EQUIVARIANT HOMOTOPY AND COHOMOLOGY THEORY

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Introduction

This volume began with Bob Piacenza's suggestion that I be the principal lecturer at an NSF/CBMS Regional Conference in Fairbanks, Alaska. That event took place in August of 1993, and the interim has seen very substantial progress in this general area of mathematics. The scope of this volume has grown accordingly.

The original focus was an introduction to equivariant algebraic topology, to stable homotopy theory, and to equivariant stable homotopy theory that was geared towards graduate students with a reasonably good understanding of nonequivariant algebraic topology. More recent material is changing the direction of the last two subjects by allowing the introduction of point-set topological algebra into stable homotopy theory, both equivariant and non-equivariant, and the last portion of the book focuses on an introduction to these new developments. There is a progression, with the later portions of the book on the whole being more difficult than the earlier portions.

Equivariant algebraic topology concerns the study of algebraic invariants of spaces with group actions. The first two chapters introduce the basic structural foundations of the subject: cellular theory, ordinary homology and cohomology theory, Eilenberg-Mac Lane G-spaces, Postnikov systems, localizations of G-spaces and completions of G-spaces. In most of this work, G can be any topological group, but we restrict attention to compact Lie groups in the rest of the book.

Chapter III, on equivariant rational homotopy theory, was written by Georgia Triantafillou. In it, she shows how to generalize Sullivan's theory of minimal models to obtain an algebraization of the homotopy category of (nilpotent) G-spaces for a finite group G. This chapter contains a first surprise: rational Hopf G-spaces need not split as products of Eilenberg-Mac Lane G-spaces. This is a hint that the calculational behavior of equivariant algebraic topology is more intricate and difficult to determine than that of the classical nonequivariant theory.

Chapter IV gives two proofs of the first main theorem of equivariant algebraic topology, which goes under the name of "Smith theory": any fixed point space of an action of a finite p-group on a mod p homology sphere is again a mod p homology sphere. One proof uses ordinary (or Bredon) equivariant cohomology and the other uses a general localization theorem in classical (or Borel) equivariant cohomology.

Parts of equivariant theory require a good deal of categorical bookkeeping, for example to keep track of fixed point data and to construct new G-spaces from diagrams of potential fixed point spaces. Some of the relevant background, such as geometric realization of simplicial spaces and the construction of homotopy colimits, is central to all of algebraic topology. These matters are dealt with in Chapter V, where Eilenberg-Mac Lane G-spaces and universal $\mathscr{F}$-spaces for families $\mathscr{F}$ of subgroups of a given group G are constructed. Special cases of such universal $\mathscr{F}$-spaces are used in Chapter VII to study the classification of equivariant bundles.

A different perspective on these matters is given in Chapter VI, which was written by Bob Piacenza. It deals with the general theory of diagrams of topological spaces, showing how to mimic classical homotopy and homology theory in categories of diagrams of topological spaces. In particular, Piacenza constructs a Quillen (closed) model category structure on any such category of diagrams and shows how these ideas lead to another way of passing from diagrams of fixed point spaces to their homotopical realization by G-spaces.

Chapter VIII combines equivariant ideas with the use of new tools in nonequivariant algebraic topology, notably Lannes' functor T in the context of unstable modules and algebras over the Steenrod algebra, to describe one of the most beautiful recent developments in algebraic topology, namely the Sullivan conjecture and its applications. While many mathematicians have contributed to this area, the main theorems are due to Haynes Miller, Gunnar Carlsson, and Jean Lannes. Although the set $[X, Y]$ of homotopy classes of based maps from a space X to a space Y is trivial to define, it is usually enormously difficult to compute. The Sullivan conjecture, in its simplest form, asserts that $[BG, X] = 0$ if G is a finite group and X is a finite CW complex. It admits substantial generalizations which lead to much more interesting calculations, for example of the set of maps [BG, BH] for suitable compact Lie groups G and H. We shall see that an understanding of equivariant classifying spaces sheds light on what these calculations are really saying. There is already a large literature in this area, and we can only give an introduction. One theme is that the Sullivan conjecture can be viewed conceptually as a calculational elaboration of Smith theory. A starting point of this approach lies in work of Bill Dwyer and Clarence Wilkerson, which first exploited the study of modules over the Steenrod algebra in the context of the localization theorem in Smith theory.

We begin the study of equivariant stable homotopy theory in Chapter IX, which gives a brief introduction of some of the main ideas. The chapter culminates with a quick conceptual proof of a conjecture of Conner: if G is a compact Lie group and X is a finite dimensional G-CW complex with finitely many orbit types such that $\tilde{H}(X;\mathbb{Z}) = 0$, then $\tilde{H}(X/G;\mathbb{Z}) = 0$. This concrete statement is a direct consequence of the seemingly esoteric assertion that ordinary equivariant cohomology with coefficients in a Mackey functor extends to a cohomology theory graded on the real representation ring $RO(G)$; this means that there are suspension isomorphisms with respect to the based spheres associated to all representations, not just trivial ones. In fact, the interplay between homotopy theory and representation theory pervades equivariant stable homotopy theory.

One manifestation of this appears in Chapter X, which was written by Stefan Waner. It explains a variant theory of G-CW complexes defined in terms of representations and uses the theory to construct the required ordinary $RO(G)$-graded cohomology theories with coefficients in Mackey functors by means of appropriate cellular cochain complexes.

Another manifestation appears in Chapter XI, which was written by Gaunce Lewis and which explains equivariant versions of the Hurewicz and Freudenthal suspension theorems. The algebraic transition from unstable to stable phenomena is gradual rather than all at once. Nonequivariantly, the homotopy groups of first loop spaces are already Abelian groups, as are stable homotopy groups. Equivariantly, stable homotopy groups are modules over the Burnside ring, but the homotopy groups of Vth loop spaces for a representation V are only modules over a partial Burnside ring determined by V. The precise form of Lewis's equivariant suspension theorem reflects this algebraic fact.

Serious work in both equivariant and nonequivariant stable homotopy theory requires a good category of "stable spaces", called spectra, in which to work.

There is a great deal of literature on this subject. The original construction of the nonequivariant stable homotopy category was due to Mike Boardman. One must make a sharp distinction between the stable homotopy category, which is fixed and unique up to equivalence, and any particular point-set level construction of it. In fact, there are quite a few constructions in the literature. However, only one of them is known to generalize to the equivariant context, and that is also the one that is the basis for the new development of point-set topological algebra in stable homotopy theory. We give an intuitive introduction to this category in Chapter XII, beginning nonequivariantly and focusing on the construction of smash products and function spectra since that is the main technical issue. We switch to the equivariant case to explain homotopy groups, the suspension isomorphism for representation spheres, and the theory of G-CW spectra. We also explain how to transform the spectra that occur "in nature" to the idealized spectra that are the objects of the stable homotopy category.

In Chapters XIII, XIV, and XV, we introduce the most important $RO(G)$-graded cohomology theories and describe the G-spectra that represent them. We begin with an axiomatic account of exactly what $RO(G)$-graded homology and cohomology theories are and a proof that all such theories are representable by G-spectra. We also discuss ring G-spectra and products in homology and cohomology theories. We show how to construct Eilenberg-Mac Lane G-spectra by representing the zeroth term of a $\mathbb{Z}$-graded cohomology theory defined by means of G-spectrum level cochains. This implies an alternative construction of ordinary $RO(G)$-graded cohomology theories with coefficients in Mackey functors.

Chapter XIV, which was written by John Greenlees, gives an introduction to equivariant K-theory. The focus is on equivariant Bott periodicity and its use to prove the Atiyah-Segal completion theorem. That theorem states that, for any compact Lie group G, the nonequivariant K-theory of the classifying space BG is isomorphic to the completion of the representation ring $R(G)$ at its augmentation ideal I. The result is of considerable importance in the applications of K-theory, and it is the prototype for a number of analogous results to be described later.

Chapter XV, which was written by Steve Costenoble, gives an introduction to equivariant cobordism. The essential new feature is that transversality fails in general, so that geometric equivariant bordism is not same as stable (or homotopical) bordism; the latter is the theory represented by the most natural equivariant generalization of the nonequivariant Thom spectrum. Costenoble also explains the use of adjacent families of subgroups to reduce the calculation of equivariant bordism to suitably related nonequivariant calculations. The equivariant results are considerably more intricate than the nonequivariant ones. While the G-spectra that represent unoriented geometric bordism and its stable analog split as products of Eilenberg Mac Lane G-spectra for finite groups of odd order, just as in the nonequivariant case, this is false for the cyclic group of order 2.

Chapters XVI–XIX describe the basic machinery and results on which all work in equivariant stable homotopy theory depends. Chapter XVI describes fixed point and orbit spectra, shows how to relate equivariant and nonequivariant homology and cohomology theories, and, more generally, shows how to relate homology and cohomology theories defined for a group G to homology and cohomology theories defined for subgroups and quotient groups of G. These results about change of groups are closely related to duality theory, and we give basic information about equivariant Spanier-Whitehead, Atiyah, and Poincaré duality.

In Chapter XVII, we discuss the Burnside ring $A(G)$. When G is finite, $A(G)$ is the Grothendieck ring associated to the semi-ring of finite G-sets. For any compact Lie group G, $A(G)$ is isomorphic to the zeroth equivariant stable homotopy group of spheres. It therefore acts on the equivariant homotopy groups $\pi_n^G(X) = \pi_n(X^G)$ of any G-spectrum X, and this implies that it acts on all homology and cohomology groups of any G-spectrum. Information about the algebraic structure of $A(G)$ leads to information about the entire stable homotopy category of G-spectra. It turns out that $A(G)$ has Krull dimension one and an easily analyzed prime ideal spectrum, making it quite a tractable ring. Algebraic analysis of localizations of $A(G)$ leads to analysis of localizations of equivariant homology and cohomology theories. For example, for a finite group G, the localization of any theory at a prime p can be calculated in terms of subquotient p-groups of G.

In Chapter XVIII, we construct transfer maps, which are basic calculational tools in equivariant and nonequivariant bundle theory, and describe their basic properties. Special cases were vital to the earlier discussion of change of groups. The deepest property is the double coset formula, and we say a little about its applications to the study of the cohomology of classifying spaces.

In Chapter XIX, we discuss several fundamental splitting theorems in equivariant stable homotopy theory. These describe the equivariant stable homotopy groups of G-spaces in terms of nonequivariant homotopy groups of fixed point spaces. These theorems lead to an analysis of the structure of the subcategory of the stable category whose objects are the suspension spectra of orbit spaces. A Mackey functor is an additive contravariant functor from this subcategory to Abelian groups, and, when G is finite, the analysis leads to a proof that this topological definition of Mackey functors is equivalent to an earlier and simpler algebraic definition. Mackey functors describe the algebraic structure that is present on the system of homotopy groups $\pi_n^H(X) = \pi_n(X^H)$ of a G-spectrum X, where H runs over the subgroups of G. The action of the Burnside ring on $\pi_n^G(X)$ is part of this structure. It is often more natural to study such systems than to focus on the individual groups. In particular, we describe algebraic induction theorems that often allow one to calculate the value of a Mackey functor on the orbit G/G from its values on the orbits G/H for certain subgroups H. Such theorems have applications in various branches of mathematics in which finite group actions appear. Again, algebraic analysis of rational Mackey functors shows that, when G is finite, rational G-spectra split as products of Eilenberg-Mac Lane G-spectra. This is false for general compact Lie groups G.

In Chapter XX, we turn to another of the most beautiful recent developments in algebraic topology: the Segal conjecture and its applications. The Segal conjecture can be viewed either as a stable analogue of the Sullivan conjecture or as the analogue in equivariant stable cohomotopy of the Atiyah-Segal completion theorem in equivariant K-theory. The original conjecture, which is just a fragment of the full result, asserts that, for a finite group G, the zeroth stable cohomotopy group of the classifying space BG is isomorphic to the completion of $A(G)$ at its augmentation ideal I. The key step in the proof of the Segal conjecture is due to Gunnar Carlsson. We explain the proof and also explain a number of generalizations of the result. One of these leads to a complete algebraic determination of the group of homotopy classes of stable maps between the classifying spaces of any two finite groups. This is analogous to the role of the Sullivan conjecture in the study of ordinary homotopy classes of maps between classifying spaces. Use of equivariant classifying spaces is much more essential here. In fact, the Segal conjecture is intrinsically a result about the I-adic completion of the sphere G-spectrum, and the application to maps between classifying spaces depends on a generalization in which the sphere G-spectrum is replaced by the suspension G-spectra of equivariant classifying spaces.

Chapter XXI is an exposition of joint work of John Greenlees and myself in which we generalize the classical Tate cohomology of finite groups and the periodic cyclic cohomology of the circle group to obtain a Tate cohomology theory associated to any given cohomology theory on G-spectra, for any compact Lie group G. This work has had a variety of applications, most strikingly to the computation of the topological cyclic homology and thus to the algebraic K-theory of number rings. While we shall not get into that application here, we shall describe the general Atiyah-Hirzebruch-Tate spectral sequences that are used in that work and we shall give a number of other applications and calculations. For example, we shall explain a complete calculation of the Tate theory associated to the equivariant K-theory of any finite group. This is an active area of research, and some of what we say at the end of this chapter is rather speculative. The Tate theory provides some of the most striking examples of equivariant phenomena illuminating nonequivariant phenomena, and it leads to interrelationships between the stable homotopy groups of spheres and the Tate cohomology of finite groups that have only begun to be explored.

Chapters XXII through XXV concern "brave new algebra", the study of point-set level topological algebra in stable homotopy theory. The desirability of such a theory was advertised by Waldhausen under the rubric of "brave new rings", hence the term "brave new algebra" for the new subject. Its starting point is the construction of a new category of spectra, the category of "S-modules", that has a smash product that is symmetric monoidal (associative, commutative, and unital up to coherent natural isomorphisms) on the point-set level. The construction is joint work of Tony Elmendorf, Igor Kriz, Mike Mandell, and myself, and it changes the nature of stable homotopy theory. Ever since its beginnings with Adams' use of stable homotopy theory to solve the Hopf invariant one problem some thirty-five years ago, most work in the field has been carried out working only "up to homotopy"; formally, this means that one is working in the stable homotopy category. For example, classically, the product on a ring spectrum is defined only up to homotopy and can be expected to be associative and commutative only up to homotopy. In the new theory, we have rings with well-defined point-set level products, and they can be expected to be strictly associative and commutative. In the associative case, we call these "S-algebras". The new theory permits constructions that have long been desired, but that have seemed to be out of reach technically: simple constructions of many of the most basic spectra in current use in algebraic topology; simple constructions of generalized universal coefficient, Künneth, and other spectral sequences; a conceptual and structured approach to Bousfield localizations of spectra, a generalized construction of topological Hochschild homology and of spectral sequences for its computation; a simultaneous generalization of the algebraic K-theory of rings and of spaces; etc. Working nonequivariantly, we shall describe the properties of the category of S-modules and shall sketch all but the last of the cited applications in Chapter XXII.

We return to the equivariant world in Chapter XXIII, which was written jointly with Elmendorf and Lewis, and sketch how the construction of the category of S_G-modules works. Here S_G denotes the sphere G-spectrum. The starting point of the construction is the "twisted half smash product", which is a spectrum level generalization of the half-smash product $X \ltimes Y = X_+ \wedge Y$ of an unbased G-space X and a based G-space Y and is perhaps the most basic construction in equivariant stable homotopy theory. Taking X to be a certain G-space $\mathscr{L}(j)$ of linear isometries, one obtains a fattened version $\mathscr{L}(j) \ltimes E_1 \wedge \cdots \wedge E_j$ of the j-fold smash product of G-spectra. Taking $j = 2$, insisting that the E_i have extra structure given by maps $\mathscr{L}(1) \ltimes E_i \longrightarrow E_i$, and quotienting out some of the fat, one obtains a commutative and associative smash product of G-spectra with actions by the monoid $\mathscr{L}(1)$; a little adjustment adds in the unit condition and gives the category of S_G-modules. The theory had its origins in the notion of an E_∞ ring spectrum introduced by Quinn, Ray, and myself over twenty years ago. Such rings were defined in terms of "operad actions" given by maps $\mathscr{L}(j) \ltimes E^j \longrightarrow E$, where E^j is the j-fold smash power of E, and it turns out that such rings are virtually the same as our new commutative S_G-algebras. The new theory makes the earlier notion much more algebraically tractable, while the older theory gives the basic examples to which the new theory can be applied.

In Chapter XXIV, which was written jointly with Greenlees, we give a series of algebraic definitions, together with their brave new algebra counterparts, and we show how these notions lead to a general approach to localization and completion theorems in equivariant stable homotopy theory. We shall see that Grothendieck's local cohomology groups are relevant to the study of localization theorems in equivariant homology and that analogs called local homology groups are relevant to the study of completion theorems in equivariant cohomology. We use these constructions to prove a general localization theorem for suitable commutative S_G-algebras R_G. Taking R to be the underlying S-algebra of R_G and taking M to be the underlying R-module of an R_G-module M_G, the theorem implies both a localization theorem for the computation of $M_*(BG)$ in terms of $M_*^G(pt)$

and a completion theorem for the computation of $M^*(BG)$ in terms of $M_G^*(pt)$. Of course, this is reminiscent of the Atiyah-Segal completion for equivariant K-theory and the Segal conjecture for equivariant cobordism. The general theorem does apply to K-theory, giving a very clean description of $K_*(BG)$, but it does not apply to cohomotopy: there the completion theorem for cohomology is true but the localization theorem for homology is false.

We are particularly interested in stable equivariant complex bordism, represented by MU_G, and modules over it. We explain in Chapter XXII how simple it is to construct all of the usual examples of MU-module spectra in the homotopical sense, such as Morava K-theory and Brown-Peterson spectra, as brave new point-set level MU-modules. We show in Chapter XXIII how to construct equivariant versions M_G as brave new MU_G-modules of all such MU-modules M, where G is any compact Lie group. We would like to apply the localization theorem of Chapter XXIV to MU_G and its module spectra, but its algebraic hypotheses are not satisfied. Nevertheless, as Greenlees and I explain in chapter XXV, the localization theorem is in fact true for MU_G when G is finite or a finite extension of a torus. The proof involves the construction of a multiplicative norm map in MU_*^G, together with a double coset formula for its computation. This depends on the fact that MU_G can be constructed in a particularly nice way, codified in the notion of a "global $\mathscr{I}_*$-functor with smash product", as a functor of G.

These results refocus attention on stable equivariant complex bordism, whose study lapsed in the early 1970's. In fact, some of the most significant calculational results obtained then were never fully documented in the literature. In Chapter XXVI, which was written by Gustavo Comezana, new and complete proofs of these results are presented, along with results on the relationship between geometric and stable equivariant complex cobordism. In particular, when G is a compact Abelian Lie group, Comezana proves that MU_*^G is a free MU_*-module on even degree generators.

In Chapter XXVI, and in a few places earlier on, complete proofs are given either because we feel that the material is inadequately treated in the published literature or because we have added new material. However, most of the material in the book is known and has been treated in full detail elsewhere. Our goal has been to present what is known in a form that is more readily accessible and assimilable, with emphasis on the main ideas and the structure of the theory and with pointers to where full details and further developments can be found.

Most sections have their own brief bibliographies at the end; thus, if an author's work is referred to in a section, the appropriate reference is given at the end of that section. There is also a general bibliography but, since it has over 200 items, I felt that easily found local references would be more helpful. With a few exceptions, the general bibliography is restricted to items actually referred to in the text, and it makes no claim to completeness. A full list of relevant and interesting papers would easily double the number of entries. I offer my apologies to authors not cited who should have been. Inevitably, the choice of topics and of material within topics has had to be very selective and idiosyncratic.

There are some general references that should be cited here (reminders of their abbreviated names will be given where they are first used). Starting with Chapter XII, references to [LMS] (= [133]) are to

L.G. Lewis, J.P. May, and M. Steinberger (with contributions by J.E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics Vol. 1213. 1986.

Most of the material in Chapter XII and in the five chapters XV–XIX is based on joint work of Gaunce Lewis and myself that is presented in perhaps excruciating detail in that rather encyclopedic volume. There are also abbreviated references in force in particular chapters: [L1]–[L3] = [128, 129, 130] in Chapter XI and [tD] = [55] in Chapter XVII.

The basic reference for the proofs of the claims in Chapters XXII and XXIII is

[EKMM] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. Preprint, 1995.

We shall also refer to the connected sequence of expository papers [73, 88, 89]

[EKMM′] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory.

[GM1] J. P. C. Greenlees and J. P. May. Completions in algebra and topology.

[GM2] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory.

These are all in the "Handbook of Algebraic Topology", edited by Ioan James, that came out in 1995. While these have considerable overlap with Chapters XXII through XXIV, we have varied the perspective and emphasis, and each exposition includes a good deal of material that is not discussed in the other. In particular, we point to the application of brave new algebra to chromatic periodicity in [GM1]: the ideas there have yet to be fully exploited and are not discussed here.

In view of the broad and disparate range of topics, we have tried very hard to make the chapters, and often even the sections, independent of one another. We have also broken the material into short and hopefully manageable chunks; only a few sections are as long as five pages, and all chapters are less than twenty-five pages long. Very few readers are likely to wish to read straight through, and the reader should be unafraid to jump directly to what he or she finds of interest. The reader should also be unintimidated by finding that he or she has insufficient background to feel comfortable with particular sections or chapters. Unfortunately, the subject of algebraic topology is particularly badly served by its textbooks. For example, none of them even mentions localizations and completions of spaces, although those have been standard tools since the early 1970's. We have tried to include enough background to give the basic ideas. Modern algebraic topology is a thriving subject, and perhaps jumping right in and having a look at some of its more recent directions may give a better perspective than trying to start at the beginning and work one's way up.

As the reader will have gathered, this book is a cooperative enterprise. Perhaps this is the right place to try to express just how enormously grateful I am to all of my friends, collaborators, and students. This book owes everything to our joint efforts over many years. When planning the Alaska conference, I invited some of my friends and collaborators to give talks that would mesh with mine and help give a reasonably coherent overview of the subject. Most of the speakers wrote up their talks and gave me license to edit them to fit into the framework of the book. Since TeX is refractory about listing authors inside a Table of Contents, I will here list those chapters that are written either solely by other authors or jointly with me.

Chapter III. Equivariant rational homotopy theory

by Georgia Triantafillou

Chapter VI. The homotopy theory of diagrams

by Robert Piacenza

Chapter X. G-CW(V) complexes and $RO(G)$-graded cohomology

by Stefan Waner

Chapter XI. The equivariant Hurewicz and suspension theorems

by L. G. Lewis Jr.

Chapter XIV. An introduction to equivariant K-theory

by J. P. C. Greenlees

Chapter XV. An introduction to equivariant cobordism by Steven Costenoble

Chapter XXI. Generalized Tate cohomology

by J. P. C. Greenlees and J. P. May

Chapter XXIII. Brave new equivariant foundations

by A. D. Elmendorf, L. G. Lewis Jr., and J. P. May

Chapter XXIV. Brave new equivariant algebra

by J. P. C. Greenlees and J. P. May

Chapter XXV. Localization and completion in complex cobordism

by J. P. C. Greenlees and J. P. May

Chapter XXVI. Some calculations in complex equivariant bordism

by Gustavo Costenoble

My deepest thanks to these people and to Stefan Jackowski and Chun-Nip Lee, who also gave talks; their topics were the subjects of their recent excellent survey papers [107] and [125] and were therefore not written up for inclusion here. I would also like to thank Jim McClure, whose many insights in this area are reflected throughout the book, and Igor Kriz, whose collaboration over the last six years has greatly influenced the more recent material. I would also like to thank my current students at Chicago — Maria Basterra, Mike Cole, Dan Isaksen, Mike Mandell, Adam Przezdziecki, Laura Scull, and Jerome Wolbert — who have helped catch many soft spots of exposition and have already made significant contributions to this general area of mathematics.

It is an especial pleasure to thank Bob Piacenza and his wife Lyric Ozburn for organizing the Alaska conference and making it a memorably pleasant occasion for all concerned. Thanks to their thoughtful arrangements, the intense all day mathematical activity took place in a wonderfully convivial and congenial atmosphere. Finally, my thanks to all of those who attended the conference and helped make the week such a pleasant mathematical occasion: thanks for bearing with me.

J. Peter May

December 31, 1995

CHAPTER I

Equivariant Cellular and Homology Theory

1. Some basic definitions and adjunctions

The objects of study in equivariant algebraic topology are spaces equipped with an action by a topological group G. That is, the subject concerns spaces X together with continuous actions $G\times X \longrightarrow X$ such that $ex = x$ and $g(g'x) = (gg')x$. Maps $f : X \longrightarrow Y$ are equivariant if $f(gx) = gf(x)$. We then say that f is a G-map. The usual constructions on spaces apply equally well in the category $G\mathscr{U}$ of G-spaces and G-maps. In particular G acts diagonally on Cartesian products of spaces and acts by conjugation on the space $\mathrm{Map}(X, Y)$ of maps from X to Y. That is, we define $g \cdot f$ by $(g \cdot f)(x) = gf(g^{-1}x)$.

As usual, we take all spaces to be compactly generated (which means that a subspace is closed if its intersection with each compact Hausdorff subspace is closed) and weak Hausdorff (which means that the diagonal $X \subset X\times X$ is a closed subset, where the product is given the compactly generated topology). Among other things, this ensures that we have a G-homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \tag{1.1}$$

for any G-spaces X, Y, and Z.

For us, subgroups of G are assumed to be closed. For $H \subset G$, we write $X^H = \{x|hx = x \text{ for } h \in H\}$. For $x \in X$, $G_x = \{h|hx = x\}$ is called the isotropy group of x. Thus $x \in X^H$ if H is contained in G_x. A good deal of the formal homotopy theory of G-spaces reduces to the ordinary homotopy theory of fixed point spaces. We let NH be the normalizer of H in G and let $WH = NH/H$. (We sometimes write N_GH and W_GH.) These "Weyl groups" appear ubiquitously in the theory. Note that X^H is a WH-space. In equivariant theory, orbits G/H play the role of points, and the set of G-maps $G/H \longrightarrow G/H$ can be identified with the group WH. We also have the orbit spaces X/H obtained by identifying points of X in the same orbit, and these too are WH-spaces. For a space K regarded as a G-space with trivial G-action, we have

$$G\mathscr{U}(K,X) \cong \mathscr{U}(K,X^G) \tag{1.2}$$

and

$$G\mathscr{U}(X,K) \cong \mathscr{U}(X/G,K). \tag{1.3}$$

If Y is an H-space, there is an induced G-space $G \times_H Y$. It is obtained from $G \times Y$ by identifying (gh, y) with (g, hy) for $g \in G$, $h \in H$, and $y \in Y$. A bit less obviously, we also have the "coinduced" G-space $\mathrm{M}ap_H(G,Y)$, which is the space of H-maps $G \longrightarrow Y$ with left action by G induced by the right action of G on itself, $(g \cdot f)(g') = f(g'g)$. For G-spaces X and H-spaces Y, we have the adjunctions

$$G\mathscr{U}(G \times_H Y, X) \cong H\mathscr{U}(Y,X) \tag{1.4}$$

and

$$H\mathscr{U}(X,Y) \cong G\mathscr{U}(X, \mathrm{Map}_H(G,Y)). \tag{1.5}$$

Moreover, for G-spaces X, we have G-homeomorphisms

$$G \times_H X \cong (G/H) \times X \tag{1.6}$$

and

$$\mathrm{Map}_H(G,X) \cong \mathrm{Map}(G/H,X). \tag{1.7}$$

For the first, the unique G-map $G \times_H X \longrightarrow (G/H) \times X$ that sends $x \in X$ to (eH, x) has inverse that sends (gH, x) to the equivalence class of $(g, g^{-1}x)$.

A homotopy between G-maps $X \longrightarrow Y$ is a homotopy $h : X \times I \longrightarrow Y$ that is a G-map, where G acts trivially on I. There results a homotopy category $hG\mathscr{U}$. Recall that a map of spaces is a weak equivalence if it induces an isomorphism of all homotopy groups. A G-map $f : X \longrightarrow Y$ is said to be a weak equivalence if $f^H : X^H \longrightarrow Y^H$ is a weak equivalence for all $H \subset G$. We let $\bar{h}G\mathscr{U}$ denote the category constructed from $hG\mathscr{U}$ by adjoining formal inverses to the weak equivalences. We shall be more precise shortly. The algebraic invariants of G-spaces that we shall be interested in will be defined on the category $\bar{h}G\mathscr{U}$.

General References

G. E. Bredon. Introduction to compact transformation groups. Academic Press. 1972.

T. tom Dieck. Transformation groups. Walter de Gruyter. 1987.

(This reference contains an extensive Bibliography.)

2. Analogs for based G-spaces

It will often be more convenient to work with based G-spaces. Basepoints are G-fixed and are generically denoted by $*$. We write X_+ for the union of a G-space X and a disjoint basepoint. The wedge, or 1-point union, of based G-spaces is denoted by $X \vee Y$. The smash product is defined by $X \wedge Y = X \times Y/X \vee Y$. We write $F(X,Y)$ for the based G-space of based maps $X \longrightarrow Y$. Then

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)). \tag{2.1}$$

We write $G\mathscr{T}$ for the category of based G-spaces, and we have

$$G\mathscr{T}(K,X) \cong \mathscr{T}(K, X^G) \tag{2.2}$$

and

$$G\mathscr{T}(X,K) \cong \mathscr{T}(X/G, K) \tag{2.3}$$

for a based space K and a based G-space X. Similarly, for a based G-space X and a based H-space Y, we have

$$G\mathscr{T}(G_+ \wedge_H Y, X) \cong H\mathscr{T}(Y,X) \tag{2.4}$$

and

$$H\mathscr{T}(X,Y) \cong G\mathscr{T}(X, F_H(G_+, Y)), \tag{2.5}$$

where $F_H(G_+,Y) = \mathrm{Map}_H(G,X)$ with the trivial map as basepoint, and we have G-homeomorphisms

$$G_+ \wedge_H X \cong (G/H)_+ \wedge X \tag{2.6}$$

and

$$F_H(G_+, X) \cong F(G/H_+, X). \tag{2.7}$$

A based homotopy between based G-maps $X \longrightarrow Y$ is given by a based G-map $X \wedge I_+ \longrightarrow Y$. Here the based cylinder $X \wedge I_+$ is obtained from $X \times I$ by collapsing the line through the basepoint of X to the basepoint. There results a homotopy category $hG\mathscr{T}$, and we construct $\bar{h}G\mathscr{T}$ by formally inverting the weak equivalences. Of course, we have analogous categories $hG\mathscr{U}$, and $\bar{h}G\mathscr{U}$ in the unbased context.

In both the based and unbased context, cofibrations and fibrations are defined exactly as in the nonequivariant context, except that all maps in sight are G-maps. Their theory goes through unchanged. A based G-space X is nondegenerately based if the inclusion $\{*\} \longrightarrow X$ is a cofibration.

3. G-CW complexes

A G-CW complex X is the union of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along attaching G-maps $G/H \times S^n \longrightarrow X^n$. Such an attaching map is determined by its restriction $S^n \longrightarrow (X^n)^H$, and this allows the inductive analysis of G-CW complexes by reduction to nonequivariant homotopy theory. Subcomplexes and relative G-CW complexes are defined in the obvious way. I will review my preferred way of developing the theory of G-CW complexes since this will serve as a model for other versions of cellular theory that we shall encounter.

We begin with the Homotopy Extension and Lifting Property. Recall that a map $f : X \longrightarrow Y$ is an n-equivalence if $\pi_q(f)$ is a bijection for $q < n$ and a surjection for $q = n$ (for any choice of basepoint). Let ν be a function from conjugacy classes of subgroups of G to the integers ≥ -1. We say that a map $e : Y \longrightarrow Z$ is a ν-equivalence if $e^H : Y^H \longrightarrow Z^H$ is a $\nu(H)$-equivalence for all H. (We allow $\nu(H) = -1$ to allow for empty fixed point spaces.) We say that a G-CW complex X has dimension ν if its cells of orbit type G/H all have dimension $\leq \nu(H)$.

Theorem 3.1 (HELP). Let A be a subcomplex of a G-CW complex X of dimension ν and let $e : Y \longrightarrow Z$ be a ν-equivalence. Suppose given maps $g : A \longrightarrow Y$, $h : A \times I \longrightarrow Z$, and $f : X \longrightarrow Z$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"'] & & X. \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Then there exist maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commute.

PROOF. We construct $\tilde{g}$ and $\tilde{h}$ on $A \cup X^n$ by induction on n. When we pass from the n-skeleton to the $(n+1)$-skeleton, we may work one cell at a time, dealing with the cells of X not in A. By considering attaching maps, we quickly reduce the proof to the case when $(X, A) = (G/H \times D^{n+1}, G/H \times S^n)$. But this case reduces directly to the nonequivariant case of (D^{n+1}, S^n). $\square$

THEOREM 3.2 (WHITEHEAD). Let $e : Y \longrightarrow Z$ be a ν-equivalence and X be a G-CW complex. Then $e_* : hG\mathscr{U}(X, Y) \longrightarrow hG\mathscr{U}(X, Z)$ is a bijection if X has dimension less than ν and a surjection if X has dimension ν.

PROOF. Apply HELP to the pair $(X, \emptyset)$ for the surjectivity. Apply HELP to the pair $(X \times I, X \times \partial I)$ for the injectivity. $\square$

COROLLARY 3.3. If $e : Y \longrightarrow Z$ is a ν-equivalence between G-CW complexes of dimension less than ν, then e is a G-homotopy equivalence.

PROOF. A map $f : Z \longrightarrow Y$ such that $e_*[f] = \mathrm{id}$ is a homotopy inverse to e. $\square$

The cellular approximation theorem works equally simply. A map $f : X \longrightarrow Y$ between G-CW complexes is said to be cellular if $f(X^n) \subset Y^n$ for all n, and similarly in the relative case.

THEOREM 3.4 (CELLULAR APPROXIMATION). Let (X, A) and (Y, B) be relative G-CW complexes, (X', A') be a subcomplex of (X, A), and $f : (X, A) \longrightarrow (Y, B)$ be a G-map whose restriction to (X', A') is cellular. Then f is homotopic rel $X' \cup A$ to a cellular map $g : (X, A) \longrightarrow (Y, B)$.

PROOF. This again reduces to the case of a single nonequivariant cell. $\square$

COROLLARY 3.5. Let X and Y be G-CW complexes. Then any G-map $f : X \longrightarrow Y$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.

PROOF. Apply the theorem in the cases $(X, \emptyset)$ and $(X \times I, X \times \partial I)$. $\square$

THEOREM 3.6. For any G-space X, there is a G-CW complex ΓX and a weak equivalence $\gamma : \Gamma X \longrightarrow X$.

PROOF. We construct an expanding sequence of G-CW complexes $\{Y_i | i \geq 0\}$ together with maps $\gamma_i : Y_i \longrightarrow X$ such that $\gamma_{i+1}|Y_i = \gamma_i$. Choose a representative map $f : S^q \longrightarrow X^H$ for each element of $\pi_q(X^H, x)$. Here q runs over the non-negative integers, H runs over the conjugacy classes of subgroups of G, and x runs over the components of X^H. Let Y_0 be the disjoint union of spaces $G/H \times S^q$, one for each chosen map f, and let γ_0 be the G-map induced by the maps f. Inductively, assume that $\gamma_i : Y_i \longrightarrow X$ has been constructed. Choose representative maps (f, g) for each pair of elements of $\pi_q((Y_i)^H, y)$ that are equalized by $\pi_q(\gamma_i)$; here again, q runs over the non-negative integers, H runs over the conjugacy classes of subgroups of G, and y runs over the components of $(Y_i)^H$. We may arrange that f and g have image in the q-skeleton of Y_i. Let Y_{i+1} be the homotopy coequalizer of the disjoint union of these pairs of maps; that is Y_{i+1} is obtained by attaching a tube $(G/H_+ \wedge S^q \times I_+$ via each chosen pair (f, g). Define γ_{i+1} by use of homotopies $h : \gamma_i f \simeq \gamma_i g$ based at $\gamma_i(y)$. It is easy to triangulate Y_{i+1} as a G-CW complex that contains Y_i as a subcomplex. Taking ΓX to be the union of the Y_i and γ to be the map induced by the γ_i, we obtain the desired weak equivalence. $\square$

The Whitehead theorem implies that the G-CW approximation ΓX is unique up to G-homotopy equivalence. If $f : X \longrightarrow X'$ is a G-map, there is a unique homotopy class of G-maps $\Gamma f : \Gamma X \longrightarrow \Gamma X'$ such that $\gamma' \circ \Gamma f \simeq f \circ \gamma$. That is, Γ becomes a functor $hG\mathscr{U} \longrightarrow hG\mathscr{U}$ such that γ is natural. A construction of Γ that is functorial even before passage to homotopy is possible (Seymour). It follows that the morphisms of the category $\bar{h}G\mathscr{U}$ can be specified by

$$\bar{h}G\mathscr{U}(X, X') = hG\mathscr{U}(\Gamma X, \Gamma X') = hG\mathscr{C}(\Gamma X, \Gamma X'), \tag{3.7}$$

where $G\mathscr{C}$ is the category of G-CW complexes and cellular maps. From now on, we shall write $[X, X']_G$ for this set, or for its based variant, depending on the context.

Almost all of this works just as well in the based context, giving a theory of "G-CW based complexes", which are required to have based attaching maps. This notion is to be distinquished from that of a based G-CW complex, which is just a G-CW complex with a G-fixed base vertex. In detail, a G-CW based complex X is the union of based sub G-spaces X^n such that X^0 is a point and X^{n+1} is obtained from X^n by attaching G-cells $G/H_+ \wedge D^{n+1}$ along based attaching G-maps $G/H_+ \wedge S^n \longrightarrow X^n$. Observe that such G-CW based complexes are G-connected in the sense that all of their fixed point spaces are non-empty and connected.

Nonequivariantly, one often starts proofs with the simple remark that it suffices to consider connected spaces. Equivariantly, this won't do; many important foundational parts of homotopy theory have only been worked out for G-connected G-spaces.

I should emphasize that G has been an arbitrary topological group in this discussion. When G is a compact Lie group — and we shall later restrict attention to such groups — there are important results saying that reasonable spaces are triangulable as G-CW complexes or have the homotopy types of G-CW complexes. It is fundamental for our later work that smooth compact G-manifolds are triangulable as finite G-CW complexes (Verona, Illman). In contrast to the nonequivariant situation, this is false for topological G-manifolds, which have the homotopy types of G-CW complexes but not necessarily finite ones. Metric G-ANR's have the homotopy types of G-CW complexes (Kwasik). Milnor's results on spaces of the homotopy type of CW complexes generalize to G-spaces (Waner). In particular, $\mathrm{Map}(X, Y)$ has the homotopy type of a G-CW complex if X is a compact G-space and Y has the homotopy type of a G-CW complex, and similarly for based function spaces.

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4. Ordinary homology and cohomology theories

Let $\mathscr{G}$ denote the category of orbit G-spaces G/H; the standard notation is $\mathscr{O}_G$. Observe that there is a G-map $f : G/H \longrightarrow G/K$ if and only if H is subconjugate to K since, if $f(eH) = gK$, then $g^{-1}Hg \subset K$. Let $h\mathscr{G}$ be the homotopy category of $\mathscr{G}$. Both $\mathscr{G}$ and $h\mathscr{G}$ play important roles and it is essential to keep the distinction in mind.

Define a coefficient system to be a contravariant functor $h\mathscr{G} \longrightarrow \mathscr{A}b$. One example to keep in mind is the system $\underline{\pi}_n(X)$ of homotopy groups of a based

G-space X : $\underline{\pi}_n(X)(G/H) = \pi_n(X^H)$. Formally, we have an evident fixed point functor $X^* : \mathscr{G} \longrightarrow \mathscr{T}$. The map $X^K \longrightarrow X^H$ induced by a G-map $f : G/H \longrightarrow G/K$ such that $f(eH) = gK$ sends x to gx. Any covariant functor $h\mathscr{T} \longrightarrow \mathscr{A}b$, such as π_n, can be composed with this functor to give a coefficient system. It should be intuitively clear that obstruction theory must be developed in terms of ordinary cohomology theories with coefficients in such coefficient systems. The appropriate theories were introduced by Bredon.

Since the category of coefficient systems is Abelian, with kernels and cokernels defined termwise, we can do homological algebra in it. Let X be a G-CW complex. We have a coefficient system

$$\underline{C}_n(X) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z}). \tag{4.1}$$

That is, the value on G/H is $H_n((X^n)^H, (X^{n-1})^H)$. The connecting homomorphisms of the triples $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H)$ specify a map

$$d : \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X)$$

of coefficient systems, and $d^2 = 0$. That is, we have a chain complex of coefficient systems $\underline{C}_*(X)$. For based G-CW complexes, we define $\tilde{\underline{C}}_*(X)$ similarly. Write $\mathrm{Hom}_{\mathscr{G}}(M, M')$ for the Abelian group of maps of coefficient systems $M \longrightarrow M'$ and define

$$C^n_G(X; M) = \mathrm{Hom}_{\mathscr{G}}(\underline{C}_n(X), M), \text{ with } \delta = \mathrm{Hom}_{\mathscr{G}}(d, \mathrm{id}). \tag{4.2}$$

Then $C^*_G(X; M)$ is a cochain complex of Abelian groups. Its homology is the Bredon cohomology of X, denoted $H^*_G(X; M)$.

To define Bredon homology, we must use covariant functors $N : h\mathscr{G} \longrightarrow \mathscr{A}b$ as coefficient systems. If $M : h\mathscr{G} \longrightarrow \mathscr{A}b$ is contravariant, we define an Abelian group

$$M \otimes_{\mathscr{G}} N = \sum M(G/H) \otimes N(G/H)/(\approx),$$

where the equivalence relation is specified by $(mf^*, n) \approx (m, f_*n)$ for a map $f : G/H \longrightarrow G/K$ and elements $m \in M(G/K)$ and $n \in N(G/H)$. Here we write contravariant actions from the right to emphasize the analogy with tensor products. Such "coends", or categorical tensor products of functors, occur very often in equivariant theory and will be formalized later. We define cellular chains by

$$C^G_n(X; N) = \underline{C}_n(X) \otimes_{\mathscr{G}} N, \text{ with } \partial = d \otimes 1. \tag{4.3}$$

Then $C^G_*(X;N)$ is a chain complex of Abelian groups. Its homology is the Bredon homology of X, denoted $H^G_*(X;N)$.

Clearly Bredon homology and cohomology are functors on the category $G\mathscr{C}$ of G-CW complexes and cellular maps. A cellular homotopy is easily seen to induce a chain homotopy of cellular chain complexes in our Abelian category of coefficient systems, so homotopic maps induce the same homomorphism on homology and cohomology with any coefficients.

The development of the properties of these theories is little different from the nonequivariant case. A key point is that $\underline{C}_*(X)$ is a projective object in the category of coefficient systems. To see this, observe that $\underline{C}_*(X)$ is a direct sum of coefficient systems of the form

$$\tilde{\underline{H}}_n(G/K_+ \wedge S^n) \cong \tilde{\underline{H}}_0(G/K_+) \cong \underline{H}_0(G/K). \tag{4.4}$$

If F denotes the free Abelian group functor on sets, then

$$\underline{H}_0(G/K)(G/H) = H_0((G/K)^H) = F\pi_0((G/K)^H) = F[G/H, G/K]_G. \tag{4.5}$$

Therefore $\mathrm{Hom}_{\mathscr{G}}(\underline{H}_0(G/K), M) \cong M(G/K)$ via $\phi \longrightarrow \phi(1_{G/K}) \in M(G/K)$. In detail, for a G-map $f : G/H \longrightarrow G/K$, we have $f = f^*(1_{G/K})$,

$$f^* : F[G/K, G/K]_G \longrightarrow F[G/H, G/K]_G,$$

so that $\phi(1_{G/K})$ determines ϕ via $\phi(f) = f^*\phi(1_{G/K})$. This calculation implies the claimed projectivity. It also implies the dimension axiom:

$$H^*_G(G/K; M) = H^0_G(G/K; M) \cong M(G/K) \tag{4.6}$$

and

$$H^G_*(G/K; N) = H^G_0(G/K; N) \cong N(G/K), \tag{4.7}$$

these giving isomorphisms of coefficient systems, of the appropriate variance, as K varies.

If A is a subcomplex of X, we obtain the relative chain complex $\underline{C}_*(X,A) = \tilde{\underline{C}}_*(X/A)$. The projectivity just proven implies the expected long exact sequences of pairs. For additivity, just note that the disjoint union of G-CW complexes is a G-CW complex. For excision, if X is the union of subcomplexes A and B, then $B/A\cap B \cong X/A$ as G-CW complexes. We take the "weak equivalence axiom" as a definition. That is, for general G-spaces X, we define

$$H^*_G(X;M) \equiv H^*_G(\Gamma X;M) \quad \text{and} \quad H^G_*(X;N) \equiv H^G_*(\Gamma X;N).$$

Our results on G-CW approximation of G-spaces and on cellular approximation of G-maps imply that these are well-defined functors on the category $\bar{h}G\mathscr{U}$. Similarly, we can approximate any pair (X, A) by a G-CW pair $(\Gamma X, \Gamma A)$. Less obviously, if $(X; A, B)$ is an excisive triad, so that X is the union of the interiors of A and B, we can approximate $(X; A, B)$ by a triad $(\Gamma X; \Gamma A, \Gamma B)$, where ΓX is the union of its subcomplexes ΓA and ΓB.

That is all there is to the construction of ordinary equivariant homology and cohomology groups satisfying the evident equivariant versions of the Eilenberg-Steenrod axioms.

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S. J. Willson. Equivariant homology theories on G-complexes. Trans. Amer. Math. Soc. 212(1975), 155-271.

5. Obstruction theory

Obstruction theory works exactly as it does nonequivariantly, and I'll just give a quick sketch. Fix $n \geq 1$. Recall that a connected space X is said to be n-simple if $\pi_1(X)$ is Abelian and acts trivially on $\pi_q(X)$ for $q \leq n$. Let (X, A) be a relative G-CW complex and let Y be a G-space such that Y^H is non-empty, connected, and n-simple if H occurs as an isotropy subgroup of $X \setminus A$. Let $f : X^n \cup A \longrightarrow Y$ be a G-map. We ask when f can be extended to X^{n+1}. Composing the attaching maps $G/H \times S^n \longrightarrow X$ of cells of $X \setminus A$ with f gives elements of $\pi_n(Y^H)$. These elements specify a well-defined cocycle

$$c_f \in C_G^{n+1}(X, A; \underline{\pi}_n(Y)),$$

and f extends to X^{n+1} if and only if $c_f = 0$. If f and f' are maps $X^n \cup A \longrightarrow Y$ and h is a homotopy rel A of the restrictions of f and f' to $X^{n-1} \cup A$, then f, f', and h together define a map

$$h(f, f') : (X \times I)^n \longrightarrow Y.$$

Applying $c_{h(f,f')}$ to cells $j \times I$, we obtain a deformation cochain

$$d_{f,f',h} \in C_G^n(X, A; \underline{\pi}_n(Y))$$

such that $\delta d_{f,f',h} = c_f - c_{f'}$. Moreover, given f and d, there exists f' that coincides with f on X^{n-1} and satisfies $d_{f,f'} = d$, where the constant homotopy h is understood. This gives the first part of the following result, and the second part is similar.

