Chapter 15 Collision Theory

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> Despite my resistance to hyperbole, the LHC [Large Hadron Collider] belongs to a world that can only be described with superlatives. It is not merely large: the LHC is the biggest machine ever built. It is not merely cold: the 1.9 kelvin (1.9 degrees Celsius above absolute zero) temperature necessary for the LHC's supercomputing magnets to operate is the coldest extended region that we know of in the universe—even colder than outer space. The magnetic field is not merely big: the superconducting dipole magnets generating a magnetic field more than 100,000 times stronger than the Earth's are the strongest magnets in industrial production ever made.
>
> And the extremes don't end there. The vacuum inside the proton-containing tubes, a 10 trillionth of an atmosphere, is the most complete vacuum over the largest region ever produced. The energy of the collisions are the highest ever generated on Earth, allowing us to study the interactions that occurred in the early universe the furthest back in time.[1]
>
> *Lisa Randall*

15.1 Introduction

When discussing conservation of momentum, we considered examples in which two objects collide and stick together, and either there are no external forces acting in some direction (or the collision was nearly instantaneous) so the component of the momentum of the system along that direction is constant. We shall now study collisions between objects in more detail. In particular we shall consider cases in which the objects do not stick together. The momentum along a certain direction may still be constant but the mechanical energy of the system may change. We will begin our analysis by considering two-particle collision. We introduce the concept of the relative velocity between two particles and show that it is independent of the choice of reference frame. We then show that the change in kinetic energy only depends on the change of the square of the relative velocity and therefore is also independent of the choice of reference frame. We will then study one- and two-dimensional collisions with zero change in potential energy. In particular we will characterize the types of collisions by the change in kinetic energy and analyze the possible outcomes of the collisions.

15.2 Reference Frames and Relative Velocities

We shall recall our definition of relative inertial reference frames. Let $\vec{\mathbf{R}}$ be the vector from the origin of frame S to the origin of reference frame S'. Denote the

[1] Randall, Lisa, *Knocking on Heaven's Door: How Physics and Scientific Thinking Illuminate the Universe and the Modern World*, Ecco, 2011.

position vector of the j^{th} particle with respect to the origin of reference frame S by $\vec{\mathbf{r}}_j$ and similarly, denote the position vector of the j^{th} particle with respect to the origin of reference frame S' by $\vec{\mathbf{r}}'_j$ (Figure 15.1).

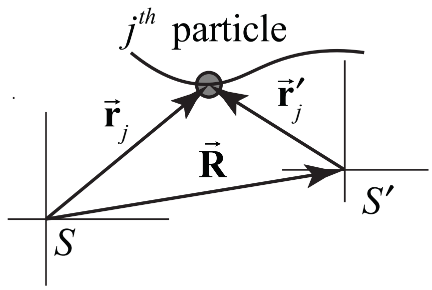


Figure 15.1 Position vector of j^{th} particle in two reference frames.

The position vectors are related by

$$\vec{\mathbf{r}}_j = \vec{\mathbf{r}}'_j + \vec{\mathbf{R}} . \tag{15.2.1}$$

The relative velocity (call this the *boost velocity*) between the two reference frames is given by

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt} . \tag{15.2.2}$$

Assume the boost velocity between the two reference frames is constant. Then, the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}} . \tag{15.2.3}$$

When Eq. (15.2.3) is satisfied, the reference frames S and S' are called *relatively inertial reference frames.*

Suppose the j^{th} particle in Figure 15.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of j^{th} particle in frame S by $\vec{\mathbf{v}}_j = d\vec{\mathbf{r}}_j / dt$, and the velocity of the same particle in frame S' by $\vec{\mathbf{v}}'_j = d\vec{\mathbf{r}}'_j / dt$. Taking derivative, the velocities of the particles in two different reference frames are related according to

$$\vec{\mathbf{v}}_j = \vec{\mathbf{v}}'_j + \vec{\mathbf{V}} . \tag{15.2.4}$$

15.2.1 Relative Velocities

Consider two particles of masses m_1 and m_2 interacting via some force (Figure 15.2).

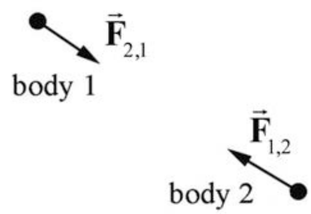


Figure 15.2 Two interacting particles

Choose a coordinate system (Figure 15.3) in which the position vector of body 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of body 2 is given by $\vec{\mathbf{r}}_2$. The *relative position* of body 1 with respect to body 2 is given by $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$.

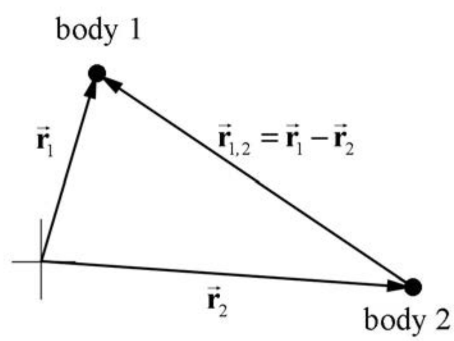


Figure 15.3 Coordinate system for two bodies.

During the course of the interaction, body 1 is displaced by $d\vec{\mathbf{r}}_1$ and body 2 is displaced by $d\vec{\mathbf{r}}_2$, so the ***relative displacement*** of the two bodies during the interaction is given by $d\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$. The ***relative velocity*** between the particles is

$$\vec{\mathbf{v}}_{1,2} = \frac{d\vec{\mathbf{r}}_{1,2}}{dt} = \frac{d\vec{\mathbf{r}}_1}{dt} - \frac{d\vec{\mathbf{r}}_2}{dt} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 . \qquad (15.2.5)$$

We shall now show that the relative velocity between the two particles is independent of the choice of reference frame providing that the reference frames are relatively inertial. The relative velocity $\vec{\mathbf{v}}'_{12}$ in reference frame S' can be determined from using Eq. (15.2.4) to express Eq. (15.2.5) in terms of the velocities in the reference frame S',

$$\vec{\mathbf{v}}_{1,2} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = (\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}) - (\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}) = \vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}'_{1,2} \qquad (15.2.6)$$

and is equal to the relative velocity in frame S.

For a two-particle interaction, the relative velocity between the two vectors is independent of the choice of relatively inertial reference frames.

15.2.2 Center-of-mass Reference Frame

Let $\vec{\mathbf{r}}_{cm}$ be the vector from the origin of frame S to the center-of-mass of the system of particles, a point that we will choose as the origin of reference frame S_{cm}, called the ***center-of-mass reference frame***. Denote the position vector of the j^{th} particle with respect to origin of reference frame S by $\vec{\mathbf{r}}_j$ and similarly, denote the position vector of the j^{th} particle with respect to origin of reference frame S_{cm} by $\vec{\mathbf{r}}'_j$ (Figure 15.4).

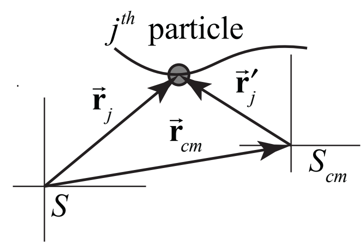


Figure 15.4 Position vector of j^{th} particle in the center-of-mass reference frame.

The position vector of the j^{th} particle in the center-of-mass frame is then given by

$$\vec{\mathbf{r}}'_j = \vec{\mathbf{r}}_j - \vec{\mathbf{r}}_{cm}. \tag{15.2.7}$$

The velocity of the j^{th} particle in the center-of-mass reference frame is then given by

$$\vec{\mathbf{v}}'_j = \vec{\mathbf{v}}_j - \vec{\mathbf{v}}_{cm}. \tag{15.2.8}$$

There are many collision problems in which the center-of-mass reference frame is the most convenient reference frame to analyze the collision.

