

STOCHASTIC OPTIMIZATION MODELS OF INSURANCE MATHEMATICS

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Abstract. The paper overviews stochastic optimization models of insurance mathematics and methods for their solution from the point of view of stochastic programming and stochastic optimal control methodology, with vector optimality criteria. The evolution of an insurance company's capital is considered in discrete time. The main random variables, which influence this evolution, are levels of payments, i.e. the ratios of paid claims to the corresponding premiums, per unit of time. The main decision variables are the structure of the insurance portfolio (the structure of the total premium) and the dividend payments. As for efficiency criteria, indicators of profitability are taken, and, as risk criteria, the probability of ruin or the recourse capital is used. The goal of optimization is to build efficiency frontiers and to find out Pareto-optimal solutions. Methods for solving these tasks are proposed.

Keywords. Insurance mathematics, risk process, ruin probability, stochastic programming, multi-criteria problems, two-stage tasks, probabilistic constraints, stochastic optimal control, mixed-integer programming, dynamic programming.

Introduction. The traditional theory of optimal insurance is based on the expected utility theory [1, 2]. An alternative approach is based on optimizing profitability and/or risk [3, 7 - 13]. As an indicator of risk, the probability of ruin (the probability of insolvency) is mostly used, which has been studied and estimated by the vast literature [5, 6, 14]. The difficulty of the arising optimization problems is due to the fact that they belong to the class of non-convex stochastic programming or stochastic optimal control problems under probabilistic constraints. The theory of stochastic programming is designed to formalize decision-making problems in conditions of stochastic uncertainty [15 - 18]. The present article shows, with model examples, how the problems of optimizing the insurance business are formulated and numerically solved according to several criteria within the framework of computational stochastic programming [19].

In contrast to the classical Kramer-Lundberg model [1 - 6], in this article, the process of stochastic evolution of an insurance company capital is considered in discrete time, that is justified by discrete-time (quarterly, annual) reporting of companies on their performance. In insurance, the main sources of randomness are insurance claims that appear at random times and have random size. In the considered discrete-time models, the main random variable is the level of insurance payments (unprofitableness), which is the ratio of payments to premiums per unit of time (quarter,

¹ International Institute for Applied Systems Analysis, Laxenburg, Austria; ermoliev@iiasa.ac.at

² V.M.Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine & Faculty of Applied Mathematics of the National Technical University "Igor Sikorsky Kyiv Polytechnic Institute"; vladimir.norikin@gmail.com

³ V.M.Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine, Kyiv, Ukraine; bogdan.norikin@gmail.com

year). The dynamics of making optimal decisions in these models is reflected in the two- and multi-stage structure of stochastic programming models, as well as in the multi-stage models of stochastic dynamic programming. The paper proposes several new methods for solving arising stochastic optimization problems. These issues are considered in more detail in [20].

The E-model of stochastic programming. Consider the following conceptual model of an insurance company. Let $u \geq 0$ denotes an insurance reserve from a certain feasibility set U , $x \in X$ be the planned volume of insurance contracts (in monetary terms) from an allowable set X , $v \geq 0$ be the insurance load, $c(x)$ be the net premium under insurance contracts (gross premium minus production costs to maintain the volume x of contracts), $\xi \geq 0$ be (random) unprofitableness of insurance business (with distribution function $F(\cdot)$), that is, a random amount of insurance claims per unit of premiums. In insurance statistics, the value ξ is called the level of payments [22]. In stochastic programming, the so-called two-stage decision-making model is considered. As applied to insurance modeling, it looks as follows: at the first stage, a deterministic decision (u, x) is made, and at the second stage, a value ξ is observed, i.e. the value of insurance claims becomes known. The purpose of the business is to maximize dividends d , which may be, depending on the setting, either deterministic or random. Let the decision on dividends be made before the insurance claims ξx are finally known. Thus, if at the first stage the company's capital was u , then at the second stage it becomes a random variable

$$f(u, x, d, \xi) = u + c(x) - d - \xi x, \quad (1)$$

which can be both positive and negative. In the latter case, they say that the company is insolvent. Within the framework of the standard stochastic programming theory, it is believed that the problem of insolvency can be resolved by borrowing capital of the size $(-\min\{0, u + c(x) - d - \xi x\})$, multiplied by the penalty coefficient $q > 0$ for the insolvency (it is natural to assume that $q \geq 1$). Then the decision-making task on values u, v, x, d consists in maximizing the total expected income (E-task):

$$F(u, x, q) = d + qE_{\xi} \min\{0, u + c(x) - d - \xi x\} \rightarrow \max_{(u, x) \in W \subseteq U \times X, d \geq 0}, \quad (2)$$

