# THE PLACE OF SYLLOGISTIC IN LOGICAL THEORY

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CONTENTS

# PREFACE

Łukasiewicz published his classic work on Aristotelian syllogistic in 1951. Unlike his book, the present monograph makes no attempt to contribute to Aristotelian scholarship, but it does aim to locate the place of syllogistic in modern logical theory. Łukasiewicz's syllogistic, which is simply the result of grafting some special axioms on to propositional logic, with their term variables having non-empty name-like expressions as substituends, does not show how syllogistic logic relates to the modern logic of general propositons, and his interpretation is in any case open to philosophical criticism. However, as a result of his pioneering study and of others' subsequent work, I believe that it is now possible to determine the place of syllogistic far more satisfactorily.

In Chapter 1 BS, a basic syllogistic system based on Aristotle's logic, is presented in natural deduction form, so avoiding the need to adjoin propositional principles (apart from *reductio*) in the manner of Łukasiewicz and Bocheński. As far as I know, the idea of presenting Aristotelian logic as a natural deduction system was first suggested by Robert Feys in a review, and I devised the system BS for teaching purposes some years ago. In the last few years natural deduction systems have been published by Corcoran and Smiley, but their concern has been with exegesis of Aristotle's text. Since I am concerned with a wider assessment of Aristotle's logic and of his contribution to the whole subject, I am less interested in representing the minutiae of Aristotelian doctrine in my basic system and more interested in presenting a system I can use to relate the syllogistic type of logic to that which has superseded it in modern times. Deductions in BS are set out in tree form, and are therefore far easier to construct and follow than deductions in the systems of Corcoran or Smiley.

Chapters 2 and 3 treat the metatheory of the basic system. The exposition is relatively informal, since I have aimed to make the whole monograph accessible to anyone with an elementary knowledge of formal logic. In any case, it will not be essential for the reader to work through all the details of metatheoretical proofs in order to understand the rest of the text.

Since Słupecki proved the completeness of Łukasiewicz's syllogistic and Wedberg proved the completeness of a system with negative terms, much more concise proofs of these results have been given by Shepherdson. Shepherdson's proofs, however, are algebraic, whereas the proofs presented here are purely logical and are derived (and simplified) from a completeness proof given by Corcoran for his syllogistic system mentioned above. The style of proofs in the metatheory and the order of exposition were chosen particularly to bring out a hitherto unnoticed but remarkable feature of Aristotle's attempt to show that his logic was complete. His approach anticipates modern Henkin-style completeness proofs. Aristotle's fundamental insights - his idea of inferences valid in virtue of their general structure and his use of variables to reveal it, his construction of what amounts to a nearly complete deductive system of logic - are well appreciated. But his attempt to show that his logic is complete has received far less attention. If it did not give a false impression of his understanding of logic, we could almost say that Aristotle came close to anticipating model theory.

For the purposes of the first four chapters, which together with the fifth are concerned with the formal rather than the philosophical aspects of the subject, I use interpretations in which term variables have substantival phrases as their substituends. This is because I wanted to employ an interpretation of the same general sort as that which Aristotle and the medievals had in mind, and which enabled the formal treatment to proceed smoothly. I also wanted to avoid prejudging the philosophical discussion of Chapters 6 and 7. This certainly does not mean that I really subscribe to a 'two-name' interpretation of general propositions: far from it. Objections to such interpretations are presented in Chapter 7, where I conclude that the terms of *A*, *E*, *I* and *O* propositions are best construed as predicative in nature.

The investigation of the nature of general propositions takes us into the heart of contemporary discussion of reference and generality, and much of Chapter 6 and 7 is concerned with critical assessment of certain doctrines of Leśniewski, Strawson and Geach, which are also of interest apart from their relevance to syllogistic.

Syllogistics with negative variables, derived from the nineteenth century enlargement of traditional logic to cope comprehensively with syllogisms containing negative terms, are dealt with in Chapter 9 as class calculi, in which manner the basic systems are construed in the preceding chapter.

The present work is limited to categorical syllogistic. There is no treatment of Aristotle's modal syllogisms, nor indeed of any of his other, more piecemeal, contributions to formal logic.

I should like to thank those who have discussed some of the contents of this monograph with me or who have made useful comments on preliminary drafts, notably my former colleagues Michael Partridge, David Braine (Aberdeen University) and Norton Nelkin (University of New Orleans); my former tutor, Christopher Kirwan (Exeter College, Oxford); and Professor Peter Geach (Leeds University). My thanks are also due to the University of Nottingham for supplying funds, and to Mrs. Eileen Long, Mrs. Rose Holman and Miss Christine Flear for typing the manuscript. And special acknowledgement is owed to Dr. Richard Cardwell for pioneering and organizing the monograph series in which the present work appears as the first volume.

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April, 1979

# CHAPTER 1

# A BASIC SYLLOGISTIC SYSTEM

## 1.1 *Aristotle's theory of the assertoric syllogism*

1.11 Until the nineteenth century Aristotle's work on the assertoric syllogism was the most notable achievement in formal logic. Of course, it by no means exhausts the interesting or important work in the field up to that time, even by Aristotle himself, but in producing a complete system of deductive inference, albeit of a very restricted sort, he undoubtedly justified Leibniz's verdict that 'l'invention de la forme des syllogismes est une des plus belles de l'esprit humain, et même des plus considérables' (1704, IV, xvii, 4, quoted by Couturat (1903) ).*

The basic syllogistic system to be presented in this chapter is based fairly closely on Aristotle's doctrine of the assertoric syllogism in the *Prior Analytics*. For the moment the system will be thought of as a logic for the four traditional types of categorical proposition, as exemplified by the following sentences:

| | Quantity | Quality | |
|---|---|---|---|
| Every man is a hypocrite | universal | affirmative | (*A*) |
| No man is a hypocrite | universal | negative | (*E*) |
| Some man is a hypocrite | particular | affirmative | (*I*) |
| Not every man is hypocrite | particular | negative | (*O*) |

A proposition will be regarded in the medieval manner as a sentence with a certain meaning and the issue of whether there exist propositions in some more abstract sense will be left open. It will be convenient also to follow the medieval practice of referring to the propositional forms by means of the letters '*A*', '*E*', '*I*', and '*O*'.

* Works referred to by author and date are listed at the back.

Such categorical propositions are to be formed by inserting an appropriate term into each of the two gaps in any of the following sentence-frames:

> Every — is a(n) —; No — is a(n) —; Some is a(n) —;
> Not every — is a(n) —.[1]

Terms may be single nouns or they may be noun phrases, as in the propositions *No Indian elephant is a cheap pet* and *Some photograph of Wittgenstein is a collector's item*; but admissible terms will be restricted to words which are count nouns or count-noun phrases. Thus a word or phrase *a* will be an admissible term iff (if and only if) you can frame a significant question of the form, 'How many *a*'s are there?'.[2] This means that our categorical propositions belong to a narrower range than the examples used by Aristotle himself, which include cases like *Some snow is white*, a sentence in which neither term passes the test. However, the restriction has certain advantages at this point in the exposition. Inferences which involve the transposition of subject and predicate terms cannot be made from propositions like *Some snow is white* without idiomatic adjustments (*Something white is snow*), whereas no such tinkering is needed in converting a sentence like *Some man is a hypocrite.*

One of Aristotle's most significant achievements was the introduction of variables, which reflects his fundamental insight that the validity of the inferences he was studying depends on their form. For term variables we shall use small roman letters: the form of *Every man is a hypocrite*, for example, will be expressed by *Every a is a b*.

[1] Cf. Christine Ladd-Franklin in Baldwin (1901-2), 2, p. 329, cited by A.N. Prior, (1976), p. 53, where further details will be found.

[2] The question need not have a definite answer, however. See David Wiggins on oily waves and crowns (1968), pp. 39-40.

1.12 Aristotle was primarily interested in those inferences in which a conclusion is drawn from two premisses which share one of their terms. These syllogisms normally contain three distinct terms, one of which appears as subject term of the conclusion and in one of the premisses (the minor term), another which appears as the predicate term of the conclusion and in the other premiss (the major term), and the middle term, which appears in each premiss. A premiss itself is called 'major' or 'minor' according as it contains the major or minor term. The definition of major and minor terms by reference to position in the conclusion is due not to Aristotle but to his commentator John Philoponus, but it seems the neatest way of understanding them. As an example we may take the syllogism:

Every hypocrite is a liar
Some man is a hypocrite
―――――――――――――
Therefore, some man is a liar

The first premiss, containing as it does the major term *liar*, is the major premiss, the second the minor premiss, and *hypocrite* is the middle term. If we replace major, minor and middle terms respectively by the variables $p$, $s$ and $m$, the form of the inference, in virtue of which it is deductively valid, may be expressed like this:

Every $m$ is a $p$
Some $s$ is an $m$
―――――――――――――
Some $s$ is a $p$

When the major and minor terms have the positions in the premisses which they have in the example above (major in predicate, minor in subject position), the syllogism is said to be in the first figure. Clearly there are three other ways of arranging the major and minor terms in the premisses, and so four figures in all, usually presented by means of the following medieval schemas:

| 1 | | 2 | | 3 | | 4 | |
|---|---|---|---|---|---|---|---|
| $m$ | $p$ | $p$ | $m$ | $m$ | $p$ | $p$ | $m$ |
| $s$ | $m$ | $s$ | $m$ | $m$ | $s$ | $m$ | $s$ |

The conclusion will always be: $s\ p$. It is unlikely, though, that Aristotle himself used schemas of this sort to work out the different figures, particularly since he does not recognize a fourth *figure* (though he did recognize the validity of such arguments and proves them).[3]

The mood of a syllogism is specified by giving the quantity and quality of its constituent propositions, with those of the major premiss standardly given first: thus the example above is in the mood *AII*. Only a minority of moods in each figure is valid. The system to be presented will enable us to prove all the valid argument patterns which Aristotle recognized, as well as the weakened moods[4] which later logicians included in their treatment. The reference list which follows gives the medieval mnemonic names whose vowels indicate these moods, with the names of the weakened moods in brackets:

First figure: Barbara, (Barbari), Celarent, (Celaront), Darii, Ferio
Second figure: Cesare, (Cesaro), Camestres, (Camestrop), Festino, Baroco
Third figure: Darapti, Disamis, Datisi, Felapton, Bocardo, Ferison
Fourth figure: Bramantip, Camenes, (Camenop), Dimaris, Fesapo, Fresison

1.13 In the *Prior Analytics* Aristotle shows how to reduce the moods of the 'less evident' second and third figures to the (unweakened) first figure moods by means of the principles of conversion and *reductio ad absurdum*. (For *reductio* see 1.14). For example, he reduces Camestres in the second figure to Celarent in the first:

[3] See Lynn Rose (1965).

[4] A mood is said to be weakened if its conclusion is weaker than it need be, i.e. if the conclusion is particular and the universal conclusion of the same quality also follows from the premisses.

> ... if $M$ belongs to all $N$, but to no $O$, then $N$ will belong to no $O$. For if $M$ belongs to no $O$, $O$ belongs to no $M$: but $M$ (as was said) belongs to all $N$: $O$ will then belong to no $N$: for the first figure has again been formed. But since the negative relation is convertible, $N$ will belong to no $O$. ($27^a 9$-14.)

Aristotle frequently expresses propositions in the manner of this quotation, a manner which was as artificial in Greek as it is in English. Expressed in the style we have adopted, Camestres has the form:

$$\frac{\begin{array}{c}\text{Every } n \text{ is an } m\\ \text{No } o \text{ is an } m\end{array}}{\text{No } o \text{ is an } n}$$

We are told that the minor premiss is simply convertible. Simple conversion involves merely transposing the subject and predicate terms, and seems to yield valid inferences in the cases of $E$ and $I$ propositions, but invalid inferences when applied to $A$ and $O$. We may set out the simple conversion of the minor premiss thus:

$$\frac{\text{No } o \text{ is an } m}{\text{No } m \text{ is an } o}\ \text{s.c.}$$

*No n is an o* is then derivable from *No m is an o* together with the major premiss of Camestres, *Every n is an m*, by means of the principle of the first figure mood Celarent:

$$\frac{\text{No } m \text{ is an } o \qquad \text{Every } n \text{ is an } m}{\text{No } n \text{ is an } o}\ \text{Celarent}$$

And the conclusion of Camestres is derivable from *No n is an o*, again by simple conversion. Putting these three steps together we get:

$$\cfrac{\cfrac{\text{No } o \text{ is an } m}{\text{No } m \text{ is an } o}\ \text{s.c.} \qquad \text{Every } n \text{ is an } m}{\cfrac{\text{No } n \text{ is an } o}{\text{No } o \text{ is an } n}\ \text{s.c.}}\ \text{Celarent}$$

What we have here is in fact a form of deduction, using the principles of simple conversion and Celarent as rules of inference. Given the principle of simple conversion, the question of the validity of the mood Camestres has been reduced to that of the validity of Celarent. In general a syllogism (a) is reduced to another syllogism (b) when the premisses of (a) deductively yield those of (b), and the conclusion of (b) in turn yields the conclusion of (a).

To take next an example from the third figure, Darapti is reduced to Darii as follows:

$$\cfrac{\text{Every } s \text{ is a } p \qquad \cfrac{\text{Every } s \text{ is an } r}{\text{Some } r \text{ is an } s}\ \text{c.p.a.}}{\text{Some } r \text{ is a } p}\ \text{Darii}$$

*Every s is an r* is converted *per accidens*, i.e. the term variables are interchanged and quantity reduced from universal to particular.

1.14 All the assertoric syllogisms which Aristotle dealt with are reducible to the first figure in the way we have described, apart from Baroco and Bocardo, which he reduces 'indirectly' by *reductio ad absurdum*. Here is his reduction of Baroco:

> if *M* belongs to all *N*, but not to some *O*, it is necessary that *N* does not belong to some *O*: for if *N* belongs to all *O*, and *M* is predicated also of all *N*, *M* must belong to all *O*: but we assumed that *M* does not belong to some *O*. ($27^a37$–$27^b1$.)

The form to be reduced is:

$$\begin{array}{l} \text{Every } n \text{ is an } m \\ \underline{\text{Not every } o \text{ is an } m} \\ \text{Not every } o \text{ is an } n \end{array}$$

(The $O$ forms have been given in the manner announced in 1.11, rather than in the 'Some — is not ...' form. See 1.3 for the explanation of this.) Aristotle supposes for the sake of argument that the conclusion is false, that is, he assumes the logical contradictory of the conclusion: *Every o is an n*. But from this assumption and the major premiss *Every n is an m* we can derive *Every o is an m* by means of the principle of the first-figure mood Barbara:

$$\frac{\text{Every } n \text{ is an } m \qquad [\text{ Every } o \text{ is an } n\,]}{\text{Every } o \text{ is an } m}\ \text{Barbara}$$

The supposition has been enclosed within square brackets. The conclusion now derived is the logical contradictory of the minor premiss, *Not every o is an m*. Since logical contradictories must have opposite truth-values, the truth of the original premisses is not compatible with the contradictory of the original conclusion: so if the premisses are true, the original conclusion must also be true. We shall display the deduction of which this is the rationale as follows:

$$\frac{\dfrac{\text{Every } n \text{ is an } m \qquad [\text{ Every } o \text{ is an } n\,]}{\text{Every } o \text{ is an } m}\ \text{Barbara} \qquad \text{Not every } o \text{ is an } m}{\text{Not every } o \text{ is an } n}\ \text{r.a.a.}$$

At the final step the contradictory of the supposition is asserted on the basis of the two (unbracketed) premisses *Every n is an m*, *Not every o is an m*. The question of the validity of Baroco has been reduced to that of the validity of Barbara, and once again we have a form of deduction, though less simple than in the previous example. As it is used here the *reductio* principle is parasitic on other principles like simple conversion and Barbara, and does not simply involve the direct derivation of a conclusion from premisses. What justifies the r.a.a. step is not simply the presence of the two contradictories above the line but the fact that one of them rests in part on the bracketed assumption. That assumption is *discharged* at the final step and so the conclusion rests only on *Every n is an m* and *Not every o is an m*.

Strictly speaking, we should not bracket any formula until it is being discharged. Thus, just before the final step, the emerging deduction should look like this:

$$\dfrac{\text{Every } n \text{ is an } m \qquad \text{Every } o \text{ is an } n}{\text{Every } o \text{ is an } m}\ \text{Barbara} \qquad\qquad \text{Not every } o \text{ is an } m$$

The three formulas *Every n is an m*, *Every o is an n* and *Not every o is an m* jointly yield the contradictories *Every o is an m*, *Not every o is an m*. The first three formulas cannot, therefore, all be true, and if any two are true the third must be false. Consequently, *any one* of the three can be discharged by the assertion of its contradictory, which will follow from the other two. The formula discharged is then bracketed to show that the formula newly derived does not rest on it.

*Reductio* arguments are, of course, a very powerful and important mode of reasoning in logic and mathematics. They are familiar from the work of Aristotle's contemporary, Euclid, and before that Zeno's arguments had more or less taken this form. As Aristotle noticed, *reductio* arguments are available for reducing all valid syllogisms, though they are not always indispensable.

The reduction of Bocardo may be set out like this:

$$\dfrac{\dfrac{[\text{Every } r \text{ is a } p] \qquad \text{Every } s \text{ is an } r}{\text{Every } s \text{ is a } p}\ \text{Barbara} \qquad\qquad \text{Not every } s \text{ is a } p}{\text{Not every } r \text{ is a } p}\ \text{r.a.a.}$$

Not only did Aristotle show how to reduce second and third figure syllogisms to the first figure, but, among other things, he showed how to reduce the first figure moods Darii and Ferio to Celarent via the second figure, and thereby showed that all syllogisms are reducible to Barbara or Celarent. Darii is treated thus:

> ... if *A* belongs to all *B*, and *B* to some *C*, it follows that *A* belongs to some *C*. For if it belonged to no *C*, and belongs to all *B*, then *B* will belong to no *C*: this we know by means of the second figure. ($29^{b}8$-11.)

Darii has the form: *Every b is an a*, *Some c is a b / Some c is an a*. Assume the contradictory of the conclusion, viz. *No c is an a*. By Camestres, already reduced to Celarent, we may derive *No c is a b*, which contradicts the minor premiss of Darii. Putting this argument together with the 'reduction' of Camestres we may express the reasoning in the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\text{No } c \text{ is an } a]}{\text{No } a \text{ is a } c}\;\text{s.c.} \qquad \text{Every } b \text{ is an } a}{\text{No } b \text{ is a } c}\;\text{Celarent}}{\text{No } c \text{ is a } b}\;\text{s.c.} \qquad\qquad \text{Some } c \text{ is a } b}{\text{Some } c \text{ is an } a}\;\text{r.a.a.}$$

*No c is an a* must be false if *Every b is an a* and *Some c is a b* are true, since the trio jointly yields a pair of contradictories. Hence *No c is an a* is discharged and its contradictory asserted on the basis of the other two formulas in the trio.

If we put together the tree for the reduction of Darapti to Darii and the last tree for the reduction of Darii to Celarent, we have the full reduction of Darapti to Celarent·

$$\dfrac{\dfrac{\dfrac{\dfrac{[\text{No } r \text{ is a } p]}{\text{No } p \text{ is an } r}\,\text{s.c.} \qquad \text{Every } s \text{ is a } p}{\text{No } s \text{ is an } r}\,\text{Celarent}}{\text{No } r \text{ is an } s}\,\text{s.c.} \qquad \dfrac{\text{Every } s \text{ is an } r}{\text{Some } r \text{ is an } s}\,\text{c.p.a.}}{\text{Some } r \text{ is a } p}\,\text{r.a.a.}$$

Using the same principles of inference — s.c., Celarent, c.p.a. and r.a.a. — it is possible to prove Darii with slightly more economy:[5]

$$\dfrac{\dfrac{\dfrac{[\text{No } r \text{ is a } p] \qquad \text{Every } s \text{ is an } r}{\text{No } s \text{ is a } p}\,\text{Celarent}}{\text{No } p \text{ is an } s}\,\text{s.c.} \qquad \dfrac{\text{Every } s \text{ is a } p}{\text{Some } p \text{ is an } s}\,\text{c.p.a.}}{\text{Some } r \text{ is a } p}\,\text{r.a.a.}$$

The moods now called figure 4 are dealt with by Aristotle at 29$^{a}$23-27 (Fesapo and Fresison) and 53$^{a}$3-12 (Bramantip, Camenes and Dimaris). The weakened moods can be demonstrated in much the same way as the others.

1.15 Not all of the principles of inference used so far are independent of one another. When introducing conversion in the second chapter of the *Prior Analytics* Aristotle demonstrates simple conversion of *I* propositions and conversion per accidens of *A* by using r.a.a. and simple conversion of *E*. The last principle is itself demonstrated 'by ecthesis', but we shall disregard this for the moment. Simple conversion of *I* is demonstrated at 25$^{a}$20-22:

[5] **Even more economy can be achieved by allowing the use of *reductio ad absurdum* when the inconsistent formulas derived are not contradictories but corresponding *A* and *E* formulas — contraries. (Aristotle himself argues like this, for example, in his proof of conversion *per accidens* at 25$^{a}$17-19 - see below.) The proof of Darapti would then have only two steps, an application of Celarent and one of r.a.a.**

> ... if some $B$ is $A$, then some of the $A$s must be $B$. For if none were, then no $B$ would be $A$.

$$\cfrac{\cfrac{[\text{No } a \text{ is a } b]}{\text{No } b \text{ is an } a}\ \text{s.c.} \qquad \text{Some } b \text{ is an } a}{\text{Some } a \text{ is a } b}\ \text{r.a.a.}$$

and c.p.a. of $A$ at $25^a17$-19:

> ... if every $B$ is $A$, then some $A$ is $B$. For if no $A$ were $B$, then no $B$ could be $A$. But we assumed that every $B$ is $A$.[6]

$$\cfrac{\cfrac{[\text{No } a \text{ is a } b]}{\text{No } b \text{ is an } a}\ \text{s.c.} \qquad \text{Every } b \text{ is an } a}{\text{Some } a \text{ is a } b}\ \text{r.a.a.}$$

Notice that the inconsistent formulas derived prior to the application of the *reductio* step are not contradictories (formulas which must have opposite truth-values) but merely contraries (inconsistent formulas which can't both be true). This form of the *reductio* principle will have the status of a derived rule in the basic system to be presented in this chapter.

What we have proved in the demonstration above are special cases of the principles s.c. of $I$ and c.p.a. of $A$ used in the tree deductions: for example, the last demonstration shows how to deduce *Some a is a b* from *Every b is an a*, but the 'principle' of c.p.a. for $A$ equally permits the deduction of *Some a is an a* from *Every a is an a*, etc. The tree deductions contain not propositions but formulas, which we are regarding as propositional forms, and the rules of inference used in these deductions enable you to derive formulas of certain patterns from other formulas of certain patterns. To

[6] **Conversion *per accidens* of $E$, not actually used by Aristotle, can be proved in a similar way.**

specify the pattern of a formula we shall use letters, $\alpha$, $\beta$, $\gamma$ as arbitrary term variables. The rule of simple conversion of $E$, for example, will be given as:

$$\frac{\text{No } \alpha \text{ is a } \beta}{\text{No } \beta \text{ is an } \alpha}$$

And to show that s.c. of $I$, for example, is a derived rule, we should really derive *Some* $\beta$ *is an* $\alpha$ from *Some* $\alpha$ *is a* $\beta$ metalogically in a proof schema.

When we set up our basic syllogistic system in the next section we shall express the constants *Every — is a —*, *No — is a —*, etc., by means of the single letters '$A$', '$E$', '$I$' and '$O$' preceding the two variables. Thus for *Every a is a b* we shall write $Aab$. This has the advantage not only of economy but also of making it easier to consider different interpretations of the system.

If the form of r.a.a. in which the *reductio* step immediately succeeds the deduction of corresponding $A$ and $E$ formulas is to be treated as a derived rule of the systems we are constructing, it seems that we need to retain c.p.a. of $A$ as a primitive rule. Because this latter rule is interderivable with the principle of subaltern inference for $A$ formulas, from $A\alpha\beta$ to $I\alpha\beta$ (sub.($A$)), we shall treat sub.($A$) as primitive, since this seems to yield a slight gain in perspicuity. We show the two are interderivable by deducing sub.($A$) using c.p.a.($A$) and vice versa:

$$\begin{array}{ll} A\alpha\beta & \\ \hline I\beta\alpha & \text{c.p.a.} \\ \hline I\alpha\beta & \text{s.c.}(I) \end{array} \qquad \begin{array}{ll} A\alpha\beta & \\ \hline I\alpha\beta & \text{sub.}(A) \\ \hline I\beta\alpha & \text{s.c.} \end{array}$$

And we shall also treat s.c.($I$), rather than s.c.($E$), as primitive. The reader can easily verify that, in a system with sub.($A$), s.c.($I$) and the narrower form of r.a.a. as primitive, s.c.($E$), c.p.a.($E$)

and sub.$(E)$ are derived rules.

It is obvious that the principles we have used are not limited to affording derivations of inference patterns with just one or two premisses and that the patterns of longer inferences ('sorites', etc.) are also derivable. As an illustration consider the following argument:

$$\begin{array}{l} \text{No socialist is a conformist} \\ \text{Some humanist is a socialist} \\ \text{Every humanist is a rationalist} \\ \text{Every rationalist is a disbeliever} \\ \text{Every disbeliever is a sceptic} \\ \hline \text{Not every sceptic is a conformist} \end{array}$$

There is no difficulty in deriving a sequent expressing the pattern of this inference within our basic system: $Elc, Ihl, Ahr, Ard, Adp \vdash Opc$. (Letters have been chosen which are suggested by the terms in the inference, but it must be emphasized that these letters are not to be thought of as short for the terms, but as variables replacing them.)

$$\dfrac{Ihl \qquad \dfrac{\dfrac{Ahr \qquad \dfrac{Ard \qquad \dfrac{Adp \quad [Apc]}{Adc}\,\text{Barbara}}{Arc}\,\text{Barbara}}{Ahc}\,\text{Barbara} \qquad \dfrac{Elc}{Ecl}\,\text{s.c.}(E)}{Ehl}\,\text{Celarent}}{Opc}\,\text{r.a.a.}$$

1.2 *Rules of 'identity'.*

Many syllogistic systems postulate that formulas of the form $A\alpha\alpha$ are necessary truths, which can be done in the present system by adding the following rule of 'identity':

$$\dfrac{*}{A\alpha\alpha}$$

This is to be understood as licensing you to write any formula of the form $A\alpha\alpha$ immediately below an asterisk written at a tip (i.e. at the top of a branch of a tree) to indicate that the formula rests on no assumptions. The principle was introduced by Leibniz, but Aristotle himself makes use of the claim that *Every b is a b* is a necessary truth in a demonstration at 68ᵃ19. Moreover, formulas of the form *Not every* $\alpha$ *is an* $\alpha$ seem to be the patterns of necessary falsehoods; and if *A* formulas are the contradictories of the corresponding *O* formulas, it follows that formulas of the form $A\alpha\alpha$ are the patterns of necessary truths. Aristotle's belief in the necessary truth of propositions of the form *Every a is an a* also commits him to the necessary truth of those of the form *Some a is an a*, which is deducible from the *A* form by subaltern inference. He is also committed to the necessary truth of the latter when, in Chapter 15 of Book II, he implies that *No c is a c* is necessarily false, for this means that its contradictory, *Some c is a c*, will be necessarily true.

The view that propositions of this last form are necessary truths seems to be a very questionable one, however, for surely the formula entails the existence of some *c* and is false if there is no *c*. Indeed it has often been thought that propositions of the form *Every a is an a* are false if there are no *a*'s. These considerations would count against adding the rule of identity given above and in favour simply of adding the weaker rule:

$$\frac{I\,\alpha\beta}{A\,\alpha\alpha}$$

So we shall distinguish a stronger system, with the full identity rule (id.$^+$), from the weaker system, which has instead the weaker rule (id.). In the opening chapters we shall be concerned principally with the weaker system.

## 1.8 *The basic systems*

We may now proceed to a formal description of a basic syllogistic system BS in which the deductions of sections 1.3 - 1.5 can be made. The interpretation of this system to be adopted in the opening chapters will be described in the next section.

*Language.*

Constants: '$A$', '$E$', '$I$', '$O$' Variables: '$a$', '$b$', '$c$' ...

Square brackets: '[ ', ']'

*Formation rule.*

A formula (wff) consists of a constant followed by just two variables. (The terms 'formula' and 'well-formed formula' - abbreviated 'wff' - will be used interchangeably.)

*Rules of inference.*[7]

(i) s.c. $\dfrac{I\alpha\beta}{I\beta\alpha}$ (ii) sub. $\dfrac{A\alpha\beta}{I\alpha\beta}$ (iii) Barbara $\dfrac{A\alpha\beta \quad A\beta\gamma}{A\alpha\gamma}$

(iv) Celarent $\dfrac{A\alpha\beta \quad E\beta\gamma}{E\alpha\gamma}$ (v) r.a.a. $\dfrac{\begin{matrix}[\varphi] \\ \psi \quad \bar{\psi}\end{matrix}}{\bar{\varphi}}$

In the statement of r.a.a. '$\varphi$' and '$\psi$' are schematic letters for arbitrary formulas and $\varphi$ is related to $\bar{\varphi}$ thus:

[7] It will be noticed that in the statement of the rules Barbara and Celarent the minor premiss schema has been given first, contrary to the usual practice. This is perhaps a little misleading, since the first vowel in the mnemonic names refers to the major premiss; but the rules are easier to remember in the form given, since occurrences of the middle term variable occur together in the middle. Indeed, this may well be connected with the fact that Aristotle more often gave the major premiss first, for he reversed the subject and predicate letters — recall how he often speaks of $B$ belonging to all $A$, etc. If we reverse the order of the term variables and give the major premiss schema first, we get $A\gamma\beta$, $A\beta\alpha \vdash A\gamma\alpha$, with the middle term letter occurring twice in the middle. See Patzig (1968), pp. 59-61.

if $\varphi = A\alpha\beta$, $\bar{\varphi} = O\alpha\beta$; if $\varphi = E\alpha\beta$, $\bar{\varphi} = I\alpha\beta$;
if $\varphi = I\alpha\beta$, $\bar{\varphi} = E\alpha\beta$; if $\varphi = O\alpha\beta$, $\bar{\varphi} = A\alpha\beta$.
(Similarly for $\psi$, $\bar{\psi}$.)

(vi) id. $\dfrac{I\alpha\beta}{A\alpha\alpha}$

For the stronger system BS$^+$ the rule id. is replaced by the stronger identity rule id.$^+$:

$$\frac{*}{A\alpha\alpha}$$

Unlike most previous formal syllogistic systems the present one is a natural-deduction system. This has the advantage not only of making deductions within the system easier to discover, but also of making it unnecessary to adjoin auxiliary principles and symbols of propositional logic not explicitly deployed by Aristotle, which would be unavoidable if the postulates were given in axiomatic form. (Since I started working on this book, natural deduction versions of syllogistic have been presented independently by Corcoran (1973) and Smiley (1973). Both defend the approach from the point of view of Aristotelian exegesis.)

Since the interpretation of the *Prior Analytics* is not a main concern of this study, it is of little consequence if the natural-deduction approach is not completely faithful to Aristotle. In fact, since he probably thought of his actual syllogisms as inferences rather than as implications, it may not be so very unfaithful, though he does seem to have reasoned *about* the syllogisms in terms of inference patterns stated in implicational form. (Cf. Prior (1962), p. 116)

For the moment deductions will be given in the form of trees. It is easy to get an intuitive grasp of the notion of a tree from the examples, but a rigorous definition will now be given. According to this, a tree is not the array of formulas but the set of points associated with them, though we shall continue often to speak loosely of the tree as if it were the proof.

