that $x_{(1)}^a \leq \ldots \leq x_{(n)}^a$. Also $A_{(i)} = \{(i), \ldots, (n)\}$, and $A_{(n+1)} = \emptyset$. We thus observe that the weights ω_i related to the criteria, which were supposed independent, have been substituted by the weights $\mu(i_1, \ldots, i_k)$ related to all the combinations of criteria, thus making possible to express dependence between some criteria.

Note however that more than one form of dependence exist. Indeed, the criteria can be mutually preferentially independent while some of them are correlated or interchangeable. Contrary to the weighted arithmetic mean, the Choquet integral (2) is able to represent dependence between criteria in many situations, whatever the form of the dependence. For this reason it has been thoroughly studied in the context of multicriteria decision problems (see e.g. [6, 7, 14, 15, 18] and the references there).

In this paper, we propose a model allowing to identify the weights of interacting criteria on the basis of learning data consisting of a partial preorder over a reference set of alternatives (prototypes) whose profiles are known, and also from semantical considerations about criteria such as a partial preorder over the set of weights related to each criterion, a partial preorder over interactions between pairs of criteria, and the knowledge of the sign of interactions between some pairs of criteria.

The main advantage of the model we propose is that a set of weights can be obtained simply by solving a linear program. Of course, these weights, once obtained, may not be used in any aggregation procedure. Their nature depends on the aggregation function used to construct them. Thus, following our model, they will be always used in a Choquet integral.

The outline of this paper is as follows. In Section 2, we recall the definition of the Choquet integral as an aggregation tool. In Section 3, we present the notion of interaction between criteria. In Section 4, we propose to restrict the model to the 2nd order, thus assuming that interaction between more than two criteria does not exist. Finally, in Section 5, we present the model, and two small examples as illustration.

Cardinality of subsets $S, T, \ldots$ will often be denoted by corresponding lower cases $s, t, \ldots$. Furthermore, we will whenever possible omit braces for singletons, e.g. writing $\mu(i)$, $S \cup i$ instead of $\mu(\{i\})$, $S \cup \{i\}$. Also, for pairs, triples, we will write ij, ijk instead of $\{i, j\}$, $\{i, j, k\}$, as for example $S \cup ijk$.

2 The Choquet integral as an aggregation operator

Until recently, the most common aggregation tool which is used in multicriteria decision making is the weighted arithmetic mean (1), with all its well-known drawbacks. Indeed, since this aggregation function is not appropriate when interacting criteria are considered, people usually tend to construct independent criteria, or criteria that are supposed to be so, causing some bias effect in evaluation.

However, this problem has been partially solved since the introduction of fuzzy integrals, a concept proposed by Sugeno [21, 22]. Those fuzzy integrals, such as the Choquet integral, are defined from the concept of fuzzy measure (also called Choquet capacity).

Definition 2.1 A (discrete) *fuzzy measure* on N is a monotonic set function $\mu : 2^N \to [0, 1]$ with $\mu(\emptyset) = 0$ and $\mu(N) = 1$. Monotonicity means that $\mu(S) \leq \mu(T)$ whenever $S \subseteq T$.

One thinks of $\mu(S)$ as the weight of importance of the subset of criteria S. Thus, in addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined.

A fuzzy measure is said to be *additive* if $\mu(S \cup T) = \mu(S) + \mu(T)$ whenever $S \cap T = \emptyset$. In this case, it suffices to define the n coefficients (weights) $\mu(1), \ldots, \mu(n)$ to define the measure entirely. In general, one needs to define the 2^n coefficients corresponding to the 2^n subsets of N.

Through the natural identification of subsets with their characteristic vector in $\{0, 1\}^n$, a fuzzy measure is seen to be a pseudo-Boolean function, that is a function $f : \{0, 1\}^n \to \mathbb{R}$. Such a function has a unique expression as a multilinear polynomial (see [11]):

$$f(x) = \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i, \qquad x \in \{0, 1\}^n. \tag{3}$$

In combinatorics, a viewed as a set function on N is called the *Möbius transform* of v (see e.g. Rota [16]), which is given by

$$a(S) = \sum_{T \subseteq S} (-1)^{s-t} \mu(T), \qquad S \subseteq N.$$

Of course, any set of 2^n coefficients $\{a(T) \mid T \subseteq N\}$ could not be the Möbius representation of a fuzzy measure. The boundary and monotonicity conditions must be ensured. In terms of the Möbius representation, those conditions are very easy to prove, see e.g. [2]:

$$\begin{cases} a(\emptyset) = 0, \qquad \sum_{T \subseteq N} a(T) = 1, \\ \sum_{T : i \in T \subseteq S} a(T) \geq 0, \qquad \forall S \subseteq N, \ \forall i \in S. \end{cases} \tag{4}$$