where E_{ξ} is the mathematical expectation operator with respect to ξ . If the constraint set $W \subseteq U \times X$ is a singleton, i.e. the values u, x are fixed, then problem (2), in essence, concerns only the choice of dividends d , paid in advance. For a fixed parameter q and upward convex function $c(x)$, problem (2) is a convex stochastic programming problem, for the solution of which

there is a wide variety of methods [15, 16, 18]. Let (u^*, x^*, d^*) be the optimal solution for problem (2). At the same time, it is possible that for some values ξ the company is insolvent, $f(u^*, x^*, d^*, \xi) < 0$, or, in other words, the probability $\Pr\{f(u^*, x^*, d^*, \xi) < 0\}$ of insolvency is greater than zero. However, by increasing the penalty ratio q , the desired reduction in this probability can be achieved [21].

The P-model of stochastic programming. There are other settings of stochastic programming problems, for example, the so-called P-problems [17]. In the notation of the previous section, we consider the following problem:

$$d \rightarrow \max_{(u,x) \in W \subseteq U \times X, d \geq 0} \quad (3)$$

subject to the probabilistic constraint

$$\Pr\{u + c(x) - \xi x - d < 0\} \leq \epsilon, \quad (4)$$

where $\Pr\{\cdot\}$ indicates the probability of the event in brackets; ϵ is the reliability parameter of the insurance business, $0 < \epsilon < 1$. In contrast to (2), model (3), (4) allows us to explicitly control the probability of insolvency, but it turns out to be non-convex. Indeed, let us denote $F(z) = \Pr\{\xi \leq z\}$ the distribution function of the random variable ξ . Then

$$\Pr\{u + c(x) - d - \xi x < 0\} = \Pr\{\xi > (u + c(x) - d) / x\} = 1 - F((u + c(x) - d) / x)$$

and thus, constraint (4) can be rewritten as follows:

$$F((u + c(x) - d) / x) \geq 1 - \epsilon. \quad (5)$$

This shows that restriction (5) and, therefore (4), is non-convex, that greatly complicates the solution of problem (3), (4).

Modeling an insurance portfolio. In insurance mathematics, a portfolio means the structure of the insurance premium, i.e. shares coming from different kinds of policies. We will describe the insurance portfolio with a vector $x \in X$ with components (x_1, \dots, x_n) , where X denotes the set of acceptable portfolios, for example, $X = \{x \geq 0 : \sum_{i=1}^n x_i \leq w\}$, with a fixed total nominal value w . Let $c(x_i)$ and ξ_i be the net income and random loss-making (level of payments) of the i -th type of insurance (for example, in annual terms), $\xi = (\xi_1, \dots, \xi_n)$. Then, similarly to (1), with the initial capital u and dividend payments d , the random capital at the end of the time unit will be

$$f(u, x, d, \xi) = u + \sum_{i=1}^n c_i(x_i) - d - \sum_{i=1}^n x_i \xi_i .$$

The tasks of optimizing the insurance portfolio are set similarly (2):

$$F(u, x, q) = d + qE_{\xi} \min\{0, f(u, x, d, \xi)\} \rightarrow \max_{(u,x) \in W \subseteq U \times X, d \geq 0} , \quad (6)$$

or similarly (3), (4):

$$d \rightarrow \max_{(u,x) \in W \subseteq U \times X, d \geq 0} , \quad (7)$$

$$\Pr \{f(u, x, d, \xi) < 0\} \leq \epsilon. \quad (8)$$

In the multidimensional case, the probability constraint (8) cannot be expressed in terms of a distribution function as in (5), therefore, problem (7), (8) requires the development of the special solution technique.

Reducing the P-model of stochastic programming to a mixed-integer optimization problem. If the random variable ξ takes on only a finite number of values (e.g., historical scenarios) $\{\xi_1, \dots, \xi_k\}$ with probabilities $\{p_1, \dots, p_k\}$, then problem (3), (4) can be equivalently reduced to a partially integer programming problem. Let $z = \{z_1, \dots, z_k\}$ be a set of Boolean variables, and M be a sufficiently large constant. Then problem (3), (4) is equivalent to the following mixed-integer programming problem:

$$d \rightarrow \max_{(u,x) \in W, d \geq 0, z \in \{0,1\}^k} , \quad (9)$$

$$\sum_{i=1}^k p_i z_i \geq 1 - \epsilon , \quad (10)$$

$$u + c(x) - d - \xi_i x \geq -M(1 - z_i), \quad i = 1, \dots, k. \quad (11)$$

Here, constraint (10) means that the total probability of those scenarios i , for which $z_i = 1$, is greater than or equal to $(1 - \epsilon)$. When $z_i = 1$, the corresponding restriction in (11) turns into the inequality $u + c(x) - d - \xi_i x \geq 0$; when $z_i = 0$, it turns into the inequality $u + c(x) - d - \xi_i x \geq -M$, and due to a large value of M , it is satisfied automatically, i.e. it is not restricting. Thus jointly, constraints (10), (11) entail the fulfillment of the probabilistic constraint (4). A similar approach to reducing a chance constrained problem to a partially integer programming problem (first proposed in [23], see also [24]) is used in [11] to optimize the insurance portfolio subject to a constraint on the probability of ruin.

Vector stochastic programming problems. A task of insurance business optimization can be considered as multi-criteria one. For example, instead of problems (2) and (3), (4), we can consider the following two-criteria (f_1, f_2) problems:

$$f_1(u, x, d) = d \rightarrow \max_{(u, x) \in W, d \geq 0}, \quad (12)$$

$$f_2(u, x, d) = -E_{\xi} \min\{0, u + c(x) - d - \xi x\} \rightarrow \min_{(u, x) \in W, d \geq 0}, \quad (13)$$

where the second criterion $f_2(u, x, d)$ means the average capital deficit at the time of ruin, and

$$f_1(u, x, d) = d \rightarrow \max_{(u, x) \in W, d \geq 0}, \quad (14)$$

$$f_3(u, x, d) = \Pr\{u + c(x) - d - \xi x < 0\} \rightarrow \min_{(u, x) \in W, d \geq 0} \quad (15)$$

The efficiency frontier of the first task (12), (13) in the plane “profitability-risk” (f_1, f_2) is defined by the function:

$$F_2(d) = \min_{(u, x) \in W} f_2(u, x, d) = \min_{(u, x) \in W} E_{\xi} \max\{0, d + \xi x - u - c(x)\}. \quad (16)$$

If $c(x)$ is a linear or concave function and the constraint set W is convex, then (16) is a convex stochastic programming problem [15, 16, 18].

The efficiency frontier of the second problem (14), (15) in the plane “dividends - the probability of ruin” (f_1, f_3) is given by the function:

$$F_3(d) = \min_{(u, x) \in W} f_3(u, x, d) = \min_{(u, x) \in W} \Pr\{u + c(x) - d - \xi x < 0\}. \quad (17)$$

This task is a non-convex stochastic programming problem. In the case of scalar ξ and x , we get

$$F_3(d) = \min_{(u, x) \in W} [1 - F((u + c(x) - d) / x)],$$

where $F(\cdot)$ is the distribution function of the random variable ξ .

If the random variable ξ takes on a finite set of values (scenarios) $\{\xi_1, \dots, \xi_n\}$ with probabilities $\{p_1, \dots, p_n\}$, then problem (17) is reduced to a partially integer programming problem similarly to [24]. Indeed, let us introduce Boolean variables $\{z_1, \dots, z_n\}$ and consider the problem:

$$\sum_{i=1}^n p_i z_i \rightarrow \max_{(u, x) \in W, z_i \in \{0, 1\}^n}$$

$$u + c(x) - d - \xi_i x \geq -M(1 - z_i), \quad i = 1, \dots, n,$$

where M is a sufficiently large constant. In this problem, the probability of non-ruin is maximized, and therefore, it is equivalent to problem (17).

A stochastic programming model with decision strategies. Let us now consider the case when the decision on the dividends payment is made after the insurance claims ξx become known. In this case, dividends $d = d(\cdot)$ can be an arbitrary measurable function of the company capital $u + c(x) - \xi x$. Then problem (3), (4) takes the following form:

$$E_{\xi} d(u + c(x) - \xi x) \rightarrow \max_{(u, x) \in W, d(\cdot) \geq 0} \quad (18)$$

subject to the chance constraint

$$\Pr \{u + c(x) - \xi x - d(u + c(x) - \xi x) < 0\} \leq \epsilon, \quad (19)$$

where the optimization is carried out not only over variables u and x but also over all measurable (by Borel) functions $d(\cdot) \geq 0$. This problem is infinite-dimensional; to solve it, it is necessary to examine all possible measurable functions $d(\cdot)$ together with all admissible values of u and x . One of possible approaches to the approximate solution of this problem is to specify a parametric form of the function $d(\cdot)$. For example, one can restrict oneself to the so-called barrier strategies $d(\cdot, y) = \max\{0, \cdot - y\}$, where y is the value of the barrier. The choice of the barrier strategy for paying dividends means that with a company's capital less than the barrier, the dividends are not taken, and with the capital more than the barrier, all capital exceeding the barrier is subtracted as dividends. Of course, other types of dividend strategies are possible, depending, generally speaking, on a finite-dimensional vector parameter. When the function $d(\cdot, y)$ is substituted into problem (18), (19), the latter turns into finite-dimensional one:

$$E_{\xi} d(u + c(x) - \xi x, y) = E_{\xi} \max\{0, (u + c(x) - \xi x - y)\} \rightarrow \max_{(u, x) \in W, y \geq 0} \quad (20)$$

subject to the chance constraint

$$\Pr \{u + c(x) - \xi x - \max\{0, (u + c(x) - \xi x - y)\} < 0\} \leq \epsilon \quad (21)$$

and is solvable by the search (evolutionary) algorithms. In the same way, vector problems (12) – (15) can be reformulated. A similar approach to solving complex dynamic problems of the dividend optimal control is used in [28].

Dynamic stochastic programming models. Let an insurance company's reserves X^t change over (discrete) time $t = 0, 1, \dots$ according to the following law:

$$X^{t+1} = X^t + c(x) - d(X^t, y) - \xi^t x, \quad X^0 = u, \quad t = 0, 1, \dots, \quad (22)$$

where x is the nominal value of the insurance portfolio; $c(x)$ denotes deterministic premiums per unit of time from the insurance portfolio; $d(X^t, y) \in [0, X^t]$ defines dividend strategy as a function of the current capital X^t and a parameter y ; $\{\xi^t, t = 0, 1, \dots\}$ are independent equally distributed (as some random variable ξ with a distribution function F) observations of the level of insurance claims payments. Process (22) is called the risk process. By the event of ruin (or insolvency) of the risk process (22), we understand such realizations of the process (22) that $X^t < 0$ at some $t > 0$. We denote $\varphi^t(u)$ the probability of non-ruin of process (22) in steps t for the initial capital $X^0 = u$:

$$\varphi^t(u) = \Pr\{X^k \geq 0, \quad 0 \leq k \leq t, \quad X^0 = u\},$$

where $\Pr\{\cdot\}$ designates the probability of the event in brackets. The functions $\varphi^t(u)$ depend not only on u and also on other parameters (x, y) of process (22) but in this context this dependence is not explicitly indicated. The sequence of functions $\{\varphi^t(\cdot), t = 0, 1, \dots\}$ satisfies the following integral relations:

$$\begin{aligned} \varphi^1(u) &= \Pr\{u + c(x) - d(u) - \xi x \geq 0\} = \\ &= \Pr\{\xi \leq (u + c(x) - d(u)) / x\} = \\ &= F((u + c(x) - d(u)) / x), \end{aligned} \quad (23)$$

$$\varphi^{t+1}(u) = E_\xi \varphi^t(u + c(x) - d(u) - \xi x), \quad t = 1, 2, \dots, \quad (24)$$

where E_ξ denotes the mathematical expectation operator (Lebesgue integral with respect to the measure induced by the random variable ξ), $F(z) = \Pr\{\xi \leq z\}$; by definition, $\varphi^t(u) = 0$ when $u < 0$, for all t .

Integral equations of insurance mathematics. The probability of non-ruin of process (22) on an infinite time interval

$$\varphi(u) = \Pr\{X^t \geq 0 \quad \forall t \geq 0, \quad X^0 = u\}$$

satisfies, under a fixed x , the (integral) equation

$$\begin{aligned} \varphi(u) &= E_\xi \varphi(u + c(x) - d(u) - \xi x) = \\ &= \int_{\{\eta: u + c(x) - d(u) - \eta x \geq 0\}} \varphi(u + c(x) - d(u) - \eta x) dF(\eta), \end{aligned} \quad (25)$$

where, by definition, $\varphi(u) = 0$ for $u < 0$. This is a linear integral equation, which always has the zero solution. However, we are interested in conditions under which there is a non-decreasing solution $\varphi(\cdot)$ such that

$$0 \leq \varphi(u) \leq 1, \quad \lim_{\{u \rightarrow +\infty\}} \varphi(u) = 1. \quad (26)$$

In [25], the properties of such equations and conditions for the existence and uniqueness of their solution are studied. Moreover, relations (23), (24), essentially, give the method of successive approximations for solving the integral equation (25).