For our purposes a *tree* is a finite set of points together with a binary relation *immediately above* satisfying the following conditions:

(i) There is a special point R called the *root* (elsewhere it is called the *origin*).

(ii) Every point except R is immediately above just one (other) point.

(iii) For any point P in the tree there is definite sequence of points, called a *branch*, from R to P in which each point but the first is immediately above its predecessor. (Obviously, in view of (ii), there will only be one such sequence.)

(iv) Immediately above a point P there is either:

(a) no point (in which case we call P a *tip*); or

(b) one point (P is a node);

(c) two points (P is a branch-point).

Provisionally, we may define a *deduction* (*derivation, proof*) in BS as an array of formulas, together with any square brackets around formulas, which satisfies the following conditions:

(i) There is a tree each point of which has just one formula associated with it, and such that every formula in the deduction is associated with (is 'at') some point.

(ii) Each formula is either at a tip or is derived from one or two formulas immediately above it in accordance with a rule of inference.

The formula at the root is said to be derived from the set of all those formulas at the tips which have not been discharged (bracketed) as a result of applying r.a.a. To say that a formula $\varphi$ is derivable from a set $\Gamma$ we shall write: $\Gamma \vdash \varphi$, sentences of this form, like

$$\{Aab,\ Abc\} \vdash Aac,$$

being known as 'sequents'.

For the stronger system BS$^+$ the definition needs some extension: a deduction will be an array of formulas and, if id.$^+$ is used, asterisks, plus any square brackets around formulas, which satisfies conditions (i') and (ii'). Condition (i') will require that every point be associated with just one wff or asterisk and that every wff or asterisk be associated with some point; and condition (ii') will state that each wff is either at a tip or immediately below an asterisk or derived from one or two wffs immediately below it in accordance with a rule of inference.

What makes the tree form of deduction so convenient for the basic system is the fact that derivable forms can always be proved by means of trees in which no assumption is associated with more than one point - what we shall call 'non-repetitive' derivations. We could, in consequence, stipulate the exclusion of other types of derivation, without weakening the deductive power of the systems. (Check the forms of derivation listed for the proof procedure in 3.3 to verify this.)

1.4 *An interpretation of BS*

If only for expository convenience we shall in the first instance interpret the four propositional forms in the manner which seems to have become standard in medieval logic. This interpretation, which we shall call 'Interpretation I', was probably adopted by William of Sherwood and also by Ockham, and is unequivocally to be found in

Buridan and Albert the Great. It has even been maintained by Manley Thomson that it was what Aristotle himself had in mind, though if so he was not consistent about it. The affirmative forms are to be interpreted as having existential import and entailing the existence of something to which the subject term applies, but existential import is denied to the negative forms. *Aab* (*Every a is a b*) is therefore regarded as true only when there is at least one *a* and all the *a*'s are *b*'s, and *Iab* (*Some a is a b*) is true when there is something which is both an *a* and a *b*. *Eab* (*No a is a b*) is true when there is nothing which is both an *a* and a *b*, which, of course, includes the case where there is no *a*. Finally, the *O* form, read here as *Not every a is a b*, will be true (i) when there is no *a* or (ii) when there is an *a* which is not a *b*. In the case of the *O* form the medieval interpretation differs significantly from that assumed in most contemporary textbooks, which usually render the *O* pattern in a manner closer to *Some a is not a b*.

Various different interpretations will be discussed later in the book, and there is no intention at this point of prejudging that discussion. It must therefore be affirmed that in adopting Interpretation I at this stage we are not putting it forward as the preferred interpretation nor are we claiming that the English sentences in which the categorical propositions are expressed are most accurately construed in this way. The term *interpretation*, it should be added, is being used throughout this book in the same way as it is used by Alonzo Church (1956).

If you want an interpretation under which BS is sound, i.e., one under which every derivable form is valid, and you do not want to depart radically from the ordinary senses of the words in which we have expressed the propositional forms, you are easily led to the present interpretation. The rule of simple conversion for *I* formulas requires the $I$ form to have existential import, for, otherwise, if there were no $a$ but some $b$, *Some a is a b* would be true but *Some b is an a* false. This means that the $A$ form must have existential import; otherwise, if there were no $a$, *Every a is a b* could be true but *Some a is a b* false. Now, if the *reductio* rule is to be sound, corresponding $A$ and $O$ forms must be contradictories, and the same goes for corresponding $I$ and $E$ forms: so if the affirmative forms have existential import, the negative forms must lack it; otherwise corresponding $A$ and $O$ (or $E$ and $I$) propositions could both be false.

The weaker identity rule, id., is clearly sound under this interpretation, since if $I\alpha\beta$ is true there is some $\alpha$ and $A\alpha\alpha$ must be true. But, as we have already pointed out, the stronger rule is unsound under this interpretation since $A\alpha\alpha$ will be false if there is no $\alpha$. The soundness of the system BS is proved formally in the next chapter (Section 2.2).

## 1.5 *On the admission of redundant premisses*

The assertoric syllogisms which Aristotle studied usually had three distinct terms. It is not surprising that he is uninterested in examples like the following, where the minor premiss is manifestly redundant and the conclusion simply repeats the major:

Every horse is a horse
Every horse is an animal
---
Every horse is an animal

Yet we ought nevertheless to regard this as an example of a syllogism in Barbara. For to say that the inference pattern *Every a is a b*, *Every b is a c*, *Therefore every a is a c* is valid is in effect to say that every inference of that pattern is valid. And the example just given is undoubtedly of that pattern, the term *horse* replacing both variables $a$ and $b$. In general, all inferences of the specific pattern

*Every a is an a*, *Every a is a b*, *Therefore every a is a b* are also of the more general pattern just given. And it is certainly possible to establish the sequent $\{Aaa, Aab\} \vdash Aab$ by the use of the rule Barbara. The fact that $Aaa$ is redundant does not prevent its being brought into the deduction.

Now although we cannot reasonably avoid admitting redundant premisses like this, it might be thought that we should none the less refuse to admit redundant premisses which cannot be brought into a deduction in this manner. Maybe we have to allow that $Aab$ is deducible from $\{Aaa, Aab\}$, but need we allow it to be deducible from $\{Acd, Aab\}$? However, we shall resist any temptation to distinguish between the two sorts of case, for it would unduly complicate the formal task of assimilating syllogistic to modern logic. It is worth noting that this distinction cannot be made for propositional calculus inferences, since any redundant premisses can be brought into a proof by using the principles exemplified by $\{P, Q\} \vdash P \,\&\, Q$ and $\{P \,\&\, Q\} \vdash P$ (see 4.3). Consider, for example, the inference

Ryle wrote *The Concept of Mind*
The moon is the earth's only natural satellite
———————————————————
Ryle wrote *The Concept of Mind*

This inference is of the form $\{P, Q\} \,/\, P$, where $Q$ is clearly redundant, and the form is derivable in virtue of the other two quoted sequents.

For present purposes let us say that an inference pattern is sound or valid iff there is no substitution instance in which the premisses are true and the conclusion false. Or, rather, to cover cases where there are no premisses: iff there is no substitution instance in which the conclusion is false but there is no false premiss.[8] Then a concrete inference exemplifying some pattern will be formally valid iff it exemplifies some valid pattern. In other words, as Aristotle puts it at 57ª36-7, 'if the conclusion is false, the premisses of the argument must be false, either all or some of them'. Now, clearly, if an inference has this property it will not lose it if we add further premisses, no matter what they are.

Applying these considerations to our systems, we redefine a deduction in such a way that any redundant premiss formulas may be admitted into the proof. Redundant formulas not brought into the tree part of the proof may be written in an initial list above the tree. Thus a proof of the sequent $\{Aab, Abc, Acd, Afg\} \vdash Aac$ will look like this:

$$Acd \quad Afg$$

$$\frac{Aab \quad Abc}{Aac}\ \text{Barbara}$$

A deduction is now an array of formulas etc. on a tree and in a (possibly empty) initial list. The statement of the rule r.a.a. has to be revised to take account of these lists: an exact statement would go:

if $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\varphi\} \vdash \overline{\chi}$, then $\Gamma \vdash \overline{\varphi}$;
and, if $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\psi\} \vdash \overline{\chi}$, then
$\Gamma \cup \{\varphi\} \vdash \overline{\psi}$ and also $\Gamma \cup \{\psi\} \vdash \overline{\varphi}$.

The Greek capitals denote (possibly empty) sets of formulas. (Strictly, a formula in a proof will always rest on a set of one or more formulas - and in $BS^+$ the set may be empty: it will be convenient, however, to go on talking of a formula resting on another formula when we mean that the other formula belongs to the set on which the first rests.)

8 More accurately still: iff every model of the premiss formulas is a model of the conclusion. But in the present case, where substitution instances are in English, our working definition has the same effect as the model - theoretic one. See Quine (1970), pp. 53-54.

We have a deduction of $\varphi$ from $\Gamma$ when $\varphi$ is at a root and $\Gamma$ contains all the undischarged formulas in the initial list and at the tips.

Actually, we should have a proof of the sequent of the example above even if the initial list were omitted from the proof. But initial lists are not entirely dispensable, since a formula discharged by r.a.a. may have to appear in one. And it is a consequence of the definition that we can derive any conclusion we wish from a pair of formulas $\varphi$ and $\bar{\varphi}$: for example, *Acd* from the set $\{Aab, Oab\}$. We simply assume *Ocd* and put it in the initial list. Now $\{Aab, Ocd\} \vdash Aab$ is established thus:

$$\begin{array}{c} Ocd \\ Aab \end{array}$$

where the initial list consists simply of *Ocd* and the tree has only one point, with which *Aab* is associated. Similarly.

$$\begin{array}{c} Ocd \\ Oab \end{array}$$

will count as a derivation of *Oab* from $\{Ocd, Oab\}$. Since *Aab*, *Oab* are corresponding *A* and *O* formulas, we may apply r.a.a. to discharge *Ocd* and derive the corresponding *A* formula, *Acd*. The full deduction is:

$$[Ocd] \qquad \text{(initial list)}$$

$$\frac{Aab \qquad Oab}{Acd}\ \text{r.a.a.} \qquad \text{(tree)}$$

Another un-Aristotelian consequence of our admission of redundant premisses arises in the stronger system BS$^+$. Formulas of the forms $A\alpha\alpha$ or *Iaa* will be derivable separately from each member of a pair $\varphi$, $\bar{\varphi}$, as for example in the following case:

$$Aab \qquad\qquad Oab \qquad\qquad \text{(initial lists)}$$

$$\frac{*}{Acc}\ \text{id.}^{+} \qquad\qquad \frac{*}{Acc}\ \text{id.}^{+} \qquad\qquad \text{(trees)}$$

These are respectively proofs of $\{Aab\} \vdash Acc$ and of $\{Oab\} \vdash Acc$. In view of Aristotle's claim in Chapter 4 of Book II that 'it is impossible that the same thing should be necessitated by the being and not-being of the same thing' ($57^{b}3$-4), it seems unlikely that he would have accepted this.

Of course, if our primary interest were the interpretation of Aristotle, we should not set up a system admitting redundant premisses, but rather the sort of system devised by Corcoran and Smiley (loc. cit. ), to which the interested reader is referred.

## 1.6 *The adequacy of BS under Interpretation I*

In the two chapters which follow we shall show that BS is both sound and complete with respect to Interpretation I. Nevertheless there are weighty reasons for dissatisfaction with the way in which the system deals with subject-predicate propositions, as compared with the modern post Fregean articulation. The restriction of terms to substantival sortal expressions which we made in 1.11 was, in fact, designed to mask these difficulties temporarily so that their philosophical discussion could be postponed until the formal treatment of the systems has been completed. We shall have occasion to discuss this philosophical issue at some length in Chapter 7, but in the meantime we pass on to the formal metatheory of BS.

## APPENDIX TO CHAPTER 1

### A NOTE ON ARISTOTELIAN EXEGESIS

The present study makes no attempt to contribute to Aristotelian exegesis. Since, however, one of the main reasons for relating syllogistic to modern logic is to throw light on the development of the subject, a few brief remarks on the exegetical issue will be added.

Although the basic systems presented in this first chapter are no more than loosely based on the *Prior Analytics*, recent work by Corcoran (1973) and Smiley (1973) to which the reader is referred, seems to show that they are a good deal more faithful to Aristotle than is Łukasiewicz's well-known system. Very roughly, Corcoran's system is like BS with formulas restricted to those with two distinct variable letters, so that his system lacks any 'identity' rule. Derivations never have initial lists and *reductio* may be used only once in a deduction, namely as the final step. According to Corcoran, the natural deduction formulation has the advantage of showing how Aristotle can treat the logic of *An. Pr.* as an underlying logic for the axiomatic sciences treated at the beginning of the *Posterior Analytics*.

Aristotle's tendency generally[9] to exclude propositions with the same subject and predicate terms conflicts with his insight that the inferences he is treating are valid in virtue of their form alone. (A similar point applies to his failure to provide for redundant premisses.) For if an inference is valid in virtue of having the form Barbara, for example, any inference of that form must be valid, including:

[9] He does not ignore them entirely, as we have seen (but cf. Corcoran (1972) p. 99).

$$\frac{Aab \qquad Aba}{Aaa}$$

There just is not going to be any system which is consistent with everything Aristotle says or implies.

Corcoran, like Smiley, interprets his system in such a way that term letters are never empty. But it can also be interpreted in the medieval manner (our Interpretation 1), with affirmative propositions having existential import and negative ones lacking it. Although there is some textual basis for each of these interpretations, the basis for the second is admittedly more tenuous. In favour of the first is Aristotle's view in the *Categories* that each secondary substance is instantiated by at least one primary substance (i.e. by at least one individual). In favour of the second is the claim put forward in Chapter 46 of *An. Pr.*, Bk. I that whereas the propositions *It is white* and *It is-not white*, are contradictories, the affirmative pair *It is white* and *It is not-white* are merely contraries and could both be false. One explanation for this would be that the second pair both have existential import, whereas only the affirmative member of the first pair does, the negative proposition lacking it. On the other hand the difference between *It is not white* and *It is not-white* can also be explained by maintaining that, while the first is true of something like a number which could not be white, the second is false of it.

The metatheory of Corcoran's system is easily developed by adapting details given in subsequent chapters for BS and $BS^+$. [10]

[10] Six other possible interpretations which are 'suggested by Aristotle's terminology or incidental remarks' are set out on pp. 64-66 of Kneale and Kneale (1962). The six in question are those numbered (1) - (5) and (7) in their text.

# CHAPTER 2

## METATHEORY OF BS:
## (I) CONSISTENCY, SOUNDNESS AND INDEPENDENCE

### 2.1 *Consistency*

Both basic systems, BS and $BS^+$, are consistent in the sense that no two wffs $\varphi$, $\bar{\varphi}$ are derivable as theorems of those systems. By a theorem we mean a formula provable as the conclusion of a derivation in which it rests on no assumptions. (Some logicians reserve the word 'proof' for derivations of theorems, but we shall not follow this practice.) The consistency results are not very strong ones, and indeed for BS the result is quite trivial, since it is easy to show that no theorems whatsoever are derivable within it. We shall demonstrate the consistency of the stronger system $BS^+$, from which the consistency of the included system BS follows immediately as a corollary.

*Metatheorem 1. No wffs $\varphi$, $\bar{\varphi}$ are derivable as theorems of $BS^+$.*

As one would expect from a cursory glance at the rules, only wffs of the forms $A\alpha\alpha$ and $I\alpha\alpha$ are derivable as theorems. To prove consistency, however, it is sufficient to show that no negative wffs are derivable as theorems, since one of any pair $\varphi$, $\bar{\varphi}$ is bound to be negative. The theorem therefore follows from the following lemma:

*Lemma: Every negative formula, $\psi$, in a proof rests on at least one negative formula.*

A formula $\varphi$ in a proof rests on the set containing:

(i) itself, if it is a wff in the initial list;

(ii) itself plus any wffs in the initial list, if it is at a tip; otherwise

(iii) all wffs not yet discharged which occur either in the initial list or at tips linked by branches to $\varphi$.

'Not yet discharged' means not discharged by the application of r.a.a. anywhere in the proff above the formula $\varphi$. When we say that a wff rests on a wff we mean that the former rests on some set containing the latter.

Thus in the proof given at the end of 1.15 *Ecl* rests on {*Elc*}, *Ehl* rests on {*Adp*, *Apc*, *Ard*, *Ahr*, *Elc*}, and *Opc* rests on {*Adb*, *Apc*, *Ard*, *Ahr*, *Ihl*, *Elc*}.

The proof of the lemma is by strong induction on the rank of the negative wff $\psi$ in a proof. The rank of a wff is defined as the length of the longest branch from the wff to a tip, unless it occurs in the initial list, in which case it is of rank 1. For example, in the following derivation *Aab*, *Abc* and *Ecd* are each of rank 1, *Aac* of rank 2, *Ead* of rank 3 and *Eda* of rank 4.

$$\cfrac{\cfrac{\cfrac{Aab \quad Abc}{Aac}\,\text{Barbara} \qquad\qquad Ecd}{Ead}\,\text{Celarent}}{Eda}\,\text{s.c.}$$

*Basis.* The formula $\psi$ is of rank 1. It is in the initial list or at a tip. Since no rule has been applied to reach this formula, it will be among the formulas on which it rests.

*Induction step.* Consider a formula $\psi$ of rank $k$ ($k$ higher than 1) and assume that every negative wff in a proof of rank lower than $k$ rests on some negative wff. We prove that it follows that the wff $\psi$ of rank $k$ must also rest on some negative wff.

Since $\psi$ is of rank higher than 1 and is negative, it must be the result of applying Celarent or r.a.a.

Celarent. $\psi$ is derived immediately from one negative and one affirmative wff. On the hypothesis of the induction the negative wff above $\psi$ rests on some negative wff. But $\psi$ rests on the union of the sets on which the two wffs immediately above it rest. (Thus, in the derivation above, *Aac* rests on {*Aab*, *Abc*} and *Ecd* rests on {*Ecd*}: *Ead* therefore rests on {*Aab*, *Abc*, *Ecd*}.) So $\psi$ also rests on some negative wff.

r.a.a. $\psi$ is immediately below one negative wff, which on the hypothesis of the induction will rest on some negative wff. The newly derived wff $\psi$ must also rest on that negative wff, unless the latter is discharged at this point. But it cannot be discharged at this point, since when a negative wff is derived by r.a.a. the wff discharged is affirmative.

So if $\psi$ is of rank 1 it rests on some negative wff. And if every negative wff of rank lower than $k$ rests on some negative wff, so does every wff $\psi$ of rank $k$. By the principle of strong mathematical induction it follows that whatever the rank of the negative wff $\psi$ it rests on some negative wff.

## 2.2 *Soundness*

A sequent is correct iff any uniform substitution which produces a false conclusion makes at least one premiss in the premiss-set false. A system is sound iff every provable sequent is correct. If a system is sound, clearly we cannot move from a set of true premisses to a false conclusion.

It is easy to show that $BS^+$ is not sound for Interpretation I. *Aaa* is derivable as a theorem of this system and has false substitution instances like *Every unicorn is a unicorn.*[1] The sequent $\vdash Aaa$ has no premisses and so a fortiori has no premises with false substituends. Sound interpretations for $BS^+$ will be considered later.

The present section is devoted to proving that the weaker system BS is sound under Interpretation I. To say that the inference from $\Gamma$ to $\varphi$ is sound we shall link designations of the premiss-set and the conclusion by means of the symbol $\Vdash$. The metatheorem to be proved may then be expressed succinctly in the following manner:

[1] On the use of such substituends see Chapter 3, where it is pointed out that their use is open to criticism. The criticism can be avoided by using the decision procedure of 3.3 to show that $\{Oaa\}$ is consistent.

*Metatheorem 2. If* $\Gamma \vdash_{BS} \varphi$, *then* $\Gamma \Vdash_{I} \varphi$.

The pattern of the proof is similar to that of the lemma above. This time we wish to show that if a wff $\varphi$ in a proof is *false* it rests on some false wff. (We shall continue to speak loosely of true and false wffs, since the meaning should be clear; in more cumbrous but more accurate language the last sentence would read: all uniform substitutions which transform a wff $\varphi$ in a proof into a false proposition transform at least one of the wffs on which it rests into a false proposition.)

Once again the proof is by strong induction, on the rank of the wff $\varphi$, defined as in the last section.

*Basis.* $\varphi$ is of rank 1. It is in the initial list or at a tip, and so will be among the wffs on which it rests.

*Induction step.* We show that, if the theorem holds for all wffs of rank lower than $k$ $(k > 1)$, it holds when $\varphi$ has the rank $k$. It is necessary to consider each rule in turn.

s.c. If the final step leading to $\varphi$ is an application of the rule s.c., $\varphi$ will have the form $I\beta\alpha$ and will appear immediately below a wff of the form $I\alpha\beta$. If $I\beta\alpha$ is false and so no $\beta$ is an $\alpha$, then no $\alpha$ is a $\beta$ and $I\alpha\beta$ is false. By the hypothesis of the induction, it will rest on some false assumption, and therefore so will $I\beta\alpha$.

sub. If the final step leading to $\varphi$ is an application of sub., $\varphi$ will have the form $I\alpha\beta$ and will appear immediately below a wff $A\alpha\beta$. If $I\alpha\beta$ is false, then either there is no $\alpha$ or there is an $\alpha$ but no $\alpha$ is a $\beta$; in either case $A\alpha\beta$ is false. On the induction hypothesis $A\alpha\beta$ will then rest on some false assumption, and so therefore will $I\alpha\beta$.

Barbara. If the final step leading to $\varphi$ is an application of Barbara, $\varphi$ will have the form $A\alpha\gamma$ and will appear immediately below wffs of the forms $A\alpha\beta$ and $A\beta\gamma$. If $A\alpha\gamma$ is false, either there is no $\alpha$, in which case $A\alpha\beta$ is also false, or there is some $\alpha$ which is not a $\gamma$, in which case it cannot be true both that every $\alpha$ is a $\beta$ and that every $\beta$ is a $\gamma$ (i.e. at least one of the wffs $A\alpha\beta$, $A\beta\gamma$ must be false). Thus, if $A\alpha\gamma$ is false, so is at least one of $A\alpha\beta$, $A\beta\gamma$ on the hypothesis of the induction one of them must therefore rest on some false assumption. Since $A\alpha\gamma$ rests on all the assumptions on which $A\alpha\beta$ and $A\beta\gamma$ rest, it too will rest on some false assumption.

Celarent. If the final step leading to $\varphi$ is an application of Celarent, $\varphi$ will have the form $E\alpha\gamma$ and appear immediately below wffs of the form $A\alpha\beta$ and $E\beta\gamma$. If $E\alpha\gamma$ is false, some $\alpha$ is a $\gamma$, so that it cannot be the case both that every $\alpha$ is a $\beta$ and that no $\beta$ is a $\gamma$ (i.e. at least one of $A\alpha\beta$, $E\beta\gamma$ must be false). On the hypothesis of the induction one of $A\alpha\beta$, $E\beta\gamma$ must rest on some false assumption, and so therefore must $E\alpha\gamma$.

id. If the final step leading to $\varphi$ is an application of the weak identity rule id., $\varphi$ will have the form $A\alpha\alpha$ and appear immediately below a wff $I\alpha\beta$. If $A\alpha\alpha$ is false, there is no $\alpha$, and so $I\alpha\beta$ will also be false. On the induction hypothesis $I\alpha\beta$ will then rest on some false assumption, and so therefore will $A\alpha\alpha$.

r.a.a. If the final step leading to $\varphi$ is an application of r.a.a., $\varphi$ will be immediately below a pair of wffs $A\alpha\beta$, $O\alpha\beta$ or a pair $E\alpha\beta$, $I\alpha\beta$. Under Interpretation I one of each of these pairs must be false, and on the induction hypothesis this false wff will rest on some false assumption. $\varphi$ will therefore also rest on the false assumption unless this is the wff which is discharged when $\varphi$ is derived, viz. the wff $\overline{\varphi}$. But if $\varphi$ is false, $\overline{\varphi}$ will be true and so cannot be the false assumption in question.

This completes the induction step. We have now shown that all wffs of rank 1 are soundly derived and that, if all wffs of rank lower than $k$ ($k > 1$) are soundly derived, then so are wffs of rank $k$. It follows by the principle of strong mathematical induction that formulas of any rank whatsoever are soundly derived. All derivable sequents, then, are valid. Q.E.D.

The consistency of BS, which we establised in the last section, can also be proved as a corollary of the metatheorem just proved. If there were some negative theorem, then we should have some derivable sequent $\vdash E\alpha\beta$ or $\vdash O\alpha\beta$, and hence it would be the case that $\Vdash E\alpha\beta$ or $\Vdash O\alpha\beta$. But wffs of both the forms in question have false substitution instances (e.g. *No triangle is a triangle*, *Not every triangle is a triangle*).[2] And we have seen that, if there are no negative theorems, the system is consistent.

In a similar way we can also go on to show that the only theorems in the stronger system $BS^+$ are of the form $A\alpha\alpha$ or $I\alpha\alpha$. There are no theorems of the form $A\alpha\beta$ or $I\alpha\beta$ ($\alpha \neq \beta$), since they have such false substitution instances as *Every triangle is a circle* and *Some triangle is a circle.*[3]

The soundness of BS also follows from the fact whose proof is indicated in Chapter 5, viz. that BS is translatable into a fragment of the predicate calculus as standardly interpreted, which is known to be sound.

## 2.3 *Independence*

The methods of the last section enable us to show that each of the six rules is independent of the others, in the sense that none of the six rules is a derived rule of the system. We take each rule in turn and produce an interpretation under which the other rules are sound but the rule under examination is not. The rule must then be independent, since

[2] Cf. footnote 1.

[3] Cf. footnote 1.

it enables you to construct proofs not possible with the other rules alone. In each case the soundness of the other rules is to be shown by means of the type of inductive proof given in the last section.

One way of proving independence is to make use of interpretations in which $A$, $E$, $I$ and $O$ signify relations between positive integers and in which the variables accordingly take numbers as values. To prove the independence of s.c., for example, we may take

$A\alpha\beta$ as $\alpha = \beta$ $\quad E\alpha\beta$ as $\alpha > \beta$ $\quad I\alpha\beta$ as $\alpha \leq \beta$ $\quad O\alpha\beta$ as $\alpha \neq \beta$.

Now, clearly, under this interpretation a simple conversion like that from $Iab$ to $Iba$ is unsound: let $a = 1$, $b = 2$ and you have:- $1 \leq 2$ (true), therefore $2 \leq 1$ (false). The other rules will, however, be found valid on this interpretation.

sub. Interpret $A\alpha\beta$ as $\alpha = \beta$, $E\alpha\beta$ as $\alpha = \beta$, $I\alpha\beta$ as $\alpha \neq \beta$ and $O\alpha\beta$ as $\alpha \neq \beta$. The invalidity of the inference from $Aab$ to $Iab$ under this interpretation is evident if you take $a = 1$, $b = 1$; the other rules are sound.

To prove the independence of Barbara we may follow Łukasiewicz (1957, p.90) and interpret $A\alpha\beta$ as $\alpha + 1 \neq \beta$, $E\alpha\beta$ as $\alpha + \beta \neq \beta + \alpha$, $I\alpha\beta$ as $\alpha + \beta = \beta + \alpha$ and $O\alpha\beta$ as $\alpha + 1 = \beta$. The falsity of the sequent $\{Aab, Abc\} \vdash Aac$ can then be demonstrated by taking $a = 2$, $b = 1$ and $c = 3$, which makes $Aab$ $2+1 \neq 1$, $Abc$ $1+1 \neq 3$ (both true) and $Aac$ $2+1 \neq 3$ (false). The other rules can be shown to be sound on on this interpretation.

A somewhat more complex interpretation seems to be necessary to prove the independence of Celarent. I propose the following:

$A\alpha\beta$ : $\alpha$ and $\beta$ do not differ by exactly 1 and $\alpha + 1 > \beta$.

$E\alpha\beta$ : $\alpha$ and $\beta$ differ by exactly 1.

$I\alpha\beta$ : $\alpha$ and $\beta$ do not differ by exactly 1.

$O\alpha\beta$ : $\alpha$ and $\beta$ differ by exactly 1 or $\alpha + 1 \leq \beta$.

On this interpretation all the rules except Celarent are sound. The invalidity of Celarent can be shown in the following way. If Celarent were sound the sequent $\{Aab, Ebc\} \vdash Eac$ would be true under the interpretation; but it is not, as is clear if you take $a$ as 4, $b$ as 1 and $c$ as 2. Then $Aab$, $Ebc$ are:

4 and 1 do not differ by exactly 1, and $4+1 > 1$

1 and 2 differ by exactly 1,

which are true. $Eac$ is the false proposition:

4 and 2 differ by exactly 1.

Both the strong[4] and the weak forms of the identity rule can be proved independent of the other rules by means of the following interpretation:

take $A\alpha\beta$ as $\alpha > \beta$, $E\alpha\beta$ as $\alpha+\beta \neq \beta+\alpha$,

$I\alpha\beta$ as $\alpha+\beta = \beta+\alpha$, $O\alpha\beta$ as $\alpha \leq \beta$.

Finally, r.a.a. can be proved independent if all four forms, $A\alpha\beta$, $E\alpha\beta$, $I\alpha\beta$ and $O\alpha\beta$ are interpreted as $\alpha = \beta$. The other rules are sound on this interpretation, but r.a.a. is not. For consider the following derivation:

$$\frac{\begin{matrix} [Oab] \\ Acd \quad Ocd \end{matrix}}{Aab}\,\text{r.a.a.}$$

If $a$, $c$ and $d$ each have the value 1 and $b$ has the value 2, the premisses $Acb$ and $Ocd$ each become the true proposition $1 = 1$ and the conclusion becomes the false proposition $1 = 2$.

[4] The independence of the strong rule id.$^+$ also follows from the fact that BS but not BS$^+$ is sound under Interpretation I.

# CHAPTER 3

## METATHEORY OF BS: (II) COMPLETENESS

### 3.1 *Aristotle's method of invalidation*

We have shown that our basic syllogistic system BS derived from Aristotle is sound if interpreted on medieval lines. The present chapter is mainly concerned with proving the converse result, that it is possible to derive every valid form expressible within the system. Aristotle himself has made an important, though relatively neglected, contribution here, and we shall begin with some examination of it.

A deductive inference is valid iff one cannot compatibly assert its premisses and deny its conclusion. For, if the premisses entail the conclusion, they must be incompatible with any proposition which is itself incompatible with that conclusion. The inference from

{Every man is a featherless biped, Every Greek is a man}

to

Every Greek is a featherless biped

is valid since the set consisting of the premisses and the contradictory of the conclusion:

{Every man is a featherless biped, Every Greek is a man,
Not every Greek is a featherless biped}

has incompatible members. In general an inference from $\{Abc, Aab\}$ to $Aac$ is valid if the set $\{Abc, Aab, Cac\}$ is (simultaneously) unsatisfiable. If we have a set for which we can find substitutions to transform all its formulas into true propositions, we have thereby shown it to be satisfiable: for example, the wffs of the set $\{Oab, Aba\}$ have as instances *Not every animal is a man* and *Every man is an animal*, and so there can be no valid inference from $\{Oab\}$ to $Oba$, since it is possible to assert $Oab$ and the contradictory of $Aba$ without incompatibility ($25^a22$–6).