Consider a system consisting of two particles, which we shall refer to as particle 1 and particle 2. We can use Eq. (15.2.8) to determine the velocities of particles 1 and 2 in the center-of-mass,

$$\vec{\mathbf{v}}'_1 = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_{cm} = \vec{\mathbf{v}}_1 - \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}(\vec{\mathbf{v}}_{1,} - \vec{\mathbf{v}}_2) = \frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2}. \tag{15.2.9}$$

where $\vec{\mathbf{v}}_{12} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$ is the relative velocity of particle 1 with respect to particle 2. A similar result holds for particle 2:

$$\vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_{cm} = \vec{\mathbf{v}}_2 - \frac{m_1\vec{\mathbf{v}}_1 + m_2\vec{\mathbf{v}}_2}{m_1 + m_2} = -\frac{m_1}{m_1 + m_2}(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2) = -\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}. \quad (15.2.10)$$

The momentum of the system the center-of-mass reference frame is zero as we expect,

$$m_1\vec{\mathbf{v}}'_1 + m_2\vec{\mathbf{v}}'_2 = \mu\vec{\mathbf{v}}_{12} - \mu\vec{\mathbf{v}}_{12} = \vec{\mathbf{0}}. \quad (15.2.11)$$

15.2.3 Kinetic Energy in the Center-of-Mass Reference Frame

The kinetic energy in the center of mass reference frame is given by

$$K_{cm} = \frac{1}{2}m_1\vec{\mathbf{v}}'_1 \cdot \vec{\mathbf{v}}'_1 + \frac{1}{2}m_2\vec{\mathbf{v}}'_2 \cdot \vec{\mathbf{v}}'_2. \quad (15.2.12)$$

We now use Eqs. (15.2.9) and (15.2.10) to rewrite the kinetic energy in terms of the relative velocity $\vec{\mathbf{v}}'_{12} = \vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2$,

$$\begin{aligned} K_{cm} &= \frac{1}{2}m_1\left(\frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2}\right)\cdot\left(\frac{\mu}{m_1}\vec{\mathbf{v}}_{1,2}\right) + \frac{1}{2}m_2\left(-\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}\right)\cdot\left(-\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,2}\right) \\ &= \frac{1}{2}\mu^2\vec{\mathbf{v}}_{1,2}\cdot\vec{\mathbf{v}}_{1,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{1}{2}\mu v_{1,2}^{\ 2} \end{aligned}. \quad (15.2.13)$$

where we used the fact that we defined the reduced mass by

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2}. \quad (15.2.14)$$

15.2.4 Change of Kinetic Energy and Relatively Inertial Reference Frames

The kinetic energy of the two particles in reference frame S is given by

$$K_S = \frac{1}{2}m_1 v_1^{\ 2} + \frac{1}{2}m_2 v_2^{\ 2}. \quad (15.2.15)$$

We can take the scalar product of Eq. (15.2.8) to rewrite Eq. (15.2.15) as

$$K_S = \frac{1}{2}m_1(\vec{\mathbf{v}}_1' + \vec{\mathbf{v}}_{cm})\cdot(\vec{\mathbf{v}}_1' + \vec{\mathbf{v}}_{cm}) + \frac{1}{2}m_2(\vec{\mathbf{v}}_2' + \vec{\mathbf{v}}_{cm})\cdot(\vec{\mathbf{v}}_2' + \vec{\mathbf{v}}_{cm})$$
$$= \frac{1}{2}m_1 v_1'^{\,2} + \frac{1}{2}m_2 v_2'^{\,2} + \frac{1}{2}(m_1 + m_2)v_{cm}^{\;2} + (m_1\vec{\mathbf{v}}_1' + m_2\vec{\mathbf{v}}_2')\cdot\vec{\mathbf{v}}_{cm} \quad . \qquad (15.2.16)$$

The last term is zero due to the fact that the momentum of the system in the center of mass reference frame is zero (Eq. (15.2.11)). Therefore Eq. (15.2.16) becomes

$$K_S = \frac{1}{2}m_1 v_1'^{\,2} + \frac{1}{2}m_2 v_2'^{\,2} + \frac{1}{2}(m_1 + m_2)v_{cm}^{\;2} \, . \qquad (15.2.17)$$

The first two terms correspond to the kinetic energy in the center of mass frame, thus the kinetic energies in the two reference frames are related by

$$K_S = K_{cm} + \frac{1}{2}(m_1 + m_2)v_{cm}^{\;2} \ . \qquad (15.2.18)$$

We now use Eq. (15.2.13) to rewrite Eq. (15.2.18) as

$$K_S = \frac{1}{2}\mu v_{1,2}^{\;2} + \frac{1}{2}(m_1 + m_2)v_{cm}^{\;2} \qquad (15.2.19)$$

Even though kinetic energy is a reference frame dependent quantity, because the second term in Eq. (15.2.19) is a constant, the change in kinetic energy in either reference frame is equal to

$$\Delta K = \frac{1}{2}\mu\left(\left(v_{1,2}^{\;2}\right)_f - \left(v_{1,2}^{\;2}\right)_i\right) \ . \qquad (15.2.20)$$

This generalizes to any two relatively inertial reference frames because the relative velocity is a reference frame independent quantity,

> *the change in kinetic energy is independent of the choice of relatively inertial reference frames.*

We showed in Appendix 13A that when two particles of masses m_1 and m_2 interact, the work done by the interaction force is equal to

$$W = \frac{1}{2}\mu\left(\left(v_{1,2}^{\;2}\right)_f - \left(v_{1,2}^{\;2}\right)_i\right). \qquad (15.2.21)$$

Hence we explicitly verified that for our two-particle system

$$W = \Delta K_{sys} \, . \qquad (15.2.22)$$

15.3 Characterizing Collisions

In a collision, the ratio of the magnitudes of the initial and final relative velocities is called the coefficient of restitution and denoted by the symbol e,

$$e = \frac{v_B}{v_A}. \tag{15.3.1}$$

If the magnitude of the relative velocity does not change during a collision, $e = 1$, then the change in kinetic energy is zero, (Eq. (15.2.21)). Collisions in which there is no change in kinetic energy are called ***elastic collisions***,

$$\Delta K = 0, \quad \text{elastic collision}. \tag{15.3.2}$$

If the magnitude of the final relative velocity is less than the magnitude of the initial relative velocity, $e < 1$, then the change in kinetic energy is negative. Collisions in which the kinetic energy decreases are called ***inelastic collisions***,

$$\Delta K < 0, \ \text{inelastic collision}. \tag{15.3.3}$$

If the two objects stick together after the collision, then the relative final velocity is zero, $e = 0$. Such collisions are called ***totally inelastic***. The change in kinetic energy can be found from Eq. (15.2.21),

$$\Delta K = -\frac{1}{2}\mu v_A^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_A^2, \quad \text{totally inelastic collision}. \tag{15.3.4}$$

If the magnitude of the final relative velocity is greater than the magnitude of the initial relative velocity, $e > 1$, then the change in kinetic energy is positive. Collisions in which the kinetic energy increases are called ***superelastic collisions***,

$$\Delta K > 0, \ \text{superelastic collision}. \tag{15.3.5}$$

15.4 One-Dimensional Collisions Between Two Objects

15.4.1 One Dimensional Elastic Collision in Laboratory Reference Frame

Consider a one-dimensional elastic collision between two objects moving in the x-direction. One object, with mass m_1 and initial x-component of the velocity $v_{1x,i}$, collides with an object of mass m_2 and initial x-component of the velocity $v_{2x,i}$. The scalar components $v_{1x,i}$ and $v_{1x,i}$ can be positive, negative or zero. No forces other than the interaction force between the objects act during the collision. After the collision, the final x-component of the velocities are $v_{1x,f}$ and $v_{2x,f}$. We call this reference frame the "laboratory reference frame".

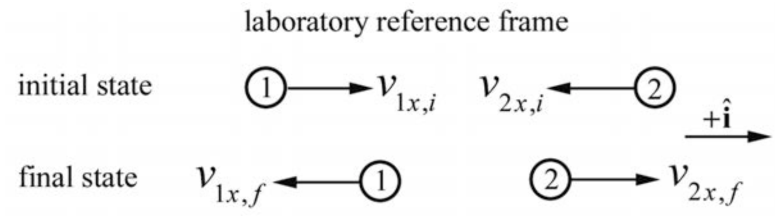


Figure 15.5 One-dimensional elastic collision, laboratory reference frame

For the collision depicted in Figure 15.5, $v_{1x,i} > 0$, $v_{2x,i} < 0$, $v_{1x,f} < 0$, and $v_{2x,f} > 0$. Because there are no external forces in the x-direction, momentum is constant in the x-direction. Equating the momentum components before and after the collision gives the relation

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 v_{2x,f} . \tag{15.4.1}$$

Because the collision is elastic, kinetic energy is constant. Equating the kinetic energy before and after the collision gives the relation

$$\frac{1}{2} m_1 v_{1x,i}^2 + \frac{1}{2} m_2 v_{2x,i}^2 = \frac{1}{2} m_1 v_{1x,f}^2 + \frac{1}{2} m_2 v_{2x,f}^2 \tag{15.4.2}$$