Definition 2.2 Let μ be a fuzzy measure on N. The (discrete) Choquet integral of a function $x : N \to \mathbb{R}$ with respect to μ is defined by

$$C_\mu(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_{(i)} \, [\mu(A_{(i)}) - \mu(A_{(i+1)})],$$

where $(\cdot)$ indicates a permutation such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$, and $A_{(n+1)} = \emptyset$.

Thus defined, the Choquet integral has very good properties for aggregation (see e.g. Grabisch [7]). For instance, it is continuous, non decreasing, comprised between min and max, stable under the same transformations of interval scales in the sense of the theory of measurement, and coincides with the weighted arithmetic mean (Lebesgue integral) as soon as the fuzzy measure is additive.

In terms of the Möbius representation, the Choquet integral is written (see [2]):

$$C_\mu(x) = \sum_{T \subseteq N} a(T) \bigwedge_{i \in T} x_i, \qquad x \in \mathbb{R}^n,$$

where $\wedge$ stands for the minimum operation. This shows that the Choquet integral is simply an extension on $\mathbb{R}^n$ of the pseudo-Boolean function (3).

In this paper, we substitute the Choquet integral to the weighted arithmetic mean whenever interacting criteria are considered.

3 The concept of interaction among criteria

The overall importance of a criterion $i \in N$ is not solely determined by the number $\mu(i)$, but also by all $\mu(S)$ such that $i \in S$. Shapley [20] has proposed a definition of a coefficient of importance, based on a set of reasonable axioms. The *importance index* or *Shapley value* of criterion i with respect to μ is defined by:

$$\phi_{\mathrm{Sh}}(i) := \sum_{T \subseteq N \setminus i} \frac{(n-t-1)! \, t!}{n!} \, [\mu(T \cup i) - \mu(T)]. \tag{5}$$

The Shapley value is a fundamental concept in game theory [20] expressing a power index. It can be interpreted as a weighted average value of the marginal contribution of element i alone in all coalitions. A basic property of the Shapley value is $\sum_{i=1}^{n} \phi_{\mathrm{Sh}}(i) = \mu(N)$.

There is in fact another common way of defining a power index, due to Banzhaf [1] (see also Dubey and Shapley [4]). The so-called *Banzhaf value*, defined as

$$\phi_{\mathrm{B}}(i) := \frac{1}{2^{n-1}} \sum_{T \subseteq N \setminus i} [\mu(T \cup i) - \mu(T)], \tag{6}$$

can be viewed as an alternative to the Shapley value.

Another interesting concept is the one of *interaction* between criteria. The fact that in general the Shapley value $\phi_{\mathrm{Sh}}(i)$ related to the criterion i is different from $\mu(i)$ shows that the criteria interact. Of course, it would be interesting to appraise the degree of interaction among any subset of criteria.

Consider a pair $\{i,j\} \subseteq N$ of criteria. The difference

$$a(ij) = \mu(ij) - \mu(i) - \mu(j)$$

seems to reflect the degree of interaction between i and j. This difference is zero when the individual importances $\mu(i)$ and $\mu(j)$ add up without interfering. In this case, there is no interaction between i and j. The difference is positive if there is a synergy effect between i and j. These two criteria then interfere in a positive way. Finally, the difference is negative in case of overlap effect between i and j. The criteria then interfere in a negative way.

As for importance, a proper definition of interaction should consider not only $\mu(i), \mu(j), \mu(ij)$ but also the measures of all subsets containing i and j. Thus the interaction between criteria i an j can be considered as the average of the marginal contributions of j in the presence of i minus the average of the marginal contributions of j in the absence of i which corresponds to the weighted sum over all combinations $T \subseteq N \setminus ij$ of

$$\mu(T \cup ij) - \mu(T \cup i) - \mu(T \cup j) + \mu(T).$$

Murofushi and Soneda [13] have proposed the following definition, borrowing concepts from multiattribute utility theory, which is very similar to the one of Shapley. The interaction index of elements i, j is defined by

$$I(ij) := \sum_{T \subseteq N \setminus ij} \frac{(n-t-2)!\, t!}{(n-1)!} \left[\mu(T \cup ij) - \mu(T \cup i) - \mu(T \cup j) + \mu(T)\right],$$

and can be interpreted as a weighted average value of the *added value* produced by putting i and j together, all coalitions being considered. When $I(ij)$ is positive (resp. negative), then the interaction between i and j is said to be positive (resp. negative).