Note that the operator $A\varphi(u) = E_\xi \varphi(u + c(x) - d(u) - \xi x)$ on the right-hand side of (25) is non-expanding with respect to the sup-norm:

$$\begin{aligned} \|A\varphi_1 - A\varphi_2\| &= \sup_{u \geq 0} |A\varphi_1(u, x) - A\varphi_2(u, x)| = \\ &= \sup_{u \geq 0} |E_\xi \varphi_1(u + c(x) - d(u) - \xi x, x) - E_\xi \varphi_2(u + c(x) - d(u) - \xi x, x)| \leq \\ &\leq E_\xi \sup_{u \geq 0} |\varphi_1(u + c(x) - d(u) - \xi x, x) - \varphi_2(u + c(x) - d(u) - \xi x, x)| \leq \\ &\leq E_\xi \sup_{y \geq 0} |\varphi_1(y, x) - \varphi_2(y, x)| = E_\xi \|\varphi_1 - \varphi_2\| = \|\varphi_1 - \varphi_2\|. \end{aligned}$$

But this operator is not contracting either; therefore, the existence and uniqueness of the solution of equation (25), as well as the convergence of the method of successive approximations (23), (24) cannot be deduced from the principle of contracting mappings.

Necessary and sufficient conditions for the existence and uniqueness of solutions to the integral equations of insurance mathematics. Let $A\varphi(u) = E_\xi \varphi(u + c(x) - d - \xi x)$ be the operator in the right hand side of (25), where x, d are fixed parameters. Suppose that the random variable ξ is bounded, $\xi \leq m$, with probability one. Let us find a function $\varphi_*(u)$ such that $A\varphi_*(u) \geq \varphi_*(u)$ of the form $\varphi_*(u) = \max\{0, 1 - e^{-L(u-u_*)}\}$, where L, u_* are searched parameters. The following estimates hold true:

$$\begin{aligned} A\varphi_*(u) &\geq \int_0^{(u+c(x)-d)/x} (1 - e^{-L(u-u_*+c(x)-d-\eta x)}) dF(\eta) = \\ &= F((u + c(x) - d) / x) - e^{-L(u-u_*)} \int_0^{(u+c(x)-d)/x} e^{L(x\eta+d-c(x))} dF(\eta) \geq \\ &\geq F((u + c(x) - d) / x) - e^{-L(u-u_*)} \int_0^{+\infty} e^{L(x\eta+d-c(x))} dF(\eta). \end{aligned} \quad (27)$$

Let us find u_*, L such that the following conditions are satisfied

$$(u_* + c(x) - d) / x \geq m, \quad \int_0^{+\infty} e^{L(x\eta+d-c(x))} dF(\eta) \leq 1,$$

Then from (27) for $u \geq u_*$, we obtain $A\varphi_*(u) \geq 1 - e^{-L(u-u_*)}$, and thus, for all $u \geq 0$, we have $A\varphi_*(u) \geq \max\{0, 1 - e^{-L(u-u_*)}\} = \varphi_*(u)$. This condition is necessary and sufficient for the existence of a solution of problem (25), (26) [25].

Let us apply the operator A to the unity function $\mathbf{1}(u) \equiv 1$:

$$A\mathbf{1}(u) = \int_0^{(u+c(x)-d)/x} \mathbf{1}(\eta) dF(\eta) = F((u+c(x)-d)/x).$$

If $F(y) < 1$ for all y , then $A\mathbf{1}(u) < 1$ for everyone $u \geq 0$. This condition is sufficient for the uniqueness of the solution to problem (25), (26). The necessary conditions for the uniqueness of a solution (of a similar type) are given in [25].

Discounted performance indicators. For process (22), we consider a series of additive functionals of utility and risk. Suppose that at each step, process (22) is characterized by an indicator $r(\cdot, \xi)$, then we define the functional that is additive along the path of the process (22):

$$V(u, x, y) = \mathbb{E} \sum_{t=0}^{\tau-1} \gamma^t r(X^t, \xi^t),$$

where the mathematical expectation \mathbb{E} is taken along all possible paths of the process, and τ is the random moment of the ruin of the process, i.e.

$$\tau = \sup\{t : \min_{0 \leq k < t} X^k \geq 0\}.$$

Note that the function $V(u, x, y)$ depends on the parameters (u, x, y) of the process (22) but does not contain any supremum or infimum operations, unlike the standard Bellman optimality function. If the function $r(\cdot)$ grows not faster than linear, then the function $V(\cdot, x, y)$ satisfies the equation [26]:

$$V(u, x, y) = \mathbb{E}_\xi r(u, \xi) + \gamma \mathbb{E}_\xi V(u + c(x) - d(u, y) - x\xi, x, y), \quad (28)$$

where $V(u, x, y) = 0$ for $u < 0$. Note that, in contrast to the standard Bellman equation, the right-hand side of (28) does not contain any supremum or infimum operations.