This manner of demonstrating the invalidity of an inference pattern is the essence of Aristotle's method. He wants to show that his system is complete in the sense of enabling him to derive all the inference patterns he regards as valid syllogistic moods. To this end he takes each of the pairs of premiss forms in the various moods and seeks to show either that the pair yields a conclusion or that any syllogistic inference from those premisses is invalid. Thus he takes the syllogistically 'non-probative'[1] pair of premisses *AE* in the first figure, $\{Aba, Ecb\}$, and provides the trio of terms *animal*, *man* and *horse* as respective substituends for $a$, $b$ and $c$, giving us the true propositions

Every man is an animal
No horse is a man

In this way he is able to show that the set $\{Aba, Ecb, Aca\}$ is satisfiable, since substitution in *Aca* yields the true proposition *Every horse is an animal*, which taken with the other two propositions gives us a trio of true propositions instantiating the wffs of the set. Now if there were a valid inference from $\{Aba, Ecb\}$ to *Eca*, the set would have to be unsatisfiable, since *Aca* is the contrary of *Eca*. And the set would also have to be unsatisfiable if there were an inference from $\{Aba, Ecb\}$ to *Oca*, since *Aca* and *Oca* are contradictories. So the demonstration of the satisfiability of the set is sufficient to show that the two inference patterns are invalid.

Aristotle makes use of similar substituends to show that *AE* in the first figure do not have affirmative consequences either. The terms *animal*, *man* and *stone* are supplied as major, middle and minor terms and their substitution for $a$, $b$, and $c$ in the set $\{Aba, Ecb, Eca\}$ generates the true propositions: *Every man is an animal*, *No stone is a*

[1] I take the term from Kneale and Kneale (1962). Barnes' word, translating Patzig (1968), is 'inconcludent', which though somewhat of a barbarism is more accurate, since proof requires more than validity (as Aristotle himself points out elsewhere).

*man, No stone is an animal.* It follows that there is no valid inference from $\{Aba, Ecb\}$ either to $Aca$ or to $Ica$.

It is important to be clear what is being asserted of an inference pattern when it is described as invalid: it is *equivalent* to the claim that it has some invalid substitution instance. (Cf. fn. 8 in Chapter 1.) One should avoid falling into the mistake of thinking that *every* substitution instance of every invalid pattern is itself invalid.[2] A pattern is valid iff every instance is valid, and so therefore a pattern is invalid iff it is not the case that every instance is valid, i.e. (since if well-formed it will have instances) iff some instance is invalid.[3]

Aristotle did not find it necessary to produce fresh substituends for every non-probative pair of premisses, since, having shown that certain pairs yielded no syllogistic consequence, he was able to argue that certain others must also be non-probative. Having shown that $AE_1$ ($AE$ in the first figure) is a non-probative pair, for example, it is easy for him to show that the same is true of $AO_1$.[4] If $AO_1$ had some syllogistic consequence, then since $E$ entails $O$, $AE_1$ would have the same consequence; but this has already been ruled out.

It is evident that the Aristotelian method of providing substituends suffices to demonstrate the completeness of the system with respect to sequents with just two premisses. Although Aristotle's own treatment falls short of actually doing this in a number of minor respects, which are briefly detailed below, there is clearly no difficulty in completing

[2] Cf. J. Willard Oliver (1967).

[3] Thus an invalid pattern may have a valid instance. For example, the pattern $Aba$, $Ecb$ / $Eca$ ($AEE$ in the first figure) has the valid instance: *Every animal is an animal, No stone is an animal; therefore no stone is an animal.* The example is valid, of course, because it also instantiates the more specific valid pattern $Aaa$, $Eba$ / $Eba$.

[4] Actually in this case Aristotle uses both methods.

the task by means of his technique (which is not to say that the method is beyond criticism, as we shall see shortly).

1. In one or two cases Aristotle thinks invalidity cannot be demonstrated by directly providing terms to satisfy the wffs of the associated set. An example is $EOO_2$: *Emo, Ono / Onm*. This pattern is invalid if the set {*Emo, Ono, Anm*} is satisfiable. Now having shown directly that $EEO_2$ is invalid, Aristotle can argue that it follows that $EOO_2$ is invalid as well. But why should it not be possible to demonstrate this directly? It seems easy to think of substituends to transform the wffs of the set into a set of true propositions:

{No bird is a man, Not every rook is a man,
Every rook is a bird}

Yet Aristotle would not be happy with the second of these propositions, on the ground that to utter it is falsely to imply that some rook *is* a man. He would prefer to confine the use of *O* propositions to cases where the corresponding *E* proposition is false. (This is presumably why he ignores the weakened moods.) The difficulty is not confined to the particular example chosen, since *Eno* will always be true for values which verify the other two formulas, *Emo, Anm* - otherwise Celarent would not be a sound rule. But we should surely not regard this as a genuine difficulty, since all that matters is that *Not every rook is a man*, or whatever *O* proposition we choose for this purpose, should be *true*, and that can scarcely be doubted. It may be a misleading sentence to utter when the corresponding *E* proposition is also true, but Aristotle can hardly deny its truth if he regards subaltern inference as a sound principle.

2. Aristotle restricts his attention to syllogistic moods in the three figures he recognizes and to patterns with three distinct variables. A pair of premisses is either shown to yield a conclusion or substituends are provided for the purpose of showing that the pair is syllogistically

non-probative. Nothing is said about a case like $AAE_1$ which, though invalid, does not have entirely non-probative premisses (since $AAA_1$, $AAI_1$ are provable).[5] Such cases can easily be dealt with by providing appropriate substituends, as can the invalid moods of the fourth figure and invalid moods with only one or two distinct variables. In the case of some valid two-premiss patterns with only two distinct variables, it is necessary to use the weak identity rule id. to prove them.

3. It is, of course, important that the propositions resulting from the proposed substitutions should actually be true. But Aristotle's examples do not always meet this requirement: he wrongly assumes the truth, for example, of the propositions *No snow is black* and *Every swan is white* (as Geach points out (1971), p. 298). Indeed, examples like the last are best avoided anyway, since, even if we knew that there had never in the past been any swans which were not white, we should probably not be in a position to know that none would ever evolve in the future. But admittedly many of Aristotle's examples are immune from this sort of objection, because they are analytically true.

What now of inferences with more than two premisses? In I 25 Aristotle seems to be arguing, in effect, that any valid inference with more than two premisses can be resolved into a chain of valid (two-premiss) syllogisms. If this argument had been successful, it would have been sufficient to prove completeness for inferences with two or more premisses, given a proof of completeness for two-premiss inferences. (For comment on Aristotle's arguments, see Smiley (1974)).

[5] No doubt he thought it unnecessary to show that $AA_1$ did not have negative consequences, since he had shown that you could derive the $A$ conclusion, which is incompatible with the corresponding $E$ and $O$ forms. And, indeed, if the derivation is sound and the premisses are themselves mutually compatible, the set $\{Aba, Acb, Aca\}$ must be satisfiable. With three distinct variables the two premiss formulas are, in fact, always mutually compatible, sharing as they do just one variable. These considerations will not provide for cases like $AIA_3$ and $AIO_3$, however. $AII_3$ (Datisi) is validly derivable, but one cannot argue immediately from this to the invalidity of $AIA_3$ and $AIO_3$.

It is clear, anyway, that we have to go beyond the *ad hoc* provision of substituends if we are to show that BS is complete in the sense that all valid sequents with any finite number of premisses are derivable, since there are infinitely many such valid sequents. We need to show that all the infinitely many sets associated with underivable patterns are satisfiable. Which sets are those, precisely? They turn out to be those sets which do not yield any pair of wffs $\varphi$ and $\overline{\varphi}$, sets which we shall call *consistent* (sets, that is, which are consistent with respect to derivability). If such a pair of wffs *is* derivable from the set we shall call that set 'inconsistent'. It is not difficult to see that $\Gamma \vdash \varphi$ iff its associated set $\Gamma \cup \{\overline{\varphi}\}$ is inconsistent. For if $\Gamma \vdash \varphi$, $\varphi$ is derivable from $\Gamma$, and $\overline{\varphi}$ is derivable from itself. And, if $\Gamma \cup \{\overline{\varphi}\}$ yields corresponding $A/O$ or $E/I$ wffs, a further step of r.a.a. will permit the derivation of $\varphi$ from $\Gamma$. So if we can show that all consistent sets are satisfiable, we can show that all underivable sequents are invalid and therefore that the system is complete. What Aristotle did - in effect - was to show certain consistent sets with three wffs were satisfiable.

It is important for the purposes of the present chapter to be clear about the distinction between the notion of (in)consistency on the one hand and of (un)satisfiability on the other. A *consistent* set has the syntactic property of failing to yield any pair of wffs $\varphi$, $\overline{\varphi}$ by means of the rules of inference. A *satisfiable* set has the semantic property of containing wffs all of which can simultaneously be turned into true propositions by means of uniform substitutions on the variables. Satisfiability, unlike consistency, is therefore relative to an interpretation. A system which is both sound and complete is one in which all and only consistent sets are satisfiable, one, that is, in which the two properties of sets of wffs are extensionally equivalent. (For convenience we shall count the empty set as satisfiable.)

It is possible to describe general ways of finding terms to satisfy the wffs of consistent sets. But the use of non-logical terms of the sort supplied by Aristotle will seem an impurity to many logicians, even when the resulting propositions are analytic; as Geach says: 'if we know a form to be invalid, it can only be through lack of ingenuity that we fail to find a counter-example to it outside a specific subject-matter, since logic applies to all subject matters alike' (1971), p. 279. The substituends used in the completeness proof which follows in the next section are consequently of a less subject-bound character, but the central idea of the proof remains Aristotle's basic insight that there is no valid inference from $\Gamma$ to $\varphi$ if the members of some instance of $\Gamma \cup \{\overline{\varphi}\}$ are mutually compatible. This is, indeed, the basic idea behind all Henkin-style proofs of completeness. The proof is adapted and simplified from Corcoran (1973).

### 3.2 *Completeness of BS*

In the present section we prove that the basic system BS is complete with respect to the interpretation of the last chapter (Interpretation I). We shall indicate in later chapters how this proof may be adapted to prove completeness with respect to somewhat different interpretations. But throughout this section 'valid' is to mean *valid under Interpretation I*, and similar remarks apply to 'satisfiable', 'unsatisfiable' and '$\Vdash$' (the symbol for semantic entailment). Since we have defined a *derivation* in the system in such a way that there are always finitely many premisses, sets of wffs mentioned in the proof are all meant to be finite.[6]

[6] Although it would be possible to extend the system to deal with inferences from infinitely many premisses and to develop the methatheory accordingly, it does not seem worthwhile for so restricted a system.

*Metatheorem 3* (Completeness). *If* $\Gamma \Vdash_I \varphi$, *then* $\Gamma \vdash_{BS} \varphi$.

*Proof.* (1) If $\Gamma \Vdash \varphi$, then $\Gamma$, $\bar{\varphi}$ is unsatisfiable.
('$\Gamma$, $\bar{\varphi}$' abbreviates '$\Gamma \cup \{\bar{\varphi}\}$'.)

(2) If $\Gamma$, $\bar{\varphi}$ is unsatisfiable, it is inconsistent.

(3) If $\Gamma$, $\bar{\varphi}$ is inconsistent, $\Gamma \vdash \varphi$.

The theorem follows from (1), (2) and (3).

(1) and (3) are easily proved.

(1) If $\Gamma \Vdash \varphi$, then there are no uniform substitutions under which $\varphi$ can come out false without at least one wff in $\Gamma$ also coming out false. Any substitutions which turn $\bar{\varphi}$ into a true proposition must turn $\varphi$ into a false one, for under Interpretation I corresponding *A/O* wffs must have opposite truth-values for uniform substitutions, and the same is true of corresponding *E/I* wffs. Hence uniform substitutions which make $\bar{\varphi}$ true must make some wff in $\Gamma$ false: accordingly no uniform substitutions can make all the wffs of the set $\Gamma$, $\varphi$ come out true, and it is unsatisfiable.

(3) If corresponding *A/O* or *E/I* wffs are derivable by the rules from $\Gamma$, $\bar{\varphi}$, then $\varphi$ is derivable from $\Gamma$ in one further step by r.a.a. (given the relation between $\varphi$ and $\bar{\varphi}$ defined on p. 16 above).

(2) follows from (indeed is equivalent to) (2): *Every consistent set of wffs is satisfiable*. Its demonstration constitutes the bulk of the completeness proof:

Let $\Delta$ be a (finite) consistent set of wffs;

V, the set of variable letters in the wffs of $\Delta$;

P(V), the power set of V (i.e. the set of subsets of V).

Then suppose that a set U($\Delta$) is formed from P(V) in the following way:

for every wff $A\alpha\beta$ in $\Delta$, each set containing the letter $\alpha$ but lacking the letter $\beta$ (each $[\alpha\beta']$) is deleted from P(V);

for every wff $E\alpha\beta$ in $\Delta$, each set containing both $\alpha$ and $\beta$ (each $[\alpha\beta]$) is deleted from P(V) (and, consequently, for every $E\alpha\alpha$ in $\Delta$, each $[\alpha]$ is deleted);

for every wff $O\alpha\alpha$ in $\Delta$, each $[\alpha]$ is deleted.

The wffs of $\Delta$ are to be turned into propositions by interpreting the constants $A$, $E$, $I$ and $O$ according to Interpretation I and making substitutions on the variables in accordance with the following prescription:

for each variable $\alpha$ substitute the corresponding term *set in* $U(\Delta)$ *containing the letter* $\alpha$, or '$[\alpha]$' for short.

*Example*:

| | |
|---|---|
| $\Delta$: | $\{Aab, Oba, Ecb, Iab, Abb, Oac, Occ\}$ |
| V: | $\{a, b, c\}$ |
| P(V): | $\{\Lambda, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ |
| U($\Delta$): | $\{\Lambda, \{b\}, \{a,b\}\}$ |

$Occ$ alone excludes all sets containing $c$, and $\{a\}$ is excluded by $Aab$, leaving three sets in U($\Delta$). It is easy to see that on Interpretation I the substitutions for variable letters in the wffs of $\Delta$ result in true propositions. $Aab$, for example, becomes Every set in U($\Delta$) containing the letter $a$ is a set in U($\Delta$) containing the letter $b$ or Every $[a]$ is a $[b]$.

We now proceed to show that the prescribed substitutions will always result in a set of true propositions, no matter what the contents of the consistent set.

Any formula of the form $E\alpha\beta$, $E\alpha\alpha$ or $O\alpha\alpha$ (*Not every* $\alpha$ *is an* $\alpha$) will be turned into a true proposition, because any set which could make it false will have been deleted from P(V) in the formation of U($\Delta$). So it remains to consider wffs in $\Delta$ of the forms $A\alpha\beta$, $I\alpha\beta$, and (where $\alpha$ and $\beta$ are distinct) $O\alpha\beta$.

1. Suppose that a wff $A\alpha\beta$ occurs in $\Delta$ and the prescribed substitutions render it false. It cannot be falsified by the presence in $U(\Delta)$ of some set $\{\alpha\beta'\}$, since all such sets will have been deleted. So its falsity must be due to the absence of sets containing $\alpha$. (Remember that $A$ wffs are to be interpreted as having existential import.)

Now consider the set $\{\alpha, \gamma_1, \ldots, \gamma_n\}$, $n \geq 1$, in P(V), where $\gamma_1, \ldots, \gamma_n$ are all the $\gamma_i$'s in V such that $\Delta \vdash A\alpha\gamma_i$. ($\beta$, of course, will be one of the $\gamma_i$'s.) We show that, since $\Delta$ is consistent, this set will not be deleted from P(V), and therefore that $A\alpha\beta$ must after all be turned into a true proposition.

The set cannot be deleted because of an $A$ wff in $\Delta$. Such a wff would have to be of the form $A\gamma\delta$ or $A\gamma_i\delta$, where $\delta$ did not belong to the set being deleted. But if a wff $A\alpha\delta \in \Delta$, $\delta$ is one of the $\gamma_i$'s. And if some $A\gamma_i\delta \in \Delta$, $\Delta$ yields $A\alpha\gamma_i$: $A\gamma_i\delta$, $A\alpha\gamma_i \vdash A\alpha\delta$ in virtue of of the rule Barbara; and so again $\delta$ is one of the $\gamma_i$'s.

Nor can the set be deleted because of some $E$ wff in $\Delta$. Four types of $E$ wff would lead to the deletion of the set, viz. $E\alpha\alpha$, $E\alpha\gamma_i$, $E\gamma_i\alpha$ or $E\gamma_i\gamma_j$ (where $i$ may or may not equal $j$). We can show, however, that the occurrence of any wff of these types would mean that the set was inconsistent. Thus $A\alpha\beta \vdash I\alpha\alpha$:

$$\frac{\dfrac{\dfrac{A\alpha\beta}{I\alpha\beta}\,\text{sub.}}{A\alpha\alpha}\,\text{id.}}{I\alpha\alpha}\,\text{sub.}$$

so that the occurrence of $E\alpha\alpha$ in $\Delta$ would make it inconsistent. Again, suppose $E\alpha\gamma_i \in \Delta$: then $\Delta$ yields $A\alpha\gamma_i$ and therefore $I\alpha\gamma_i$, making the set inconsistent. The same is true if $E\gamma_i\alpha \in \Delta$, since $E\gamma_i\alpha \vdash E\alpha\gamma_i$ in virtue of s.c. ($E$). Finally, suppose $E\gamma_i\gamma_j \in \Delta$. Then $\Delta \vdash A\alpha\gamma_i$. $A\alpha\gamma_i$, $E\gamma_i\gamma_j \vdash E\alpha\gamma_j$ owing to Celarent, and therefore $\Delta \vdash O\alpha\gamma_j$. But $\Delta$ will also yield $A\alpha\gamma_j$ and so be inconsistent.

Nor can the set be deleted because of some $O$ wff in $\Delta$. This would have to be of the form $O\alpha\alpha$ or $O\gamma_i\gamma_i$, If it were $O\alpha\alpha$, $\Delta$ would be inconsistent since $A\alpha\beta$ yields $A\alpha\alpha$. If it were $O\gamma_i\gamma_i$, $\Delta$ would yield $A\alpha\gamma_i$, which in turn yields $A\gamma_i\gamma_i$:

$$\dfrac{\dfrac{A\alpha\gamma_i}{I\gamma_i\alpha}\ \text{c.p.a.}}{A\gamma_i\gamma_i}\ \text{id.}$$

Thus if $A\alpha\beta$ belongs to a consistent set $\Delta$, substitution according to the prescription must produce a true proposition. And since none of the argument above for the case of $A\alpha\beta$ depends on $\alpha$'s being distinct from $\beta$, this conclusion applies equally to the special case of $A\alpha\alpha$.

2. Suppose a wff $I\alpha\beta$ occurs in $\Delta$ and the prescribed substitutions render it false. Then there is no $[\alpha\beta]$ in $U(\Delta)$.

Now consider the set $\{\alpha, \beta, \gamma_1 \ldots, \gamma_n\}$, $n \geq 0$, in P(V), where $\gamma_1, \ldots, \gamma_n$ are all the $\gamma_i$'s in V such that $\Delta \vdash A\alpha\gamma_i$ or $\Delta \vdash A\beta\gamma_i$. If $I\alpha\beta$ is to turn out false, the contents of $\Delta$ must require the deletion of this set.

No $A$ wff can result in its deletion, since, if $A\alpha\delta$ occurs in $\Delta$, $\delta$ will be one of the $\gamma_i$'s; and similarly if $A\beta\delta$ occurs in $\Delta$. And if some $A\gamma_i\delta$ occurs in $\Delta$, $\Delta$ yields $A\alpha\gamma_i$ or $A\beta\gamma_i$, which together with $A\gamma_i\delta$ yields $A\alpha\delta$ or $A\beta\delta$ so that once again $\delta$ will be one of the $\gamma_i$'s.

If the set is deleted, it must, then, be due to some negative wff in $\Delta$. We list below all the possible types of $E$ and $O$ wff which could have this effect, together with indications of the reason why in each case the wff can occur in $\Delta$ only on pain of $\Delta$'s inconsistency.

$E\alpha\alpha$. $\quad$ $I\alpha\beta$ yields $I\alpha\alpha$.

$E\beta\beta$. $\quad$ $I\alpha\beta$ yields $I\beta\beta$ (by s.c., id., sub.).

$E\alpha\beta$.

$E\beta\alpha$, which yields $E\alpha\beta$.

$E\alpha\gamma_i$. Then either $\Delta \vdash A\alpha\gamma_i$, and so $\Delta \vdash I\alpha\gamma_i$; or $\Delta \vdash A\beta\gamma_i$, which together with $E\gamma_i\alpha$ (derivable by s.c.$(E)$ from $E\alpha\gamma_i$) yields $E\beta\alpha$, and so $E\alpha\beta$.

$E\gamma_i\alpha$. Then either $\Delta \vdash A\alpha\gamma_i$, and so $\Delta \vdash I\gamma_i\alpha$ in virtue of c.p.a.; or $\Delta \vdash A\beta\gamma_i$, which together with $E\gamma_i\alpha$ yields $E\beta\alpha$, and so $E\alpha\beta$.

$\left.\begin{array}{l} E\beta\gamma_i. \\ E\gamma_i\beta. \end{array}\right\}$ Reasons parallel to the last two cases.

$E\gamma_i\gamma_j$ (including cases where $i = j$). Either: $\Delta \vdash A\alpha\gamma_i$, which with $E\gamma_i\gamma_j$ yields $E\alpha\gamma_j$. $\Delta$ will also yield $A\alpha\gamma_j$ (and so $I\alpha\gamma_j$), or it will yield $A\beta\gamma_j$, from which it will be possible, with $E\alpha\gamma_j$, to derive $E\alpha\beta$. Or: $\Delta \vdash A\beta\gamma_i$, which with $E\gamma_i\gamma_j$ yields $E\beta\gamma_j$. $\Delta$ will also yield $A\beta\gamma_j$ (and so $I\beta\gamma_j$) or it will yield $A\alpha\gamma_j$, which with $E\beta\gamma_j$ yields $E\alpha\beta$.

$O\alpha\alpha$. $I\alpha\beta \vdash A\alpha\alpha$.

$O\beta\beta$. $I\alpha\beta \vdash A\beta\beta$.

$O\gamma_i\gamma_i$. $\Delta \vdash A\alpha\gamma_i$ or $A\beta\gamma_i$, both of which yield $A\gamma_i\gamma_i$.

Thus $I\alpha\beta$ can be deleted neither by an affirmative nor by a negative wff, and substitution in $I\alpha\beta$ according to the prescription must produce a true proposition. Since none of the argument above depends on $\alpha$'s being distinct from $\beta$, the conclusion applies equally to the special case of $I\alpha\alpha$.

3. Suppose a wff $O\alpha\beta$ occurs in $\Delta$ and that $\alpha \neq \beta$. If the prescribed substitutions make it false, U($\Delta$) must contain $[\alpha]$ (since $O\alpha\beta$ lacks existential import and is true if U($\Delta$) lacks $[\alpha]$) but no $[\alpha\beta']$.

Consider the set $\{\alpha, \gamma_1, \ldots, \gamma_n\}$, $n \geq 0$, where $\gamma_1, \ldots, \gamma_n$ are all the $\gamma_i$'s in V such that $\Delta \vdash A\alpha\gamma_i$. $\beta$ cannot be one of the $\gamma_i$'s, since $\Delta$ would then yield $A\alpha\beta$ and so be inconsistent. So, if $O\alpha\beta$ is transformed into a false proposition, the set under consideration will have to be deleted. But, for reasons similar to those given under 1 above, it cannot be deleted by an $A$ wff.

So it will have to be deleted by a negative wff. We show that none of the negative wffs which could have this effect can belong to $\Delta$ without making it inconsistent, unless it also excludes every $[\alpha]$ and so verifies the substituend of $O\alpha\beta$.

| | |
|---|---|
| $E\alpha\alpha$. | Would exclude every $[\alpha]$. |
| $E\alpha\gamma_i$, | $\Delta \vdash A\alpha\gamma_i$, which yields $I\alpha\gamma_i$. |
| $E\gamma_i\alpha$. | $\Delta \vdash A\alpha\gamma_i$, which yields $I\gamma_i\alpha$. |
| $E\gamma_i\gamma_j$ | (including cases where $i = j$). $\Delta \vdash A\alpha\gamma_i$, which with $E\gamma_i\gamma_j$ yields $E\alpha\gamma_j$. But $\Delta \vdash A\alpha\gamma_j$, which yields $I\alpha\gamma_j$. |
| $O\alpha\alpha$. | Would exclude every $[\alpha]$. |
| $O\gamma_i\gamma_i$. | $\Delta \vdash A\alpha\gamma_i$, which yields $A\gamma_i\gamma_i$. |

This concludes the proof of Metatheorem 3.

### 3.3 *Decision and proof procedures for BS*

The completeness proof of the last section gives us a method of demonstrating the satisfiability of any (finite) consistent set. We now prove, as a corollary of Metatheorem 2 of the last chapter, that every satisfiable set is consistent.

Suppose $\Delta$ is a satisfiable set which is inconsistent: then some pair $\varphi$, $\bar{\varphi}$ can be derived from the set. *Ex hypothesi* there is some uniform way of substituting for the variables in the wffs of the set to to make them all come out true under Interpretation I. Substitutions in $\varphi$, $\bar{\varphi}$ uniform with these must render one of the pair false. But then the system would be unsound, contradicting Metatheorem 2.

So we may conclude that a set is consistent iff it is satisfiable, and so inconsistent iff unsatisfiable. Since every consistent set is satisfiable by the substitutional procedure of the last section, a set is satisfiable iff it is satisfiable by that method. This means that we have an effective procedure for identifying unsatisfiable sets, and consequently for identifying correct sequents. This decision procedure will often prove impracticable to operate, however, since if there is any appreciable number of different variables, say 10, in the formulas of the set being tested, P(V) will have very many members: 10 variables in V mean $2^{10}$ (1,024) members of P(V), which will consequently take rather a long time to construct without the help of a computer.

*Illustrations*

1. Does *Oac* follow from $\{Oab, Abc\}$?

Test the set $\{Oab, Abc, Aac\}$ for satisfiability. V and P(V) are as in the example on p. 43. U($\Delta$) is:

$$\{\Lambda, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\},$$

which verifies the set under the prescribed substitutions. The inference is therefore invalid.

2. Does *Oac* follow from $\{Oab, Acb\}$?

Test the set $\{Oab, Acb, Aac\}$. U($\Delta$) is:

$$\{\Lambda, \{b\}, \{b,c\}, \{a,b,c\}\},$$

of which it is false that not every $[a]$ is a $[b]$. The inference is therefore valid.

It is possible, however, to give a fairly brief list of the types of inconsistent/unsatisfiable set, and to show how to derive corresponding $A/O$ or $E/I$ wffs from them. This not only gives us a simpler decision procedure but also a proof procedure, that is, a mechanical way of generating a proof for any derivable sequent. For, since a sequent $\Gamma \vdash \varphi$ is derivable iff $\Gamma, \bar{\varphi}$ is inconsistent and we have a method of deriving a pair $\psi$, $\bar{\psi}$ from any inconsistent set, an additional step of r.a.a. will complete the derivation of $\varphi$ from $\Gamma$.

In listing the types of inconsistent/unsatisfiable set the notion of a *chain of* A wffs is employed. An $\alpha$-$\beta$ chain is either the single wff $A\alpha\beta$ or a series of two or more $A$ wffs $A\alpha\gamma_1, \ldots, A\gamma_i\gamma_{i+1}, \ldots A\gamma_n\beta$, $n \geq 2$. A set is said to contain such a chain when it contains all the wffs in such a chain. Thus a set which contains the wffs $Aac$, $Acd$, $Adb$, $Abe$ contains an $a$-$e$ chain.

A chain whose last wff has $\beta$ as predicate variable is a chain *to* $\beta$. If there is an $\alpha$-$\beta$ chain we shall say $\alpha$ *is chained to* $\beta$.

The following are the inconsistent cases, with indications, where necessary, of the way to make the derivations of inconsistent formulas:

(1) The set contains some negative wff $E\alpha\alpha$ or $O\alpha\alpha$ as well as some affirmative wff with a variable letter $\alpha$. $O\alpha\alpha$ is derivable from $E\alpha\alpha$ or from itself, and $A\alpha\alpha$ from $A\alpha\beta$ or $A\beta\alpha$ or $I\alpha\beta$ or $I\beta\alpha$. For example:

$$\dfrac{\dfrac{A\alpha\beta}{I\alpha\beta}\ \text{sub.}}{A\alpha\alpha}\ \text{id.} \qquad\qquad \dfrac{\dfrac{A\beta\alpha}{I\alpha\beta}\ \text{c.p.a.}}{A\alpha\alpha}\ \text{id.}$$

(2) The set contains some negative wff $O\alpha\beta$, $\alpha \neq \beta$ and there is an $\alpha$-$\beta$ chain. $A\alpha\beta$ is derivable from the $\alpha$-$\beta$ chain by successive applications of Barbara.

(3) The set contains some wff $E\alpha\beta$ ($\alpha \neq \beta$) with an $\alpha$–$\beta$ chain. $A\alpha\beta$ is derivable from the chain, and so therefore is $I\alpha\beta$.

(4) The set contains a wff $E\alpha\beta$ ($\alpha \neq \beta$) with a $\beta$–$\alpha$ chain. $A\beta\alpha$, and hence $I\alpha\beta$, is derivable from the chain.

(5) The set contains a wff $E\alpha\beta$ ($\alpha \neq \beta$) with $\gamma$–$\alpha$ and $\gamma$–$\beta$ chains. $A\gamma\alpha$, $A\gamma\beta$ are derivable from the respective chains. Then $E\gamma\beta$, $I\gamma\beta$ are derivable as follows:

$$\frac{A\gamma\alpha \quad E\alpha\beta}{E\gamma\beta}\ \text{Celarent} \qquad\qquad \frac{A\gamma\beta}{I\gamma\beta}\ \text{sub.}$$

(6) The set contains some wff $E\alpha\beta$ ($\alpha \neq \beta$) and a wff $I\alpha\beta$ or $I\beta\alpha$.

(7) There is a wff $E\alpha\beta$ ($\alpha \neq \beta$) and $I\alpha\gamma$ or $I\gamma\alpha$ with a $\gamma$–$\beta$ chain. $E\alpha\beta$ yields $E\beta\alpha$, which with $A\gamma\beta$ (derivable from the chain) yields $E\gamma\alpha$. $I\gamma\alpha$ is derivable, either from $I\gamma\alpha$ or from itself.

(8) There is a wff $E\alpha\beta$ ($\alpha \neq \beta$) and $I\beta\gamma$ or $I\gamma\beta$ with a $\gamma$–$\alpha$ chain. The chain yields $A\gamma\alpha$, which with $E\alpha\beta$ yields $E\beta\alpha$. $I\gamma\beta$ is also derivable, either from $I\beta\gamma$ or from itself.