THEOREM 5.1. For $f : X^n \cup A \longrightarrow Y$, the restriction of f to $X^{n-1} \cup A$ extends to a map $X^{n+1} \cup A \longrightarrow Y$ if and only if $[c_f] = 0$ in $H^{n+1}_G(X, A; \underline{\pi}_n(Y))$. Given maps $f, f' : X^n \longrightarrow Y$ and a homotopy rel A of their restrictions to $X^{n-1} \cup A$, there is an obstruction in $H^n_G(X, A; \underline{\pi}_n(Y))$ that vanishes if and only if the restriction of the given homotopy to $X^{n-2} \cup A$ extends to a homotopy $f \simeq f'$ rel A.

G. Bredon. Equivariant cohomology theories. Springer Lecture Notes in Mathematics Vol 34. 1967.

6. Universal coefficient spectral sequences

While easy to define, Bredon cohomology is hard to compute. However, we do have universal coefficient spectral sequences, which we describe next.

Let W_0H be the component of the identity element of WH and define a coefficient system $\underline{J}_*(X)$ by

$$\underline{J}_*(X)(G/H) = \underline{H}_*(X^H/W_0H; \mathbb{Z}). \tag{6.1}$$

Thus $\underline{J}_*(X)$ coincides with the obvious coefficient system $\underline{H}_*(X)$ if G is discrete. We claim that, if G is a compact Lie group, then $\underline{J}_*(X)$ is the coefficient system that is obtained by taking the homology of $\underline{C}_*(X)$. The point is that a Lie theoretic argument shows that

$$\pi_0((G/K)^H) \cong (G/K)^H/W_0H.$$

We deduce that the filtration of X^H/W_0H induced by the filtration of X gives rise to the chain complex $\underline{C}_*(X)(G/H)$.

We can construct an injective resolution Q^* of the coefficient system M and form $\mathrm{Hom}_{\mathscr{G}}(\underline{C}_*(X), Q^*)$. This is a bicomplex with total differential the sum of the differentials induced by those of $\underline{C}_*(X)$ and of Q^*. It admits two filtrations. Using one of them, the differential on E_0 comes from the differential on Q^*, and $E_1^{p,q}$ is $\mathrm{Ext}^{p,q}_{\mathscr{G}}(\underline{C}_*(X), M)$. Since $\underline{C}_*(X)$ is projective, the higher Ext groups are zero, and E_1 reduces to $C^*_G(X; M)$. Thus $E_2 = E_\infty = H^*_G(X; M)$, and our bicomplex computes Bredon cohomology. Filtering the other way, the differential on E_0 comes from the differential on $\underline{C}_*(X)$, and we can identify E_2. Using a projective resolution of N, we obtain an analogous homology spectral sequence.

THEOREM 6.2. Let G be either discrete or a compact Lie group and let X be a G-CW complex. There are universal coefficient spectral sequences

$$E_2^{p,q} = \mathrm{Ext}_{\mathscr{G}}^{p,q}(\underline{J}_*(X), M) \Longrightarrow H_G^n(X; M)$$

and

$$E_{p,q}^2 = \mathrm{Tor}_{p,q}^{\mathscr{G}}(\underline{J}_*(X), N) \Longrightarrow H_n^G(X; N).$$

We should say something about change of groups and about products in cohomology, but it would take us too far afield to go into detail. For the first, we simply note that, for $H \subset G$, we can obtain H-coefficient systems from G-coefficient systems via the functor $\mathscr{H} \to \mathscr{G}$ that sends H/K to $G/K = G \times_H H/K$. For the second, we note that, for groups H and G, projections give a functor from the orbit category of $H \times G$ to the product of the orbit categories of H and of G, so that we can tensor an H-coefficient system and a G-coefficient system to obtain an $(H \times G)$-coefficient system. When $H = G$, we can then apply change of groups to the diagonal inclusion $G \subset G \times G$. The resulting pairings of coefficient systems allow us to define cup products exactly as in ordinary cohomology, using cellular approximations of the diagonal maps of G-CW complexes.

G. Bredon. Equivariant cohomology theories. Springer Lecture Notes in Mathematics Vol 34. 1967.

S. J. Willson. Equivariant homology theories on G-complexes. Trans. Amer. Math. Soc. 212(1975), 155-271.

CHAPTER II

Postnikov Systems, Localization, and Completion

1. Eilenberg-MacLane G-spaces and Postnikov systems

Let M be a coefficient system. An Eilenberg-Mac Lane G-space $K(M, n)$ is a G-space of the homotopy type of a G-CW complex such that

$$\underline{\pi}_q(K(M,n)) = \begin{cases} M & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

While our interest is in Abelian group-valued coefficient systems, we can allow M to be set-valued if $n = 0$ and group-valued if $n = 1$. I will give an explicit construction later. Ordinary cohomology theories are characterized by the usual axioms, and, by checking the axioms, it is easily verified that the reduced cohomology of based G-spaces X is represented in the form

$$\tilde{H}^n_G(X; M) \cong [X, K(M, n)]_G, \tag{1.1}$$

where homotopy classes of based maps (in $\bar{h}G\mathscr{T}$) are understood.

Recall that a connected space X is said to be simple if $\pi_1 A$ is Abelian and acts trivially on $\pi_n(X)$ for $n \geq 2$. More generally, a connected space X is said to be nilpotent if $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_n(X)$ for $n \geq 2$. A G-connected G-space X is said to be simple if each X^H is simple. A G-connected G-space X is said to be nilpotent if each X^H is nilpotent and, for each $n \geq 1$, the orders of nilpotency of the $\pi_1(X^H)$-groups $\pi_n(X^H)$ have a common bound. We shall restrict our sketch proofs to simple G-spaces, for simplicity, in the next few sections, but everything that we shall say about their Postnikov towers and about localization and completion generalizes readily to the case of nilpotent G-spaces. The only difference is that each homotopy group system must be built up in finitely many stages, rather than all at once.

A Postnikov system for a based simple G-space X consists of based G-maps

$$\alpha_n : X \longrightarrow X_n \quad \text{and} \quad p_{n+1} : X_{n+1} \longrightarrow X_n$$

for $n \geq 0$ such that X_0 is a point, α_n induces an isomorphism $\underline{\pi}_q(X) \longrightarrow \underline{\pi}_q(X_n)$ for $q \leq n$, $p_{n+1}\alpha_{n+1} = \alpha_n$, and p_{n+1} is the G-fibration induced from the path space fibration over a $K(\underline{\pi}_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \longrightarrow K(\underline{\pi}_{n+1}(X), n+2)$. It follows that $X_1 = K(\underline{\pi}_1(X), 1)$ and that $\underline{\pi}_q(X_n) = 0$ for $q > n$. Our requirement that Eilenberg-Mac Lane G-spaces have the homotopy types of G-CW complexes ensures that each X_n has the homotopy type of a G-CW complex. The maps α_n induce a weak equivalence $X \longrightarrow \lim X_n$, but the inverse limit generally will not have the homotopy type of a G-CW complex. Just as nonequivariantly, the k-invariants that specify the tower are to be regarded as cohomology classes

$$k^{n+2} \in H^{n+2}_G(X_n; \underline{\pi}_{n+1}(X)).$$

These classes together with the homotopy group systems $\underline{\pi}_n(X)$ specify the weak homotopy type of X. On passage to H-fixed points, a Postnikov system for X gives a Postnikov system for X^H. We outline the proof of the following standard result since there is no complete published proof and my favorite nonequivariant proof has also not been published. The result generalizes to nilpotent G-spaces.

THEOREM 1.2. A simple G-space X of the homotopy type of a G-CW complex has a Postnikov tower.

PROOF. Assume inductively that $\alpha_n : X \longrightarrow X_n$ has been constructed. By the homotopy excision theorem applied to fixed point spaces, we see that the cofiber $C(\alpha_n)$ is $(n+1)$-connected and satisfies

$$\underline{\pi}_{n+2}(C\alpha_n) = \underline{\pi}_{n+1}(X).$$

More precisely, the canonical map $F(\alpha_n) \longrightarrow \Omega C(\alpha_n)$ induces an isomorphism on $\underline{\pi}_q$ for $q \leq n+1$. We construct

$$j : C(\alpha_n) \longrightarrow K(\underline{\pi}_{n+1}(X), n+2)$$

by inductively attaching cells to $C(\alpha_n)$ to kill its higher homotopy groups. We take the composite of j and the inclusion $X_n \subset C(\alpha_n)$ to be the k-invariant k^{n+2}. By our definition of a Postnikov tower, X_{n+1} must be the homotopy fiber of k^{n+2}. Its points are pairs (ω, x) consisting of a path $\omega : I \longrightarrow K(\underline{\pi}_{n+1}(X), n+2)$ and a point $x \in X_n$ such that $\omega(0) = *$ and $\omega(1) = k^{n+2}(x)$. The map $p_{n+1} : X_{n+1} \longrightarrow X_n$ is given by $p_{n+1}(\omega, x) = x$, and the map $\alpha_{n+1} : X \longrightarrow X_{n+1}$ is given by $\alpha_{n+1}(x) = (\omega(x), \alpha_n(x))$, where $\omega(x)(t) = j(x, 1-t)$, $(x, 1-t)$ being a point on the cone $CX \subset C(\alpha_n)$. Clearly $p_{n+1}\alpha_{n+1} = \alpha_n$. It is evident that α_{n+1} induces an isomorphism on $\underline{\pi}_q$ for $q \leq n$, and a diagram chase shows that this also holds for $q = n+1$. $\square$

2. Summary: localizations of spaces

Nonequivariantly, localization at a prime p or at a set of primes T is a standard first step in homotopy theory. We quickly review some of the basic theory. Say that a map $f : X \longrightarrow Y$ is a T-cohomology isomorphism if

$$f^* : H^*(Y; A) \longrightarrow H^*(X; A)$$

is an isomorphism for all T-local Abelian groups A.

THEOREM 2.1. The following properties of a nilpotent space Z are equivalent. When they hold, Z is said to be T-local.

(a) Each $\pi_n(Z)$ is T-local.
(b) If $f : X \longrightarrow Y$ is a T-cohomology isomorphism, then $f^* : [Y, Z] \longrightarrow [X, Z]$ is a bijection.
(c) The integral homology of Z is T-local.

THEOREM 2.2. Let X be a nilpotent space. The following properties of a map $\lambda : X \longrightarrow X_T$ from X to a T-local space X_T are equivalent. There is one and, up to homotopy, only one such map λ. It is called the localization of X at T.

(a) $\lambda^* : [X_T, Z] \longrightarrow [X, Z]$ is a bijection for all T-local spaces Z.
(b) λ is a T-cohomology isomorphism.
(c) $\lambda_* : \pi_*(X) \longrightarrow \pi_*(X_T)$ is localization at T.
(d) $\lambda_* : H_*(X; \mathbb{Z}) \to H_*(X_T; \mathbb{Z})$ is localization at T.

A. K. Bousfield and D. M. Kan. Homotopy limits, completions, and localizations. Springer Lecture Notes in Mathematics Vol 304. 1972.
D. Sullivan. The genetics of homotopy theory and the Adams conjecture. Annals of Math. 100(1974), 1-79.

3. Localizations of G-spaces

Now let G be a compact Lie group. Say that a G-map $f : X \longrightarrow Y$ is a T-cohomology isomorphism if

$$f^* : H^*_G(Y;M) \longrightarrow H^*_G(X;M)$$

is an isomorphism for all T-local coefficient systems M.

THEOREM 3.1. The following properties of a nilpotent G-space Z are equivalent. When they hold, Z is said to be T-local.

(a) Each Z^H is T-local.

(b) If $f : X \longrightarrow Y$ is a T-cohomology isomorphism, then

$$f^* : [Y,Z]_G \longrightarrow [X,Z]_G$$

is a bijection.

THEOREM 3.2. Let X be a nilpotent G-space. The following properties of a map $\lambda : X \longrightarrow X_T$ from X to a T-local G-space X_T are equivalent. There is one and, up to homotopy, only one such map λ. It is called the localization of X at T.

(a) $\lambda^* : [X_T, Z] \longrightarrow [X, Z]$ is a bijection for all T-local G-spaces Z.

(b) λ is a T-cohomology isomorphism.

(c) Each $\lambda^H : X^H \longrightarrow (X_T)^H$ is localization at T.

PROOFS. We restrict attention to simple G-spaces. Assuming (a) in Theorem 3.1, we may replace Z by a weakly equivalent Postnikov tower and we may assume that the G-spaces X and Y given in (b) are G-CW complexes, so that we are dealing with actual homotopy classes of maps. Then (a) implies (b) by a word-for-word dualization of our proof of the Whitehead theorem. Conversely, (b) implies (a) since the specialization of (b) to T-cohomology isomorphisms of the form $G/H_+ \wedge f$, where $f : X \longrightarrow Y$ is a nonequivariant T-cohomology isomorphism, implies (b) of Theorem 2.1. In Theorem 3.2, (a) implies (b) by letting Z run through $K(M,n)$'s, and (b) implies (a) by Theorem 3.1. Let $\mathbb{Z}_T$ be the localization of $\mathbb{Z}$ at T. One sees that (c) implies (b) by applying the universal coefficient spectral sequence of I.6.2, taken with homology and coefficient systems tensored with $\mathbb{Z}_T$. The maps λ^H induce isomorphisms on homology with coefficients in $\mathbb{Z}_T$, and one can deduce (with some work in the general compact Lie case) that they therefore induce an isomorphism $\underline{J}_*(X;\mathbb{Z}_T) \longrightarrow \underline{J}_*(X_T;\mathbb{Z}_T)$. Since the universal property (a) implies uniqueness, to complete the proof we need only construct a map λ that satisfies (c). For this, we may assume that X is a Postnikov tower, and we localize its terms inductively by localizing k-invariants and comparing fibration sequences. The starting point is just the observation that the algebraic localization $M \longrightarrow M_T = M \otimes \mathbb{Z}_T$ of coefficient systems induces localization maps $\lambda : K(M, n) \longrightarrow K(M_T, n)$. The relevant diagram is:

$$\begin{tikzcd}
K(\underline{\pi}_{n+1}(X), n+1) \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & K(\underline{\pi}_{n+1}(X), n+2) \arrow[d] \\
K(\underline{\pi}_{n+1}(X)_T, n+1) \arrow[r] & (X_{n+1})_T \arrow[r] & (X_{n+1})_T \arrow[r] & K(\underline{\pi}_{n+1}(X)_T, n+2).
\end{tikzcd}$$

We construct the right square by localizing the k-invariant, we define $(X_{n+1})_T$ to be the fiber of the localized k-invariant, and we obtain $X_{n+1} \longrightarrow (X_{n+1})_T$ making the middle square commute and the left square homotopy commute by standard fiber sequence arguments. $\square$

J. P. May. The dual Whitehead theorems. London Math. Soc. Lecture Note Series Vol 86, 1983, 46-54.

J. P. May, J. McClure, and G. V. Triantafillou. Equivariant localization. Bull. London Math. Soc. 14(1982), 223-230.

4. Summary: completions of spaces

Completion at a prime p or at a set of primes T is another standard first step in homotopy theory. Since completion at T is the product of the completions at p for $p \in T$, we restrict to the case of a single prime. We first review some of the nonequivariant theory. The algebra we begin with is a preview of algebra to come later in our discussion of completions of G-spectra at ideals of the Burnside ring.

The p-adic completion functor, $\hat{A}_p = \lim(A/p^n A)$, is neither left nor right exact in general, and it has left derived functors L_0 and L_1. If

$$0 \longrightarrow F' \longrightarrow F \longrightarrow A \longrightarrow 0$$

is a free resolution of A, then $L_0 A$ and $L_1 A$ are the cokernel and kernel of $\hat{F'}_p \longrightarrow \hat{F}_p$, and there results a natural map $\eta : A \longrightarrow L_0 A$. The higher left derived functors are zero, and a short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

gives rise to a six term exact sequence

$$0 \longrightarrow L_1 A' \longrightarrow L_1 A \longrightarrow L_1 A'' \longrightarrow L_0 A' \longrightarrow L_0 A \longrightarrow L_0 A'' \longrightarrow 0.$$

If $L_1A = 0$, then we call $\eta : A \longrightarrow L_0A$ the "p-completion" of A. It must not to be confused with the p-adic completion. We say that A is "p-complete" if $L_1A = 0$ and η is an isomorphism. The groups L_0A, L_1A, and $\hat{A}_p$ are p-complete for any Abelian group A. While derived functors give the best conceptual descriptions of L_0A and L_1A, there are more easily calculable descriptions. Let $\mathbb{Z}/p^\infty$ be the colimit of the sequence of homomorphisms $p : \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n+1}$. Then $\mathbb{Z}/p^\infty \cong \mathbb{Z}[p^{-1}]/\mathbb{Z}$ and there are natural isomorphisms

$$L_0(A) \cong \mathrm{Ext}(\mathbb{Z}/p^\infty, A) \quad \text{and} \quad L_1(A) \cong \mathrm{Hom}(\mathbb{Z}/p^\infty, A).$$

There is also a natural short exact sequence

$$0 \longrightarrow \lim{}^1 \mathrm{Hom}(\mathbb{Z}/p^n, A) \longrightarrow L_0A \longrightarrow \hat{A}_p \longrightarrow 0.$$

In particular, $L_1A = 0$ and $L_0A \cong \hat{A}_p$ if the p-torsion of A is of bounded order.

Say that a map $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism if

$$f^* : H^*(Y; A) \longrightarrow H^*(X; A)$$

is an isomorphism for all p-complete Abelian groups A. This holds if and only if it holds for all $\mathbb{F}_p$-vector spaces A, and this in turn holds if and only if $f_* : H_*(X; \mathbb{F}_p) \longrightarrow H_*(Y; \mathbb{F}_p)$ is an isomorphism, where $\mathbb{F}_p$ is the field with p elements. While this homological characterization is essential to our proofs, we prefer to emphasize cohomology.

THEOREM 4.1. The following properties of a nilpotent space Z are equivalent. When they hold, Z is said to be p-complete.

(a) Each $\pi_n(Z)$ is p-complete.
(b) If $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism, then $f^* : [Y, Z] \longrightarrow [X, Z]$ is a bijection.

THEOREM 4.2. Let X be a nilpotent space. The following properties of a map $\gamma : X \longrightarrow \hat{X}_p$ from X to a p-complete space $\hat{X}_p$ are equivalent. There is one and, up to homotopy, only one such map γ. It is called the completion of X at p.

(a) $\gamma^* : [\hat{X}_p, Z] \longrightarrow [X, Z]$ is a bijection for all p-complete spaces Z.
(b) γ is a $\hat{p}$-cohomology isomorphism.

For $n \geq 1$, there is a natural and splittable short exact sequence

$$0 \longrightarrow L_0\pi_n(X) \longrightarrow \pi_n(\hat{X}_p) \longrightarrow L_1\pi_{n-1}(X) \longrightarrow 0.$$

If $L_1\pi_*(X) = 0$, then γ is also characterized by

(c) $\gamma_* : \pi_*(X) \longrightarrow \pi_*(\hat{X}_p)$ is completion at p.

A. K. Bousfield and D. M. Kan. Homotopy limits, completions, and localizations. Springer Lecture Notes in Mathematics, Vol. 304. 1972.
D. Sullivan. The genetics of homotopy theory and the Adams conjecture. Annals of Math. 100(1974), 1-79.

5. Completions of G-spaces

Now let G be a compact Lie group. Say that a G-map $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism if

$$f^* : H^*_G(Y; M) \longrightarrow H^*_G(X; M)$$

is an isomorphism for all p-complete coefficient systems M. This will hold if each $f^H : X^H \longrightarrow Y^H$ is a $\hat{p}$-cohomology isomorphism by another application of the universal coefficients spectral sequence, with a little work in the general compact Lie case to handle $\underline{J}_*(f)$.

THEOREM 5.1. The following properties of a nilpotent G-space Z are equivalent. When they hold, Z is said to be p-complete.

(a) Each Z^H is p-complete.
(b) If $f : X \longrightarrow Y$ is a $\hat{p}$-cohomology isomorphism, then $f^* : [Y, Z]_G \longrightarrow [X, Z]_G$ is a bijection.

THEOREM 5.2. Let X be a nilpotent G-space. The following properties of a map $\gamma : X \longrightarrow \hat{X}_p$ from X to a p-complete G-space $\hat{X}_p$ are equivalent. There is one and, up to homotopy, only one such map γ. It is called the completion of X at p.

(a) $\gamma^* : [\hat{X}_p, Z] \longrightarrow [X, Z]$ is a bijection for all p-complete G-spaces Z.
(b) γ is a $\hat{p}$-cohomology isomorphism.
(c) Each $\gamma^H : X^H \longrightarrow (\hat{X}_p)^H$ is completion at p.

For $n \geq 1$, there is a natural short exact sequence

$$0 \longrightarrow L_0\underline{\pi}_n(X) \longrightarrow \underline{\pi}_n(\hat{X}_p) \longrightarrow L_1\underline{\pi}_{n-1}(X) \longrightarrow 0.$$

PROOFS. The proofs are the same as those of Theorems 3.1 and 3.2, except that completions of Eilenberg-Mac Lane G-spaces are not Eilenberg-Mac Lane G-spaces in general. For a coefficient system M, $\eta : M \longrightarrow L_0M$ induces p-completions $K(M, n) \longrightarrow K(L_0M, n)$ whenever $L_1M = 0$. For the general case, let FM be the coefficient system obtained by applying the free Abelian group functor to M regarded as a set-valued functor. There results a natural epimorphism $FM \longrightarrow M$ of coefficient systems. Let $F'M$ be its kernel. Since L_1 vanishes on free modules, we can construct the completion of $K(M,n)$ at p via the following diagram of fibrations:

$$\begin{tikzcd}
K(FM,n) \arrow[r] \arrow[d] & K(M,n) \arrow[r] \arrow[d] & K(F'M,n+1) \arrow[r] \arrow[d] & K(FM,n+1) \arrow[d] \\
K(L_0FM,n) \arrow[r] & K(M,n)^{\wedge}_p \arrow[r] & K(L_0F'M,n+1) \arrow[r] & K(L_0FM,n+1).
\end{tikzcd}$$

That is, $K(M,n)^{\wedge}_p$ is the fiber of $K(L_0F'M,n+1) \longrightarrow K(L_0FM,n+1)$. It is complete since its homotopy group systems are complete. The map $K(M,n) \longrightarrow K(M,n)^{\wedge}_p$ is a $\hat{p}$-cohomology isomorphism because its fixed point maps are so, by the Serre spectral sequence. $\square$

J. P. May. Equivariant completion. Bull. London Math. Soc. 14(1982), 231-237.

CHAPTER III

Equivariant Rational Homotopy Theory

by Georgia Triantafillou

1. Summary: the theory of minimal models

Let G be a finite group. In this chapter, we summarize our work on the algebraicization of rational G-homotopy theory.

To simplify the statements we assume simply connected spaces throughout the chapter. The theory can be extended to the nilpotent case in a straightforward manner. We recall that by rationalizing a space X, we approximate it by a space X_0 the homotopy groups of which are equal to $\pi_*(X) \otimes \mathbb{Q}$, thus neglecting the torsion. The advantage of doing so is that rational homotopy theory is determined completely by algebraic invariants, as was shown by Quillen and later by Sullivan. Our theory is analogous to Sullivan's theory of minimal models, which we now review. For our purposes we prefer Sullivan's approach because of its computational advantage and its relation to geometry by use of differential forms.

The algebraic invariants that determine the rational homotopy type are certain algebras that we call DGA's. By definition a DGA is a graded, commutative, associative algebra with unit over the rationals, with differential $d : A^n \to A^{n+1}$ for $n \geq 0$. We say that A is *connected* if $H^0(A) = \mathbb{Q}$ and *simply connected* if, in addition, $H^1(A) = 0$. Again we assume that all DGA's in sight are connected and simply connected. A map of DGA's is said to be a *quasi-isomorphism* if it induces an isomorphism on cohomology.

Certain DGA's, the so called minimal ones, play a special role to be described below. A DGA $\mathcal{M}$ is said to be *minimal* if it is free and its differential is decomposable. Freeness means that $\mathcal{M}$ is the tensor product of a polynomial algebra generated by elements of even degree and an exterior algebra generated by elements of odd degree. Decomposability of the differential means that $d(\mathcal{M}) \subseteq \mathcal{M}^+ \wedge \mathcal{M}^+$, where $\mathcal{M}^+$ is the set of positive degree elements of $\mathcal{M}$.

There is an algebraic notion of homotopy between maps of DGA's that mirrors the topological notion. Let $\mathbb{Q}(t, dt)$ be the free DGA on two generators t and dt of degree 0 and 1 respectively with $d(t) = dt$.

DEFINITION 1.1. Two morphisms $f, g : \mathcal{A} \to \mathcal{B}$ are homotopic if there is a map $H : \mathcal{A} \to \mathcal{B} \otimes \mathbb{Q}(t, dt)$ such that $e_0 \circ H = f$ and $e_1 \circ H = g$, where e_0 is the projection $t = 0, dt = 0$ and e_1 the projection $t = 1, dt = 0$.

The basic example of a DGA in the theory is the PL De Rham algebra $\mathcal{E}_X$ of a simplicial complex X, which is constructed as follows. Let

$$\sigma^n = \Delta^n = \{(t_0, t_1, \dots, t_n) | 0 \leq t_i \leq 1, \Sigma_{i=0}^n t_i = 1\}$$

be an n-simplex of X canonically embedded in $\mathbb{R}^{n+1}$. A polynomial form of degree p on σ^n is an expression

$$\sum_I f_I(t_0, \dots, t_n) dt_{i_1} \wedge \cdots \wedge dt_{i_p},$$

where $I = \{i_1, \dots, i_p\}$ and f_I is a polynomial with coefficients in $\mathbb{Q}$. A global PL (piecewise linear) form on X is a collection of polynomial forms, one for each simplex of X, which coincide on common faces. The set of PL forms of X is the DGA $\mathcal{E}_X$. A version of the classical de Rham theorem holds, namely that

$$H^*(\mathcal{E}_X) = H^*(X; \mathbb{Q}).$$

We have the following facts.

THEOREM 1.2. A quasi-isomorphism between minimal DGA's is an isomorphism.

THEOREM 1.3. If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of DGA's and $\mathcal{M}$ is a minimal DGA, then $f_* : [\mathcal{M}, \mathcal{A}] \to [\mathcal{M}, \mathcal{B}]$ is an isomorphism.

THEOREM 1.4. For any simply connected DGA $\mathcal{A}$ there is a minimal DGA $\mathcal{M}$ and a quasi-isomorphism $\rho : \mathcal{M} \to \mathcal{A}$. Moreover $\mathcal{M}$ is unique up to (non-canonical) isomorphism, namely if $\rho' : \mathcal{M}' \to \mathcal{A}$ is another quasi- isomorphism then there is an isomorphism $e : \mathcal{M} \to \mathcal{M}'$ such that $\rho' \circ e$ and ρ are homotopic.

Here $\mathcal{M}$ is said to be the *minimal model* of $\mathcal{A}$. The minimal model $\mathcal{M}_X$ of the PL de Rham algebra $\mathcal{E}_X$ of a simply connected space X is called the minimal model of X.

THEOREM 1.5. The correspondance $X \to \mathcal{M}_X$ induces a bijection between rational homotopy types of simplicial complexes on the one hand and isomorphism classes of minimal DGA's on the other.

More precisely, assuming X is a rational space, the homotopy groups $\pi_n(X)$ of X are isomorphic to $Q(\mathcal{M}_X)_n$, where $Q(\mathcal{M}) \equiv \mathcal{M}^+/\mathcal{M}^+ \wedge \mathcal{M}^+$ is the space of indecomposables of $\mathcal{M}$. The nth stage X_n of the Postnikov tower of X has $\mathcal{M}_X(n)$ as its minimal model, where $\mathcal{M}(n)$ denotes the subalgebra of $\mathcal{M}$ that is generated by the elements of degree $\leq n$. The k-invariant $k^{n+2} \in H^{n+2}(X_n, \pi_{n+1}(X))$, which can be represented as a map $\pi_{n+1}(X)^* \to H^{n+2}(X_n)$, is determined by the differential $d : Q(\mathcal{M}_X)_{n+1} \to H^{n+2}(\mathcal{M}_X(n))$. These properties enable the inductive construction of a rational space that realizes a given minimal algebra.

On the morphism level we have

THEOREM 1.6. If Y is a rational space then

$$[X, Y] \equiv [\mathcal{M}_Y, \mathcal{M}_X].$$

We warn here that the minimal model, though very useful computationally, is not a functor. In particular a map of spaces induces a map of the corresponding minimal models only up to homotopy.

D. Quillen. Rational Homotopy Theory, Ann of Math. 90(1968), 205-295.
D. Sullivan. Infinitesimal Computations in Topology, Publ. Math. IHES 47(1978), 269-332.

2. Equivariant minimal models

For finite groups G an analogous theory can be developed for G-rational homotopy types of G-simplicial complexes. For simplicity we assume throughout that the spaces X are G-connected and G-simply connected, which means that each fixed point space X^H is connected and simply connected; however, the theory works just as well for G-nilpotent spaces. In fact, by work of B. Fine, the theory can be extended in such a way that no fixed base point and no connectivity assumption on the fixed point sets are required.

Let Vec_G be the category of rational coefficient systems and Vec_G^* the category of covariant functors from $\mathcal{G}$ to rational vector spaces. Our invariants for determining

G-rational homotopy types are functors of a special type from $\mathcal{G}$ into DGA's, which we now describe.

DEFINITION 2.1. A system of DGA's is a covariant functor from $\mathcal{G}$ to simply connected DGA's such that the underlying functor in Vec_G^* is injective.

The injective objects of Vec_G^* or, equivalently, the projective rational coefficient systems can be characterized as follows.

THEOREM 2.2. (i) For $H \subseteq G$ and a WH-representation V, there is a projective coefficient system $\underline{V} \in Vec_G$ such that

$$\underline{V}(G/K) = \mathbb{Q}[(G/H)^K] \otimes_{\mathbb{Q}[WH]} V,$$

where the first factor is the vector space generated by the set $(G/H)^K$.
(ii) Every projective coefficient system is a direct sum of such $\underline{V}$'s.

The basic system of DGA's in the theory is the system of de Rham algebras $\mathcal{E}_{X^H}$ of the fixed point sets X^H of a G-simplicial complex X. We denote this system by $\underline{\mathcal{E}}_X$. It is crucial to realize that $\underline{\mathcal{E}}_X$ is injective and that this property is central to the theory. The injectivity of $\underline{\mathcal{E}}_X$ can be shown by utilising the splitting of X into its orbit types.

We note that $\mathcal{E}_X$ together with the induced G-action determine a minimal algebra equipped with a G-action. However there are in general many G-rational homotopy types of G-simplicial complexes that realize this minimal G-algebra. In order to have unique spacial realizations we need to take into account the algebraic data of all fixed point sets, which leads us to systems of DGA's.

Define the cohomology of a system $\mathcal{A}$ of DGA's with respect to a covariant coefficient system $N \in Vec_G^*$ to be the cohomology of the cochain complex $Hom_{\mathcal{G}}(N, \mathcal{A})$. An equivariant de Rham theorem follows by use of the universal coefficients spectral sequence.

THEOREM 2.3. For $M \in Vec_G$ with dual $M^* \in Vec_G^*$,

$$H_G^*(X; M) \equiv H_G^*(\underline{\mathcal{E}}_X; M^*).$$

The lack of functoriality of the minimal model of a space complicates the construction of equivariant minimal models. We cannot, for instance, define "the system of minimal models" $\mathcal{M}_{X^H}$ of the fixed point sets of a G-complex X. It turns out that the right definition of minimal models in the equivariant context is the following.

DEFINITION 2.4. A system $\mathcal{M}$ of DGA's is said to be minimal if

(i) each algebra $\mathcal{M}(G/H)$ is free commutative,
(ii) the DGA $\mathcal{M}(G/G)$ is minimal, and
(iii) the differential on each $\mathcal{M}(G/H)$ is decomposable when restricted to the intersection of the kernels of the maps $\mathcal{M}(G/H) \to \mathcal{M}(G/K)$ induced by proper inclusions $H \subset K$.

One can think of *(ii)* as an "initial condition" and of *(iii)* as the minimality condition that guarantees the uniqueness of equivariant minimal models. As in the nonequivariant case, minimal systems are classified by their cohomology.

THEOREM 2.5. A quasi-isomorphism between minimal systems of DGA's is an isomorphism.

Also, Theorems 1.3, 1.4, 1.5, and 1.6 have equivariant counterparts.

THEOREM 2.6. If $\mathcal{A}$ is a system of DGA's, then there is a quasi-isomorphism $f : \mathcal{M} \to \mathcal{A}$, where $\mathcal{M}$ is a minimal system. Moreover $\mathcal{M}$ is unique up to (non-canonical) isomorphism.

This result provides the existence of equivariant minimal models. Unlike the nonequivariant case the proof is rather involved and is based on a careful investigation of the universal coefficients spectral sequence. We define the *equivariant minimal model* $\mathcal{M}_X$ of a G-simplicial complex X to be the minimal system of DGA's that is quasi-isomorphic to the system of de Rham algebras $\underline{\mathcal{E}}_X$.

A notion of homotopy can be defined for systems of DGA's. If $\mathcal{A}$ is a system of DGA's we denote by $\mathcal{A} \otimes \mathbb{Q}(t, dt)$ the functor

$$\mathcal{A} \otimes \mathbb{Q}(t, dt)(G/H) = \mathcal{A}(G/H) \otimes \mathbb{Q}(t, dt).$$

It can be shown that this functor is injective and therefore it forms a system of DGA's. Homotopy of maps of systems of DGA's can now be defined in the obvious way suggested by the nonequivariant case. Let $[\mathcal{A}, \mathcal{B}]_G$ denote homotopy classes of maps of systems.

THEOREM 2.7. If $f : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism of systems of DGA's and $\mathcal{M}$ is a minimal system of DGA's, then

$$f^* : [\mathcal{M}, \mathcal{A}]_G \to [\mathcal{M}, \mathcal{B}]_G$$

is an isomorphism.

The equivariant minimal model determines the rational G-homotopy type of a G-space, namely

THEOREM 2.8. The correspondence $X \to \mathcal{M}_X$ induces a bijection between rational G-homotopy types of G-simplicial complexes on the one hand and isomorphism classes of minimal systems of DGA's on the other.

More precisely, there is a filtration of minimal subsystems of DGA's

$$\cdots \subseteq \mathcal{M}_X(n) \subseteq \mathcal{M}_X(n+1) \subseteq \cdots \subseteq \mathcal{M}_X$$

such that each stage is the equivariant minimal model of the equivariant Postnikov term X_n of the space X. The system of rational homotopy groups of the fixed point sets $\underline{\pi}_n(X) \otimes \mathbb{Q}$ and the rational equivariant k-invariants can also be read from the model $\mathcal{M}_X$. This makes the inductive construction of the Postnikov decomposition of the rationalization X_0 possible if the equivariant minimal model is given.

On the morphism level we have

THEOREM 2.9. If Y is a rational G-simplicial complex then there is a bijection

$$[X, Y] \equiv [\mathcal{M}_Y, \mathcal{M}_X].$$

G. Triantafillou. Äquivariante Rationale Homotopietheorie, Bonner Math. Schriften Vol. 110. 1978.
G. Triantafillou. Equivariant minimal models. Trans. Amer. Math. Soc. 274(1982), 509-532.

3. Rational equivariant Hopf spaces

In spite of the conceptual analogy of the equivariant theory to the nonequivariant one, the calculations in the equivariant case are much more subtle and can yield surprising results. We illustrate this by describing our work on rational Hopf G-spaces. It is a basic feature of nonequivariant homotopy theory that the rational Hopf spaces split as products of Eilenberg-Mac Lane spaces. The equivariant analogue is false. By a Hopf G-space we mean a based G-space X together with a G-map $X \times X \to X$ such that the base point is a two-sided unit for the product. Examples include Lie groups K with a G-action such that G is a finite subgroup of the inner automorphisms of K, and loop spaces $\Omega(X)$ of G-spaces based at a G-fixed point of X.

THEOREM 3.1. Let X be a G-connected rational Hopf G-space of finite type. If G is cyclic of prime power order, then X splits as a product of Eilenberg-Mac Lane G-spaces. If $G = \mathbb{Z}_p \times \mathbb{Z}_q$ for distinct primes p and q, then there are counterexamples to this statement.

Outline of proof: In this outline we suppress the technical part of the proof which is quite extensive. As in the nonequivariant case, the n^{th} term X_n of a Postnikov tower of X is a Hopf G-space. Moreover the k-invariant $k^{n+2} \in H^{n+2}(X_n; \underline{\pi}_{n+1}(X))$ is a primitive element. This means that

$$m^*(k^{n+2}) = (p_1)^*(k^{n+2}) + (p_2)^*(k^{n+2}),$$

in $H^{n+2}(X_n \times X_n; \underline{\pi}_{n+1}(X))$, where m is the product and the p_i are the projections.

The difference in the two cases $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_p \times \mathbb{Z}_q$ stems from the fact that rational coefficient systems for these groups have different projective dimensions. Indeed, systems for $\mathbb{Z}_{p^k}$ have projective dimension at most 1, whereas there are rational coefficient systems for $\mathbb{Z}_p \times \mathbb{Z}_q$ of projective dimension 2. Using this fact about $\mathbb{Z}_{p^k}$ we can compute inductively the equivariant minimal model of each Postnikov term X_n and its cohomology. In particular we show that all non-zero elements of $H_G^{n+2}(X_n; \underline{\pi}_{n+1}(X))$ are decomposable and therefore non-primitive.

In the case of $\mathbb{Z}_p \times \mathbb{Z}_q$ we construct counterexamples which are 2-stage Postnikov systems with primitive k-invariant. As in the nonequivariant case, if X has only two non-vanishing homotopy group systems, then the primitivity of the unique k-invariant is a sufficient condition for X to be a Hopf G-space. By construction, the two systems of homotopy groups $\underline{\pi}_n(X)$ and $\underline{\pi}_{n+1}(X)$ are as follows. The groups $\pi_n(X^H)$ are zero for all proper subgroups H and $\pi_n(X^{\mathbb{Z}_p \times \mathbb{Z}_q}) = \mathbb{Z}$. The groups $\pi_{n+1}(X^H)$ are zero for all nontrivial subgroups H and $\pi_{n+1}(X) = \mathbb{Z}$. The first coefficient system has projective dimension 2. This and the universal coefficients spectral sequence yields $H_G^{n+2}(X_n; \underline{\pi}_{n+1}(X)) = \mathbb{Z}$. Moreover all non-zero elements of this group are primitive. This gives an infinite choice of primitive k-invariants and therefore an infinite collection of rationally distinct Hopf G-spaces which do not split rationally into products of Eilenberg-Mac Lane G-spaces.

The counterexamples X constructed in the theorem are infinite loop G-spaces in the sense that there are G-spaces E_n and homotopy equivalences $E_n \to \Omega E_{n+1}$, with $X = E_0$. For the more sophisticated notion of infinite loop G-spaces where indexing over the representation ring of G is used, no such pathological behavior is possible.

As a final comment we mention that the theory of equivariant minimal models has been used by my collaborators and myself to obtain aplications of a more geometric nature, like the classification of a large class of G-manifolds up to finite ambiguity and the equivariant formality of G-Kähler manifolds.

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CHAPTER IV

Smith Theory

1. Smith theory via Bredon cohomology

We shall explain two approaches to the classical results of P.A. Smith. We begin with the statement. Let G be a finite p-group and let X be a finite dimensional G-CW complex such that $H^*(X;\mathbb{F}_p)$ is a finite dimensional vector space, where $\mathbb{F}_p$ denotes the field with p elements. All cohomology will have coefficients in $\mathbb{F}_p$ here.

THEOREM 1.1. If X is a mod p cohomology n-sphere, then X^G is empty or is a mod p cohomology m-sphere for some $m \leq n$. If p is odd, then $n-m$ is even and X^G is non-empty if n is even.

If H is a non-trivial normal subgroup of G, then $X^G = (X^H)^{G/H}$. By induction on the order of G, Theorem 1.1 will be true in general if it is true when $G = \mathbb{Z}/p$ is the cyclic group of order p. Our first proof is an almost trivial exercise in the use of Bredon cohomology. We restrict attention to $G = \mathbb{Z}/p$, but we do not assume that X is a mod p cohomology sphere until we put things together at the end. Observe that an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of coefficient systems give rise to a long exact sequence

(1.2)
$$\cdots \longrightarrow H^q_G(X;L) \longrightarrow H^q_G(X;M) \longrightarrow H^q_G(X;N) \longrightarrow H^{q+1}_G(X;L) \longrightarrow \cdots .$$

Let $FX = X/X^G$. The action of G on FX is free away from the basepoint. There are coefficient systems L, M, and N such that

$$H^q_G(X;L) \cong \tilde{H}^q(FX/G),$$
$$H^q_G(X;M) \cong H^q(X),$$

and

$$H^q_G(X;N) \cong H^q(X^G).$$

To determine L, M, and N, we need only calculate the right sides when $q=0$ and X is an orbit, that is, $X = G$ or $X = *$. We find:

$$\begin{aligned} L(G) &= \mathbb{F}_p & L(*) &= 0 \\ M(G) &= \mathbb{F}_p[G] & M(*) &= \mathbb{F}_p \\ N(G) &= 0 & N(*) &= \mathbb{F}_p. \end{aligned}$$

Let I be the augmentation ideal of the group ring $\mathbb{F}_p[G]$, and let I^n denote both the n^{th} power of I and the coefficient system whose value on G is I^n and whose value on $*$ is zero. Then $I^{p-1} = L$. It is easy to check that we have exact sequences of coefficient systems

$$0 \longrightarrow I \longrightarrow M \longrightarrow L \oplus N \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow M \longrightarrow I \oplus N \longrightarrow 0.$$

These exact sequences coincide if $p=2$. By (1.2), they give rise to long exact sequences

$$\cdots \longrightarrow H^q_G(X;I) \longrightarrow H^q(X) \longrightarrow \tilde{H}^q(FX/G) \oplus H^q(X^G) \longrightarrow H^{q+1}_G(X;I) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \tilde{H}^q(FX/G) \longrightarrow H^q(X) \longrightarrow H^q_G(X;I) \oplus H^q(X^G) \longrightarrow \tilde{H}^{q+1}(FX/G) \longrightarrow \cdots .$$

Define

$$a_q = \dim \tilde{H}^q(FX/G),\ \bar{a}_q = \dim H^q_G(X;I),\ b_q = \dim H^q(X),\ c_q = \dim H^q(X^G).$$

Note that $a_q = \bar{a}_q$ if $p=2$. We read off the inequalities

$$a_q + c_q \le b_q + \bar{a}_{q+1} \quad \text{and} \quad \bar{a}_q + c_q \le b_q + a_{q+1}.$$

Iteratively, these imply the following inequality for $q \ge 0$ and $r \ge 0$.

$$(1.3) \qquad a_q + c_q + c_{q+1} + \cdots + c_{q+r} \le b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1},$$

where r is odd if $p > 2$. In particular, with $q = 0$ and r large,

$$\sum c_q \leq \sum b_q. \tag{1.4}$$

Using the further short exact sequences

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow L \longrightarrow 0, 1 \leq n \leq p-1,$$

we can also read off the the Euler characteristic formula

$$\chi(X) = \chi(X^G) + p\tilde{\chi}(FX/G). \tag{1.5}$$

First proof of Theorem 1.1. Here $\sum b_q = 2$, hence $\sum c_q \leq 2$. The case $\sum c_q = 1$ is ruled out by the congruence $\chi(X) \equiv \chi(X^G) \mod p$; when $p > 2$, this congruence also implies that $n - m$ is even and that X^G is non-empty if n is even. Taking $q = n+1$ and r large in (1.3), we see that m cannot be greater than n. □

J. P. May. A generalization of Smith theory. Proc. Amer. Math. Soc. 101 (1987), 728-730.
P. A. Smith. Transformations of finite period. Annals of Math. 39 (1938), 127-164.

2. Borel cohomology, localization, and Smith theory

Let EG be a free contractible G-space. For a G-space X, the Borel construction on X is the orbit space $EG \times_G X$ and the Borel homology and cohomology of X (with coefficients in an Abelian group A) are defined to be the nonequivariant homology and cohomology of this space. For reasons to be made clear later, the Borel construction is also called the "homotopy orbit space" and is sometimes denoted X_{hG}. People not focused on equivariant algebraic topology very often refer to Borel cohomology as "equivariant cohomology." We can relate it to Bredon cohomology in a simple way. Let $\underline{A}$ denote the constant coefficient system at A. Since the orbit spaces $(G/H)/G$ are points, we see immediately from the axioms that $H^*_G(X; \underline{A})$ is isomorphic to $H^*(X/G; A)$, and similarly in homology. Therefore

$$H_*(EG \times_G X; A) \cong H^q_G(EG \times X; \underline{A}) \quad \text{and} \quad H^*(EG \times_G X; A) \cong H^*_G(EG \times X; \underline{A}).$$

Observe that the Borel cohomology of a point is the cohomology of the classifying space $BG = EG/G$. In this section, we shall use the notation

$$H^*_G(X) = H^*(EG \times_G X),$$

standard in much of the literature.

Here we fix a prime p and understand mod p coefficients. If X is a based G-space, we let $\tilde{H}^*_G(X)$ be the kernel of $H^*_G(X) \longrightarrow H^*_G(*) = H^*(BG)$. Equivalently,

$$\tilde{H}^*_G(X) = \tilde{H}^*(EG_+ \wedge_G X).$$

Because G acts freely on EG, it acts freely on $EG \times X$. Therefore, by the Whitehead theorem, if $f : X \longrightarrow Y$ is a G-map between G-CW complexes that is a nonequivariant homotopy equivalence, then

$$1 \times f : EG \times X \longrightarrow EG \times Y$$

is a G-homotopy equivalence and therefore

$$1 \times_G f : EG \times_G X \longrightarrow EG \times_G Y$$

is a homotopy equivalence. At first sight, it seems unreasonable to expect $EG \times_G X$ to carry much information about X^G, but it does.

