12. **The universal enveloping algebra of a Lie algebra**

12.1. **The definition of the universal enveloping algebra.** Let V be a vector space over a field $\mathbf{k}$. Recall that the **tensor algebra** of V is the $\mathbb{Z}$-graded associative algebra $TV := \oplus_{n\geq 0} V^{\otimes n}$ (with $\deg(V^{\otimes n}) = n$), with multiplication given by $a \cdot b = a \otimes b$ for $a \in V^{\otimes m}$ and $b \in V^{\otimes n}$. If $\{x_i\}$ is a basis of V then TV is just the free algebra with generators x_i (i.e., without any relations). Its basis consists of various words in the letters x_i.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$.

Definition 12.1. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of $T\mathfrak{g}$ by the ideal I generated by the elements $xy - yx - [x, y]$, $x, y \in \mathfrak{g}$.

Recall that any associative algebra A is also a Lie algebra with operation $[a, b] := ab - ba$. The following proposition follows immediately from the definition of $U(\mathfrak{g})$.

Proposition 12.2. *(i) Let $J \subset T\mathfrak{g}$ be an ideal, and $\rho : \mathfrak{g} \to T\mathfrak{g}/J$ the natural linear map. Then ρ is a homomorphism of Lie algebras if and only if $J \supset I$, so that $T\mathfrak{g}/J$ is a quotient of $T\mathfrak{g}/I = U(\mathfrak{g})$. In other words, $U(\mathfrak{g})$ is the largest quotient of $T\mathfrak{g}$ for which ρ is a homomorphism of Lie algebras.*

(ii) (universal property of $U(\mathfrak{g})$) Let A be any associative algebra over $\mathbf{k}$. Then the map

$$\mathrm{Hom}_{\mathrm{associative}}(U(\mathfrak{g}), A) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A)$$

given by $\phi \mapsto \phi \circ \rho$ is a bijection.

Part (ii) of this proposition implies that any Lie algebra map $\psi : \mathfrak{g} \to A$ can be uniquely extended to an associative algebra map $\phi : U(\mathfrak{g}) \to A$ so that $\psi = \phi \circ \rho$. This is the universal property of $U(\mathfrak{g})$ which justifies the term "universal enveloping algebra".

In particular, it follows that a representation of $\mathfrak{g}$ on a vector space V is the same thing as an algebra map $U(\mathfrak{g}) \to \mathrm{End}(V)$ (i.e., a representation of $U(\mathfrak{g})$ on V). Thus, to understand the representation theory of $\mathfrak{g}$, it is helpful to understand the structure of $U(\mathfrak{g})$; for example, every central element $C \in U(\mathfrak{g})$ gives rise to a morphism of representations $V \to V$ (note that this has already come in handy in studying representations of $\mathfrak{sl}_2$).

In terms of the basis $\{x_i\}$ of $\mathfrak{g}$, we can write the bracket as

$$[x_i, x_j] = \sum_k c_{ij}^k x_k,$$

where $c_{ij}^k \in \mathbf{k}$ are the **structure constants**. Then the algebra $U(\mathfrak{g})$ can be described as the quotient of the free algebra $\mathbf{k}\langle\{x_i\}\rangle$ by the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k.$$

Example 12.3. 1. If $\mathfrak{g}$ is abelian (i.e., $c_{ij}^k = 0$) then $U(\mathfrak{g}) = S\mathfrak{g} = \mathbf{k}[\{x_i\}]$ is the symmetric algebra of $\mathfrak{g}$, $S\mathfrak{g} = \oplus_{n\geq 0} S^n\mathfrak{g}$, which in terms of the basis is the polynomial algebra in x_i.

2. $U(\mathfrak{sl}_2(\mathbf{k}))$ is generated by e, f, h with defining relations

$$he - eh = 2e,\ hf - fh = -2f,\ ef - fe = h.$$

Recall that $\mathfrak{g}$ acts on $T\mathfrak{g}$ by derivations via the adjoint action. Moreover, using the Jacobi identity, we have

$$\mathrm{ad}z(xy - yx - [x,y]) = [z,x]y + x[z,y] - [z,y]x - y[z,x] - [z,[x,y]] =$$
$$([z,x]y - y[z,x] - [[z,x],y]) + (x[z,y] - [z,y]x - [x,[z,y]]).$$

Thus $\mathrm{ad}z(I) \subset I$, and hence the action of $\mathfrak{g}$ on $T\mathfrak{g}$ descends to its action on $U(\mathfrak{g})$ by derivations (also called the adjoint action). It is easy to see that these derivations are in fact inner:

$$\mathrm{ad}z(a) = za - az$$

for $a \in U(\mathfrak{g})$ (although this is not so for $T\mathfrak{g}$). Indeed, it suffices to note that this holds for $a \in \mathfrak{g}$ by the definition of $U(\mathfrak{g})$.