(9) There is a wff $E\alpha\beta$ ($\alpha \neq \beta$) and $I\gamma\delta$ or $I\delta\gamma$ with a $\gamma$–$\alpha$ and a $\delta$–$\beta$ chain. $A\gamma\alpha$ and $A\delta\beta$ are derivable from the chains, and then $E\delta\gamma$ and $I\delta\gamma$ are derivable as follows:

$$\dfrac{A\delta\beta \qquad \dfrac{\dfrac{A\gamma\alpha \quad E\alpha\beta}{E\gamma\beta}\ \text{Celarent}}{E\beta\gamma}\ \text{s.c. }(E)}{E\delta\gamma}\ \text{Celarent} \qquad\qquad \dfrac{(I\gamma\delta)}{I\delta\gamma}\ \text{s.c.}$$

*Illustrations*

1. Is $Ogb$ a consequence of (and therefore derivable from) the set

$$\{Eab,\ Icd,\ Ocd,\ Ace,\ Aea,\ Adf,\ Afg\}?$$

Consider the set {*Eab*, *Icd*, *Ocd*, *Ace*, *Aea*, *Adf*, *Afg*, *Agb*}. The chains to *a* are:

*Aea*

**Ace*, *Aea*,

and the chains to *b*:

*Agb*

*Afg*, *Agb*

**Adf*, *Afg*, *Agb*.

The starred chain to *a* begins with the variable *c* and the starred chain to *b* begins with the variable *d*, and these two variables are linked in the formula *Icd*. The set therefore falls under case (9). *Edc*, *Idc* are derivable from the set in the following way:

$$\dfrac{Icd}{Idc}\,\text{s.c.} \qquad \dfrac{\dfrac{\dfrac{Adf \quad Afg}{Adg}\,\text{Barbara} \qquad Agb}{Adb}\,\text{Barbara} \qquad \dfrac{\dfrac{\dfrac{Ace \quad Aea}{Aca}\,\text{Barbara} \qquad Eab}{Ecb}\,\text{Celarent}}{Ebc}\,\text{s.c. }(E)}{Edc}\,\text{Celarent}$$

A proof of the original inference pattern, which has been shown to be both valid and derivable, is then obtained by applying r.a.a. and discharging *Agb* to derive *Ogb*. *Ocd* and *Egh* are written above the tree in an initial list.

2. Is *Ogb* a consequence of (and therefore derivable from) the set

{*Eab*, *Icd*, *Ocd*, *Ace*, *Aea*, *Ahf*, *Afg*}?

Consider the set {*Eab*, *Icd*, *Ocd*, *Ace*, *Aea*, *Ahf*, *Afg*, *Agb*}. The chains to *a* are the same as those in the first illustration and those to *b* are the same except that the longest chain begins with *Ahf* (instead of *Adf*). There is no negative formula with two occurrences of the same variable letter, so that the set does not fall under case (1). There is a wff *Ocd*, but no *c*–*d* chain: so it does not fall under case (2). Nor is there any *a*–*b* or $b$-$a$ chain: cases (3) and (4) are ruled out. No chain to $a$ shares a variable with any chain to $b$: this rules out case (5). There is no wff *Iab* or *Iba*, so ruling out case (6). There is no *I* wff containing either $a$ or $b$, thus ruling out (7) and (8). And although there is a wff *Icd* with a $c$-$a$ chain, there is no $d$-$b$ chain, so that case (9) is also ruled out, and we may conclude that the set is consistent and the original inference pattern neither valid nor derivable.

It is, of course, one thing to describe the decision and proof procedure and another to prove that it is adequate as such a procedure. A proof is given in the Appendix which follows.

## APPENDIX TO CHAPTER 3

### PROOF THAT THE PROCEDURE OF 3.3 IS AN ADEQUATE DECISION AND PROOF PROCEDURE

As a preliminary we prove two lemmas, the first relating to the model set U($\Delta$) described in 3.2:

*Lemma 1.* *If a set of affirmative wffs excludes every set* $[\alpha_1, \ldots, \alpha_n, \beta']$ *( where no* $\alpha_i = \beta$*), then for some* $\alpha_i$ *the set contains an* $\alpha_i$-$\beta$ *chain.*

*Proof.* Consider the set Z, $\{\alpha_1, \ldots, \alpha_n,\ \gamma_1, \ldots \gamma_k\}$, $n \geq 1$, $k \geq 0$, where $\gamma_1, \ldots, \gamma_k$ are all the $\gamma_i$'s chained to some $\alpha_i$ in the set of affirmative wffs. Then *either* $\beta \in$ Z, in which case $\beta$ will be one of the $\gamma_i$'s and so have an $\alpha_i$ chained to it;

*or* $\beta \notin$ Z, in which case, on the hypothesis of the lemma, Z is excluded by some affirmative wff A$\delta\eta$, $\delta \in$ Z, $\eta \notin$ Z. $\delta$ is either an $\alpha_i$ or has an $\alpha_i$ chained to it, so that an $\alpha_i$ is also chained to $\eta$, which is therefore one of the $\gamma_i$'s. Consequently $\eta \in$ Z. Contradiction.

Hence $\beta$ must belong to Z and be one of the $\gamma_i$'s to which an $\alpha_i$ is chained. Q.E.D.

*Lemma 2.* *If a wff* $\varphi$ *is derivable from a set of* $\Gamma$ *containing more than one negative wff, then it is derivable from some subset of* $\Gamma$ *containing at most one of the negative wffs.*

This is proved by showing:- that every affirmative wff in a proof which rests on a set $\Gamma$ is derivable from a subset of $\Gamma$ which either contains only affirmative wffs or is an inconsistent set just one member of which is negative; and that every negative wff in a proof which rests on a set $\Gamma$ is derivable from a subset of $\Gamma$ just one member of which is negative. (Thesis T.)

The proof is by strong induction on the rank of the wff $\varphi$ in a proof. (For the notion of rank see 2.1.

*Basis*. The formula is of rank 1, i.e. in the initial list or at a tip. The set $\Gamma$ on which it rests will therefore include the wff itself. But it is also derivable from itself alone, and hence from a subset of $\Gamma$. (If it is affirmative, that subset is an all-affirmative subset; if it is negative, the subset has only one member - its sole member - which is negative.)

*Induction step*. If the thesis (T) holds for all wffs in a proof of rank lower than $k$ ($k > 1$), it holds for wffs of rank $k$.

A wff of rank higher than 1 will be derived in a proof from a wff or wffs immediately above it by means of one of the rules of inference. We take each rule in turn.

s.c. $\varphi$ is immediately below just one affirmative wff. If $\varphi$ rests in the proof on the set $\Gamma$, then so does the wff immediately above it. By the hypothesis of the induction, that wff is derivable from an all-affirmative subset of $\Gamma$ or an inconsistent subset of $\Gamma$ with just one negative member. But such a derivation can be continued by an application of s.c. to derive $\varphi$ from that same subset.

sub. and id. The argument is exactly parallel to that for s.c.

Barbara. $\varphi$ is immediately below two affirmative wffs. If $\varphi$ rests in the proof on the set $\Gamma$, then the affirmative wffs immediately above it rest on subsets $\Delta$ and Z which exhaust $\Gamma$. By the hypothesis of the induction they are either both derivable from all-affirmative subsets of $\Delta$ and Z, or at least one is derivable from an inconsistent subset of $\Delta$ or Z just one member of which is negative. If both are derivable from all-affirmative subsets of $\Delta$, Z respectively, then the application of Barbara to the conclusions of the two derivations will result in a derivation of $\varphi$ from the union of the subsets of $\Delta$ and Z, which will itself be an all-affirmative subset of $\Gamma$. If at least one of the two wffs is derivable from an inconsistent subset of $\Delta$ or Z just one member of which is negative, then $\varphi$ is derivable from the same set, since any wff is derivable from an inconsistent set. In this case, then, $\varphi$ is derivable from an inconsistent subset of $\Gamma$ with just one negative member.

Celarent. $\varphi$ is immediately below one affirmative and one negative wff. If $\varphi$ rests on $\Gamma$, the affirmative wff on $\Delta$ and the negative on Z, $\Delta \cup Z = \Gamma$. By the hypothesis of the induction *either*: (i) the affirmative wff is derivable from an all-affirmative subset of $\Gamma$ and the negative from a subset of Z with just one negative member. By Celarent $\varphi$ is then derivable from the union of these two subsets, which is a subset of $\Gamma$ with just one negative member. *Or*: (ii) the affirmative wff is derivable from an inconsistent subset of $\Gamma$ with just one negative member; but since such a set yields any wff, $\varphi$ is also derivable from it. And this set is also a subset of $\Gamma$ with just one negative member.

r.a.a. This case is of some complexity. $\varphi$ is immediately below one affirmative and one negative wff. If $\varphi$ rests on $\Gamma$, the affirmative wff on $\Delta$ and the negative on Z, then $\Delta \cup Z = \Gamma, \overline{\varphi}$.

(i) Suppose the affirmative wff above $\varphi$ is derivable from an all-affirmative subset of $\Delta$. On the hypothesis of the induction the negative wff above $\varphi$ is derivable from a subset of Z with just one negative wff.

If $\overline{\varphi}$ belongs to either of these subsets, the two derivations can be extended by discharging $\overline{\varphi}$ to derive $\varphi$ from the remaining wffs of the subset by r.a.a. If $\varphi$ is affirmative, $\overline{\varphi}$ is negative, so that the remaining wffs will form an all-affirmative subset of $\Gamma$. If $\varphi$ is negative,

$\bar{\varphi}$ is affirmative, and the remaining wffs will form a subset of $\Gamma$ with just one negative wff.

If $\bar{\varphi}$ belongs to neither of the subsets, their union constitutes an inconsistent subset of $\Gamma$ with just one negative wff. Any wff is derivable from an inconsistent set; so *a fortiori* $\varphi$ is derivable from it.

(ii) Suppose the affirmative wff above $\varphi$ is derivable from an inconsistent subset of $\Delta$ with just one negative wff.

If $\bar{\varphi}$ belongs to this subset, $\varphi$ is derivable from its remaining wffs by r.a.a. If $\varphi$ is affirmative, $\bar{\varphi}$ is negative and the remaining wffs form an all-affirmative subset of $\Gamma$. If $\varphi$ is negative, $\bar{\varphi}$ is affirmative and the remaining wffs form a subset of $\Gamma$ with just one negative wff.

If $\bar{\varphi}$ does not belong to the inconsistent subset, that subset is also a subset of $\Gamma$. $\varphi$ is derivable from it, since any wff is derivable from an inconsistent set.

This completes the proof of the lemma. The lemma means that, if there is more than one negative premiss, all but one are redundant. In fact, it can be shown by similar means that there is never more than one negative wff at the tips of a tree if the set of wffs at the tips is consistent, a result related to the traditional rule that no syllogistic conclusion may be drawn from two negative premisses.

*Metatheorem 4. The procedure of 3.3 is an adequate decision and proof procedure for BS.*

The theorem follows if we can show that the following list of inconsistent sets is exhaustive, i.e. that all other types of set are consistent.

(1) A set containing some wff $E\alpha\alpha$ or $O\alpha\alpha$ and some affirmative wff with a variable $\alpha$.

(2) A set with some $O\alpha\beta$ $(\alpha \neq \beta)$ and an $\alpha$-$\beta$ chain.

(3) A set with some $E\alpha\beta$ ($\alpha \neq \beta$) and an $\alpha$-$\beta$ chain.

(4) A set with some $E\alpha\beta$ ($\alpha \neq \beta$) and a $\beta$-$\alpha$ chain.

(5) A set with some $E\alpha\beta$ ($\alpha \neq \beta$) and $\gamma$-$\alpha$ and $\gamma$-$\beta$ chains.

(6) A set with $E\alpha\beta$ ($\alpha \neq \beta$) and either $I\alpha\beta$ or $I\beta\alpha$.

(7) A set with $E\alpha\beta$ ($\alpha \neq \beta$), either $I\alpha\gamma$ or $I\gamma\alpha$, and a $\gamma$-$\beta$ chain.

(8) A set with $E\alpha\beta$ ($\alpha \neq \beta$), either $I\beta\gamma$ or $I\gamma\beta$, and a $\gamma$-$\alpha$ chain.

(9) A set with $E\alpha\beta$ ($\alpha \neq \beta$), either $I\gamma\delta$ or $I\delta\gamma$, and $\gamma$-$\alpha$ and $\delta$-$\beta$ chains.

*Case I.* Sets with no more than one negative wff. We show that every set of this sort which is of none of the types listed above is satisfiable. Since we have shown that every satisfiable set is consistent, it follows that every Case I set which belongs to none of the nine listed types is consistent. We use the substitution methods of the completeness proof in 3.2.

(a) The set consists only of affirmatives. V cannot be deleted by any affirmative wff, since no affirmative wff will have a predicate letter not in V. Substitutions according to the prescriptions in 3.2 must therefore always result in true propositions, since V verifies all *I* propositions and, after the prescribed deletions from P(V), *A* propositions can be falsified only if there is no set with the subject variable of the replaced *A* wff. But V will contain all such letters.

(b) There is one negative wff.

(i) The negative wff is $E\alpha\alpha$ or $O\alpha\alpha$. The set is of type (1) unless $\alpha$ is absent from all the other wffs. But then V $-$ $\{\alpha\}$ remains undeleted and verifies any affirmative propositions. The substituend of the negative wff is automatically verified as a result of the deletion process.[1]

[1]Consequently, if the negative wff is the only wff $\epsilon$ $\Delta$, $\Delta$ is satisfiable.

(ii) The negative wff is $O\alpha\beta$ ($\alpha \neq \beta$). Either the set is of type (2) or there is no $\alpha$-$\beta$ chain. No affirmative wff can turn out false, since V cannot be deleted ($O\alpha\beta$ deletes no wffs from P(V)). Suppose now that there is no $\alpha$-$\beta$ chain. By Lemma 1 there is some undeleted set $[\alpha\beta']$, which verifies the substituend of $O\alpha\beta$. So if there is no $\alpha$-$\beta$ chain, the set is satisfiable.[1]

In (iii)-(ix) $E\alpha\beta$ ($\alpha \neq \beta$) is the negative wff and its substituend is automatically verified by the deletion process.[1] Any wff $A\gamma\gamma$ or $I\gamma\gamma$ in the set is verified by $\{\gamma\}$, which cannot be deleted either by an affirmative wff, or by an $E$ wff in which the two variable letters are distinct. It remains to consider affirmative wffs with two distinct variables.

(iii) Suppose $A\alpha\gamma$ occurs in the set and its substituend is false. Then there is no set $[\alpha]$ in $U(\Delta)$ and so no set $[\alpha\beta']$. This set cannot be excluded by $E\alpha\beta$, and so it must be excluded by the affirmative wffs alone. But then, by Lemma 1, there is an $\alpha$-$\beta$ chain and the set is of type (3). So if it is not of type (3) the substituend of $A\alpha\gamma$ is true; and *a fortiori* if it is of none of the nine types the substituend is true.

(iv) Suppose $A\beta\gamma$ occurs in the set and its substituend is false. Then there is no set $[\beta]$ in $U(\Delta)$ and so no set $[\beta\alpha']$. This set cannot be excluded by $E\alpha\beta$, and so it must be excluded by the affirmative wffs alone. But then, by Lemma 1, there is a $\beta$-$\alpha$ chain and the set is of type (4). So if it is not of type (4) the substituend of $A\beta\gamma$ is true; and *a fortiori* if it is of none of the nine types the substituend is true.

(v) Suppose $A\gamma\delta$[2] ($\gamma \neq \alpha$, $\gamma \neq \beta$) occurs in the set and its substituend is false. Then there is no set $[\gamma]$ in $U(\Delta)$ and so no set $[\gamma\alpha']$ nor $[\gamma\beta']$. These last two sorts of set cannot be excluded by $E\alpha\beta$, and so must be excluded by the affirmative wffs alone. But

[2]Including $A\gamma\alpha$, $A\gamma\beta$ as special cases.

then, by Lemma 1, there is a $\gamma$-$\alpha$ chain and a $\gamma$-$\beta$ chain and the set is of type (5). So if the set is of none of the nine types the substituend of $A\gamma\delta$ is true.

(vi) If $I\alpha\beta$ or $I\beta\alpha$ occurs in the set it is of type (6).

(vii) Suppose $I\alpha\gamma$ or $I\gamma\alpha$ $(\gamma \neq \alpha,\ \gamma \neq \beta)$ occurs in the set and its substituend is false. Then there is no set $[\alpha\gamma]$ in $\mathrm{U}(\Delta)$ and so no $[\alpha\gamma\beta']$. Then, by Lemma 1, there is either an $\alpha$-$\beta$ chain and the set is of type (3) or a $\gamma$-$\beta$ chain and the set is of type (7). So if the set is of none the nine types the substituend is true.

(viii) Suppose $I\beta\gamma$ or $I\gamma\beta$ $(\gamma \neq \alpha,\ \gamma \neq \beta)$ occurs in the set and its substituend is false. Then there is no set $[\beta\gamma]$ in $\mathrm{U}(\Delta)$ and so no $[\beta\gamma\alpha']$. Then, by Lemma 1, there is either a $\beta$-$\alpha$ chain and the set is of type (4) or a $\gamma$-$\alpha$ chain and the set is of type (8). So if the set is of none of the nine types the substituend is true.

(ix) Suppose, finally, that $I\gamma\delta$ or $I\delta\gamma$, neither variable identical with $\alpha$ or $\beta$, occurs in the set and its substituend is false. Then there is no set $[\gamma\delta]$ in $\mathrm{U}(\Delta)$ and so no $[\gamma\delta\alpha']$ and no $[\gamma\delta\beta']$. By Lemma 1, there is a $\gamma$-$\alpha$ or $\delta$-$\alpha$ chain and a $\gamma$-$\beta$ or $\delta$-$\beta$ chain, and the set is accordingly of type (5) or type (9). So if the set is of none of these nine types the substituend is true.

Every satisfiable set is consistent. So every Case I set which is of none of the nine listed types is consistent.

*Case II.* Sets with more than one negative wff.

(a) All the wffs are negative. $E$ wffs and wffs of the form $O\alpha\alpha$ are automatically replaced by true propositions. Suppose a wff $O\alpha\beta$ $(\alpha \neq \beta)$ is replaced by a false proposition. Then there is no wff $E\alpha\alpha$ or $O\alpha\alpha$, since either of these would exclude every set $[\alpha]$ and so make the substituend of $O\alpha\beta$ true. But then the set $\{\alpha\}$ cannot be deleted, which makes the substituend of $O\alpha\beta$ true. Contradiction. Hence if all the wffs are negative, they will be replaced by true propositions, and an all-negative set must be consistent.

(b) At least one wff is affirmative. Such a set is inconsistent only if it has an inconsistent subset consisting of just one of the negative and all of the affirmative wffs.

*Proof.* Suppose $\varphi$ is one of the affirmative wffs. Then if the set is inconsistent, the wff $\overline{\varphi}$ is also derivable from the set, since any wff is derivable from an inconsistent set. By Lemma 2, if $\overline{\varphi}$ is derivable from the set, it is derivable from a subset containing no more than one (in fact just one) of the negative wffs, and therefore from a subset containing just one of the negatives and all of the affirmatives. This last subset will include $\varphi$ and yield $\overline{\varphi}$, and so will be inconsistent. Consequently, if a set $\Delta$ with at least one affirmative and at least two negatives is inconsistent, it has an inconsistent subset with just one negative. Now if $\Delta$ is of none of the nine types listed above, it has no subset of any of the nine types. Consequently, it has no inconsistent subset, and must be consistent.

Cases I and II exhaust the possibilities. So every set which is of none of the nine listed types is consistent, which means that the list of inconsistent types is exhaustive.

As was shown in Chapter 3, it follows that the procedures of 3.3 constitute decision procedures both for derivability and validity, as well as proof procedures, for BS. Q.E.D.

*Other decision procedures*

The decision procedure just validated is probably the most practicable we have. But for the sake of completeness other procedures will be mentioned.

Since it is clear from the indications in 3.3 for deriving inconsistent wffs from the various inconsistent sets that such wffs are derivable by means of the non-discharge rules (i.e. those rules distinct from r.a.a.), it is obvious that there is also the following decision procedure. To determine whether $\varphi$ is a consequence of $\Gamma$, form the set $\Gamma$, $\overline{\varphi}$. Take each wff in turn and apply s.c.($I$) where it yields a new wff. Repeat for sub. ($A$) and id.$^{+}$. Then take each pair of wffs from the (possibly) enlarged set and apply Barbara and Celarent wherever they yield new wffs. Continue to add wffs by repeating the whole process on the growing pool as often as possible. The procedure is bound to come to an end, since the non-discharge rules cannot introduce any new variable and there are only finitely many wffs composed of the variables of the wffs of a finite set (and so only finitely many wffs derivable from the wffs of a finite set by means of the non-discharge rules). The set will be inconsistent (and therefore unsatisfiable) iff some pair, $\psi$, $\overline{\psi}$ is derived; and the inference from $\Gamma$ to $\varphi$ will be valid iff the set being tested is inconsistent. If a pair $\psi$, $\overline{\psi}$ is derived, one step of r.a.a. will complete a derivation of $\varphi$ from $\Gamma$ (so we have another proof procedure too).

Secondly, the arithmetical decision procedure given by Ivo Thomas (1952) for Łukasiewicz's syllogistic system is easily adapted to BS.

Further procedures are available if inference patterns are translated into monadic predicate logic. (See Chapter 5.)

## CHAPTER 4

## SYLLOGISTIC AND PROPOSITIONAL LOGIC

### 4.1 *Trees and linear proofs*

For the moment we shall abandon the compact tree arrangement of proofs for the more usual linear presentation, consisting of a vertical list of formulas flanked by columns giving the information which in tree derivations is largely supplied by the layout. The relation between the two modes of presentation can easily be grasped from the following example of corresponding tree and linear proofs:

$$\dfrac{\dfrac{\dfrac{Aab \quad Abc}{Aac}\,\text{Barbara} \qquad Ecd}{Ead}\,\text{Celarent}}{Eda}\,\text{s.c. }(E)$$

In linear form the proof will look like this:

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $Aab$ | As. |
| $\{2\}$ | (2) | $Abc$ | As. |
| $\{1,2\}$ | (3) | $Aac$ | 1,2 Barbara |
| $\{4\}$ | (4) | $Ecd$ | As. |
| $\{1,2,4\}$ | (5) | $Ead$ | 3,4 Celarent |
| $\{1,2,4\}$ | (6) | $Eda$ | 5 s.c.$(E)$ |

Within each line the left-hand column specifies the set of assumption(s) on which the formula in the central column rests, assumptions which in a tree proof appear as undischarged formulas at the tips or in the initial list. In this linear deduction wffs (1), (2) and (4) are assumptions and rest on themselves, and so their own number is entered in the left-hand column and 'As.' for *assumption* is entered in the right-hand column. Each wff which results from the application of a non-discharge rule, that is any rule of the system apart from r.a.a., rests on the assumptions which support the formula(s) it is immediately derived from. The numbers of any wffs from which the wff on the line is immediately derived are entered in the right-hand column together with the name of the rule being used.

For a tree proof which includes an application of r.a.a. we may take the following example, repeated from Chapter 1:

$$\dfrac{\dfrac{[Apr] \quad Arb}{Apb}\,\text{Barbara} \qquad\qquad Opb}{Opr}\,\text{r.a.a.}$$

In linear form it comes out as:

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $[Apr]$ | As. |
| $\{2\}$ | (2) | $Arb$ | As. |
| $\{1,2\}$ | (3) | $Apb$ | 1,2 Barbara |
| $\{4\}$ | (4) | $Opb$ | As. |
| $\{2,4\}$ | (5) | $Opr$ | 1, 3 & 4 r.a.a. |

The formula derived by r.a.a. rests on all the assumptions on which *Apb* and *Opb* rest, apart from *Apr*, which is discharged. In view of the presence of the left-hand column, the square brackets round discharged assumptions could now be dispensed with, but it will do no harm to keep them - though they need not be mentioned in the definition of a linear deduction.

A linear deduction in the system may be defined as a finite sequence of consecutively numbered lines each of which is made up of (the designation of) a set of assumption numbers (determined as indicated above) and a wff which is either an assumption or results from the application of one of the rules of inference.

The left-hand columns in our examples therefore count as integral parts of the proofs, whereas the right-hand columns are merely additional descriptive apparatus. We have a deduction of $\varphi$ from $\Gamma$ when and only when

(i) $\varphi$ is on the last line, and

(ii) every assumption on which $\varphi$ rests (as indicated by the assumption numbers of that line) belongs to $\Gamma$.

It is obvious enough that any tree proof in BS (or BS$^+$) can be re-written in linear form (e.g. by first writing every wff of rank 1, then every wff of rank 2, and so on), and every linear proof re-written in tree form. It is less evident that every sequent which has a linear proof has a proof in the form of a *non-repetitive tree* (see 1.3, p. 18), but this can be proved without much difficulty. The proof is sketched below. The linear presentations of proofs will make it easier to consider the formal relationship of the basic syllogistic to modern logic, which cannot be presented conveniently in the tree form defined in Chapter 1 unless we use one of Gentzen's calculi of sequents.

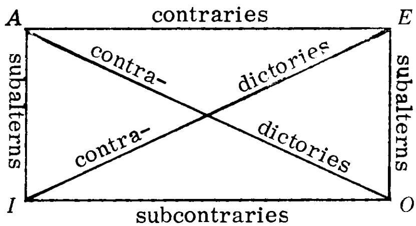
*Metatheorem 5. There is a linear derivation of a sequent in BS iff there is a non-repetitive tree derivation of that sequent in BS.*

*Sketch of proof.* From what we have just shown, any non-repetitive tree proof can be re-written in linear form.

To show the converse, suppose that there were a linear proof of a sequent which has no proof in the form of a non-repetitive tree. We have indicated on pp. 49-50 how to show that a sequent of BS has a tree proof iff it has a non-repetitive tree proof. By Theorem 3 a sequent that has no tree proof is invalid; so a sequent that has no *non-repetitive* tree proof is invalid. In that case some invalid sequent would have a linear proof. But it can be shown that all sequents derivable in a linear manner are valid, by adapting the proof of soundness in 2.2. The rank of a formula is redefined as its reference number, and the words 'wff(s) immediately above $\varphi$' replaced by 'wff(s) from which $\varphi$ has been immediately derived', etc.[1]

## 4.2 *The Square of Opposition*

The basic systems are not rich enough to prove all of the Square of Opposition relations which Aristotle discusses in *De Interpretatione*, these relations are conveniently summarized in the familiar diagram (Aristotle does not himself use the terms 'subaltern' and 'subcontrary'):




We already have rules (sub.(A) and sub.(E)) to reflect the relations of subalternation. In order to express the other relations in inferential terms we introduce an operator for propositional negation, '—', together with two double negation rules (DN):

$$\text{(i)}\quad \frac{\varphi}{--\varphi} \qquad\qquad \text{(ii)}\quad \frac{--\varphi}{\varphi}$$

Their soundness is evident for the usual truth-functional interpretation of '—'. The formation rule for the enlarged system must be extended by adding that $\ulcorner -\varphi \urcorner$ is well-formed whenever $\ulcorner \varphi \urcorner$ is. We also modify the system by deleting '*E*' and '*O*' from the list of primitive constants and reintroducing them as constants defined in terms of '*A*' and '*I*':

$$\text{Df.:} \quad E\alpha\beta = -I\alpha\beta$$

$$\text{Df.:} \quad O\alpha\beta = -A\alpha\beta$$

[1] A similar result is provable for the stronger system $BS^+$, given an interpretation for which it can be proved sound - see Chapter 5.

(Definitions of this sort were originally proposed by Leibniz.) The medieval interpretation was tailored to ensure the possibility of such definitions: with affirmative wffs having existential import and negative wffs lacking it, *No $\alpha$ is a $\beta$* is equivalent to *It is not the case that some $\alpha$ is a $\beta$*, and both $O\alpha\beta$ and $-A\alpha\beta$ will be read as *Not every $\alpha$ is a $\beta$*.

It is now very easy to prove the inferential analogues of the logical relations of the Square. Two formulas are contradictory iff every pair of propositions instantiating them has one true and one false member. So we want to show that an $A$ wff is interderivable with the negation of the corresponding $O$, $-A$ with $O$, and similarly that $E$ is interderivable with $-I$, and $-E$ with $I$. Here is a sample proof, the rest being left to the reader:

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $A\alpha\beta$ | As. |
| $\{1\}$ | (2) | $--A\alpha\beta$ | 1 DN |
| $\{1\}$ | (3) | $-O\alpha\beta$ | 2 df. |

We shall say that two formulas are contraries iff the members of instantiating pairs are never both true. We want to show that $A\alpha\beta \vdash -E\alpha\beta$ and that $E\alpha\beta \vdash -A\alpha\beta$. We prove the former and leave the other to the reader:

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $A\alpha\beta$ | As. |
| $\{1\}$ | (2) | $I\alpha\beta$ | 1 sub. |
| $\{1\}$ | (3) | $--I\alpha\beta$ | 2 DN |
| $\{1\}$ | (4) | $-E\alpha\beta$ | 3 df. |

We shall say that two formulas are subcontraries iff the members of instantiating pairs are never both false. So we want to show that $-I\alpha\beta \vdash O\alpha\beta$ and $-O\alpha\beta \vdash I\alpha\beta$.

We shall call the newly enlarged version of BS, EBS (and of $BS^+$, $EBS^+$). All the results stated in this section follow from the fact that EBS is both sound and complete with respect to Interpretation I plus the truth-functional interpretation of negation (see 4.4).

(On the definitions of contrariety and subcontrariety used here all contradictories are both contraries and subcontraries. It is customary to define contraries by adding that two contraries may both be false, but if a pair of formulas includes a necessary formula or proposition this will not be the case. Similarly, it is customary to define subcontraries by including the requirement that two subcontraries may both be true, which once more cannot be satisfied if one of them is necessarily false. The point is made by David H. Sanford (1968), p. 65. Lemmon (1965) defines contraries and subcontraries as we do, and so does Strawson (1951).

Now under Interpretation I no formula will either be necessary or necessarily false, so that under the customary definition *A* and *E formulas* will always be contraries and *I* and *O formulas* will always be subcontraries. (Once again we are speaking loosely when we speak of true or false formulas; cf. parenthetical remark p. 30.) But this will not be true of propositions instantiating them, e.g. the conjoint falsity of *Every triangle is a triangle* and *No triangle is a triangle* is not a logical possibility (in a broad sense of 'logical'), since the former is a necessary truth; nor is the conjoint truth of *Some triangle is a triangle* and *Not every triangle is a triangle*. This will be so despite the attribution of existential import to affirmative propositions, provided that we grant the 'necessary existence' of triangles. Moreover, on any interpretation under which $A\alpha\alpha$ and $I\alpha\alpha$ are necessarily true formulas, it will not even be possible to hold that corresponding *A* and *E formulas* are contraries nor that corresponding *I* and *O* formulas are subcontraries, if we define these terms in the customary manner.)

When we confine our consideration to elementary propositions of the four categorical forms, there is no great harm in talking of '*the* contrary' or '*the* subcontrary' of a proposition, just as we talk of '*the* contradictory'. But it should be noticed that a proposition may have more than one non-equivalent contrary or subcontrary, whereas it may have no more than one non-equivalent contradictory. (Cf. Geach (1971), pp. 70-74.) For example, *No man is a hypocrite and every man is debtor* and *No man is a hypocrite but not every man is a debtor* are both contrary to *Every man is a hypocrite.*

4.3 *The basic systems adjoined to the propositional calculus*

The double negation rules are principles of a more fundamental sort than the non-discharge rules of the basic systems. They belong to the logic of unanalysed propositions, which is nowadays presented in the various versions of the propositional (sentential, statement) calculus. We shall use a natural-deduction version of the calculus similar to the one in E. J. Lemmon (1965) (which is virtually Gentzen's system NK), but unlike Lemmon we shall take only negation and conjunction as primitive propositional operators ('—' and '&'). As under the usual interpretation of the calculus these are to be understood in a purely truth-functional way, so that $-\varphi$ is true when $\varphi$ is false and false when $\varphi$ is true; and $\varphi$ & $\psi$ is true when and only when each conjunct is true. '$P$', '$Q$', '$R$'...are used as propositional variables (ranging over propositions of any sort) and the formation rules are as usual for such systems.