The interaction index among a combination S of criteria has been introduced by Grabisch [8] as a natural extension of the case $|S| = 2$. The *Shapley interaction index* related to μ, is defined by

$$I_{\mathrm{Sh}}(S) := \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!} \sum_{L \subseteq S} (-1)^{s-l} \mu(L \cup T), \qquad S \subseteq N,$$

that is, in terms of the Möbius representation [8],

$$I_{\mathrm{Sh}}(S) = \sum_{T \supseteq S} \frac{1}{t-s+1}\, a(T), \qquad S \subseteq N.$$

Viewed as a set function, the Shapley interaction index coincides on singletons with the Shapley value (5).

Roubens [17] developed a parallel notion of interaction index, based on the Banzhaf value (6): the *Banzhaf interaction index*, defined by

$$I_{\mathrm{B}}(S) := \frac{1}{2^{n-s}} \sum_{T \subseteq N \setminus S} \sum_{L \subseteq S} (-1)^{s-l} \mu(L \cup T), \qquad S \subseteq N,$$

that is, in terms of the Möbius representation [17],

$$I_{\mathrm{B}}(S) = \sum_{T \supseteq S} (\frac{1}{2})^{t-s} \, a(T), \qquad S \subseteq N.$$

It should be noted that the interaction indices I_{B} and I_{Sh} have been axiomatically characterized by Grabisch and Roubens [10].

The concept of interaction between elements is not really new. It had already been introduced in statistical analysis of factorial experiments, where the main effects (average contributions) of a number of different factors are investigated simultaneously. The interactions among factors have then been defined to model a degree of dependence between them (see e.g. [3, Chapter 5]).

4 The 2-order model

We know that a problem involving n criteria requires 2^n coefficients in $[0,1]$ in order to define the fuzzy measure μ on every subset. Of course, a decision maker is not able to give such an amount of information. Moreover, the meaning of the numbers $\mu(S)$ and $a(S)$ is not always clear for the decision maker.

To overcome this problem, Grabisch [8] proposed to use the concept of *k-order fuzzy measure*. Looking at the polynomial expression of a fuzzy measure (3), one can notice that additive measures have a linear representation $f(x) = \sum_{i=1}^{n} a(i)\, x_i$. By extension, we may think of a fuzzy measure having a polynomial representation of degree 2, or 3, or any fixed integer k. Such a fuzzy measure is naturally called k-order fuzzy measure since it represents a k-order approximation of its polynomial expression in the neighborhood of the origin.

We now confine to the 2-order case, which seems to be the most interesting in practical applications, since it permits to model interaction between criteria while remaining very simple. Indeed, only $n + \binom{n}{2} = \frac{n(n+1)}{2}$ coefficients are required to define the fuzzy measure, namely the coefficients

$$\begin{aligned} \mu(i) &= a(i), \quad i \in N, \\ \mu(ij) &= a(i) + a(j) + a(ij), \quad \{i,j\} \subseteq N. \end{aligned}$$

The other coefficients are then given by:

$$\mu(S) = \sum_{i \in S} a(i) + \sum_{\{i,j\} \subseteq S} a(ij), \qquad S \subseteq N, \ |S| \geq 2.$$

Note that the 2-order case is equivalent to assume that the Shapley and Banzhaf interaction indices are zero for subsets of at least 3 elements. In this case, the Choquet integral becomes

$$C_\mu(x) = \sum_{i \in N} a(i)\, x_i + \sum_{\{i,j\} \subseteq N} a(ij)\, (x_i \wedge x_j), \qquad x \in \mathbb{R}^n. \tag{7}$$

Moreover, the interaction indices coincide ($I_{\mathrm{Sh}} = I_{\mathrm{B}} = I$) and we have immediately:

$$\begin{aligned} I(i) &= a(i) + \frac{1}{2} \sum_{j \in N \setminus i} a(ij), \qquad i \in N, \\ I(ij) &= a(ij), \qquad i, j \in N, \end{aligned}$$

and $I(S) = 0$ for all $S \subseteq N$, $|S| > 2$.