If $r(\cdot)$ is upper semicontinuous, $c(x)$ and $d(u, y)$ are continuous in their arguments, then $V(u, x, y)$ is upper semicontinuous, and for each fixed pair (x, y) , its values can be found by the method of successive approximations [26]:

$$V^{k+1}(u, x, y) = r(u) + \gamma \mathbb{E}_\xi V^k(u + c(x) - d(u, y) - \xi x, x, y), \quad V^0(u, x, y) = 0, \quad k = 0, 1, \dots \quad (29)$$

For example, if

$$r_0(X^t) = \begin{cases} d(X^t, y), & X^t \geq 0, \\ 0, & X^t < 0, \end{cases}$$

then the average discounted dividends until the ruin are expressed as follows

$$V_0(u, x, y) = \mathbb{E} \sum_{t=0}^{\tau-1} \gamma^t d(X^t, y),$$

and the function $V_0(\cdot, x, y)$ satisfies the equation

$$V_0(u, x, y) = d(u, y) + \gamma \mathbb{E}_\xi V_0(u + c(x) - d(u, y) - x\xi, x, y).$$

If

$$r_1(X^t) = \begin{cases} 1, & X^t \geq 0, \\ 0, & X^t < 0, \end{cases}$$

then the average discounted lifetime of the process

$$V_1(u, x, y) = \mathbb{E} \sum_{t=0}^{\tau-1} \gamma^t,$$

satisfies the equation

$$V_1(u, x, y) = 1 + \gamma \mathbb{E}_\xi V_1(u + c(x) - d(u, y) - x\xi, x, y);$$

If

$$r_2(X^t, \xi^t) = \begin{cases} \mathbf{1}_{\{X^t + c(x) - d(X^t, y) - \xi^t x < 0\}}, & X^t \geq 0, \\ 0, & X^t < 0, \end{cases}$$

then the (discounted) probability of ruin,

$$V_2(u, x, y) = \mathbb{E} \sum_{t=0}^{\tau-1} \gamma^t \mathbf{1}_{\{u + c(x) - d(u, y) - \xi^t x < 0\}}$$

satisfies the equation

$$\begin{aligned} V_2(u, x, y) &= \mathbb{E}_\xi \mathbf{1}_{\{u + c(x) - d(u, y) - \xi x < 0\}} + \gamma \mathbb{E}_\xi V_2(u + c(x) - d(u, y) - \xi x, x, y) = \\ &= (1 - F((u + c(x) - d(u, y)) / x) + \\ &\quad + \gamma \mathbb{E}_\xi V_2(u + c(x) - d(u, y) - \xi x, x, y), \end{aligned}$$

where $F(z) = \Pr\{\xi \leq z\}$.

Vector optimal control of risk processes. The task of vector optimal control of the risk process (22) is to search for non-dominated values of the vector indicator $\vec{V}(u, x, y) = \{V_i(u, x, y), i = 0, 1, \dots\}$:

$$\vec{V}(u, x, y) \rightarrow \text{extr}_{(u, x) \in W, y \in Y}, \quad (30)$$

as well as the corresponding Pareto-optimal values of parameters (u, x, y) , where u denotes the initial value of the risk process (22), x describes the structure of the insurance portfolio, and the parameter y is responsible for choosing a dividend strategy. For example, from heuristic

considerations, for the numerical approximation of the Pareto optimal boundary, the so-called barrier-proportional strategies for managing dividend payments are considered, i.e. $d(x, y) = y_1 \max\{0, x - y_2\}$, $y = (y_1, y_2)$, $y_1 \in [0, 1]$, $y_2 \geq 0$.

The complexity of this problem lies in the fact that, firstly, the indicators $V_i(u, x, y)$ themselves are not explicitly known but are solutions of the corresponding integral equations; secondly, these indicators can be non-convex functions; and thirdly, the corresponding Pareto-optimal set can have very complex structure.

The values of the functions $V_i(u, x, y)$ can be found by the method of successive approximations (29) or by the method of statistical simulations, in particular, by their parallel versions.

For small dimensions of the parameter vector (u, x, y) , problem (30) can be approximately solved by the discrete approximation of sets W, Y by finite sets W^N, Y^N and solving the discrete vector optimization problem:

$$\vec{V}(u, x, y) \rightarrow \text{extr}_{(u, x) \in W^N, y \in Y^N},$$

A similar method for solving vector optimization problems was considered in [27], its convergence was studied in [28 - 31].