The rules of our propositional calculus are, with one exception, exceedingly obvious and simple principles about negation and conjunction. The exception is effectively a generalization of the *reductio* rule we have been using in BS, and its presence as the only non-trivial rule of a complete version of the propositional calculus (PC) indicates how powerful a principle it is. For each primitive operator there is an introduction and elimination rule(s):

&I $\dfrac{\varphi \quad \psi}{(\varphi \ \& \ \psi)}$ &E $\dfrac{(\varphi \ \& \ \psi)}{\varphi}$ $\dfrac{(\varphi \ \& \ \psi)}{\psi}$

DN elimination $\dfrac{--\varphi}{\varphi}$ RAA $\dfrac{\begin{matrix}[\varphi] \\ \psi \quad -\psi\end{matrix}}{-\varphi}$

(The schema for RAA is actually short for the rules: if $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\varphi\} \vdash -\chi$, then $\Gamma \vdash -\varphi$; and if $\Gamma \cup \{\varphi\} \vdash \chi$ and $\Gamma \cup \{\psi\} \vdash -\chi$, then $\Gamma \cup \{\varphi\} \vdash -\psi$ and also $\Gamma \cup \{\psi\} \vdash -\varphi$.)

The operators for *or* and *if* are introduced as defined constants by means of definitions:

$$\text{Df.:} \quad (\varphi \vee \psi) = -(-\varphi \ \& \ -\psi)$$

$$\text{Df.:} \quad (\varphi \rightarrow \psi) = -(\varphi \ \& \ -\psi)$$

(In practice outermost brackets will be dropped according to the usual custom.)

A deduction in the system is defined as for EBS. As an illustration we give a proof of the principle known since medieval times as *modus ponendo ponens*: $P \rightarrow Q, P \vdash Q$.

| | | |
|---|---|---|
| {1} | (1) $P \rightarrow Q$ | As. |
| {2} | (2) $P$ | As. |
| {3} | (3) $[-Q]$ | As. |
| {1} | (4) $-(P \ \& \ -Q)$ | 1 df. $\rightarrow$ |
| {2,3} | (5) $P \ \& \ -Q$ | 2,3 &I |
| {1,2} | (6) $--Q$ | 3,4 & 5 RAA |
| {1,2} | (7) $Q$ | 6 DN |

Notice that RAA operates in the same way as the similar rule of EBS.

Suppose we now adjoin EBS to the propositional calculus (PC) to give us a system EBS + PC. The rules r.a.a. and DN introduction then become assimilated as derived rules. Since $E$ is defined as $-I$, and $O$ as $-A$, and as we indicated in the last section $A$ and $I$ are interderivable with $-O$ and $-E$ respectively, $\overline{\varphi} = -\varphi$, and r.a.a. is more or less a special case of the general propositional *reductio* rule.

The primitive rules of the new system are therefore the following:

&I, &E, RAA, DN(elim), s.c., sub., Barbara, Celarent, id.

An economy can be effected in this basis by replacing s.c. and Celarent by the single rule Datisi:

$$\frac{I\beta\alpha \quad A\beta\gamma}{I\alpha\gamma}$$

To show that this replacement does not affect the deductive power of the system we derive s.c. and Celarent within the revised system and Datisi within the original version (using the linear form metalogically):

s.c.

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $I\alpha\beta$ | As. |
| $\{1\}$ | (2) | $A\alpha\alpha$ | 1 id. |
| $\{1\}$ | (3) | $I\beta\alpha$ | 1,2 Datisi |

Celarent

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $A\alpha\beta$ | As. |
| $\{2\}$ | (2) | $E\beta\gamma$ | As. |
| $\{3\}$ | (3) | $[I\alpha\gamma]$ | As. |
| $\{1,3\}$ | (4) | $I\gamma\beta$ | 1,3 Datisi |

| | | |
|---|---|---|
| $\{1,3\}$ | (5) $I\beta\gamma$ | 4 s.c. |
| $\{2\}$ | (6) $-I\beta\gamma$ | 2 df. $E$ |
| $\{1,2\}$ | (7) $-I\alpha\gamma$ | 3, 5 & 6 RAA |
| $\{1,2\}$ | (8) $E\alpha\gamma$ | 7 df. $E$ |

Datisi

| | | |
|---|---|---|
| $\{1\}$ | (1) $A\beta\gamma$ | As. |
| $\{2\}$ | (2) $[-I\alpha\gamma]$ | As. |
| $\{2\}$ | (3) $E\alpha\gamma$ | 2 df. $E$ |
| $\{2\}$ | (4) $E\gamma\alpha$ | 3 s.c.($E$) |
| $\{1,2\}$ | (5) $E\beta\alpha$ | 1,4 Celarent |
| $\{1,2\}$ | (6) $-I\beta\alpha$ | 5 df. $E$ |
| $\{7\}$ | (7) $I\beta\alpha$ | As. |
| $\{1,7\}$ | (8) $--I\alpha\gamma$ | 2,6 & 7 RAA |
| $\{1,7\}$ | (9) $I\alpha\gamma$ | 8 DN |

The syllogistic rules of the more economical system are therefore: sub., Barbara, Datisi, id.

The same economy can, of course, be made for EBS$^+$ + PC. And in this system the rule sub. is interchangeable with the identity rule id.$^+$ ($I$):

$$\frac{*}{I\alpha\alpha}$$

It is obvious that the latter is a derived rule of the original system, and sub. can be derived in the revised system in the following manner:

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $A\alpha\beta$ | As. |
| $\Lambda$ | (2) | $I\alpha\alpha$ | id.$^+$ $(I)$ |
| $\{1\}$ | (3) | $I\alpha\beta$ | 1,2 Datisi |
| $\Lambda$ | (4) | $A\alpha\alpha$ | id.$^+$ |
| $\{1\}$ | (5) | $I\beta\alpha$ | 3,4 Datisi |
| $\{1\}$ | (6) | $I\alpha\beta$ | 5 s.c. |

It is now easy to see that EBS$^+$ + PC is equivalent to the well-known system of Łukasiewicz, which consists of the following special axioms adjoined to an axiomatic version of PC (with substitution and *modus ponens* as rules of inference):

| | | | |
|---|---|---|---|
| (i) | $Aaa$ | (iii) | $(Abc \;\&\; Aab) \rightarrow Aac$ |
| (ii) | $Iaa$ | (iv) | $(Abc \;\&\; Iba) \rightarrow Iac$ |

and definitions of the negative constants in terms of the affirmatives.[2] These special axioms correspond to our rules id.$^+$, id.$^+$ $(I)$, Barbara and Datisi and are easily derivable as theorems of EBS$^+$ + PC once we have proved the deduction theorem for our version of the propositional calculus:

*Metatheorem 6. If* $\Gamma, \varphi \vdash_{PC} \psi$, *then* $\Gamma \vdash_{PC} \varphi \rightarrow \psi$.

| | | | | |
|---|---|---|---|---|
| | | ⋮ | | |
| | $\{n\}$ | $(n)$ | $\varphi \;\&\; -\psi$ | As. |
| | $\{n\}$ | $(n+1)$ | $\varphi$ | $n$, &E |
| | $\{n\}$ | $(n+2)$ | $-\psi$ | $n$, &E |
| Now suppose that | $\Gamma$, $n$ | $(n+3)$ | $\psi$ | $\ldots, n+1$ |
| then | $\Gamma$ | $(n+4)$ | $-(\varphi \;\&\; -\psi)$ | $n$, $n+2$ & $n+3$, RAA[3] |
| | $\Gamma$ | $(n+5)$ | $\varphi \rightarrow \psi$ | $n+4$, df. $\rightarrow$ |

[2]Łukasiewicz (1957) interprets his syllogistic as a theory of object-language sentences: that is to say, his variables range over terms ('stone', 'man' etc.) rather than individuals. Furthermore, the terms are treated as names (see 7.1 below).
[3]See exact statement of RAA on p.69.

Axiom (iii), for example, is now proved in the following way:

| | | | |
|---|---|---|---|
| {1} | (1) | $Abc \ \& \ Aab$ | As. |
| {1} | (2) | $Abc$ | 1 &E |
| {1} | (3) | $Aab$ | 1 &E |
| {1} | (4) | $Aac$ | 2,3 Barbara |
| $\Lambda$ | (5) | $(Abc \ \& \ Aac) \rightarrow Aac$ | 1-4 Met. 6 |

Conversely, it is not difficult to establish the syllogistic rules of $EBS^+$ + PC as derived rules of Łukasiewicz's system (granted that natural deduction derivations are permitted within it).

Łukasiewicz has shown that his syllogistic axioms are independent of one another and of his PC axioms, and his methods can very easily be adapted to our system $EBS^+$ + PC (see pp. 89-90 of Łukasiewicz (1957) and cf. 2.3 above).

Another system which features prominently in the literature is Bocheński's CS, a system which is in fact contained within Łukasiewicz's. Bocheński has Ferio in place of Datisi (there are various possible replacements for Datisi) and adjoins only a fragment of PC - only those propositional principles he needs in order to prove the syllogistic ones in the axiomatic system. Moreover he has only three term variables, since he does not seem to be concerned with providing for the patterns of arguments with more than two premisses. The system CS appears rather unwieldy for the limited task it was designed to perform. (For details see Bocheński (1948).)

If EBS is given Interpretation I and PC is interpreted in the usual manner, EBS + PC is sound, i.e. all sequents derivable in the system are valid. The proof of this is a routine extension of the proof of the soundness of BS given in 2.2. The consistency of the system follows easily from this. The consistency of the stronger system $EBS^+$ + PC can be rapidly established in the manner of Łukasiewicz (1957), p. 89, bearing in mind the correspondence between his axioms and our rules. Łukasiewicz interprets each affirmative wff $A\alpha\beta$ and $I\alpha\beta$ as a propositional wff $(\varphi \rightarrow \varphi)$ & $(\psi \rightarrow \psi)$. Then all his axioms, both Aristotelian and propositional, are tautologies and his rules of inference, substitution and *modus ponens*, preserve this property. (The consistency of EBS + PC is of course a corollary of the result that EBS$^+$ + PC is consistent.)

4.4 *Completeness of EBS + PC*

By an argument similar to the one at the beginning of section 3.2 we can show that EBS and EBS + PC are complete with respect to Interpretation I etc. if we can show that every consistent set of wffs in those systems is satisfiable.

*Metatheorem 7. EBS + PC is complete with respect to Interpretation I combined with the usual interpretation of propositional logic.*

*Proof.* Let $\Delta$ be a consistent set of wffs in the primitive notation of the system, e.g. $\{P, Iab \mathbin{\&} -Aab, -(Q \mathbin{\&} -Aab)\}$. The term *elementary wff* will be used for single propositional letters, their negations and uncompounded positive Aristotelian wffs.

Suppose, now, that an inverted tree is formed by writing the wffs of $\Delta$ one under the other and transforming non-elementary wffs by successive applications of the following transformation rules:

(i) DN $\dfrac{--\varphi}{\varphi}$ (ii) Separation $\dfrac{\varphi \mathbin{\&} \psi}{\begin{matrix}\varphi \\ \psi\end{matrix}}$ (iii) Branching $-(\varphi \mathbin{\&} \psi)$ branching to $-\varphi$ and $-\psi$

(iv) $\dfrac{-A\alpha\beta}{O\alpha\beta}$; $\dfrac{-E\alpha\beta}{I\alpha\beta}$; $\dfrac{-I\alpha\beta}{E\alpha\beta}$; $\dfrac{-O\alpha\beta}{A\alpha\beta}$

It is easy to check that the procedure must terminate in elementary wffs (proof by induction on the number of connectives in a wff).

*Example.* $\Delta = \{P,\ Iab\ \&\ -Aab,\ -(Q\ \&\ -Aab)\}$

Inverted tree: (1) $P$

(2) $Iab\ \&\ -Aab$

(3) $-(Q\ \&\ -Aab)$

$Iab$ from (2)

$-Aab$ from (2)

$$\begin{array}{ccc} & Oab & \\ / & & \backslash \\ --Q & & --Aab \end{array}$$ from (3) by Branching

$Aab$

Let us call the list of wffs of $\Delta$ at the top of the inverted tree *the initial segment.* We can show, by strong induction on the length of a branch which includes the initial segment, that, if the initial segment is consistent, so is at least one branch of the whole tree.[4] Essentially, this is a matter of showing that each of the transformation rules preserves consistency down at least one branch. This is very easy in the case of (i), (ii) and (iv), since they are primitive or derived rules of inference of EBS + PC. Nor is it particularly difficult in the case of (iii), Branching. Suppose that $-(\varphi\ \&\ \psi)$ is consistent but that both $-\varphi$ and $-\psi$ are inconsistent. Then $\varphi$ and $\psi$ will be theorems, and so therefore will $\varphi\ \&\ \psi$. Hence $-(\varphi\ \&\ \psi) \vdash (\varphi\ \&\ \psi)\ \&\ -(\varphi\ \&\ \psi)$, i.e. $-(\varphi\ \&\ \psi)$ will be inconsistent, contrary to hypothesis. Therefore, if $-(\varphi\ \&\ \psi)$ is consistent, at least one of the wffs $\varphi$, $\psi$ is consistent.

[4] The induction hypothesis may be formulated as follows: if the first $k$ wffs on a branch, B, include the initial segment and form a consistent set, then either that branch terminates at the $k$th wff, or there is some $k+1$th wff on a branch which ramifies from or continues B, the first $k+1$ wffs of which form a consistent set.

In the example given above, the left-hand branch proves to be consistent but the other branch is not, containing as it does both $—Aab$ and $Aab$.

Consider next the subset of elementary wffs of the leftmost consistent branch (the set $\{Iab,\ Oab,\ —Q\}$ in the example). This set is simultaneously satisfiable, since the subset of Aristotelian wffs is satisfiable (as shown in 3.2); and the value *true* may be assigned to each of the propositional wffs, because they do not include any propositional letter and its negation and no interpretation of the Aristotelian wffs will exclude any truth-value assignment to the propositional wffs.

Finally, we need to show that, if the set of elementary wffs on a branch is satisfiable, the set of *all* the wffs on that branch is satisfiable, and so therefore is the subset $\Delta$ of wffs in the initial segment. This may be done by strong induction on the height of the branch, and will involve taking each transformation rule in turn and showing that truth is preserved in an upward direction on the branch - which is simply a matter of elementary truth-table considerations.

The construction of the inverted tree may also serve as a *decision procedure*. $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\overline{\varphi}\}$ is inconsistent. Let $\Delta = \Gamma \cup \{\overline{\varphi}\}$. $\Delta$ will be inconsistent iff every branch contains an inconsistent pair of elementary wffs (e.g. $Q$, $—Q$; or $Eab$, $Iab$; or $Abc$, $Obc$).[5]

[5] In Chapter 5 we indicate how to modify the proof of 3.2 to show that $BS^+$ is complete with respect to a certain interpretation. This may be used to generate a decision procedure for $BS^+$ and consequently for $EBS^+ + PC$, since the proof of Metatheorem 7 is easily adapted to demonstrate the completeness of the latter with respect to that interpretation. Łukasiewicz shows in Chapter 5 (1957) that his equivalent system is decidable, using notions specifically designed to demonstrate decidability, viz. those of 'rejection' and what he calls 'deductive equivalence', notions which can of course be extended beyond syllogistic logic. We have shown, however, that more orthodox methods are adequate for the metatheory of syllogistic systems. Oddly enough, although Łukasiewicz raises the question of the completeness of his system in his book (p. 98) he never actually proves it, though he mentions Słupecki's result.

The syllogistic systems which incorporate PC are not entirely felicitous assimilations of syllogistic to modern logic. Consisting as they do of PC plus (in their more economical versions) four syllogistic rules, Barbara, Datisi, with sub. and id. for the weaker, and id.$^{+}$ and id.$^{+}$ $(I)$ for the stronger system, they are little more than graftings of syllogistic nearly intact on to the logic of unanalysed propositions. They enable us to perform simple inferences turning on *every*, *no* and *some*, as well as the usual propositional inferences, but the 'quantificational inferences' permitted are very limited. From a formal point of view, it would seem more satisfactory if we could assimilate syllogistic to that part of modern logic which deals comprehensively with *every*, *some* etc., namely predicate logic. We turn to this task in the next chapter.

## CHAPTER 5

## SYLLOGISTIC AND PREDICATE LOGIC

### 5.1 *The predicate calculus and Interpretation I of BS*

The version of the predicate calculus to be used throughout most of this chapter is once again similar to the system to be found in Lemmon (1965), to which the reader is referred for a more detailed explanation. Our version differs in four major respects: (i) the PC basis on which it is built is the PC system of the last chapter; (ii) the existential quantifier rules are replaced by a definition of the existential quantifier in terms of the universal; (iii) there is only one sort of name, no distinction being made between Lemmon's proper and arbitrary names; and (iv) all its predicate letters are monadic.

In addition to the symbolism of the propositional calculus there are the following symbols:

predicate letters: $F, G, H, \ldots$ names: $m, n, \ldots$

individual variables: $x, y, z, \ldots$

$Fm$, for example, is to be thought of as expressing the form of propositions like *Mount Everest is snow-capped.* $(x)Fx$ is true when and only when $Fx$ is true for all values of $x$, and expresses the form of propositions like *Everything is snow-capped.* We add the following formation rules, using 't' as a schematic letter for a name, 'v' as a schematic letter for an individual variable:

Any predicate letter followed by a single name is a wff.

If $\varphi(t/v)$ is well-formed, then so is $(v)\varphi(v)$.

$\varphi(v)$ is a formula containing v in which all occurrences of v, but no occurrences of any other variable, are free, and $\varphi(t/v)$ is the formula obtained by replacing every occurrence of v in $\varphi(v)$ by the name t.

These rules are to be added to the conventional formation rules for PC, and there are to be no other wffs. The result is a slightly unorthodox set of formation rules for a version of the predicate calculus, since they proscribe expressions with free variables or vacuous quantifiers, and expressions like $(x)(Fx \rightarrow \exists xGx)$ in which a quantifier occurs within the scope of another quantifier with the same variable. These simplifications taken over from Lemmon do not diminish the expressive power of the system and will not prove formally inconvenient in the present context.

*Rules.* 1. Universal elimination (UE). If $\Gamma \vdash (v)\varphi(v)$, then $\Gamma \vdash \varphi(t/v)$.

2. Universal introduction (UI). If $\Gamma \vdash \varphi(t)$, where $\varphi(t)$ is a wff containing t but not v, and t does not occur in any wff in $\Gamma$, then $\Gamma \vdash (v)\varphi(v/t)$.

The first rule is intuitively easy to accept, but the second is a little more difficult, licensing as it does a move from $\varphi(t)$ to its universal generalisation $(v)\varphi(v)$. Very crudely it may be justified like this: if $\varphi(t)$ follows from premises which, making no mention of the individual denoted by t, give no special information about it, then any conclusion $\varphi(s/t)$, which is the same as $\varphi(t)$ except that it has an s wherever that formula has t, should also follow from the same premisses; in which case $(v)\varphi(v)$ should also be a consequence of those premisses.

*Definition of* '$\exists$'. Df.: $\exists v\varphi(v) = -(v)-\varphi(v)$.

It is usual, following Frege and Russell, to express a universal affirmative form *Every a is a b* in this calculus by means of a formula like $(x)-(Fx \;\&\; -Gx)$ (equivalent by definition to $(x)(Fx \rightarrow Gx)$); that is, to construe *Every a is a b* as 'Nothing which is an a is not a b'. *Some a is a b* is rendered, for example, as $\exists x(Fx \;\&\; Gx)$. Now the latter conforms with Interpretation I for *I* formulas and clearly the predicate calculus formula is simply convertible to $\exists x(Gx \;\&\; Fx)$ (see below). But in translation the *A* form lacks existential import, and, if we are to follow the medieval interpretation, this must be restored by adding an existential conjunct, giving us $(x)(Fx \rightarrow Gx)$ & $\exists xFx$ and thereby validating subaltern inference. R.a.a. requires that the *E* and *O* forms be the contradictories of *A* and *I*, and so we may translate formulas of BS into the predicate calculus symbolism according to the following prescription (where $\varphi$, $\psi$ are schematic letters for predicate letters):

| | | |
|---|---|---|
| $A\alpha\beta$ | becomes | $(v)(\varphi v \rightarrow \psi v)$ & $\exists v \varphi v$ |
| $E\alpha\beta$ | | $-\exists v(\varphi v$ & $\psi v)$ |
| $I\alpha\beta$ | | $\exists v(\varphi v$ & $\psi v)$ |
| $O\alpha\beta$ | | $-((v)(\varphi v \rightarrow \psi v)$ & $\exists v\varphi v)$, which is interderivable with $\exists v(\varphi v$ & $-\psi v)$ v $-\exists v\varphi v$ |

BS then becomes a fragment of the monadic predicate calculus, as can readily be established by showing that the primitive rules of BS are derived rules of the calculus. In translation the rules of BS become:

| | |
|---|---|
| s.c. | $\{\exists v(\varphi v$ & $\psi v\} \vdash \exists v(\psi v$ & $\varphi v)$ |
| sub. | $\{(v)(\varphi v \rightarrow \psi v)$ & $\exists v\varphi v\} \vdash \exists v(\varphi v$ & $\psi v)$ |
| Barbara | $\{(v)(\varphi v \rightarrow \psi v)$ & $\exists v\varphi v$, $(v)(\psi v \rightarrow \chi v)$ & $\exists v\psi v\} \vdash$ $(v)(\varphi v \rightarrow \chi v)$ & $\exists v\varphi v$ |
| Celarent | $\{(v)(\varphi v \rightarrow \psi v)$ & $\exists v\varphi v$, $-\exists v(\psi v$ & $\chi v)\} \vdash -\exists v(\varphi v$ & $\chi v)$ |
| r.a.a. | becomes a special case of RAA |
| id. | $\{\exists v(\varphi v$ & $\psi v)\} \vdash (v)(\varphi v \rightarrow \varphi v)$ & $\exists v\varphi v$ |

Anyone who wishes to prove these sequents within the monadic calculus described in the present section will find his task greatly eased if he makes use of the following result:

*Metatheorem 8.* *If* $\Gamma$, $\varphi(t/v) \vdash \psi(t/v)$ *and* t *does not occur in* $\psi(v)$ *or in any wff of* $\Gamma$, *then* $\Gamma$, $\exists v\varphi(v) \vdash \exists v\psi(v)$.

| | | | | |
|---|---|---|---|---|
| Suppose that | | | $\vdots$ | |
| | $\Gamma \cup \{\varphi(t/v)\}$ | $(n)$ | $\psi(t/v)$ | |
| | $\{n+1\}$ | $(n+1)$ | $(v)—\psi(v)$ | As. |
| | $\{n+1\}$ | $(n+2)$ | $—\psi(t/v)$ | $n+1$, UE |
| | $\Gamma \cup \{n+1\}$ | $(n+3)$ | $—\varphi(t/v)$ | ..., $n$ & $n+2$, RAA |
| | $\{n+4\}$ | $(n+4)$ | $\exists v\varphi(v)$ | As. |
| | $\{n+4\}$ | $(n+5)$ | $—(v)—\varphi(v)$ | $n+4$, df. $\exists$ |
| | $\Gamma \cup \{n+1\}$ | $(n+6)$ | $(v)—\varphi(v)$ | $n+3$, UI |
| | $\Gamma \cup \{n+4\}$ | $(n+7)$ | $—(v)—\psi(v)$ | $n+1$, $n+5$ & $n+6$, RAA |
| Then | $\Gamma \cup \{n+4\}$ | $(n+8)$ | $\exists v\psi(v)$ | $n+7$, df. $\exists$ |

This metatheorem enables us to prove the translated version of s.c. for example, simply by proving the schema $\varphi t$ & $\psi t \vdash \psi t$ & $\varphi t$. Given the soundness of the predicate calculus under its standard interpretation, the translation of BS into a fragment of it furnishes a further proof of the soundness of BS.

## 5.2 *An artificial interpretation of BS⁺*

Unlike BS, $BS^+$ is not sound under the above translation scheme plus the standard interpretation of monadic predicate logic. Since $(v)(\varphi v \rightarrow \varphi v)$ & $\exists v\varphi v$ has invalid instances under that interpretation, $A\alpha\alpha$ has such instances when we add the translation scheme; hence the strong identity rule id.$^+$ of $BS^+$ permits derivation of invalid formulas. We can, however, modify and elaborate the translation until we reach the rather artificial and unnatural scheme which follows:[1]

[1]There is at least one sound alternative, but it is no less artificial.

The interpretation thus generated, which gives existential import to the negative wffs rather than the affirmatives, renders $BS^+$ sound and makes all the translated rules of that system including id.$^+$ derived rules of the predicate calculus.

| | |
|---|---|
| $A\alpha\beta$ | $(v)((\varphi v \rightarrow \psi v) \ \& \ (\psi v \rightarrow \varphi v))$ |
| $E\alpha\beta$ | $-\exists v(\varphi v \ \& \ \psi v) \ \& \ \exists v \varphi v \ \& \ \exists v \psi v$ |
| $I\alpha\beta$ | $\exists v(\varphi v \ \& \ \psi v) \ v \ -\exists v \varphi v \ v \ -\exists v \psi v$ |
| $O\alpha\beta$ | $\exists v(\varphi v \ \& \ -\psi v) \ v \ \exists v(\psi v \ \& \ -\varphi v)$ |

5.3 *Non-empty terms*

If the artificial interpretation given in the last section were the best that could be done for $BS^+$, one would have to conclude that the system, and its extensions, $EBS^+$, $EBS^+$ + PC and Łukasiewicz's system, had no interesting interpretation. However, Łukasiewicz has pointed out that his system is sound if term variables are allowed to range only over non-empty terms.

One way of representing this interpretation (Interpretation II) in the predicate calculus reflects the device of adding *constantiae* to be found in John of St. Thomas. *A*, *E*, *I* and *O* formulas are translated in the Frege/Russell manner ($A\alpha\beta = (v)(\varphi v \rightarrow \psi v)$, $O\alpha\beta = \exists v(\varphi v \ \& \ -\psi v)$, etc.) and existential premisses are added to inferences where necessary. Thus, in translation, the rule of subaltern inference becomes:
$\{(v)(\varphi v \rightarrow \psi v), \exists v \varphi v\} \vdash \exists v(\varphi v \ \& \ \psi v)$. To take a further example, the sequent $\{Aab, Abc\} \vdash Iac$ becomes:
$\{(x)(Fx \rightarrow Gx), (x)(Gx \rightarrow Hx), \exists x Fx\} \vdash \exists x(Fx \ \& \ Hx)$. $\exists x Fx$ is added (as a constantia) to validate the sequent. Such an addition is required with valid syllogistic moods whenever a particular conclusion is drawn from two universal premisses (of which there are nine cases). In $EBS^+$ the inferences from $A\alpha\beta$ to $-E\alpha\beta$, from $E\alpha\beta$ to $-A\alpha\beta$ etc. must also be

treated in the same way. This manner of translating syllogistic into the predicate calculus is favoured by Alonzo Church (1972) and E. J. Lemmon (1965), and has the merit of making the extra-logical existential assumptions explicit. Nevertheless it turns the immediate subaltern inference into a mediate one, and no longer represents $A$ and $E$ forms as intrinsically contrary nor $I$ and $O$ as intrinsically subcontrary. Of course, if an axiom schema $\vdash \exists v \varphi v$ were introduced, the addition of existential *constantiae* would no longer be necessary and these objections would be avoided; but only at the cost of adding a highly implausible postulate. It can scarcely be accepted as a truth of logic that every predicate has application, that every describable sort of thing is actually to be found in the world: indeed, it is clear that it is just false. However, suppose we introduce a special predicate letter for predicates which do have application, where the use of such a letter is to carry the commitment of application. Then the axiom schema $\vdash \exists v \mathscr{F} v$ (where $\mathscr{F}$ is a schematic special-predicate letter) could be introduced and constantiae again avoided. It is easy to check that, whichever of these techniques is adopted, the translated rules of $BS^+$ are derivable. Objections to this mode of translation will be considered in 5.5 when we evaluate translation into many-sorted logic.

$BS^+$ is complete under Interpretation II. (Its soundness is a corollary of its translatability into the predicate calculus, given the soundness of the latter.) Completeness can be proved by modifying the proof of 3.2 in the manner sketched below.

*Metatheorem 9. $BS^+$ is complete under Interpretation II.*

To obtain the model set $U(\Delta)$ from the set of wffs $\Delta$:

for each $A\alpha\beta$ delete each $[\alpha\beta]$ ;

for each $E\alpha\beta$ delete each $[\alpha\beta]$.

(1) First show that, for every $\alpha \epsilon$ V, there is some $[\alpha]$ in U($\Delta$). Consider $\{\alpha, \gamma_1, \ldots, \gamma_n\}$, $n \geq 0$, in P(V), where $\gamma_1, \ldots, \gamma_n$ are all the $\gamma_i$'s in V such that $\Delta \vdash A\alpha\gamma_i$. We can show that this set is undeletable if $\Delta$ is consistent, in much the same way as on pp. 44-5 of 3.2.

All formulas of the forms $A\alpha\beta$, $E\alpha\beta$ are therefore verified by the model set.

(2) Every formula in $\Delta$ of the form $I\alpha\beta$ is verified. The argument is similar to the argument on pp. 45-6 except that it is not necessary to consider the possibility of deletion resulting from O formulas.

(3) Suppose a formula $O\alpha\beta$ occurs in $\Delta$. If it is false, every set $[\alpha\beta']$ must be absent from U($\Delta$). Consider the set $\{\alpha, \gamma_1, \ldots, \gamma_n\}$, where $\gamma_1, \ldots, \gamma_n$ are all the $\gamma_i$'s in V such that $\Delta \vdash A\alpha\gamma_i$. $\beta$ cannot be one of the $\gamma_i$'s, since $\Delta$ would then yield $A\alpha\beta$ and so be inconsistent. Its deletion by any $A$ formula is ruled out by the usual considerations. The only other possibilities are deletion by some wff $E\alpha\gamma_i$ or $E\gamma_i\alpha$. But $\Delta \vdash A\alpha\gamma_i$, which yields both $I\alpha\gamma_i$ and $I\gamma_i\alpha$. So $O\alpha\beta$ can be false only if $\Delta$ is inconsistent.

5.4 *Many-sorted predicate logic*

From a formal point of view a very attractive way of translating a system like BS$^+$ into a fragment of the predicate calculus is the one devised by Timothy Smiley (1961). In ordinary predicate logic provable sequents are correct for all domains which are not empty. Thus $\exists x(Fx \vee - Fx)$ is a theorem of the system: there is something which either has $F$ or lacks it, which would clearly not be true of the empty domain. (Free logics, which apply equally to the empty domain, have been devised in recent years by logicians who feel that the assumption of a non-empty domain is an extra-logical one, but such systems need not concern us here.) Smiley has presented a many-sorted logic, in which individual variables range over possibly different domains, and the non-emptiness of each domain reflects the non-emptiness of the terms in $BS^+$ under Interpretation II. A variation of his system will now be described.