Rewrite these Eqs. (15.4.1) and (15.4.2) as

$$m_1 (v_{1x,i} - v_{1x,f}) = m_2 (v_{2x,f} - v_{2x,i}) \tag{15.4.3}$$

$$m_1 (v_{1x,i}^2 - v_{1x,f}^2) = m_2 (v_{2x,f}^2 - v_{2x,i}^2) . \tag{15.4.4}$$

Eq. (15.4.4) can be written as

$$m_1 (v_{1x,i} - v_{1x,f})(v_{1x,i} + v_{1x,f}) = m_2 (v_{2x,f} - v_{2x,i})(v_{2x,f} + v_{2x,i}) . \tag{15.4.5}$$

Divide Eq. (15.4.4) by Eq. (15.4.3), yielding

$$v_{1x,i} + v_{1x,f} = v_{2x,i} + v_{2x,f} . \tag{15.4.6}$$

Eq. (15.4.6) may be rewritten as

$$v_{1x,i} - v_{2x,i} = v_{2x,f} - v_{1x,f} . \tag{15.4.7}$$

Recall that the relative velocity between the two objects is defined to be

$$\vec{\mathbf{v}}^{\text{rel}} \equiv \vec{\mathbf{v}}_{1,2} \equiv \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 . \tag{15.4.8}$$

where we used the superscript "rel" to remind ourselves that the velocity is a relative velocity (and to simplify our notation). Thus $v_{x,i}^{\text{rel}} = v_{1x,i} - v_{2x,i}$ is the initial x-component of the relative velocity, and $v_{x,f}^{\text{rel}} = v_{1x,f} - v_{2x,f}$ is the final x-component of the relative velocity. Therefore Eq. (15.4.7) states that during the interaction the initial relative velocity is equal to the negative of the final relative velocity

$$\vec{\mathbf{v}}_i^{\text{rel}} = -\vec{\mathbf{v}}_f^{\text{rel}}, \quad (1-\text{dimensional energy-momentum prinicple}). \tag{15.4.9}$$

Consequently the initial and final relative speeds are equal. We shall call this relationship between the relative initial and final velocities the **one-dimensional energy-momentum principle** because we have combined these two principles to realize this result. The energy-momentum principle is independent of the masses of the colliding particles.

Although we derived this result explicitly, we have already shown that the change in kinetic energy for a two-particle interaction (Eq. (15.2.20)), in our simplified notation is given by

$$\Delta K = \frac{1}{2}\mu((v^{\text{rel}})_f^2 - (v^{\text{rel}})_i^2) \tag{15.4.10}$$

Therefore for an elastic collision where $\Delta K = 0$, the square of the relative speed remains constant

$$(v^{\text{rel}})_f^2 = (v^{\text{rel}})_i^2 \; . \tag{15.4.11}$$

For a one-dimensional collision, the magnitude of the relative speed remains constant but the direction changes by 180°.

We can now solve for the final x-component of the velocities, $v_{1x,f}$ and $v_{2x,f}$, as follows. Eq. (15.4.7) may be rewritten as

$$v_{2x,f} = v_{1x,f} + v_{1x,i} - v_{2x,i} \, . \tag{15.4.12}$$

Now substitute Eq. (15.4.12) into Eq. (15.4.1) yielding

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 (v_{1x,f} + v_{1x,i} - v_{2x,i}) \, . \tag{15.4.13}$$

Solving Eq. (15.4.13) for $v_{1x,f}$ involves some algebra and yields

$$v_{1x,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1x,i} + \frac{2\,m_2}{m_1 + m_2} v_{2x,i} \, . \tag{15.4.14}$$

To find $v_{2x,f}$, rewrite Eq. (15.4.7) as

$$v_{1x,f} = v_{2x,f} - v_{1x,i} + v_{2x,i}\,. \tag{15.4.15}$$

Now substitute Eq. (15.4.15) into Eq. (15.4.1) yielding

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 (v_{2x,f} - v_{1x,i} + v_{2x,i}) v_{1x,f} + m_2 v_{2x,f}\,. \tag{15.4.16}$$

We can solve Eq. (15.4.16) for $v_{2x,f}$ and determine that

$$v_{2x,f} = v_{2x,i}\frac{m_2 - m_1}{m_2 + m_1} + v_{1x,i}\frac{2\,m_1}{m_2 + m_1}\,. \tag{15.4.17}$$

Consider what happens in the limits $m_1 >> m_2$ in Eq. (15.4.14). Then

$$v_{1x,f} \to v_{1x,i} + \frac{2}{m_1} m_2 v_{2x,i}\,; \tag{15.4.18}$$

the more massive object's velocity component is only slightly changed by an amount proportional to the less massive object's x-component of momentum. Similarly, the less massive object's final velocity approaches

$$v_{2x,f} \to -v_{2x,i} + 2v_{1x,i} = v_{1x,i} + v_{1x,i} - v_{2x,i}\,. \tag{15.4.19}$$

We can rewrite this as

$$v_{2x,f} - v_{1x,i} = v_{1x,i} - v_{2x,i} = v_{x,i}^{\text{rel}}. \tag{15.4.20}$$

i.e. the less massive object "rebounds" with the same speed relative to the more massive object which barely changed its speed.

If the objects are identical, or have the same mass, Eqs. (15.4.14) and (15.4.17) become

$$v_{1x,f} = v_{2x,i}, \quad v_{2x,f} = v_{1x,i}\,; \tag{15.4.21}$$

the objects have exchanged x-components of velocities, and unless we could somehow distinguish the objects, we might not be able to tell if there was a collision at all.

15.4.2 One-Dimensional Collision Between Two Objects – Center-of-Mass Reference Frame

We analyzed the one-dimensional elastic collision (Figure 15.5) in Section 15.4.1 in the laboratory reference frame. Now let's view the collision from the center-of-mass (CM) frame. The x-component of velocity of the center-of-mass is

$$v_{x,\mathrm{cm}} = \frac{m_1 v_{1x,i} + m_2 v_{2x,i}}{m_1 + m_2}. \tag{15.4.22}$$

With respect to the center-of-mass, the x-components of the velocities of the objects are

$$\begin{aligned} v'_{1x,i} &= v_{1x,i} - v_{x,\mathrm{cm}} = (v_{1x,i} - v_{2x,i})\frac{m_2}{m_1 + m_2} \\ v'_{2x,i} &= v_{2x,i} - v_{x,\mathrm{cm}} = (v_{2x,i} - v_{1x,i})\frac{m_1}{m_1 + m_2}. \end{aligned} \tag{15.4.23}$$

In the CM frame the momentum of the system is zero before the collision and hence the momentum of the system is zero after the collision. For an elastic collision, the only way for both momentum and kinetic energy to be the same before and after the collision is either the objects have the same velocity (a miss) or to reverse the direction of the velocities as shown in Figure 15.6.

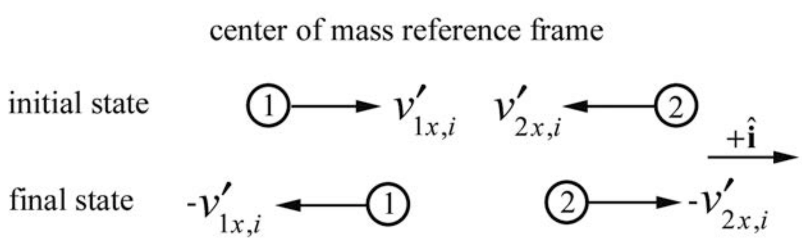


Figure 15.6 One-dimensional elastic collision in center-of-mass reference frame

In the CM frame, the final x-components of the velocities are

$$\begin{aligned} v'_{1x,f} &= -v'_{1x,i} = (v_{2x,i} - v_{1x,i})\frac{m_2}{m_1 + m_2} \\ v'_{2x,f} &= -v'_{2x,i} = (v_{2x,i} - v_{1x,i})\frac{m_1}{m_1 + m_2}. \end{aligned} \tag{15.4.24}$$

The final x-components of the velocities in the "laboratory frame" are then given by

$$v_{1x,f} = v'_{1x,f} + v_{x,\text{cm}}$$
$$= (v_{2x,i} - v_{1x,i})\frac{m_2}{m_1 + m_2} + \frac{m_1 v_{1x,i} + m_2 v_{2x,i}}{m_1 + m_2} \qquad (15.4.25)$$
$$= v_{1x,i}\frac{m_1 - m_2}{m_1 + m_2} + v_{2x,i}\frac{2m_2}{m_1 + m_2}$$

as in Eq. (15.4.14) and a similar calculation reproduces Eq. (15.4.17).