Notice that $I(i), a(i) \in [0, 1]$ for all $i \in N$. Moreover, Roubens [17] and Grabisch [8] proved that $I(ij) \in [-1, 1]$ for all $i, j \in N$.

Finally, in this context conditions (4) for the coefficients $a(\emptyset)$, $a(i)$ ($i \in N$), $a(ij)$ ($i, j \in N$) to define a fuzzy measure become:

$$\begin{cases} a(\emptyset) = 0, \\ \sum_{i \in N} a(i) + \sum_{\{i,j\} \subseteq N} a(ij) = 1, \\ a(i) \geq 0, \quad \forall i \in N, \\ a(i) + \sum_{j \in T} a(ij) \geq 0, \quad \forall i \in N, \forall T \subseteq N \setminus i. \end{cases} \tag{8}$$

5 Identification of weights

We address now the problem of identification of weights of interacting criteria. More precisely, we are interested in finding a 2-order fuzzy measure on the basis of a partial ranking over a reference set of prototypes and some semantical considerations about criteria. These latter ones could be the following.

- *Importance of criteria.* This can be properly done by giving a partial preorder on N, representing a ranking of the weights $\mu(i)$, $i \in N$. One can also imagine that some exact values be given.

- *Interaction between criteria.* The interaction index $I(ij) = a(ij)$ is suitable for this. One can give a partial preorder on the set of pairs of criteria. The sign of each interaction $a(ij)$ can also be given, or even exact values.

- *Symmetric criteria.* Two criteria i and j are symmetric if they can be exchanged without changing the aggregation mode. Then $\mu(T \cup i) = \mu(T \cup j)$ for all $T \subseteq N \setminus ij$. This reduces the number of coefficients.

- *Veto and favor effects* [9]. A criterion $i \in N$ is said to be a *veto* for a decision problem modelled by the Choquet integral if $\mathcal{C}_\mu(x) \leq x_i$ for all $x \in \mathbb{R}^n$. This mean that a bad score on criterion i will lead to a bad global score, whatever the values of the other scores. To model such an effect, it suffices to take a fuzzy measure μ such that $\mu(S) = 0$ whenever $i \notin S$. Similarly, criterion i is said to be a *favor* if $\mathcal{C}_\mu(x) \geq x_i$ for all $x \in \mathbb{R}^n$. Here, a good score on i will lead to a good global score, whatever the values of the remaining scores. To obtain this, it suffices to take μ such that $\mu(S) = 1$ whenever $i \in S$.

We thus suppose that we have at our disposal an expert or decision maker who is able to tell the relative importance of criteria, and the kind of interaction between them, if any. In fact, in practical applications the decision maker is able to give information on the weights $\mu(i)$ and interaction indices much more easily than to assess directly the values of the fuzzy measure. It is thus important to ask the decision maker the good questions that will allow to identify the fuzzy measure (elicitation from the decision maker).

Formally, the input data of the problem can be summarized as follows:

- The set A of alternatives and the set N of criteria,
- A table of individual scores (utilities) x_i^a given on a same interval scale $X \subseteq \mathbb{R}$,
- A partial preorder $\succeq_A$ on A (ranking of alternatives),
- A partial preorder $\succeq_N$ on N (ranking of criteria),
- A partial preorder $\succeq_P$ on the set of pairs of criteria (ranking of interaction indices),
- The sign of interaction between some pairs of criteria $a(ij) : > 0, = 0, < 0$.

All these data can be formulated in terms of linear equalities or inequalities linking the unknown "weights" μ. The model then consists in finding a feasible 2-order fuzzy measure. Thus we are faced with a linear constraints satisfaction problem. Note that strict inequalities can be converted into vague inequalities by introducing a positive slack variable as the following immediate proposition shows.