In the case of the ordinary predicate calculus we suppose a single non-empty domain over which the individual variables $x$, $y$, $z$ etc. range. In the many-sorted calculus we shall have indefinitely many different sorts of individual variable, each sort associated with one of the various domains. (The domains for different sorts need not all be different, and where they differ they may overlap.) We shall use a different letter of the alphabet for each sort of variable. For example, we might take $a$, $a^*$, $a^{**}$, $a^{***}$...to range over men, $b$, $b^*$, $b^{**}$, $b^{***}$..., to range over hypocrites, and so on. $(a)Ca$ would mean that every man has the property $C$, and $(b)Cb$ that every hypocrite has the property $C$. In order to avoid breaking out into exotic symbolism, we use some of the roman letters already used in the basic syllogistics: but the use of $a$, $b$, $c$... as sorted individual variables as well as term variables should not be taken to imply that their roles in the respective systems are precisely the same.

There will be a sortal predicate letter corresponding to each sort of individual variable. We shall have '$A$' corresponding to the variables in $a$, '$B$' corresponding to the variables in $b$, and so on. If the $a$'s range over men, '$A$' will be understood as 'is a man'; if the $b$'s range over hypocrites, '$B$' will be understood as 'is a hypocrite', and so on.

Just as the single domain for the ordinary predicate calculus has to be non-empty, so each of the domains for this many-sorted calculus will be non-empty, since it duplicates ordinary predicate logic for each of the many domains.

Finally, there will be many sorts of name, one for each domain. Take the domain of then and suppose the variables in $a$ to range over it. Then (at least some of) the members of the domain are to be named by the letters $a_1$, $a_2$, $a_3$,.... Similarly, if the variables $b$, $b^*$, $b^{**}$,... range over hypocrites, a hypocrite will be named by each of the following: $b_1$, $b_2$, $b_3$,.... Different names, whether of the same or of a different sort, may name the same individual. Whereas, then, small letters (possibly followed by one or more asterisks) *without* numerical subscripts are *variables*, small letters *with* numerical subscripts are *names*.

The language of the system will therefore consist of propositional calculus symbols plus:

| sorted individual variables (bound) | sorted names | sortal predicates |
|---|---|---|
| $a$, $a^*$, $a^{**}$, $a^{***}$,... | $a_1$, $a_2$, $a_3$, $a_4$,... | $A$ |
| $b$, $b^*$, $b^{**}$, $b^{***}$,... | $b_1$, $b_2$, $b_3$, $b_4$,... | $B$ |
| $c$, $c^*$, $c^{**}$, $c^{***}$,... | $c_1$, $c_2$, $c_3$, $c_4$,... | $C$ |
| . | . | . |
| . | . | . |
| . | . | . |

To illustrate the intended interpretation of the symbolism here are some paradigms:

| | *Logical Form* |
|---|---|
| Every man is a hypocrite | $(a)Ba$ |
| No man is a hypocrite | $-\exists aBa$, or, equivalently, $(a)-Ba$ |
| Some man is a hypocrite | $\exists aBa$ |
| Not every man is a hypocrite | $-(a)Ba$, or equivalently, $\exists a-Ba$ |

In general, a formula $A\alpha\beta$ will be translated into one of the form (v)Uv and a formula $I\alpha\beta$ into one of the form $\exists$vUv; $O$ and $E$ formulas will be translated into the corresponding negations.

Now for the rules. The elimination rule for the universal quantifier needs a little modification from its counterpart in the single-sorted system. For example, we do not want to be able to derive $Ca_1$ from $(b)Cb$ alone, for this would be instantiated by the inference

Every hypocrite is a liar
Therefore (the man) Jones is a liar

To turn this into a valid deductive inference we obviously need the additional premiss

Jones is a hypocrite

So, although we do not want to be able to derive $Ca_1$ from $(b)Cb$, we do want to be able to derive it from that premiss supplemented with the premiss $Ba_1$. Thus the UE rule takes the following form:

If $\Gamma \vdash (v)\varphi(v)$ and $\Gamma \vdash V(v_i)$, then $\Gamma \vdash \varphi(v_i/v)$

Again, the rule UI needs modification as a result of the introduction of different sorts of variable. Universal introduction is confined to the case where a name is replaced by a variable *of the same sort*:

If $\Gamma \vdash \varphi(v_i)$ and $v_i$ does not occur in any wff in $\Gamma$
nor v in $\varphi(v_i)$, then $\Gamma \vdash (v)\varphi(v/v_i)$

'$\exists$' is defined as before in terms of the universal quantifier.

An additional rule is needed to reflect the relation between sorted names and their corresponding sortal predicates, a rule which we shall call the 'sortal rule' (SR):

$$\frac{*}{Vv_i}$$

This rules gives us indefinitely many theorems like $Aa_1$, $Bb_5$, $Kk_{27}$, where the name is of the same sort as the predicate. UI then secures the connection between sorted variables and their corresponding sortal predicates and generates theorems like $(a)Aa$, $(e)Ee$. The addition of SR also gives rise to the useful derived rule $\{(v)\varphi(v)\} \vdash \varphi(v_i/v)$, which we shall call 'UE(S)'. For, given $(v)\varphi(v)$, we have $Vv_i$ by SR, and hence $\varphi(v_i/v)$ by UE.

It is now possible to prove all the rules of $BS^+$ as derived rules of this calculus. r.a.a. becomes an obvious special case of RAA and no more need be said about it. On translation the other rules of $BS^+$ become the following:

| | |
|---|---|
| s.c. | $\{\exists uVu\} \vdash \exists vUv$, which is shorthand for |
| | $\{-(u)-Vu\} \vdash -(v)-Uv$ |
| sub. | $\{(u)Vu\} \vdash \exists uVu$, $= \{(u)Vu\} \vdash -(u)-Vu$ |
| Barbara | $\{(u)Vu, (v)Wv\} \vdash (u)Wu$ |
| Celarent | $\{(u)Vu, -\exists vWv\} \vdash -\exists uWu$ |
| id.$^+$ | $\Lambda \vdash (v)Vv$ |

We give proof schemas of all of these except Celarent, the proof of which is similar to that of Barbara (though slightly longer).

| | | | | |
|---|---|---|---|---|
| s.c. | $\{1\}$ | (1) | $[(v)-Uv]$ | As. |
| | $\{2\}$ | (2) | $[Vu_1]$ | As. |
| | $\{1,2\}$ | (3) | $-Uu_1$ | 1,2 UE |
| | $\Lambda$ | (4) | $Uu_1$ | SR |
| | $\{1\}$ | (5) | $-Vu_1$ | 2, 3 & 4 RAA |
| | $\{1\}$ | (6) | $(u)-Vu$ | 5 UI |
| | $\{7\}$ | (7) | $-(u)-VU$ | As. |
| | $\{7\}$ | (8) | $-(v)-Uv$ | 1, 6 & 7 RAA |

| | | | | |
|---|---|---|---|---|
| sub. | $\{1\}$ | (1) | (u)Vu | As. |
| | $\{2\}$ | (2) | (u)—Vu | As. |
| | $\{1\}$ | (3) | $Vu_1$ | 1 UE(S) |
| | $\{2\}$ | (4) | $-Vu_1$ | 2 UE(S) |
| | $\{1\}$ | (5) | —(u)—Vu | 2, 3 & 4 RAA |
| Barbara | $\{1\}$ | (1) | (u)Vu | As. |
| | $\{2\}$ | (2) | (v)Wv | As. |
| | $\{1\}$ | (3) | $Vu_1$ | 1 UE(S) |
| | $\{1,2\}$ | (4) | $Wu_1$ | 2,3 UE |
| | $\{1,2\}$ | (5) | (u)Wu | 4 UI |
| id.$^+$ | $\Lambda$ | (1) | $Vv_1$ | SR |
| | $\Lambda$ | (2) | (v)Vv | 2 UI |

Since $\exists$vVv is derivable from (v)Vv, we can thereby prove that each domain is non-empty.

5.5 *Objections to essentially non-empty terms*

It has been pointed out (for examples, by the Kneales) that a logic for non-empty terms is narrower, less comprehensive, than one for unrestricted terms; but that is not the principal objection to this interpretation. Essentially, the main objection is that in English and similar natural languages noun expressions like *man* and *white rabbit* are not *necessarily*-non-empty terms, so that the logic cannot really be applied in the suggested way to natural language. It is true, for example, that the inference from *Aab* to *Iab* is 'sound' under an interpretation which restricts the substituends of *a* to terms which are in fact non-empty, for it will then be impossible to produce a substitution instance with a true premiss and a false conclusion, but it is a mistake to think that if *A* sentences are to be interpreted as lacking existential import the second of the following inferences is a genuine entailment:

| | |
|---|---|
| Every unicorn is a quadruped | Every man is a hypocrite |
| So, some unicorn is a quadruped | So, some man is a hypocrite |

If *A* sentences lack existential import the first of these is not a genuine entailment, because the *A* premiss is true whereas the *I* conclusion is false. But then the second fails to be an entailment too, since although there are in fact men it is logically possible that there should be none, and so logically possible (granted that *A* lacks existential import) that *Every man is a hypocrite* should be true and *Some man is a hypocrite* false. It is all very well to say that the system will never take us from true premisses to a false conclusion provided we use terms which are in fact non-empty; the non-emptiness of our terms will (except perhaps in special cases) be an extra-logical matter which will have to be established empirically, and such extra-logical facts as that men exist should be stated in extra premisses. The modified single-sorted predicate system with special script predicate letters for non-empty predicates is open to a similar objection. Ordinary predicates - in English, at any rate - do not come with their non-emptiness built into their meaning (with the possible exception of special cases like 'is coloured or not coloured', though even these cases are debatable).

It is only an aspect of the same objection to the suggested use of the many-sorted system that it presupposes that it is a necessary truth that there is something which is an *A*, something which is a *B*, and so on. It just is not necessarily true that there are men, hypocrites, horses, stones, etc.

Of course, this consideration does nothing to undermine the value of Smiley's interpretation for exegetical purposes, and there is indeed some reason to think that Aristotle himself did regard terms as essentially non-empty. (See p.26 above, and Corcoran (1972), p. 104.)

5.6 *A syllogistic system for the Brentano/Frege/Russell interpretation of general categoricals*

The following system goes over into a fragment of the ordinary single-sorted calculus if $A$, $E$ formulas are translated in the style of Brentano, Frege and Russell as lacking existential import, while $I$ and $O$ formulas have it.

In $BS^+$ replace sub. by the two rules:

$$\frac{I\alpha\beta}{I\alpha\alpha} \qquad \frac{O\alpha\beta}{I\alpha\alpha} \qquad \text{(cf. Shepherdson (1956).)}$$

This system can be proved complete after the manner of 3.2 with respect to an interpretation which allows terms to be empty, and takes $A$ and $E$ as true if the subject term is empty and $I$ and $O$ as false. (*Metatheorem 10.*)

*Sketch of proof.* The model set $U(\Delta)$ is to be formed as follows:

for every $A\alpha\beta$ in $\Delta$, delete each $[\alpha\beta']$ in $P(V)$;

for every $E\alpha\beta$ in $\Delta$, delete each $[\alpha\beta]$ in $P(V)$.

Universal formulas are automatically verified and only two cases have to be considered, formulas of the forms $I\alpha\beta$ and $O\alpha\beta$.

(1) Suppose that, with $\Delta$ consistent, some wff in $\Delta$ of the form $I\alpha\beta$ is false. Then there is no set $[\alpha\beta]$ in $U(\Delta)$. Show this by considering the set $\{\alpha,\beta,\gamma_1\ldots,\gamma_n\}$, where $\gamma_1,\ldots\gamma_n$ are, as usual, all the $\gamma_i$'s in V such that $\Delta \vdash A\alpha\gamma_i$ or $\Delta \vdash A\beta\gamma_i$. The set is undeletable by an $A$ wff for the reasons given in 3.2 and can also be shown to be undeletable by an $E$ wff. (Consider $E\alpha\alpha$, $E\beta\beta$, $E\alpha\beta$, $E\beta\alpha$, $E\alpha\gamma_i$, $E\gamma_i\alpha$, $E\beta\gamma_j$, $E\gamma_i\beta$, $E\gamma_i\gamma_j$.)

(2) Suppose that, with $\Delta$ consistent, some wff in $\Delta$ of the form $O\alpha\beta$ $(\alpha \neq \beta)$ is false. Then $U(\Delta)$ will contain no set $[\alpha\beta']$. But the set

$\{\alpha, \gamma_1, \ldots, \gamma_n\}$, formed in the usual way, will prove undeletable, both by $A$ wffs, for the usual reasons, and by $E$ wffs. Consider $E\alpha\alpha$, $E\alpha\gamma_i$, $E\gamma_i\alpha$, $E\gamma_i\gamma_j$.)

No set $\Delta$ containing $O\alpha\alpha$ is consistent, since $\Delta \vdash A\alpha\alpha$ by id.$^+$.

With no rule of subaltern inference, there will be no derived rules of conversion *per accidens* for $A$ and $E$ formulas. (Under the Brentano interpretation $A$ and $E$ also cease to be contraries, $I$ and $O$ to be subcontraries.)

5.7 *Multiple generality*

None of the basic systems, even when adjoined to the propositional calculus, is as comprehensive a system as monadic predicate calculus. Nor, unlike the monadic calculus, can they be extended into a polyadic system capable of expressing relations. Take, for example, the following relational sort of inference considered by Ockham:

> All men are animals; Socrates sees a man; therefore,
> Socrates sees an animal

the form of which can be expressed in the (polyadic) predicate calculus thus:

$$\{(x)(Mx \rightarrow Ax),\ \exists x(Mx \;\&\; Ssx)\} \Vdash \exists x(Ax \;\&\; Ssx)$$

But the most spectacular advance by modern over syllogistic logic is the capacity of the full predicate calculus to cope successfully with multiple generality. Geach (1962) has examined and criticized the medieval attempt to deal with multiple generality by means of different types of *suppositio* (reference), and it is beyond doubt that the problem was not solved until Frege. The range of inferences expressible in our language which can be expressed with merely syllogistic resources is narrow in the extreme compared with those which can be expressed within the predicate calculus. Syllogistic cannot begin to express and distinguish between the following pair of inferences devised by Geach (1971, p.102ff.), for example, the first of which is clearly valid and the second invalid:

(A) (1) Any thing that counts as the personal property of a tribesman is suitable to offer to a guest by way of hospitality

(2) One thing that counts as the personal property of a tribesman is that tribesman's wife

(3) So she is suitable to offer to a guest by way of hospitality

(b) (4) Any woman whom every tribesman admires is beautiful by European standards

(5) One woman whom every tribesman admires is that tribesman's wife

(6) So she is beautiful by European standards

Within the predicate calculus the patterns of these arguments can be given in the following way (though Geach himself would not actually approve of these analyses - see 7.2 below):

(A) (1) $(x)(y)((Tx \;\&\; Byx) \rightarrow Sy)$

(2) $(x)(y)((Tx \;\&\; Mxy) \rightarrow Byx)$

(3) $(x)(y)((Tx \;\&\; Mxy) \rightarrow Sy)$

(B) (4) $(x)((Wx \;\&\; (y)(Ty \rightarrow Ayx)) \rightarrow Ex)$

(5) $(y)(Ty \rightarrow \exists x(Wx \;\&\; Mxy \;\&\; Ayx))$

(6) $(y)(Ty \rightarrow \exists x(Wx \;\&\; Mxy \;\&\; Ex))$

To whatever extent natural language is streamlined when re-expressed by means of predicate calculus symbolism, the latter is still an incomparably more powerful tool for the analysis of inferences than syllogistic. Moreover, there is clearly no hope of developing syllogistic into a language capable of expressing mathematics. The axioms of set theory can only be expressed in a symbolism which can cope with multiple generality. Consider, for example, the power set axiom (variables here ranging over sets): $(x)\exists y(z)(z\in y \leftrightarrow z\subseteq x)$. This implies that the set of subsets of any set is itself a set, which is clearly distinguishable from the false claim that there is one and the same set which is the power set of every set, implied by $\exists y(x)(z)(z\in y \leftrightarrow z\subseteq x)$.

Frege found that the traditional manner of analysing sentences stood in the way of developing means of expressing multiple generality. Instead of treating 'Socrates is a philosopher' as composed of subject-copula-predicate-term, he found it better to treat the copula as part of the predicate. The predicate was thought of as referring to a function, of which the reference of the subject term was the argument. For accounts of Frege's invention, rightly called by the Kneales (1962, p. 511) 'one of the greatest intellectual inventions of the nineteenth century', the reader is referred to Anthony Kenny, (1973, Chapter 2), P. T. Geach (1971, 1.1) and Michael Dummett (1973, Chapter 2). In a later chapter we shall turn to a consideration of subject and predicate and a comparison between the traditional and predicate calculus treatments of them. Meanwhile, in the next chapter we shall look at a rather different type of interpretation of traditional logic, which has received extensive attention in recent years.

## APPENDIX TO CHAPTER 5
## SINGULAR TERMS

Smiley has compared certain proofs in the many-sorted logic with Aristotle's use of 'exposed' terms in his proofs by *ecthesis*. Smiley has free variables where we have names, but we may make a similar comparison.

In his treatment of assertoric logic Aristotle indicates ecthetic proofs in four cases: simple conversion of *E*, Darapti, Datisi and Bocardo, though no details are given in the case of Datisi. We shall take Darapti first, the mood in which *Irp* follows from $\{Asp, Asr\}$. Having shown that it is provable both by direct and indirect reduction, he describes a third method of proof:

> ...if both *P* and *R* belong to all *S*, should one of the *S*s, e.g. *N*, be taken, both *P* and *R* will belong to this, and thus *P* will belong to some *R*. ($28^{a}22$)

It is very natural, at least using the Oxford translation, to take *N* as a singular term and construe the argument in the following way. Every *s* is a *p* and every *s* is an *r*; if one *s*, call it '*N*', is taken, then it will be both a *p* and an *r*: hence some *r* is a *p*. Such an interpretation, though it gives us a simple and perspicuous proof, has been vigorously disputed in recent times by Łukasiewicz (1957, pp. 59-67) and Patzig (1968, pp. 156-68), the latter providing further textual evidence to support Łukasiewicz's case. They interpret the exposed term as a general term and claim that Aristotle is arguing somewhat as follows. If every *s* is both a *p* and an *r*, then for some term '*n*', every *n* is both a *p* and an *r*, and so therefore some *r* is a *p*. Aristotle's ecthetic proofs are claimed to be significant, not for the role they play in his logic - which they regard as quite incidental - but because the notion of existential quantification is implicit in them.

Now even though it now seems an unlikely interpretation of the text, it seems clear that the proof using *N* as a singular term is both valid and perspicuous provided that *A* propositions are taken as having existential import. Łukasiewicz (1957) criticizes Aristotle's commentator Alexander for taking *N* as a singular term and construing the proof as an empirical one depending on the perception of an individual, which Łukasiewicz says is 'not sufficient for a logical proof' (p. 62). But, clearly, one can use *N* in the proof as a singular term without perceiving its referent, and the demonstration does not have to be construed as empirical. Even though Aristotle probably did not intend his exposed terms as singular terms, it is interesting to follow up this interpretation.

Before presenting the analogue of this proof in the many-sorted calculus, we prove the following derived principle of the calculus:

P1 $\{Uv_i \ \& \ Vv_i\} \vdash \exists vUv$

Intuitively this is very clear: if some individual, $v_i$, is both a u and a v, then some v is a u. This is the principle relied on in the ecthetic argument if $N$ is taken as a singular term.

| | | | |
|---|---|---|---|
| {1} | (1) | $[(v)-Uv]$ | As. |
| {2} | (2) | $Uv_i \ \& \ Vv_i$ | As. |
| {2} | (3) | $Uv_i$ | 2 &E |
| {2} | (4) | $Vv_i$ | 2 &E |
| {1,2} | (5) | $-Uv_i$ | 1, 4 UE |
| {2} | (6) | $-(v)-Uv$ | 1, 3 & 5 RAA |
| {2} | (7) | $\exists vUv$ | 6 df. $\exists$ |

Now, using this principle we can give the proof of (an instance of) Darapti in the following form:

| | | | |
|---|---|---|---|
| {1} | (1) | $(s)Ps$ | As. |
| {2} | (2) | $(s)Rs$ | As. |
| {1} | (3) | $Ps_1$ | 1 UE(S) |
| {2} | (4) | $Rs_1$ | 2 UE(S) |
| {1,2} | (5) | $Ps_1 \ \& \ Rs_1$ | 3,4 &I |
| {1,2} | (6) | $\exists rPr$ | 5 Pl |

The sorted name here, $s_1$, is the analogue of $N$ taken as a singular term.

The ecthetic proof of the simple conversion of $E$ is given at $25^a$ 15-17:

> If no $B$ is $A$, neither can any $A$ be $B$. For if some $A$ (say $C$) were $B$, it would not be true that no $B$ is $A$: for $C$ is a $B$.

If we take $C$ as a singular term it is again possible to mimic the proof in the many-sorted system, though we need first to state another derived principle (which we shall leave to the reader to prove):

P2 If $\Gamma$, $\varphi(v_i/v) \vdash \psi$, and $v_i$ does not occur in $\varphi(v)$, $\psi$ or any wff of $\Gamma$, then $\Gamma$, $\exists v\varphi(v) \vdash \psi$

This is, in fact, Lemmon's rule of existential elimination, and for further illustrations and an explanation of its use the reader is referred to pp. 112-16 of his book. The proof of $\{(b)-Ab\} \vdash (a)-Ba$ (a typical instance of s.c.($E$) can now be given in the following form (it is not claimed, of course, that these are necessarily the simplest, most straightforward proofs of the sequents in this calculus):

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $[\exists aBa]$ | As. |
| $\{2\}$ | (2) | $Ba_1$ | As. |
| $\Lambda$ | (3) | $Aa_1$ | SR |
| $\{2\}$ | (4) | $\exists bAb$ | 2,3 P1 |
| $\{1\}$ | (5) | $\exists bAb$ | 1, 2-4 P2 |
| $\{1\}$ | (6) | $-(b)-Ab$ | 5 df.$\exists$ |
| $\{7\}$ | (7) | $(b)-Ab$ | As. |
| $\{7\}$ | (8) | $-\exists aBa$ | 1, 6 & 7 RAA |
| $\{7\}$ | (9) | $--(a)-Ba$ | 8 df.$\exists$ |
| $\{7\}$ | (10) | $(a)-Ba$ | 9 DN |

Ecthesis is concisely indicated as a second way of proving (an instance of) Bocardo ($\{Osp, Asr\} \vdash Orp$) at $28^b$ 20–22.

> Proof is possible without reduction *ad impossibile*, if one of the $Ss$ be taken to which $P$ does not belong.

Using a singular term, the argument would presumably be reconstructed like this. Some $s$ is not a $p$ but every $s$ is an $r$. So take one of the $s$'s, say $s_1$, which is not a $p$: it must be an $r$. Then you have an $r$ which is not a $p$. Its analogue in the many-sorted system uses P2 and a principle similar to P1:

$$\text{P3} \qquad \{-\mathrm{Uv}_i \ \& \ \mathrm{Vv}_i\} \vdash \exists \mathrm{v}-\mathrm{Uv}$$

The derivation of P3 is also similar to that of P1. We may now give the following proof of Bocardo:

| | | | |
|---|---|---|---|
| $\{1\}$ | (1) | $\exists s-Ps$ | As. |
| $\{2\}$ | (2) | $(s)Rs$ | As. |
| $\{3\}$ | (3) | $[-Ps_1]$ | As. |
| $\{2\}$ | (4) | $Rs_1$ | 2 UE (S) |
| $\{2,3\}$ | (5) | $-Ps_1 \ \& \ Rs_1$ | 3,4 A1 |
| $\{2,3\}$ | (6) | $\exists r-Pr$ | 5 P3 |
| $\{1,2\}$ | (7) | $\exists r-Pr$ | 1, 2-6 P2 |

It is possible to introduce singular terms into syllogistic in the form of sorted names along lines suggested by the foregoing. Again we use small roman letters with numerical subscripts as sorted names. '$U$' and '$\bar{U}$' are introduced as further constants and the formation rule extended to allow formulas of the forms $U\alpha_i\beta$ and $\bar{U}\alpha_i\beta$ for $\alpha_i$ *is a* $\beta$ and $\alpha_i$ *is not a* $\beta$ respectively. The system, $BS^+$ with names, has the following primitive rules, the original rules of $BS^+$ becoming derived rules:

(i) $$\frac{U\alpha_i\beta \qquad U\alpha_i\gamma}{I\beta\gamma}$$ (Cf. P1)

(ii) $$\frac{U\alpha_i\beta \qquad \bar{U}\alpha_i\gamma}{O\beta\gamma}$$ (Cf. P3)

(iii) $$\frac{I\alpha\beta \qquad \begin{matrix}[U\alpha_i\beta] \\ \varphi\end{matrix}}{\varphi}$$

(iv) $$\frac{O\alpha\beta \qquad \begin{matrix}[\bar{U}\alpha_i\beta] \\ \varphi\end{matrix}}{\varphi}$$

provided in the case of both rules (iii) and (iv) that $\varphi$ does not contain $\alpha_i$.

(v) $$\frac{}{U\alpha_i\alpha}$$ (Cf. SR)

(vi) r.a.a. - as for $BS^+$, but add that:

if $\varphi = U\alpha_i\beta$, $\bar{\varphi} = \bar{U}\alpha_i\beta$;

if $\varphi = \bar{U}\alpha_i\beta$, $\bar{\varphi} = U\alpha_i\beta$.

The asymmetry between subject and predicate in singular propositions with names as subjects is reflected by the fact that names are allowed only in subject place.

The whole of the original basic syllogistic $BS^+$ is now derivable on the the basis of these quasi-ecthetic principles and r.a.a. As examples we give the derivations of s.c. and id.$^+$.

| | | | | |
|---|---|---|---|---|
| s.c. | $\{1\}$ | (1) | $I\alpha\beta$ | As. |
| | $\{2\}$ | (2) | $U\alpha_1\beta$ | As. |
| | $\Lambda$ | (3) | $U\alpha_1\alpha$ | Rule (v) |
| | $\{2\}$ | (4) | $I\beta\alpha$ | 2, 3 Rule (i) |
| | $\{1\}$ | (5) | $I\beta\alpha$ | 1, 2-4 Rule (iii) |
| id.$^+$ | $\{1\}$ | (1) | $[O\alpha\alpha]$ | As. |
| | $\{2\}$ | (2) | $[\bar{U}\alpha_1\alpha]$ | As. |
| | $\Lambda$ | (3) | $U\alpha_1\alpha$ | Rule (v) |
| | $\{2\}$ | (4) | $A\alpha\alpha$ | 1, 2 & 3 r.a.a. |
| | $\Lambda$ | (5) | $A\alpha\alpha$ | 1, 2-4 Rule (iv) |

# CHAPTER 6

## STRAWSON'S INTERPRETATION AND THE QUESTION OF EXISTENTIAL IMPORT

### 6.1 *Strawson's interpretation of traditional logic*

A categorical proposition with subject term *a* will be said to have existential import iff it is a logically necessary condition of its truth that there is (was/will be) an *a*. If existential import is understood in this way, Strawson's interpretation, to be found in his article 'On Referring' (1950) and in his book *Introduction to Logical Theory* (*ILT*) (1952), attributes such import to all four forms. (*O* propositions are given in the more customary manner: 'Some ... is not ...') and predicates are not restricted to the form *is a* + noun (phrase).) Under his interpretation, which is supposed to be faithful to the ordinary meanings of the categorical sentences - a claim we shall consider in 6.3 - the whole of traditional logic with negative terms is supposed to hold good. We may consider it as an interpretation of EBS$^{+}$ (since we shall see that it makes id.$^{+}$ come out valid), and not wait until the introduction of negative terms in Chapter 9, because it is now clear that Strawson was actually wrong in thinking that traditional logic with *negative* terms is sound under his interpretation.

According to Strawson, a statement in one of the categorical forms is true or false only if its subject term has application. On p. 177 of *ILT* he says:

> We are to imagine that every logical rule of the system, when expressed in terms of truth and falsity, is preceded by the phrase 'assuming that the statements concerned are either true or false, then ...'. Thus the rule that *A* is the contradictory of *O* states that, if corresponding statements of the *A* and *O* forms both have truth-values, then they must have

opposite truth-values; the rule that $A$ entails $I$ states that, if corresponding statements of these forms have truth-values, then if the statement of the $A$ form is true, the statement of the $I$ form is true, and so on.

Under this interpretation the semantic analogue of the rule id.$^+$ will read: 'Assuming that the statement expressed by $A\,\alpha\alpha$ is true or false, then it is true on the basis of no further assumptions', so that id.$^+$ will turn out sound. (There is reason to think that Strawson did not intend it to do so — see the note by T.J. Smiley (1967) — but no matter.) And it is easy to see that the other rules of EBS$^+$ are validated as well.

In traditional logic with negative terms one of the principles of inversion, $E\alpha\beta \vdash I\,\alpha'\beta$, is made invalid.[1] ($\alpha'$ is to be read here as *non*-$\alpha$.) The following set verifies $Eab$ (interpreted as *No* [$a$] *is a* [$b$]) and falsifies $Ia'b$ (*Some* [$a'$] *is a* [$b$]):

$$\{\Lambda, \{a\}\}$$

Or, to take a less formal counter-example:

No elephant is a unicorn (true)
Therefore, some non-elephant is a unicorn (false)

Someone attracted by Strawson's interpretation of the English sentences might respond that this was so much the worse for the traditional logic of negative terms, which erroneously regarded inversion of $E$ as a valid step. (We shall have more to say about negative terms later.)

It is true that Strawson has not been consistent in the accounts he has given of entailment in the writings we are referring to. Sometimes, for example, he seems to adopt a more conventional account:

[1] See chapter 9. Smiley makes the point in his note.

> It is self-contradictory to conjoin S with the denial of S' if S' is a necessary condition of the truth, simply, of S ... The relation between S and S' in this case is that S entails S'. (*ILT*, p. 175. See also p. 212.)

In other words, S entails S' iff S' must be true whenever S is true (i.e. iff S strictly implies S'). But this will not do if Strawson wants to save traditional logic, since it fails to validate simple conversion of *E*: consider

(1) No elephant is a unicorn

(2) No unicorn is an elephant

Strawson must regard (1) as true, since its existential presupposition is satisfied, and (2) as truth-valueless, since its existential presupposition is false. There can be little doubt, therefore, that his intended account of entailment is more accurately given above in the quotation from p. 177. On that account S entails S' iff it cannot be the case that S' is false and S true.[2]

Yet there are serious objections to that account of entailment. In the first place transitivity breaks down in what is surely an unacceptable manner. This can be shown by the following example, which would have to count as an entailment on the Strawsonian account:

(3) No dog is a unicorn

(4) (In the U.K.) every animal which is not a dog may be kept without a Post Office licence

Therefore, (5) some unicorn may be kept without a Post Office licence

[2] It is not clear whether he would want to add the additional proviso that if S' is false S must be false too, as Hart (1951) does in his similar account of the Square of Opposition. Smiley thinks Strawson implies this on p. 213 of *ILT*.