15.5 Worked Examples

Example 15.1 Elastic One-Dimensional Collision Between Two Objects

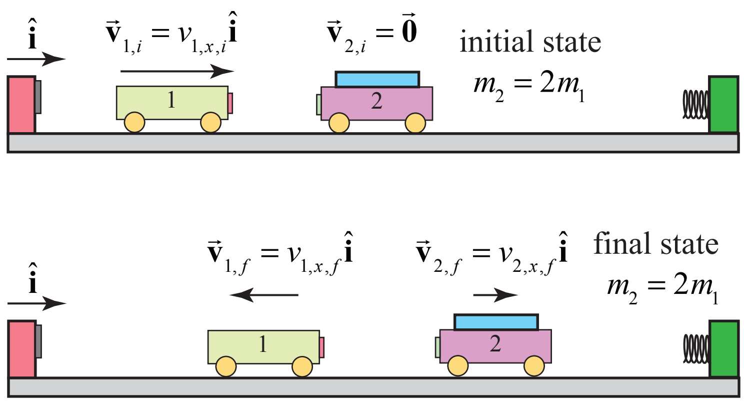


Figure 15.7 Elastic collision between two non-identical carts

Consider the elastic collision of two carts along a track; the incident cart 1 has mass m_1 and moves with initial speed $v_{1,i}$. The target cart has mass $m_2 = 2m_1$ and is initially at rest, $v_{2,i} = 0$, (Figure 15.7). Immediately after the collision, the incident cart has final speed $v_{1,f}$ and the target cart has final speed $v_{2,f}$. Calculate the final x-component of the velocities of the carts as a function of the initial speed $v_{1,i}$.

Solution The momentum flow diagram for the objects before (initial state) and after (final state) the collision are shown in Figure 15.7. We can immediately use our results above with $m_2 = 2m_1$ and $v_{2,i} = 0$. The final x-component of velocity of cart 1 is given by Eq. (15.4.14), where we use $v_{1x,i} = v_{1,i}$

$$v_{1x,f} = -\frac{1}{3}v_{1,i}\,. \tag{15.5.1}$$

The final x-component of velocity of cart 2 is given by Eq. (15.4.17)

$$v_{2x,f} = \frac{2}{3}v_{1,i}\,. \tag{15.5.2}$$

Example 15.2 The Dissipation of Kinetic Energy in a Completely Inelastic Collision Between Two Objects

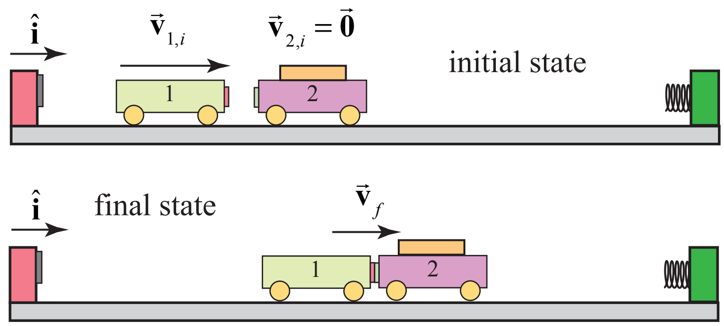


Figure 15.7b Inelastic collision between two non-identical carts

An incident cart of mass m_1 and initial speed $v_{1,i}$ collides completely inelastically with a cart of mass m_2 that is initially at rest (Figure 15.7b). There are no external forces acting on the objects in the direction of the collision. Find $\Delta K / K_{\text{initial}} = (K_{\text{final}} - K_{\text{initial}}) / K_{\text{initial}}$.

Solution: In the absence of any net force on the system consisting of the two carts, the momentum after the collision will be the same as before the collision. After the collision the carts will move in the direction of the initial velocity of the incident cart with a common speed v_f found from applying the momentum condition

$$\begin{gathered} m_1 v_{1,i} = (m_1 + m_2)v_f \Rightarrow \\ v_f = \frac{m_1}{m_1 + m_2}v_{1,i}\,. \end{gathered} \tag{15.5.3}$$

The initial relative speed is $v_i^{\text{rel}} = v_{1,i}$. The final relative velocity is zero because the carts stick together so using Eq. (15.3.4), the change in kinetic energy is

$$\Delta K = -\frac{1}{2}\mu(v_i^{rel})^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_{1,i}^2\,. \tag{15.5.4}$$

The ratio of the change in kinetic energy to the initial kinetic energy is then

$$\Delta K / K_{\text{initial}} = -\frac{m_2}{m_1 + m_2}. \qquad (15.5.5)$$

As a check, we can calculate the change in kinetic energy via

$$\begin{aligned}\Delta K = (K_f - K_i) &= \frac{1}{2}(m_1 + m_2)v_f^2 - \frac{1}{2}v_{1,i}^2 \\ &= \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}\right)^2 v_{1,i}^2 - \frac{1}{2}v_{1,i}^2 \\ &= \left(\frac{m_1}{m_1 + m_2} - 1\right)\left(\frac{1}{2}m_1 v_{1,i}^2\right) = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_{1,i}^2.\end{aligned} \qquad (15.5.6)$$

in agreement with Eq. (15.5.4).

Example 15.3 Bouncing Superballs

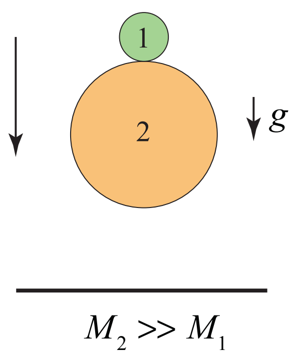
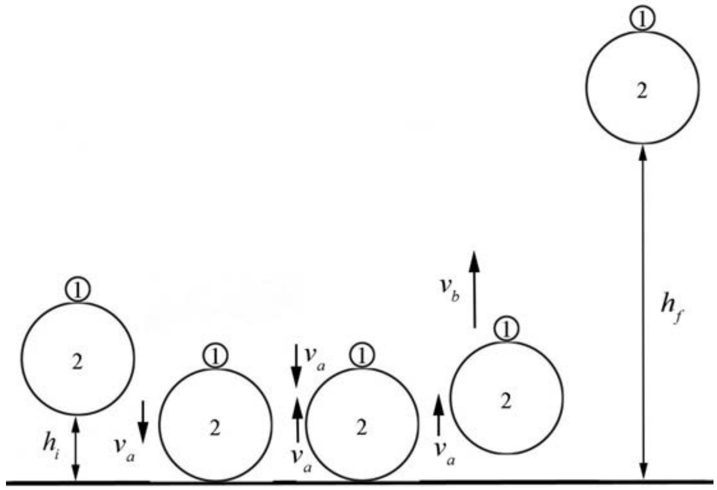


Figure 15.8b Two superballs dropping

Consider two balls that are dropped from a height h_i above the ground, one on top of the other (Figure 15.8). Ball 1 is on top and has mass M_1, and ball 2 is underneath and has mass M_2 with $M_2 >> M_1$. Assume that there is no loss of kinetic energy during all collisions. Ball 2 first collides with the ground and rebounds. Then, as ball 2`starts to move upward, it collides with the ball 1 which is still moving downwards (figure below left). How high will ball 1 rebound in the air? Hint: consider this collision as seen by an observer moving upward with the same speed as the ball 2 has after it collides with ground. What speed does ball 1 have in this reference frame after it collides with the ball 2?

Solution

The system consists of the two balls and the earth. There are five special states for this motion shown in the figure below.



part a)

Initial State: the balls are released from rest at a height h_i above the ground.

State A: the balls just reach the ground with speed $v_a = \sqrt{2gh_i}$. This follows from $\Delta E_{mech} = 0 \Rightarrow \Delta K = -\Delta U$. Thus $(1/2)mv_a^2 - 0 = -mg\Delta h = mgh_i \Rightarrow v_a = \sqrt{2gh_i}$.

State B: immediately before the collision of the balls. Ball 2 has collided with the ground and reversed direction with the same speed, v_a, but ball 1 is still moving downward with speed v_a.

State C: immediately after the collision of the balls. Because we are assuming that $m_2 >> m_1$, ball 2 does not change its speed as a result of the collision so it is still moving upward with speed v_a. As a result of the collision, ball 1 moves upward with speed v_b.