We now assume that G is an elementary Abelian p-group, $G = (\mathbb{Z}/p)^n$ for some n, and that X is a finite dimensional G-CW complex. We shall describe how to use Borel cohomology to determine the mod p cohomology of X^G as an algebra over the Steenrod algebra, and we shall sketch another proof of Theorem 1.1. Our starting point is the localization theorem.

Since $G = (\mathbb{Z}/p)^n$, $H^*(BG)$ is a polynomial algebra on n generators of degree one if $p = 2$ and is the tensor product of an exterior algebra on n generators of degree one and the polynomial algebra on their Bocksteins if $p > 2$. Let S be the multiplicative subset of $H^*(BG)$ generated by the non-zero elements of degree one if $p = 2$ and by the non-zero images of Bocksteins of degree two if $p > 2$.

Theorem 2.1 (Localization). For a finite dimensional G-CW complex X, the inclusion $i : X^G \longrightarrow X$ induces an isomorphism

$$i^* : S^{-1}H^*_G(X) \longrightarrow S^{-1}H^*_G(X^G).$$

Proof. Let $FX = X/X^G$. By the cofiber sequence $X^G_+ \longrightarrow X_+ \longrightarrow FX$, it suffices to show that $S^{-1}\tilde{H}^*_G(FX) = 0$. Here FX is a finite dimensional G-CW complex and $(FX)^G = *$. By induction up skeleta, it suffices to show that $S^{-1}\tilde{H}^*_G(Y) = 0$ when Y is a wedge of copies of $G/H_+ \wedge S^q$ for some $H \neq G$, and such a wedge can be rewritten as $Y = G/H_+ \wedge K$, where K is a wedge of copies of S^q. Since $EG \times_G (G/H) = EG/H$ is a model for BH, we see that $EG_+ \wedge_G Y = BH_+ \wedge K$. At least one element of S restricts to zero in $H^*(BH)$, and this implies that $S^{-1}\tilde{H}^*_G(Y) = 0$.

Localization theorems of this general sort appear ubiquitously in equivariant theory. As here, the proofs of such results reduce to the study of orbits by general nonsense arguments, and the specifics of the situation are then used to determine what happens on orbits. When $n = 1$, we can be a little more precise.

LEMMA 2.2. If $G = \mathbb{Z}/p$ and $\dim X = r$, then $i^* : H^q_G(X) \longrightarrow H^q_G(X^G)$ is an isomorphism for $q > r$.

PROOF. It suffices to show that $\tilde{H}^*_G(FX) = 0$ for $q > r$. Since FX is G-free away from its basepoint, the projection $EG_+ \longrightarrow S^0$ induces a G-homotopy equivalence $EG_+ \wedge FX \longrightarrow FX$ and therefore a homotopy equivalence $EG_+ \wedge_G FX \longrightarrow FX/G$. Obviously $\dim(FX/G) \le r$. □

Since G acts trivially on X^G, $EG \times_G X^G = BG \times X^G$.

SECOND PROOF OF THEOREM 1.1. Take $G = \mathbb{Z}/p$ and let X be a mod p homology n-sphere. We assume that X^G is non-empty. The Serre spectral sequence of the bundle $EG \times_G X \longrightarrow BG$ converges from

$$H^*(G; H^*(X)) = H^*(BG) \otimes H^*(X)$$

to $H^*_G(X)$. Since a fixed point of X gives a section, $E_2 = E_\infty$. Therefore $\tilde{H}^*_G(X)$ is a free $H^*(BG)$-module on one generator of degree n and, in high degrees, this must be isomorphic to

$$\tilde{H}^*_G(X^G) = H^*(BG_+ \wedge X^G) = H^*(BG) \otimes \tilde{H}^*(X^G).$$

By a trivial dimension count, this can only happen if X^G is a mod p cohomology m-sphere for some m. Naturality arguments from the $H^*(BG)$-module structure show that m must be less than n and must be congruent to n mod 2 if $p > 2$. To see that X^G is non-empty if $p > 2$ and n is even, one assumes that X^G is empty and deduces from the multiplicative structure of the spectral sequence that X cannot be finite dimensional. □

Returning to the context of the localization theorem, one would like to retrieve $H^*(X^G)$ algebraically from $S^{-1}H^*_G(X)$. As a matter of algebra, $S^{-1}H^*_G(X)$ inherits a structure of algebra over the mod p Steenrod algebra A from $H^*_G(X)$. However, it no longer satisfies the instability conditions that are satisfied in the cohomology of spaces. For any A-module M, the subset of elements that do satisfy these conditions form a submodule $\mathrm{Un}(M)$. Obviously the localization map

$$H^*(BG) \otimes H^*(X^G) \cong H^*_G(X^G) \longrightarrow S^{-1}H^*_G(X^G) \cong S^{-1}H^*_G(X)$$

takes values in $\mathrm{Un}(S^{-1}H^*_G(X))$. By a purely algebraic analysis, using basic information about the Steenrod operations, Dwyer and Wilkerson proved the following remarkable result. (They assume that X is finite, but the argument still works when X is finite dimensional.)

Theorem 2.3. For any elementary Abelian p-group G and any finite dimensional G-CW complex X,

$$H^*(BG) \otimes H^*(X^G) \longrightarrow \mathrm{Un}(S^{-1}H^*_G(X))$$

is an isomorphism of A-algebras and $H^*(BG)$-modules. Therefore

$$H^*(X^G) \cong \mathbb{F}_p \otimes_{H^*(BG)} \mathrm{Un}(S^{-1}H^*_G(X)).$$

We will come back to this point when we talk about the Sullivan conjecture.

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CHAPTER V

Categorical Constructions; Equivariant Applications

1. Coends and geometric realization

We pause to introduce some categorical and topological constructs that are used ubiquitously in both equivariant and nonequivariant homotopy theory. They will be needed in a number of later places. We are particularly interested in homotopy colimits. These are examples of geometric realizations of spaces, which in turn are examples of coends, which in turn are examples of coequalizers.

Let Λ be a small category and let $\mathscr{C}$ be a category that has all colimits. Write $\coprod$ for the categorical coproduct in $\mathscr{C}$. The coequalizer $C(f, f')$ of maps $f, f' : X \longrightarrow Y$ is a map $g : Y \longrightarrow C(f, f')$ such that $gf = gf'$ and g is universal with this property. It can be constructed as the pushout in the following diagram, where $\nabla = 1 + 1$ is the folding map:

$$\begin{tikzcd} X \coprod X \arrow[r, "f+f'"] \arrow[d, "\nabla"'] & Y \arrow[d, "g"] \\ X \arrow[r] & C(f, f'). \end{tikzcd}$$

Coends are categorical generalizations of tensor products. Given a functor $F : \Lambda^{op} \times \Lambda \longrightarrow \mathscr{C}$, the coend

$$\int^{\Lambda} F(n, n)$$

is defined to be the coequalizer of the maps

$$f, f' : \coprod_{\phi : m \to n} F(n, m) \longrightarrow \coprod F(n, n)$$

whose restrictions to the ϕth summand are

$$F(\phi,\mathrm{id}) : F(n,m) \longrightarrow F(m,m) \ \text{ and } \ F(Id,\phi) : F(n,m) \longrightarrow F(n,n),$$

respectively. It satisfies a universal property like that of tensor products. If the objects of $F(n,n)$ have points that can be written in the form of "tensors" $x \otimes y$, then the coend is obtained from the coproduct of the $F(n,n)$ by identifying $x\phi \otimes y$ with $x \otimes \phi y$ whenever this makes sense. Here ϕ is a map in Λ, contravariant actions are written from the right, and covariant actions are written from the left.

Dually, if $\mathscr{C}$ has limits, a functor $F : \Lambda^{op} \times \Lambda \longrightarrow \mathscr{C}$ has an end

$$\int^{\Lambda} F(n,n).$$

It is defined to be the equalizer, $E(f,f')$, of the maps

$$f, f' : \prod F(n) \longrightarrow \prod_{\phi : m \to n} F(m,n)$$

whose projections to the ϕth factor are

$$F(\mathrm{id},\phi) : F(m,m) \longrightarrow F(m,n) \ \text{ and } \ F(\phi,\mathrm{id}) : F(n,n) \longrightarrow F(m,n).$$

Recall that a simplicial object in a category $\mathscr{C}$ is a contravariant functor $\triangle \longrightarrow \mathscr{C}$, where $\triangle$ is the category of sets $\underline{n} = \{0,1,2,\ldots,n\}$ and monotonic maps. Using the usual face and degeneracy maps, we obtain a covariant functor $\triangle_*\colon \triangle \longrightarrow \mathscr{U}$ that sends $\underline{n}$ to the standard topological n-simplex $\triangle_n$. For a simplicial space $X_* : \triangle \longrightarrow \mathscr{U}$, we have the product functor

$$X_* \times \triangle_*\colon \triangle^{op} \times \triangle \longrightarrow \mathscr{U}.$$

Define the geometric realization of X_* to be the coend

$$|X_*| = \int^{\triangle} X_n \times \triangle_n. \tag{1.1}$$

If X_* is a simplicial based space, so that all its face and degeneracy maps are basepoint preserving, then all points of each subspace $\{*\} \times \triangle_n$ are identified to the point $(*,1) \in X_0 \times \triangle_0$ in the construction of $|X_*|$, hence

$$|X_*| = \int^{\triangle} X_n \wedge (\triangle_n)_+. \tag{1.2}$$

If X_* is a simplicial G-space, then $|X_*|$ inherits a G-action such that

$$|X_*|^H = |X_*^H| \text{ for all } H \subset G. \tag{1.3}$$

S. Mac Lane. Categories for the Working Mathematician. Springer. 1976.
J. P. May. The Geometry of Iterated Loop Spaces (§11). Springer Lecture Notes Vol 271. 1972.

2. Homotopy colimits and limits

Let $\mathscr{D}$ be any small topological category. We understand $\mathscr{D}$ to have a discrete object set and to have spaces of maps $d \to d'$ such that composition is continuous. Let $B_n(\mathscr{D})$ be the set of n-tuples $\underline{f} = (f_1, \dots, f_n)$ of composable arrows of $\mathscr{D}$, depicted

$$d_0 \xleftarrow{f_1} d_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} d_n.$$

Here $B_0(\mathscr{D})$ is the set of objects of $\mathscr{D}$ and $B_n(\mathscr{D})$ is topologized as a subspace of the n-fold product of the total morphism space $\coprod \mathscr{D}(d, d')$. With zeroth and last face given by deleting the zeroth or last arrow of n-tuples $\underline{f}$ (or by taking the source or target of f_1 if $n = 1$) and with the remaining face and degeneracy operations given by composition or by insertion of identity maps in the appropriate position, $B_*(\mathscr{D})$ is a simplicial set called the nerve of $\mathscr{D}$. Its geometric realization is the classifying space $B\mathscr{D}$. If $\mathscr{D}$ has a single object d, then $G = \mathscr{D}(d, d)$ is a topological monoid (= associative Hopf space with unit) and $B\mathscr{D} = BG$ is its classifying space.

We can now define the two-sided categorical bar construction. It will specialize to give homotopy colimits. Let $T : \mathscr{D} \longrightarrow \mathscr{U}$ be a continuous contravariant functor. This means that each $T(d)$ is a space and each function $T : \mathscr{D}(d, d') \longrightarrow \mathscr{U}(T(d'), T(d))$ is continuous. Let $S : \mathscr{D} \longrightarrow \mathscr{U}$ be a continuous covariant functor. We define

$$B(T, \mathscr{D}, S) = |B_*(T, \mathscr{D}, S)|. \tag{2.1}$$

Here $B_*(T, \mathscr{D}, S)$ is the simplicial space whose set of n-simplices is

$$\{(t, \underline{f}, s) | t \in T(d_0), \underline{f} \in B_n(\mathscr{D}), \text{ and } s \in S(d_n)\},$$

topologized as a subspace of the product $(\coprod T(d)) \times (\coprod \mathscr{D}(d, d'))^n \times (\coprod S(d))$; $B_0(T, \mathscr{D}, S) = \coprod T(d) \times S(d)$. The zeroth and last face use the evaluation of the functors T or S; the remaining faces and the degeneracies are defined like those of $B_*\mathscr{D}$.

Since the coend of $T \times S : \mathscr{D}^{op} \times \mathscr{D} \longrightarrow \mathscr{U}$ is exactly the coequalizer of $d_0, d_1 : B_1(T, \mathscr{D}, S) \longrightarrow B_0(T, \mathscr{D}, S)$, we obtain a natural map

$$\epsilon : B(T, \mathscr{D}, S) \longrightarrow \int^{\mathscr{D}} T(d) \times S(d) \equiv T \otimes_{\mathscr{D}} S. \tag{2.2}$$

It is obtained by using iterated compositions to map$B_*(T,\mathscr{D},S)$to the constant simplicial space at the cited coend, which we denote by $T\otimes_{\mathscr{D}} S$.

Let $\mathscr{D}_e$ be the covariant functor represented by an object e of $\mathscr{D}$, so that $\mathscr{D}_e(d)=\mathscr{D}(e,d)$. Then ϵ reduces to a map

$$\varepsilon : B(T,\mathscr{D},\mathscr{D}_e) \longrightarrow T(e),$$

and this map is a homotopy equivalence. In fact, using the identity map of e, we obtain an inclusion $\eta : T(e) \longrightarrow B(T,\mathscr{D},\mathscr{D}_e)$ such that $\varepsilon\eta = 1$ and a simplicial deformation $\eta\varepsilon \simeq \mathrm{id}$. There is a left–right symmetric analogue.

If the functor S takes values in $G\mathscr{U}$, then $B_*(T,\mathscr{D},S)$ is a simplicial G-space and $B(T,\mathscr{D},S)$ is a G-space such that

$$B(T,\mathscr{D},S)^H = B(T,\mathscr{D},S^H). \tag{2.3}$$

We define the homotopy colimit of our covariant functor S by

$$\operatorname{Hocolim} S = B(*,\mathscr{D},S), \tag{2.4}$$

where $* : \mathscr{D} \longrightarrow \mathscr{U}$ is the trivial functor to a 1-point space. Here the coend on the right of (1.5) is exactly the ordinary colimit of S. Thus we have

$$\varepsilon : \operatorname{hocolim} S \longrightarrow \operatorname{colim} S. \tag{2.5}$$

When G is a group regarded as category with a single object and X is a (left) G-space regarded as a covariant functor, the homotopy colimit of X is the "homotopy orbit space" $EG\times_G X = X/hG$, and ε is the natural map $X/hG \longrightarrow X/G$.

Our preferred definition of homotopy limits is precisely dual. We have a cosimplicial space $C_*(T,\mathscr{D},S)$, the two-sided cobar construction. Its set of n-cosimplices is the product over all $\underline{f}\in B_n(\mathscr{D})$ of the spaces $T(d_0)\times S(d_n)$, topologized as a subspace of $\operatorname{Map}(B_n(\mathscr{D}),\coprod T(d)\times S(d'))$. The $\underline{f}$th coordinates of the cofaces and codegeneracies with target $C_n(T,\mathscr{D},S)$ are obtained by projecting onto the coordinate of their source that is indexed by the corresponding face or degeneracy applied to $\underline{f}$, except that, for the zeroth and last coface, we must compose with

$$T(f_1)\times \mathrm{id} : T(d_0)\times S(d_n) \longrightarrow T(d_1)\times S(d_n)$$

or

$$\mathrm{id}\times S(f_n) : T(d_0)\times S(d_n) \longrightarrow T(d_0)\times S(d_{n-1}).$$

We define the geometric realization, or totalization, "TotY_*" of a cosimplicial space Y_* to be the end

$$\mathrm{Tot}Y_* = \int_{\triangle} \mathrm{Map}(\triangle_n, Y_n). \tag{2.6}$$

Here we are using the evident functor $\triangle^{op} \times \triangle \longrightarrow \mathscr{U}$ that sends (m,n) to $\mathrm{Map}(\triangle_m, Y_n)$. If Y_* takes values in based spaces, we may rewrite this as

$$\mathrm{Tot}Y_* = \int_{\triangle} F((\triangle_n)_+, Y_n). \tag{2.7}$$

We then define

$$C(T, \mathscr{D}, S) = \mathrm{Tot}C_*(T, \mathscr{D}, S), \tag{2.8}$$

and we have a natural map

$$\eta : \int^{\mathscr{D}} T(d) \times S(d) \longrightarrow C(T, \mathscr{D}, S). \tag{2.9}$$

We define the homotopy limit of our contravariant functor $T : \mathscr{D} \longrightarrow \mathscr{U}$ to be

$$\mathrm{Holim}\, T = \mathrm{Tot}C_*(T, \mathscr{D}, *), \tag{2.10}$$

and we see that η specializes to give a natural map

$$\eta : \lim T \longrightarrow \mathrm{holim}\, T. \tag{2.11}$$

When G is a group regarded as a category with a single object and X is a (right) G-space regarded as a contravariant functor, the homotopy limit of X is the "homotopy fixed point space" of G-maps $EG \longrightarrow X$,

$$\mathrm{Map}_G(EG, X) = \mathrm{Map}(EG, X)^G = X^{hG},$$

and η is the natural map $X^G \longrightarrow X^{hG}$ that sends a fixed point to the constant function at that point. This map is the object of study of the Sullivan conjecture.

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J. P. May. Classifying spaces and fibrations (§12). Memoirs Amer. Math. Soc. No. 155, 1975.

3. Elmendorf's theorem on diagrams of fixed point spaces

Recall that $\mathscr{G}$ is the category of orbit spaces. We shall regard $\mathscr{G}$ as a topological category with a discrete set of objects. We write $[G/H]$ for a typical object, to avoid confusing it with the G-space G/H. The space of morphisms $[G/H] \longrightarrow [G/K]$ is the space of G-maps $G/H \longrightarrow G/K$, and this space may be identified with $(G/K)^H$. Define a $\mathscr{G}$-space to be a continuous contravariant functor $\mathscr{G} \longrightarrow \mathscr{U}$. A map of $\mathscr{G}$-spaces is a natural transformation, and we write $\mathscr{G}\mathscr{U}$ for the category of G-spaces. We shall compare this category with $G\mathscr{U}$. We have already observed that a G-space X gives a $\mathscr{G}$-space X^*, and we write

$$\Phi : G\mathscr{U} \longrightarrow \mathscr{G}\mathscr{U}$$

for the functor that sends X to X^*. We wish to determine how much information the functor Φ loses.

By the definition of $\underline{C}_*(X)$, it is clear that the ordinary homology and cohomology of X depend only on ΦX. If $T : \mathscr{G} \longrightarrow \mathscr{U}$ is a $\mathscr{G}$-space such that each $T(G/H)$ is a CW-complex and each $T(G/K) \longrightarrow T(G/H)$ is a cellular map, then we can define $H^*_G(T; M)$ exactly as we defined $H^*_G(X; M)$. Note, however, that unless G is discrete, X^H will not inherit a structure of a CW-complex from a G-CW complex X. Indeed, for compact Lie groups, we saw that it was not quite the functor X^* that was relevant to ordinary cohomology, but rather the functor that sends G/H to X^H/W_0H.

There is an obvious way that $\mathscr{G}$-spaces determine G-spaces.

Lemma 3.1. Define a functor $\Theta : \mathscr{G}\mathscr{U} \longrightarrow G\mathscr{U}$ by $\Theta T = T(G/e)$, with the G-maps $G/e \longrightarrow G/e$ inducing the action. Then Θ is left adjoint to Φ,

$$\mathscr{G}\mathscr{U}(T, \Phi X) \cong G\mathscr{U}(\Theta T, X).$$

Proof. Clearly $\Theta\Phi X = X$. The quotient map $G \longrightarrow G/H$ induces a map $\eta : T(G/H) \longrightarrow T(G/e)^H$, and these maps together specify a natural map $\eta : T \longrightarrow \Phi\Theta T$. Passage from $\phi : T \longrightarrow \Phi X$ to $\Theta\phi : \Theta T \longrightarrow X$ is a bijection whose inverse sends $f : \Theta T \longrightarrow X$ to $\Phi f \circ \eta$. $\square$

The following result of Elmendorf shows that $\mathscr{G}$-spaces determine G-spaces in a less obvious way. In fact, up to homotopy, any $\mathscr{G}$-space can be realized as the fixed point system of a G-space and, up to homotopy, the functor Φ has a right adjoint as well as a left adjoint. Note that we can form the product $T \times K$ of a $\mathscr{G}$-space

T and a space K by setting $(T \times K)(G/H) = T(G/H) \times K$. In particular, $T \times I$ is defined, and we have a notion of homotopy between maps of $\mathscr{G}$-spaces. Write $[T, T']_{\mathscr{G}}$ for the set of homotopy classes of maps $T \longrightarrow T'$.

THEOREM 3.2 (ELMENDORF). There is a functor $\Psi : \mathscr{G}\mathscr{U} \longrightarrow G\mathscr{U}$ and a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{id}$ such that each $\varepsilon : (\Psi T)^H \longrightarrow T(G/H)$ is a homotopy equivalence. If X has the homotopy type of a G-CW complex, then there is a natural bijection

$$[X, \Psi T]_G \cong [\Phi X, T]_{\mathscr{G}}.$$

PROOF. Let $S : \mathscr{G} \longrightarrow G\mathscr{U}$ be the covariant functor that sends the object $[G/H]$ to the G-space G/H. On morphisms, it is given by identity maps

$$\mathscr{G}([G/H], [G/K]) \longrightarrow G\mathscr{U}(G/H, G/K).$$

For a $\mathscr{G}$-space T, define ΨT to be the G-space $B(T, \mathscr{G}, S)$. We have

$$S^H[G/K] = (G/K)^H = G\mathscr{U}(G/H, G/K) = \mathscr{G}([G/H], [G/K]),$$

and (2.2) and (2.3) give homotopy equivalences $\varepsilon : (\Psi T)^H \longrightarrow T(G/H)$ that define a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{id}$. Clearly

$$\Theta\varepsilon : \Psi T = \Theta\Phi\Psi T \longrightarrow \Theta T$$

is a weak equivalence of G-spaces for any T. With $T = \Phi X$, this gives a weak equivalence $\Theta\varepsilon : \Psi\Phi X \longrightarrow X$. We can check that $\Psi\Phi X$ has the homotopy type of a G-CW complex if X does. Therefore $\Theta\varepsilon$ is an equivalence, and we choose a homotopy inverse $(\Theta\varepsilon)^{-1}$. Define

$$\alpha : [X, \Psi T]_G \longrightarrow [\Phi X, T]_{\mathscr{G}} \quad \text{and} \quad \beta : [\Phi X, T]_{\mathscr{G}} \longrightarrow [X, \Psi T]_G$$

by $\alpha(f) = \varepsilon \circ \Phi f$ and $\beta(\phi) = \Psi\phi \circ (\Theta\varepsilon)^{-1}$. Easy diagram chases show that $\alpha\beta(\phi) \simeq \phi$ and $\beta\alpha(f) \simeq (\Psi\varepsilon) \circ (\Theta\varepsilon)^{-1} \circ f$. Since $\Psi\varepsilon$ is a weak equivalence, the Whitehead theorem gives that $\beta\alpha$ is a bijection. It follows formally that α and β are inverse bijections. $\square$

A. D. Elmendorf. Systems of fixed point sets. Trans. Amer. Math. Soc. 277(1983), 275-284.

4. Eilenberg-Mac Lane G-spaces and universal $\mathscr{F}$-spaces

We give some important applications of this construction, starting with the construction of equivariant Eilenberg-Mac Lane spaces that we promised earlier.

EXAMPLE 4.1. Let B be the classifying space functor from topological monoids to spaces. It is product-preserving, and it therefore gives an Abelian topological group when applied to an Abelian topological group. If π is a discrete Abelian group, then the n-fold iterate $B^n\pi$ is a $K(\pi, n)$. A coefficient system $M : h\mathscr{G} \longrightarrow \mathscr{A}b$ may be regarded as a continuous functor $\mathscr{G} \longrightarrow \mathscr{U}$ (with discrete values). We may compose with B^n to obtain a $\mathscr{G}$-space $B^n \circ M$. In view of the equivalences $\varepsilon : \Psi(B^n \circ M)^H \longrightarrow K(M(G/H), n)$, $\Psi(B^n \circ M)$ is a $K(M, n)$. Theorem 3.2 gives a homotopical description of ordinary cohomology in terms of maps of $\mathscr{G}$-spaces:

$$\tilde{H}^n_G(X; M) \cong [X, K(M, n)]_G \cong [\Phi X, B^n \circ M]_{\mathscr{G}}.$$

In interpreting this, one must remember that the right side concerns homotopy classes of genuine natural transformations $\Phi X \longrightarrow B^n M$, and not just natural transformations in the homotopy category. The latter would be directly computable in terms of nonequivariant comology.

EXAMPLE 4.2. If M is a contravariant functor from $h\mathscr{G}$ to (not necessarily Abelian) groups, then we can regard $B \circ M$ as a $\mathscr{G}$-space and so obtain an Eilenberg-Mac Lane G-space $K(M, 1) = \Psi(B \circ M)$.

EXAMPLE 4.3. A set-valued functor M on $h\mathscr{G}$ is the same thing as a continuous set-valued functor on $\mathscr{G}$. Applying Ψ to such an M, we obtain an Eilenberg-Mac Lane G-space $K(M, 0)$. Its fixed point spaces $K(M, 0)^H$ are homotopy equivalent to the discrete spaces $M(G/H)$, but the G-space $K(M, 0)$ generally has non-trivial cohomology groups in arbitrarily high dimension. For set-valued coefficient systems M and M', let $Nat_{\mathscr{G}}(M, M')$ be the set of natural transformations $M \longrightarrow M'$. Then Theorem 3.2 and the discreteness of M give isomorphisms

$$[X, K(M, 0)]_G \cong [\Phi X, M]_{\mathscr{G}} \cong Nat_{\mathscr{G}}(\underline{\pi}_0(X), M). \tag{4.4}$$

This may seem frivolous at first sight, but in fact the spaces $K(M, 0)$ are central to equivariant homotopy theory. For example, we shall see later that the isomorphisms just given specialize to give a classification theorem for equivariant bundles — and to reprove the classical classification of nonequivariant bundles. The relevant $K(M, 0)$'s are special cases of those in the following basic definition.

DEFINITION 4.5. A family $\mathscr{F}$ in G is a set of subgroups of G that is closed under subconjugacy: if $H \in \mathscr{F}$ and $g^{-1}Kg \subset H$, then $K \in \mathscr{F}$. An $\mathscr{F}$-space is a G-space all of whose isotropy groups are in $\mathscr{F}$. Define a functor $\underline{\mathscr{F}} : h\mathscr{G} \longrightarrow Sets$ by sending G/H to the 1-point set if $H \in \mathscr{F}$ and to the empty set if $H \notin \mathscr{F}$. Define the universal $\mathscr{F}$-space $E\mathscr{F}$ to be $\Psi\underline{\mathscr{F}}$. It is universal in the sense that, for an $\mathscr{F}$-space X of the homotopy type of a G-CW complex, there is one and, up to homotopy, only one G-map $X \longrightarrow E\mathscr{F}$. Define the classifying space of the family $\mathscr{F}$ to be the orbit space $B\mathscr{F} = E\mathscr{F}/G$.

In thinking about this example, it should be remembered that there are no maps from a non-empty set to the empty set. In particular, there are no G-maps $X \longrightarrow E\mathscr{F}$ if X is not an $\mathscr{F}$-space. This also shows that the functor $\underline{\mathscr{F}}$ only makes sense if the given set $\mathscr{F}$ of subgroups of G is a family. We augment the definition with the following relative version. It will become very important later.

DEFINITION 4.6. For a subfamily $\mathscr{F}$ of a family $\mathscr{F}'$, define $E(\mathscr{F}', \mathscr{F})$ to be the cofiber of the based G-map (unique up to homotopy) $E\mathscr{F}_+ \longrightarrow E\mathscr{F}'_+$. Let $\mathscr{A}\ell\ell$ be the family of all subgroups of G, and let $\tilde{E}\mathscr{F} = E(\mathscr{A}\ell\ell, \mathscr{F})$. Since $E\mathscr{A}\ell\ell$ is G-contractible, $\tilde{E}\mathscr{F}$ is equivalent to the unreduced suspension of $E\mathscr{F}$ with one of the cone points as basepoint. The space $(\tilde{E}\mathscr{F})^H$ is contractible if $H \in \mathscr{F}$ and is the two-point space S^0 if $H \notin \mathscr{F}$. For $\mathscr{F} \subset \mathscr{F}'$, the G-space $E(\mathscr{F}', \mathscr{F})$ is equivalent to $E\mathscr{F}'_+ \wedge \tilde{E}\mathscr{F}$.

The following observation will become valuable when we examine the structure of equivariant classifying spaces.

LEMMA 4.7. Let $\mathscr{F}$ be a family in G and H be a subgroup of G.

(a) Regarded as an H-space, $E\mathscr{F}$ is $E(\mathscr{F}|H)$, where

$$\mathscr{F}|H = \{K | K \in \mathscr{F} \text{ and } K \subset H\}.$$

(b) If $H \in \mathscr{F}$, then, regarded as a WH-space, $(E\mathscr{F})^H$ is $E(\mathscr{F}^H)$, where

$$\mathscr{F}^H = \{L | L = K/H \text{ for some } K \in \mathscr{F} \text{ such that } H \subset K \subset NH\}.$$

The classical example is $\mathscr{F} = \{e\}$. An $\{e\}$-space X is a G-space all of whose isotropy groups are trivial. That is, X is a free G-space. Then $EG \equiv E\{e\}$ is exactly the standard example of a free contractible G-space, and the quotient map $\pi : EG \longrightarrow BG$ is a principal G-bundle. Given the result that pullbacks of bundles along homotopic maps are homotopic, we have already proven that π is universal.

Indeed, if $p : E \longrightarrow B$ is a principal G-bundle, we have a unique homotopy class of G-maps $\tilde{f} : E \longrightarrow EG$. The map $f : B \longrightarrow BG$ that is obtained by passage to orbits from $\tilde{f}$ is the classifying map of p. Certainly p is equivalent to the bundle obtained by pulling π back along f.

When G is discrete, the ordinary homology and cohomology of the G-spaces $E\mathscr{F}$ admit descriptions as Ext groups, generalizing the classical identification of the homology and cohomology of groups with the homology and cohomology of $K(\pi, 1)$'s. This can be seen from the projectivity of the cellular chains $\underline{C}_*(E\mathscr{F})$ and inspection of definitions or by collapse of the universal coefficients spectral sequences. Write $\mathbb{Z}[\underline{\mathscr{F}}]$ for the free Abelian group functor composed with the functor $\underline{\mathscr{F}}$.

Proposition 4.8. Let G be discrete. For a covariant coefficient system N and a contravariant coefficient system M,

$$H_*^G(E\mathscr{F}; N) = \mathrm{Tor}_*^{\mathscr{G}}(\mathbb{Z}[\underline{\mathscr{F}}]; N) \quad \text{and} \quad H_G^*(E\mathscr{F}; M) = \mathrm{Ext}_{\mathscr{G}}^*(\mathbb{Z}[\underline{\mathscr{F}}]; M).$$

CHAPTER VI

The Homotopy Theory of Diagrams

by Robert J. Piacenza

1. Elementary homotopy theory of diagrams

A substantial portion of the homotopy, homology, and cohomology theory of G-spaces X depends only on the underlying diagram of fixed point spaces $\Phi X : \mathscr{G} \longrightarrow \mathscr{U}$. There is a vast and growing literature in which the homotopy theory of spaces is generalized to a homotopy theory of diagrams of spaces that are indexed on arbitrary small indexing categories. The purpose of this chapter is to outline this theory and to demonstrate the connection between diagrams and equivariant theory. A very partial list of sources for further reading is given at the end of this section.

Throughout the chapter, we let $\mathscr{U}$ be the cartesian category of compactly generated weak Hausdorff spaces and let J be a small topological category over $\mathscr{U}$ with discrete object space. Define $\mathscr{U}^J$ to be the category of continuous contravariant $\mathscr{U}$-valued functors on J. Its objects are called either diagrams or J-spaces; its morphisms, which are natural transformations, are called J-maps. Note that $\mathscr{U}^J$ is a topological category: its hom sets are spaces and composition is continuous.

Let I be the unit interval in $\mathscr{U}$. If X and Y are diagrams, then a homotopy from X to Y is a J-map $H : I \times X \longrightarrow Y$, where $I \times X$ is the diagram defined on objects $j \in |J|$ by $(I \times X)(j) = I \times X(j)$ and similarly for morphisms of J. In the usual way homotopy defines an equivalence relation on the J-maps that gives rise to the quotient homotopy category $h\mathscr{U}^J$. We denote the homotopy classes of J-maps from X to Y by $h\mathscr{U}^J(X,Y)$, abbreviated $h(X,Y)$. An isomorphism in

$h\mathscr{U}^J$ will be called a homotopy equivalence.

A J-map is called a J-cofibration if it has the J homotopy extension property, abbreviated $J - HEP$. The basic facts about cofibrations in $\mathscr{U}$ apply readily to J-cofibrations.

The following standard results for spaces are inherited by the category $\mathscr{U}^J$.

Theorem 1.1 (Invariance of pushouts). Suppose given a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] \arrow[drr, "\alpha"] & B \arrow[d] \arrow[drr, "\beta"] & & \\
X \arrow[r] \arrow[drr, "\gamma"'] & Y \arrow[drr, "\delta"] & A' \arrow[r, "f'"] \arrow[d, "i'"] & B' \arrow[d] \\
 & & X' \arrow[r] & Y'
\end{tikzcd}$$

in which i and i' are J-cofibrations, f and f' are arbitrary J-maps, α, β, and γ are homotopy equivalences, and the front and back faces are pushouts. Then the induced map δ on pushouts is also a homotopy equivalence.

Theorem 1.2 (Invariance of colimits over cofibrations). Suppose given a homotopy commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r, "i_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "i_1"] \arrow[d, "f^1"] & \cdots \arrow[r] & X^k \arrow[r, "i_k"] \arrow[d, "f^k"] & \cdots \\
Y^0 \arrow[r, "j_0"] & Y^1 \arrow[r, "j_1"] & \cdots \arrow[r] & Y^k \arrow[r, "j_k"] & \cdots
\end{tikzcd}$$

in $\mathscr{U}^J$ where the i_k and j_k are J-cofibrations and the f^k are homotopy equivalences. Then the map $\operatorname{colim}_k f^k : \operatorname{colim}_k X^k \longrightarrow \operatorname{colim}_k Y^k$ is a homotopy equivalence.

The reader will readily accept that other such standard results in the homotopy theory of spaces carry over directly to the homotopy theory of diagrams.

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2. Homotopy Groups

Let I^n be the topological n-cube and ∂I^n its boundary. For an object $j \in |J|$, let $\underline{j} \in \mathscr{U}^J$ denote the associated represented functor; its value on an object k is the space $\mathscr{U}^J(k, j)$.

Definition 2.1. By a pair (X, Y) in $\mathscr{U}^J$, we mean a J-space X together with a sub J-space Y. Morphisms of pairs are defined in the obvious way. Similar definitions apply to triples, n-ads, etc. Let $\phi : \underline{j} \longrightarrow Y$ be a morphism in $\mathscr{U}^J$. By the Yoneda lemma, ϕ is completely determined by the point $\phi(\mathrm{id}_j) = y_0 \in Y(j)$. For each $n \geq 0$, define

$$\pi_n^j(X, Y, \phi) = h((I^n, \partial I^n, \{0\}) \times \underline{j}, (X, Y, Y))$$

where $y_0 = \phi(id_j) \in Y(j)$ serves as a basepoint, and all homotopies are homotopies of triples relative to ϕ. The reader may formulate a similar definition for the absolute case $\pi_n^j(X, \phi)$. For $n = 0$ we adopt the convention that $I^0 = \{0, 1\}$ and $\partial I^0 = \{0\}$ and proceed as above. These constructions extend to covariant functors on $\mathscr{U}^J$. From now on, we shall often drop ϕ from the notation $\pi_n^j(X, Y, \phi)$.

The following proposition follows immediately from the Yoneda lemma.

Proposition 2.2. There are natural isomorphisms $\pi_n^j(X) \cong \pi_n(X(j))$ and $\pi_n^j(X, Y) \cong \pi_n(X(j), Y(j))$ that preserve the group structures when $n \geq 1$ (in the absolute case; the relative case requires $n \geq 2$).

As a direct consequence of Proposition 2.2 we obtain the usual long exact sequences.

PROPOSITION 2.3. For (X, Y) and j as in Definition 2.1, there exist natural boundary maps ∂ and long exact sequences

$$\cdots \longrightarrow \pi^j_{n+1}(X, Y) \xrightarrow{\partial} \pi^j_n(Y) \longrightarrow \pi^j_n(X) \longrightarrow \cdots \longrightarrow \pi^j_0(Y) \longrightarrow \pi^j_0(X)$$

of groups up to $\pi^j_1(Y)$ and pointed sets thereafter.

DEFINITION 2.4. A map $e : (X, Y) \longrightarrow (X', Y')$ of pairs in $\mathscr{U}^J$ is said to be an n-equivalence if $e(j) : (X(j), Y(j)) \longrightarrow (X'(j), Y'(j))$ is an n-equivalence in $\mathscr{U}$ for each $j \in |J|$. A map e is said to be a weak equivalence if it is an n-equivalence for each $n \geq 0$. Observe that e is an n-equivalence if for every $j \in |J|$ and $\phi : j \longrightarrow Y$, $e_* : \pi^j_p(X, Y, \phi) \longrightarrow \pi^j_p(X', Y', e\phi)$ is an isomorphism for $0 \leq p < n$ and an epimorphism for $p = n$. The reader may easily formulate similar definitions for J-maps $e : X \longrightarrow X'$ (the absolute case).

3. Cellular Theory

In this section we adapt May's preferred approach to the classical theory of CW complexes to develop a theory of J-CW complexes.

Let D^{n+1} be the topological $(n+1)$-disk and S^n the topological n-sphere. Of course, these spaces are homeomorphic to I^{n+1} and ∂I^{n+1} respectively. We shall construct cell complexes over J by the process of attaching cells of the form $D^{n+1} \times \underline{j}$ by attaching morphisms with domain $S^n \times \underline{j}$.

DEFINITION 3.1. A J-complex is an object X of $\mathscr{U}^J$ with a decomposition $X = \operatorname{colim}_{p \geq 0} X^p$ where

$$X^0 = \coprod_{\alpha \in A_0} D^{n_\alpha} \times \underline{j}_\alpha$$

and, inductively,

$$X^p = X^{p-1} \bigcup_f \Big(\coprod_{\alpha \in A_p} D^{n_\alpha} \times \underline{j}_\alpha \Big)$$

for some attaching J-map $f : \coprod_{\alpha \in A_p} S^{n_\alpha - 1} \times \underline{j}_\alpha \longrightarrow X^{p-1}$; here, for each $p \geq 0$, $\{j_\alpha \mid \alpha \in A_p\}$ is a set of objects of J. We call X a J-CW complex if X is a J-complex such that $n_\alpha = p$ for all $p \geq 0$ and $\alpha \in A_p$.

Now J-subcomplexes and relative J-complexes are defined in the obvious way. We adopt the standard terminology for CW-complexes for J-CW-complexes without further comment.

The following technical lemma reduces directly to its space level analog.

LEMMA 3.2. Suppose that $e : Y \longrightarrow Z$ is an n-equivalence. Then we can complete the following diagram in $\mathscr{U}^J$:

$$\begin{tikzcd}
S^{n-1} \times \underline{j} \arrow[rr, "i_0"] \arrow[dd] & & S^{n-1} \times I \times \underline{j} \arrow[dl, "h"'] \arrow[dd] & & S^{n-1} \times \underline{j} \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"'] & \\
D^n \times \underline{j} \arrow[ur, "f"] \arrow[rr, "i_0"'] & & D^n \times I \times \underline{j} \arrow[ul, dashed, "\tilde{h}"'] & & D^n \times \underline{j}. \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

From here, we proceed exactly as in I§3 to obtain the following results.

THEOREM 3.3 (J-HELP). If (X, A) is a relative J-CW complex of dimension $\leq n$ and $e : Y \longrightarrow Z$ is an n-equivalence, then we can complete the following diagram in $\mathscr{U}^J$:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & A \times I \arrow[dl] \arrow[dd] & & A \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur] \arrow[rr] & & X \times I \arrow[ul, dashed] & & X. \arrow[ll] \arrow[ul, dashed]
\end{tikzcd}$$

THEOREM 3.4 (WHITEHEAD). Let $e : Y \longrightarrow Z$ be an n-equivalence and X be a J-CW complex. Then $e_* : h(X, Y) \longrightarrow h(X, Z)$ is a bijection if X has dimension less than n and a surjection if X has dimension n. If e is a weak equivalence, then $e_* : h(X, Y) \longrightarrow h(X, Z)$ is a bijection for all X.

COROLLARY 3.5. If $e : Y \longrightarrow Z$ is an n-equivalence between J-CW complexes of dimension less than n, then e is a J-homotopy equivalence. If e is a weak equivalence between J-CW complexes, then e is a J-homotopy equivalence.

THEOREM 3.6 (CELLULAR APPROXIMATION). Let (X, A) and (Y, B) be relative J-CW complexes, (X', A') be a subcomplex of (X, A), and $f : (X, A) \longrightarrow (Y, B)$ be a map of pairs in $\mathscr{U}^J$ whose restriction to (X', A') is cellular. Then f is homotopic rel $X' \cup A$ to a cellular map $g : (X, A) \longrightarrow (Y, B)$.

COROLLARY 3.7. Let X and Y be J-CW complexes. Then any J-map $f : X \longrightarrow Y$ is homotopic to a cellular J-map, and any two homotopic cellular J-maps are cellularly homotopic.

Next we discuss the local properties of J-CW complexes. First we develop some preliminary concepts. Let X be a J-space and, for each $j \in |J|$, let $t_j : X(j) \longrightarrow \operatorname{colim}_J X$ be the natural map of $X(j)$ into the colimit. Observe that, for each morphism $s : i \longrightarrow j$ of J, $t_j = t_i \circ X(s)$. For each subspace $A \subseteq \operatorname{colim}_J X$, we define $\check{A}(j) = t_j^{-1}(A)$; for each morphism $s : i \longrightarrow j$ of J, we define $\check{A}(s) : \check{A}(j) \longrightarrow \check{A}(i)$ to be the restriction of $X(s)$. (As usual, we apply the k-ification functor to ensure that all spaces defined above are compactly generated.) One quickly checks that $\check{A}$ is a J-space, that $\operatorname{colim}_J \check{A} = A$, and that there is a natural inclusion $\check{A} \longrightarrow X$. To simplify notation, we write

$$X/J = \operatorname{colim}_J X$$

from now on.

Definition 3.8. A pair $(X, \check{A})$ is a J-neighborhood retract pair (abbreviated J-NR pair) if there exists an open subset U of X/J such that $A \subseteq U$ and a retraction $r : \check{U} \longrightarrow \check{A}$. A pair $(X, \check{A})$ is a J-neighborhood deformation retract pair (abbreviated J-NDR pair) if $(X, \check{A})$ is a J-NR pair and the J-map r is a J-deformation retraction.

Let X be a J-CW complex. The functor colim_J sends the J-space $A \times \underline{j}$ determined by a space A and object j to the space A, and it preserves colimits. Therefore the cellular decomposition of X determines a natural structure of a CW complex on X/J; its attaching maps are the images under the functor colim_J of the attaching J-maps of X. One may also check that if A is a subcomplex of X/J, then $\check{A}$ has a natural structure of a subcomplex of X. In particular, if A^p is the p-skeleton of X/J, then $\check{A}^p = X^p$ is the p-skeleton of X.

Proposition 3.9 (Local contractibility). Let X be a J-CW complex and $A = \{a\}$ be a point of X/J. Then there is an object $j \in |J|$ such that $\check{A} \cong \underline{j}$, and $(X, \check{A})$ is a J-NDR pair.

Proof. Let a be in the p-skeleton but not in the $(p-1)$-skeleton of X/J. Then there is a unique attaching map $f : S^{p-1} \times \underline{j} \longrightarrow X^{p-1}$ such that a is in the interior of D^p. It follows that $\check{A} \cong \underline{j}$. To construct the required neighborhood U, first take an open ball U_1 contained in the interior of B_p and centered at a. Then U_1 is a neighborhood in $(X/J)^p$ that contracts to A. One then extends U_1 inductively cell by cell by the usual space level procedure to construct the required neighborhood U. $\square$

PROPOSITION 3.10. Let X be a J-CW complex and A be a subcomplex of X/J. Then $(X, \check{A})$ is a J-NDR pair.

PROOF. It follows from J-HELP that $\check{A} \longrightarrow X$ is a J-cofibration. Just as on the space level, a J-cofibration is the inclusion of a J-NDR pair. $\square$

R. J. Piacenza. Homotopy theory of diagrams and CW-complexes over a category. Canadian J. Math. 43(1991), 814-824.

4. The homology and cohomology theory of diagrams

The ordinary homology and cohomology theories of I§3 are special cases of a construction that applies to the category $\mathscr{U}^J$ for any J. The difference is that the theory in I§3 started with G-CW complexes and then passed to the associated diagrams defined on the orbit category of G, whereas we here exploit the theory of J-CW complexes. There is again a vast literature on the cohomology of diagrams, some relevant references being listed in Section 1.

Define a J-coefficient system to be a continuous contravariant functor $M : J \longrightarrow \mathscr{A}b$. Continuity ensures that M factors through the homotopy category hJ. Let $\mathscr{A}b^{hJ}$ be the category of J-coefficient systems. It is an Abelian category, and we can do homological algebra in it. As in I§4, a covariant homotopy invariant functor $\mathscr{U} \longrightarrow \mathscr{A}b$ induces a functor from J-spaces to J-coefficient systems by composition; we name such functors by underlining the name of the given functor. Of course, we also have the notion of a covariant J-coefficient system.

Let (X, A) be a relative J-CW complex with n skeleton X^n and observe that

$$X^n/X^{n-1} = (\coprod_{j_\alpha} D^n \times \underline{j}_\alpha)/(\coprod_{j_\alpha} S^{n-1} \times \underline{j}_\alpha) \cong S^n \wedge (\underline{j}_\alpha)_+, \tag{4.1}$$

where the $+$ indicates the addition of disjoint basepoints. Define a chain complex $\underline{C}_*(X, A)$ in $\mathscr{A}b^J$, called the J-cellular chains of (X, A), by setting

$$\underline{C}_n(X, A) = \underline{H}_n(X^n, X^{n-1}; \mathbb{Z}). \tag{4.2}$$

The connecting homomorphisms of the triples (X^n, X^{n-1}, X^{n-2}) specify the differential

$$d : \underline{C}_n(X, A) \longrightarrow \underline{C}_{n-1}(X, A). \tag{4.3}$$

Clearly (5.1) implies that

$$\underline{C}_n(X,A)(j) = \sum_{j_\alpha} \underline{\tilde{H}}_0(J(j,j_\alpha)_+;\mathbb{Z}). \tag{4.4}$$

The construction is functorial with respect to cellular maps $(X,A) \longrightarrow (Y,B)$.