Thus we get

Proposition 12.4. *The center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ coincides with the subalgebra of invariants $U(\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$.*

Example 12.5. The Casimir operator $C = 2fe + \frac{h^2}{2} + h$ which we used to study representations of $\mathfrak{g} = \mathfrak{sl}_2$ is in fact a central element of $U(\mathfrak{g})$.

12.2. **Graded and filtered algebras.** Recall that a $\mathbb{Z}_{\geq 0}$-**filtered** algebra is an algebra A equipped with a filtration

$$0 = F_{-1}A \subset F_0A \subset F_1A \subset ... \subset F_nA \subset ...$$

such that $1 \in F_0A$, $\cup_{n\geq 0} F_nA = A$ and $F_iA \cdot F_jA \subset F_{i+j}A$. In particular, if A is generated by $\{x_\alpha\}$ then a filtration on A can be obtained by declaring x_α to be of degree 1; i.e., $F_nA = (F_1A)^n$ is the span of all words in x_α of degree $\leq n$.

If $A = \oplus_{i\geq 0} A_i$ is $\mathbb{Z}_{\geq 0}$-graded then we can define a filtration on A by setting $F_nA := \oplus_{i=0}^n A_i$; however, not any filtered algebra is obtained in this way, and having a filtration is a weaker condition than having a grading. Still, if A is a filtered algebra, we can define its **associated graded algebra** $\mathrm{gr}(A) := \oplus_{n\geq 0}\mathrm{gr}_n(A)$ (also denoted $\mathrm{gr}A$), where $\mathrm{gr}_n(A) := F_nA/F_{n-1}A$. The multiplication in $\mathrm{gr}(A)$ is given by the "leading terms" of multiplication in A: for $a \in \mathrm{gr}_i(A),\ b \in \mathrm{gr}_j(A)$, pick their representatives $\widetilde{a} \in F_iA,\ \widetilde{b} \in F_jA$ and let ab be the projection of $\widetilde{a}\widetilde{b}$ to $\mathrm{gr}_{i+j}(A)$.

Proposition 12.6. *If* $\mathrm{gr}(A)$ *is a domain (has no zero divisors) then so is* A.

Exercise 12.7. Prove Proposition 12.6.

Example 12.8. Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$. Define a filtration[11] on $U(\mathfrak{g})$ by setting $\deg(\mathfrak{g}) = 1$. Thus $F_nU(\mathfrak{g})$ is the image of $\oplus_{i=0}^n \mathfrak{g}^{\otimes i} \subset T\mathfrak{g}$. Note that since

$$xy - yx = [x, y],\ x \in \mathfrak{g},$$

we have $[F_iU(\mathfrak{g}), F_jU(\mathfrak{g})] \subset F_{i+j-1}U(\mathfrak{g})$. Thus, $\mathrm{gr}U(\mathfrak{g})$ is commutative; in other words, we have a surjective algebra morphism

$$\phi : S\mathfrak{g} \to \mathrm{gr}U(\mathfrak{g}).$$

12.3. **The coproduct of $U(\mathfrak{g})$.** For a vector space $\mathfrak{g}$ define the algebra homomorphism $\Delta : T\mathfrak{g} \to T\mathfrak{g} \otimes T\mathfrak{g}$ given for $x \in \mathfrak{g} \subset T\mathfrak{g}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ (it exists and is unique since $T\mathfrak{g}$ is freely generated by $\mathfrak{g}$).