Final State: ball 1 reaches a maximum height $h_f = v_b{}^2 / 2g$ above the ground. This again follows from $\Delta K = -\Delta U \Rightarrow 0 - (1/2)mv_b^2 = -mg\Delta h = -mgh_f \Rightarrow h_f = v_b^2 / 2g$.

Choice of Reference Frame:

As indicated in the hint above, this collision is best analyzed from the reference frame of an observer moving upward with speed v_a, the speed of ball 2 just after it rebounded with the ground. In this frame immediately, before the collision, ball 1 is moving downward with a speed v'_b that is twice the speed seen by an observer at rest on the ground (lab reference frame).

$$v'_a = 2v_a \tag{15.5.7}$$

The mass of ball 2 is much larger than the mass of ball 1, $m_2 >> m_1$. This enables us to consider the collision (between States B and C) to be equivalent to ball 1 bouncing off a hard wall, while ball 2 experiences virtually no recoil. Hence ball 2 remains at rest in the reference frame moving upwards with speed v_a with respect to observer at rest on ground. Before the collision, ball 1 has speed $v'_a = 2v_a$. Since there is no loss of kinetic energy during the collision, the result of the collision is that ball 1 changes direction but maintains the same speed,

$$v'_b = 2v_a\,. \tag{15.5.8}$$

However, according to an observer at rest on the ground, after the collision ball 1 is moving upwards with speed

$$v_b = 2v_a + v_a = 3v_a\,. \tag{15.5.9}$$

While rebounding, the mechanical energy of the smaller superball is constant (we consider the smaller superball and the Earth as a system) hence between State C and the Final State,

$$\Delta K + \Delta U = 0\,. \tag{15.5.10}$$

The change in kinetic energy is

$$\Delta K = -\frac{1}{2}m_1(3v_a)^2\,. \tag{15.5.11}$$

The change in potential energy is

$$\Delta U = m_1 g h_f\,. \tag{15.5.12}$$

So the condition that mechanical energy is constant (Equation (15.5.10)) is now

$$-\frac{1}{2}m_1(3v_{1a})^2 + m_1 g h_f = 0\,. \tag{15.5.13}$$

We can rewrite Equation (15.5.13) as

$$m_1 g h_f = 9\frac{1}{2}m_1(v_a)^2\,. \tag{15.5.14}$$

Recall that we can also use the fact that the mechanical energy doesn't change between the Initial State and State A yielding an equation similar to Eq. (15.5.14),

$$m_1 g h_i = \frac{1}{2} m_1 (v_a)^2 . \qquad (15.5.15)$$

Now substitute the expression for the kinetic energy in Eq. (15.5.15) into Eq. (15.5.14) yielding

$$m_1 g h_f = 9 m_1 g h_i . \qquad (15.5.16)$$

Thus ball 1 reaches a maximum height

$$h_f = 9 h_i . \qquad (15.5.17)$$

15.6 Two Dimensional Elastic Collisions

15.6.1 Two-dimensional Elastic Collision in Laboratory Reference Frame

Consider the elastic collision between two particles in which we neglect any external forces on the system consisting of the two particles. Particle 1 of mass m_1 is initially moving with velocity $\vec{\mathbf{v}}_{1,i}$ and collides elastically with a particle 2 of mass m_2 that is initially at rest. We shall refer to the reference frame in which one particle is at rest, 'the target', as the **laboratory reference frame**. After the collision particle 1 moves with velocity $\vec{\mathbf{v}}_{1,f}$ and particle 2 moves with velocity $\vec{\mathbf{v}}_{2,f}$, (Figure 15.9). The angles $\theta_{1,f}$ and $\theta_{2,f}$ that the particles make with the positive forward direction of particle 1 are called the **laboratory scattering angles**.

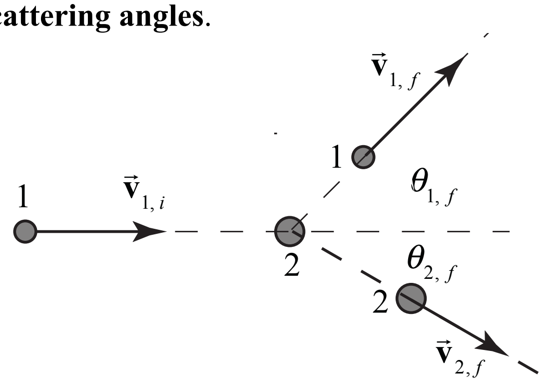


Figure 15.9 Two-dimensional collision in laboratory reference frame

Generally the initial velocity $\vec{\mathbf{v}}_{1,i}$ of particle 1 is known and we would like to determine the final velocities $\vec{\mathbf{v}}_{1,f}$ and $\vec{\mathbf{v}}_{2,f}$, which requires finding the magnitudes and directions of each of these vectors, $v_{1,f}$, $v_{2,f}$, $\theta_{1,f}$, and $\theta_{2,f}$. These quantities are related by the two equations describing the constancy of momentum, and the one equation describing constancy of the kinetic energy. Therefore there is one degree of freedom that we must specify in order to determine the outcome of the collision. In what follows we shall express our results for $v_{1,f}$, $v_{2,f}$, and $\theta_{2,f}$ in terms of $v_{1,i}$ and $\theta_{1,f}$.

The components of the total momentum $\vec{\mathbf{p}}_i^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,i} + m_2\vec{\mathbf{v}}_{2,i}$ in the initial state are given by

$$\begin{aligned} p_{x,i}^{\text{sys}} &= m_1 v_{1,i} \\ p_{y,i}^{\text{sys}} &= 0. \end{aligned} \tag{15.6.1}$$

The components of the momentum $\vec{\mathbf{p}}_f^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f}$ in the final state are given by

$$\begin{aligned} p_{x,f}^{\text{sys}} &= m_1 v_{1,f}\cos\theta_{1,f} + m_2 v_{2,f}\cos\theta_{2,f} \\ p_{y,f}^{\text{sys}} &= m_1 v_{1,f}\sin\theta_{1,f} - m_2 v_{2,f}\sin\theta_{2,f}. \end{aligned} \tag{15.6.2}$$

There are no any external forces acting on the system, so each component of the total momentum remains constant during the collision,

$$p_{x,i}^{\text{sys}} = p_{x,f}^{\text{sys}} \tag{15.6.3}$$

$$p_{y,i}^{\text{sys}} = p_{y,f}^{\text{sys}}. \tag{15.6.4}$$

Eqs. (15.6.3) and (15.6.4) become

$$m_1 v_{1,i} = m_1 v_{1,f}\cos\theta_{1,f} + m_2 v_{2,f}\cos\theta_{2,f}, \tag{15.6.5}$$

$$0 = m_1 v_{1,f}\sin\theta_{1,f} - m_2 v_{2,f}\sin\theta_{2,f}. \tag{15.6.6}$$

The collision is elastic and therefore the system kinetic energy of is constant

$$K_i^{\text{sys}} = K_f^{\text{sys}}. \tag{15.6.7}$$

Using the given information, Eq. (15.6.7) becomes

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_2 v_{2,f}^2. \tag{15.6.8}$$

Rewrite the expressions in Eqs. (15.6.5) and (15.6.6) as

$$m_2 v_{2,f}\cos\theta_{2,f} = m_1(v_{1,i} - v_{1,f}\cos\theta_{1,f}), \tag{15.6.9}$$

$$m_2 v_{2,f} \sin\theta_{2,f} = m_1 v_{1,f} \sin\theta_{1,f} \ . \qquad (15.6.10)$$

Square each of the expressions in Eqs. (15.6.9) and (15.6.10), add them together and use the identity $\cos^2\theta + \sin^2\theta = 1$ yielding

$$v_{2,f}^2 = \frac{m_1^2}{m_2^2}(v_{1,i}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2) \, . \qquad (15.6.11)$$

Substituting Eq. (15.6.11) into Eq. (15.6.8) yields

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}\frac{m_1^2}{m_2}(v_{1,i}^2 - 2v_{1,i}\,v_{1,f}\cos\theta_{1,f} + v_{1,f}^2) \, . \qquad (15.6.12)$$

Eq. (15.6.12) simplifies to

$$0 = \left(1 + \frac{m_1}{m_2}\right)v_{1,f}^2 - \frac{m_1}{m_2}2v_{1,i}\,v_{1,f}\cos\theta_{1,f} - \left(1 - \frac{m_1}{m_2}\right)v_{1,i}^2 \, , \qquad (15.6.13)$$