For a covariant J-coefficient system N, define the cellular chain complex of (X,A) with coefficients N by

$$C_*(X,A;N) = \underline{C}_*(X;A) \otimes_J N, \tag{4.5}$$

where the tensor product on the right is interpreted as the coend over J. Passing to homology, we obtain the cellular homology $H_*(X,A;N)$.

For a contravariant J-coefficient system M, define the cellular cochain complex of (X,A) with coefficients M by

$$C^*(X,A;M) = \mathrm{Hom}_J(\underline{C}_*(X;A), M). \tag{4.6}$$

Passing to cohomology, we obtain the cellular cohomology $H^*(X,A;M)$.

THEOREM 4.7. Cellular homology and cohomology for pairs of J-CW complexes satisfy the standard Eilenberg-Steenrod axioms, suitably reformulated for diagrams.

REMARK 4.8. We may extend the cellular theory to arbitrary pairs of diagrams by means of cellular approximations; see Proposition 4.6. That is, we extend our homology and cohomology theories to theories that carry weak equivalences to isomorphisms. We may also adapt Ilman's construction of equivariant singular theory to construct a singular theory for diagrams. Of course, the singular theory is isomorphic to the cellular theory on the category of J-CW complexes.

S. Illman. Equivariant singular homology and cohomology. Memoirs Amer. Math. Soc. No. 156. 1975.

5. The closed model structure on $\mathscr{U}^J$

Just as the category of spaces has a (closed) model structure in the sense of Quillen, so does the category of G-spaces for any G. This point of view has not been taken earlier since the conclusions are obvious to the experts and perhaps not very helpful to the novice on a first reading. However, since the homotopical properties of categories of diagrams are likely to be less familiar than those of the category of spaces, it is valuable to understand how they inherit model structures from the standard model structure on $\mathscr{U}$, which is the special case of the trivial category J in the definitions here. We use the name q-fibration and q-cofibration for the model structure fibrations and cofibrations to avoid confusion with other kinds of fibrations and cofibrations. The weak equivalences of the model structure will be the weak equivalences that we have already defined; an acyclic q-fibration is one that is a weak equivalence, and similarly for acyclic q-cofibrations. Consider diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

The map g has the left lifting property (LLP) with respect to f if one can always fill in the dotted arrow. The right lifting property (RLP) is defined dually.

Definition 5.1. A J-map $f : X \longrightarrow Y$ is a q-fibration if $f(j) : Y(j) \longrightarrow X(j)$ is a Serre fibration for each object $j \in |J|$. Observe that f is a q-fibration if f has the homotopy lifting property for all objects of the form $I^n \times \underline{j}$. A map $g : A \longrightarrow B$ is a q-cofibration if it has the LLP with respect to all acyclic q-cofibrations.

Theorem 5.2. *With the structure just defined, $\mathscr{U}^J$ is a model category.*

Proof. Just as as for spaces, one quickly checks Quillen's axioms, using the factorization lemma below to verify the factorization axiom $\underline{M2}$. □

As for spaces, the proof leads directly to the following characterizations of q-cofibrations and of acyclic q-fibrations.

Corollary 5.3. *A J-map $g : A \longrightarrow B$ is a q-cofibration if and only if it is a retract of the inclusion $A' \longrightarrow B'$ of a relative J-complex (B', A').*

Corollary 5.4. *A J-map $f : X \longrightarrow Y$ is an acyclic q-fibration if and only if it has the RLP with respect to each q-cofibration $S^n \times \underline{j} \longrightarrow D^{n+1} \times \underline{j}$.*

Lemma 5.5 (Quillen's factorization lemma). *Any J-map $f : X \longrightarrow Y$ can be factored as $f = p \circ g$, where g is a q-cofibration and p is an acyclic q-fibration.*

PROOF. We construct a diagram

$$\begin{tikzcd} X \arrow[r, "g_0"] \arrow[dr, "f"'] & Z^0 \arrow[r, "g_1"] \arrow[d, "p_0"] & Z^1 \arrow[r] & \cdots \\ & Y & & \end{tikzcd}$$

as follows. Let $Z^{-1} = X$ and $p_{-1} = f$. Having obtained Z^{n-1}, consider the set of all diagrams of the form

$$\begin{tikzcd} S^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "t_\alpha"] \arrow[d] & Z^{n-1} \arrow[d, "p_{n-1}"] \\ D^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "s_\alpha"'] & Y. \end{tikzcd}$$

Forming the coproduct over all of the left vertical arrows, we may define $g_n : Z^{n-1} \longrightarrow Z^n$ by the pushout diagram

$$\begin{tikzcd} \coprod S^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "\coprod t_\alpha"] \arrow[d] & Z^{n-1} \arrow[d, "p_{n-1}"] \\ \coprod D^{q_\alpha} \times \underline{j}_\alpha \arrow[r, "\coprod s_\alpha"'] & Z^n. \end{tikzcd}$$

We have allowed the zero dimensional pair $(D^0, S^{-1}) = (\{pt\}, \emptyset)$ in this construction. Define $p_n : Z^n \longrightarrow Y$ by pushing out along p_{n-1} and the coproduct of the maps s_α. Then let

$$Z = \operatorname{colim} Z^n, \quad p = \operatorname{colim} p_n, \quad \text{and} \quad g = \operatorname{colim} g_n g_{n-1} \cdots g_0.$$

One may check that g has the LLP with respect to each acyclic q-fibration and, by the "small object argument" based on the compactness of the D^n, that p is an acyclic q-fibration. $\square$

Let $\bar{h}\mathscr{U}^J$ be the localization of $h\mathscr{U}^J$ obtained by formally inverting the weak equivalences. The model structure implies that $\bar{h}\mathscr{U}^J$ is equivalent to the homotopy category of J-CW complexes, as we indicate next.

LEMMA 5.6. Let $X = \operatorname{colim} X_n$ taken over a sequence of J-cofibrations such that each X_n has the homotopy type of a J-CW complex. Then X has the homotopy type of a J-CW complex.

PROOF. Up to homotopy, we may approximate the sequence by a sequence of J-CW complexes and cellular inclusions; we then use the homotopy invariance of colimits (Theorem 1.2). $\square$

The following proposition follows easily.

PROPOSITION 5.7. Each J-complex is of the homotopy type of a J-CW complex.

THEOREM 5.8 (APPROXIMATION THEOREM). There is a functor $\Gamma : \mathscr{U}^J \longrightarrow \mathscr{U}^J$ and a natural transformation $\gamma : \Gamma \longrightarrow \mathrm{id}$ such that, for each $X \in \mathscr{U}^J$, ΓX is a J-complex and $\gamma : \Gamma X \longrightarrow X$ is an acyclic q-fibration.

PROOF. Applying Lemma 5.3 to the inclusion of the empty set in X, we obtain an acyclic q-fibration $\gamma : \Gamma X \longrightarrow X$. By the explicit construction, we see that ΓX is a J-complex, Γ is a functor, and γ is a natural transformation. $\square$

The following corollary is immediate from the previous two results.

COROLLARY 5.9. The category $\bar{h}\mathscr{U}^J$ is equivalent to the homotopy category of J-CW complexes.

D. G. Quillen. Homotopical Algebra. Springer Lecture Notes in Mathematics Vol. 43. 1967.
W. G. Dwyer and J. Spalinsky. Homotopy theories and model categories. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 73-126.

6. Another proof of Elmendorf's theorem

The theory of diagrams leads to an alternative proof of Elmendorf's theorem V.3.2, one which gives a precise cellular perspective and illustrates the force of model category techniques. We adopt the notations of V§3.

Observe that the fixed point diagram functor Φ from G-spaces to $\mathscr{G}$-spaces carries $X \times G/H$ to $X \times \underline{G/H}$ for a space X regarded as a G-trivial G-space. Thus it preserves cells. It also preserves the pushouts relevant to cellular theory.

LEMMA 6.1. If

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is a pushout of G-spaces in which i is a closed inclusion, then

$$\begin{tikzcd} \Phi A \arrow[r] \arrow[d, "\Phi i"'] & \Phi X \arrow[d] \\ \Phi B \arrow[r] & Y \end{tikzcd}$$

is a pushout of $\mathscr{G}$-spaces.

PROOF. Stripping away the topology we see that this holds on the set level since every G-set is a coproduct of orbits. One may then check that the topologies agree. $\square$

THEOREM 6.2. Each $\mathscr{G}$-complex (or $\mathscr{G}$-CW complex) $Y \in \mathscr{G}\mathscr{U}$ is isomorphic to ΦX for some G-complex (or G-CW complex) X. Therefore Φ is an isomorphism between the category of G-complexes (or $\mathscr{G}$-CW complexes) and the category of $\mathscr{G}$-complexes (or $\mathscr{G}$-CW complexes).

PROOF. The functor Φ carries G-complexes to $\mathscr{G}$-complexes since it preserves cells, the relevant pushouts, and ascending unions. The assertion follows since Φ is full and faithful: inductively, the attaching maps of Y are in the image of Φ. $\square$

This leads to our alternative version of V.3.2.

THEOREM 6.3 (ELMENDORF). There is a functor $\Psi : \mathscr{G}\mathscr{U} \longrightarrow G\mathscr{U}$ and a natural transformation $\varepsilon : \Phi\Psi \longrightarrow \mathrm{id}$ such that ΨX is a G-complex, $\Phi\Psi X$ is a $\mathscr{G}$-complex, and $\varepsilon : \Phi\Psi X \longrightarrow X$ is a weak equivalence of G-spaces for each $\mathscr{G}$-space X. Therefore Φ and Ψ induce an equivalence of categories between $\bar{h}\mathscr{G}\mathscr{U}$ and $\bar{h}G\mathscr{U}$.

PROOF. We construct the functor Ψ and transformation ε by using the functor Γ and transformation p given in Theorem 5.7 on the level of diagrams and using Theorem 6.2 to transport from $\mathscr{G}$-complexes to G-complexes. The result follows from the cited results and Corollary 5.8. $\square$

COROLLARY 6.4. Let Y be a G-space of the homotopy type of a G-CW complex. Then, for any $\mathscr{G}$-space X,

$$hG\mathscr{U}(Y, \Psi X) \cong h\mathscr{G}\mathscr{U}(\Phi Y, X) \cong \bar{h}(\Phi Y, X).$$

PROOF. This follows from Theorem 6.3 and generalities about model categories. $\square$

In turn, this implies the following comparison with the original form, V.3.2, of Elmendorf's theorem.

Corollary 6.5. Write Ψ' and ε' for the constructions given in V.3.2. For a $\mathscr{G}$-space X, there is a weak equivalence of G-spaces $\xi : \Psi X \longrightarrow \Psi' X$ such that ξ is natural up to homotopy and the following diagram commutes up to homotopy:

$$\begin{tikzcd} \Phi\Psi X \arrow[rr, "\Phi\xi"] \arrow[dr, "\varepsilon"'] & & \Phi\Psi' X \arrow[dl, "\varepsilon'"] \\ & X. & \end{tikzcd}$$

CHAPTER VII

Equivariant Bundle theory and Classifying Spaces

1. The definition of equivariant bundles

Equivariant bundle theory can be developed at various levels of generality. We assume given a subgroup Π of a compact Lie group Γ. We set $G = \Gamma/\Pi$, and we let $q : \Gamma \longrightarrow G$ be the quotient homomorphism. That is, we consider an extension of compact Lie groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Many sources restrict attention to split extensions, but we see little point in that. By far the most interesting case is $\Gamma = G \times \Pi$. When Π is $O(n)$ or $U(n)$, this case will lead to real and complex equivariant K-theory.

Define a principal $(\Pi; \Gamma)$-bundle to be the projection to orbits $p : E \longrightarrow E/\Pi = B$ of a Π-free Γ-space E. Note that G acts on the base space B. Let F be a Γ-space. By a G-bundle with structural group Π, total group Γ, and fiber F, we mean the projection $E \times_\Pi F \longrightarrow B$ induced by a principal $(\Pi; \Gamma)$-bundle E; E is called the associated principal bundle. Although we prefer to think of bundles this way, it is not hard to give an intrinsic characterization of when a G-map $Y \longrightarrow B$ that is a Π-bundle with fiber F is such a $(\Pi; \Gamma)$-bundle.

When $\Gamma = G \times \Pi$, we shall refer to (G, Π)-bundles rather than to $(\Pi; G \times \Pi)$-bundles. Here it is usual to require the fiber F be a Π-space. A principal (G, Π)-bundle E has actions by G and Π that commute with one another; it is usual to write the action of Π on the right and the action of G on the left. Equivariant vector bundles fit into this framework: a $(G, O(n))$-bundle with fiber $\mathbb{R}^n$ is called an n-plane G-bundle, and similarly in the complex case. The tangent and normal bundles of a smooth G-manifold give examples.

EXAMPLE 1.1. A finite G-cover $p : Y \longrightarrow B$ is a G-map that is also a finite cover. Such a map is necessarily a (G, Σ_n)-bundle with fiber the Σ_n-set $F = \{1, \ldots, n\}$. Its associated principal (G, Σ_n)-bundle E is the subspace of $\mathrm{Map}(F, Y)$ consisting of the bijections onto fibers of p.

Classical bundle theory readily generalizes to the equivariant context, and we content ourselves with a very brief summary of some of the main points. A principal $(\Pi; \Gamma)$-bundle is said to be trivial if it is equivalent to a bundle of the form

$$q \times \mathrm{id} : \Gamma \times_\Lambda U \longrightarrow G \times_H U,$$

where $H \subset G$, $\Lambda \subset \Gamma$, $\Lambda \cap \Pi = e$, q maps Λ isomorphically onto H, and U is an H-space regarded as a Λ-space by pullback along q. Provided that E and therefore also B are completely regular, a principal $(\Pi; \Gamma)$-bundle $p : E \longrightarrow B$ is locally trivial. If, in addition, B is paracompact, then p is numerable. Numerable $(\Pi; \Gamma)$-bundles satisfy the equivariant bundle covering homotopy property, and a numerable bundle E over $B \times I$ is equivalent to the bundle $(E \times \{0\}) \times I$. Therefore the pullbacks of a numerable $(\Pi; \Gamma)$-bundle along homotopic G-maps are equivalent.

R. K. Lashof. Equivariant bundles. Ill. J. Math. 26(1982).
R. K. Lashof and J. P. May. Generalized equivariant bundles. Bulletin de la Société Mathématique de Belgique 38(1986), 265-271.
L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. (IV§1.) Springer Lecture Notes in Mathematics Vol. 1213. 1986.

2. The classification of equivariant bundles

Let $\mathscr{B}(\Pi; \Gamma)(X)$ be the set of equivalence classes of principal $(\Pi; \Gamma)$-bundles with base G-space X. We assume that X has the homotopy type of a G-CW complex, and we check that this implies that any bundle over X has the homotopy type of a Γ-CW complex. Then Elmendorf's theorem, V.3.2, specializes to give a classification theorem for principal $(\Pi; \Gamma)$-bundles.

DEFINITION 2.1. Define $\mathscr{F}(\Pi; \Gamma)$ to be the family of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and observe that an $\mathscr{F}(\Pi; \Gamma)$-space is the same thing as a Π-free Γ-space. Write

$$E(\Pi; \Gamma) = E\mathscr{F}(\Pi; \Gamma) \quad \text{and} \quad B(\Pi; \Gamma) = E(\Pi; \Gamma)/\Pi,$$

and let

$$\pi : E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$$

be the resulting principal $(\Pi;\Gamma)$-bundle. In the case $\Gamma = G \times \Pi$, write $\mathscr{F}_G(\Pi) = \mathscr{F}(\Pi; G \times \Pi)$,

$$E_G(\Pi) = E(\Pi; G \times \Pi) \quad \text{and} \quad B_G(\Pi) = B(\Pi; G \times \Pi).$$

Observe that, since $E(\Pi;\Gamma)$ is a contractible space, $B(\Pi;\Gamma)$ is a model for $B\Pi$ that carries a particular action by G.

THEOREM 2.2. The bundle $\pi : E(\Pi;\Gamma) \longrightarrow B(\Pi;\Gamma)$ is universal . That is, pullback of π along G-maps $X \longrightarrow B(\Pi;\Gamma)$ gives a bijection

$$[X, B(\Pi;\Gamma)]_G \longrightarrow \mathscr{B}(\Pi;\Gamma)(X).$$

It is crucial to the utility of this result to understand the fixed point structure of $B(\Pi;\Gamma)$. For any principal $(\Pi;\Gamma)$-bundle $p : E \longrightarrow B$ and any $H \subset G$, one can check that B^H is the disjoint union of the spaces $p(E^\Lambda)$, where Λ runs over the Π-conjugacy classes of subgroups $\Lambda \subset \Gamma$ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$. Define

$$\Pi^\Lambda \equiv \Pi \cap N_\Gamma \Lambda = \Pi \cap Z_\Gamma \Lambda, \tag{2.3}$$

where $Z_\Gamma \Lambda$ is the centralizer of Λ in Γ; the equality here is an easy observation. Then E^Λ is a principal $(\Pi^\Lambda; W_\Gamma \Lambda)$-bundle and $p(E^\Lambda)$ is its base space. We can go on to analyze the structure of B^H as a $W_G H$-space. In the case of the universal bundle, we can determine the structure of E^Λ by use of IV.4.7. Putting things together, we arrive at the following conclusion.

THEOREM 2.4. For a subgroup H of G,

$$B(\Pi;\Gamma)^H = \coprod B(\Pi^\Lambda),$$

where the union runs over the Π-conjugacy classes of subgroups Λ of Γ such that $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$; as a $W_G H$-space,

$$B(\Pi;\Gamma)^H = \coprod W_G H \times_{V(\Lambda)} B(\Pi^\Lambda; W_\Gamma \Lambda),$$

where the union runs over the $q^{-1}(N_G H)$-conjugacy classes of such groups Λ and $V(\Lambda) = W_\Gamma \Lambda / \Pi^\Lambda$ is the image of $W_\Gamma \Lambda$ in $W_G H$.

Here, by use of Lie group theory, $V(\Lambda)$ has finite index in W_GH.

Specializing to $\Gamma = G\times\Pi$, we see that the subgroups Λ of Γ such that $\Lambda\cap\Pi = e$ are exactly the twisted diagonal subgroups

$$\Delta(\rho) = \{(h,\rho(h))|h\in H\}, \tag{2.5}$$

where H is a subgroup of G and $\rho : H \longrightarrow \Pi$ is a homomorphism. Let $N(\rho) = N_{G\times\Pi}\Delta(\rho)$ and observe that

$$N(\rho) = \{(g,\pi)|g\in N_GH \;\text{ and }\; \pi\rho(h)\pi^{-1} = \rho(ghg^{-1}) \text{ for all } h\in H\}.$$

Therefore $\Pi\cap N(\rho)$ coincides with the centralizer

$$\Pi^\rho = \{\pi|\pi\rho(h) = \rho(h)\pi \text{ for all } h\in H\}. \tag{2.6}$$

Let

$$W(\rho) = W_{G\times\Pi}\Delta(\rho) \;\text{ and }\; V(\rho) = W(\rho)/\Pi^\rho \subset W_GH.$$

As usual, let $\mathrm{Rep}(G,\Pi)$ denote the set of Π-conjugacy classes of homomorphisms $G\longrightarrow\Pi$. Define an action of the group W_GH on the set $\mathrm{Rep}(H,\Pi)$ by letting $(gH)\rho$ be the conjugacy class of $g\cdot\rho$, where, for $g\in N_GH$, $g\cdot\rho : H\longrightarrow\Pi$ is the homomorphism specified by $(g\cdot\rho)(h) = \rho(g^{-1}hg)$. Observe that the isotropy group of (ρ) is $V(\rho)$.

THEOREM 2.7. For a subgroup H of G,

$$(B_G\Pi)^H = \coprod B(\Pi^\rho),$$

where the union runs over $(\rho)\in\mathrm{Rep}(H,\Pi)$; as a W_GH-space,

$$(B_G\Pi)^H = \coprod W_GH\times_{V(\rho)} B(\Pi^\rho; W(\rho)),$$

where the union runs over the orbit set $\mathrm{Rep}(H,\Pi)/W_GH$.

It is important to observe that the group $W(\rho)$ need not split as a product $V(\rho)\times\Pi^\rho$ in general. Therefore, in order to fully understand the classifying G-spaces for (G,Π)-bundles, one is forced to study the classifying spaces for the more general kind of bundles that we have introduced. These are complicated objects, and their study is in a primitive state. In particular, rather little is known about equivariant characteristic classes. Such classes are understood in Borel cohomology, however. By the universal property of $E(\Pi;\Gamma)$, there is a Γ-map $E\Gamma\longrightarrow E(\Pi;\Gamma)$, which is unique up to homotopy. The induced G-map $E\Gamma/\Pi\longrightarrow B(\Pi;\Gamma)$ is a nonequivariant equivalence and so induces an isomorphism on Borel cohomology. The projection $EG \times E\Gamma \longrightarrow E\Gamma$ is clearly a Γ-homotopy equivalence, and it induces an equivalence

$$EG \times_G (E\Gamma/\Pi) = (EG \times E\Gamma)/\Gamma \longrightarrow E\Gamma/\Gamma = B\Gamma.$$

This already implies the following calculation. We again denote Borel cohomology by H_G^* for the moment.

THEOREM 2.8. With any coefficients, $H_G^*(B(\Pi;\Gamma)) \cong H^*(B\Gamma)$. With field coefficients, $H_G^*(B_G\Pi) \cong H^*(BG) \otimes H^*(B\Pi)$ as an $H^*(BG)$-module.

The interpretation is that the Borel cohomology characteristic classes of a principal (G,Π)-bundle E over X are determined by the $H^*(BG)$-module structure on $H_G^*(X)$ together with the nonequivariant characteristic classes of the Π-bundle $EG \times_G E$ over $EG \times_G X$.

We shall later see that generalized versions of the Atiyah-Segal completion theorem and of the Segal conjecture give calculations of the characteristic classes of (G,Π)-bundles in equivariant K-theory and in equivariant cohomotopy.

R. K. Lashof and J. P. May. Generalized equivariant bundles. Bulletin de la Société Mathématique de Belgique 38(1986), 265-271.

J. P. May. Characteristic classes in Borel cohomology. J. Pure and Applied Algebra 44(1987), 287-289.

3. Some examples of classifying spaces

It is often valuable to have alternative descriptions of universal bundles. We have Grassmannian models when Π is an orthogonal or unitary group. These lead to good models for the classifying spaces for equivariant K-theory, and, just as nonequivariantly, they are useful for the proof of equivariant versions of the Thom cobordism theorem.

EXAMPLE 3.1. For a real inner product G-space V, let $BO(n,V)$ be the G-space of n-planes in V and let $EO(n,V)$ be the G-space whose points are pairs consisting of an n-plane π in V and a vector $v \in \pi$. The map $EO(n,V) \longrightarrow BO(n,V)$ that sends (π, v) to π is a real n-plane G-bundle. Provided that V is large enough, say the direct sum of infinitely many copies of each irreducible real representation of G, p is a universal real n-plane G-bundle. A similar construction works in the complex case.

Clearly a principal $(\Pi;\Gamma)$-bundle E is universal if and only if E^{Λ} is contractible for $\Lambda \in \mathscr{F}(\Pi;\Gamma)$. Using the fact that the space of G-maps from a free G-CW complex to a nonequivariantly contractible G-space is contractible, one can use this criterion to obtain a simple model that has particularly good naturality properties. Regard EG as a Γ-space via $q : \Gamma \longrightarrow G$ and define

$$\mathrm{Sec}(EG, E\Gamma) \subset \mathrm{Map}(EG, E\Gamma)$$

to be the sub Γ-space consisting of those maps $f : EG \longrightarrow E\Gamma$ such that the composite of $Eq : E\Gamma \longrightarrow EG$ and f is the identity map. Note that

$$\mathrm{Sec}(EG, E(G \times \Pi)) = \mathrm{Map}(EG, E\Pi)$$

since $E(G \times \Pi)$ is homeomorphic to $EG \times E\Pi$.

THEOREM 3.2. The Γ-space $\mathrm{Sec}(EG, E\Gamma)$ is a universal principal $(\Pi;\Gamma)$-bundle and therefore the G-space $\mathrm{Sec}(EG, E\Gamma)/\Pi$ is a model for $B(\Pi;\Gamma)$. In particular, the $G\times\Pi$-space $\mathrm{Map}(EG, E\Pi)$ is a universal principal (G,Π)-bundle and therefore the G-space $\mathrm{Map}(EG, E\Pi)/\Pi$ is a model for $B_G\Pi$.

Since we are interested in maps from G-CW complexes into classifying spaces, the fact that these models need not have the homotopy types of G-CW complexes need not concern us.

Observe that the map $\pi : E\Gamma \longrightarrow B\Gamma$ induces a natural G-map

$$\alpha : B(\Pi;\Gamma) = \mathrm{Sec}(EG, E\Gamma)/\Pi \longrightarrow \mathrm{Sec}(EG, B\Gamma), \tag{3.3}$$

where $\mathrm{Sec}(EG, B\Gamma)$ is the G-space of maps $f : EG \longrightarrow B\Gamma$ such that the composite of f and $Bq : B\Gamma \longrightarrow BG$ is $\pi : EG \longrightarrow BG$. With $\Gamma = G \times \Pi$, this map is

$$\alpha : B_G\Pi \longrightarrow \mathrm{Map}(EG, B\Pi). \tag{3.4}$$

These maps have bundle theoretic interpretations. Restricting for simplicity to the case $\Gamma = G \times \Pi$, let

$$\mathscr{B}_G(\Pi)(X) \cong [X, B_G\Pi]_G$$

be the set of equivalence classes of (G,Π)-bundles over X and let $\mathscr{B}(\Pi)(X)$ be the set of equivalence classes of nonequivariant Π-bundles over X. By adjunction, a G-map $X \longrightarrow \mathrm{Map}(EG, B\Pi)$ is the same as a map $EG\times_G X \longrightarrow B\Pi$. Thus the represented equivalent of α is the Borel construction on bundles that was relevant to Theorem 2.8; it gives

$$\mathscr{B}_G(\Pi)(X) \longrightarrow \mathscr{B}(\Pi)(EG \times_G X).$$

It is important to know how much information this construction loses, hence it is important to know how near α is to being an equivalence. Elementary covering space theory gives the following result.

PROPOSITION 3.5. If Γ is discrete, then the G-map α of (3.3) is a homeomorphism. If Π, but not necessarily G, is discrete, then the G-map α of (3.4) is a homeomorphism.

An Abelian compact Lie group is the product of a finite Abelian group and a torus. Using ordinary cohomology to study the finite factor and continuous cohomology to handle the torus factor, Lashof, May, and Segal proved another result along these lines.

THEOREM 3.6. If G is a compact Lie group and Π is an Abelian compact Lie group, then the G-map $\alpha : B_G\Pi \longrightarrow \mathrm{Map}(EG, B\Pi)$ is a weak equivalence.

Consequences of the Sullivan conjecture will tell us much more about these maps. To see this, we will need to know the behavior of the maps α on fixed point spaces. We have determined the fixed point spaces $B(\Pi;\Gamma)^H$, and it is clear that

$$\mathrm{Sec}(EG, B\Gamma)^H = \mathrm{Sec}(BH, B\Gamma)$$

is the space of maps $f : BH \longrightarrow B\Gamma$ such that

$$Bq \circ f = Bi : BH \longrightarrow BG,$$

where $i : H \longrightarrow G$ is the inclusion and we take Bi to be the quotient map $EG/H \longrightarrow EG/G$. In particular,

$$\mathrm{Sec}(BH, BG \times B\Pi) = \mathrm{Map}(BH, B\Pi).$$

LEMMA 3.7. Let $\Lambda \subset \Gamma$ satisfy $\Lambda \cap \Pi = e$ and $q(\Lambda) = H$. Define a homomorphism $\mu : H \times \Pi^\Lambda \longrightarrow \Gamma$ by $\mu(q(\lambda), \pi) = \lambda\pi$ and observe that $q \circ \mu = i \circ \pi_1 : H \times \Pi^\Lambda \longrightarrow G$. The restriction of

$$\alpha^H : B(\Pi;\Gamma)^H \longrightarrow \mathrm{Sec}(BH, B\Gamma)$$

to $B(\Pi^\Lambda)$ is the adjoint of the classifying map

$$B\mu : BH \times B(\Pi^\Lambda) = B(H \times \Pi^\Lambda) \longrightarrow B\Gamma.$$

Therefore, if $\Gamma = G \times \Pi$, the restriction of

$$\alpha^H : (B_G\Pi)^H \longrightarrow \mathrm{Map}(BH, B\Pi)$$

to $B(\Pi^\rho)$, $\rho : H \longrightarrow \Pi$, is the adjoint of the map of classifying spaces

$$B\nu : BH \times B(\Pi\rho) = B(H \times \Pi^\rho) \longrightarrow B\Pi,$$

where $\nu : H \times \Pi^\rho \longrightarrow \Pi$ is defined by $\nu(h, \pi) = \rho(h)\pi$.

Consider what happens on components. In nonequivariant homotopy theory, maps between the classifying spaces of compact Lie groups have been studied for many years. One focus has been the question of when passage to classifying maps

$$B : \mathrm{Rep}(G, \Pi) \longrightarrow [BG, B\Pi]$$

is a bijection. We now see that, for $H \subset G$, a map $BH \longrightarrow B\Pi$ not in the image of B corresponds to a principal Π-bundle over BH that does not come from a principal (G, Π)-bundle over an orbit G/H. The equivariant results above imply that there are no such exotic maps if Π is either finite or Abelian. The Sullivan conjecture will give information about general compact Lie groups Π under restrictions on G.

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CHAPTER VIII

The Sullivan Conjecture

1. Statements of versions of the Sullivan conjecture

We defined the homotopy orbit space of a G-space X to be

$$X_{hG} = EG \times_G X,$$

and we defined the homotopy fixed point space of X dually:

$$X^{hG} = \mathrm{Map}(EG, X)^G = \mathrm{Map}_G(EG, X)$$

is the space of G-maps $EG \longrightarrow X$. The projection $EG \longrightarrow *$ induces

$$X^G = \mathrm{Map}(*, X)^G \longrightarrow \mathrm{Map}(EG, X) = X^{hG}.$$

It sends a fixed point to the constant map $EG \longrightarrow X$ at that fixed point. It is very natural to ask how close this map is to being a homotopy equivalence. Thinking equivariantly, it is even more natural to ask how close the G-map

$$\eta : X = \mathrm{Map}(*, X) \longrightarrow \mathrm{Map}(EG, X)$$

is to being a G-homotopy equivalence. Since a G-map $f : X \longrightarrow Y$ that is a nonequivariant equivalence induces a weak equivalence of G-spaces

$$\mathrm{Map}(W, Y) \longrightarrow \mathrm{Map}(W, X)$$

for any free G-CW complex W, such as EG, one cannot expect η to be an equivalence in general. Very little is known about this question for general finite groups. However, for finite p-groups G, to which we restrict ourselves unless we specify otherwise, the Sullivan conjecture gives a beautiful answer. We agree to work in the categories $\bar{h}\mathscr{U}$ and $\bar{h}G\mathscr{U}$, implicitly applying CW approximation. This allows us to ignore the distinction between weak and genuine equivalences.

THEOREM 1.1 (GENERALIZED SULLIVAN CONJECTURE). Let X be a nilpotent finite G-CW complex. Then the natural G-map

$$\hat{X}_p \longrightarrow \mathrm{Map}(EG, \hat{X}_p)$$

is an equivalence.

The hypothesis that X be nilpotent can be removed by applying the Bousfield-Kan simplicial completion on fixed point spaces and then assembling these completed fixed point spaces to a global G-completion by means of Elmendorf's construction. This equivariant interpretation of the Sullivan conjecture was noticed by Haeberly, who also gave some information for finite groups that are not p-groups. Looking at fixed points under $H \subset G$ and noting that EG is a model for EH, we see that the result immediately reduces to the fixed point space level.

THEOREM 1.2 (MILLER, CARLSSON, LANNES). Let X be a nilpotent finite G-CW complex, where G is a finite p-group. Then the natural map

$$(X^G)^{\wedge}_p \cong (\hat{X}_p)^G \longrightarrow \mathrm{Map}(EG, \hat{X}_p)^G = (\hat{X}_p)^{hG}$$

is an equivalence.

Again, the nilpotence hypothesis is unnecessary provided that one understands $\hat{X}_p$ to mean the Bousfield-Kan completion of X, which generalizes the nilpotent completion that we defined, and takes $(X^G)^{\wedge}_p$ and not $(\hat{X}_p)^G$ as the source: there is a natural map

$$(X^G)^{\wedge}_p \longrightarrow (\hat{X}_p)^G,$$

but it is not an equivalence in general. When G acts trivially on X, the result was first proven by Miller, and he deduced the following powerful consequence.

THEOREM 1.3 (MILLER). Let G be a discrete group such that all of its finitely generated subgroups are finite and let X be a connected finite dimensional CW complex. Then $\pi_*F(BG, X) = 0$.

To deduce this from Theorem 1.2, one first observes that any map $BG \longrightarrow X$ induces the trivial map of fundamental groups and so lifts to the universal cover, while a map $\Sigma^n BG \longrightarrow X$ for $n > 0$ trivially lifts to the universal cover. Thus one can assume that X is simply connected. Note that this reduction depends on the fact that we are here working with finite dimensional and not just finite complexes, and one must generalize Theorem 1.2 accordingly; this seems to require trivial action on X. One then applies an inductive argument to reduce to the case $G = \mathbb{Z}/p$. Here the weak equivalence $\hat{X}_p \longrightarrow \mathrm{Map}(BG, \hat{X}_p)$ implies that $\pi_* F(BG, \hat{X}_p) = 0$, and this implies that $\pi_* F(BG, X) = 0$.

The general case of Theorem 1.2 reduces immediately to the case when $G = \mathbb{Z}/p$, by induction on the order of G. To see this, consider an extension

$$1 \longrightarrow C \longrightarrow G \longrightarrow J \longrightarrow 1,$$

where C is cyclic of order p. For any G-space Y, $(Y^{hC})^{hJ}$ is equivalent to Y^{hG}. In fact, by passing to G-fixed points by first passing to C-fixed points and then to J-fixed points, we obtain a homeomorphism

$$\mathrm{Map}(EJ \times EG, Y)^G \cong \mathrm{Map}(EJ, \mathrm{Map}(EG, Y)^C)^J.$$

Since $EJ \times EG$ is a free contractible G-space and EG is a free contractible C-space, this gives the stated equivalence of homotopy fixed point spaces. The equivalence $(X^C)^{\wedge}_p \longrightarrow (\hat{X}_p)^{hC}$ is a J-map, hence it induces an equivalence on passage to J-homotopy fixed point spaces, and the map of Theorem 1.2 coincides with the composite equivalence

$$(X^G)^{\wedge}_p = ((X^C)^J)^{\wedge}_p \longrightarrow ((X^C)^{\wedge}_p)^{hJ} \longrightarrow (\hat{X}_p)^{hC})^{hJ} \cong (\hat{X}_p)^{hG}.$$

When $G = \mathbb{Z}/p$, Theorem 1.2 was proven independently by Lannes and Miller, using nonequivariant techniques, and by Carlsson, using equivariant techniques. Lannes later gave a variant of his original proof that generalizes the result, uses equivariant ideas, and enjoys a pleasant conceptual relationship to Smith theory. We shall sketch that proof in the following three sections.

There is a basic principle in equivariant topology to the effect that, when working at a prime p, results that hold for p-groups can be generalized to p-toral groups G, which are extensions of the form

$$1 \longrightarrow T \longrightarrow G \longrightarrow \pi \longrightarrow 1.$$

The point is that the circle group can be approximated by the union σ_∞ of its p-subgroups σ_n of $(p^n)th$ roots of unity, and an r-torus T can be approximated by the union τ_∞ of its p-subgroups $\tau_n = (\sigma_n)^r$. It is not hard to see that the map $B\tau_\infty \longrightarrow BT$ induces an isomorphism on mod p homology. Using this basic idea, Notbohm generalized Theorem 1.2 to p-toral groups.

THEOREM 1.4 (NOTBOHM). The generalized Sullivan conjecture, Theorem 1.2, remains true as stated when G is a p-toral group.

Technically, this still works using Bousfield-Kan completion for "p-good" G-spaces X, for which $X \longrightarrow \hat{X}_p$ is a mod p equivalence.

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2. Algebraic preliminaries: Lannes' functors T and Fix

Let V be an elementary Abelian p-group, fixed throughout this section and the next. It would suffice to restrict attention to $V = \mathbb{Z}/p$. The notation V indicates that we think of V ambiguously as both a vector space over $\mathbb{F}_p$ and a group that will act as symmetries of spaces. We refer back to IV.2.3, which gave

$$H^*(X^V) \cong \mathbb{F}_p \otimes_{H^*(BV)} Un(S^{-1}H^*_V(X)) \tag{2.1}$$

for a finite dimensional V-CW complex X.

We begin by describing this in more conceptual algebraic terms. In this section, we let $\mathscr{U}$ be the category of unstable modules over the mod p Steenrod algebra A and let $\mathscr{K}$ be the category of unstable A-algebras. Thus the mod p cohomology of any space is in $\mathscr{K}$. We shall abbreviate notation by setting $H = H^*(BV)$. The celebrated functor $T : \mathscr{U} \longrightarrow \mathscr{U}$ introduced by Lannes is the left adjoint of $H \otimes (\cdot)$: for unstable A-modules M and N,

$$\mathscr{U}(TM, N) \cong \mathscr{U}(M, H \otimes N). \tag{2.2}$$

Observe that the adjoint of the map $M = \mathbb{F}_p \otimes M \longrightarrow H \otimes M$ induced by the unit of H gives a natural A-map $\pi : TM \longrightarrow M$. The key properties of the functor T are as follows.

(2.3) The functor T is exact and commutes with suspension.

(2.4) The functor T commutes with tensor products.

This property implies that if M is an unstable A-algebra, then so is TM. The resulting functor $T : \mathscr{K} \longrightarrow \mathscr{K}$ is also left adjoint to $H \otimes (\cdot)$: for unstable A-algebras M and N,

$$\mathscr{K}(TM, N) \cong \mathscr{K}(M, H \otimes N). \tag{2.5}$$

The Borel cohomology $H^*_V(X)$ is both an unstable A-algebra and an H-module. The action of H is given by a map of A-modules, and the bundle map

$$EV \times_V X \longrightarrow BV$$

induces a map $H \longrightarrow H^*_V(X)$ of unstable A-algebras. We codify these structures in algebraic definitions. Thus let $H\mathscr{U}$ be the category of unstable A-modules M together with an H-module structure given by an A-map $H \otimes M \longrightarrow M$. For such an H-A-module M, define an unstable A-module $\mathrm{Fix}(M)$ by

$$\mathrm{Fix}(M) = \mathbb{F}_p \otimes_{TH} TM \cong \mathbb{F}_p \otimes_H (H \otimes_{TH} TM). \tag{2.6}$$

The notation "Fix" anticipates a connection with (2.1). Here we have used (2.4) to give that TH is an augmented A-algebra and that TM is a TH-module; TH acts on H through the adjoint $TH \longrightarrow H$ of the coproduct $\psi : H \longrightarrow H \otimes H$. We have another adjunction. For unstable H-A-modules M and unstable A-modules N, we have

$$\mathscr{U}(\mathrm{Fix}(M), N) \cong H\mathscr{U}(M, H \otimes N). \tag{2.7}$$

Comparing the adjunctions (2.2) and (2.7), we easily find that, for an unstable A-module M,

$$\mathrm{Fix}(H \otimes M) \cong TM \quad \text{as unstable } A\text{-modules.} \tag{2.8}$$

Less obviously, one can also construct a natural isomorphism

$$H \otimes_{TH} TM \cong H \otimes \mathrm{Fix}(M) \quad \text{as unstable } H\text{-}A\text{-modules.} \tag{2.9}$$

The functor Fix has properties just like those of T.

(2.10) $\quad$ $\mathrm{Fix} : H\mathscr{U} \longrightarrow \mathscr{U}$ is exact and commutes with suspension.

The appropriate tensor product in $H\mathscr{U}$ is $M \otimes_H N$.

(2.11)
There is a natural isomorphism $\mathrm{Fix}(M \otimes_H N) \cong \mathrm{Fix}(M) \otimes \mathrm{Fix}(N)$.

Define $H\backslash\mathscr{K}$ to be the category of unstable A-algebras under H. If M is an unstable A-algebra under H, then its product factors through $M \otimes_H M$ and we deduce from (2.11) that $\mathrm{Fix}(M)$ is an unstable A-algebra. If M is an unstable A-algebra, then (2.8) is an isomorphism of unstable A-algebras. If M is an unstable A-algebra under H, then the isomorphism (2.9) is one of unstable A-algebras under H. We now reach the adjunction that we really want. For an unstable A-algebra M under H and an unstable A-algebra N,

$$\mathscr{K}(\mathrm{Fix}(M), N) \cong (H\backslash\mathscr{K})(M, H \otimes N). \tag{2.12}$$

3. Lannes' generalization of the Sullivan conjecture

Returning to topology, let X be a V-space. Abbreviate

$$\mathrm{Fix}^*_V(X) = \mathrm{Fix}(H^*_V(X)).$$

This is a cohomology theory on V-spaces. The inclusion $i : X^V \longrightarrow X$ induces a natural map

$$j : \mathrm{Fix}^*_V(X) \longrightarrow \mathrm{Fix}^*_V(X^V) \cong TH^*(X^V) \longrightarrow H^*(X^V).$$

Here the middle isomorphism is implied by (2.8) and the last map is an instance of the natural map $\pi : TM \longrightarrow M$. The map j specifies a transformation of cohomology theories on X. By a check on V-spaces of the form $V/W_+ \wedge K$, one finds that, if X is a finite dimensional V-CW complex, then

(3.1) $\quad$ $j : \mathrm{Fix}^*_V(X) \longrightarrow H^*(X^V)$ is an isomorphism.

An alternative proof using the localization theorem is possible. In fact, this must be the case: the only way to reconcile (2.1) and (3.1) is to have an algebraic isomorphism

$$\mathrm{Fix}(M) \cong \mathbb{F}_p \otimes_H Un(S^{-1}M) \tag{3.2}$$

for reasonable M. As a matter of algebra, Dwyer and Wilkerson prove that there is an isomorphism of H-A-algebras

$$H \otimes_{TH} TM \cong Un(S^{-1}M) \tag{3.3}$$

for any unstable H-A-algebra M that is finitely generated as an H-module. Tensoring over H with $\mathbb{F}_p$, this gives (3.2). Combined with (2.9), this gives an entirely algebraic version of the isomorphism

$$H^*(BV) \otimes H^*(X^V) \cong Un(S^{-1}H^*_V(X_{hV}))$$

of IV.2.3. Here, if $M = H^*_V(X)$ is finitely generated over H, the isomorphism (3.2) agrees with that obtained by combining (2.1) and (3.1). Thus we may view (3.1) as another reformulation of Smith theory. This reformulation is at the heart of the Sullivan conjecture, which is a corollary of the following theorem.

THEOREM 3.4 (LANNES). *Let X be a V-space whose cohomology is of finite type and let Z be a space (with trivial V-action) whose cohomology is of finite type. Let $\omega : EV \times Z \longrightarrow X$ be a V-map. Then the homomorphism of unstable A-algebras*

$$\omega^\# : \mathrm{Fix}^*_V(X) \longrightarrow H^*(Z)$$

induced by ω is an isomorphism if and only if the map

$$\tilde{\omega} : \hat{Z}_p \longrightarrow (\hat{X}_p)^{hV}$$

induced by ω is an equivalence.

The map ω determines and is determined by a map

$$\omega' : BV \times Z \longrightarrow EV \times_V X = X_{hV}$$

of bundles over BV. The map $\omega^\#$ of the theorem is the adjoint via (2.12) of the map under H induced on cohomology by ω'. The map ω induces a map $EV \times \hat{Z}_p \longrightarrow \hat{X}_p$, and the map $\tilde{\omega}$ of the theorem is its adjoint.

To prove the Sullivan conjecture, we take $Z = X^V$ and take $\omega : EG \times X^V \longrightarrow X$ to be the adjoint of the canonical map $X^V \longrightarrow X^{hV}$. Then $\omega^\#$ is the isomorphism j of (3.1), and $\tilde{\omega} : (X^V)^\wedge_p \longrightarrow (\hat{X}_p)^{hV}$ is the map that Theorem 1.2 claims to be an equivalence. Thus we see the Sullivan conjecture as a natural elaboration of Smith theory.

Theorem 3.4 has other applications. In the Sullivan conjecture, we applied it to obtain homotopical information from cohomological information, but its converse implication is also of interest. Taking $Z = X^{hV}$ and letting $\omega : EV \times X^{hV} \longrightarrow X$ be the evaluation map, the theorem specializes to give the following result.

THEOREM 3.5. Let X be a V-space such that the cohomologies of X and of X^{hV} are of finite type. Then the canonical map

$$\mathrm{Fix}^*_V(X) \longrightarrow H^*(X^{hV})$$

is an isomorphism of unstable A-algebras if and only if the canonical map

$$(X^{hV})\hat{_p} \longrightarrow (\hat{X}_p)^{hV}$$

is an equivalence.

When both X and X^{hV} are p-complete, so that $(X^{hV})\hat{_p} \longrightarrow (\hat{X}_p)^{hV}$ is the identity, we conclude that $H^*(X^{hV})$ is calculable as $\mathrm{Fix}^*_V(X)$. This is the starting point for remarkable work of Dwyer and Wilkerson in which they redevelop a great deal of Lie group theory in a homotopical context of p-complete finite loop spaces.

If we specialize to spaces without actions and use (2.8), we get the following nonequivariant version of Theorem 3.4.

THEOREM 3.6. Let Y and Z be spaces with cohomology of finite type and let $\omega : BV \times Z \longrightarrow Y$ be a map. Then the homomorphism of unstable A-algebras $\omega^{\#} : TH^*(Y) \longrightarrow H^*(Z)$ induced by ω is an isomorphism if and only if the map $\tilde{\omega} : \hat{Z}_p \longrightarrow \mathrm{Map}(BV, \hat{Y}_p)$ is an equivalence.