Lemma 12.9. *If* $\mathfrak{g}$ *is a Lie algebra then the kernel* I *of the map* $T\mathfrak{g} \to U(\mathfrak{g})$ *satisfies the property* $\Delta(I) \subset I \otimes T\mathfrak{g} + T\mathfrak{g} \otimes I \subset T\mathfrak{g} \otimes T\mathfrak{g}$. *Thus* Δ *descends to an algebra homomorphism* $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Proof. For $x, y \in \mathfrak{g}$ and $a = a(x, y) := xy - yx - [x, y]$ we have $\Delta(a) = a \otimes 1 + 1 \otimes a$. The lemma follows since the ideal I is generated by elements of the form $a(x, y)$. □

The homomorphism Δ is called the **coproduct** (of $T\mathfrak{g}$ or $U(\mathfrak{g})$).

Example 12.10. Let $\mathfrak{g} = V$ be abelian (a vector space). Then $U(\mathfrak{g}) = SV$, which for $\dim V < \infty$ can be viewed as the algebra of polynomial functions on V^*. Similarly, $SV \otimes SV$ is the algebra of polynomial functions on $V^* \times V^*$. In terms of this identification, we have $\Delta(f)(x, y) = f(x + y)$.

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[11]The grading on $T\mathfrak{g}$ does not descend to $U(\mathfrak{g})$, in general, since the relation $xy - yx = [x, y]$ is not homogeneous: the right hand side has degree 1 while the left hand side has degree 2. So $U(\mathfrak{g})$ is not graded but is only filtered.

12.4. **Differential operators on manifolds and Lie groups.** We have seen in Subsection 5.2 that a vector field on a manifold X is the same thing as a derivation of the algebra $O(U)$ for every open set $U \subset X$ compatible with restriction maps $O(U) \to O(V)$ for $V \subset U$; in particular, on every U we have $[\mathbf{v}, m_f] = m_{\mathbf{v}(f)}$ where $f \in O(U)$ and $m_f : O(U) \to O(U)$ is the operator of multiplication by $f \in O(U)$. Thus if also $g \in O(U)$ then $[[\mathbf{v}, m_f], m_g] = 0$. Conversely, if A is an endomorphism of the space $O(U)$ for every open $U \subset X$ compatible with restriction maps and $[[A, m_f], m_g] = 0$ for any $f, g \in O(U)$ then $A = \mathbf{v} + m_h$ for a unique vector field $\mathbf{v}$ and regular function h on X (check this!). This gives rise to the following generalization of the notion of a vector field.

Definition 12.11. (Grothendieck) A **differential operator** of order $\le N$ on X is an endomorphism of the space $O(U)$ for every open set $U \subset X$ compatible with restriction maps $O(U) \to O(V)$ for $V \subset U$ such that for any $f_0, ..., f_N \in O(U)$ one has

$$[...[[A, f_0], f_1], ..., f_N] = 0.$$

It is easy to show that the latter condition is equivalent to the classical condition for a differential operator of order $\le N$: in local coordinates (x_i) on a chart $U \subset X$ the operator A looks like

$$A = \sum_{k=0}^{N} \sum_{i_1 \le ... \le i_k} F_{i_1,...,i_k} \frac{\partial^k}{\partial x_{i_1} ... \partial x_{i_k}},$$

where $F_{i_1,...,i_k} \in O(U)$ (check this!). The space of such operators is denoted by $D_N(X)$. Thus we have a nested sequence of spaces

$$O(X) = D_0(X) \subset D_1(X) \subset ... \subset D_N(X) \subset ...$$

The nested union $\cup_{N \ge 0} D_N(X)$ is a filtered associative algebra called the **algebra of differential operators on** X and denoted by $D(X)$.

Now suppose that a Lie group G with Lie algebra $\mathfrak{g}$ acts on X. Then we have a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(X)$, which can be viewed as a Lie algebra homomorphism $\mathfrak{g} \to D(X)$. Thus by the universal property of the universal enveloping algebra, we obtain an associative algebra homomorphism $\xi : U(\mathfrak{g}) \to D(X)$. Moreover, this homomorphism preserves filtrations.

For example, if $X = G$ and G acts by right translations, then the corresponding map $\mathfrak{g} \to \mathrm{Vect}(G)$ identifies $\mathfrak{g}$ with the Lie algebra $\mathrm{Vect}_L(G)$ of left-invariant vector fields on G. Thus the map $\xi : U(\mathfrak{g}) \to D(G)$ lands in the subalgebra $D_L(G)$ of left-invariant differential operators on G.

Exercise 12.12. Show that the map $\xi : U(\mathfrak{g}) \to D_L(G)$ is a filtered algebra isomorphism.