Let $\alpha = m_1 / m_2$ then Eq. (15.6.13) can be written as

$$0 = (1+\alpha)v_{1,f}^2 - 2\alpha v_{1,i}\,v_{1,f}\cos\theta_{1,f} - (1-\alpha)v_{1,i}^2 \, , \qquad (15.6.14)$$

The solution to this quadratic equation is given by

$$v_{1,f} = \frac{\alpha v_{1,i}\cos\theta_{1,f} \pm \left(\alpha^2 v_{1,i}^2\cos^2\theta_{1,f} + (1-\alpha^2)v_{1,i}^2\right)^{1/2}}{(1+\alpha)} \, . \qquad (15.6.15)$$

Divide Eq. (15.6.10) by Eq. (15.6.9):

$$\frac{v_{2,f}\sin\theta_{2,f}}{v_{2,f}\cos\theta_{2,f}} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}} \, . \qquad (15.6.16)$$

Eq. (15.6.16) simplifies to

$$\tan\theta_{2,f} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}} \, . \qquad (15.6.17)$$

The relationship between the scattering angles in Eq. (15.6.17) is independent of the masses of the colliding particles. Thus the scattering angle for particle 2 is

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right) \qquad (15.6.18)$$

We can now use Eq. (15.6.10) to find an expression for the final velocity of particle 2

$$v_{2,f} = \frac{\alpha v_{1,f}\sin\theta_{1,f}}{\sin\theta_{2,f}} \,. \qquad (15.6.19)$$

Example 15.5 Elastic Two-dimensional collision of identical particles

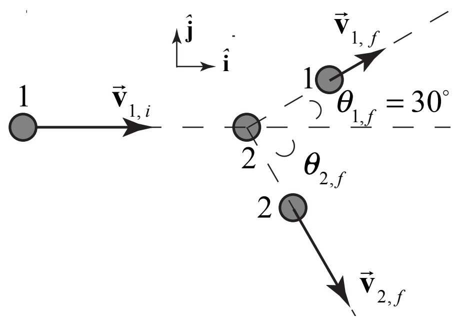


Figure 15.10 Momentum flow diagram for two-dimensional elastic collision

Object 1 with mass m_1 is initially moving with a speed $v_{1,i} = 3.0\,\text{m}\cdot\text{s}^{-1}$ and collides elastically with object 2 that has the same mass, $m_2 = m_1$, and is initially at rest. After the collision, object 1 moves with an unknown speed $v_{1,f}$ at an angle $\theta_{1,f}$ with respect to its initial direction of motion and object 2 moves with an unknown speed $v_{2,f}$, at an unknown angle $\theta_{2,f}$ (as shown in the Figure 15.10). Find the final speeds of each of the objects and the angle $\theta_{2,f}$.

Solution: Because the masses are equal, $\alpha = 1$. We are given that $v_{1,i} = 3.0\,\text{m}\cdot\text{s}^{-1}$ and $\theta_{1,f} = 30^{o}$. Hence Eq. (15.6.14) reduces to

$$v_{1,f} = v_{1,i}\cos\theta_{1,f} = (3.0\,\text{m}\cdot\text{s}^{-1})\cos 30^{\circ} = 2.6\,\text{m}\cdot\text{s}^{-1} \,. \qquad (15.6.20)$$

Substituting Eq. (15.6.20) in Eq. (15.6.17) yields

$$\theta_{2,f} = \tan^{-1}\left(\frac{v_{1,f}\sin\theta_{1,f}}{v_{1,i} - v_{1,f}\cos\theta_{1,f}}\right)$$

$$\theta_{2,f} = \tan^{-1}\left(\frac{(2.6\ \text{m}\cdot\text{s}^{-1})\sin(30^\circ)}{3.0\ \text{m}\cdot\text{s}^{-1} - (2.6\ \text{m}\cdot\text{s}^{-1})\cos(30^\circ)}\right) \tag{15.6.21}$$

$$= 60^\circ.$$

The above results for $v_{1,f}$ and $\theta_{2,f}$ may be substituted into either of the expressions in Eq. (15.6.9), or Eq. (15.6.11), to find $v_{2,f} = 1.5\,\text{m}\cdot\text{s}^{-1}$. Eq. (15.6.11) also has the solution $v_{2,f} = 0$, which would correspond to the incident particle missing the target completely.

Before going on, the fact that $\theta_{1,f} + \theta_{2,f} = 90^\circ$, that is, the objects move away from the collision point at right angles, is not a coincidence. A vector derivation is presented in Example 15.6. We can see this result algebraically from the above result. Substituting Eq. (15.6.20) $v_{1,f} = v_{1,i}\cos\theta_{1,f}$ in Eq. (15.6.17) yields

$$\tan\theta_{2,f} = \frac{\cos\theta_{1,f}\sin\theta_{1,f}}{1 - \cos\theta_{1,f}^2} = \cot\theta_{1,f} = \tan(90^\circ - \theta_{1,f}); \tag{15.6.22}$$

showing that $\theta_{1,f} + \theta_{2,f} = 90^\circ$, the angles $\theta_{1,f}$ and $\theta_{2,f}$ are complements.

Example 15.6 Two-dimensional elastic collision between particles of equal mass

Show that the equal mass particles emerge from a two-dimensional elastic collision at right angles by making explicit use of the fact that momentum is a vector quantity.

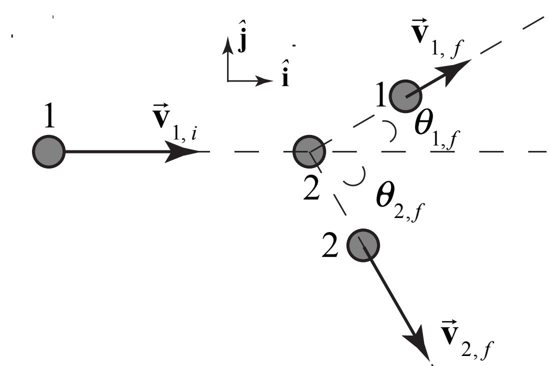


Figure 15.11 Elastic scattering of identical particles

Solution: Choose a reference frame in which particle 2 is initially at rest (Figure 15.11). There are no external forces acting on the two objects during the collision (the collision forces are all internal), therefore momentum is constant

$$\vec{\mathbf{p}}_i^{\text{sys}} = \vec{\mathbf{p}}_f^{\text{sys}} , \tag{15.6.23}$$

which becomes

$$m_1 \vec{\mathbf{v}}_{1,i} = m_1 \vec{\mathbf{v}}_{1,f} + m_1 \vec{\mathbf{v}}_{2,f} . \tag{15.6.24}$$

Eq. (15.6.24) simplifies to

$$\vec{\mathbf{v}}_{1,i} = \vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f} . \tag{15.6.25}$$

Recall the vector identity that the square of the speed is given by the dot product $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = v^2$. With this identity in mind, we take the dot product of each side of Eq. (15.6.25) with itself,

$$\begin{aligned} \vec{\mathbf{v}}_{1,i} \cdot \vec{\mathbf{v}}_{1,i} &= (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \cdot (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \\ &= \vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{1,f} + 2\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} + \vec{\mathbf{v}}_{2,f} \cdot \vec{\mathbf{v}}_{2,f} . \end{aligned} \tag{15.6.26}$$

This becomes

$$v_{1,i}^2 = v_{1,f}^2 + 2\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} + v_{2,f}^2 . \tag{15.6.27}$$

Recall that kinetic energy is the same before and after an elastic collision, and the masses of the two objects are equal, so constancy of energy, (Eq. (15.4.2)) simplifies to

$$v_{1,i}^2 = v_{1,f}^2 + v_{2,f}^2 . \tag{15.6.28}$$

Comparing Eq. (15.6.27) to Eq. (15.6.28), we see that

$$\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} = 0 . \tag{15.6.29}$$

The dot product of two nonzero vectors is zero when the two vectors are at right angles to each other justifying our claim that the collision particles emerge at right angles to each other.

Example 15.7 Two dimensional collision between particles of unequal mass

Particle 1 of mass m_1, initially moving in the positive x-direction (to the right in the figure below) with speed $v_{1,i}$, collides with particle 2 of mass $m_2 = m_1 / 3$, which is initially moving in the opposite direction (Figure 15.12) with an unknown speed $v_{2,i}$. Assume that the total external force acting on the particles is zero. Do not assume the collision is elastic. After the collision, particle 1 moves with speed $v_{1,f} = v_{1,i} / 2$ in the negative y-direction. After the collision, particle 2 moves with an unknown speed $v_{2,f}$, at an angle $\theta_{2,f} = 45^o$ with respect to the positive x-direction. (i) Determine the initial speed $v_{2,i}$ of particle 2 and the final speed $v_{2,f}$ of particle 2 in terms of $v_{1,i}$. (ii) Is the collision elastic?