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4. Sketch proof of Lannes' theorem

We briefly sketch the strategy of the proof of Theorem 3.4. The first step is to reduce it to the nonequivariant version given in Theorem 3.6. It is easy to see that, for a group G and G-space Y, we have an identification

$$(4.1) \qquad Y^{hG} \equiv \mathrm{Map}_G(EG, Y) = \mathrm{Sec}(BG, EG \times_G Y) \equiv \mathrm{Sec}(BG, Y_{hG}),$$

where the right side is the space of sections of the bundle $Y_{hG} \longrightarrow BG$. Let $\mathrm{Map}(BG, BG)_1$ denote the component of the identity map and $\mathrm{Map}(BG, Y_{hG})_1$

denote the space of maps whose projection to BG is homotopic to the identity. We have a fibration

$$\mathrm{Map}(BG, Y_{hG})_1 \longrightarrow \mathrm{Map}(BG, BG)_1$$

with fiber Y^{hG} over the identity map.

Now return to $G = V$. Here easy inspections of homotopy groups show that evaluation at a basepoint gives an equivalence

$$\varepsilon : \mathrm{Map}(BV, BV)_1 \longrightarrow BV$$

and that the composition action of $\mathrm{Map}(BV, BV)_1$ on $\mathrm{Map}(BV, Y_{hV})_1$ induces an equivalence

$$Y^{hV} \times \mathrm{Map}(BV, BV)_1 \longrightarrow \mathrm{Map}(BV, Y_{hV})_1.$$

For a V-space X, the natural map $EV \times \hat{X}_p \longrightarrow (EV \times X)\hat{_p}$ induces a natural map $(\hat{X}_p)_{hV} \longrightarrow (X_{hV})\hat{_p}$, and this map is an equivalence. By (3.7), the map $\tilde{\omega}$ of Theorem 3.4 may be viewed as a map

$$\hat{Z}_p \longrightarrow \mathrm{Sec}(BV, (\hat{X}_p)_{hV}). \tag{4.2}$$

The map ω determines a map $EV \times \hat{Z}_p \longrightarrow \hat{X}_p$, and this in turn determines and is determined by a map

$$BV \times \hat{Z}_p \longrightarrow (\hat{X}_p)_{hV} \tag{4.3}$$

of bundles over BG. The map (3.8) is the composite map of fibers in the following diagram of fibrations

$$\begin{tikzcd}
\hat{Z}_p \arrow[r] \arrow[d] & \mathrm{Map}(BV, \hat{Z}_p) \arrow[r] \arrow[d] & \mathrm{Sec}(BV, (\hat{X}_p)_{hV} \arrow[d] \\
BV \times \hat{Z}_p \arrow[r] \arrow[d] & \mathrm{Map}(BV, BV \times \hat{Z}_p)_1 \arrow[r] \arrow[d] & \mathrm{Map}(BV, (\hat{X}_p)_{hV})_1 \arrow[d] \\
BV \arrow[r] & \mathrm{Map}(BV, BV)_1 \arrow[r, "="] & \mathrm{Map}(BV, BV)_1.
\end{tikzcd}$$

The left map of fibrations is determined by a chosen homotopy inverse to $\varepsilon : \mathrm{Map}(BV, BV)_1 \longrightarrow BV$ and the inclusion of $\hat{Z}_p$ in $\mathrm{Map}(BV, \hat{Z}_p)$ as the subspace of constant functions. Clearly the middle composite is an equivalence if and only if $\tilde{\omega}$ is an equivalence. Applying Theorem 3.6 with Z replaced by $BV \times Z$, Y

taken to be X_{hV} and ω replaced by the adjoint $\nu : BV \times BV \times Z \longrightarrow X_{hV}$ of the composite map

$$BV \times Z \longrightarrow \mathrm{Map}(BV, BV \times Z)_1 \longrightarrow \mathrm{Map}(BV, X_{hV})$$

defined as in the middle row, but before applying completions, we find that the middle composite is an equivalence if and only if the induced map $\nu^{\#} : TH^*(X_{hV}) \longrightarrow H \otimes H^*(Z)$ is an isomorphism. Now (2.9) gives an isomorphism

$$H \otimes_{TH} TH^*(X_{hV}) \cong H \otimes \mathrm{Fix}(H^*(X_{hV}))$$

of unstable H-A-algebras. Its explicit construction parallels the topology in such a way that the map $\omega^{\#} : \mathrm{Fix}_V^*(X) \longrightarrow H^*(Z)$ agrees with $H \otimes_H \nu^{\#}$. This allows us to deduce that $\nu^{\#}$ is an isomorphism if and only if $\omega^{\#}$ is an isomorphism.

It remains to say something about the proof of Theorem 3.6. Since this is nonequivariant topology of the sort that requires us to join with those who use the word "space" to mean "simplicial set", we shall say very little. For a map $\phi : M \longrightarrow N$ of unstable A-algebras, there are certain algebraic functors that one may call $\mathrm{Ext}^{s,t}_{\mathscr{K}}(M, N; \phi)$; for fixed t, they are the left derived functors of a certain functor of derivations $\mathrm{Der}^t_{\mathscr{K}}(\cdot, N; \cdot)$ that is defined on the category of unstable A-algebras over N. The relevance of the functor T comes from the fact that its defining adjunction leads to natural isomorphisms

$$\mathrm{Ext}^{s,t}_{\mathscr{K}}(TM, N; \tilde{\phi}) \cong \mathrm{Ext}^{s,t}_{\mathscr{K}}(M, H \otimes N; \phi)$$

for a map $\phi : M \longrightarrow H \otimes N$ of unstable A-algebras with adjoint $\tilde{\phi}$.

There is an unstable Adams spectral sequence, due originally to Bousfield and Kan. However, the relevant version is a generalization due to Bousfield. For a map $f : X \longrightarrow \hat{Y}_p$, it starts from

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{K}}(H^*(Y), H^*(X); f^*),$$

and it converges (in total degree $t-s$) to $\pi_*(\mathrm{Map}(X, \hat{Y}_p); f)$. Under the hypotheses of Theorem 3.6, the map $\tilde{\omega} : \hat{Z}_p \longrightarrow \mathrm{Map}(BV, \hat{Y}_p)$ induces a map of spectral sequences (for any base point of Z) that is given on the E_2-level by the map

$$\mathrm{Ext}^{s,t}_{\mathscr{K}}(H^*(Z), \mathbb{F}_p) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{K}}(TH^*(Y), \mathbb{F}_p) \cong \mathrm{Ext}^{s,t}_{\mathscr{K}}(H^*(Y), H)$$

induced by $\omega^{\#} : TH^*(Y) \longrightarrow H^*(Z)$. With due care of detail, the deduction that $\tilde{\omega}$ is an equivalence if $\omega^{\#}$ is an isomorphism follows by a comparison of spectral sequences argument. The converse implication is shown by a detailed inductive analysis of the spectral sequence.

An alternative procedure for processing Lannes' algebra to obtain the topological conclusion of Theorem 3.6 has been given by Morel. Using a topological interpretation of the functor T in terms of the continuous cohomology of pro-p-spaces, together with a comparison of Sullivan's p-adic completion functor with that of Bousfield and Kan, he manages to circumvent use of the Bousfield-Kan unstable Adams spectral sequence and thus to avoid use of heavy simplicial machinery.

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5. Maps between classifying spaces

We shall sketch the explanation given by Lannes in a talk at Chicago of how his Theorem 3.6 applies to give a version of results of Dwyer and Zabrodsky that apply the Sullivan conjecture to the study of maps between classifying spaces. Although these authors apparently were not aware of the connection with equivariant bundle theory, what is at issue is precisely the map

$$\alpha^G : \coprod B(\Pi^\rho) = B_G(\Pi)^G \longrightarrow \mathrm{Map}(EG, B\Pi)^G = \mathrm{Map}(BG, B\Pi)$$

that we described in VII.3.7; here the coproduct runs over $(\rho) \in \mathrm{Rep}(G, \Pi)$. The relevant theorem of Lannes is as follows.

Theorem 5.1 (Lannes). *If G is an elementary Abelian p-group and Π is a compact Lie group, then the map*

$$\coprod B(\Pi^\rho)^{\wedge}_p \longrightarrow \mathrm{Map}(BG, (B\Pi)^{\wedge}_p)$$

induced by α^G is an equivalence.

It should be possible to deduce inductively that the result holds in this form for any finite p-group. The original version of Dwyer and Zabrodsky is somewhat different and in some respects a little stronger, although it seems possible to deduce much of one from the other. We say that a map $f : X \longrightarrow Y$ is a "mod p equivalence" if it induces an isomorphism on mod p homology. We say that f is a "strong mod p equivalence" if it satisfies the following conditions.

(i) f induces an isomorphism $\pi_0(X) \longrightarrow \pi_0(Y)$;
(ii) f induces an isomorphism $\pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ for any $x \in X$;
(iii) f induces an isomorphism

$$H_*(\tilde{X}_x, \mathbb{F}_p) \longrightarrow H_*(\tilde{Y}_{f(x)}, \mathbb{F}_p)$$

for any $x \in X$, where $\tilde{X}_x$ and $\tilde{Y}_{f(x)}$ are the universal covers of the components of X and Y that contain x and $f(x)$.

Say that a G-map $f : X \longrightarrow Y$ is a (strong) mod p equivalence if $f^H : X^H \longrightarrow Y^H$ is a (strong) mod p equivalence for each $H \subset G$. In view of VII.3.7, the following statements are equivariant reinterpretations of nonequivariant results of Dwyer and Zabrodsky and Notbohm. In nonequivariant terms, when $\Gamma = G \times \Pi$, their results are statements about the map α^G above.

Theorem 5.2 (Dwyer and Zabrodsky). *If Π is a normal subgroup of a compact Lie group Γ and $G = \Gamma/\Pi$ is a finite p-group, then the G-map $\alpha : B(\Pi; \Gamma) \longrightarrow \mathrm{Sec}(EG, B\Gamma)$ is a strong mod p equivalence.*

Actually, Dwyer and Zabrodsky give the result in this generality for $G = \mathbb{Z}/p$, and they give an inductive scheme to prove the general case when $\Gamma = G \times \Pi$. However, their inductive scheme works just as well to handle the case of general extensions. Their result was generalized to p-toral groups by Notbohm.

Theorem 5.3 (Notbohm). *If Π is a normal subgroup of a compact Lie group Γ and $G = \Gamma/\Pi$ is a p-toral group, then the G-map $\alpha : B(\Pi; \Gamma) \longrightarrow \mathrm{Sec}(EG, B\Gamma)$ is a mod p equivalence.*

However, α need not a strong mod p equivalence in this case: the components of α^H induce injections but not surjections on the fundamental groups of corresponding components.

These results are some of the starting points for beautiful work of Jackowski, McClure, and Oliver, and others, on maps between classifying spaces; these authors have given an excellent survey of the state of the art on this topic.

Lannes' deduces Theorem 4.1 from Theorem 3.6 by taking $Z = \coprod B\Pi^\rho$ and $Y = B\Pi$. The map ω is then the sum of the classifying maps of the homomorphisms $\nu : V \times \Pi^\rho \longrightarrow \Pi$ specified in VII.3.7. The deduction is based on the case $X = *$ of the following calculation.

THEOREM 5.4. Let X be a finite Π-CW complex. Then the natural map

$$TH^*_\Pi(X) \longrightarrow \prod H^*_{\Pi^\rho}(X^{\rho(V)})$$

is an isomorphism, where the product runs over $(\rho) \in \mathrm{Rep}(V, \Pi)$.

PROOF. The proof is an adaptation of methods of Quillen. Embed Π in $U(n)$ for some large n and let F be the G-space $U(n)/S$, where S is a maximal elementary Abelian subgroup of $U(n)$. Quillen shows that the evident diagram of projections

$$X \times F \times F \rightrightarrows X \times F \longrightarrow X$$

induces an equalizer diagram

$$H^*_\Pi(X) \longrightarrow H^*_\Pi(X \times F) \rightrightarrows H^*_\Pi(X \times F \times F).$$

Let

$$j^*(X) = TH^*_\Pi(X)$$

and

$$k^*(X) = \prod_{(\rho)\in \mathrm{Rep}(V,\Pi)} H^*_{\Pi^\rho}(X^{\rho(V)}).$$

These are both Π-cohomology theories in X. Applied to our original diagram of projections, both give equalizers, the first because the functor T is exact and the second by an elaboration of Quillen's argument. We have an induced map from the equalizer diagram for j^* to that for k^*. The isotropy subgroups of the finite Π-CW complexes $X \times F$ and $X \times F \times F$ are elementary Abelian, and it therefore suffices to show that the map

$$TH^*(BW) \cong j^*(\Pi/W) \longrightarrow k^*(\Pi/W) \cong \prod_{(\rho)\in \mathrm{Rep}(V,\Pi)} H^*(E\Pi^\rho \times_{\Pi^\rho} (\Pi/W)^{\rho(V)})$$

is an isomorphism when W is an elementary Abelian subgroup of Π. I learned the details of how to see this from Nick Kuhn. He has shown that T enjoys the property

$$TH^*(BW) \cong \prod_{(\sigma)\in \mathrm{Rep}(V,W)} H^*(BW), \qquad (5.5)$$

and the map in cohomology that we wish to show is an isomorphism is in fact induced by a homeomorphism

$$\coprod_{(\rho)\in \mathrm{Rep}(V,\Pi)} E\Pi^{\rho} \times_{\Pi^{\rho}} (\Pi/W)^{\rho(V)} \longrightarrow \coprod_{(\sigma)\in \mathrm{Rep}(V,W)} BW. \tag{5.6}$$

To see the homeomorphism, note that Π acts on the disjoint union over $\rho \in \mathrm{Hom}(V,\Pi)$ of the spaces $(\Pi/W)^{\rho(V)}$; π sends a point $\pi' W$ fixed by $\rho(V)$ to the point $\pi\pi' W$ fixed by the π-conjugate of ρ. It is not hard to check that, as Π-spaces,

$$\coprod_{(\rho)\in \mathrm{Rep}(V,\Pi)} \Pi \times_{\Pi^{\rho}} (\Pi/W)^{\rho(V)} \cong \coprod_{\rho\in \mathrm{Hom}(V,\Pi)} (\Pi/W)^{\rho(V)} \cong \coprod_{\sigma\in \mathrm{Hom}(V,W)} \Pi/W.$$

Taking $E\Pi$ as a model for each $E\Pi^{\rho(V)}$, this implies the required homeomorphism. $\square$

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CHAPTER IX

An introduction to equivariant stable homotopy

$\mathscr{M}_G(V)$

1. G-spheres in homotopy theory

What is a G-sphereΓ In our work so far, we have only used spheres S^n, which have trivial action by G. Clearly this is contrary to the equivariant spirit of our work. The full richness of equivariant homotopy and homology theory comes from the interplay of homotopy theory and representation theory that arises from the consideration of spheres with non-trivial actions by G. In principle, it might seem reasonable to allow arbitrary G-actions. However, a closer inspection of the role of spheres in nonequivariant topology, both in manifold theory and in homotopy theory, gives the intuition that we should restrict to the linear spheres that arise from representations. Throughout the rest of the book, we shall generally use the term "representation of G" or sometimes "G-module" to mean a finite dimensional real inner product space with a given smooth action of G through linear isometries. We may think of V as a homomorphism of Lie groups $\rho : G \longrightarrow O(V)$. This convention contradicts standard usage, in which representations are defined to be isomorphism classes.

For a representation V, we have the unit sphere $S(V)$, the unit disk $D(V)$, and the one-point compactification S^V; G acts trivially on the point at infinity, which is taken as the basepoint of S^V. The based G-spheres S^V will be central to virtually everything that we do from now on. We agree to think of n as standing for $\mathbb{R}^n$ with trivial G-action, so that S^n is a special case of our definition. For a based G-space X, we write

$$\Sigma^V X = X \wedge S^V \quad \text{and} \quad \Omega^V X = F(S^V, X).$$

Of course, Σ^V is left adjoint to Ω^V.

When do we use trivial spheres and when do we use representation spheresΓ This is a subtle question, and in some of our work the answer may well seem counterintuitive. In defining weak equivalences of G-spaces, we only used homotopy groups defined in terms of trivial spheres, and that is unquestionably the right choice in view of the Whitehead theorem for G-CW complexes. Nevertheless, there are homotopy groups defined in terms of representation spheres, and they often play an important role, although more often implicit than explicit. We may think of a G-representation V as an H-representation for any $H \subseteq G$. For a based G-space X, we define

(1.1) $$\pi_V^H(X) = [S^V, X]_H \cong [G_+ \wedge_H S^V, X]_G.$$

Here the brackets denote based homotopy classes of based maps, with the appropriate equivariance. For a pair (X, A) of based G-spaces, we form the usual homotopy fiber Fi of the inclusion $i : A \longrightarrow X$, and we define

(1.2) $$\pi_{V+1}^H(X, A) = \pi_V^H(Fi).$$

It is natural to separate out the trivial and non-trivial parts of representations. Thus we let $V(H)$ denote the orthogonal complement in V of the fixed point space V^H. We then have the long exact sequence

(1.3)
$$\cdots \longrightarrow \pi_{V(H)+n}^H(X) \longrightarrow \pi_{V(H)+n}^H(X, A) \longrightarrow \cdots \longrightarrow \pi_{V(H)}^H(A) \longrightarrow \pi_{V(H)}^H(X)$$

of groups up to $\pi_{V(H)+1}^H(X)$ and of pointed sets thereafter.

Waner will develop a G-CW theory adapted to a given representation V in the next chapter, and Lewis will use it to study the Freudenthal suspension theorem for these homotopy groups in the chapter that follows. There is a more elementary standard form of the Freudenthal suspension theorem, due first to Hauschild, that suffices for many purposes. Just as nonequivariantly, it is proven by studying the adjoint map $\eta : Y \longrightarrow \Omega^V \Sigma^V Y$. Here one proceeds by reduction to the nonequivariant case and use of obstruction theory. Recall the notion of a ν-equivalence from I§3, where ν is a function from conjugacy classes of subgroups of G to the integers greater than or equal to -1. Define the connectivity function $c^*(Y)$ of a G-space Y by letting $c^H(Y)$ be the connectivity of Y^H for $H \subseteq G$; we set $c^H(Y) = -1$ if Y^H is not path connected.

THEOREM 1.4 (FREUDENTHAL SUSPENSION). The map $\eta : Y \longrightarrow \Omega^V\Sigma^V Y$ is a ν-equivalence if ν satisfies the following two conditions:

(1) $\nu(H) \leq 2c^H(Y) + 1$ for all subgroups H with $V^H \neq 0$, and
(2) $\nu(H) \leq c^K(Y)$ for all pairs of subgroups $K \subseteq H$ with $V^K \neq V^H$.

Therefore the suspension map

$$\Sigma^V : [X, Y]_G \longrightarrow [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if $dim(X^H) \leq \nu(H)$ for all H, and bijective if $dim(X^H) \leq \nu(H) - 1$.

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2. G-Universes and stable G-maps

We next explain how to stabilize homotopy groups and, more generally, sets of homotopy classes of maps between G-spaces. There are several ways to make this precise. The most convenient is that based on the use of universes.

DEFINITION 2.1. A G-universe U is a countable direct sum of representations such that U contains a trivial representation and contains each of its sub-representations infinitely often. Thus U can be written as a direct sum of subspaces $(V_i)^\infty$, where $\{V_i\}$ runs over a set of distinct irreducible representations of G. We say that a universe U is complete if, up to isomorphism, it contains every irreducible representation of G. If G is finite, one example is V^∞, where V is the regular representation of G. We say that a universe is trivial if it contains only the trivial irreducible representation. One example is U^G for a complete universe U. A finite dimensional sub G-space of a universe U is said to be an indexing space in U.

We should emphasize right away that, as soon as we start talking seriously about stable objects, or "spectra", the notion of a universe will become important even in the nonequivariant case.

We can now give a first definition of the set $\{X, Y\}_G$ of stable maps between based G-spaces X and Y.

DEFINITION 2.2. Let U be a complete G-universe. For a finite based G-CW complex X and any based G-space Y, define

$$\{X, Y\}_G = \mathrm{colim}_V [\Sigma^V X, \Sigma^V Y]_G,$$

where V runs through the indexing spaces in U and the colimit is taken over the functions

$$[\Sigma^V X, \Sigma^V Y]_G \longrightarrow [\Sigma^W X, \Sigma^W Y]_G, \quad V \subset W,$$

that are obtained by sending a map $\Sigma^V X \longrightarrow \Sigma^V Y$ to its smash product with the identity map of S^{W-V}.

When G is finite and X is finite dimensional, the Freudenthal suspension theorem implies that if we suspend by a sufficiently large representation, then all subsequent suspensions will be isomorphisms.

COROLLARY 2.3. If G is finite and X is finite dimensional, there is a representation $V_0 = V_0(X)$ such that, for any representation V,

$$\Sigma^V : [\Sigma^{V_0} X, \Sigma^{V_0} Y]_G \xrightarrow{\cong} [\Sigma^{V_0 \oplus V} X, \Sigma^{V_0 \oplus V} Y]_G$$

is an isomorphism.

Let X and Y be finite G-CW complexes. If G is finite, the stable value

$$[\Sigma^{V_0} X, \Sigma^{V_0} Y]_G = \{X, Y\}_G$$

is a finitely generated abelian group. However, if G is a compact Lie group and X has infinite isotropy groups, there is usually no representation V_0 for which all further suspensions Σ^V are isomorphisms, and $\{X, Y\}_G$ is usually not finitely generated.

REMARK 2.4. The groups $\{S^V, X\}_G$ are called equivariant stable homotopy groups of X and are sometimes denoted $\omega_V^G(X)$. However, it is more usual to denote them by $\pi_V^G(X)$, relying on context to resolve the ambiguity between stable and unstable homotopy groups.

The definition of $\{X, Y\}_G$ just given is not the right definition for an infinite complex X. Observe that

$$[\Sigma^V X, \Sigma^V Y]_G \cong [X, \Omega^V \Sigma^V Y]_G.$$

DEFINITION 2.5. Let U be a complete G-universe. For a based G-space X, define

$$QX = \operatorname{colim}_V \Omega^V \Sigma^V X,$$

where V runs over the indexing spaces in U and the colimit is taken over the maps

$$\Omega^V \Sigma^V X \longrightarrow \Omega^W \Sigma^W X, \quad V \subset W,$$

that are obtained by sending a map $S^V \longrightarrow X \wedge S^V$ to its smash product with the identity map of S^{W-V}. Observe that the maps of the colimit system are inclusions.

LEMMA 2.6. Fix an indexing space V in U. For based G-spaces X, there is a natural homeomorphism

$$QX \cong \Omega^V Q\Sigma^V X.$$

PROOF. Clearly QX is homeomorphic to $\mathrm{colim}_{W\supset V}\,\Omega^W\Sigma^W X$, and similarly for $Q\Sigma^V X$. By the compactness of S^V and the evident isomorphisms of functors $\Sigma^V\Sigma^{W-V} \cong \Sigma^W$ and $\Omega^V\Omega^{W-V} \cong \Omega^W$ for $V \subseteq W$,

$$\mathrm{colim}\,\Omega^W\Sigma^W X \cong \mathrm{colim}\,\Omega^V\Omega^{W-V}\Sigma^{W-V}\Sigma^V X \cong \Omega^V\,\mathrm{colim}\,\Omega^{W-V}\Sigma^{W-V}\Sigma^V X,$$

where the colimits are taken over $W \supseteq V$. The conclusion follows. $\square$

LEMMA 2.7. If X is a finite G-CW complex, then

$$\{X, Y\}_G \cong [X, QY]_G.$$

PROOF. This is immediate from the compactness of X, which ensures that

$$[X, QY]_G \cong \mathrm{colim}_V [X, \Omega^V\Sigma^V Y]_G. \quad \square$$

For infinite complexes X, it is $[X, QY]_G$ that gives the right notion of the stable maps from X to Y. We shall return to this point in Chapter XII, where we introduce the stable homotopy category of spectra.

3. Euler characteristic and transfer G-maps

We here introduce some fundamentally important examples of stable maps that require the use of representations for their definitions. The Euler characteristic and transfer maps defined here will appear at increasing levels of sophistication and generality as we go on.

Let M be a smooth closed G-manifold. We may embed M in a representation V, say with normal bundle ν. We may then embed a copy of ν as a tubular neighborhood of M in V. Just as for nonequivariant bundles, the Thom complex $T\xi$ of a G-vector bundle ξ is constructed by forming the fiberwise one-point compactification of the bundle, letting G act trivially on the points at infinity, and then identifying all of the points at infinity to a single G-fixed basepoint $*$. We then have the Pontrjagin-Thom map

$$t : S^V \longrightarrow T\nu.$$

It is the based G-map obtained by mapping the tubular neighborhood isomorphically onto ν and mapping all points not in the tubular neighborhood to the basepoint $*$. The inclusion of ν in $\tau_M \oplus \nu$, where τ_M is the tangent bundle of M, induces a based G-map

$$e : T\nu \longrightarrow T(\tau_M \oplus \nu) \cong M_+ \wedge S^V.$$

The composite of these two maps is the "transfer map"

$$\tau(M) = e \circ t : S^V \longrightarrow \Sigma^V M_+ \tag{3.1}$$

associated to the projection $M \longrightarrow \{pt\}$, which we think of as a trivial G-bundle. Of course, this projection induces a map

$$\xi : \Sigma^V M_+ \longrightarrow \Sigma^V S^0 \cong S^V.$$

We define the Euler characteristic of M to be the based G-map

$$\chi(M) = \xi \circ \tau(M) : S^V \longrightarrow S^V. \tag{3.2}$$

The name comes from the fact that if we ignore the action of G and regard $\chi(M)$ as a nonequivariant map of spheres, then its degree is just the classical Euler characteristic of M. The proof is an interesting exercise in classical algebraic topology, but the fact will become clear from our later more conceptual description of these maps. In fact, from the point of view that we will explain in XV§1, this map *is* the Euler characteristic of M, by definition.

Since V is not well-defined — we just chose some V large enough that we could embed M in it — it is most natural to regard the transfer and Euler characteristics as stable maps

$$\tau(M) \in \{S^0, M_+\}_G \quad \text{and} \quad \chi(M) \in \{S^0, S^0\}_G. \tag{3.3}$$

Observe that, when $M = G/H$, the map $\tau(G/H)$ of (3.1) can be written as the composite

$$\tau(G/H) : S^V \xrightarrow{t} G_+ \wedge_H S^W \xrightarrow{e} G_+ \wedge_H S^V \cong (G/H)_+ \wedge S^V, \tag{3.4}$$

where W is the complement of the image in V of the tangent plane $L(H)$ at the identity coset and e is the extension to a G-map of the H-map obtained by smashing the inclusion $S^0 \longrightarrow S^{L(H)}$ with S^W. The unlabelled isomorphism is given by I.2.6.

More generally, for subgroups $K \subset H$ of G, there is a stable transfer G-map $\tau(\pi) : G/K_+ \longrightarrow G/H_+$ associated to the projection $G/H \longrightarrow G/K$. In fact, we may view π as the extension to a G-map

$$G \times_K (K/H) \longrightarrow G/K$$

of the projection $K/H \longrightarrow \{pt\}$, and we may construct the transfer K-map $\tau(K/H)$ starting from an embedding of K/H in a G-representation V regarded as a K-representation by restriction. We then define $\tau(\pi)$ to be the map

(3.5)
$$\tau(\pi) : G/K_+ \wedge S^V \cong G_+ \wedge_K S^V \longrightarrow G_+ \wedge_K (K/H_+ \wedge S^V) \cong G/H_+ \wedge S^V,$$

where the isomorphisms are given by I.2.6 and the arrow is the extension of the K-map $\tau(K/H)$ to a G-map. Note that any G-map $f : G/K_+ \longrightarrow G/H_+$ is the composite of a conjugation isomorphism $c_g : G/K \longrightarrow G/g^{-1}Kg$ and the projection induced by an inclusion $g^{-1}Kg \subset H$. We let $\tau(c_g) = c_{g^{-1}}$. With these definitions, we obtain a contravariantly functorial assignment of stable transfer maps $\tau(f)$ to G-maps f between orbits. Of course, such G-maps may themselves be regarded as stable G-maps between orbits.

4. Mackey functors and coMackey functors

We are headed towards the notions of $RO(G)$-graded homology and cohomology theories, but we start by describing what the coefficients of such theories will look like in the case of "ordinary" $RO(G)$-graded theories.

Recall that the ordinary homology and the ordinary cohomology of G-spaces are defined in terms of covariant and contravariant coefficient systems, which are functors from the homotopy category $h\mathscr{G}$ of orbits to the category $\mathbb{A}b$ of Abelian groups. Let $\mathscr{A}_G$ denote the category that is obtained from $h\mathscr{G}$ by applying the free Abelian group functor to morphisms. Thus $\mathscr{A}_G(G/H, G/K)$ is the free Abelian group generated by $h\mathscr{G}(G/H, G/K)$. Then coefficient systems are the same as additive functors $\mathscr{A}_G \longrightarrow \mathscr{A}b$.

Now imagine what the stable analog might be. It is clear that the sets $\{X, Y\}_G$ are already Abelian groups.

DEFINITION 4.1. Define the Burnside category $\mathscr{B}_G$ to have objects the orbit spaces G/H and to have morphisms

$$\mathscr{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G,$$

with the evident composition. We shall also refer to $\mathscr{B}_G$ as the stable orbit category. Observe that it is an "$\mathscr{A}b$-category": its Hom sets are Abelian groups and composition is bilinear.

We must explain the name "Burnside". The zeroth equivariant stable homotopy group of spheres or equivariant "zero stem" $\{S^0, S^0\}_G$ is a ring under composition. We shall denote this ring by B_G for the moment. It is a fundamental insight of Segal that, if G is finite, then B_G is isomorphic to the Burnside ring $A(G)$. Here $A(G)$ is defined to be the Grothendieck ring of isomorphism classes of finite G-sets with addition and multiplication given by disjoint union and Cartesian product. For a compact Lie group G, tom Dieck generalized this description of B_G by defining the appropriate generalization of the Burnside ring. In this case, $A(G)$ is defined to be the ring of equivalence classes of smooth closed G-manifolds, where two such manifolds are said to be equivalent if they have the same Euler characteristic in B_G; again, addition and multiplication are given by disjoint union and Cartesian product. An exposition will be given in XVII§2.

Definition 4.2. A covariant or contravariant stable coefficient system is a covariant or contravariant additive functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$. A *contravariant* stable coefficient system is called a *Mackey functor*. A *covariant* stable coefficient system is called a *coMackey functor*.

When G is finite, Dress first introduced Mackey functors, using an entirely different but equivalent definition, to study induction theorems in representation theory. We shall explain the equivalence of definitions in XIX§3. The classical examples of Mackey functors are the representation ring and Burnside ring Mackey functors, which send G/H to $R(H)$ or $A(H)$. The generalization to compact Lie groups was first defined and exploited by Lewis, McClure, and myself.

Observe that we obtain an additive functor $\mathscr{A}_G \longrightarrow \mathscr{B}_G$ by sending the homotopy class of a G-map $f : G/H \longrightarrow G/K$ to the corresponding stable map. Therefore a (covariant or contravariant) stable coefficient system has an underlying ordinary coefficient system. Said another way, stable coefficient systems can be viewed as given by additional structure on underlying ordinary coefficient systems.

What is the additional structureΓ Viewed as a stable map, $\tau(G/H)$ is a morphism $G/G \longrightarrow G/H$ in the category $\mathscr{B}_G$, and, more generally, so is $\tau(f)$ for any G-map $f : G/H \longrightarrow G/K$. We shall see in XIX§3 that every morphism of the category $\mathscr{B}_G$ is a composite of stable G-maps of the form f or $\tau(f)$. That is, the extra structure is given by transfer maps. When G is finite, we shall explain algebraically how composites of such maps are computed. In the general compact Lie case, such composites are quite hard to describe. For this reason, it is also quite hard to construct Mackey functors algebraically. However, we have the following concrete example. It may not seem particularly interesting at first sight, but we shall shortly use it to prove an important result called the Conner conjecture.

PROPOSITION 4.3. Let G be any compact Lie group. There is a unique Mackey functor $\underline{\mathbb{Z}} : \mathscr{B}_G \longrightarrow \mathscr{A}b$ such that the underlying coefficient system of $\underline{\mathbb{Z}}$ is constant at $\mathbb{Z}$ and the homomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ induced by the stable transfer map $G/K_+ \longrightarrow G/H_+$ associated to an inclusion $H \subset K$ is multiplication by the Euler characteristic $\chi(K/H)$.

PROOF. In XIX§3, we shall give a complete additive calculation of the morphisms of $\mathscr{B}_G$, from which the uniqueness will be clear. The problem is to show that the given specifications are compatible with composition. We do this indirectly. As already noted, we have the Burnside Mackey functor $\underline{A}$. Thought of topologically, its value on G/H is

$$\{G/H_+, S^0\}_G \cong \{S^0, S^0\}_H = B_H,$$

and the contravariant functoriality is clear from this description. Define another Mackey functor $\underline{I}$ by letting $\underline{I}(G/H)$ be the augmentation ideal of $A(H)$. Thought of topologically, its value on G/H is the kernel of the map

$$\{G/H_+, S^0\}_G \longrightarrow \{G_+, S^0\}_G \cong \mathbb{Z}$$

induced by the G-map $G \longrightarrow G/H$ that sends the identity element e to the coset eH. Using XIX.3.2 and the definition of Burnside rings in terms of Euler characteristics, one can check that $\underline{I}$ is a subfunctor of $\underline{A}$. A key point is the identity

$$\chi(Y)\chi(H/K) = \chi(H \times_K Y)$$

of nonequivariant Euler classes for $H \subset K$ and H-spaces Y. One can then define $\underline{\mathbb{Z}}$ to be the quotient Mackey functor $\underline{A}/\underline{I}$; the desired Euler characteristic formula can be deduced from the formula just cited. □

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5. $RO(G)$-graded homology and cohomology

We shall be precise about how to define $RO(G)$-graded homology and cohomology theories in XIII§1. Here we give an intuitive description. The basic idea is that if we understand G-spheres to be representation spheres S^V, then we must understand the suspension axiom to allow suspension by such spheres. This forces us to grade on representations. However, the standard term "$RO(G)$-grading" is a technical misnomer since the real representation ring $RO(G)$ is defined in terms of isomorphism classes of representations, and this is too imprecise to allow the control of "signs" (which must be interpreted as units in the Burnside ring of G).

Thus, intuitively, a reduced $RO(G)$-graded homology theory $\tilde{E}^G_*$ defined on based G-spaces X consists of functors $\tilde{E}^G_\alpha : \bar{h}G\mathscr{T} \longrightarrow \mathscr{A}b$ for all $\alpha \in RO(G)$ together with suitably compatible natural suspension isomorphisms

$$\tilde{E}^G_\alpha(X) \cong \tilde{E}^G_{\alpha+V}(\Sigma^V X)$$

for all G-representations V. We require each $\tilde{E}^G_\alpha$ to carry cofibration sequences $A \longrightarrow X \longrightarrow X/A$ of based G-spaces to three term exact sequences and to carry wedges to direct sums. We have combined the homotopy and weak equivalence axioms in the statement that the $\tilde{E}^G_\alpha$ are defined on $\bar{h}G\mathscr{T}$.

For each representation V with $V^G = 0$, it follows by use of the suspension isomorphism for S^1 that the groups $\{\tilde{E}^G_{V+n} | n \in \mathbb{Z}\}$ give a reduced $\mathbb{Z}$-graded homology theory in the sense that the evident equivariant analogs of the Eilenberg-Steenrod axioms, other than the dimension axiom, are satisfied. Taking $V = 0$, this gives the underlying $\mathbb{Z}$-graded homology theory of the given $RO(G)$-graded theory. We could elaborate by defining unreduced theories, showing how to construct unreduced theories from reduced ones by adjoining disjoint basepoints and defining appropriate relative groups, and showing that unreduced theories give rise to reduced ones in the usual fashion. However, we concentrate on the essential new feature, which is the suspension axiom for general representations V.

Of course, we have a precisely similar definition of an $RO(G)$-graded cohomology theory. There are two quite different philosophies about these $RO(G)$-graded theories. One may view them as the right context in which to formulate calculations. For example, there are calculations of Lewis that show that the cohomology of a space may have an elegant algebraic description in $RO(G)$-graded cohomology that is completely obscured when one looks only at the $\mathbb{Z}$-graded part of the relevant theory. In contrast, one may view $RO(G)$-gradability as a tool for the study of the $\mathbb{Z}$-graded parts of theories. Our proof of the Conner conjecture in the next section will be a direct application of that philosophy.

When can the $\mathbb{Z}$-graded cohomology theory with coefficients in a coefficient system M be extended to an $RO(G)$-graded cohomology theory? If we are given such an extension, then the transfer maps $\tau(G/H) : S^V \longrightarrow G/H_+ \wedge S^V$ of (3.4) will induce transfer homomorphisms

(5.1)
$$\begin{array}{c} \tilde{H}^n_H(X; M|H) \cong \tilde{H}^{v+n}_G(\Sigma^V(G/H_+ \wedge X); M) \\ \Big\downarrow \\ \tilde{H}^n_G(X; M) \cong \tilde{H}^{v+n}_G(\Sigma^V X; M). \end{array}$$

Taking $n = 0$ and $X = S^0$, we obtain a transfer homomorphism $M(G/H) \longrightarrow M(G/G)$. An elaboration of this argument shows that the coefficient system M must extend to a Mackey functor. It is a pleasant fact that this necessary condition is sufficient.

THEOREM 5.2. Let G be a compact Lie group and let M and N be a contravariant and a covariant coefficient system The ordinary cohomology theory $\tilde{H}^*_G(-; M)$ extends to an $RO(G)$-graded cohomology theory if and only if M extends to a Mackey functor. The ordinary homology theory $\tilde{H}^G_*(-; N)$ extends to an $RO(G)$-graded homology theory if and only if N extends to a coMackey functor.

We shall later explain two very different proofs. Waner will describe a chain level construction in terms of G-CW(V) complexes in the next chapter. I will describe a spectrum level construction of the representing Eilenberg-Mac Lane G-spectra in XIII§4.

L. G. Lewis, Jr. The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/p$ actions. Springer Lecture Notes in Mathematics Vol. 1361, 1988, 53-122.

6. The Conner conjecture

To illustrate the force of $RO(G)$-gradability, we explain how the results stated in the previous two sections directly imply the following conjecture of Conner.

THEOREM 6.1 (CONNER CONJECTURE). Let G be a compact Lie group and let X be a finite dimensional G-space with finitely many orbit types. Let A be any Abelian group. If $\tilde{H}^*(X; A) = 0$, then $\tilde{H}^*(X/G; A) = 0$.

This was first proven by Oliver, using Čech cohomology and wholly different techniques. It was known early on that the conjecture would hold if one could construct a suitable transfer map.

THEOREM 6.2. Let X be any G-space and let $\pi : X/H \longrightarrow X/G$ be the natural projection, where $H \subset G$. For $n \geq 0$, there is a natural transfer homomorphism

$$\tau : \tilde{H}^n(X/H; A) \longrightarrow \tilde{H}^n(X/G; A)$$

such that $\tau \circ \pi^*$ is multiplication by the Euler characteristic $\chi(G/H)$.

PROOF. Tensoring the Mackey functor $\underline{\mathbb{Z}}$ of Proposition 4.3 with A, we obtain a Mackey functor $\underline{A}$ whose underlying coefficient system is constant at A. The map $\underline{A}(G/H) \longrightarrow \underline{A}(G/G)$ associated to the stable transfer map $G/G_+ \longrightarrow G/H_+$ is multiplication by $\chi(G/H)$. As we observed in our first treatment of Smith theory (IV§1), ordinary G-cohomology with coefficients in a constant coefficient system is the same as ordinary nonequivariant cohomology on orbit spaces:

$$H^n(X/H; A) \cong H^n_H(X; \underline{A}|H) \quad \text{and} \quad H^n(X/G; R) \cong H^n_G(X; \underline{A}).$$

Taking $M = \underline{A}$, (5.1) already displays the required transfer map. The formula for $\tau \circ \pi^*$ follows formally, but it can also be derived from the fact that the equivariant Euler characteristic

$$S^V \longrightarrow G/H_+ \wedge S^V \longrightarrow S^V,$$

regarded as a nonequivariant map, has degree $\chi(G/H)$. $\square$

How does the Conner conjecture follow? Conner himself proved it when G is a finite extension of a torus, the methods being induction and use of Smith theory — one proves that both X^G and X/G are A-acyclic. For example, the result for a torus reduces immediately to the result for a circle. Here the "finitely many orbit types" hypothesis implies that $X^G = X^C$ for C cyclic of large enough order, so that we really are in the realm where Smith theory can be applied. Assuming that the result holds when G is a finite extension of a torus, let N be the normalizer of a maximal torus in G. Then N is a finite extension of a torus and $\chi(G/N) = 1$. The composite

$$\tau \circ \pi^* : \tilde{H}^n(X/G; A) \longrightarrow \tilde{H}^n(X/N; A) \longrightarrow \tilde{H}^n(X/G; A)$$

is the identity, and that's all there is to it.

P. Conner. Retraction properties of the orbit space of a compact topological transformation group. Duke Math. J. 27(1960), 341-357.

G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$-graded cohomology. Bulletin Amer. Math. Soc. 4(1981), 208-212.

R. Oliver. A proof of the Conner conjecture. Annals of Math. 103(1976), 637-644.

CHAPTER X

G-CW(V) complexes and $RO(G)$-graded cohomology

by Stefan Waner

1. Motivation for cellular theories based on representations

If a compact Lie group G acts smoothly on a smooth manifold M then the action is locally orthogonal. That is, for each $x \in M$ there is a G_x-invariant neighborhood U of x diffeomorphic to the open unit disc in a representation V of G_x. Moreover, writing G_x as H, if $L(H)$ is the tangent representation of H at $eH \in G/H$, then $L(H)$ is a summand of V. (Of course, $L(H) = 0$ if G is finite.) It follows that the G-orbit of x has a neighborhood diffeomorphic to $G \times_H D(V - L(H))$, where $V - L(H)$ is the orthogonal complement of $L(H)$ in V.

The above remarks seem to suggest that one ought to consider G-complexes modeled by cells of this form. On the other hand, it has been established by Bredon and others that ordinary G-CW complexes seem to suffice for practical purposes. These are G-complexes with "cells" of the form $G/H \times D^n$, where G acts trivially on D^n. Basically, the local neighborhoods $G \times_H D(V - L(H))$ can be G-triangulated into cells of the above form, so it would seem that there is no need to consider anything more elaborate than G-CW complexes. But there are some theoretical difficulties:

(1) Duality doesn't work. That is, the cellular chains obtained from G-CW structures on smooth G-manifolds do not exhibit Poincaré duality. The geometric reason for this is that the dual of an n-dimensional G-cell $G/H \times D^n$ is not a G-cell. The dual cell to a zero dimensional cell G/H is defined as its star in the first barycentric subdivision, while the duals of higher dimensional cells are

intersections of such stars. In general, the dual of a G-cell $G/H \times D^n$ has the form $G \times_H D(V - L(H) - \mathbb{R}^n)$, where V is the local representation at eH. This really forces our hand.

(2) One has the result, due to various authors (Lewis, May, McLure, Waner) that, if M is a Mackey functor, then Bredon cohomology with coefficients in M extends to an $RO(G)$-graded cohomology theory. This will be treated from the stable homotopy category point of view later in the book. The question then is: what is the geometric representation of the cells in dimension V? In particular, can we write the Vth cohomology group in terms of the cohomology of a cellular cochain complex?

The purpose of this chapter is to outline the basic theory of cell complexes modeled on representations of G, and to use them to construct explicit models of ordinary $RO(G)$-graded cohomology in which Poincaré duality holds for certain classes of G-manifolds. For reasons of clarity, only complexes modeled on a single representation V of G will be discussed. The more elaborate theory in which V is allowed to vary is already completed as joint work with Costenoble and May, and some of it has appeared in papers of Costenoble and myself. Roughly speaking, whatever works for a single representation generalizes to the more elaborate case.

When G is not finite, there appear to be two theories of G-CW(V) complexes. The one that I will concentrate on will be the one that is *not* dual to the usual G-CW theory (on suitable G-manifolds), but that does work as a cellular theory and gives rise to ordinary $RO(G)$-graded cohomology. To make amends, we will very briefly indicate the present state of the variant that gives the true dual theory.

S. R. Costenoble and S. Waner. The equivariant Thom isomorphism theorem. Pacific J. Math. 152(1992), 21-39.

S. R. Costenoble and S. Waner. Equivariant Poincaré duality. Michigan Math. J. 39(1992).

2. G-CW(V) complexes

Let V be a fixed given orthogonal representation of G and write dim $V = |V|$. To understand the definitions that follow, it is useful to keep in mind the following observation, whose easy inductive proof will be left to the reader.

LEMMA 2.1. Let $H_n \subset H_{n-1} \subset \cdots \subset H_0 = G$ be a strictly increasing chain of subgroups of G such that each H_i occurs as the isotropy subgroup of some point in V (the point 0 having isotropy group G). Then, as a representation of H_n, V contains a trivial representation of dimension n.

For $H \subseteq G$, we let $V(H)$ denote the orthogonal complement of V^H in V. If W is an H-module, we let $D(W)$ and $S(W)$ denote the unit disc and sphere in W.

DEFINITION 2.2. A G-CW(V) complex is a G-space X with a decomposition $X = \mathrm{colim}_n X^n$ such that X^0 is a disjoint union of G-orbits of the form G/H, where H acts trivially on V, and X^n is obtained from X^{n-1} by attaching "cells" $G \times_H D(V(H) \oplus \mathbb{R}^t)$, where $|V(H)| + t = n$, along attaching G-maps

$$G \times_H S(V + \mathbb{R}^t) \longrightarrow X^{n-1}.$$

A map $f : X \longrightarrow Y$ between G-CW(V) complexes is cellular if $f(X^n) \subset Y^n$ for all n, and the notions of skeleta, dimension, subcomplex, relative G-CW(V) complex, and so on are defined as one would expect from the classical case $V = 0$.

REMARKS 2.3. (i) Although imprecise, it is convenient to think of $V(H) \oplus \mathbb{R}^t$ as $V + \mathbb{R}^s$, where $|V| + s = n$ and thus $|V^H| + s = t$; here s may be negative, but then the definition implies that $|V^H| \geq -s$ for all subgroups H occurring in the decomposition.
(ii) The stipulation on the dimension implies that the cell $G \times_H D(V(H) \oplus \mathbb{R}^t)$ is an $(n + dim\, G/H)$-dimensional G-manifold.

The last observation explains why the definition does not give the true dual theory when G has positive dimension. The following variant rectifies this. However, this theory has not yet been worked out thoroughly or extended to deal with varying representations, although we suspect that all works well.