Proposition 5.1 *$x \in \mathbb{R}^n$ is a solution of the linear system*

$$\begin{cases} \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, & i = 1, \ldots, p, \\ \sum_{j=1}^{n} c_{ij}\, x_j < d_i, & i = 1, \ldots, q, \end{cases}$$

if and only if there exists $\varepsilon > 0$ such that

$$\begin{cases} \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, & i = 1, \ldots, p, \\ \sum_{j=1}^{n} c_{ij}\, x_j \leq d_i - \varepsilon, & i = 1, \ldots, q. \end{cases}$$

In particular, a solution exists if and only if the following linear program

$$\textit{maximize} \quad z = \varepsilon$$

$$\textit{subject to} \quad \begin{cases} \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, & i = 1, \ldots, p, \\ \sum_{j=1}^{n} c_{ij}\, x_j \leq d_i - \varepsilon, & i = 1, \ldots, q, \end{cases}$$

has an optimal solution $x^* \in \mathbb{R}^n$ with an optimal value $\varepsilon^* > 0$. In this case, x^* is a solution of the first system.

Thus, the problem of finding a 2-order fuzzy measure can be formalized with the help of a linear program. It is obvious that the poorer the input information, the bigger the solution set. Hence, it is desirable that the information is as complete as possible. However, if this information contains incoherences then the solution set could be empty. Note however that an empty solution set could also be due to an incompatibility between the given information and the assumption that the fuzzy measure is of order 2. In this case, it can be useful to consider a 3-order fuzzy measure or, if necessary, a fuzzy measure of higher order.

Written in terms of the Möbius representation, a model for identifying weights could be given as follows:

$$
\begin{array}{l}
\text{maximize } z = \varepsilon \\[1em]
\text{subject to} \\[0.5em]
\left.\begin{array}{ll}
C(a) - C(b) \geq \delta + \varepsilon & \text{if } a \succ_A b \\
-\delta \leq C(a) - C(b) \leq \delta & \text{if } a \sim_A b
\end{array}\right\} \text{partial semiorder with threshold } \delta \\
\left.\begin{array}{ll}
a(i) - a(j) \geq \varepsilon & \text{if } i \succ_N j \\
a(i) = a(j) & \text{if } i \sim_N j
\end{array}\right\} \text{ranking of criteria (weights on singletons)} \\
\left.\begin{array}{ll}
a(ij) - a(kl) \geq \varepsilon & \text{if } ij \succ_P kl \\
a(ij) = a(kl) & \text{if } ij \sim_P kl
\end{array}\right\} \text{ranking of pairs of criteria (interactions)} \\
\left.\begin{array}{ll}
a(ij) \geq \varepsilon \text{ (resp. } \leq -\varepsilon) & \text{if } a(ij) > 0 \text{ (resp. } < 0) \\
a(ij) = 0 & \text{if } a(ij) = 0
\end{array}\right\} \text{sign of some interactions} \\
\left.\begin{array}{ll}
\sum_{i\in N} a(i) + \sum_{\{i,j\}\subseteq N} a(ij) = 1 & \\
a(i) \geq 0 & \forall i \in N \\
a(i) + \sum_{j\in T} a(ij) \geq 0 & \forall i \in N,\ \forall T \subseteq N \setminus i
\end{array}\right\} \begin{array}{l}\text{boundary and}\\ \text{monotonicity}\\ \text{conditions (8)}\end{array} \\
\left. C(a) = \sum_{i\in N} a(i)\, x_i^a + \sum_{\{i,j\}\subseteq N} a(ij)\, [x_i^a \wedge x_j^a] \quad \forall a \in A \right\} \text{definition of } \mathcal{C}_\mu
\end{array}
$$

It seems natural to assume that the ranking over A is translated into a partial semiorder over the set of the global evaluations given by the Choquet integral. This partial semiorder has a fixed preference threshold δ, which can be tuned as wished. Such a threshold level should be reached by the difference between global scores to consider that one object should be significantly preferred to another object.

Let us comment on the scale used to define the utilities. Since the Choquet integral is stable under the same admissible transformations of interval scales, using utilities on a $[0, 100]$ scale or on $[-2, 3]$ scale has no influence on the ranking of alternatives. Now suppose that the utilities x_i^a are defined in $X = [0, 1]$. Changing this scale into $[p, q]$, with $p < q$, and translating the utilities in the appropriate way amounts to replacing only the first set of constraints by

$$C(a) - C(b) \geq \delta + \frac{\varepsilon}{q - p} \quad \text{if } a \succ_A b. \tag{9}$$

It is clear that if ε is missing in the other constraints then the optimal solution of the linear program depends on no scale transformation. Otherwise, if ε is present both in constraints (9) and the other constraints then the optimal solution can be sensitive to any scale transformation, although the feasibility of the system remains unaltered. In this case, since $a(i) \in [0, 1]$ and $a(ij) \in [-1, 1]$ for all $i, j \in N$, we necessarily have $\varepsilon \in [0, 2]$. We then can make the following theoretical observations:

- If $q-p$ is large (for example $q-p=100$) then $\varepsilon/(q-p)$ is small and the global evaluations $C(a)$ have good chances to be not very contrasted in the optimal solution.
- If $q-p$ is small (for example $q-p=0.01$) then, multiplying inequation (9) by $q-p$, we see that the optimal value ε^* will be small too; this implies that the weights $a(i)$ or the interactions $a(ij)$ will be not very contrasted.