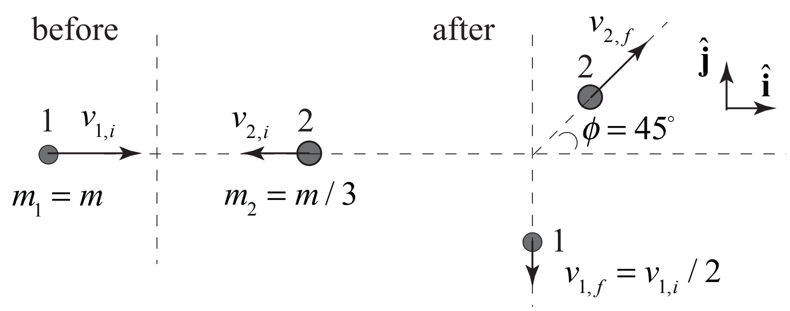


Figure 15.12 Two-dimensional collision between particles of unequal mass

Solution: We choose as our system the two particles. We are given that $v_{1,f} = v_{1,i}/2$. We apply the two momentum conditions,

$$m_1 v_{1,i} - (m_1/3)v_{2,i} = (m_1/3)\, v_{2,f}(\sqrt{2}/2) \qquad (15.6.30)$$

$$0 = m_1 v_{1,f} - (m_1/3)\, v_{2,f}(\sqrt{2}/2)\ . \qquad (15.6.31)$$

Solve Eq. (15.6.31) for $v_{2,f}$

$$v_{2,f} = 3\sqrt{2} v_{1,f} = \frac{3\sqrt{2}}{2} v_{1,i} \qquad (15.6.32)$$

Substitute Eq. (15.6.32) into Eq. (15.6.30) and solve for $v_{2,i}$

$$v_{2,i} = (3/2) v_{1,i}\ . \qquad (15.6.33)$$

The initial kinetic energy is then

$$K_i = \frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2}(m_1/3) v_{2,i}^2 = \frac{7}{8} m_1 v_{1,i}^2 . \qquad (15.6.34)$$

The final kinetic energy is

$$K_f = \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 = \frac{1}{8} m_1 v_{1,i}^2 + \frac{3}{4} m_1 v_{1,i}^2 = \frac{7}{8} m_1 v_{1,i}^2\ . \qquad (15.6.35)$$

Comparing our results, we see that kinetic energy is constant so the collision is elastic.

15.7 Two-Dimensional Collisions in Center-of-Mass Reference Frame

15.7.1 Two-Dimensional Collision in Center-of-Mass Reference Frame

Consider the elastic collision between two particles in the laboratory reference frame (Figure 15.9). Particle 1 of mass m_1 is initially moving with velocity $\vec{\mathbf{v}}_{1,i}$ and collides elastically with a particle 2 of mass m_2 that is initially at rest. After the collision the particle 1 moves with velocity $\vec{\mathbf{v}}_{1,f}$ and particle 2 moves with velocity $\vec{\mathbf{v}}_{2,f}$. In section 15.7.1 we determined how to find $v_{1,f}$, $v_{2,f}$, and $\theta_{2,f}$ in terms of $v_{1,i}$ and $\theta_{2,f}$. We shall now analyze the collision in the center-of-mass reference frame, which is boosted form the laboratory frame by the velocity of center-of-mass given by

$$\vec{\mathbf{v}}_{cm} = \frac{m_1 \vec{\mathbf{v}}_{1,i}}{m_1 + m_2}. \qquad (15.6.36)$$

Because we assumed that there are no external forces acting on the system, the center-of-mass velocity remains constant during the interaction.

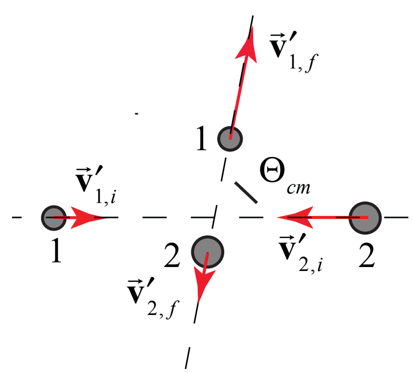


Figure 15.13 Two-dimensional elastic collision in center-of-mass reference frame

Recall the velocities of particles 1 and 2 in the center-of-mass frame are given by (Eq.,(15.2.9) and (15.2.10)). In the center-of-mass reference frame the velocities of the two incoming particles are in opposite directions, as are the velocities of the two outgoing particles after the collision (Figure 15.13). The angle Θ_{cm} between the incoming and outgoing velocities is called the **center-of-mass scattering angle**.

15.7.2 Scattering in the Center-of-Mass Reference Frame

Consider a collision between particle 1 of mass m_1 and velocity $\vec{\mathbf{v}}_{1,i}$ and particle 2 of mass m_2 at rest in the laboratory frame. Particle 1 is scattered elastically through a scattering angle Θ in the center-of-mass frame. The center-of-mass velocity is given by

$$\vec{\mathbf{v}}_{cm} = \frac{m_1 \vec{\mathbf{v}}_{1,i}}{m_1 + m_2}. \tag{15.6.37}$$

In the center-of-mass frame, the momentum of the system of two particles is zero

$$\vec{\mathbf{0}} = m_1 \vec{\mathbf{v}}'_{1,i} + m_2 \vec{\mathbf{v}}'_{2,i} = m_1 \vec{\mathbf{v}}'_{1,f} + m_2 \vec{\mathbf{v}}'_{2,f} \ . \tag{15.6.38}$$

Therefore

$$\vec{\mathbf{v}}'_{1,i} = -\frac{m_2}{m_1} \vec{\mathbf{v}}'_{2,i} \ . \tag{15.6.39}$$

$$\vec{\mathbf{v}}'_{1,f} = -\frac{m_2}{m_1} \vec{\mathbf{v}}'_{2,f} \tag{15.6.40}$$

The energy condition in the center-of-mass frame is

$$\frac{1}{2} m_1 v'^2_{1,i} + \frac{1}{2} m_2 v'^2_{2,i} = \frac{1}{2} m_1 v'^2_{1,f} + \frac{1}{2} m_2 v'^2_{2,f} \,. \tag{15.6.41}$$

Substituting Eqs. (15.6.39) and (15.6.40) into Eq. (15.6.41) yields

$$v'_{1,i} = v'_{1,f} \,. \tag{15.6.42}$$

(we are only considering magnitudes). Therefore

$$v'_{2,i} = v'_{2,f} \,. \tag{15.6.43}$$

Because the magnitude of the velocity of a particle in the center-of-mass reference frame is proportional to the relative velocity of the two particles, Eqs. (15.6.42) and (15.6.43) imply that the magnitude of the relative velocity also does not change

$$\left| \vec{\mathbf{v}}'_{1,2,i} \right| = \left| \vec{\mathbf{v}}'_{1,2,f} \right|, \tag{15.6.44}$$

verifying our earlier result that for an elastic collision the relative speed remains the same, (Eq. (15.2.20)). However the direction of the relative velocity is rotated by the center-of-mass scattering angle Θ_{cm}. This generalizes the energy-momentum principle to two dimensions. Recall that the relative velocity is independent of the reference frame,

$$\vec{\mathbf{v}}_{1,i} - \vec{\mathbf{v}}_{2,i} = \vec{\mathbf{v}}'_{1,i} - \vec{\mathbf{v}}'_{2,i} \tag{15.6.45}$$

In the laboratory reference frame $\vec{\mathbf{v}}_{2,i} = \vec{\mathbf{0}}$, hence the initial relative velocity is $\vec{\mathbf{v}}'_{1,2,i} = \vec{\mathbf{v}}_{1,2,i} = \vec{\mathbf{v}}_{1,i}$, and the velocities in the center-of-mass frame of the particles are then

$$\vec{\mathbf{v}}'_{1,i} = \frac{\mu}{m_1}\vec{\mathbf{v}}_{1,i} \tag{15.6.46}$$

$$\vec{\mathbf{v}}'_{2,i} = -\frac{\mu}{m_2}\vec{\mathbf{v}}_{1,i} \quad . \tag{15.6.47}$$

Therefore the magnitudes of the final velocities in the center-of-mass frame are

$$v'_{1,f} = v'_{1,i} = \frac{\mu}{m_1}v'_{1,2,i} = \frac{\mu}{m_1}v_{1,2,i} = \frac{\mu}{m_1}v_{1,i} . \tag{15.6.48}$$

$$v'_{2,f} = v'_{2,i} = \frac{\mu}{m_2}v'_{1,2,i} = \frac{\mu}{m_2}v_{1,2,i} = \frac{\mu}{m_2}v_{1,i} . \tag{15.6.49}$$

Example 15.8 Scattering in the Lab and CM Frames

Particle 1 of mass m_1 and velocity $\vec{\mathbf{v}}_{1,i}$ by a particle of mass m_2 at rest in the laboratory frame is scattered elastically through a scattering angle Θ in the center of mass frame, (Figure 15.14). Find (i) the scattering angle of the incoming particle in the laboratory frame, (ii) the magnitude of the final velocity of the incoming particle in the laboratory reference frame, and (iii) the fractional loss of kinetic energy of the incoming particle.