VARIANT 2.4. Let G be an infinite compact Lie group. There is a variant definition of a G-CW(V) complex which differs from the definition given in that we require X^0 to be a disjoint union of finite orbits G/H such that H acts trivially on V and we attach cells of the form $G \times_H D((V - L(H)) + \mathbb{R}^s)$, where $|V| + s = n$, when constructing X^n from X^{n-1}. Here $L(H)$ is the tangent representation of G/H at eH, and the definition implies that $L(H)$ is contained in $V|_H$ for all subgroups H occurring in the decomposition. With these stipulations on dimensions, the n-cells that we attach are n-dimensional G-manifolds.

Part of our motivation comes from consideration of G-manifolds that are locally modeled on a single representation.

Definition 2.5. A smooth G-manifold M has dimension V if, for each $x \in M$, there is a G_x-invariant neighborhood U of x that is diffeomorphic to the open unit disc in the restriction of V to G_x. It follows that $L(H)$ embeds in $V|_H$ and the orbit Gx has a neighborhood of the form $G \times_H D(V - (L(H))$. Any smooth G-manifold M each of whose fixed point sets is non-empty and connected must have dimension V, where V is the tangent representation at any G-fixed point. More generally, M has dimension $V - i$ for a positive integer $i \leq |V|$ if, for each $x \in M$, G_x acts on V with an i-dimensional trivial summand and there is a G_x-invariant neighborhood U of x that is G_x-diffeomorphic to the open unit disc in $V - \mathbb{R}^i$. Thus, if M has dimension V, then ∂M has dimension $V - 1$. For example, $D(V)$ is a V-dimensional manifold and $S(V)$ is a $(V-1)$-dimensional manifold.

When G is finite, G-manifolds of dimension V and their bordism theories were first discussed by Pulikowski and Kosniowski; I later carried the study further. By a theorem of Stong, if G is finite of odd order, then any G-manifold is cobordant to a sum of G-manifolds of the form $G \times_H N$, where N has dimension W for some H-module W.

The classical theory of dual cell decompositions of smooth manifolds (for which see Seifert and Threlfall) generalizes to V-manifolds. We shall not go into the definitions needed to make this precise. The intuition comes from equivariant Spanier-Whitehead and Atiyah duality, which will be discussed in XVI§§7-8. If a closed smooth G-manifold M embeds in V, then M_+ is V-dual to the Thom space $T\nu$ of the normal bundle of the embedding. In the case $M = G/H$, this normal bundle is $T\nu = G_+ \wedge_H S^{V-L(H)}$.

Proposition 2.6. If G is finite, then we obtain a G-CW(V) structure on a $(V - i)$-dimensional manifold M by passage to dual cells from an ordinary G-CW structure. With the variant definition of a G-CW(V) complex, the statement remains true for general compact Lie groups G.

From now on, we restrict attention to our first definition of a G-CW(V) complex.

Lemma 2.7. If X is a G-CW complex, then $X \times D(V)$ has the structure of a G-CW(V) complex under the usual product structure. Therefore, for any V, any G-CW complex is G-homotopy equivalent to a G-CW(V) complex.

Proposition 2.8. For any V, a G-space has the G-homotopy type of a G-CW complex if and only if it has the G-homotopy type of a G-CW(V) complex.

The lemma gives the forward implication in the case of finite G. The case for general compact Lie groups is harder, and we need to use the equivariant version of Brown's construction to give a brute force weak G-approximation by a G-CW(V) complex. That this approximation is in fact a G-homotopy equivalence then follows from the converse and the G-Whitehead theorem. For the converse, if X is a G-CW(V) complex, then X is a colimit of spaces of the G-homotopy type of G-CW complexes, and thus X is also such a homotopy type by a telescope argument and the homotopy invariance of colimits.

PROPOSITION 2.9. If X and Y have, respectively, a G-CW(V) and G-CW(W) structure, then $X \times Y$ has a G-CW($V \oplus W$) structure.

C. Kosniowski. A note on $RO(G)$-graded G-bordism. Quart J. Math. Oxford 26(1975), 411-419.

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H. Seifert and W. Threlfall. A Textbook of Topology (translation). Academic Press. 1980.

R. E. Stong. Unoriented bordism and actions of finite groups. Memoirs A.M.S. No. 103. 1970.

Equivariant $RO(G)$-graded bordism theories. Topology and its Applications. 17(1984), 1-26.

3. Homotopy theory of G-CW(V) complexes

We now do a little homotopy theory. Since we are using representations to define attaching maps, it is reasonable to consider the homotopy groups that were defined in terms of representations in IX.1.1.

DEFINITION 3.1. A G-space X is V-connected if X^H is $|V^H|$-connected for each closed subgroup $H \subseteq G$. Let $e : X \longrightarrow Y$ be a G-map and let n be an integer. Then e is a $(V+n)$-equivalence if, for each $H \subset G$ and each choice of basepoint in X^H, $e_* : \pi^H_{V(H)+q}(X) \longrightarrow \pi^H_{V(H)+q}(Y)$ is an isomorphism if $q \leq |V^H| + n - 1$ and an epimorphism if $q \leq |V^H| + n$.

THEOREM 3.2 (HELP). Let $e : Y \longrightarrow Z$ be a $(V+n)$-equivalence and let (X, A) be a relative G-CW(V) complex of dimension $\leq |V| + n$. Then we can complete the following homotopy extension and lifting diagram:

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A\times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr, "i_0"'] \arrow[ur, "f"] & & X\times I \arrow[ul, dashed, "\tilde{h}"'] & & X. \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Sketch of Proof. We extend the G-maps g and h cell-by-cell and work inductively. This reduces the problem to the special case where $A = G \times_H S(W)$ and $X = G \times_H D(W)$. The pair (X, A) then has the structure of a relative G-CW complex with G-cells of the form $G/K \times D^r$ with $r \leq |W^K| \leq |V^K| + n$ and K subconjugate to H. Since e^K is a $(|V^K| + n)$-equivalence, this allows us to apply the HELP theorem of ordinary G-homotopy theory to complete the proof. $\square$

Theorem 3.3 (G-CW(V) Whitehead). Let $e : Y \longrightarrow Z$ be a $(V + n)$-equivalence and let X be a G-CW(V) complex. Then $e_* : [X, Y]_G \longrightarrow [X, Z]_G$ (unbased G-homotopy classes) is an isomorphism if $\dim X < n + |V|$ and an epimorphism if $\dim X = n + |V|$. Moreover the conclusion remains true if $n = \infty$.

Proof. As usual, apply HELP to the pair $(X, \emptyset)$ for surjectivity and to the pair $(X \times I, X \times \partial I)$ for injectivity. $\square$

Theorem 3.4 (Cellular Approximation). Every G-map $f : X \longrightarrow Y$ of G-CW(V) complexes is G-homotopic to a cellular map. If f is already cellular on a subcomplex A, then the homotopy can be taken relative to A.

Sketch of proof. One easily shows that the inclusion $i : Y^n \longrightarrow Y$ is a $(V + n - |V|)$-equivalence, and HELP then applies inductively to push X^n into Y^n and give the required homotopy. $\square$

Theorem 3.5. For any G-space X, there is a G-CW(V) complex $?\,X$ and a weak equivalence $\gamma : ?\,X \longrightarrow X$.

Sketch of proof. In view of Proposition 2.8, this follows directly from the analog for ordinary G-CW complexes. $\square$

4. Ordinary $RO(G)$-graded homology and cohomology

Recall the discussion of stable coefficient systems, alias Mackey and coMackey functors, from IX§4. The algebra of stable coefficient systems works in the same way as the algebra of coefficient systems discussed in I§3. The categories of Mackey functors and of coMackey functors are Abelian. If M and N are, respectively, Mackey and coMackey functors, we have the coend or tensor product $M \otimes_{\mathscr{B}_G} N$. If M and M' are Mackey functors, we have the group of natural transformations $\mathrm{Hom}_{\mathscr{B}_G}(M, M')$.

Observe that, for any based G-spaces X and Y, we have a Mackey functor $\underline{\{X,Y\}}_G$ with values

$$\underline{\{X,Y\}}_G(G/H) = \{G/H_+ \wedge X, Y\}_G.$$

The contravariant functoriality is given by composition in the evident way.

Definition 4.1. Let X be a G-CW(V) complex. Define a chain complex $\underline{C}_*^V(X)$ in the Abelian category of Mackey functors as follows. Let

$$\underline{C}_n^V(X) = \underline{\left\{S^{V-|V|+n}, X^n/X^{n-1}\right\}}_G.$$

This is the stable H-homotopy group of X^n/X^{n-1} in dimension $V - |V| + n$. Let

$$d_n : \underline{C}_n^V(X) \longrightarrow \underline{C}_{n-1}^V(X)$$

be the stable connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}).

Observe that X^n/X^{n-1} is the wedge over the n-cells of X of G-spaces of the form $G/H_+ \wedge S^{V-|V|+n}$ and that $\underline{C}_n^V(X)$ is the direct sum of corresponding free Mackey functors represented by the objects G/H.

Definition 4.2. Let X be a G-CW(V) complex. For a Mackey functor M, define the ordinary cohomology of X with coefficients in M to be

$$H_G^{V+n}(X; M) = H^{|V|+n}\, \mathrm{Hom}_{\mathscr{B}_G}(\underline{C}_*^V(X), M).$$

For a coMackey functor N, define the ordinary homology of X with coefficients in N to be

$$H_{V+n}^G(X; N) = H_{|V|+n}(\underline{C}_*^V(X) \otimes_{\mathscr{B}_G} N).$$

Precisely similar definitions apply to give relative homology and cohomology groups for relative G-CW(V) complexes (X, A). In the special case when A is a subcomplex of X, $\underline{C}_*^V(X, A)$ is isomorphic to $\underline{C}_*^V(X)/\underline{C}_*^V(A)$, and we obtain the expected long exact sequences. If $* \in X$ is a G-fixed basepoint and $(X, *)$ is a relative G-CW(V) complex, we define the reduced homology and cohomology of X by

$$\tilde{H}_G^{V+n}(X; M) = H_G^{V+n}(X, *; M) \quad \text{and} \quad \tilde{H}^G_{V+n}(X; N) = H^G_{V+n}(X, *; N).$$

Observe, however, that $*$ cannot be a vertex of X unless G acts trivially on V, by our limitation on the orbits G/H that are allowed in the zero skeleta of G-CW(V) complexes.

Using cellular approximation, homology and cohomology are seen to be functorial on the homotopy category of G-CW(V) complexes. We extend the definition to arbitrary G-spaces by using approximations by weakly equivalent G-CW(V) complexes. The definitions for pairs extend similarly. Finally, we extend the grading to all of $RO(G)$ by setting

$$\tilde{H}_G^{W-V+n}(X; M) = \tilde{H}_G^{W+n}(\Sigma^V X; M)$$

and

$$\tilde{H}^G_{W-V+n}(X; N) = \tilde{H}^G_{W+n}(\Sigma^V X; N).$$

We easily deduce from a relative version of Proposition 2.9 that, for a relative G-CW(W) complex $(X, *)$ and any representation V, $(\Sigma^V X, *)$ inherits a structure of relative G-CW($V \oplus W$) complex such that the W-cellular chain complex of $(X, *)$ is isomorphic to the $(V \oplus W)$-cellular chain complex of $(\Sigma^V X, *)$, with an appropriate shift of dimensions. This gives isomorphisms

$$\tilde{H}_G^{W+n}(X; M) \cong \tilde{H}_G^{V \oplus W+n}(\Sigma^V X)$$

and

$$\tilde{H}^G_{W+n}(X; M) \cong \tilde{H}^G_{V \oplus W+n}(\Sigma^V X).$$

It is quite tedious, but not difficult, to verify the precise axioms for $RO(G)$-graded homology and cohomology theories from the definitions just indicated. The alternative construction by stable homotopy category techniques in XIII§4 is less tedious, but perhaps less intuitive.

REMARKS 4.3. (1) There is a twisted version of the theory, where the twisting is taken over the fundamental groupoid of X.

(2) As already indicated, this theory also extends to a theory graded on representations of the fundamental groupoids of G-spaces. Roughly, such a representation assigns a representation to each component of each fixed point set in an appropriately coherent fashion. We also have a twisted version of this fancier theory.

(3) In the untwisted theory given above, Poincaré duality and the Thom isomorphism theorem hold for oriented V-manifolds. These are V-manifolds whose tangent bundles admit orientations in the geometric sense. They possess fundamental classes in dimension V.

(4) There is also a version of the Hurewicz theorem, which Lewis will discuss in the next chapter.

(5) There is an unpublished theory of equivariant Chern classes which live in off-integral dimensions, but this theory is not yet well-understood.

(6) The cohomology of a point is highly nontrivial, since there is no dimension axiom away from integer gradings. Indeed, among other applications related to ordinary cohomology, I have a curious result to the effect that if you localize the cohomology of a point by inverting a Chern class in dimension $V - |V|$, where V contains a free G-orbit, then you get the cohomology of BG.

REMARK 4.4. The chain level construction just sketched has applications to manifold theory. Since Poincaré duality works for this theory (V-manifolds have fundamental classes in the twisted theory), Costenoble and I have been able to use it to obtain a workable definition of Poincaré duality spaces and to prove a $\pi - \pi$ theorem for such spaces, giving a criterion for a G-CW complex to have the G-homotopy type of a G-manifold in the presence of suitable "gap hypotheses" on the homotopy groups of its fixed point spaces. We have also extended this to the case of simple G-homotopy theory, since it turns out that Poincaré duality is given by a simple chain equivalence, just as in the nonequivariant case. Thus we can say when a G-CW complex has the simple G-homotopy type of a G-manifold.

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CHAPTER XI

The equivariant Hurewicz and Suspension Theorems

by L. Gaunce Lewis, Jr.

1. Background on the classical theorems

We begin by recalling the statements of two basic theorems in nonequivariant homotopy theory. The first of these is the very familiar Hurewicz Theorem.

THEOREM A. If Y is a simply connected space and $n \geq 2$, then the following are equivalent:

(i) $H_k(Y;\mathbb{Z}) = 0$ for all $k < n$.

(ii) $\pi_k Y = 0$ for all $k < n$.

Moreover, either of these implies that the Hurewicz homomorphism

$$h : \pi_n Y \rightarrow H_n(Y;\mathbb{Z})$$

is an isomorphism.

There is, of course, an extension of this theorem that describes the relation between $\pi_1 Y$ and $H_1(Y;\mathbb{Z})$, but we shall here restrict attention to the simply connected case, in both nonequivariant and equivariant homotopy theory, to avoid some unpleasant technicalities that obscure the central issues. The Hurewicz theorem is important because it describes the basic connection between the two most commonly used functors in algebraic topology. It allows us to convert information about homology groups, which are relatively easy to compute, into information about homotopy groups, which are much harder to compute but also much more useful.

The second theorem is the Freudenthal suspension theorem.

THEOREM B. Let Y be an n-connected space, where $n \geq 1$, and let X be a finite CW complex. Then the suspension map

$$\sigma : [X, Y] \rightarrow [\Sigma X, \Sigma Y]$$

is surjective if $dim\, X \leq 2n + 1$ and bijective if $dim\, X \leq 2n$.

Historically, this result grew out of Freudenthal's study of the homotopy groups of spheres. His original version of this result merely gave conditions on m and n under which the suspension map

$$\sigma : \pi_n S^m \rightarrow \pi_{n+1} S^{m+1}$$

was surjective or bijective. This initial result was rather quickly extended to one giving conditions under which the suspension map

$$\sigma : \pi_n Y \rightarrow \pi_{n+1} \Sigma Y$$

was surjective or bijective. Eventually, the result was generalized to Theorem B. As with the Hurewicz Theorem, this result allows us to compare a well-behaved object that we have some hope of understanding with an apparently less well-behaved one. The point here is that $[\Sigma X, \Sigma Y]$ is a group and, if we suspend it once more, it becomes an abelian group. On the other hand, $[X, Y]$ need only be a pointed set. As a vague general principle, which will be made more precise later, the more we suspend a space, the more algebraic tools (like group structures) we gain for the study of the space. The Freudenthal result allows us to convert information that we obtain working in the more structured setting of objects that have been repeatedly suspended into information about the original, unsuspended, objects.

These two basic theorems are actually quite closely related. If one constructs homology using Eilenberg-Mac Lane spaces, then the Hurewicz theorem follows directly from the suspension theorem and the simple observation that the Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$ in dimension n associated to the group $\mathbb{Z}$ has a CW structure in which the bottom cell is a sphere in dimension n and in which there are no $(n+1)$-cells. The Hurewicz map itself is derived from the inclusion of this bottom cell. If one thinks of homology in terms of the Eilenberg-Mac Lane spectrum $K\mathbb{Z}$ associated to the group $\mathbb{Z}$, then the Hurewicz theorem follows even more directly from the suspension theorem and the observation that $K\mathbb{Z}$ has a CW structure in which the bottom cell is a copy of the zero sphere and in which there are no 1-cells.

We shall discuss the equivariant analogues of these two theorems in this chapter. Full details and more general versions of the results are given in the first two of the following three papers; we shall occasionally refer to these papers by number, and a little guide to them is given in a scholium at the end of the chapter.

[L1] L. G. Lewis, Jr., Equivariant Eilenberg-Mac Lane spaces and the equivariant Seifert-van Kampen and suspension theorems. Topology and its Applications 48 (1992), 25-61.
[L2] L. G. Lewis, Jr., The equivariant Hurewicz map. Trans. Amer. Math. Soc., 329 (1992), 433-472.
[L3] L. G. Lewis, Jr., Change of universe functors in equivariant stable homotopy theory. Fund. Math. To appear.

2. Formulation of the problem and counterexamples

Throughout the chapter, we assume that G is a compact Lie group and that the spaces considered are left G-spaces. There are two issues that come up immediately when one starts thinking about generalizing these basic theorems to the equivariant context. The first is how one should measure the connectivity of G-spaces. There are two solutions to this problem. The first is the notion of V-connectivity that Stefan Waner introduced in the previous chapter. This notion focuses on a single G-representation V and measures the connectivity of a G-space Y as seen through the "eyes" of that representation. The other notion of equivariant connectivity is less dependent on individual representations and somewhat less exotic in its definition. It too has already been introduced earlier, but we recall the definition.

Definition 2.1. (a) A *dimension function* ν is a function from the set of conjugacy classes of subgroups of G to the integers ≥ -1. Write n^* for the dimension function that takes the value n at each H. Associated to any G-representation V is the dimension function $|V^*|$ whose value at K is the real dimension of the K-fixed subspace V^K of V.

(b) Let ν be a dimension function. Then a G-space Y is *G-ν-connected* if, for each subgroup K of G, the fixed point space Y^K is $\nu(K)$-connected. The based G-space Y is *homologically G-ν-connected* if, for every subgroup K of G and every integer m with $0 \leq m \leq \nu(K)$, the equivariant homology group $\tilde{H}^K_m Y$ is zero. A G-space Y is *G-connected* if it is G-0^*-connected. A G-space is *simply G-connected* if it is G-1^*-connected. The prefix "G-" will be deleted from the notation whenever the omission should not lead to confusion.

(c) Define the connectivity function c^*Y of a G-space Y by letting c^KY be the connectivity of the space Y^K for each subgroup K of G. Define $c^KY = -1$ if Y^K is not path connected.

A basic result of Waner indicates that the two rather different measures of equivariant connectivity that we have described are intimately related.

LEMMA 2.2. Let Y be a G-space and V be a G-representation. Then the space Y is V-connected if and only if it is $|V^*|$-connected.

Because of this lemma, we will use the terms V-connected and $|V^*|$-connected interchangeably.

The second issue that comes up immediately is what sort of suspensions one wishes to allow in the equivariant context and, intimately tied to that, how one grades equivariant homotopy and homology groups. The point here is that one may define ΣY to be $Y \wedge S^1$. Therefore, in the equivariant context, if V is a G-representation and S^V is its one-point compactification (with G acting trivially on the point at infinity, which is taken to be the basepoint), then it is natural to think of $Y \wedge S^V$ as the suspension $\Sigma^V Y$ of Y by V. With this viewpoint, it is natural to want an equivariant suspension theorem which describes the map

$$\sigma_V : [X, Y]_G \rightarrow [\Sigma^V X, \Sigma^V Y]_G.$$

Moreover, since, in the nonequivariant context, $\pi_n Y$ is just $[S^n, Y]$, it is natural to regard $[S^V, Y]_G$ as the V^{th} homotopy group (or set) $\pi_V^G Y$. Thus, we would like to have a V^{th} homology group $H_V^G Y$, an equivariant Hurewicz map

$$h : \pi_V^G Y \rightarrow H_V^G Y,$$

and an equivariant Hurewicz theorem that tells us when this map is an isomorphism. The previous chapter has already given one construction of $H_V^G Y$, and Chapter XIII will give another. The precise definition of the map h is given in [L2], but it should become apparent from the discussion of the relationship between equivariant spectra and equivariant homology to be given later.

We must still resolve the issue of what coefficients should be used for this homology group since it is very important in the nonequivariant Hurewicz Theorem that integral coefficients be used. Burnside ring coefficients turn out to be the appropriate ones, essentially because the equivariant zero stem is the Burnside ring.

It should be fairly clear that the sort of equivariant suspension theorem that we would like to have would be something along the lines of:

"THEOREM". Let Y be a simply G-connected space, X be a finite G-CW complex, and V be a G-representation. Then the suspension map

$$\sigma_V : [X, Y]_G \to [\Sigma^V X, \Sigma^V Y]_G$$

is surjective if, for every subgroup K of G, $dim\, X^K \leq 2c^K Y + 1$ and is bijective if, for every subgroup K, $dim\, X^K \leq 2c^K Y$.

Unfortunately, this result is wildly false. For example, let $G = \mathbb{Z}/2$, $n \geq 3$, and V be the real one-dimensional sign representation of G. Then our proposed "Theorem" would require that the maps

$$\sigma_V : [S^n, S^n]_G \to [S^{n+V}, S^{n+V}]_G$$

and

$$\sigma_V : [S^{n+V}, \Sigma^{n+V} G_+]_G \to [S^{n+2V}, \Sigma^{n+2V} G_+]_G$$

be isomorphisms. However, simple calculations give that

$$[S^n, S^n]_G = \mathbb{Z} \quad \text{and} \quad [S^{n+V}, S^{n+V}]_G = \mathbb{Z}^2,$$

$$[S^{n+V}, \Sigma^{n+V} G_+]_G = \mathbb{Z}^2 \quad \text{and} \quad [S^{n+2V}, \Sigma^{n+2V} G_+]_G = \mathbb{Z}.$$

Thus, the first of the two maps above can't be surjective and the second can't be injective. In fact, calculations for arbitrary groups G and low-dimensional nontrivial G-representations V and W suggest that the suspension map

$$\sigma_W : [S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G$$

is almost never an isomorphism. The restriction of "low dimension" is essential here because, as we have seen in IX.2.3, if G is finite and V contains enough copies of the regular representation of G, then σ_W is an isomorphism for any G-representation W. Similar calculations of equivariant homotopy and homology groups suggest rather quickly that there is no simple generalization of the Hurewicz theorem to the equivariant context.

One way to save the equivariant suspension theorem is to insert additional hypotheses, as in IX.1.4. The inequalities required there between the dimension of Y^H and the connectivity of Y^K when $K \subseteq H$ with $V^K \neq V^H$ tend to be quite restrictive and hard to verify. Thus, what we intend to discuss is another approach to generalizing the Hurewicz and suspension theorems to the equivariant context. For this alternative approach, we must revert to the earlier form of the suspension theorem which deals only with the suspension of homotopy groups.

3. An oversimplified description of the results

Hereafter, in discussing the suspension map

$$\sigma_W : [X, Y]_G \to [\Sigma^W X, \Sigma^W Y]_G,$$

we will consider only the case in which $X = S^V$ for some G-representation V. As a matter of convenience, we will assume that the representation V contains at least two copies of the one-dimensional trivial G-representation. This ensures that the set $\pi_V^G Y$ is an abelian group. The motivation for the alternative approach is that, even though the suspension map

$$\sigma_W : [S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G$$

is rather badly behaved, we can, at least in theory, compute exactly what it does. Thus, it is reasonable to ask if our understanding of this map can be used to shed some light on the suspension map

$$\sigma_W : \pi_V^G Y = [S^V, Y]_G \to [S^{V+W}, \Sigma^W Y]_G = \pi_{V+W}^G \Sigma^W Y$$

for any suitably connected G-space Y.

A feeling for the sort of result that we should expect is best conveyed by a slight oversimplification of the actual result. The set $[S^V, S^V]_G$ is a ring under composition. Here the right distributivity law depends on the fact that V contains two copies of $\mathbb{R}$ and uses IX.1.4, which ensures that every element of $[S^V, S^V]_G$ is a suspension. Moreover,

$$\sigma_W : [S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G$$

is a ring homomorphism. For any based G-space Y, the abelian groups $\pi_V^G Y$ and $\pi_{V+W}^G \Sigma^W Y$ may be regarded as modules over $[S^V, S^V]_G$ and $[S^{V+W}, S^{V+W}]_G$, respectively. If $\pi_{V+W}^G \Sigma^W Y$ is regarded as a $[S^V, S^V]_G$-module via the ring homomorphism

$$[S^V, S^V]_G \to [S^{V+W}, S^{V+W}]_G,$$

then the map

$$\sigma_W : \pi_V^G Y \to \pi_{V+W}^G \Sigma^W Y$$

is a $[S^V, S^V]_G$-module homomorphism. The usual change of rings functor converts the $[S^V, S^V]_G$-module $\pi_V^G Y$ into the $[S^{V+W}, S^{V+W}]_G$-module

$$\pi_V^G Y \otimes_{[S^V, S^V]_G} [S^{V+W}, S^{V+W}]_G.$$

The homomorphism σ_W induces an $[S^{V+W}, S^{V+W}]_G$-module homomorphism

$$\widehat{\sigma}_W : \ \pi_V^G Y \otimes_{[S^V,S^V]_G} [S^{V+W}, S^{V+W}]_G \ \ \to \ \ \pi_{V+W}^G \Sigma^W Y.$$

The alternative suspension theorem should, in this oversimplified form, assert that the map $\widehat{\sigma}_W$, rather than σ_W, is an isomorphism or epimorphism.

We would also like to obtain an equivariant Hurewicz theorem along the same lines. Again, to convey some intuition for what we hope to prove, we begin with an oversimplified version of the desired theorem. If one has a sufficiently slick definition of the homology group $H_V^G Y$, then it is obvious that this group is a module over the ring $[S^V, S^V]_G$. Moreover, there is an equivariant Hurewicz map

$$h \ : \ \pi_V^G Y \ \to \ H_V^G Y$$

that is a $[S^V, S^V]_G$-module homomorphism. However, the group $H_V^G Y$ carries a far richer structure than just that of a $[S^V, S^V]_G$-module. For any G-representation W, there is a homology suspension isomorphism $H_V^G Y \cong H_{V+W}^G \Sigma^W Y$. Here, our assumption that V contains at least two copies of the trivial representation removes the need to worry about reduced and unreduced homology. This isomorphism indicates that $H_V^G Y$ actually carries the structure of a $[S^{V+W}, S^{V+W}]_G$-module. A bit of fiddling with the definitions indicates that the $[S^V, S^V]_G$-module structure on $H_V^G Y$ is just that obtained by restricting the $[S^{V+W}, S^{V+W}]_G$-module structure along the ring homomorphism

$$\sigma_W : \ [S^V, S^V]_G \ \ \to \ \ [S^{V+W}, S^{V+W}]_G.$$

Since this is true for every G-representation W, what we have on $H_V^G Y$ is a coherent family of $[S^{V+W}, S^{V+W}]_G$-module structures for all possible representations W. This suggests that we introduce a new ring in which we let W go to infinity. This ring ought to be defined as some sort of colimit of the rings $[S^{V+W}, S^{V+W}]_G$, where W ranges over all possible finite-dimensional representations of G.

As was explained in IX§§3,4, we use a complete G-universe U to make this colimit precise. With the notations there, the ring structure on $B_G = \{S^0, S^0\}_G$ is that inherited from the ring structures on the $[S^V, S^V]_G$. Since U is complete, it contains a copy of every representation V. Selecting one of these copies, we obtain a ring homomorphism

$$\sigma_\infty : \ [S^V, S^V]_G \ \ \to \ \ B_G.$$

It can be shown that σ_∞ is actually independent of the choice of the copy of V in U. It follows from our observation about the module structures on $H_V^G Y$ that $H_V^G Y$ carries the structure of a B_G-module. Moreover, its natural $[S^V, S^V]_G$-module structure is just that obtained by restricting the B_G-module structure along σ_∞. The Hurewicz map

$$h : \pi_V^G Y \to H_V^G Y$$

induces a map

$$\widehat{h} : \pi_V^G Y \otimes_{[S^V, S^V]_G} B_G \to H_V^G Y$$

of B_G-modules. In this oversimplified outline form, our equivariant Hurewicz theorem gives conditions under which the map $\widehat{h}$, rather than the map h, is an isomorphism.

The proposed equivariant suspension and Hurewicz theorems may seem more reasonable if one considers the nonequivariant Hurewicz theorem in dimension 1. This result asserts that, if Y is connected, then the map $h : \pi_1 Y \to H_1 Y$ induces an isomorphism between $H_1 Y$ and the abelianization of $\pi_1 Y$. We are encountering the same sort of phenomenon in the equivariant context—that is, we are trying to compare two objects which carry rather different structures. The two objects become isomorphic when we modify the less well-structured one to have the same sort of structure as that carried by the nicer object.

4. The statements of the theorems

The oversimplification in the introduction to our two theorems comes from the fact that, in order to understand the maps

$$\sigma_W : \pi_V^G Y \to \pi_{V+W}^G \Sigma^W Y$$

and

$$h : \pi_V^G Y \to H_V^G Y$$

fully, one must look not only at the group $\pi_V^G Y$, but also at the groups $\pi_V^K Y$ for all the subgroups K of G. The maps $\widehat{\sigma}_W$ and $\widehat{h}$ constructed in the rough sketch of our results do not take into account the influence that the groups $\pi_V^K Y$ have on the maps σ_W and h. In order to take this influence into account, we must replace the ring $[S^V, S^V]_G$ with a small Ab-category $\mathscr{B}_G(V)$ and replace the module $\pi_V^G Y$ with a contravariant additive functor $\underline{\pi}_V^G Y$ from $\mathscr{B}_G(V)$ into the category Ab of abelian groups. The category $\mathscr{B}_G(V)$ and the functor $\underline{\pi}_V^G Y$ should be regarded as bookkeeping devices that allow us to keep track of the influence of the groups $\pi_V^K Y$ on the maps σ_W and h.

Recall the definitions of the Burnside category $\mathscr{B}_G$ and of Mackey functors from IX.4.1 and IX.4.2.

DEFINITION 4.1. (a) Let V be a finite-dimensional representation of G that contains at least two copies of the trivial representation. The *V-Burnside category* $\mathscr{B}_G(V)$ has as its objects the orbits G/K. The set of morphisms from G/K to G/J in $\mathscr{B}_G(V)$ is $[\Sigma^V G/K_+, \Sigma^V G/J_+]_G$. Note that the morphism sets of $\mathscr{B}_G(V)$ are abelian groups.

(b) If V and W are G-representations of G, then suspension gives a functor

$$s : \mathscr{B}_G(V) \;\to\; \mathscr{B}_G(V+W)$$

that is the identity on objects. Moreover, any inclusion of V into the G-universe U gives a functor

$$s_\infty : \mathscr{B}_G(V) \;\to\; \mathscr{B}_G$$

that is also the identity on objects. It can be shown that the functor s_∞ is independent of the choice of the copy of V in U.

Motivated by the interpretation of contravariant additive functors $\mathscr{B}_G \longrightarrow \mathscr{A}b$ as Mackey functors, we refer to contravariant additive functors $\mathscr{B}_G(V) \longrightarrow \mathscr{A}b$ as V-Mackey functors for any compact Lie group G and G-representation V. The category of V-Mackey functors and natural transformations between such is denoted $\mathscr{M}_G(V)$. The category of Mackey functors is denoted $\mathscr{M}_G$.

EXAMPLES 4.2. (a) If V is a representation of G that contains at least two copies of the trivial representation and Y is a G-space, then the homotopy group $\pi_V^G Y$ can be extended to a V-Mackey functor $\underline{\pi}_V^G Y$. For $K \le G$, we define $(\underline{\pi}_V^G Y)(G/K)$ to be the group

$$[\Sigma^V G/K_+, Y]_G \cong [S^V, Y]_K = \pi_V^K Y.$$

The effect of a morphism f in $\mathscr{B}_G(V)(G/K, G/J) = [\Sigma^V G/K_+, \Sigma^V G/J_+]_G$ on $(\underline{\pi}_V^G Y)(G/J)$ is just that of precomposition by f.

(b) If V is a G-representation and Y is a G-space, then the homology group $H_V^G Y$ can be extended to a Mackey functor $\underline{H}_V^G Y$. If $K \le G$, then

$$(\underline{H}_V^G Y)(G/K) = H_V^K Y.$$

The functoriality of $\underline{H}_V^G Y$ on $\mathscr{B}_G$ will be apparent from the spectrum level construction of XIII§4.

Our actual equivariant suspension and Hurewicz theorems describe the relations among the functors $\underline{\pi}_V^G Y$, $\underline{\pi}_{V+W}^G \Sigma^W Y$, and $\underline{H}_V^G Y$. In order to state these theorems, we must introduce the change of category functors that replace the change of ring functors that were used in the intuitive presentation of our results.

DEFINITION 4.3. (a) Precomposition by the functors s and s_∞ of Definition 4.1 gives functors

$$s^* : \mathscr{M}_G(V+W) \rightarrow \mathscr{M}_G(V)$$

and

$$s^*_\infty : \mathscr{M}_G \rightarrow \mathscr{M}_G(V).$$

These functors have left adjoints

$$s_* : \mathscr{M}_G(V) \rightarrow \mathscr{M}_G(V+W)$$

and

$$s^\infty_* : \mathscr{M}_G(V) \rightarrow \mathscr{M}_G$$

that are given categorically by left Kan extension.

(b) The suspension maps

$$\sigma^K_W : \ \pi^K_V Y \ \rightarrow \ \pi^K_{V+W}\Sigma^W Y,$$

as K varies over the subgroups of G, fit together to form a natural transformation

$$\sigma_W : \ \underline{\pi}^G_V Y \ \rightarrow \ s^*\underline{\pi}^G_{V+W}\Sigma^W Y.$$

The adjoint of this map under the $(s_*\,, s^*)$-adjunction is denoted

$$\widetilde{\sigma}_W : \ s_*\underline{\pi}^G_V Y \ \rightarrow \ \underline{\pi}^G_{V+W}\Sigma^W Y.$$

(c) The Hurewicz maps

$$h^K_W : \ \pi^K_V Y \ \rightarrow \ H^K_V Y,$$

as K varies over the subgroups of G, fit together to form a natural transformation

$$h : \ \underline{\pi}^G_V Y \ \rightarrow \ s^*_\infty \underline{H}^G_V Y.$$

The adjoint of this map under the $(s^\infty_*\,, s^*_\infty)$-adjunction is denoted

$$\widetilde{h} : \ s^\infty_* \underline{\pi}^G_V Y \ \rightarrow \ \underline{H}^G_V Y.$$

It is the maps $\widetilde{\sigma}_W$ and $\widetilde{h}$ that play the role in the precise statements of our Hurewicz and suspension theorems that was played by the maps $\widehat{\sigma}_W$ and $\widehat{h}$ in our intuitive sketch of these results.

THEOREM 4.4 (HUREWICZ). Let Y be a based G-CW complex and let V be a representation of G that contains at least two copies of the trivial representation. Then the following two conditions are equivalent.

(i) Y is $|(V-1)^*|$-connected.

(ii) Y is simply G-connected and homologically $|(V-1)^*|$-connected.

Moreover, if W is any representation of G such that $2^* \leq |W^*| \leq |V^*|$, then either of these conditions implies that the map

$$\widetilde{h} : s_*^\infty \underline{\pi}_W^G Y \to \underline{H}_W^G Y$$

is an isomorphism and that both $\underline{\pi}_W^G Y$ and $\underline{H}_W^G Y$ are zero if $|W^*| < |V^*|$.

THEOREM 4.5 (FREUDENTHAL SUSPENSION). Let V and W be representations of G and let Y be a based G-CW complex. If V contains at least two copies of the trivial representation and Y is $|(V-1)^*|$-connected, then the suspension map

$$\widetilde{\sigma}_W : \; s_* \underline{\pi}_V^G Y \quad \to \quad \underline{\pi}_{V+W}^G \Sigma^W Y$$

is an isomorphism.

There are several ways in which these two theorems are a bit disappointing. One of the most obvious is that, in our anticipated applications, we expect to be able to compute $\underline{H}_W^G Y$ and $\underline{\pi}_{V+W}^G \Sigma^W Y$, and we want to derive information about $\underline{\pi}_V^G Y$ from these computations. The presence of the functors s_*^∞ and s_* would seem to make it difficult to learn much about $\underline{\pi}_V^G Y$ in this fashion. However, the following lemma ensures that we can, at least, detect the vanishing of $\underline{\pi}_V^G Y$ with these two theorems.

LEMMA 4.6. Let V be a representation of G that contains at least two copies of the trivial representation and M be a V-Mackey functor. Then the following are equivalent:

(i) $M = 0$.
(ii) $s_* M = 0$ for any representation W of G.
(iii) $s_*^\infty M = 0$.

Moreover, the explicit descriptions of the functors s_* and s_*^∞ given in [L1, L2] can be used to extract some information about $\underline{\pi}_V^G Y$ from a knowledge of $s_* \underline{\pi}_V^G Y$ or $s_*^\infty \underline{\pi}_V^G Y$ even in the cases where $\underline{\pi}_V^G Y$ does not vanish.

A second disappointment in these two theorems is that they say nothing about the case in which V contains only one copy of the trivial representation. In this context, $\pi_V^G Y$ need not be an abelian group, but one would expect generalizations of our two theorems which relate the abelianization of $\pi_V^G Y$ to $H_W^G Y$ and $\pi_{V+W}^G \Sigma^W Y$ (or more precisely, to $\underline{H}_W^G Y$ and $\underline{\pi}_{V+W}^G \Sigma^W Y$). Generalizations of this form are given in [L1]. They are omitted here because including them would require introducing some unpleasant technicalities that would only obscure the central ideas.

A third disappointment is that, in our suspension theorem, Y is required to be $|(V-1)^*|$-connected, whereas one would expect that connectivity on the order of $|V^*|/2$ would suffice. There are counterexamples (see [L1]) which show that there is no simple way to weaken this connectivity condition on Y. The source of this problem is that the functor s_* is not exact. It is therefore able to capture the effects of suspension only in the lowest dimensions. There is, however, a spectral sequence whose E^2-term is formed from the homotopy groups of Y. This spectral sequence converges to the homotopy groups of $\Sigma^W Y$ in the range of dimensions that one would expect based on the connectivity restrictions in Theorem B; see [L3].

A further disappointing aspect of our suspension theorem is that it applies only to the homotopy groups $\pi_V^G Y$ and not to the set $[X, Y]_G$ of G-homotopy classes of G-maps out of an arbitrary space X. This restriction seems to be unavoidable in the equivariant context.

5. Sketch proofs of the theorems

We turn now to the matter of proving our two theorems. The equivariant Hurewicz theorem follows almost trivially from the equivariant suspension theorem if one is willing to use a little equivariant stable homotopy theory. We will devote our attention to the proof of the suspension theorem. The best way to gain insight into the proof is to look at a rather nonstandard proof of a special case of the corresponding nonequivariant result. This nonstandard proof uses nothing more than two rather simple facts about Eilenberg-Mac Lane spaces and a simple lemma from category theory.

Recall that, if n is a positive integer and M is an abelian group, then the Eilenberg-Mac Lane space $K(M, n)$ is a CW-complex such that $\pi_n K(M, n) = M$ and $\pi_j K(M, n) = 0$ for $j \neq n$. This property characterizes $K(M, n)$ up to homotopy. The first fact that we need about Eilenberg-Mac Lane spaces is that, for any positive integer n and any abelian group M, $\Omega K_{n+1} M \simeq K(M, n)$. This fact follows immediately from a computation of the homotopy groups of $\Omega K(M, n+1)$. If X is any based space, then taking n^{th} homotopy groups gives a map

$$\pi : [X, K(M, n)] \longrightarrow \hom(\pi_n X, \pi_n K(M, n)) = \hom(\pi_n X, M)$$

from the set $[X, K(M, n)]$ of based homotopy classes of maps from X into $K(M, n)$ to the set $\hom(\pi_n X, M)$ of group homomorphisms from $\pi_n X$ to M. Since the Eilenberg-Mac Lane space $K(M, n)$ represents cohomology in dimension n with

M coefficients, the set $[X, K(M,n)]$ is just $H^n(X; M)$. It follows easily from the nonequivariant Hurewicz theorem and the universal coefficient theorem that the map π is an isomorphism if X is an $(n-1)$-connected CW-complex. Homotopy theorists use this observation on a regular basis.

For our proof of the nonequivariant suspension theorem, we need a categorical interpretation of this result. Let $\mathscr{W}_n$ be the category of $(n-1)$-connected based spaces that have the homotopy types of CW-complexes, and let $h\mathscr{W}_n$ be the associated (based) homotopy category. Then the assignment of the Eilenberg-Mac Lane space $K(M,n)$ to the abelian group M gives a functor $K(-,n)$ from the category Ab of abelian groups to the category $h\mathscr{W}_n$. On the other hand, taking n^{th} homotopy groups gives a functor π_n from $h\mathscr{W}_n$ to Ab. Our assertion that the map π above is an isomorphism when X is $(n-1)$-connected translates formally into the categorical assertion that the functor $K(-,n)$ is right adjoint to the functor π_n. This adjunction is the second fact about Eilenberg-Mac Lane spaces that we need.

Now consider the diagram of categories and functors

$$\begin{tikzcd}
 & \mathscr{A}b \arrow[dl, shift left, "K(-,n)"] \arrow[dr, shift left, "K(-,n+1)"] & \\
h\mathscr{W}_n \arrow[ur, shift left, "\pi_n"] \arrow[rr, shift left, "\Sigma"] & & h\mathscr{W}_{n+1} \arrow[ul, shift left, "\pi_{n+1}"] \arrow[ll, shift left, "\Omega"]
\end{tikzcd}$$

The functor Σ is left adjoint to the functor Ω. Thus, we have two functors, $\Omega K(-,n+1)$ and $K(-,n)$, from Ab to $h\mathscr{W}_n$ with left adjoints $\pi_{n+1}\circ\Sigma$ and π_n, respectively. The homotopy equivalences $\Omega K_{n+1}M \simeq K(M,n)$ fit together to give a natural isomorphism between the functors $\Omega K(-,n+1)$ and $K(-,n)$. The following easy lemma from category theory allows us to convert this natural isomorphism into a nonequivariant suspension theorem.

LEMMA 5.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, $R_1, R_2 : \mathscr{C} \to \mathscr{D}$ be functors from $\mathscr{C}$ to $\mathscr{D}$, and $L_1, L_2 : \mathscr{D} \to \mathscr{C}$ be functors from $\mathscr{D}$ to $\mathscr{C}$ such that L_i is left adjoint to R_i. Then there is a one-to-one correspondence between natural transformations $\tau : R_1 \to R_2$ and natural transformations $\tilde{\tau} : L_2 \to L_1$. Moreover, the natural tranformation $\tau : R_1 \to R_2$ is a natural isomorphism if and only if the associated natural transformation $\tilde{\tau} : L_2 \to L_1$ is a natural isomorphism.

The lemma gives us a natural isomorphism $\pi_n Y \longrightarrow \pi_{n+1}\Sigma Y$ for $(n-1)$-connected spaces Y of the homotopy types of CW-complexes. By examining the proof of the lemma and chasing a few diagrams, it is possible to see that this isomorphism is, in fact, the usual suspension map $\sigma : \pi_n Y \longrightarrow \pi_{n+1}\Sigma Y$.

This nonequivariant suspension theorem is, of course, substantially weaker than Theorem B because it requires much more connectivity of Y and because it applies only to the homotopy group $\pi_n Y$ rather than to an arbitrary set $[X, Y]$ of homotopy classes of maps. However, counterexamples exist which show that limitations of this sort are an essential part of an equivariant suspension theorem. Thus, our alternative approach to proving the nonequivariant suspension theorem is an ideal approach to proving the equivariant theorem.

Let V be a representation of G that contains at least two copies of the trivial representation. Let $\mathscr{W}_G(V)$ be the category of based $|(V-1)^*|$-connected G-spaces that have the G-homotopy types of G-CW complexes, and let $h\mathscr{W}_G(V)$ be the associated homotopy category; its morphisms are based G-homotopy classes of based G-maps.

To prove our equivariant suspension theorem, we must associate an Eilenberg-Mac Lane space $K_G(M,V)$ to each V-Mackey functor M in such a way that we obtain a functor from $\mathscr{M}_G(V)$ to $h\mathscr{W}_G(V)$. We must show that this functor is right adjoint to the functor $\underline{\pi}_V^G : h\mathscr{W}_G(V) \rightarrow \mathscr{M}_G(V)$. Then we must demonstrate that, if N is a $(V+W)$-Mackey functor, there is a G-homotopy equivalence $\Omega^W K_G(N, V+W) \simeq K_G(s^*N, V)$. Here, the functor s^* enters in a way that no analogous functor appears in the nonequivariant case because, in the equivariant case, the functors $\underline{\pi}_V^G$ and $\underline{\pi}_{V+W}^G$ land in different categories, whereas the functors π_n and π_{n+1} both produce abelian groups in the nonequivariant case. Now consider the diagram

$$\begin{tikzcd}
\mathscr{M}_G(V) \arrow[r, shift left, "s_*"] \arrow[d, shift left, leftarrow, "\underline{\pi}_V^G"'] & \mathscr{M}_G(V+W) \arrow[l, shift left, "s^*"] \arrow[d, shift left, leftarrow, "\underline{\pi}_{V+W}^G"'] \\
h\mathscr{W}_G(V) \arrow[r, shift left, "\Sigma^W"] \arrow[u, shift left, leftarrow, "K_G(-,V)"'] & h\mathscr{W}_G(V+W) \arrow[l, shift left, "\Omega^W"] \arrow[u, shift left, leftarrow, "K_G(-,V+W)"']
\end{tikzcd}$$

of categories and functors. The composites $\Omega^W K_G(-, V+W)$ and $K_G(s^*-, V)$ have left adjoints $\underline{\pi}_{V+W}^G \Sigma^W$ and $s_*\underline{\pi}_V^G$ respectively. Thus the natural isomorphism $\Omega^W K_G(-, V+W) \longrightarrow K_G(s^*-, V)$ that is derived from our G-homotopy equivalences $\Omega^W K_G(N, V+W) \simeq K_G(s^*N, V)$ implies a natural isomorphism $s_*\underline{\pi}_V^G \longrightarrow \underline{\pi}_{V+W}^G \Sigma^W$. Again, a bit of diagram chasing confirms that this isomorphism is just the standard suspension map

$$\widetilde{\sigma}_W : s_*\underline{\pi}_V^G \longrightarrow \underline{\pi}_{V+W}^G \Sigma^W.$$

It is easy enough to say what a V-Eilenberg-Mac Lane space ought to be.