Taking these facts into account, it seems that a reasonable compromise would be to take $q-p=1$ and hence to define the utilities on the unit interval $[0,1]$.

The value of the threshold δ must also be chosen carefully. Indeed, the larger δ, the smaller ε^*. A too large δ can even make the program infeasible. We will not suggest any rule to fix δ. It is better to compare the solutions obtained with different values of δ.

Let us make a last observation. It might happen that some equalities are very constraining and make the program infeasible. In this case, these equalities can be relaxed into some strict inequalities in a coherent way. For example, the equality in the conditions $a(1)=a(2)>a(3)$ could be relaxed into the following strict inequalities:

$$\left.\begin{matrix} a(1) \\ a(2) \end{matrix}\right\} > a(3) \quad \text{and} \quad |a(1)-a(2)| < \left\{\begin{matrix} |a(1)-a(3)| \\ |a(2)-a(3)| \end{matrix}\right.$$

that is

$$\begin{cases} a(1) > a(3) \\ a(2) > a(3) \\ 2\,a(1) - a(2) - a(3) > 0 \\ -a(1) + 2\,a(2) - a(3) > 0 \end{cases}$$

In order to illustrate the model, we now present three small examples. They are constructed in such a way that no weighted arithmetic mean can be used as a utility function. The first one is similar to Example 1.1. But here commensurable utilities are available.

Example 5.1 Consider the problem of ranking cooks on the basis of their capacity for preparing three dishes: frogs'legs (FL), steak tartare (ST), and stuffed clams (SC). Four cooks a,b,c,d acting as prototypes are evaluated as follows (marks are multiplied by 20):

cook	FL	ST	SC
a	18	15	19
b	15	18	19
c	15	18	11
d	18	15	11

The decision maker is asked to express its advice by giving a ranking over $A=\{a,b,c,d\}$. Of course, he/she immediately suggests $a \succ_A d$ and $b \succ_A c$. However, these preferences do not contribute to anything since they naturally follow from the monotonicity of the Choquet integral. The decision maker realizes that the other comparisons are not so obvious since the associated profiles interlace. He/she then proposes the following reasoning: when a cook is renowned for his stuffed clams, it is preferable that he/she is also better in cooking frogs'legs than steak tartare, so $a \succ_A b$. However, when a cook badly prepares stuffed clams, it is more important that he/she is better in preparing steak tartare than frogs'legs, and so $c \succ_A d$.

Now, the question arises: does there exist an additive model leading to this partial ranking? Let $\omega_1, \omega_2, \omega_3$ represent the weights of criteria FL, ST, SC respectively. By using the weighted arithmetic mean (1) as a utility function, we obtain:

$$\begin{aligned} a \succ_A b &\Leftrightarrow \omega_1 > \omega_2, \\ c \succ_A d &\Leftrightarrow \omega_1 < \omega_2. \end{aligned}$$

We immediately observe that no weighted arithmetic mean can yield the proposed ranking. This is not surprising since clearly the criteria are not mutually preferentially independent. Thus, it is necessary to take into account the interactions between criteria.

By extending the weighted arithmetic mean to the 2-order Choquet integral (7), we are led to the following conditions:

$$\begin{aligned} a \succ_A b &\Leftrightarrow 0.15\, a(1) - 0.15\, a(2) + 0.15\, a(13) - 0.15\, a(23) > 0, \\ b \succ_A c &\Leftrightarrow 0.4\, a(3) + 0.2\, a(13) + 0.35\, a(23) > 0, \\ c \succ_A d &\Leftrightarrow -0.15\, a(1) + 0.15\, a(2) > 0. \end{aligned}$$

Of course, these three conditions imply $a \succ_A d$.