On any view both premisses are true, while on Strawson's view the conclusion is neither true nor false. If, however, it did have a truth-value and there *were* a unicorn, it would, on the first premiss, be an animal but not a dog, and, given the truth of the second premiss, an animal you could keep without a licence. The conclusion would be true, and on Strawson's account it is therefore jointly entailed by (3) and (4). Now, on Strawson's account - and on any reasonable view - (5) entails the somewhat clumsily expressed statement:

> (6) Something that may be kept without a Post Office licence is a unicorn

which on Strawson's view is false, and so cannot be entailed by (3) and (4). Some logicians have wanted to define a notion of entailment which is not unrestrictedly transitive, but in such instances transitivity is sacrificed for the sake of avoiding the Lewis paradoxes and in any case breaks down only when necessary or impossible propositions are involved. (For some account of this see the final essay in G.H. von Wright (1957).)

Even more serious is the following consideration. As Strawson himself clearly recognizes on p. 13 of *ILT*, logical inference should never take us from true premisses to conclusions which are not true:

> Though inferring, proving, arguing have different purposes, they seem usually to have also the common purpose of connecting truths with truths. The validity of the steps is, in general, prized for the sake of the truth of the conclusions to which they lead.

Strawson's account of entailment is unacceptable since it divorces entailment from inference in this respect. We just do not want to have true premisses entailing conclusions which are not true. Once we know that our premisses are true, we should not have to appeal to further extra-logical considerations to determine whether their logical consequences are true. Some other definitions of entailment do not allow that every inference corresponds to an entailment, since they restrict the latter's transitivity; but to have entailments which do not correspond to valid logical inferences is quite unacceptable. Hence, we should reject the claim that (5) above is jointly entailed by (3) and (4). The more conventional account of entailment quoted from p. 175 of *ILT* was much better, since it was not open to this criticism; but, as we have seen, it cannot be used to save traditional logic in a Strawsonian manner. To the extent that it worked, Strawson's interpretation depended on a redefinition of semantic entailment which is wholly unacceptable to a logician.

One of Strawson's motives for wanting to interpret syllogistic in the manner described was to underpin his treatment of the grammatical subject expressions of categorical sentences as genuine logical subjects. Here he thinks traditional logic is superior to predicate logic, which construes such grammatical subjects as logically predicative. Yet surely this is a serious mistake. On Strawson's view, to say 'Some mammal is a sea-dweller' is to refer to a particular mammal and say of it that it is a sea-dweller. But to which mammal? The one the speaker is thinking of? But he may not be thinking of any mammal in particular. And, even if he were, his statement would still be true if the mammal he was thinking of were not a sea-dweller, provided at least one other mammal lived in the sea. And, if the statement were false, it would not be false simply because the particular mammal thought of by the speaker was not a sea-dweller, but because no mammal was.

### 6.2 *Existential import and presupposition*

We may, of course, consider Strawson's view of the existential import of categorical statements quite irrespective of whether he has provided an acceptable interpretation of traditional logic. Although he accepts that, at least in many ordinary cases, all four types have existential import as we have defined it, he would deny that the statements expressed by those sentences *entailed* their existential commitments. Rather, they are held to presuppose them. According to *ILT* S presupposes S' iff S can be true or false only if S' is true. But in this case S' cannot be false when S is true, and so, on the account of entailment attributed to Strawson above, if S presupposes S' it also entails it.[3]

It may appear that we can deny that presupposition is a case of entailment by following Hart (see footnote 2 above) and stipulating that S entails S' iff (i) S' cannot be false when S is true and (ii) S must be false if S' is false. The extra condition, (ii), however, proves far too strong. It would disqualify as an entailment the relation from, for example,

(7) Some unicorn is an elephant

to

(8) Some elephant is a unicorn

since the Strawsonian view requires that (8) be false while (7) is neither true nor false.[4]

[3] Cf. G. Nerlich (1965); R. Montague (1969).

[4] Moreover, use of the revised definition to make presupposing and entailing mutually exclusive also has the following embarrassing consequence. Consider

(P1) Some famous one-handed pianist is still giving concerts and anyone still giving concerts must exist.

(P1') There is (exists) a famous one-handed pianist.

Arguably (the statement expressed by) (P1) entails (that expressed by) (P1'). Now suppose that (P1') expresses a false statement: then surely at least one of the conjuncts of (P1) must also express a false statement. But it cannot be the second conjunct, since that expresses a necessary truth. And on the Strawsonian view the first does not express a statement with truth-value. Once again, the only way out would be an unpalatable restriction on the transitivity of entailment. (It is therefore no consolation that the argument on pp. 96-7 would *not* show that entailment as defined by Hart was non-transitive.)

So there does not seem any obviously plausible way of denying that presupposition is a special case of entailment: what seemed to be a distinctive way of dealing with existential import proves not to be so distinctive after all.

Possibly the most that can be said, then, is that the existential implication of a categorical statement expressed by a sentence of one of the four forms is a distinguishable species of entailment. We may say that S *presupposes* S' if[5] both S and its negation entail S'. Now given that the pairs *A/O* and *E/I* contain contradictories, it does follow that, if statements of each of the four forms entail that the subject term has reference (to put it rather loosely), they presuppose it in the sense just defined.

The views discussed in the present and previous sections are those defended by Strawson from 1950 to 1954. It may be thought unfair to subject that position to as detailed a scrutiny as we have given it here, on the ground that it is insufficiently worked out. But, even if this is so, it has nevertheless received a good deal of attention and figured in many an introductory logic course, for which reason alone it could scarcely be ignored.

In a more recent paper ('Identifying Reference and Truth-values', 1964, reprinted in his (1971)) Strawson has made some concessions, but he still wants to insist that the statement that some mammal is a land-dweller, for example, presupposes and does not entail that there is a mammal. He thinks that such a view stands irrespective of whether we regard the former statement as lacking a truth-value when its presupposition is false. In one respect this does indeed make his position less vulnerable, since the thesis that *A*, *E*, *I* and *O*

[5] *Sic.* The 'if' is not to be understood as 'if and only if', since the condition which follows is proposed only as a sufficient one.

propositions lack truth-value when their subject-terms are empty is one which is open to question. For example, the following proposition is surely true:

> If there are unicorns in Loch Ness, then some unicorns are very elusive creatures

Yet if its consequent lacks a truth-value, it seems a little odd to ascribe one to the whole proposition (or, if Strawson would prefer it, to regard the conditional sentence as expressing a true statement). The trouble with the revised position of 1964 is that it is impossible to assess it, since we no longer have any clear account of the notion of presupposition, the definition of which previously depended on the possibility of truth-valueless utterances of significant sentences.

### 6.3 *Existential import: some further considerations*

Under both Strawson's interpretation and Interpretation I affirmative categoricals have existential import; but how faithful is such an interpretation to the actual meaning of those sentences? We shall continue to include in our discussion sentences in which the predicate has an adjective or a verbal phrase after the copula, since the significance of the applicative particles *every* and *some* can hardly be claimed to differ according to which of the three forms the predicate has.

On the face of it, is seems self-contradictory to say, 'Every man is a hypocrite but there are no men'. However, whether this is so or not, there are other propositions of the *A* form which evidently lack existential import, like Bradley's example (1) *Every trespasser will be prosecuted*, the exhibition of which does not prove to be an empty threat merely in the event of there being no one who trespasses. Moreover, it strictly implies the proposition (2) *Every blue-skinned trespasser will be prosecuted.* Now suppose there are trespassers and each one is prosecuted. Then the first proposition is uncontroversially true. Suppose also, unsurprisingly enough, that none of the trespassers is blue-skinned. Then, if the first proposition does indeed strictly imply the second, the second must also be true, despite its lack of existential import.

This does not necessarily mean that the Frege/Russell analysis ('There is no one who is a trespasser but who will not be prosecuted', etc.) is to be adopted. For on this analysis the first proposition is true if there are no trespassers and so is *Every trespasser will avoid prosecution*. It would be natural to hold that further grounds are needed for the truth of the first proposition than merely the absence of trespassers, e.g. that the owners had made a firm and irrevocable decision to prosecute anyone who trespassed on their land.

However, this objection to the Frege/Russell analysis is by no means conclusive. Keynes himself notes that '[as] regards the ordinary usage of language there can be no doubt that we seldom do as a matter of fact make predications about non-existent subjects. For such predications would in general have little utility or interest for us' (1906, p. 235). In *A* sentences like those under consideration, the existential implication might be one of those conversational implicatures arising out of pragmatic conventions which Grice has drawn our attention to.[6] The Frege/Russell analysis of those sentences could then stand.

It is clear anyhow that the system BS cannot be accepted as a logic for all of the *A* sentences beginning with *Every*, since at least some of them lack existential import. And where sentences beginning with *Every* lack existential import, the corresponding sentences beginning with *Any*, *Anyone*, *Anybody* and *All* will surely lack it too.

[6] See H.P. Grice (1961) and (1975), L.J. Cohen (1971), R.M. Hare (1971).

(Try modifying the examples (1) and (2) above.) The same applies, I think, even to *Each*, in spite of the claims about *every* and *each* made by Zeno Vendler (1962): *Each trespasser will be prosecuted* no more entails that there will be any trespassers than does *Every trespasser*....

The position with *I* propositions is relatively uncontroversial, and it seems difficult to affirm the consistency of *Some a is F but there is no a* or its more idiomatic plural form. It is not to the point to cite an example like *Some ghosts are unfriendly*. which might occur in a story or conversation about a story; for in those contexts it *would* be self-contradictory to say that some ghosts were friendly but there were no ghosts. The following example from an old textbook is scarcely more disturbing:

> (3) Some of the cruisers for which plans were made in the last budget are not being constructed

For once again it does not seem possible consistently to say that some of the planned cruisers are not being constructed but that no cruisers were planned. In any case, the semantics of sentences containing clauses governed by psychological phrases ('plans were made') are notoriously problematic, and since those sentences clearly require special treatment, it is quite reasonable in the present context to restrict our discussion to straightforward 'non-intentional' sentences.

Since particular affirmative sentences have existential import and at least some universal affirmatives lack it (or lack it in some contexts), BS is not always an appropriate logic for affirmatives. It may even be true that only in special cases, where the existential commitment is explicitly added, or strictly implied by some special subject phrases (as in 'All of the cruisers which were built ...'), is that logic an appropriate one. Only in those cases, perhaps, are subaltern inference and c.p.a. valid (and *A* and *E* forms contrary).

By contrast with Strawson's interpretation, Interpretation I gives no existential import to *E* propositions. *No* can certainly replace *Every* in (1) or (2) without introducing such import, and there is indeed some plausibility in the claim that the form *No a is F because there is no a* is unrestrictedly self-consistent. Keynes cites such examples as:

(4) No unicorns have ever been seen

(5) No satisfactory solution of the problem of squaring the circle has ever been published

which are true simply because there are no unicorns and because there is no way of squaring the circle. Once again, in those cases where existential import might seem to be carried (*No mammal is invertebrate*?), it may be possible to explain the implication as a non-logical conversational implicature. To say that no *a F's* when one knew that there was no *a* would be to violate the pragmatic principle not to say something weak when one is in a position to say something stronger. It may, then, be the case that existential import is carried by universal negative propositions only in cases like:

(6) None of the planes became operational

Finally, in the case of *O* propositions, it seems that, if *Some a F's* has existential import, then so does *Some a does not F*. This is why, when we wanted to interpret *O* in accordance with the practice of medieval logicians like Buridan, we used the form 'Not every ...'. But it is not at all certain that this reading secures the intended result either, as can be seen by adding *Not* to the beginning of (1) or (2). Doesn't *Not everyone who passes this examination is lucky* logically imply that there will be some successful though luckless candidate? Certainly *if* an *A* sentence like *Every whale is a mammal* has existential import it is hard to deny it to *Not every whale is a mammal*.

For compare a case where the special nature of the subject term gives rise to an undeniable existential entailment:

(7) Every ship they built had flaws
(8) Not every ship they built had flaws

If we propose to interpret BS in accordance with Interpretation I, then, the English sentence forms we have used in previous chapters are not wholly suitable. A sentence may need to be supplemented with an existential conjunct, or $O$ sentences with a disjunct cancelling existential import (*Not every a is a b or there is no a*). And if, as seems reasonable, the unmodified $A$, $E$, $I$ and $O$ sentences are to be analysed in the Frege/Russell manner, we emerge with an interpretation under which neither BS nor BS$^+$ is sound. (The syllogistic which fits this interpretation was given at the end of the last chapter.)

Anyone interested in the question of the existential import of general categorical propositions will have found the treatment here sketchy, over-simple and deliberately inconclusive. I believe that the issue is of too little importance to have warranted all the attention logicians have given it, and I have therefore not attempted to add significantly to their discussions. But a lucid and thorough treatment will be found in Keynes (1906), an elegant historical review by Alonzo Church (1965), and further summaries and references are given (regrettably without their virtues of style and organization) by J.S. Wu (1969).

## CHAPTER 7

## SUBJECT AND PREDICATE

In the last chapter we generalized the interpretation of *Aab* to include not only propositions of the form 'Every — is a —', but all those of the form 'Every — + predicate', where the predicate might consist of 'is' + adjective or verb + object expression; and the interpretations of *E*, *I* and *O* froms were extended in the same way. The artificial restriction to substantival subject and predicate terms in the preceding chapters did not reflect the intentions of Aristotle or his successors, nor was it meant to be offered as a philosophically satisfactory interpretation. It was adopted to enable the discussion of the formal logic to proceed as smoothly as possible while still articulating propositions in the traditional manner. It is the purpose of this chapter to endorse the Fregean treatment of general propositions which superseded the tradional analysis, an issue which we left open in order not to prejudge the present discussion.

The 'subject' and 'predicate' terms, *a* and *b* in *Aab* etc., are treated in syllogistic logic on a par. We shall consider construing them first as names, next as general names and predicates respectively, then both as predicative terms and finally, in the following chapters, as class terms. It is the first that is suggested by the traditional logicians' propensity for distorting such propositions as *Every king is wise* and *Some Scot wrote a novel* into *Every king is a wise man* and *Some Scot is a novelist*. However, if subject and predicate terms have different logical roles, if, as Geach argues, such sentences are to be construed as having the form name + predicate, then syllogistic logic must be seen as a hopelessly inappropriate logic for such sentences, not merely a very restricted one.

## 7.1 *The two-name theory and Leśniewski's Ontology*

The essential features of a two-name account can be brought out by considering the translation of syllogistic into the interpreted formal system which its author Leśniewski calls 'Ontology'. This system has a primitive functor, $\epsilon$, which takes two names as argument-expressions. The category of names is a generously comprehensive one, including not only ordinary proper names, but also adjectives like *white* and *wise*, and simple and complex substantival terms like *the King* and *a man who writes a novel*. There is a name-forming functor 'trm< >' to form names from verbs:

$$(a)(\varphi)(a \ \epsilon \ \mathrm{trm}\langle\varphi\rangle \longleftrightarrow (a \ \epsilon \ a \ \& \ \varphi(a)))$$

$\varphi$ is a proposition-forming functor taking one name as argument-expression. Three types of name are distinguished: (i) unshared names, which each designate a single object, (ii) shared names, each designating more than one object, and (iii) fictitious names (*round square*, *unicorn*, *Vulcan*), designating no object at all.

Even *nothing* turns out to be a name, introduced by the contextual definition:

$$(a)(a \ \epsilon \ \Lambda \longleftrightarrow (a \ \epsilon \ a \ \& \ a \not\epsilon \ a))$$

Together with the axiom to be given below, this yields $(a)\Lambda \ \epsilon \ a$. One may already get the impression that the notion of a name has been stretched beyond breaking point, an impression which is confirmed when we see that *Nothing is a unicorn* ($\Lambda \ \epsilon$ *unicorn*) must turn out false. It would no doubt be possible, however, to develop a two-name theory without this incongruous feature.

Leśniewski's system of Ontology is subjoined to his Protothetic, a generalized propositional calculus. Universal and existential quantifiers may bind both names and proposition-forming functors.

'$a \in b$' is to be interpreted as giving the form of a proposition which is true iff the substituend of $a$ is an unshared name whose bearer is designated by the name which substitutes for $b$. $b$ may stand in place of an unshared or a shared name. 'The Monarch is the daughter of George VI' contains two unshared names and is true because the bearer of each name is one and the same person. 'The Monarch is a woman' contains an unshared and a shared name and is true because the bearer of the unshared name 'the Monarch' is among those objects designated by the shared name 'a woman'. Hence Leśniewski's original (1920) axiom for Ontology, which is the following rather lengthy formula:

$$(a)(b)[a \in b \leftrightarrow (\exists c\ c \in a \ \&\ (c)(c \in a \rightarrow c \in b)\ \&$$
$$(c)(d)(c \in a \ \&\ d \in a \rightarrow c \in d)]$$

That is, '$a \in b$' is true iff (1) there is an $a$, and (2) every $a$ is $b$, and (3) there is at most one $a$.

Suppose we now take '$Aab$' as an abbreviation for

$$(c)(c \in a \rightarrow c \in b)\ \&\ \exists c\ c \in a$$

'$Iab$' as an abbreviation for

$$\exists c(c \in a \ \&\ c \in b)$$

and '$Oab$', '$Eab$' as abbreviating their respective negations. It is then a straightforward matter to verify that the first syllogistic system of Chapter I, BS, becomes a fragment of Ontology.

The subject and predicate terms of categorical propositions now become predicate-names to the right of the copula. Despite their predicate position they are nevertheless construed as names, shared, unshared or fictitious. Universal affirmative propositions are true iff every object designated by the first name is also designated by the second. Similarly $I$ propositions are true iff some object is designated by both of them. The truth conditions for $E$ and $O$ propositions can be stated analogously.

To construe both subject and predicate terms of a categorical proposition as names in this way is entirely unsatisfactory. In the first place, as Geach has shown, it is a mistake to treat complex terms in general as unitary referring expressions. So at best we are limited to simple names. But even with such a limitation, it is counter-intuitive to treat the predicate terms as names, and there are well-known Fregean reasons for not doing so. This leaves us with simple singular subject terms in universal categoricals as the only remaining candidates for namehood, and undermines the interpretation given by Ontology. These objections will be developed first, and in the next section we shall consider the view that simple subject terms are names.

To admit complex terms as substituends of name variables involves treating them as logical units: thus *Every man who writes a novel earns money* is construed as *Every man-who-writes-a-novel is a man-who-earns-money*. At first sight this may seem merely unidiomatic English, replaceable in any case by *Every novelist is a money-earner*, in which the first term is no longer complex. But Geach has shown that phrases like *man-who-writes-a-novel* have no more logical unity than the italicized phrase in

The philosopher whose most famous pupil was *Plato was tall*

Try, for example, replacing 'man who writes a novel' by the single word 'novelist' in *Every man who writes a novel is paid for it*: this results in the incomplete proposition *Every novelist is paid for it*. Geach's full argument is given on pp. 116 - 118 of his (1962), where he shows that there is no plausible way of construing such complex terms to give them true logical unity.

Even simple names appear ill-suited to combine with a copula to form a predicate. If we regard the expression *a man* in *Some prime minister is not a man* as a name, it would seem natural to ask 'Which man? Which man is it that some prime minister is not?' But this would be to forget that, on Leśniewski's view, *a man* is a shared name, and so there is no implicit claim that such a question should be appropriate. Nevertheless it is perhaps a consideration which casts doubt on the very idea of a shared name.

The decisive reasons for rejecting the articulation of the predicate into copula + name were provided by Frege, whose treatment of (simple) predicates as single units and replacement of the subject-predicate analysis of general categoricals by articulation in terms of function and argument(s) was after all the key to the breakthrough in formal logic which he achieved a century ago. Consider the proposition *Marilyn loved Marilyn*. The formal structure this proposition shares with *Arthur loved Marilyn* could, it may seem, be expressed in terms of the analysis: subject + copula + predicate name (*a lover of Marilyn*). Even the structure it shares with *Marilyn loved Yves* could be exhibited by means of the sentence-frame ' — was a-person-loved-by-Marilyn'. But the form it shares with *Arthur loved Arthur* defies such an analysis. The relation expressed in both *Marilyn loved Marilyn* and *Arthur loved Arthur*, in virtue of which what is said of Marilyn in the first is the same as what is said of Arthur in the second, is not adequately expressed by the word 'loved' alone, or by any isolable part of the sentence. Frege can treat both sentences as supplying arguments for the function '$\zeta$ loved $\zeta$'. This analysis has no place for the copula and no *unique* subject term. The form shared by *Marilyn loved Marilyn* and *Marilyn loved Yves* can be expressed by '(Marilyn-loved) $\zeta$'; and that shared by the first sentence and *Arthur loved Marilyn* by '$\zeta$ (loved-Marilyn)', where the argument-place is marked by $\zeta$

and the function-expression occurs in brackets. The Fregean analysis, unlike the analysis we are criticizing, is adequate, then, for all three cases. 'In this', said Frege, 'I faithfully follow the example of the formula language of mathematics, a language to which one would do violence if he were to distinguish between subject and predicate in it' ((1879), p. 12. Cf. also Geach (1975), pp. 140ff.).

Consequently, the two-name theory is better abandoned even for general categoricals whose terms are both simple.

7.2 *General names as logical subjects*

If the subject term of an *A*, *E*, *I* or *O* proposition is simple, and determines its reference as a whole without contribution from its components, Geach regards it as a name. Thus in the proposition *Some river caused widespread flooding* 'river' is a name which combines with the predicate 'Some — caused widespread flooding'. On this view it functions as a common name, and is said to refer not to any individual river but to all rivers. The sense it which it refers to all rivers is elucidated by its truth conditions: it is true iff the Thames caused widespread flooding or the Seine did or the Danube did or ... and so on. That is, it is true iff there is some true disjunctive proposition containing as disjuncts all the propositions of the form '$x$ caused widespread flooding', where $x$ is replaced by the name of some individual river. (Presumably we must suppose that every river has a name.)

In place of the Fregean analysis of the proposition,

Something is a river and it caused widespread flooding

Geach would construe it in the following manner:

As regards some river, it caused widespread flooding

since he would deny that being the same river is equivalent to being the same thing and being a river. On his view 'same' should be treated like 'one': and being one river is certainly not equivalent to being one thing and being a river.

In (1962), pp. 149-151, he offers the following supporting argument. Consider the propositions:

(1) Heraclitus bathed in some river yesterday and bathed in the same river today

(2) Whatever is a river is water

(3) Heraclitus bathed in some water yesterday and bathed in the same water today

It seems clear that (3) is not entailed by (1) and (2); if (1) and (2) were true, (3) would almost certainly be false.

But now consider the apparent translations of (1) - (3) into the symbolism of ordinary predicate logic:

(1*) $\exists x$($x$ is a river & Heraclitus bathed in $x$ yesterday & Heraclitus bathed in $x$ today)

(2*) $(x)$($x$ is a river $\rightarrow x$ is water)

(3*) $\exists x$($x$ is water & Heraclitus bathed in $x$ yesterday & Heraclitus bathed in $x$ today)

(3*) seemingly follows from (1*) and (2*) by unimpeachable principles of logic. Geach argues that at least one of these must therefore be rejected as an inadequate translation, and chooses to reject both (1*) and (3*) on the grounds that they treat being the same river as equivalent to being the same (something or other) and being a river, and being the same water as equivalent to being the same and being water.

However, it is clear that (3*) will follow from (1*) and (2*) only if the predicate 'is water' is univocal. (3*) translates (3) only if its predicate 'is water' means '= water' ('= some mass of water'?), but as Wiggins (1968) and others have pointed out it is quite implausible to regard that as its meaning in a proposition like (2). In (2) it surely means 'is composed/made up of water'.

Even if it remains unobjectionable to treat a word like 'river' in a proposition like *Some river caused widespread flooding* as a common name, we may conclude, then, that it is not compulsory. The Fregean treatment is not undermined by Geach's argument.

Moreover, if we were to treat simple singular terms in grammatical subject position in *A*, *E*, *I* and *O* propositions as common names, those propositions would be construed differently from their counterparts with plural subject terms. *Some rivers caused widespread flooding*, *All rivers caused widespread flooding* etc. would still be analysed in the Fregean manner.[1]

## 7.3 *The two-predicate analysis*

Frege's treatment of traditional categorical propositions therefore emerges as the best of those considered in this chapter. Interpreting Aristotelian wffs on these lines will mean that their variables are regarded as *predicate* variables and that BS and EBS are construed as fragments of monadic predicate logic. (For details see Chapter 5 above.) However, *A*, *E*, *I* and *O* propositions whose terms are complex present a problem. Their predicate logic paraphrases will dissipate many of the complex terms, whereas Aristotelian wffs give their forms as if such terms were logical units. Since syllogistic logic cannot exploit the finer structure of such propositions, it will

[1] Strawson (1974, pp. 66–72) has considered (and prefers to reject) the treatment of both singular and plural subject terms as general names.

never actually lead us astray - we shall not be able to use it to validate unsound inferences, at least not as a result of the way it articulates propositions. But, if he uses this two-predicate interpretation, the purist may well prefer to confine the substituends of variables in Aristotelian wffs to simple predicate expressions with a genuine logical unity.

CHAPTER 8

CLASSES

## 8.1 *Class interpretations*

In our treatment of the metatheory of syllogistic so far there is one significant respect in which the interpretations given differ from those in more formal modern treatments. For example, a wff $A\alpha\beta$ has been interpreted as *Every* $[\alpha]$ *is a* $[\beta]$ ('Every class containing the letter $\alpha$ is a class containing the letter $\beta$' - see p. 43 above), whereas the standard procedure would be to define $A\alpha\beta$ as true iff the class of $[\alpha]$'s $\subseteq$ the class of $[\beta]$'s. (The terms *class* and *set* are used interchangeably throughout this monograph.) The standard extensional interpretation is quite clearly adequate - though it was not adopted originally, since I wished to stress the affinity with Aristotle's own method: it is evident that *Every* $[\alpha]$ *is a* $[\beta]$ will be true provided, and only provided, that the class of sets containing $\alpha$ is included in the class of sets containing $\beta$. For the purpose of basic formal logic (unless studied in a purely formalistic way) it is the truth-value of propositions that matters, not their sense; or, rather, their sense matters only in so far as it determines truth-value, for validity is in effect defined in terms of truth-value. As Frege put it in an important article unpublished during his lifetime:

> it is of no concern to logic how thoughts follow from thoughts without reference to truth-value,...the step from thought to truth-value, from sense to reference, must invariably be made;...the laws of logic are primarily laws in the realm of reference [truth-value][1] and relate only indirectly to sense. ('Ausführungen über Sinn und Bedeutung', in Frege (1969), p. 128ff. My translation.)

[1] It will be recalled that Frege held that, if all its components had reference, a proposition referred to a truth-value.

The considerations just mentioned naturally suggest that basic syllogistic systems may be regarded as systems for the inclusion and intersection of classes. The completeness proof for $BS^+$ can easily be modified to yield the result that it is complete with respect to the inclusion and intersection of *non-empty* classes (see 5.2); and the completeness proof sketched for the Brentano system in 5.6 can be adapted in the same way to show that the system is complete with respect to *unrestricted* (i.e. both empty and non-empty) classes.[2] This is simply because, for example,

Every whale is a mammal

and

The class of whales $\subseteq$ the class of mammals

are logically equivalent, if the former is taken as lacking existential import. They necessarily have the same truth-value, whether or not the class of whales or mammals is empty. Again, *Some whale is a mammal* is logically equivalent to *The class of whales intersects with the class of mammals.* (The symbol 'o' will be used for *intersects.*) No doubt it is equivalences like these which have lead so many textbook writers (not to mention George Boole) to treat the subject and predicate terms of traditional propositions as if they were the names of classes, and to use schemas like 'All S is P' whose substitution instances are so plainly ungrammatical ('All whales is (*sic*) a mammal', etc.) If *whales* or *all whales* referred to a class, one could substitute the class description for the subject term in *Every whale is a mammal* without affecting its truth-value. But *The class of whales is a mammal* is certainly not true, even if it is meaningful. Still less is every whale the class of mammals. Parallel points can be made with *E*, *I* and *O* propositions.[3]

[2] $BS^+$ is clearly unsound for this latter interpretation. For example, *Iaa* is a theorem of $BS^+$, but the intersection of the empty set with itself has no members.

[3] Even as able a philosopher as Hilary Putnam is guilty of confusing terms with class names, though he doesn't intrude the solecistic 'is'. (1972, p. 9.)

Strictly speaking, perhaps we should not treat expressions like 'the class of mammals' as *names*, since they are syntactically complex and the remarks about complex terms in Chapter 7 apply to them. So let us suppose that the substituends of $a$ and $b$ in $Aab$, for example are simple names of the classes described by such complex expressions as 'the class of mammals'. (See Geach (1962), p. 121.)

It will be recalled that when $BS^+$ is subjoined to the propositional calculus we obtain a system equivalent to Łukasiewicz's. Considered as a logic of classes this system appears a good deal less *ad hoc* than under Interpretation II or its author's, which both give it the character of a grafting of syllogistic, nearly intact, on to propositional logic.

When syllogistic wffs are given a class interpretion, they may be re-expressed in class symbolism as follows:

$$A\alpha\beta = \alpha \subseteq \beta \qquad E\alpha\beta = -\alpha \circ \beta$$

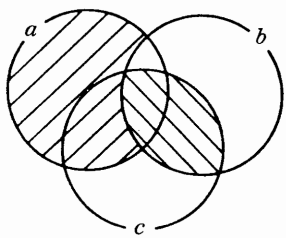
$$I\alpha\beta = \alpha \circ \beta \qquad O\alpha\beta = \alpha \nsubseteq \beta$$

'$\subseteq$' expresses class inclusion ($\alpha$ being included in $\beta$ iff there is no member of $\alpha$ which does not belong to $\beta$), 'o' class intersection ($\alpha \circ \beta$ iff $\alpha$, $\beta$, have some member in common) and '$\alpha \nsubseteq \beta$' abbreviates '$-\alpha \subseteq \beta$'. Thus '$Aab$' is now to be interpreted as '$a$ is included in $b$', where '$a$', '$b$' are variables whose substituends are simple names of classes.

## 8.2 *The use of diagrams*

Most logic textbooks since the late nineteen century which contain any treatment of the syllogism introduce Venn diagrams as a means of testing for validity. This tends to reinforce the inclination to treat subject and predicate terms as denotations of classes, though in view of the equivalences indicated above their use is quite justified. However it is less misleading to confine their use to testing inferences whose constituent propositions are explicitly about classes.

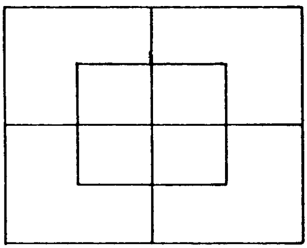
Inferences concerning the relations of inclusion and intersection between (one), two or three unrestricted sets can be demonstrated by means of Venn diagrams in which each set is represented by one of up to three interlocking circles. For example the inference $a \subseteq b$, $-b \circ c$ / $-a \circ c$ (the class of analogue of Celarent) is shown to be valid by the following Venn diagram, in which $a \subseteq b$ is represented by shading out that area of the $a$ circle which lies outside the $b$ circle and $-b \circ c$ is represented by shading out the area common to the $b$ and $c$ circles. In consequence the whole of the area common to the $a$ and $c$ circles has been shaded out, meaning that $-a \circ c$.

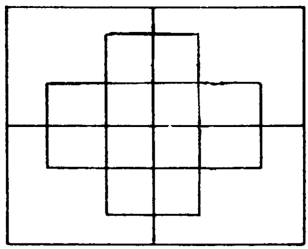



(Where two sets intersect, a cross is put in the sub-area representing their intersection.)