Definition 5.2. Let V be a representation of G that contains at least two copies of the trivial representation and M be a V-Mackey functor. A *V-Eilenberg-Mac Lane space* $K_G(M, V)$ is a based, $|(V-1)^*|$-connected G-space $K_G(M, V)$ of the G-homotopy type of a G-CW complex such that $\underline{\pi}_V^G K_G(M, V) = M$ and $\underline{\pi}_{V+k}^G K_G(M, V) = 0$ for $k > 0$.

The problem is to show that such spaces exist, that the assignment of $K_G(M, V)$ to M gives a functor from $\mathscr{M}_G(V)$ to $h\mathscr{W}_G(V)$, and that this functor is right adjoint to $\underline{\pi}_V^G$. In order to fill in these details, we utilize a variant of the G-CW(V) complexes that Waner described in the previous chapter. Waner worked with unbased complexes and adjoined his cells using unbased maps. The variant with which we must work is that of based complexes formed using based attaching maps. We take our cells to be the cones on spheres of the form $\Sigma^{V+k}G/K_+$, where $k \geq -1$ and K runs over the (closed) subgroups of G. A based G-CW(V) complex is then a G-space Y together with a sequence $\{Y^k\}_{k\geq -1}$ of closed subspaces such that Y^{-1} is a point, Y^{k+1} is the cofibre of a based map $\lambda : \bigvee_{j\in J_k} \Sigma^{V+k}G/K_j \to Y^k$ for some indexing set J_k and some collection $\{K_j\}_{j\in J_k}$ of subgroups of G, and Y is the colimit of the Y^k.

There is a general theory of abstract CW complexes that applies to spaces constructed in this form. This theory ensures that G-CW(V) complexes have all the nice properties that one might expect. For us, their most important properties are that they have the homotopy types of G-CW complexes, that they are $|(V-1)^*|$-connected, and that they can be used to approximate, up to weak G-equivalence, any G-space that is $|(V-1)^*|$-connected. Using G-CW(V) complexes, one can construct a V-Eilenberg-Mac Lane space $K_G(M, V)$ for any V-Mackey functor M by attaching cells of the form $C\Sigma^{V+k}G/K_+$ in exactly the same way that one constructs ordinary, nonequivariant, Eilenberg-Mac Lane spaces by attaching ordinary cells.

As in the nonequivariant context, there is a map

$$\pi : [X, K_G(M, V)]_G \longrightarrow \hom(\underline{\pi}_V^G X, \underline{\pi}_V^G K(M, n)) = \hom(\underline{\pi}_V^G X, M)$$

given by taking V^{th} homotopy "groups". Here, hom means the set of natural transformations between two functors in $\mathscr{M}_G(V)$. If X is $|(V-1)^*|$-connected, then it can be approximated by a G-CW(V) complex. This approximation can be used to show that the map π is an isomorphism. We proved the analogous result in the nonequivariant context using the Hurewicz theorem and the universal coefficient theorem. It can, however, be just as easily proved by using a CW approximation to X and arguing inductively up the skeleton of the approximation.

From here, the second approach to the nonequivariant result generalizes without any trouble to the equivariant context. The fact that π is an isomorphism when X is $|(V-1)^*|$-connected can be used to show that the assignment of $K_G(M,V)$ to M gives a functor and that this functor is right adjoint to $\underline{\pi}_V^G$. It can also be used to construct a G-homotopy equivalence between $\Omega^W K_G(N, V+W) \simeq K_G(s^*N, V)$ for any $(V+W)$-Mackey functor N. This completes the proof of the equivariant suspension theorem.

Scholium 5.3. This presentation has been an overview of the papers [L1] and [L2]. Reference [L1] provides full details on everything that has been said here about the equivariant suspension theorem. It includes a careful treatment of based G-CW(V) complexes and of V-Eilenberg-Mac Lane spaces. In that paper, V is assumed to have at least one copy, rather than at least two copies, of the trivial representation. Thus the theorems in [L1] are more general in that they describe the effects of the presence of a nontrivial fundamental group on the suspension and Hurewicz maps. However, this extra generality necessitates several unpleasant technical complications in the arguments that obscure the basic simplicity of the ideas. Reference [L2] is an older paper and in some respects obsolete. Its most important results, the absolute and relative unstable Hurewicz theorems (Theorems 1.7 and 1.8), are restated in a better and more general form as Theorems 2.8 and 2.9 of [L1]. The improved versions of these theorems take into account the results in [L1] dealing with the case in which V contains only one copy of the trivial representation. On the positive side, [L2] contains a description of the structure of the categories $\mathscr{B}_G(V)$ and of the functors s_* and s_*^∞. It contains the proof of Lemma 2.2 above on the equivalence of V- and $|V^*|$-connectivity in the case when G is a compact Lie group; Waner proved this result only for finite groups. Lemma 4.6 above on the vanishing of s_*M and $s_*^\infty M$ is also proved in [L2]. The definitions of the absolute and relative stable and unstable Hurewicz maps are contained in [L2]. The proof of the stable Hurewicz isomorphism theorem in section 2 of [L2] is a simple application of some of the basic techniques in equivariant stable homotopy theory that will be covered in later chapters. Going over that argument is a good way to solidify one's grasp on these basic tricks. Reference [L2] also contains a description of the process for deriving the relative Hurewicz theorem from the absolute Hurewicz theorem. All of the other arguments in [L2], and especially those in sections 5 and 6, are correct but obsolete. I developed them before I became aware of the basic connection between equivariant Eilenberg-Mac Lane spaces and the equivariant suspension theorem. The results presented in section 6 of [L2] are presented in a better and more general form in [L3], which is, essentially, an extension of [L1] from the realm of equivariant unstable homotopy theory to that of equivariant stable homotopy theory.

CHAPTER XII

The Equivariant Stable Homotopy Category

1. An introductory overview

Let us start nonequivariantly. As the home of stable phenomena, the subject of stable homotopy theory includes all of homology and cohomology theory. Over thirty years ago, it became apparent that very significant benefits would accrue if one could work in an additive triangulated category whose objects were "stable spaces", or "spectra", a central point being that the translation from topology to algebra through such tools as the Adams spectral sequence would become far smoother and more structured. Here "triangulated" means that one has a suspension functor that is an equivalence of categories, together with cofibration sequences that satisfy all of the standard properties.

The essential point is to have a smash product that is associative, commutative, and unital up to coherent natural isomorphisms, with unit the sphere spectrum S. A category with such a product is said to be "symmetric monoidal". This structure allows one to transport algebraic notions such as ring and module into stable homotopy theory. Thus, in the stable homotopy category of spectra — which we shall denote by $\bar{h}\mathscr{S}$ — a ring is just a spectrum R together with a product $\phi : R \wedge R \longrightarrow R$ and unit $\eta : S \longrightarrow R$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} S \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge R \arrow[d, "\phi"] & R \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\ & R & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} R \wedge R \wedge R \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & R \wedge R \arrow[d, "\phi"] \\ R \wedge R \arrow[r, "\phi"] & R. \end{tikzcd}$$

The unlabelled isomorphisms are canonical isomorphisms giving the unital property, and we have suppressed associativity isomorphisms in the second diagram. Similarly, there is a transposition isomorphism $\tau : E \wedge F \longrightarrow F \wedge E$ in $\bar{h}\mathscr{S}$, and R is said to be commutative if the following diagram commutes in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} R \wedge R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R \wedge R \arrow[dl, "\phi"] \\ & R. & \end{tikzcd}$$

A left R-module is a spectrum M together with a map $\mu : R \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}\mathscr{S}$:

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge M \arrow[d, "\mu"] \\ & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} R \wedge R \wedge M \arrow[r, "1\wedge\mu"] \arrow[d, "\phi\wedge 1"'] & R \wedge M \arrow[d, "\mu"] \\ R \wedge M \arrow[r, "\mu"] & M. \end{tikzcd}$$

Over twenty years ago, it became apparent that it would be of great value to have more precisely structured notions of ring and module, with good properties before passage to homotopy. For example, when one is working in $\bar{h}\mathscr{S}$ it is not even true that the cofiber of a map of R-modules is an R-module, so that one does not have a triangulated category of R-modules. More deeply, when R is commutative, one would like to be able to construct a smash product $M \wedge_R N$ of R-modules. Quinn, Ray, and I defined such structured ring spectra in 1972. Elmendorf and I, and independently Robinson, defined such structured module spectra around 1983. However, the problem just posed was not fully solved until after the Alaska conference, in work of Elmendorf, Kriz, Mandell, and myself. We shall return to this later.

For now, let us just say that the technical problems focus on the construction of an associative and commutative smash product of spectra. Before June of 1993, I would have said that it was not possible to construct such a product on a category that has all colimits and limits and whose associated homotopy category is equivalent to the stable homotopy category. We now have such a construction, and it actually gives a point-set level symmetric monoidal category.

However, it is not a totally new construction. Rather, it is a natural extension of the approach to the stable category $\bar{h}\mathscr{S}$ that Lewis and I developed in the early 1980's. Even from the viewpoint of classical nonequivariant stable homotopy theory, this approach has very significant advantages over any of its predecessors.

What is especially relevant to us is that it is the only approach that extends effortlessly to the equivariant context, giving a good stable homotopy category of G-spectra for any compact Lie group G. Moreover, for a great deal of the homotopical theory, the new point-set level construction offers no advantages over the original Lewis-May theory: the latter is by no means rendered obsolete by the new theory.

From an expository point of view this raises a conundrum. The only real defect of the Lewis-May approach is that the only published account is in the general equivariant context, with emphasis on those details that are special to that setting. Therefore, despite the theme of this book, I will first outline some features of the theory that are nearly identical in the nonequivariant and equivariant contexts, returning later to a discussion of significant equivariant points. I will follow in part an unpublished exposition of the Lewis-May category due to Jim McClure. A comparison with earlier approaches and full details of definitions and proofs may be found in the encyclopedic first reference below. The second reference contains important technical refinements of the theory, as well as the new theory of highly structured ring and module spectra. The third reference gives a brief general overview of the theory that the reader may find helpful. We shall often refer to these as [LMS], [EKMM], and [EKMM′].

General References

[LMS] L. G. Lewis, Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics. Vol. 1213. 1986.

[EKMM] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Rings, modules, and algebras in stable homotopy theory. Preprint, 1995.

[EKMM′] A. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory. In “Handbook of Algebraic Topology”, edited by I.M. James. North Holland, 1995, pp 213-254.

2. Prespectra and spectra

The simplest relevant notion is that of a *prespectrum* E. The naive version is a sequence of based spaces E_n, $n \geq 0$, and based maps

$$\sigma_n : \Sigma E_n \longrightarrow E_{n+1}.$$

A map $D \longrightarrow E$ of prespectra is a sequence of maps $D_n \longrightarrow E_n$ that commute with the structure maps σ_n. The structure maps have adjoints

$$\tilde{\sigma}_n : E_n \longrightarrow \Omega E_{n+1},$$

and it is customary to say that E is an Ω-spectrum if these maps are equivalences. While this is the right kind of spectrum for representing cohomology theories on spaces, we shall make little use of this concept. By a *spectrum*, we mean a prespectrum for which the adjoints $\tilde{\sigma}_n$ are *homeomorphisms*. (The insistence on homeomorphisms goes back to a 1969 paper of mine that initiated the present approach to stable homotopy theory.) In particular, for us, an "Ω-spectrum" need not be a spectrum: henceforward, we use the more accurate term Ω-prespectrum for this notion.

One advantage of our definition of a spectrum is that the obvious forgetful functor from spectra to prespectra — call it ℓ — has a left adjoint spectrification functor L such that the canonical map $L\ell E \longrightarrow E$ is an isomorphism. Thus there is a formal analogy between L and the passage from presheaves to sheaves, which is the reason for the term "prespectrum". The category of spectra has limits, which are formed in the obvious way by taking the limit for each n separately. It also has colimits. These are formed on the prespectrum level by taking the colimit for each n separately; the spectrum level colimit is then obtained by applying L.

The central technical issue that must be faced in any version of the category of spectra is how to define the smash product of two prespectra $\{D_n\}$ and $\{E_n\}$. Any such construction must begin with the naive bi-indexed smash product $\{D_m \wedge E_n\}$. The problem arises of how to convert it back into a singly indexed object in some good way. It is an instructive exercise to attempt to do this directly. One quickly gets entangled in permutations of suspension coordinates. Let us think of a circle as the one-point compactification of $\mathbb{R}$ and the sphere S^n as the one-point compactification of $\mathbb{R}^n$. Then the iterated structure maps $\Sigma^n E_m = E_m \wedge S^n \longrightarrow E_{m+n}$ seem to involve $\mathbb{R}^n$ as the last n coordinates in $\mathbb{R}^{m+n}$. This is literally true if we consider the sphere prespectrum $\{S^n\}$ with identity structural maps. This suggests that our entanglement really concerns changes of basis. If so, then we all know the solution: do our linear algebra in a coordinate-free setting, choosing bases only when it is convenient and avoiding doing so when it is inconvenient.

Let $\mathbb{R}^\infty$ denote the union of the $\mathbb{R}^n$, $n \geq 0$. This is a space whose elements are sequences of real numbers, all but finitely many of which are zero. We give it the evident inner product. By a universe U, we mean an inner product space isomorphic to $\mathbb{R}^\infty$. If V is a finite dimensional subspace of U, we refer to V as an indexing space in U, and we write S^V for the one-point compactification of V, which is a based sphere. We write $\Sigma^V X$ for $X \wedge S^V$ and $\Omega^V X$ for $F(S^V, X)$.

By a prespectrum indexed on U, we mean a family of based spaces EV, one for each indexing space V in U, together with structure maps

$$\sigma_{V,W} : \Sigma^{W-V} EV \longrightarrow EW$$

whenever $V \subset W$, where $W - V$ denotes the orthogonal complement of V in W. We require $\sigma_{V,V} = \mathrm{Id}$, and we require the evident transitivity diagram to commute for $V \subset W \subset Z$:

$$\begin{tikzcd}
\Sigma^{Z-W}\Sigma^{W-V} EV \arrow[r] \arrow[d, "\cong"'] & \Sigma^{Z-W} EW \arrow[d] \\
\Sigma^{Z-V} EV \arrow[r] & EZ.
\end{tikzcd}$$

We call E a spectrum indexed on U if the adjoints

$$\tilde{\sigma} : EV \longrightarrow \Omega^{W-V} EW$$

of the structural maps are homeomorphisms. As before, the forgetful functor ℓ from spectra to prespectra has a left adjoint spectrification functor L that leaves spectra unchanged. We denote the categories of prespectra and spectra indexed on U by $\mathscr{P}U$ and $\mathscr{S}U$. When U is fixed and understood, we abbreviate notation to $\mathscr{P}$ and $\mathscr{S}$.

If $U = \mathbb{R}^\infty$ and E is a spectrum indexed on U, we obtain a spectrum in our original sense by setting $E_n = E\mathbb{R}^n$. Conversely, if $\{E_n\}$ is a spectrum in our original sense, we obtain a spectrum indexed on U by setting $EV = \Omega^{\mathbb{R}^n - V} E_n$, where n is minimal such that $V \subset \mathbb{R}^n$. It is easy to work out what the structural maps must be. This gives an isomorphism between our new category of spectra indexed on U and our original category of sequentially indexed spectra.

More generally, it often happens that a spectrum or prespectrum is naturally indexed on some other cofinal set $\mathscr{A}$ of indexing spaces in U. Here cofinality means that every indexing space V is contained in some $A \in \mathscr{A}$; it is convenient to also require that $\{0\} \in \mathscr{A}$. We write $\mathscr{P}\mathscr{A}$ and $\mathscr{S}\mathscr{A}$ for the categories of prespectra and spectra indexed on $\mathscr{A}$. On the spectrum level, all of the categories $\mathscr{S}\mathscr{A}$ are isomorphic since we can extend a spectrum indexed on $\mathscr{A}$ to a spectrum indexed on all indexing spaces V in U by the method that we just described for the case $\mathscr{A} = \{\mathbb{R}^n\}$.

J. P. May. Categories of spectra and infinite loop spaces. Springer Lecture Notes in Mathematics Vol. 99. 1969, 448–479.

3. Smash products

We can now define a smash product. Given prespectra E and E' indexed on universes U and U', we form the collection $\{EV \wedge E'V'\}$, where V and V' run through the indexing spaces in U and U', respectively. With the evident structure maps, this is a prespectrum indexed on the set of indexing spaces in $U \oplus U'$ that are of the form $V \oplus V'$. If we start with spectra E and E', we can apply the functor L to get to a spectrum indexed on this set, and we can then extend the result to a spectrum indexed on all indexing spaces in $U \oplus U'$. We thereby obtain the "external smash product" of E and E',

$$E \wedge E' \in \mathscr{S}(U \oplus U').$$

Thus, if $U = U'$, then two-fold smash products are indexed on U^2, three-fold smash products are indexed on U^3, and so on.

This external smash product is associative up to isomorphism,

$$(E \wedge E') \wedge E'' \cong E \wedge (E' \wedge E'').$$

This is evident on the prespectrum level. It follows on the spectrum level by a formal argument of a sort that pervades the theory. One need only show that, for prespectra D and D',

$$L(\ell L(D) \wedge D') \cong L(D \wedge D') \cong L(D \wedge \ell L(D')).$$

Conceptually, these are commutation relations between functors that are left adjoints, and they will hold if and only if the corresponding commutation relations are valid for the right adjoints. We shall soon write down the right adjoint function spectra functors. They turn out to be so simple and explicit that it is altogether trivial to check the required commutation relations relating them and the right adjoint ℓ.

The external smash product is very nearly commutative, but to see this we need another observation. If $f : U \longrightarrow U'$ is a linear isometric isomorphism, then we obtain an isomorphism of categories $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$ via

$$(f^*E')(V) = E'(fV).$$

Its inverse is $f_* = (f^{-1})^*$. If $\tau : U \oplus U' \longrightarrow U' \oplus U$ is the transposition, then the commutativity isomorphism of the smash product is

$$E' \wedge E \cong \tau_*(E \wedge E').$$

Analogously, the associativity isomorphism implicitly used the obvious isomorphism of universes $(U \oplus U') \oplus U'' \cong U \oplus (U' \oplus U'')$.

What about unity? We would like $E \wedge S$ to be isomorphic to E, but this doesn't make sense on the face of it since these spectra are indexed on different universes. However, for a based space X and a prespectrum E, we have a prespectrum $E \wedge X$ with

$$(E \wedge X)(V) = EV \wedge X.$$

If we start with a spectrum E and apply L, we obtain a spectrum $E \wedge X$. It is quite often useful to think of based spaces as spectra indexed on the universe $\{0\}$. This makes good sense on the face of our definitions, and we have $E \wedge S^0 \cong E$, where S^0 means the space S^0.

Of course, this is not adequate, and we have still not addressed our original problem about bi-indexed smash products: we have only given it a bit more formal structure. To solve these problems, we go back to our "change of universe functors" $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$. Clearly, to define f^*, the map $f : U \longrightarrow U'$ need only be a linear isometry, not necessarily an isomorphism. While a general linear isometry f need not be an isomorphism, it is a monomorphism. For a prespectrum $E \in \mathscr{P}U$, we can define a prespectrum $f_*E \in U'$ by

$$(f_*E)(V') = EV \wedge S^{V'-fV}, \quad \text{where} \quad V = f^{-1}(V' \cap f(U)). \tag{3.1}$$

Its structure maps are induced from those of E via the isomorphisms

$$EV \wedge S^{V'-fV} \wedge S^{W'-V'} \cong EV \wedge S^{W-V} \wedge S^{W'-fW}. \tag{3.2}$$

As usual, we use the functor L to extend to a functor $f_* : \mathscr{S}U \longrightarrow \mathscr{S}U'$. As is easily verified on the prespectrum level and follows formally on the spectrum level, the inverse isomorphisms that we had in the case of isomorphisms generalize to adjunctions in the case of isometries:

$$\mathscr{S}U'(f_*E, E') \cong \mathscr{S}U(E, f^*E'). \tag{3.3}$$

How does this help us? Let $\mathscr{I}(U, U')$ denote the set of linear isometries $U \longrightarrow U'$. If V is an indexing space in U, then $\mathscr{I}(V, U')$ has an evident metric topology, and we give $\mathscr{I}(U, U')$ the topology of the union. It is vital — and not hard to prove — that $\mathscr{I}(U, U')$ is in fact a contractible space. As we shall explain later, this can be used to prove a version of the following result (which is slightly misstated for clarity in this sketch of ideas).

THEOREM 3.4. Any two linear isometries $U \longrightarrow U'$ induce canonically and coherently weakly equivalent functors $\mathscr{S}U \longrightarrow \mathscr{S}U'$.

We have not yet defined weak equivalences, nor have we defined the stable category. A map $f : D \longrightarrow E$ of spectra is said to be a weak equivalence if each of its component maps $DV \longrightarrow EV$ is a weak equivalence. Since the smash product of a spectrum and a space is defined, we have cylinders $E \wedge I_+$ and thus a notion of homotopy in $\mathscr{S}U$. We let $h\mathscr{S}U$ be the resulting homotopy category, and we let $\bar{h}\mathscr{S}U$ be the category that is obtained from $h\mathscr{S}U$ by adjoining formal inverses to the weak equivalences. We shall be more explicit later.

This is our stable category, and we proceed to define its smash product. We choose a linear isometry $f : U^2 \longrightarrow U$. For spectra E and E' indexed on U, we define an internal smash product $f_*(E \wedge E') \in \mathscr{S}U$. Up to canonical isomorphism in the stable category $\bar{h}\mathscr{S}U$, $f_*(E \wedge E')$ is independent of the choice of f. For associativity, we have

$$f_*(E \wedge f_*(E' \wedge E'')) \cong (f(1 \oplus f))_*(E \wedge E' \wedge E'') \simeq (f(f \oplus 1))_* \cong f_*(f_*(E \wedge E') \wedge E'').$$

Here we write $\cong$ for isomorphisms that hold on the point-set level and $\simeq$ for isomorphisms in the category $\bar{h}\mathscr{S}U$. For commutativity,

$$f_*(E' \wedge E) \cong f_*\tau_*(E \wedge E') \cong (f\tau)_*(E \wedge E') \simeq f_*(E \wedge E').$$

For a space X, we have a suspension prespectrum $\{\Sigma^V X\}$ whose structure maps are identity maps. We let $\Sigma^\infty X = L\{\Sigma^V X\}$. In this case, the construction of L is quite concrete, and we find that

$$\Sigma^\infty X = \{Q\Sigma^V X\}, \quad \text{where} \quad QY = \bigcup \Omega^W \Omega^W Y. \tag{3.5}$$

This gives the suspension spectrum functor $\Sigma^\infty : \mathscr{T} \longrightarrow \mathscr{S}U$. It has a right adjoint Ω^∞ which sends a spectrum E to the space $E_0 = E\{0\}$:

$$\mathscr{S}U(\Sigma^\infty X, E) \cong \mathscr{T}(X, \Omega^\infty E). \tag{3.6}$$

The functor Q is the same as $\Omega^\infty\Sigma^\infty$. For a linear isometry $f : U \longrightarrow U'$, we have

$$f_*\Sigma^\infty X \cong \Sigma^\infty X \tag{3.7}$$

since, trivially, $\Omega^\infty f^* E' = E'_0 = \Omega^\infty E'$. A space equivalent to E_0 for some spectrum E is called an infinite loop space.

Remember that we can think of the category $\mathscr{T}$ of based spaces as the category $\mathscr{S}\{0\}$ of spectra indexed on the universe $\{0\}$. With this interpretation, Ω^∞ coincides with i^*, where $i : \{0\} \longrightarrow U$ is the inclusion. Therefore, by the uniqueness of adjoints, $\Sigma^\infty X$ is isomorphic to $i_* X$. Let $i_1 : U \longrightarrow U^2$ be the inclusion of U as the first summand in $U \oplus U$. The unity isomorphism of the smash product is the case $X = S^0$ of the following isomorphism in $\bar{h}\mathscr{S}U$:

(3.8)

$$f_*(E \wedge \Sigma^\infty X) \cong f_*(i_1)_*(E \wedge X) \cong (f \circ i_1)_*(E \wedge X) \simeq 1_*(E \wedge X) = E \wedge X.$$

We conclude that, up to natural isomorphisms that are implied by Theorem 3.4 and elementary inspections, the stable category $\bar{h}\mathscr{S}U$ is symmetric monoidal with respect to the internal smash product $f_*(E \wedge E')$ for any choice of linear isometry $f : U^2 \longrightarrow U$. It is customary, once this has been proven, to write $E \wedge E'$ to mean this internal smash product, relying on context to distinguish it from the external product.

4. Function spectra

We must define the function spectra that give the right adjoints of our various kinds of smash products. For a space X and a spectrum E, the function spectrum $F(X, E)$ is given by

$$F(X, E)(V) = F(X, EV).$$

Note that this is a spectrum as it stands, without use of the functor L. We have the isomorphism

$$F(E \wedge X, E') \cong F(E, F(X, E'))$$

and the adjunction

$$\mathscr{S}U(E \wedge X, E') \cong \mathscr{T}(X, \mathscr{S}U(E, E')) \cong \mathscr{S}U(E, F(X, E')), \tag{4.1}$$

where the set of maps $E \longrightarrow E'$ is topologized as a subspace of the product over all indexing spaces V of the spaces $F(EV, E'V)$. As an example of the use of right adjoints to obtain information about left adjoints, we have isomorphisms

$$(\Sigma^\infty X) \wedge Y \cong \Sigma^\infty(X \wedge Y) \cong X \wedge (\Sigma^\infty Y). \tag{4.2}$$

For the first, the two displayed functors of X both have right adjoint

$$F(Y, E)_0 = F(Y, E_0).$$

More generally, for universes U and U' and for spectra $E' \in \mathscr{S}U'$ and $E'' \in \mathscr{S}(U \oplus U')$, we define an external function spectrum

$$F(E', E'') \in \mathscr{S}U$$

as follows. For an indexing space V in U, define $E''[V] \in \mathscr{S}U'$ by

$$E''[V](V') = E''(V \oplus V').$$

The structural homeomorphisms are induced by some of those of E'', and others give a system of isomorphisms $E''[V] \longrightarrow \Omega^{W-V} E''[W]$. Define

$$F(E', E'')(V) = \mathscr{S}U'(E', E''[V]).$$

We have the adjunction

$$\mathscr{S}(U \oplus U')(E \wedge E', E'') \cong \mathscr{S}U(E, F(E', E'')). \tag{4.3}$$

When $E' = \Sigma^\infty Y$, $\mathscr{S}U'(E', E''[V]) \cong \mathscr{T}(Y, E''(V))$. Thus, if $i_1 : U \longrightarrow U \oplus U'$ is the inclusion, then

$$F(\Sigma^\infty Y, E'') \cong F(Y, (i_1)^* E'').$$

By adjunction, this implies the first of the following two isomorphisms:

$$(i_1)_*((\Sigma^\infty X) \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y \cong (i_2)_*(X \wedge (\Sigma^\infty Y)). \tag{4.4}$$

When $U = U'$ and $f : U^2 \longrightarrow U$ is a linear isometry, we obtain the internal function spectrum $F(E', f^*E) \in \mathscr{S}U$ for spectra $E, E' \in \mathscr{S}U$. Up to canonical isomorphism in $\bar{h}\mathscr{S}U$, it is independent of the choice of f. For spectra all indexed on U, we have the composite adjunction

$$\mathscr{S}U(f_*(E \wedge E'), E'') \cong \mathscr{S}U(E, F(E', f^* E'')). \tag{4.5}$$

Again, it is customary to abuse notation by also writing $F(E', E)$ for the internal function spectrum, relying on the context for clarity. By combining the three isomorphisms (3.7), (4.2), and (4.4) — all of which were proven by trivial inspections of right adjoints — we obtain the following non-obvious isomorphism for internal smash products.

$$\Sigma^\infty (X \wedge Y) \cong (\Sigma^\infty X) \wedge (\Sigma^\infty Y). \tag{4.6}$$

Generalized a bit, this will be seen to determine the structure of smash products of CW spectra.

5. The equivariant case

We now begin working equivariantly, and we have a punch line: we were led to the framework above by nonequivariant considerations about smash products, and yet the framework is ideally suited to equivariant considerations. Let G be a compact Lie group and recall the discussion of G-spheres and G-universes from IX§§1,2. On the understanding that every space in sight is a G-space and every map in sight is a G-map, the definitions and results above apply verbatim to give the basic definitions and properties of the stable category of G-spectra. For a given G-universe U, we write $G\mathscr{S}U$for the resulting category of G-spectra, $hG\mathscr{S}U$ for its homotopy category, and $\bar{h}G\mathscr{S}U$ for the stable homotopy category that results by adjoining inverses to the weak equivalences.

The only caveat is that $\mathscr{I}(U,U')$ is understood to be the G-space of linear isometries, with G acting by conjugation, and not the space of G-linear isometries. If the G-universes U and U' are isomorphic — which means that they contain the same irreducible representations — then $\mathscr{I}(U,U')$ is G-contractible, and therefore its subspace $\mathscr{I}(U,U')^G$ of G-linear isometries is contractible.

We already see something new in the equivariant context: we have different stable categories of G-spectra depending on the isomorphism type of the underlying universe. This fact will play a vital role in the theory. Remember that a G-universe U is said to be complete if it contains every irreducible representation and trivial if it contains only the trivial irreducible representation. We sometimes refer to G-spectra indexed on a complete G-universe U as genuine G-spectra. We always refer to G-spectra indexed on a trivial G-universe, such as U^G, as naive G-spectra; they are essentially just spectra $\{E_n\}$ of the sort we first defined, but with each E_n a G-space and each structure map a G-map. We have concomitant notions of genuine and naive infinite loop G-spaces. The inclusion $i : U^G \longrightarrow U$ gives us an adjoint pair of functors relating naive and genuine G-spectra:

$$G\mathscr{S}U(i_*E, E') \cong G\mathscr{S}U^G(E, i^*E'). \tag{5.1}$$

We will see that naive G-spectra represent $\mathbb{Z}$-graded cohomology theories, whereas genuine G-spectra represent cohomology theories graded over the real representation ring $RO(G)$. Before getting to this, however, we must complete our development of the stable category.

6. Spheres and homotopy groups

We have deliberately taken a more or less geodesic route to smash products and function spectra, and we have left aside a number of other matters that must be considered. At the risk of obscuring the true simplicity of the nonequivariant theory, we work with G-spectra indexed on a fixed G-universe U from now on in this chapter. We write $G\mathscr{S}$ for $G\mathscr{S}U$. Since G will act on everything in sight, we often omit the prefix, writing spectra for G-spectra and so on.

We shall shortly define G-CW spectra in terms of sphere spectra and their cones, which provide cells. We shall deduce properties of G-CW spectra, such as HELP, by reducing to the case of a single cell and there applying an adjunction to reduce to the G-space level. For this, spheres must be defined in terms of suitable left adjoint functors from spaces to spectra. For $n \geq 0$, there is no problem: we take $\underline{S}^n = \Sigma^\infty S^n$. We shall later write S^n ambiguously for both the sphere space and the sphere spectrum, relying on context for clarity, but we had better be pedantic at first.

We also need negative dimensional spheres. We will define them in terms of shift desuspension functors, and these functors will also serve to clarify the relationship between spectra and their component spaces. Generalizing Ω^∞, define a functor

$$\Omega^\infty_V : G\mathscr{S} \longrightarrow G\mathscr{T}$$

by $\Omega^\infty_V = EV$ for an indexing space V in U. The functor Ω^∞_V has a left adjoint shift desuspension functor

$$\Sigma^\infty_V : G\mathscr{S} \longrightarrow G\mathscr{T}.$$

The spectrum $\Sigma^\infty_V X$ is $L\{\Sigma^{W-V}X\}$. Here the prespectrum $\{\Sigma^{W-V}X\}$ has Wth space Σ^{W-V} if $V \subset W$ and a point otherwise; if $V \subset W \subset Z$, then the corresponding structure map is the evident identification

$$\Sigma^{Z-W}\Sigma^{W-V}X \cong \Sigma^{Z-V}X.$$

The Vth space of $\Sigma^\infty_V X$ is the zeroth space QX of $\Sigma^\infty X$. It is easy to check the prespectrum level version of the claimed adjunction, and the spectrum level adjunction follows:

$$G\mathscr{S}(\Sigma^\infty_V X, E) \cong G\mathscr{T}(X, \Omega^\infty_V E). \tag{6.1}$$

Exactly as in (4.2) and (4.6), we have natural isomorphisms

$$(\Sigma^\infty_V X) \wedge Y \cong \Sigma^\infty_V(X \wedge Y) \cong X \wedge (\Sigma^\infty_V Y) \tag{6.2}$$

and, for the internal smash product,

$$\Sigma^\infty_{V+W}(X \wedge Y) \cong \Sigma^\infty_V X \wedge \Sigma^\infty_W Y \quad \text{if} \quad V \cap W = \{0\}. \tag{6.3}$$

Another check of right adjoints gives the relation

$$\Sigma^\infty_V X \cong \Sigma^\infty_W \Sigma^{W-V} X \quad \text{if} \quad V \subset W. \tag{6.4}$$

It is not hard to see that any spectrum E can be written as the colimit of the shift desuspensions of its component spaces. That is,

$$E \cong \operatorname{colim} \Sigma^\infty_V EV, \tag{6.5}$$

where the colimit is taken over the maps

$$\Sigma^\infty_W \sigma : \Sigma^\infty_V EV \cong \Sigma^\infty_W(\Sigma^{W-V} EV) \longrightarrow \Sigma^\infty_W EW.$$

Let us write U in the form $U = U^G \oplus U'$ and fix an identification of U^G with $\mathbb{R}^\infty$. We abbreviate notation by writing Ω^∞_n and Σ^∞_n when $V = \mathbb{R}^n$. Now define $\underline{S}^{-n} = \Sigma^\infty_n S^0$ for $n > 0$. The reader will notice that we can generalize our definitions to obtain sphere spectra $\underline{S}^V$ and $\underline{S}^{-V}$ for any indexing space V. We can even define spheres $\underline{S}^{V-W} = \Sigma^\infty_W S^V$. We shall need such generality later. However, in developing G-CW theory, it turns out to be appropriate to restrict attention to the spheres $\underline{S}^n$ for integers n. Theorem 6.8 will explain why.

In view of our slogan that orbits are the equivariant analogues of points, we also consider all spectra

$$\underline{S}^n_H \equiv G/H_+ \wedge \underline{S}^n, \;\; H \subset G \quad \text{and} \quad n \in \mathbb{Z}, \tag{6.6}$$

as spheres. By (6.2), $\underline{S}^n_H \cong \Sigma^\infty(G/H_+ \wedge S^n)$ if $n \geq 0$ and $\underline{S}^n_H \cong \Sigma^\infty_n G/H_+$ if $n < 0$. We shall be more systematic about change of groups later, but we prefer to minimize such equivariant considerations in this section. We define the homotopy group systems of G-spectra by setting

$$\pi^H_n(E) = \underline{\pi}_n(E)(G/H) = hG\mathscr{S}(\underline{S}^n_H, E). \tag{6.7}$$

Let $\mathscr{B}_G U$ be the homotopy category of orbit spectra $\underline{S}^0_H = \Sigma^\infty G/H_+$; we generally abbreviate the names of its objects to G/H. This is an additive category, as will become clear shortly, and $\underline{\pi}_n(E)$ is an additive contravariant functor $\mathscr{B}_G U \longrightarrow \mathscr{A}b$. Recall from IX§4 that such functors are called Mackey functors when the universe U is complete. They play a fundamentally important role in equivariant theory, both in algebra and topology, and we shall return to them later. For now, however, we shall concentrate on the individual homotopy groups $\pi_n^H(E)$. We shall later reinterpret these as homotopy groups $\pi_n(E^H)$ of fixed point spectra, but that too can wait.

The following theorem should be viewed as saying that a weak equivalence of G-spectra really is a weak equivalence of G-spectra. Recall that we defined a weak equivalence $f : D \longrightarrow E$ to be a G-map such that each space level G-map $fV : DV \longrightarrow EV$ is a weak equivalence. In setting up CW-theory, which logically should precede the following theorem, one must mean a weak equivalence to be a map that induces an isomorphism on all of the homotopy groups π_n^H of (6.7).

THEOREM 6.8. *Let $f : E \longrightarrow E'$ be a map of G-spectra. Then each component map $fV : EV \longrightarrow E'V$ is a weak equivalence of G-spaces if and only if $f_* : \pi_n^H E \longrightarrow \pi_n^H E'$ is an isomorphism for all $H \subset G$ and all integers n.*

By our adjunctions, we have

(6.9) $$\pi_n^H(E) \cong \pi_n((E_0)^H) \text{ if } n \geq 0 \quad\text{and}\quad \pi_n^H(E) \cong \pi_0((E\mathbb{R}^n)^H) \text{ if } n < 0.$$

Therefore, nonequivariantly, the theorem is a tautological triviality. Equivariantly, the forward implication is trivial but the backward implication says that if each $E\mathbb{R}^n \longrightarrow E'\mathbb{R}^n$ is a weak equivalence, then each $EV \longrightarrow E'V$ is also a weak equivalence. Thus it says that information at the trivial representations in U is somehow capturing information at all other representations in U. Its validity justifies the development of G-CW theory in terms of just the sphere spectra of integral dimensions.

We sketch the proof, which goes by induction. We want to prove that each map $f_* : \pi_*(EV)^H \longrightarrow \pi_*(E'V)^H$ is an isomorphism. Since G contains no infinite descending chains of (closed) subgroups, we may assume that f_* is an isomorphism for all proper subgroups of H. An auxiliary argument shows that we may assume that $V^H = \{0\}$. We then use the cofibration $S(V)_+ \longrightarrow D(V)_+ \longrightarrow S^V$, where $S(V)$ and $D(V)$ are the unit sphere and unit ball in V and thus $D(V)_+ \simeq S^0$. Applying $f : F(\cdot, EV)^H \longrightarrow F(\cdot, E'V)^H$ to this cofibration, we obtain a comparison of fibration sequences. On one end, this is

$$f_0 : (\Omega^V EV)^H = (E_0)^H \longrightarrow (E'_0)^H = (\Omega^V E'V)^H,$$

which is given to be a weak equivalence. On the other end, we can triangulate $S(V)$ as an H-CW complex with cells of orbit type H/K, where K is a proper subgroup of H. We can then use change of groups and the inductive hypothesis to deduce that f induces a weak equivalence on this end too. Modulo an extra argument to handle π_0, we can conclude that the middle map $f : (EV)^H \longrightarrow (E'V)^H$ is a weak equivalence.

7. G-CW spectra

Before getting to CW theory, we must say something about compactness, which plays an important role. A compact spectrum is one of the form $\Sigma^\infty_V X$ for some indexing space V and compact space X. Since a map of spectra with domain $\Sigma^\infty_V X$ is determined by a map of spaces with domain X, facts about maps out of compact spaces imply the corresponding facts about maps out of compact spectra. For example, if E is the union of an expanding sequence of subspectra E_i, then

$$G\mathscr{S}(\Sigma^\infty_V X, E) \cong \operatorname{colim} G\mathscr{S}(\Sigma^\infty_V X, E_i). \tag{7.1}$$

The following lemma clarifies the relationship between space level and spectrum level maps. Recall the isomorphisms of (6.4).

Lemma 7.2. Let $f : \Sigma^\infty_V X \longrightarrow \Sigma^\infty_W Y$ be a map of G-spectra, where X is compact. Then, for a large enough indexing space Z, there is a map $g : \Sigma^{Z-V} X \longrightarrow \Sigma^{Z-W} Y$ of G-spaces such that f coincides with

$$\Sigma^\infty_Z g : \Sigma^\infty_V X \cong \Sigma^\infty_Z(\Sigma^{Z-V} X) \longrightarrow \Sigma^\infty_Z(\Sigma^{Z-W} Y) \cong \Sigma^\infty_W Y.$$

This result shows how to calculate the full subcategory of the stable category consisting of those G-spectra of the form $\Sigma^\infty_V X$ for some indexing space V and finite G-CW complex X in space level terms. It can be viewed as giving an equivariant reformulation of the Spanier-Whitehead S-category. In particular, we have the following consistency statement with the definitions of IX§2.

Proposition 7.3. For a finite based G-CW complex X and a based G-space Y,

$$\{X, Y\}_G \cong [\Sigma^\infty X, \Sigma^\infty Y]_G.$$

From here, the development of CW theory is essentially the same equivariantly as nonequivariantly, and essentially the same on the spectrum level as on the space level. The only novelty is that, because we have homotopy groups in negative degrees, we must use two filtrations. Older readers may see more novelty. In contrast with earlier treatments, our CW theory is developed on the spectrum level and has nothing whatever to do with any possible cell structures on the component spaces of spectra. I view the use of space level cell structures in this context as an obsolete historical detour that serves no useful mathematical purpose.

Let $CE = E \wedge I$ denote the cone on a G-spectrum E.

Definition 7.4. A G-cell spectrum is a spectrum $E \in G\mathscr{S}$ that is the union of an expanding sequence of subspectra E_n, $n \geq 0$, such that E_0 is the trivial spectrum (each of its component spaces is a point) and E_{n+1} is obtained from E_n by attaching G-cells $C\underline{S}^q_H \cong G/H_+ \wedge C\underline{S}^q$ along attaching G-maps $\underline{S}^q_H \longrightarrow E_n$. Cell subspectra, or "subcomplexes", are defined in the evident way. A G-CW spectrum is a G-cell spectrum each of whose attaching maps $\underline{S}^q_H \longrightarrow E_n$ factors through a subcomplex that contains only cells of dimension at most q. The n-skeleton E^n is then defined to be the union of the cells of dimension at most n.

Lemma 7.5. *A map from a compact spectrum to a cell spectrum factors through a finite subcomplex. Any cell spectrum is the union of its finite subcomplexes.*

The filtration $\{E_n\}$ is called the sequential filtration. It records the order in which cells are attached, and it can be chosen in many different ways. In fact, using the lemma, we see that by changing the sequential filtration on the domain, any map between cell spectra can be arranged to preserve the sequential filtration. Using this filtration, we find that the inductive proofs of the following results that we sketched on the space level work in exactly the same way on the spectrum level. We leave it to the reader to formulate their more precise "dimension ν" versions.

Theorem 7.6 (HELP). *Let A be a subcomplex of a G-CW spectrum D and let $e : E \longrightarrow E'$ be a weak equivalence. Suppose given maps $g : A \longrightarrow E$, $h : A \wedge I_+ \longrightarrow E'$, and $f : D \longrightarrow E'$ such that $eg = hi_1$ and $fi = hi_0$ in the following diagram:*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \wedge I_+ \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd, "i"] \\
& E' & & E \arrow[ll, "e"'] & \\
D \arrow[ur, "f"] \arrow[rr, "i_0"'] & & D \wedge I_+ \arrow[ul, dashed, "\tilde{h}"] & & D \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Then there exist maps $\tilde{g}$ and $\tilde{h}$ that make the diagram commute.

Theorem 7.7 (Whitehead). Let $e : E \longrightarrow E'$ be a weak equivalence and D be a G-CW spectrum. Then $e_* : hG\mathscr{S}(D, E) \longrightarrow hG\mathscr{S}(D, E')$ is a bijection.

Corollary 7.8. If $e : E \longrightarrow E'$ is a weak equivalence between G-CW spectra, then e is a G-homotopy equivalence.

Theorem 7.9 (Cellular Approximation). Let (D, A) and (E, B) be relative G-CW spectra, (D', A') be a subcomplex of (D, A), and $f : (D, A) \longrightarrow (E, B)$ be a G-map whose restriction to (D', A') is cellular. Then f is homotopic rel $D' \cup A$ to a cellular map $g : (D, A) \longrightarrow (E, B)$.

Corollary 7.10. Let D and E be G-CW spectra. Then any G-map $f : D \longrightarrow E$ is homotopic to a cellular map, and any two homotopic cellular maps are cellularly homotopic.

Theorem 7.11. For any G-spectrum E, there is a G-CW spectrum $?E$ and a weak equivalence $\gamma : ?E \longrightarrow E$.

Exactly as on the space level, it follows from the Whitehead theorem that ? extends to a functor $hG\mathscr{S} \longrightarrow hG\mathscr{C}$, where $G\mathscr{C}$ is here the category of G-CW spectra and cellular maps, and the morphisms of the stable category $\bar{h}G\mathscr{S}$ can be specified by

$$\bar{h}G\mathscr{S}(E, E') = hG\mathscr{S}(?E, ?E') = hG\mathscr{C}(?E, ?E'). \tag{7.12}$$

From now on, we shall write $[E, E']_G$ for this set. Again, ? gives an equivalence of categories $\bar{h}G\mathscr{S} \longrightarrow hG\mathscr{C}$.

We should say something about the transport of functors F on $G\mathscr{S}$ to the category $\bar{h}G\mathscr{S}$. All of our functors preserve homotopies, but not all of them preserve weak equivalences. If F does not preserve weak equivalences, then, on the stable category level, we understand F to mean the functor induced by the composite $F \circ ?$, a functor which preserves weak equivalences by converting them to genuine equivalences.

For this and other reasons, it is quite important to understand when functors preserve CW-homotopy types and when they preserve weak equivalences. These questions are related. In a general categorical context, a left adjoint preserves CW-homotopy types if and only if its right adjoint preserves weak equivalences. When these equivalent conditions hold, the induced functors on the categories obtained by inverting the weak equivalences are again adjoint.

For example, since Ω_V^∞ preserves weak equivalences (with the correct logical order, by Theorem 6.8), Σ_V^∞ preserves CW homotopy types. Of course, since our left adjoints preserve colimits and smash products with spaces, their behavior on CW spectra is determined by their behavior on spheres. Since Σ_n^∞ clearly preserves spheres, it carries G-CW based complexes (with based attaching maps) to G-CW spectra. This focuses attention on a significant difference between the equivariant and nonequivariant contexts. In both, a CW spectrum is the colimit of its finite subcomplexes. Nonequivariantly, Lemma 7.2 implies that any finite CW spectrum is isomorphic to $\Sigma_n^\infty X$ for some n and some finite CW complex X. Equivariantly, this is only true up to homotopy type. It would be true up to isomorphism if we allowed non-trivial representations as the domains of attaching maps in our definitions of G-CW complexes and spectra. We have seen that such a theory of "G-CW(V)-complexes" is convenient and appropriate on the space level, but it seems to serve no useful purpose on the spectrum level.