Now, a solution of the problem, if any, will be given by an optimal solution of the following linear program ($N = \{1, 2, 3\}$):

maximize $z = \varepsilon$

subject to

$$\begin{cases} 0.15\, a(1) - 0.15\, a(2) + 0.15\, a(13) - 0.15\, a(23) \geq \delta + \varepsilon & \\ 0.4\, a(3) + 0.2\, a(13) + 0.35\, a(23) \geq \delta + \varepsilon & \\ -0.15\, a(1) + 0.15\, a(2) \geq \delta + \varepsilon & \\ a(1) + a(2) + a(3) + a(12) + a(13) + a(23) = 1 & \\ a(i) \geq 0 & i \in N \\ a(i) + a(ij) \geq 0 & i, j \in N \\ a(i) + a(ij) + a(ik) \geq 0 & i, j, k \in N. \end{cases}$$

Using an appropriate software, the following solution was obtained (for $\delta = 0.05$ fixed):

- Objective function: $\varepsilon = 0.025$ (note that $\varepsilon = 0$ if $\delta = 0.075$)
- Weights $a(i)$ $(= \mu(i))$ and Shapley value $I(i)$:

	FL	ST	SC
$a(i)$	0	0.5	0.5
$I(i)$	0.25	0.25	0.5

- Interaction indices $a(ij)$:

	ST	SC
FL	0	0.5
ST		-0.5

- The global evaluations $C = \mathcal{C}_\mu$:

	a	b	c	d
$C(\cdot)$	0.925	0.85	0.725	0.65
$20\,C(\cdot)$	18.5	17	14.5	13

Of course, we must be very cautious when one wants to get general conclusions only from the obtained solution. Indeed, it is not clear at all that there is no solution such that $a(23) > 0$. All that the model does is to find a 2-order fuzzy measure that is coherent with the available information. Thus, the interpretation of such a solution could be irrelevant.

Example 5.2 Consider the problem of the evaluation of students in an institute of Mathematics with respect to three subjects: linear algebra (Al), calculus (Ca), and statistics (St). Suppose that the institute is oriented towards statistics. More precisely, suppose that the decision maker suggests the following partial ranking of the criteria:

$$\mathrm{St} \succ_N \begin{cases} \mathrm{Al} \\ \mathrm{Ca} \end{cases}$$

For instance, the weights of subjects could be proportional to 2, 2 and 3, respectively.

Three students a, b, c have been evaluated as follows (marks are multiplied by 20):

student	Al	Ca	St	global evaluation (weighted arithmetic mean)
a	12	12	19	15
b	16	16	15	15.57143
c	19	19	12	16

The decision maker then reasons as follows: if a student is excellent at statistics (mark of at least 18) then he/she is excellent, whatever the marks obtained elsewhere. However, if he/she is not excellent at statistics then it is necessary to take into account the mark obtained in the other courses, using e.g. the above weighted arithmetic mean.

Consequently, on the basis of the available profiles, the decision maker proposes the following ranking:

$$a \succ_A c \succ_A b.$$

Let us show that the additive model is not appropriate for this example. Let $\omega_1, \omega_2, \omega_3$ represent the weights of criteria Al, Ca, St respectively. By using the weighted arithmetic mean as a utility function, we get:

$$\begin{aligned} a \succ_A c \quad &\Leftrightarrow \quad \omega_1 + \omega_2 - \omega_3 < 0, \\ c \succ_A b \quad &\Leftrightarrow \quad \omega_1 + \omega_2 - \omega_3 > 0. \end{aligned}$$

Now, let us turn to the 2-order model, which takes into account the interactions between pairs of criteria. On this matter, the decision maker has observed that there is some overlap between statistics and calculus courses. Hence, we may assume that there exists a negative interaction between them:

$$a(\mathrm{St\ Ca}) < 0.$$

Putting all these data into a linear program as explained above, and then solving it with the help of a software, the following solution was obtained (for $\delta = 0.025$ fixed):

- Objective function: $\varepsilon = 0.055769$
- Weights $a(i)$ and Shapley value $I(i)$:

	Al	Ca	St
$a(i)$	0.7135	0.05577	1
$I(i)$	0.3567	0.02788	0.6154

- Interaction indices $a(ij)$:

	Ca	St
Al	0	-0.7135
Ca		-0.05577

- The global evaluations $C = \mathcal{C}_\mu$:

	a	b	c
$C(\cdot)$	0.95	0.7885	0.8692
$20\,C(\cdot)$	19	15.77	17.384

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