Conclusions. The paper gives an overview of conceptual optimization models of the insurance business, based on the stochastic programming paradigm. The functioning of insurance companies is described by a random process of capital evolution in discrete time. The main random factors in evolution are insurance claims. The main characteristics of the functioning of companies are indicators of average efficiency, for example, expected profitability, and risk indicators, for example, the probability of insolvency or the amount of the required borrowed capital. One-, two-, and multi-stage models, as well as multi-criteria settings, are considered. The complexity of the resulting optimization problems is caused not only by the presence of random parameters but also by the non-convexity of the problems. Also, the performance indicators of insurance companies themselves are not explicitly known but are either multidimensional integrals such as mathematical expectations, or solutions to the integral equations of insurance mathematics. All these circumstances turn the task of optimizing insurance activity into a difficult computational problem. The paper proposes approaches to solving these problems based on the methods of stochastic programming, integer programming, multicriteria optimization, and dynamic programming.

References

1. Kaas, R., Goovaerts, M., Dhaene, J., Denuit M. (2001) . Modern Actuarial Risk Theory. Boston: Kluwer Academic Publishers, 2001. 309 p.
2. Bowers, N.L., Gerber, H.U., Hickman J.C., Jones, D.A., and Nesbitt, C.J. (1997). Actuarial mathematics. Chicago: Society of actuaries.
3. de Finetti, B. (1957). Su un' impostazione alternativa della teoria collettiva del rischio. Transactions of the XV-th International Congress of Actuaries 2. P. 433-443.
4. Beard, R.E., Pentikainen, T., and Pesonen, E. (1984). Risk theory. The stochastic basis of insurance. 3-rd ed. London, New York: Chapman and Hall, 1984. 408 p.
5. Asmussen, S., and Albrecher, H. (2010). Ruin probabilities. Second Edition. New Jersey, London: World Scientific. 602 p.
6. Korolev, V.Yu., Bening, B.E., Shorgin, S.Ya. Matematicheskiye osnovy teorii riska (Mathematical Foundations of the Risk Theory). 2-nd ed. Moscow, FIZMATLIT, 2011. 620 p. (In Russian).
7. Nakonechnyi, A.N. (1996). Optimization of risk processes. *Cybernetics and Systems Analysis*. Vol. 32, No. 5, pp. 641-646. DOI: <https://doi.org/10.1007/BF02367767>
8. Lyubchenko, G.I., and A. N. Nakonechnyi, A.N. (1998). Optimization methods for compound poisson risk processes. *Cybernetics and Systems Analysis*, Vol. 34, No. 2, pp. 230-237. DOI: <https://doi.org/10.1007/BF02742072>
9. Bayraktar, E., and Young, V. (2008). Maximizing utility of consumption subject to a constraint on the probability of lifetime ruin. *Finance and Research Letters*. Vol. 5(4), pp. 204-212. DOI: 10.1016/j.frl.2008.08.002
10. Schmidli, H. (2008). Stochastic control in insurance. London: Springer-Verlag. 254 p.
11. Norkin, B.V. (2011). Ob optimizatsii portfelya strakhovykh dogovorov (On optimizing the portfolio of insurance contracts). *Prikladna statistika. Aktuarna ta finansova matematika*. (Donetsk National University, Ukraine). No. 1-2, pp. 197-203. (In Russian). Access (31.05.2019): https://www.researchgate.net/publication/260869967_On_optimization_of_insurance_portfolio
12. Norkin, B.V. (2011). Mathematical models for insurance business optimization. *Cybernetics and Systems Analysis*, Vol. 47, No. 1, pp. 117-133. DOI: <https://doi.org/10.1007/s10559-011-9295-5>
13. Gromov, A.N. (2013). Optimal'naya strategiya perestrakhovaniya i investirovaniya (Optimal reinsurance and investment strategy). *Vestnik moskovskogo universiteta. Seriya 1, Matematika. Mekhanika*. Issue 2, pp. 6-12. Access (31.05.2019): <https://cyberleninka.ru/article/v/optimalnaya-strategiya-perestrahovaniya-i-investirovaniya>
14. Leonenko, M.M., Mishura, Yu.S., Parkhomenko, Ya.M., and Yadrenko, M.Y. (1995). Teoretyko-ymovirnisni ta statystychni metody v ekonometrytsi ta finansoviy matematytsi

(Theoretical-Probabilistic and Statistical Methods in Econometrics and Financial Mathematics). Kyiv: Informtehnika. 380 p. (In Ukrainian).

15. Shapiro, A., Dentcheva D., and Ruszczyński, A. (2009). Lectures on stochastic programming: Modeling and theory. Philadelphia: SIAM. 442 p.