The completeness proof shows that an inference of the circumscribed type about the inclusion and intersection of unrestricted classes is valid iff it is valid in a domain of $2^n$ individuals, where $n$ = the number of class variables in the wffs of the inference schema. Thus a Venn diagram with three interlocking circles provides a topological model for the case of three classes. (Each of the eight sub-areas, which include the area outside all three circles, can be taken as representing a unit class.) Consequently, it seems to me wrong to deny, as Mendelson does in his (1970), that Venn diagrams are 'tools of rigorous mathematical proof'. Their weakness lies in their limitation to simple cases, i.e. to cases involving only a small number of sets, though methods of generalising them may be found in the literature.

Although cases involving four classes may be dealt with by means of a diagram with three circles and an ellipse interlocking appropriately, or with four interlocking ellipses (see article 'Logic Diagrams' in Edwards (1967), Vol. 5), this proves an unduly complex way of testing the validity of simple inferences. Lewis Carroll's diagrams in which rectangles overlap to delineate the required sub-areas turn out to be notably simpler. The frameworks for cases involving three and four classes respectively are:





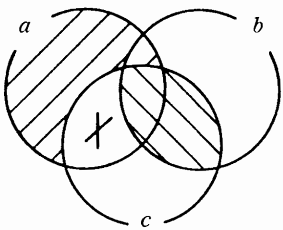
(Carroll (1893), p. 39; Geach (1976), pp. 56-60.)

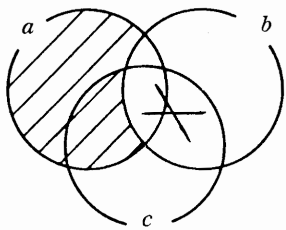
Venn and Carroll diagrams are interpreted in such a way that a blank sub-area represents a class which may or may not be empty. For testing inferences concerning non-empty classes, a cross may be inserted at will in the unshaded part of a circle, though care must be taken to ensure that it covers every blank sub-area of that circle.[4] Of the following two inferences the first is valid, the second invalid, and all classes are to be regarded as Aristotelian (non-empty):

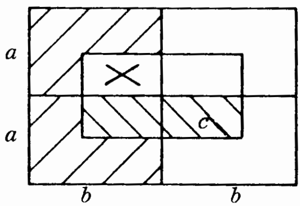
$$\text{(i)} \quad -b \circ c,\ a \subseteq c;\ \text{therefore } a \subseteq b;$$

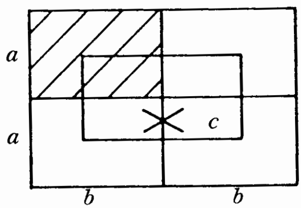
$$\text{(ii)} \quad a \subseteq b,\ b \circ c;\ \text{therefore } a \circ c.$$

Here are Venn and Carroll diagrams for each one. Note that the cross in the diagrams for (ii) means that the area it covers represents a non-empty class; it does not mean that each constituent sub-area necessarily represents a non-empty class.




[4]Another well-known type of diagram, due to the eighteenth century Swiss mathematician Leonhardt Euler, is appropriate for non-empty classes, since each interlocking circle represents a class with membership. These diagrams are useful for representing the five possible relations of inclusion and intersection between two sets, but the validation of inferences involving three classes can entail many different diagrams, and with more than three classes diagramming becomes impracticably complex.

The soundness of $BS^+$ for the interpretation in terms of non-empty classes, and of the Brentano system for unrestricted classes, can be shown by using Venn (or Carroll) diagrams within the framework of a proof by induction on the length of the longest branch of a tree deduction, provided that it is supplemented with an argument to cover the case of r.a.a. The use of Venn diagrams to show the validity of rules like s.c. and Celarent is a very simple matter and may be carried out quite easily by the reader.

# CHAPTER 9

## SYLLOGISTIC WITH NEGATIVE VARIABLES

### 9.1 *Basic syllogistic with negative variables*

If we enlarge syllogistic to include negated variables, $a'$, $b'$, $c'$ etc., then, if *Every man is intelligent* is a substitution instance of $Aab$, the corresponding instance of $Aa'b'$ will be *Every non-man is non-intelligent*. Syllogistic systems with negative variables may be obtained by adding the following definitions to BS and BS$^+$:

$$E\alpha\beta = A\alpha\beta' \qquad O\alpha\beta = I\alpha\beta'$$

together with a rule permitting the substitution of $\alpha$ for $\alpha''$ and vice versa.

Aristotle debarred himself from such a comprehensive treatment of propositions with negative terms by rejecting the inference from $Eab$ to $Aab'$ (one of the forms now known as 'obversion'), although he admitted the converse. It looks as if he confused negative terms like *non-intelligent* with terms like *unintelligent*, for the inference from *No number is intelligent* to *Every number is unintelligent* is certainly invalid, while the validity of its converse is at least defensible. (Cf. *An. Pr*. $51^b8$.) Systematic treatment of the logical relations which justify immediate inferences like obversion and contraposition described below was initiated in the fifth century by Boethius and perfected - by de Morgan - only in the nineteenth. (See Prior (1962), pp. 126-131.)

The objections to the treatment of words like *man* and *intelligent* as terms which may occupy both subject and predicate position indifferently apply no less, of course, to negative terms. The feeling of uneasiness induced by such artificial expressions as *non-man* and *non-intelligent* may be relieved, however, by construals which avoid them. $A\alpha'\beta'$ may be read as *Only* $\alpha$*s are* $\beta$*s*, and $A\alpha\beta'$ and

$I\alpha\beta'$ in the same way as their defined equivalents $E\alpha\beta$ and $O\alpha\beta$, *No* $\alpha$ *is* $\beta$ and *Some* $\alpha$ *is not* $\beta$. $A\alpha'\beta$, $I\alpha'\beta$ and $I\alpha'\beta'$ may be rendered, a little more awkwardly, as *Only* $\alpha$*s are not* $\beta$*s*, *Not only* $\alpha$*s are* $\beta$*s* and *Not only* $\alpha$*s are not* $\beta$*s* (for example).

We cannot extend Interpretation I (of BS) to BS with negatives (BSn), since, as the reader may easily verify, simple conversion of $I\alpha\beta'$ (= $O\alpha\beta$) is unsound, and the principle of Barbara fails for the case where the variable common to both premisses is negative. A sound interpretation of $BS^{+}n$ can be obtained if we take the variables to range over non-empty non-universal terms (see Keynes (1906), Part II, Ch. IV), but we have already had occasion to object to this type of interpretation (Interpretation II, 5.3 above).

A natural and unobjectionable interpretation of $BS^{+}n$ can, however, be given in terms of classes, namely the extension of the interpretation of $BS^{+}$ given in the last chapter. $BS^{+}n$ is both sound and complete when interpreted as a theory of inclusion, intersection and complementation of 'Aristotelian' classes, that is, classes which are neither empty nor universal. (Cf. Keynes (1906), loc. cit. Obviously, we cannot exclude empty classes without also excluding universal, since the empty class is the complement of the universal.)

This interpretation should be distinguished from the analogue of Strawson's for classes. Under the latter, for example, the 'predicate' variable of an $E$ formula need not range over restricted classes. Otto Bird (1964) confuses the two on p. 76, and compounds this by also confusing the interpretation we are now considering with the following (of the syllogistic part of $BS^{+}n$ subjoined to propositional logic - p. 77):

$$A\,\alpha\beta \ldots \alpha, \beta, \alpha', \beta' \neq 0 \rightarrow \alpha\beta' = 0;$$

$$E\,\alpha\beta \ldots \alpha, \beta, \alpha', \beta' \neq 0 \rightarrow \alpha\beta = 0;$$

$$I\,\alpha\beta \ldots \alpha, \beta, \alpha', \beta' \neq 0 \rightarrow \alpha\beta \neq 0;$$

$$O\alpha\beta \ldots \alpha, \beta, \alpha', \beta' \neq 0 \rightarrow \alpha\beta' \neq 0.$$

This makes any proposition in which either class expression denotes the empty class vacuously true, as well as making r.a.a. invalid. The first of these consequences can be avoided by changing $\rightarrow$ to &, but r.a.a. will still fail, and id.$^+$ (though not the weaker id.) will also become invalid.

We now set out the system BS$^+$n. With the expansion of the language of BS$^+$ the formation rule is to be modified in an obvious way. The definition of '' and $E$ make Celarent a special case of Barbara: $A\,\alpha\beta,\ A\beta\gamma' \vdash A\,\alpha\gamma'$, so that Celarent may be dropped from the list of primitive rules. The prime, ', is to be interpreted as indicating the complement of a class, $A$ will signify class inclusion and $I$ intersection.

*BS$^+$n*

*Language*. As for BS$^+$, but with the addition of the prime.

*Formation rule*. Call a variable letter followed by finitely many primes, e.g. $a'$, $a''$, $b'''''$, a *negative* variable letter.[1] Then a well-formed formula consists of a constant followed by two variable letters, either one being positive or negative.

[1] For present purposes, then, a negative variable not definitionally equivalent to a negative variable with a single prime will be equivalent to a positive variable. (Thus $a''$ is a negative variable equivalent to the positive variable $a$).

*Rules of inference.*

($\alpha, \beta, \gamma$, are schematic positive or negative variables.)

s.c. $\dfrac{I\,\alpha\beta}{I\,\beta\alpha}$ sub. $\dfrac{A\,\alpha\beta}{I\,\alpha\beta}$ Barbara $\dfrac{A\,\alpha\beta \quad A\beta\,\gamma}{A\,\alpha\gamma}$ r.a.a. $\dfrac{\begin{array}{c}[\varphi] \\ \psi \quad \overline{\psi}\end{array}}{\overline{\varphi}}$

id.[+] $\dfrac{*}{A\,\alpha\alpha}$

(As before, $\varphi$, $\overline{\varphi}$ etc. are corresponding $A$, $O$ or $E$, $I$ wffs, but either member of the pair may be a defined equivalent.)

*Definitions, etc.*

$E\alpha\beta = A\alpha\beta'$ ; $O\alpha\beta = I\,\alpha\beta'$ . Replacement rule: $\alpha$ for $\alpha''$, $\alpha''$ for $\alpha$.

*Some derived principles.*

Celarent - see above.

Obversion.[2] (i) $\dfrac{A\,\alpha\beta}{E\,\alpha\beta'}$ (ii) $\dfrac{E\,\alpha\beta}{A\,\alpha\beta'}$ (iii) $\dfrac{I\,\alpha\beta}{O\alpha\beta'}$ (iv) $\dfrac{O\alpha\beta}{I\alpha\beta'}$

Derivations of the principles of obversion are obvious. In general, the obverse of a formula is formed by taking the contrary, if the formula is universal, or subcontrary, if it is particular, and negating its second variable.

Inversion. $\dfrac{A\,\alpha\beta}{I\beta'\alpha'}$ $\dfrac{E\alpha\beta}{O\beta'\alpha'}$

$(= O\beta'\alpha)$ $(= I\beta'\alpha)$

[2] Cf. *De. Int.* $20^a20$–3. **Aristotle rejects both (ii) and (iv). See above on his rejection of (ii).**

We give the derivation of the first one, as an example, using the derived rule of s.c.$(E)$:

$$\frac{\dfrac{\dfrac{\dfrac{A\alpha\beta}{E\alpha\beta'}\text{ obversion}}{E\beta'\alpha}\text{ s.c. }(E)}{A\beta'\alpha'}\text{ obversion}}{I\beta'\alpha'}\text{ sub.}$$

In general, the full inverse of a (universal) formula is formed by taking its contradictory, transposing the variables and adding a prime to the first variable. Thus $A\alpha\beta$ becomes $O\beta'\alpha$. Alternatively, reduce the quantity, and transpose and negate the variables (which yields the definitional equivalents of the inverses).

Contraposition.[3] $\dfrac{A\alpha\beta}{A\beta'\alpha'}$ $\dfrac{O\alpha\beta}{O\beta'\alpha'}$

Contraposition, valid for $A$ and $O$, consists in transposing and negating the variables. We give the derivation of the first case:

$$\frac{\dfrac{\dfrac{\dfrac{A\alpha\beta}{A\alpha\beta''}\text{ rep.''}}{E\alpha\beta'}\text{ df. }E}{E\beta'\alpha}\text{ s.c.}(E)}{A\beta'\alpha'}\text{ df. }E$$

Contraposition of $A$ may replace s.c.$(I)$ as a primitive rule. Greater economy in the postulates may be achieved by replacing s.c. and Barbara by the single rule Datisi.[4]

[3] Cf. Aristotle's *Topics*, $113^b$.

[4] Sub. could be replaced by the rule $\dfrac{*}{I\alpha\alpha}$. Cf. Shepherdson (1956).

The notion of a derivation is defined as for $BS^+$, except that mention must also be made of the definitions and replacement rule as two-way inference licences.

*Class interpretation of* $BS^+n$.

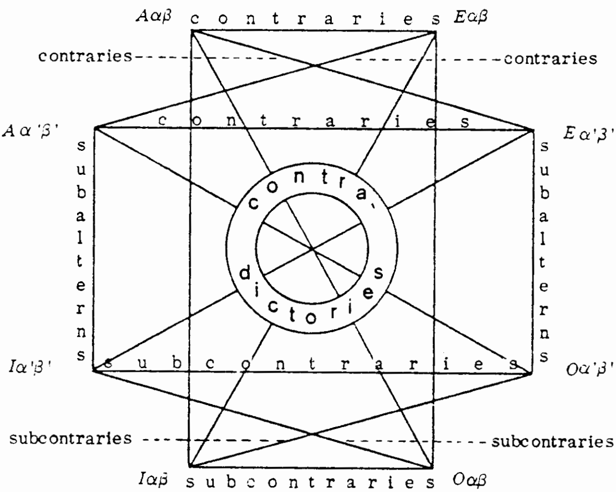
$\alpha'$ is the complement of $\alpha$.

$A\alpha\beta$ : $\alpha$ is included in $\beta$ ($\alpha \subseteq \beta$).

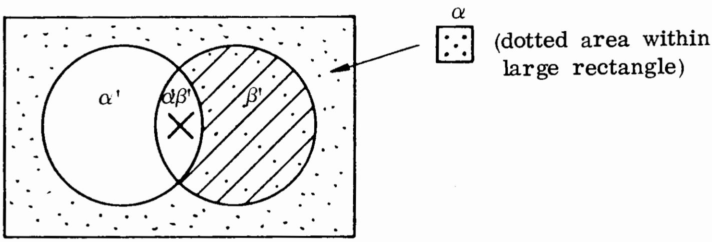
$I\alpha\beta$ : $\alpha$ intersects $\beta$ ($\alpha \text{ o } \beta$).

All classes are Aristotelian.

The following diagram (Cf. Keynes (1906), p. 144) summarises the principal relations of opposition between the formulas. Any variable, positive or negative, may be substituted for $\alpha$ or $\beta$, and any resulting double primes eliminated at will. A formula at one end of an upper horizontal or oblique line is contrary to the formula at the other end, and, similarly, a formula at one end of a lower horizontal or oblique line is subcontrary to the one at the other end. A formula at the top of a vertical line entails the one at the bottom. And, finally, a formula at one end of a diagonal is the contradictory of the formula at the other end.



Venn diagrams may be used to check simple inferences if a rectangle is added around the circles to represent the universal class. The diagram below demonstrates the inversion of $A\alpha\beta$ to $O\beta'\alpha$, the dotted area representing the class $\alpha$. Given that $\beta'$ has some member, that member does not belong to $\alpha$, so that $\beta'$ is not included in $\alpha$.




Venn diagram for inversion of $A\alpha\beta$ to $O\beta'\alpha$.

($\beta'$ is non-null.)

9.2 *Metatheory of* $BS^{+}n$

The metatheorems of this section relate to the interpretation just presented: the system is to be understood as a theory of complementation, inclusion and intersection of Aristotelian classes.

*Metatheorem 11.* (Soundness.) *If* $\Gamma \vdash_{BS^{+}n} \varphi$, *then* $\Gamma \Vdash \varphi$.

*Proof* in the style of the soundness proof for BS in 2.2. Alternatively, translate the symbolism of $BS^{+}n$ into that of the predicate calculus, so that $Aab'$, for example, becomes $(x)(Ax \rightarrow -Bx)$, $Iab$ becomes $\exists x(Ax \ \& \ Bx)$, and $Ax$ is understood as *x belongs to the class* $a$, $-Bx$ as $x$ *belongs to the class* $b'$ (= $x$ *does not belong to* $b$). The translated rules are then easily shown to be derived rules of the predicate calculus, provided that an existential premiss is supplied in the cases of sub. and id.$^{+}$. Since the monadic predicate calculus is known to be sound for the interpretation just indicated, and $BS^{+}_{n}$ can be translated into a fragment of it, the syllogistic system must be sound too.[5]

*Metatheorem* 12. (Completeness.) *If* $\Gamma \Vdash \varphi$, *then* $\Gamma \vdash_{BS^{+}_{n}} \varphi$. By the reasoning of 3.2, it suffices to show that

*every consistent set of wffs*, $\Delta$, *is* (*simultaneously*) *satisfiable*.

Let $\Delta$ be consistent, and let $\Delta^{-}$ be the set of wffs formed from $\Delta$ by replacing *E* and *O* wffs with their defined equivalents and eliminating all pairs of primes. Clearly $\Delta$ is consistent iff $\Delta^{-}$ is consistent, and satisfiable iff $\Delta^{-}$ is.

Let V be: the set of all variable letters in the wffs in $\Delta^{-}$.

W: the union of V with the set of all the complements of the letters in V. (For convenience we use the term *complement* to apply in an obvious way to letters as well as to classes. $\alpha$ is to count as the complement of $\alpha'$, and $\alpha^{c}$ will, for example, denote that complement of $\alpha$ which has no double primes.)[6]

Q(W): the set of all the subsets of W which contain as members just one variable from each of the pairs of complementary variables in W. For example, if $\Delta$ is $\{Eab', Ia'b''. Oba'\}$, $\Delta^{-}$ is $\{Aab, Ia'b, Iba\}$, $V = \{a, a', b\}$, $W = \{a, a', b, b'\}$ and $Q(W) = \{\{a, b\}, \{a, b'\}, \{a', b\}, \{a', b'\}\}$.

[5] Yet another alternative is to use Venn diagrams in the manner indicated in the last chapter.

[6] Hitherto we have used [ $\alpha\beta'$ ] to mean: *set containing* $\alpha$ *but not* $\beta$. Now that $\alpha'$ is being used for the complement of $\alpha$, we must read [ $\alpha\beta'$ ] as: *set containing both* $\alpha$ *and* $\beta'$. If $\beta$ is positive, of course, the two italicized phrases are extensionally equivalent, since, by the construction of Q(W), all and only those constituent sets lacking $\beta$ must contain $\beta'$.

Now $\Delta^{-}$ is satisfiable iff the (finite) maximal consistent set $\Delta^{-*}$ is satisfiable. This set is formed from $\Delta^{-}$, in a standard way as follows. List in some order all possible wffs, $\varphi_1, \ldots, \varphi_n$ with variable letters in W, and, starting with $\Delta^{-}$, take each $\varphi_i$ in turn and add it to the set provided the addition preserves consistency. This maximal set has the usual properties:

(i) just one member of each pair of contradictories, $\varphi_i$, $\bar{\varphi}_i$ belongs to the set;

(ii) for each $\varphi_i$, $\varphi_i \in \Delta^{-*}$ iff $\Delta^{-*} \vdash \varphi_i$.

(i) If $\Delta^{-*}$ contained both $\varphi_i$ and $\bar{\varphi}_i$, it would be inconsistent. If it contained neither, both $\Delta^{-*} \cup \{\varphi_i\}$ and $\Delta^{-*} \cup \{\bar{\varphi}_i\}$ would be inconsistent, so that, by r.a.a., both $\bar{\phi}_i$ and $\phi_i$ would be derivable from $\Delta^{-*}$, which would again be inconsistent.

(ii) If $\varphi_i \in \Delta^{-*}$, then (trivially) $\Delta^{-*} \vdash \varphi_i$.

If $\varphi_i \notin \Delta^{-*}$, then $\Delta^{-*} \cup \{\varphi_i\}$ is inconsistent, and $\bar{\varphi}_i$ is derivable from it. $\phi_i$ cannot then be derivable if $\Delta^{-*}$ is consistent.

To each variable $\alpha$ we now assign the corresponding class of $[\alpha]$'s in the model set, $U(\Delta^{-*})$, which is formed from Q(W) by deleting every $[\alpha]$ which lacks $\beta$ for every wff $A\alpha\beta \in \Delta^{-*}$. ($\alpha, \beta$, may be positive or negative. In particular $A\alpha\gamma^c$ deletes every $[\alpha\gamma]$, $A\alpha\alpha'$ deletes $[\alpha]$ and $A\alpha'\alpha$ every $[\alpha']$.)

Suppose a wff $I\alpha\beta \in \Delta^{-*}$. ($\beta \neq \alpha^c$. Note that $I\alpha\alpha^c$ is inconsistent.) We show that there is a class $[\alpha\beta] \in U(\Delta^{-*})$, by induction on the number, $n$, of complementary pairs in W.

*Basis.* $n = 1$. W contains just two variables, one the complement of the other. Then the $I$ wff has the form $I\alpha\alpha$. $\{\alpha\}$ belongs to the model set, since only $A\alpha\alpha^c$ could delete it, and the presence of such a wff in $\Delta^{-*}$ would make it inconsistent.

*Induction step*. If, when W has just $n$ complementary pairs, there is an $[\alpha\beta] \in U(\bar{\Delta}^*)$ when $I\alpha\beta \in \bar{\Delta}^*$, then this also holds true when W has $n$+1 complementary pairs.

So, by the induction hypothesis, the model contains some set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$, $1 \leq m \leq n$, where $\gamma_1 = \alpha = \beta$ or $\gamma_1 = \alpha$, $\gamma_2 = \beta$, which is not deleted by any $A$ wff in $\bar{\Delta}^*$. Consider the case where W has $n$+1 complementary pairs, so that, in addition to $\gamma_1, \ldots, \gamma_n$ and their complements, there are also the variables $\gamma_{n+1}$ and $\gamma^c_{n+1}$. It will not be possible to delate both the sets

$$\{\gamma_1, \gamma_2, \ldots, \gamma_m, \gamma_{n+1}\}$$
$$\{\gamma_1, \gamma_2, \ldots . \gamma_m, \gamma^c_{n+1}\}$$

For if it were, $\bar{\Delta}^*$ would contain wffs $A\gamma_i\gamma_{n+1}$ and $A\gamma_j\gamma^c_{n+1}$, for some $\gamma_i$, $\gamma_j$, $i$, $j$, $\leq m$. (The sets could also be deleted by their contrapositives, but, if their contrapositives belong to the maximal set, then so do they.) By contraposition and Barbara these two wffs yield $A\gamma_i\gamma^c_j$, which would therefore also belong to $\bar{\Delta}^*$. But this would delete the set $\{\gamma_1, \gamma_2, \ldots \gamma_m\}$ when W contains only $n$ complementary pairs, which contradicts the induction hypothesis.

Hence, if $I\alpha\beta \in \bar{\Delta}^*$, there is an $[\alpha\beta]$ in $U(\bar{\Delta}^*)$, and the classes of $[\alpha]$'s and $[\beta]$'s are obviously non-null and (by the construction of $U(\bar{\Delta}^*)$) non-universal.

Suppose next that a wff $A\alpha\beta \in \bar{\Delta}^*$ $(\alpha \neq \beta^c)$. Then so does $I\alpha\beta$ (by sub.) and the model set contains an $[\alpha\beta]$. The construction of $U(\bar{\Delta}^*)$ then guarantees its truth in the model.

Therefore both the particular and the universal wffs of $\bar{\Delta}^*$ are true in the model, and *a fortiori* so are all the wffs of its subset $\bar{\Delta}^*$.

*Decision procedures*

The decision procedures given in Chapter 3 (p. 47ff.) for BS may be adapted for $BS^+n$.[7] To determine whether a set of wffs, $\Delta$, is satisfiable (or consistent), we test the corresponding set, $\Delta^-$. Appropriate changes must be made in the clauses listing the types of unsatisfiable (inconsistent) sets.

For the use of Venn diagrams in cases with no more than three different variables, see the end of the last section.

## 9.3 *Unrestricted classes*

The restriction to Aristotelian classes can be removed by substituting the following rule for sub.:

$$\frac{I\alpha\beta}{I\alpha\alpha}$$

($A\alpha\alpha' \,/\, A\alpha\beta$ would do as well.) This gives us a system which is a sound and complete theory of the complementation, inclusion and intersection of unrestricted classes (classes which may be null or universal.)

*Metatheorem 13. Completeness of modified $BS^+n$.*

*Sketch of proof.* Construct $U(\Delta^{-*})$ as before. The deletions from Q(W) give a model set which verifies every $A$ wff, $A\alpha\beta$, in the maximal consistent set, since no class with $\alpha$ lacks $\beta$. If every $[\alpha]$ is deleted, the class of $[\alpha]$'s is the null set, and $A\alpha\beta$ is verified because the null set is included in every set.

$I\alpha\alpha'$ and $I\alpha'\alpha$ are still both inconsistent, of course. And, although $A\alpha\alpha'$ ceases to be inconsistent with $A\alpha'\alpha$, the former is not consistent with $I\alpha\alpha$ nor the latter with $I\alpha'\alpha'$. Once again it

[7] If you use the decision procedure based on the completeness proof, you need only construct the set $\Delta^-$. Indeed the proof of completeness can be formulated - rather less economically - without recourse to maximal sets.

can be shown that there will always be an $[\alpha\beta]$ in $U(\bar{\Delta}^*)$ to verify any $I\alpha\beta$ in the maximal set. End of sketch.

It should be noticed that the usual translation into the standard predicate calculus fails to preserve the soundness of the interpretation, since $U(\bar{\Delta}^*)$ may be empty. If $\bar{\Delta}^*$ contains two wffs of the form $A\alpha\alpha'$, $A\alpha'\alpha$, every set in Q(W) will be deleted and the model set will be empty. To obtain a system interpretable only in terms of a non-null domain, we may add to the last system the rule

$$\frac{A\,\alpha\alpha'}{I\,\alpha'\alpha'}$$

If $\alpha \subseteq \alpha', \alpha'$ o $\alpha'$: i.e. if $\alpha$ is null, $\alpha'$ is non-null.

The new rule makes $A\alpha\alpha'$ and $A\alpha'\alpha$ mutually inconsistent, so that the maximal set will contain just one of them. If $A\alpha\alpha' \notin \bar{\Delta}^*$, some $[\alpha']$ will be undeletable from Q(W); and if $A\alpha'\alpha \notin \bar{\Delta}^*$, some $[\alpha]$ will be undeletable. Either way the model set will be non-null.

## 9.4 *Complex 'terms' and Boolean algebra*

Complex 'term' variables may be formed by using the symbols $\cup$ and $\cap$ to form expressions for the union and intersection of classes. Single letters continue to be class variables, and if $\alpha$, $\beta$, are class wffs, so are $\alpha'$, $(\alpha \cup \beta)$ and $(\alpha \cap \beta)$.

To interpret these formulas as ranging only over Aristotelian classes would trivialize syllogistic. For, since the intersection of any two classes would have to be non-null, both $\alpha \cap \beta$ and $\alpha \cap \beta'$ would be non-null, and consequently $I\alpha\beta$ would always be true and $A\alpha\beta$ always false. Then any inference with a 'particular' conclusion or 'universal' premiss would be valid. Indeed, unless the formation of terms like $\alpha \cap \alpha'$ were expressly forbidden, any adequate system would have $I\alpha\alpha'$ ( $= O\alpha\alpha$) as a theorem and would accordingly be inconsistent in the sense of 2.2.

The only reasonable interpretation of such an extended syllogistic will therefore be in terms of unrestricted classes. The last system discussed in the previous section ($BS^+n$ modified with $I\alpha\beta$ / $I\alpha\alpha$ in place of sub., and with the extra rule $A\alpha\alpha'$ / $I\alpha'\alpha'$) may be extended to a Boolean algebra of classes by admitting complex terms and making the following additions (the object language being further enlarged to include $=$ and 0):

1. $\dfrac{A\alpha\beta \quad A\beta\alpha}{\alpha = \beta}$ $\qquad$ $\dfrac{\alpha = \beta}{A\alpha\beta}$ $\qquad$ $\dfrac{\alpha = \beta}{A\beta\alpha}$

2. Dfs. $A\alpha\beta =_{df.} \alpha \cap \beta = \alpha \quad I\alpha\beta =_{df.} \alpha \cap \beta \neq 0.$

3. $\left.\begin{array}{ll} \alpha \cap \alpha' = 0 & \alpha \cup 0 = \alpha \\ \alpha \cap \beta = \beta \cap \alpha & \alpha \cap (\beta \cap \gamma) = (\alpha \cap \beta) \cap \gamma \end{array}\right\}$ Axioms.

4. Dfs. $1 =_{df.} 0'$ $\qquad$ $\alpha \cup \beta =_{df.} (\alpha' \cap \beta')'.$

In their present context the rules under 1. give the essential logical properties of identity. The whole system, like any of those in this chapter, can be subjoined to the full propositional calculus. (Cf. Łukasiewicz's syllogistic, which seems far less *ad hoc* when the syllogistic component is interpreted in terms of Aristotelian classes.)

It is not claimed that the formulation of Boolean class logic given here is superior in elegance, still less in economy, to other versions. It is merely described in order to illustrate the now familiar fact that syllogistic, extended in the most plausible way to embrace complex terms, is under its class interpretation a mere fragment of Boolean class algebra. Nevertheless, this brings us to the historical threshold of contemporary logic, since it was Boole's development of the logic of classes and his treatment of it as an abstract system that were among the most significant steps culminating in the modern breakthrough to be achieved by Frege.

## PRINCIPAL SYLLOGISTIC SYSTEMS DISCUSSED IN THE TEXT

BS Rules: s.c., sub., Barbara, Celarent, r.a.a., id.
Sound and complete under Interpretation I (only affirmatives have existential import).
(See 7.1 for interpretation in terms of Ontology.)

$BS^+$ Rules: as for BS. except that $id.^+$ replaces id.
Sound and complete under Interpretation II (all 'term' variables non-empty).
Sound and complete interpreted as a system for the inclusion and intersection of Aristotelian classes.

Brentano/Frege/Russell system
Rules: s.c., $\frac{I\alpha\beta}{I\alpha\alpha}$, $\frac{O\alpha\beta}{O\alpha\alpha}$, Barbara, Celarent, r.a.a., $id.^+$.
Sound and complete for interpretation under which only particulars have existential import.
Sound and complete interpreted as a system for the inclusion and intersection of unrestricted classes.

EBS + PC
&, $-$, $A$, $I$ primitive; v, $\rightarrow$, $E$, $O$ defined.
Rules: &-introduction, &-elimination, DN-elimination,
RAA, s.c., sub., Barbara, Celarent, $id.^+$.
(Datisi may replace s.c. and Celarent.)

$EBS^+$ + PC (equivalent to Lukasiewicz's syllogistic)
Rules: as for EBS + PC, except that $id.^+$ or $*/I\alpha\alpha$ replaces id.

$BS^+n$ $A$, $I$ primitive. Primed variables introduced. $E$, $O$ defined.
Replacement rule: $\alpha''$ for $\alpha$, and vice versa.
Rules: s.c., sub., Barbara, r.a.a.
Sound and complete interpreted as a system for the inclusion, intersection and complementation of Aristotelian classes.

$BS^+n$ modified to produce a system sound and complete when interpreted in terms of the inclusion, intersection and complementation of unrestricted classes.
Rules: as for $BS^+n$, except that $\frac{I\alpha\beta}{I\alpha\alpha}$ replaces sub. (Cf. Brentano system).

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