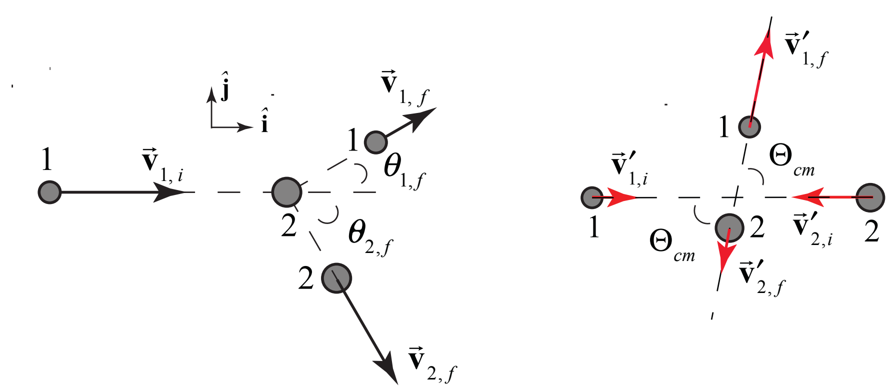


Figure 15.14 Scattering in the laboratory and center-of-mass reference frames

Solution:

i) In order to determine the center-of-mass scattering angle we use the transformation law for velocities

$$\vec{\mathbf{v}}'_{1,f} = \vec{\mathbf{v}}_{1,f} - \vec{\mathbf{v}}_{cm} \ . \tag{15.6.50}$$

In Figure 15.15 we show the collision in the center-of-mass frame along with the laboratory frame final velocities and scattering angles.

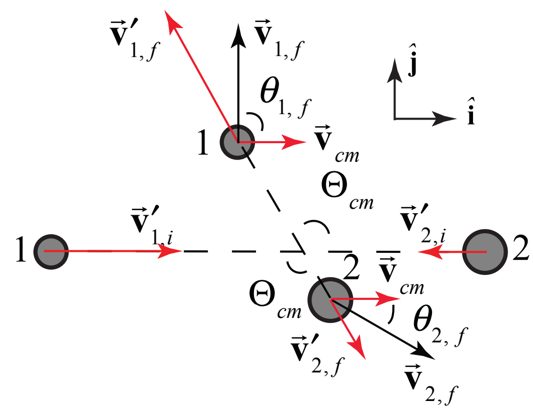


Figure 15.15 Final velocities of colliding particles

Vector decomposition of Eq. (15.6.50) yields

$$v_{1,f}\cos\theta_{1,i} = v'_{1,f}\cos\Theta_{cm} - v_{cm}\,, \tag{15.6.51}$$

$$v_{1,f}\sin\theta_{1,i} = v'_{1,f}\sin\Theta_{cm} \ . \tag{15.6.52}$$

where we choose as our directions the horizontal and vertical Divide Eq. (15.6.52) by (15.6.51) yields

$$\tan\theta_{1,i} = \frac{v_{1,f}\sin\theta_{1,i}}{v_{1,f}\cos\theta_{1,i}} = \frac{v'_{1,f}\sin\Theta_{cm}}{v'_{1,f}\cos\Theta_{cm} - v_{cm}} \tag{15.6.53}$$

Because $v'_{1,i} = v'_{1,f}$, we can rewrite Eq. (15.6.53) as

$$\tan\theta_{1,i} = \frac{v'_{1,i}\sin\Theta_{cm}}{v'_{1,i}\cos\Theta_{cm} - v_{cm}} \tag{15.6.54}$$

We now substitute Eqs. (15.6.48) and $v_{cm} = m_1 v_{1,i} / (m_1 + m_2)$ into Eq. (15.6.54) yielding

$$\tan\theta_{1,i} = \frac{m_2 \sin\Theta_{cm}}{\cos\Theta_{cm} - m_1 / m_2} \quad . \tag{15.6.55}$$

Thus in the laboratory frame particle 1 scatters by an angle

$$\theta_{1,i} = \tan^{-1}\left(\frac{m_2 \sin\Theta_{cm}}{\cos\Theta_{cm} - m_1 / m_2} \right) \quad . \tag{15.6.56}$$

ii) We can calculate the square of the final velocity in the laboratory frame

$$\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{1,f} = (\vec{\mathbf{v}}'_{1,f} + \vec{\mathbf{v}}_{cm}) \cdot (\vec{\mathbf{v}}'_{1,f} + \vec{\mathbf{v}}_{cm}) \quad . \tag{15.6.57}$$

which becomes

$$v_{1,f}^{\ 2} = v'^{\ 2}_{1,f} + 2\vec{\mathbf{v}}'_{1,f} \cdot \vec{\mathbf{v}}_{cm} + v_{cm}^{\ 2} = v'^{\ 2}_{1,f} + 2v'_{1,f} v_{cm} \cos\Theta_{cm} + v_{cm}^{\ 2} \quad . \tag{15.6.58}$$

We use the fact that $v'_{1,f} = v'_{1,i} = (\mu / m_1) v_{1,2,i} = (\mu / m_1) v_{1,i} = (m_2 / m_1 + m_2) v_{1,i}$ to rewrite Eq. (15.6.58) as

$$v_{1,f}^{\ 2} = \left(\frac{m_2}{m_1 + m_2} \right)^2 v_{1,i}^{\ 2} + 2\frac{m_2 m_1}{(m_1 + m_2)^2} v_{1,i} \cos\Theta_{cm} + \frac{m_1^{\ 2}}{(m_1 + m_2)^2} v_{1,i}^{\ 2} \quad . \tag{15.6.59}$$

Thus

$$v_{1,f} = \frac{\left(m_2^{\ 2} + 2m_2 m_1 \cos\Theta_{cm} + m_1^{\ 2} \right)^{1/2}}{m_1 + m_2} v_{1,i} \quad . \tag{15.6.60}$$

(iii) The fractional change in the kinetic energy of particle 1 in the laboratory frame is given by

$$\frac{K_{1,f} - K_{1,i}}{K_{1,i}} = \frac{v_{1,f}^{\ 2} - v_{1,i}^{\ 2}}{v_{1,i}^{\ 2}} = \frac{m_2^{\ 2} + 2m_2 m_1 \cos\Theta_{cm} + m_1^{\ 2}}{(m_1 + m_2)^2} - 1 = \frac{2m_2 m_1 (\cos\Theta_{cm} - 1)}{(m_1 + m_2)^2} \quad . \tag{15.6.61}$$

We can also determine the scattering angle Θ_{cm} in the center-of-mass reference frame from the scattering angle $\theta_{1,i}$ of particle 1 in the laboratory. We now rewrite the momentum relations as

$$v_{1,f} \cos\theta_{1,i} + v_{cm} = v'_{1,f} \cos\Theta_{cm} , \tag{15.6.62}$$

$$v_{1,f} \sin\theta_{1,i} = v'_{1,f} \sin\Theta_{cm} \quad . \tag{15.6.63}$$

In a similar fashion to the above argument, we have that

$$\tan\Theta_{cm} = \frac{v_{1,f}\sin\theta_{1,f}}{v_{1,f}\cos\theta_{1,f} + v_{cm}}. \qquad (15.6.64)$$

Recall from our analysis of the collision in the laboratory frame that if we specify one of the four parameters $v_{1,f}$, $v_{2,f}$, $\theta_{1,f}$, or $v_{1,f}$, then we can solve for the other three in terms of the initial parameters $v_{1,i}$ and $v_{2,i}$. With that caveat, we can use Eq. (15.6.64) to determine Θ_{cm}.