16. Ermoliev, Yu.M. (1976). Metody stokhasticheskogo programmirovaniya (Methods of stochastic programming). Moscow: Nauka. 240 p. (In Russian).

17. Kan, Yu.S., Kibzun, A.N. (2009). Zadachi stokhasticheskogo programmirovaniya s veroyatnostnymi kriteriyami (Problems of stochastic programming with probabilistic criteria). Moscow: FIZMATLIT. 372 p.

18. Yudin, D.B. (1979). Zadachi i metody stokhasticheskogo programmirovaniya (Problems and methods of stochastic programming). Moscow: Sovetskoe radio. 392 p.

19. Powell W.B. Clearing the Jungle of Stochastic Optimization / W.B. Powell // In: *INFORMS Tutorials in Operations Research*. Published online: 27 Oct 2014; 109-137. DOI: <http://dx.doi.org/10.1287/educ.2014.0128>

20. Norkin, B.V. (2015). Chyslovi metody rozv'yazannya stokhastychnykh zadach optymizatsiyi u strakhoviy matematytsi (Numerical methods for solving stochastic optimization problems in insurance mathematics). *Avtoref. dys. ... doktora fiz.-mat. nauk* (An Extended Doctor Thesis). Kyiv: V.M.Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine. 36 p.

21. Ermoliev, Y.M., Ermolieva, T.Y., MacDonald, G., and Norkin, V.I. (2000). Stochastic Optimization of Insurance Portfolios for Managing Exposure to Catastrophic Risks. *Annals of Operations Research*. Vol. 99, pp. 207-225. DOI: <https://doi.org/10.1023/A:1019244405392>

22. FORINSURER. Online insurance magazine in Ukraine. Access: <http://forinsurer.com/>

23. Ruszczyński, A. (2002). Probabilistic programming with discrete distributions and precedence constrained knapsack polyhedral. *Math. Program.* Vol. 93, pp. 195-215. DOI: [10.1007/s10107-002-0337-7](https://doi.org/10.1007/s10107-002-0337-7)

24. Norkin, V.I., Boyko, S.V. (2012). Safety-first portfolio selection. *Cybernetics and Systems Analysis*. Vol. 48, No. 2, pp. 180-191. DOI: <https://doi.org/10.1007/s10559-012-9396-9>

25. Norkin, B.V. (2006). Necessary and sufficient conditions of existence and uniqueness of solutions to integral equations of actuarial mathematics. *Cybernetics and Systems Analysis*. Vol. 42, No. 5, pp. 743-749. DOI: <https://doi.org/10.1007/s10559-006-0113-4>

26. Norkin, B.V. (2014). Stochastic Optimal Control of Risk Processes with Lipschitz Payoff Functions. *Cybernetics and Systems Analysis*. Vol. 50, No. 5, pp. 774-787. DOI: <https://doi.org/10.1007/s10559-014-9668-7>

27. Sobol, I.M., Statnikov, R.B. (2006). Vybory optimal'nykh parametrov v zadachakh so mnogimi kriteriyami (The choice of optimal parameters in problems with many criteria). 2nd ed. Moscow: Drofa. 176 p.

28. Norikin, B.V. (2014). Random Search of Multicriterion Optimum in Insurance. Theoretical and Applied Aspects of Cybernetics. Proceedings of the 4-th International Scientific Conference of Students and Young Scientists (TAAC-2011. Kyiv, Ukraine, November 24-28, 2014). - Kyiv: Bukrek. P.176-187. Access (31.05.2019):

https://www.researchgate.net/publication/301764924_Random_Search_of_Multicriterion_Optimum_in_Insurance

29. Norikin, B.V. (2014) Sample approximations of multiobjective stochastic optimization problems [Electronic publication]. *Optimization online*, November 2014. Access (31.05.2019): http://www.optimization-online.org/DB_HTML/2014/11/4655.html

30. Norikin, B.V. (2015). Statistical approximation of multicriteria stochastic optimization problems. *Dopovidi Nacionalnoi akademii nauk Ukrainy (Proceedings of the National Academy of Sciences of Ukraine)*. No. 4, pp. 35-41. DOI: <https://doi.org/10.15407/dopovidi2015.04.035>

31. Norikin, B.V. (2015). On the approximation of vector optimization problems. *Kibernetika i vychislitel'naya tekhnika (Cybernetics and computing technics, Kyiv, Ukraine)*. Issue 179, pp. 35-42. Access (31.05.2019): <http://dspace.nbuv.gov.ua/bitstream/handle/123456789/86145/03-Norkin.pdf?sequence=1>