Along these lines, we point out an important consequence of (6.3). It implies that the smash product of spheres $\underline{S}_H^m$ and $\underline{S}_J^n$ is $(G/H \times G/J)_+ \wedge \underline{S}^{m+n}$. When G is finite, we can use double cosets to describe $G/H \times G/J$ as a disjoint union of orbits G/K. This allows us to deduce that the smash product of G-CW spectra is a G-CW spectrum. For general compact Lie groups G, we can only deduce that the smash product of G-CW spectra has the homotopy type of a G-CW spectrum.

8. Stability of the stable category

The observant reader will object that we have called $\bar{h}G\mathscr{S}$ the "stable category", but that we haven't given a shred of justification. As usual, we write $\Sigma^V E = E \wedge S^V$ and $\Omega^V E = F(S^V, E)$.

THEOREM 8.1. For all indexing spaces V in U, the natural maps

$$\eta : E \longrightarrow \Omega^V \Sigma^V E \quad \text{and} \quad \varepsilon : \Sigma^V \Omega^V E \longrightarrow E$$

are isomorphisms in $\bar{h}G\mathscr{S}$. Therefore Ω^V and Σ^V are inverse self-equivalences of $\bar{h}G\mathscr{S}$.

Thus we can desuspend by any representations that are in U. Once this is proven, it is convenient to write Σ^{-V} for Ω^V. There are several possible proofs, all of which depend on Theorem 6.8: that is the crux of the matter, and this means that the result is trivial in the nonequivariant context. In fact, once we have Theorem 6.8, we have that the functor Σ^∞_V preserves G-CW homotopy types. Using (6.2), (6.4), and the unit equivalence for the smash product, we obtain

$$E \simeq E \wedge \underline{S}^0 \cong E \wedge \Sigma^\infty_V S^V \cong E \wedge (\Sigma^\infty_V S^0 \wedge S^V).$$

This proves that the functor Σ^V is an equivalence of categories. By playing with adjoints, we see that Ω^V must be its inverse. Observe that this proof is independent of the Freudenthal suspension theorem. This argument and (6.2) give the following important consistency relations, where we now drop the underline from our notation for sphere spectra:

$$(8.2) \qquad \Omega^V E \simeq E \wedge S^{-V} \text{ and } \Sigma^\infty_V X \cong X \wedge S^{-V}, \text{ where } S^{-V} \equiv \Sigma^\infty_V S^0.$$

Since all universes contain $\mathbb{R}$, all G-spectra are equivalent to suspensions. This implies that $\bar{h}G\mathscr{S}$ is an additive category, and it is now straightforward to prove that $\bar{h}G\mathscr{S}$ is triangulated. In fact, it has two triangulations, by cofibrations and fibrations, that differ only by signs. We have already seen that it is symmetric monoidal under the smash product and that it has well-behaved function spectra. We have established a good framework in which to do equivariant stable homotopy theory, and we shall say more about how to exploit it as we go on.

9. Getting into the stable category

The stable category is an ideal world, and the obvious question that arises next is how one gets from the prespectra that occur "in nature" to objects in this category. Of course, our prespectra are all encompassing, since we assumed nothing about their constituent spaces and structure maps, and we do have the left adjoint $L : G\mathscr{P} \longrightarrow G\mathscr{S}$. However, this is a theoretical tool: its good formal properties come at the price of losing control over homotopical information. We need an alternative way of getting into the stable category, one that retains homotopical information.

We first need to say a little more about the functor L. If the adjoint structure maps $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V} EW$ of a prespectrum E are inclusions, then $(LE)(V)$ is just the union over $W \supset V$ of the spaces $\Omega^{W-V} EW$. Taking $W = V$, we obtain an inclusion $\eta : EV \longrightarrow (LE)(V)$, and these maps specify a map of prespectra. If, further, each $\tilde{\sigma}$ is a cofibration and an equivalence, then each map η is an equivalence.

Thus we seek to transform given prespectra into spacewise equivalent ones whose adjoint structural maps are cofibrations. The spacewise equivalence property will ensure that Ω-prespectra are transported to Ω-prespectra. It is more natural to consider cofibration conditions on the structure maps $\sigma : \Sigma^{W-V}EV \longrightarrow EW$, and we say that a prespectrum E is "Σ-cofibrant" if each σ is a cofibration. If E is a Σ-cofibrant prespectrum and if each EV has cofibered diagonal, in the sense that the diagonal map $EV \longrightarrow EV \times EV$ is a cofibration, then each adjoint map $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V}EW$ is a cofibration, as desired.

Observe that no non-trivial spectrum can be Σ-cofibrant as a spectrum since the structure maps σ of spectra are surjections rather than injections. We say that a spectrum is "tame" if it is homotopy equivalent to LE for some Σ-cofibrant prespectrum E. The importance of this condition was only recognized during the work of Elmendorf, Kriz, Mandell, and myself on structured ring spectra. Its use leads to key technical improvements of [EKMM] over [LMS]. For example, the sharpest versions of Theorems 3.4 and 8.1 read as follows.

Theorem 9.1. *Let $\mathscr{S}_tU$ be the full subcategory of tame spectra indexed on U. Then any two linear isometries $U \longrightarrow U'$ induce canonically and coherently equivalent functors $h\mathscr{S}_tU \longrightarrow h\mathscr{S}_tU'$. The maps $\eta : E \longrightarrow \Omega\Sigma E$ and $\varepsilon : \Sigma\Omega E \longrightarrow E$ are homotopy equivalences of spectra when E is tame.*

Moreover, analogously to (6.5), but much more usefully, if E is a Σ-cofibrant prespectrum, then

$$LE \cong \operatorname{colim} \Sigma_V^\infty EV, \tag{9.2}$$

where the maps of the colimit system are the cofibrations

$$\Sigma_W^\infty \sigma : \Sigma_V^\infty EV \cong \Sigma_W^\infty(\Sigma^{W-V}EV) \longrightarrow \Sigma_W^\infty EW.$$

Here the prespectrum level colimit is already a spectrum, so that the colimit is constructed directly, without use of the functor L. Given a G-spectrum E', there results a valuable $\lim^1$ exact sequence

$$0 \longrightarrow \lim{}^1[\Sigma EV, E'V]_G \longrightarrow [LE, E']_G \longrightarrow \lim[EV, E'V]_G \longrightarrow 0 \tag{9.3}$$

for the calculation of maps in $\bar{h}G\mathscr{S}$ in terms of maps in $\bar{h}G\mathscr{T}$.

To avoid nuisance about inverting weak equivalences here, we introduce an equivariant version of the classical CW prespectra.

Definition 9.4. A G-CW prespectrum is a Σ-cofibrant G-prespectrum E such that each EV has cofibered diagonal and is of the homotopy type of a G-CW complex.

We can insist on actual G-CW complexes, but it would not be reasonable to ask for cellular structure maps. We have the following reassuring result relating this notion to our notion of a G-CW spectrum.

PROPOSITION 9.5. If E is a G-CW prespectrum, then LE has the homotopy type of a G-CW spectrum. If E is a G-CW spectrum, then each component space EV has the homotopy type of a G-CW complex.

Now return to our original question of how to get into the stable category. The kind of maps of prespectra that we are interested in here are "weak maps" $D \longrightarrow E$, whose components $DV \longrightarrow EV$ are only required to be compatible up to homotopy with the structural maps. If D is Σ-cofibrant, then any weak map is spacewise homotopic to a genuine map. The inverse limit term of (9.3) is given by weak maps, which represent maps between cohomology theories on spaces, and its $\lim^1$ term measures the difference between weak maps and genuine maps, which represent maps between cohomology theories on spectra.

Applying G-CW approximation spacewise, using I.3.6, we can replace any G-prespectrum E by a spacewise weakly equivalent G-prespectrum $?E$ whose component spaces are G-CW complexes and therefore have cofibered diagonal maps. However, the structure maps, which come from the Whitehead theorem and are only defined up to homotopy, need not be cofibrations. The following "cylinder construction" converts a G-prespectrum E whose spaces are of the homotopy types of G-CW complexes and have cofibered diagonals into a spacewise equivalent G-CW prespectrum KE. Both constructions are functorial on weak maps.

The composite $K?$ carries an arbitrary G-prespectrum E to a spacewise equivalent G-CW prespectrum. By Proposition 9.5, $LK?E$ has the homotopy type of a G-CW spectrum. In sum, the composite $LK?$ provides a canonical passage from G-prespectra to G-CW spectra that is functorial up to weak homotopy and preserves all homotopical information in the given G-prespectra.

The version of the cylinder construction presented in [LMS] is rather clumsy. The following version is due independently to Elmendorf and Hesselholt. It enjoys much more precise properties, details of which are given in [EKMM].

CONSTRUCTION 9.6 (CYLINDER CONSTRUCTION). Let E be a G-prespectrum indexed on U. Define KE as follows. For an indexing space V, let $\underline{V}$ be the category of subspaces $V' \subset V$ and inclusions. Define a functor E_V from $\underline{V}$ to

G-spaces by letting $E_V(V') = \Sigma^{V-V'} EV'$. For an inclusion $V'' \longrightarrow V'$,

$$V - V'' = (V - V') \oplus (V' - V'')$$

and $\sigma : \Sigma^{V'-V''} EV'' \longrightarrow EV'$ induces $E_V(V'') \longrightarrow E_V(V')$. Define

$$(KE)(V) = \operatorname{hocolim} E_V.$$

An inclusion $i : V \longrightarrow W$ induces a functor $\underline{i} : \underline{V} \longrightarrow \underline{W}$, the functor Σ^{W-V} commutes with homotopy colimits, and we have an evident isomorphism $\Sigma^{W-V} E_V \cong E\underline{i}$ of functors $\underline{V} \longrightarrow \underline{W}$. Therefore $\underline{i}$ induces a map

$$\sigma : \Sigma^{W-V} \operatorname{hocolim} E_V \cong \operatorname{hocolim} \Sigma^{W-V} E_V \cong \operatorname{hocolim} E\underline{i} \longrightarrow \operatorname{hocolim} E_W.$$

One can check that this map is a cofibration. Thus, with these structural maps, KE is a Σ-cofibrant prespectrum. The structural maps $\sigma : E_V V' \longrightarrow EV$ specify a natural transformation to the constant functor at EV and so induce a map $r : (KE)(V) \longrightarrow EV$, and these maps r specify a map of prespectra. Regarding the object V as a trivial subcategory of $\underline{V}$, we obtain $j : EV \longrightarrow (KE)(V)$. Clearly $rj = \mathrm{Id}$, and $jr \simeq \mathrm{Id}$ via a canonical homotopy since V is a terminal object of $\underline{V}$. The maps j specify a weak map of prespectra, via canonical homotopies. Clearly K is functorial and homotopy-preserving, and r is natural. If each space EV has the homotopy type of a G-CW complex, then so does each $(KE)(V)$, and similarly for the cofibered diagonals condition.

A striking property of this construction is that it commutes with smash products: if E and E' are prespectra indexed on U and U', then $KE \wedge KE'$ is isomorphic over $E \wedge E'$ to $K(E \wedge E')$.

CHAPTER XIII

$RO(G)$-graded homology and cohomology theories

1. Axioms for RO(G)-graded cohomology theories

Switching to a homological point of view, we now consider $RO(G)$-graded homology and cohomology theories. There are several ways to be precise about this, and there are several ways to be imprecise. The latter are better represented in the literature than the former. As we have already said, no matter how things are set up, "$RO(G)$-graded" is technically a misnomer since one cannot think of representations as isomorphism classes and still keep track of signs. We give a formal axiomatic definition here and connect it up with G-spectra in the next section.

From now on, we shall usually restrict attention to reduced homology and cohomology theories and shall write them without a tilde. Of course, a $\mathbb{Z}$-graded homology or cohomology theory on G-spaces is required to satisfy the redundant axioms: homotopy invariance, suspension isomorphism, exactness on cofiber sequences, additivity on wedges, and invariance under weak equivalence. Here exactness only requires that a cofiber sequence $X \longrightarrow Y \longrightarrow Z$ be sent to a three term exact sequence in each degree. The homotopy and weak equivalence axioms say that the theory is defined on $\bar{h}G\mathscr{T}$. Such theories determine and are determined by unreduced theories that satisfy the Eilenberg-Steenrod axioms, minus the dimension axiom. Since

$$k_G^{-n}(X) \cong k_G^0(\Sigma^n X),$$

only the non-negative degree parts of a theory need be specified, and a non-negative integer n corresponds to $\mathbb{R}^n$. Indexing on $\mathbb{Z}$ amounts to either choosing a basis for $\mathbb{R}^\infty$ or, equivalently, choosing a skeleton of a suitable category of trivial representations.

Now assume given a G-universe U, say $U = \oplus(V_i)^\infty$ for some sequence of distinct irreducible representations V_i with $V_1 = \mathbb{R}$. An $RO(G;U)$-graded theory can be thought of as graded on the free Abelian group on basis elements corresponding to the V_i. It is equivalent to grade on the skeleton of a category of representations embeddable in U, or to grade on this entire category. The last approach seems to be preferable when considering change of groups, so we will adopt it.

Thus let $\mathscr{R}O(G;U)$ be the category whose objects are the representations embeddable in U and whose morphisms $V \longrightarrow W$ are the G-linear isometric isomorphisms. Say that two such maps are homotopic if their associated based G-maps $S^V \longrightarrow S^W$ are *stably* homotopic, and let $h\mathscr{R}O(G;U)$ be the resulting homotopy category. For each W, we have an evident functor

$$\Sigma^W : \mathscr{R}O(G;U) \times \bar{h}G\mathscr{T} \longrightarrow \mathscr{R}O(G;U) \times \bar{h}G\mathscr{T}$$

that sends (V, X) to $(V \oplus W, \Sigma^W X)$.

Definition 1.1. An $RO(G;U)$-graded cohomology theory is a functor

$$E_G^* : h\mathscr{R}O(G,U) \times (\bar{h}G\mathscr{T})^{op} \longrightarrow \mathscr{A}b,$$

written $(V, X) \longrightarrow E_G^V(X)$ on objects and similarly on morphisms, together with natural isomorphisms $\sigma^W : E_G^* \longrightarrow E_G^* \circ \Sigma^W$, written

$$\sigma^W : E_G^V(X) \longrightarrow E_G^{V\oplus W}(\Sigma^W X),$$

such that the following axioms are satisfied.

(1) For each representation V, the functor E_G^V is exact on cofiber sequences and sends wedges to products.

(2) If $\alpha : W \longrightarrow W'$ is a map in $\mathscr{R}O(G,U)$, then the following diagram commutes:

$$\begin{tikzcd}
E_G^V(X) \arrow[r, "\sigma^W"] \arrow[d, "\sigma^{W'}"'] & E_G^{V\oplus W}(\Sigma^W X) \arrow[d, "E_G^{\mathrm{id}\,\oplus\alpha}(\mathrm{id})"] \\
E_G^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "(\Sigma^\alpha\,\mathrm{id})^*"'] & E_G^{V\oplus W'}(\Sigma^W X).
\end{tikzcd}$$

(3) $\sigma^0 = \mathrm{id}$ and the σ are transitive in the sense that the following diagram commutes for each pair of representations (W, Z):

$$\begin{tikzcd} E_G^V(X) \arrow[rr, "\sigma^W"] \arrow[dr, "\sigma^{W\oplus Z}"'] & & E_G^{V\oplus W}(\Sigma^W X) \arrow[dl, "\sigma^Z"] \\ & E_G^{V\oplus W\oplus Z}(\Sigma^{W\oplus Z}X). & \end{tikzcd}$$

We extend a theory so defined to "formal differences $V \ominus W$" for any pair of representations (V, W) by setting

$$E_G^{V\ominus W}(X) = E_G^V(\Sigma^W X).$$

We use the symbol $\ominus$ to avoid confusion with either orthogonal complement or difference in the representation ring. Rigorously, we are thinking of $V \ominus W$ as an object of the category $h\mathscr{R}O(G;U) \times h\mathscr{R}O(G;U)^{op}$, and, for each X, we have defined a functor from this category to the category of Abelian groups.

The representation group $RO(G;U)$ relative to the given universe U is obtained by passage to equivalence classes from the set of formal differences $V \ominus W$, where $V \ominus W$ is equivalent to $V' \ominus W'$ if there is a G-linear isometric isomorphism

$$\alpha : V \oplus W' \longrightarrow V' \oplus W;$$

$RO(G;U)$ is a ring if tensor products of representations embeddable in U are embeddable in U.

When interpreting $RO(G;U)$-graded cohomology theories, we must keep track of the choice of α, and we see that a given α determines the explicit isomorphism displayed as the unlabelled arrow in the diagram of isomorphisms

$$\begin{tikzcd} E_G^V(\Sigma^W X) \arrow[r, "\sigma^{W'}"] \arrow[d] & E_G^{V\oplus W'}(\Sigma^{W\oplus W'}X) \arrow[d, "E_G^{\alpha}(\Sigma^{\tau}\,\mathrm{id})"] \\ E_G^{V'}(\Sigma^{W'}X) \arrow[r, "\sigma^W"'] & E_G^{V'\oplus W}(\Sigma^{W'\oplus W}X), \end{tikzcd}$$

where $\tau : W \oplus W' \longrightarrow W' \oplus W$ is the transposition isomorphism.

If $V^G = 0$, write $V \oplus \mathbb{R}^n = V + n$. Axiom (1) ensures that, for each such V, the E_G^{V+n} and σ^1 define a $\mathbb{Z}$-graded cohomology theory. Axiom (2), together with some easy category theory, ensures that we obtain complete information if we restrict attention to one object in each isomorphism class of representations, that is, if we restrict to any skeleton of the category $\mathscr{R}O(G;U)$. One can even restrict further to a skeleton of its homotopy category. We shall say more about this in the next section.

We can replace the category $\bar{h}G\mathscr{T}$ of based G-spaces by the category $\bar{h}G\mathscr{S}U$ of G-spectra in the definition just given and so define an $RO(G;U)$-graded cohomology theory on G-spectra. Observe that, by our definition of the category $\mathscr{R}O(G;U)$, the isomorphism type of the functor E_G^V depends only on the stable homotopy type of the G-sphere S^V. Such stable homotopy types have been classified by tom Dieck.

We have the evident dual axioms for $RO(G;U)$-graded homology theories on G-spaces or G-spectra. The only point that needs to be mentioned is that homology theories must be given by contravariant functors on $\mathscr{R}O(G;U)$ in order to make sense of the homological counterpart of Axiom (2).

T. tom Dieck. Transformation groups and representation theory. Springer Lecture Notes in Mathematics. Vol. 766. 1979.

2. Representing $RO(G)$-graded theories by G-spectra

With our categorical definition of $RO(G;U)$-graded cohomology theories, it is not obvious that they are represented by G-spectra. We show that they are in this and the following section, first showing how to obtain an $RO(G;U)$-graded theory from a G-spectrum and then showing how to obtain a G-spectrum from an $RO(G;U)$-graded theory. Since I find the equivariant forms of these results in the literature to be unsatisfactory, I shall go into some detail. The problem is to pass from indexing spaces to general representations embeddable in our given universe U, and the idea is to make explicit structure that is implicit in the notion of a G-spectrum and then exploit standard categorical techniques. We begin with some of the latter.

Let $\mathscr{I}O(G;U)$ and $h\mathscr{I}O(G;U)$ be the full subcategories of $\mathscr{R}O(G;U)$ and $h\mathscr{R}O(G;U)$ whose objects are the indexing spaces in U, let

$$\Psi : \mathscr{I}O(G;U) \longrightarrow \mathscr{R}O(G;U)$$

be the inclusion, and also write Ψ for the inclusion $h\mathscr{I}O(G;U) \longrightarrow h\mathscr{R}O(G;U)$. For each representation V that is embeddable in U, choose an indexing space ΦV in U and a G-linear isomorphism $\phi_V : V \longrightarrow \Phi V$. If V is itself an indexing space in U, choose $\Phi V = V$ and let ϕ_V be the identity map. Extend Φ to a functor

$$\Phi : \mathscr{R}O(G;U) \longrightarrow \mathscr{I}O(G;U)$$

by letting $\Phi\alpha$, $\alpha : V \longrightarrow V'$, be the composite

$$\Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\alpha} V' \xrightarrow{\phi_{V'}} \Phi V'.$$

Then $\Phi \circ \Psi = \mathrm{Id}$ and the ϕ_V define a natural isomorphism $\mathrm{Id} \longrightarrow \Psi \circ \Phi$. This equivalence of categories induces an equivalence of categories between $h\mathscr{I}O(G;U)$ and $h\mathscr{R}O(G;U)$. A functor F from $h\mathscr{I}O(G;U)$ to any category $\mathscr{C}$ extends to the functor $F\Phi$ from $h\mathscr{R}O(G;U)$ to $\mathscr{C}$, and we agree to write F instead of $F\Phi$ for such an extended functor.

LEMMA 2.1. Let E be an ΩG-prespectrum. Then E gives the object function of a functor $E : h\mathscr{R}O(G;U) \longrightarrow \bar{h}G\mathscr{T}$.

PROOF. By the observations above, it suffices to define E as a functor on $h\mathscr{I}O(G;U)$. Suppose given indexing spaces V and V' in U and a G-linear isomorphism $\alpha : V \longrightarrow V'$. Choose an indexing space W large enough that it contains both V and V' and that $W - V$ and $W - V'$ both contain copies of representations isomorphic to V and thus to V'. Then there is an isomorphism $\beta : W - V \longrightarrow W' - V'$ such that

$$\beta \wedge \alpha : S^W \cong S^{W-V} \wedge S^V \longrightarrow S^{W-V'} \wedge S^{V'} \cong S^W$$

is stably homotopic to the identity. (For the verification, one relates smash product to composition product in the zero stem $\pi_0^G(S^0)$, exactly as in nonequivariant stable homotopy theory.) Then define $E\alpha : EV \longrightarrow EV'$ to be the composite

$$EV \xrightarrow{\tilde{\sigma}} \Omega^{W-V} EW \xrightarrow{\Omega^{\beta^{-1}}} \Omega^{W'-V'} EW \xrightarrow{\sigma^{-1}} EV'.$$

It is not hard to check that this construction takes stably homotopic maps α and α' to homotopic maps $E\alpha$ and $E\alpha'$ and that the construction is functorial on $\mathscr{I}O(G;U)$. $\square$

PROPOSITION 2.2. An Ω-G-prespectrum E indexed on a universe U represents an $RO(G;U)$-graded cohomology theory E_G^* on based G-spaces.

PROOF. For a representation V that embeds in U, define

$$E_G^V(X) = [X, E\Phi V]_G.$$

For each $\alpha : V \longrightarrow V'$, define

$$E_G^\alpha(X) = [X, E\Phi\alpha]_G.$$

This gives us the required functor

$$E_G^* : h\mathscr{R}O(G,U) \times (\bar{h}G\mathscr{T})^{op} \longrightarrow \mathscr{A}b,$$

and it is obvious that Axiom (1) of Definition 1.1 is satisfied.

Next, suppose given representations V and W that embed in U. We may write

$$\Phi(V \oplus W) = V' + W',$$

where $V' = \phi_{V\oplus W}(V)$ and $W' = \phi_{V\oplus W}(W)$. There result isomorphisms

$$\iota_V : \Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\phi'_V} V' \quad \text{and} \quad \iota_W : \Phi W \xrightarrow{\phi_W^{-1}} W \xrightarrow{\phi'_W} W',$$

where $\phi'_V = \phi_{V\oplus W}|_V$ and $\phi'_W = \phi_{V\oplus W}|_W$. Define

$$\sigma^W : E_G^V(X) \longrightarrow E_G^{V\oplus W}(\Sigma^W X)$$

by the commutativity of the following diagram:

$$\begin{tikzcd}
E_G^V X = [X, E\Phi V]_G \arrow[rr, "{[\mathrm{id}, E\iota_V]}"] \arrow[dd, "\sigma^W"'] & & {[X, EV']_G} \arrow[d, "{[\mathrm{id},\tilde{\sigma}]}"] \\
 & & {[X, \Omega^{W'} E(V'+W')]_G} \arrow[d, "\cong"] \\
E_G^{V\oplus W}(\Sigma^W X) = [\Sigma^W X, E\Phi(V\oplus W)]_G & & {[\Sigma^{W'} X, E(V'\oplus W')]_G.} \arrow[ll, "{[\Sigma^{\phi'_W}\mathrm{id}, \mathrm{id}]}"']
\end{tikzcd}$$

Diagram chases from the definitions demonstrate that Σ^W is natural, that the diagram of Axiom (2) of Definition 1.1 commutes, and that the transitivity diagram of Axiom 3 commutes because of the transitivity condition that we gave as part of the definition of a G-prespectrum. $\square$

There is an analog for homology theories.

A slight variant of the proof above could be obtained by first replacing the given Ω-G-prespectrum by a spacewise equivalent G-spectrum indexed on U and then specializing the following result to suspension G-spectra. Recall that, for an indexing space V, we have the shift desuspension functor Σ^∞_V from based G-spaces to G-spectra. It is left adjoint to the Vth space functor:

$$[\Sigma^\infty_V X, E]_G \cong [X, EV]_G. \tag{2.3}$$

DEFINITION 2.4. For a formal difference $V \ominus W$ of representations of G that embed in U, define the sphere G-spectrum $S^{V\ominus W}$ by

$$S^{V\ominus W} = \Sigma^\infty_{\Phi W} S^V, \tag{2.5}$$

where $\Phi : \mathscr{R}O(G;U) \longrightarrow \mathscr{I}O(G;U)$ is the equivalence of categories constructed above.

PROPOSITION 2.6. *A G-spectrum E indexed on U determines an $RO(G;U)$-graded homology theory E^G_* and an $RO(G;U)$-graded cohomology theory E^*_G on G-spectra.*

PROOF. For G-spectra X and formal differences $V \ominus W$ of representations that embed in U, we define

$$E^G_{V\ominus W}(X) = [S^{V\ominus W}, E \wedge X]_G \tag{2.7}$$

and

$$E^{V\ominus W}_G(X) = [S^{W\ominus V} \wedge X, E]_G = [S^{W\ominus V}, F(X,E)]_G. \tag{2.8}$$

Of course, in cohomology, to verify the axioms, we may as well restrict attention to the case $W = 0$, and similarly in homology. Obviously, the verification reduces to the study of the properties of the G-spheres $\Sigma_V S^0$, or of the functors Σ_V. First, we need functoriality on $\mathscr{R}O(G;U)$, but this is immediate from (2.3) and the functoriality of the EV given by Lemma 2.1. With the notations of the previous proof, we obtain the σ^W from the composite isomorphism of functors

$$\Sigma^\infty_{\Phi V} \cong \Sigma^\infty_{V'} \cong \Sigma^{W'}\Sigma^\infty_{V'+W'} \cong \Sigma^W \Sigma^\infty_{\Phi(V\oplus W)},$$

where the three isomorphisms are given by use of ι_V, passage to adjoints from the homeomorphism $\tilde\sigma : EV' \longrightarrow \Omega^{W'}E(V'+W')$, and use of ϕ'_W. From here, the verification of the axioms is straightforward. $\square$

3. Brown's theorem and $RO(G)$-graded cohomology

We next show that, conversely, all $RO(G)$-graded cohomology theories on based G-spaces are represented by Ω-G-prespectra and all theories on G-spectra are represented by G-spectra. We then discuss the situation in homology, which is considerably more subtle equivariantly than nonequivariantly.

We first record Brown's representability theorem. Brown's categorical proof applies just as well equivariantly as nonequivariantly, on both the space and the spectrum level. Recall that homotopy pushouts are double mapping cylinders and that weak pullbacks satisfy the existence but not the uniqueness property of pullbacks. Recall that a G-space X is said to be G-connected if each of its fixed point spaces X^H is non-empty and connected.

THEOREM 3.1 (BROWN). *A contravariant set-valued functor k on the homotopy category of G-connected based G-CW complexes is representable in the form $kX \cong [X,K]_G$ for a based G-CW complex K if and only if k satisfies the wedge and Mayer-Vietoris axioms: k takes wedges to products and takes homotopy pushouts to weak pullbacks. The same statement holds for the homotopy category of G-CW spectra indexed on U for any G-universe U.*

COROLLARY 3.2. *An $RO(G;U)$-graded cohomology theory E^*_G on based G-spaces is represented by an Ω-G-prespectrum indexed on U.*

PROOF. Restricting attention to G-connected based G-spaces, which is harmless in view of the suspension axiom for trivial representations, we see that (1) of Definition 1.1 implies the Mayer-Vietoris and wedge axioms that are needed to apply Brown's representability theorem. This gives that E^V_G is represented by a G-CW complex EV for each indexing space V in U. If $V \subset W$, then the suspension isomorphism

$$\sigma^{W-V} : E^V_G(X) \cong E^W_G(\Sigma^{W-V}X)$$

is represented by a homotopy equivalence $\tilde{\sigma} : EV \longrightarrow \Omega^{W-V}EW$. The transitivity of the given system of suspension isomorphisms only gives that the structural maps are transitive up to homotopy, whereas the definition of a G-prespectrum requires that the structural maps be transitive on the point-set level. If we restrict to a cofinal sequence of indexing spaces, then we can use transitivity to define the structural weak equivalences for non-consecutive terms of the sequence. We can then interpolate using loop spaces to construct a representing Ω-G-prespectrum indexed on all indexing spaces. □

We emphasize a different point of view of the spectrum level analog. In fact, we shall exploit the following result to construct ordinary $RO(G)$-graded cohomology theories in the next section.

COROLLARY 3.3. A $\mathbb{Z}$-graded cohomology theory on G-spectra indexed on U is represented by a G-spectrum indexed on U and therefore extends to an $RO(G;U)$-graded cohomology theory on G-spectra indexed on U.

PROOF. Since the loop and suspension functors are inverse equivalences on the stable category $\bar{h}G\mathscr{S}U$, we can reconstruct the given theory from its zeroth term, and Brown's theorem applies to represent the zeroth term. □

We showed in the previous chapter that an Ω-G-prespectrum determines a spacewise equivalent G-spectrum, so that a cohomology theory on based G-spaces extends to a cohomology theory on G-spectra. The extension is unique up to non-unique isomorphism, where the non-uniqueness is measured by the $\lim^1$ term in (XII.9.3).

Adams proved a variant of Brown's representability theorem for functors defined only on connected finite CW complexes, removing a countability hypothesis that was present in an earlier version due to Brown. This result also generalizes to the equivariant context, with the same proof as Adams' original one.

THEOREM 3.4 (ADAMS). A contravariant group-valued functor k defined on the homotopy category of G-connected finite based G-CW complexes is representable in the form $kX \cong [X, K]_G$ for some G-CW spectrum K if and only if k converts finite wedges to direct products and converts homotopy pushouts to weak pullbacks of underlying sets. The same statement holds for the homotopy category of finite G-CW spectra.

Here the representing G-CW spectrum K is usually infinite and is unique only up to non-canonical equivalence. More precisely, maps $g, g' : Y \longrightarrow Y'$ are said to be weakly homotopic if gf is homotopic to $g'f$ for any map $f : X \to Y$ defined on a finite G-CW spectrum X, and K is unique up to isomorphism in the resulting weak homotopy category of G-CW spectra.

Nonequivariantly, we pass from here to the representation of homology theories by use of Spanier-Whitehead duality. A finite CW spectrum X has a dual DX

that is also a finite CW spectrum. Given a homology theory E_* on based spaces or on spectra, we obtain a dual cohomology theory on finite X by setting

$$E^n(X) = E_{-n}(DX).$$

We then argue as above that this cohomology theory on finite X is representable by a spectrum E, and we deduce by duality that E also represents the originally given homology theory.

Equivariantly, this argument works for a complete G-universe U, but it does not work for a general universe. The problem is that, as we shall see later, only those orbit spectra $\Sigma^\infty G/H_+$ such that G/H embeds equivariantly in U have well-behaved duals. For example, if the universe U is trivial, then inspection of definitions shows that $F(G/H_+, S) = S$ for all $H \subseteq G$, where S is the sphere spectrum with trivial G-action. Thus X is not equivalent to DDX in general and we cannot hope to recover $E_*(X)$ as $E^*(DX)$.

Corollary 3.5. If U is a complete G-universe, then an $RO(G;U)$-graded homology theory on based G-spaces or on G-spectra is representable.

From now on, unless explicitly stated otherwise, we take our given universe U to be complete, and we write $RO(G) = RO(G;U)$. As shown by long experience in nonequivariant homotopy theory, even if one's primary interest is in spaces, the best way to study homology and cohomology theories is to work on the spectrum level, exploiting the virtues of the stable homotopy category.

J. F. Adams. A variant of E. H. Brown's representability theorem. Topology, 10(1971), 185–198.
E. H. Brown, Jr. Cohomology theories. Annals of Math. 75(1962), 467–484.
E. H. Brown, Jr. Abstract homotopy theory. Trans. Amer. Math. Soc. 119(1965), 79–85.

4. Equivariant Eilenberg-MacLane spectra

From the topological point of view, a coefficient system is a contravariant additive functor from the stable category of naive orbit spectra to Abelian groups. In fact, it is easy to see that the group of stable maps $G/H_+ \longrightarrow G/K_+$ in the naive sense is the free Abelian group on the set of G-maps $G/H \longrightarrow G/K$.

Recall from IX§4 that a Mackey functor is defined to be an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$. Clearly the Burnside category $\mathscr{B} = \mathscr{B}_G$ introduced there is just the full subcategory of the stable category whose objects are the orbit spectra $\Sigma^\infty G/H_+$. The only difference is that, when defining $\mathscr{B}_G$, we abbreviated the names of objects to G/H.

From this point of view, the forgetful functor that takes a Mackey functor to a coefficient system is obtained by pullback along the functor i^* from the stable category of genuine orbit spectra to the stable category of naive orbit spectra. In X§4, Waner described a space level construction of an $RO(G)$-graded cohomology theory with coefficients in a Mackey functor M that extends the ordinary $\mathbb{Z}$-graded cohomology theory determined by its underlying coefficient system i^*M. We shall here give a more sophisticated, and I think more elegant and conceptual, spectrum level construction of such "ordinary" $RO(G)$-graded cohomology theories, and similarly for homology.

Our strategy is to construct a genuine Eilenberg-MacLane G-spectrum $HM = K(M,0)$ to represent our theory. Just as nonequivariantly, an Eilenberg-Mac Lane G-spectrum HM is one such that $\underline{\pi_n}(HM) = 0$ for $n \neq 0$. Of course, $\underline{\pi}_0(HM) = M$ must be a Mackey functor since that is true of $\underline{\pi}_n(E)$ for any n and any G-spectrum E. We shall explain the following result.

Theorem 4.1. For a Mackey functor M, there is an Eilenberg-MacLane G-spectrum HM such that $\underline{\pi}_0(HM) = M$. It is unique up to isomorphism in $\bar{h}G\mathscr{S}$. For Mackey functors M and M', $[HM, HM']_G$ is the group of maps of Mackey functors $M \longrightarrow M'$.

There are several possible proofs. For example, one can exploit projective resolutions of Mackey functors. The proof that we shall give is the original one of Lewis, McClure, and myself, which I find rather amusing.

What is amusing is that, motivated by the desire to construct an $RO(G)$-graded cohomology theory, we instead construct a $\mathbb{Z}$-graded theory. However, this is a $\mathbb{Z}$-graded theory defined on G-spectra. As observed in Corollary 4.3, it can be represented and therefore extends to an $RO(G)$-graded theory. The representing G-spectrum is the desired Eilenberg-MacLane G-spectrum HM. What is also amusing is that the details that we shall use to construct the desired cohomology theories are virtually identical to those that we used to construct ordinary theories in the first place.

We start with G-CW spectra X. They have skeletal filtrations, and we define Mackey-functor valued cellular chains by setting

$$\underline{C}_n(X) = \underline{\pi}_n(X^n/X^{n-1}). \tag{4.2}$$

We used homology groups in I§4, but, aside from nuisance with the cases $n = 0$

and $n = 1$, we could equally well have used homotopy groups. Of course, X^n/X^{n-1} is a wedge of n-sphere G-spectra $S^n_H \simeq G/H_+ \wedge S^n$. We see that the $\underline{C}_n(X)$ are projective objects of the Abelian category of Mackey functors by essentially the same argument that we used in I§4. As there, the connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}) specifies a map of Mackey functors

$$d : \underline{C}_n(X) \longrightarrow \underline{C}_{n-1}(X),$$

and $d^2 = 0$. Write $\mathrm{Hom}_{\mathscr{B}}(M, M')$ for the Abelian group of maps of Mackey functors $M \longrightarrow M'$. For a Mackey functor M, define

$$C^n_G(X; M) = \mathrm{Hom}_{\mathscr{B}}(\underline{C}_n(X), M), \quad \text{with} \quad \delta = \mathrm{Hom}_{\mathscr{B}}(d, \mathrm{Id}). \tag{4.3}$$

Then $C^*_G(X; M)$ is a cochain complex of Abelian groups. We denote its homology by $H^*_G(X; M)$.

The evident cellular versions of the homotopy, exactness, wedge, and excision axioms admit exactly the same quick derivations as on the space level, and we use G-CW approximation to extend from G-CW spectra to general G-spectra: we have a $\mathbb{Z}$-graded cohomology theory on $\bar{h}\mathscr{G}\mathscr{S}$. It satisfies the dimension axiom

$$H^*_G(S^0_H; M) = H^0_G(S^0_H; M) = M(G/H), \tag{4.4}$$

these giving isomorphisms of Mackey functors. The zeroth term is represented by a G-spectrum HM, and we read off its homotopy group Mackey functors directly from (4.4):

$$\underline{\pi}_0(HM) = M \quad \text{and} \quad \underline{\pi}_n(HM) = 0 \text{ if } \ n \neq 0.$$

The uniqueness of HM is evident, and the calculation of $[HM, HM']_G$ follows easily from the functoriality in M of the theories $H^*_G(X; M)$.

We should observe that spectrum level obstruction theory works exactly as on the space level, modulo connectivity assumptions to ensure that one has a dimension in which to start inductions.

For G-spaces X, we now have two meanings in sight for the notation $H^n_G(X; M)$: we can regard our Mackey functor as a coefficient system and take ordinary cohomology as in I§4, or we can take our newly constructed cohomology. We know by the axiomatic characterization of ordinary cohomology that these must in fact be isomorphic, but it is instructive to check this directly. At least after a single suspension, we can approximate any G-space by a weakly equivalent G-CW based complex, with based attaching maps. The functor Σ^∞ takes G-CW based complexes to G-CW spectra, and we find that the two chain complexes in sight are isomorphic. Alternatively, we can check on the represented level:

$$[\Sigma^\infty X, \Sigma^n HM]_G \cong [X, \Omega^\infty \Sigma^n HM]_G \cong [X, K(M,n)]_G.$$

What about homology? Recall that a coMackey functor is a covariant functor $N : \mathscr{B} \longrightarrow \mathscr{A}b$. Using the usual coend construction, we define

$$C_n^G(X;N) = \underline{C}_*(X) \otimes_{\mathscr{B}} N, \quad \text{with} \quad \partial = d \otimes \mathrm{Id}. \tag{4.5}$$

Then $C_*^G(X;N)$ is a chain complex of Abelian groups. We denote its homology by $H_*^G(X;N)$. Again, the verification of the axioms for a $\mathbb{Z}$-graded homology theory on $\bar{h}\mathscr{G}\mathscr{S}$ is immediate. The dimension axiom now reads

$$H_*^G(S_H^0;N) = H_0^G(S_H^0;N) = N(G/H). \tag{4.6}$$

We define a cohomology theory on finite G-spectra X by

$$H_G^*(X;N) = H_{-*}^G(DX;N). \tag{4.7}$$

Applying Adams' variant of the Brown representability theorem, we obtain a G-spectrum JN that represents this cohomology theory. For finite X, we obtain

$$H_*^G(X;N) = H_G^{-*}(DX;N) \cong [DX, JN]_G^{-*} \cong [S, JN \wedge X]_*^G = JN_*^G(X).$$

Thus JN represents the $\mathbb{Z}$-graded homology theory that we started with and extends it to an $RO(G)$-graded theory. We again see that, on G-spaces X, $H_*^G(X;N)$ agrees with the homology of X with coefficients in the underlying covariant coefficient system of N, as defined in I§4.

What are the homotopy groups of JN? The answer must be

$$\pi_n^H(JN) = H_n^G(D(G/H_+);N).$$

For finite G, orbits are self-dual and the resulting isomorphism of the stable orbit category with its opposite category induces the evident self-duality of the algebraically defined category of Mackey functors to be discussed in XIX§3. This allows us to conclude that

$$JN = H(N^*),$$

where N^* is the Mackey functor dual to the coMackey functor N.

For general compact Lie groups, however, the dual of G/H_+ is $G \ltimes_H S^{-L(H)}$, and it is not easy to calculate the homotopy groups of JN. This G-spectrum is bounded below, but it is not connective. We must learn to live with the fact that we have two quite different kinds of Eilenberg-MacLane G-spectra, one that is suitable for representing "ordinary" cohomology and the other that is suitable for representing "ordinary" homology.

G. Lewis, J. P. May, and J. McClure. Ordinary $RO(G)$-graded cohomology. Bulletin Amer. Math. Soc. 4(1981), 208-212.

5. Ring G-spectra and products

Given our precise definition of $RO(G)$-graded theories and our understanding of their representation by G-spectra, the formal apparatus of products in homology and cohomology theories can be developed in a straightforward manner and is little different from the nonequivariant case in classical lectures of Adams. However, in that early work, Adams did not take full advantage of the stable homotopy category. We here recall briefly the basic definitions from the equivariant treatment in [LMS, III§3].

There are four basic products to consider, two external products and two slant products. The reader should be warned that the treatment of slant products in the literature is inconsistent, at best, and often just plain wrong. These four products come from the following four natural maps in $\bar{h}G\mathscr{S}$; all variables are G-spectra.

(5.1)
$$X \wedge E \wedge X' \wedge E' \xrightarrow{\mathrm{id} \wedge \tau \wedge \mathrm{id}} X \wedge X' \wedge E \wedge E'$$

(5.2)
$$F(X, E) \wedge F(X', E') \xrightarrow{\wedge} F(X \wedge X', E \wedge E')$$

(5.3)
$$\begin{tikzcd} F(X \wedge X', E) \wedge X \wedge E' \arrow[r, "/"] \arrow[d, "\cong"'] & F(X', E \wedge E') \\ F(X, F(X', E)) \wedge X \wedge E' \arrow[r, "\varepsilon \wedge \mathrm{id}"'] & F(X', E) \wedge E' \arrow[u, "\nu"'] \end{tikzcd}$$

(5.4)
$$\begin{tikzcd} X \wedge X' \wedge E \wedge F(X, E') \arrow[rr, "\backslash"] \arrow[dr, "\tau"'] & & X' \wedge E \wedge E' \\ & X' \wedge E \wedge F(X, E') \wedge X \arrow[ur, "\mathrm{id} \wedge \mathrm{id} \wedge \varepsilon"'] & \end{tikzcd}$$

The τ are transposition maps and the ε are evaluation maps. The map ν can be described formally, but it is perhaps best understood by pretending that F means Hom and $\wedge$ means $\otimes$ over a commutative ring and writing down the obvious analog. Categorically, such coherence maps are present in any symmetric monoidal category with an internal hom functor. A categorical coherence theorem asserts that any suitably well formulated diagram involving these transformations will commute.

On passage to homotopy groups, these maps give rise to four products in $RO(G)$-graded homology and cohomology. With our details on $RO(G)$-grading, we leave it as an exercise for the reader to check exactly how the grading behaves.

$$E^G_*(X) \otimes E'^G_*(X') \longrightarrow (E \wedge E')^G_*(X \wedge X') \tag{5.5}$$

$$E^*_G(X) \otimes E'^*_G(X') \longrightarrow (E \wedge E')^*_G(X \wedge X') \tag{5.6}$$

$$/ : E^*_G(X \wedge X') \otimes E'^G_*(X) \longrightarrow (E \wedge E')^*_G(X') \tag{5.7}$$

$$\backslash : E^G_*(X \wedge X') \otimes E'^*_G(X) \longrightarrow (E \wedge E')^G_*(X') \tag{5.8}$$

A ring G-spectrum E is one with a product $\phi : E \wedge E \longrightarrow E$ and a unit map $\eta : S \longrightarrow E$ such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$\begin{tikzcd} S \wedge E \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & E \wedge E \arrow[d, "\phi"] & E \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\ & E & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \wedge E \wedge E \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & E \wedge E \arrow[d, "\phi"] \\ E \wedge E \arrow[r, "\phi"] & E. \end{tikzcd}$$

The unlabelled equivalences are canonical isomorphisms in $\bar{h}G\mathscr{S}$ that give the unital property, and we have suppressed such an associativity isomorphism in the second diagram. Of course, there is a weaker notion in which associativity is not required; E is commutative if the following diagram commutes in $\bar{h}G\mathscr{S}$:

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & E \wedge E \arrow[dl, "\phi"] \\ & E. & \end{tikzcd}$$

An E-module is a spectrum M together with a map $\mu : E \wedge M \longrightarrow M$ such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & E \wedge M \arrow[d, "\mu"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \wedge E \wedge M \arrow[r, "1\wedge\mu"] \arrow[d, "\phi\wedge 1"'] & E \wedge M \arrow[d, "\mu"] \\ E \wedge M \arrow[r, "\mu"] & M. \end{tikzcd}$$

We obtain various further products by composing the four external products displayed above with the multiplication of a ring spectrum or with its action on a module spectrum. If $X = X'$ is a based G-space (or rather its suspension spectrum), we obtain internal products by composing with the reduced diagonal $\Delta : X \longrightarrow X \wedge X$. Of course, it is more usual to think in terms of unbased spaces, but then we adjoin a disjoint basepoint. In particular, for a ring G-spectrum E and a based G-space X, we obtain the cup and cap products

$$\cup : E_G^*(X) \otimes E_G^*(X) \longrightarrow E_G^*(X) \tag{5.9}$$

and

$$\cap : E_*^G(X) \otimes E_G^*(X) \longrightarrow E_*^G(X) \tag{5.10}$$

from the external products $\wedge$ and $\backslash$.

It is natural to ask when HM is a ring G-spectrum. In fact, in common with all such categories of additive functors, the category of Mackey functors has an internal tensor product (see Mitchell). In the present topological context, we can define it simply by setting

$$M \otimes M' = \underline{\pi}_0(HM \wedge HM').$$

There results a notion of a pairing $M \otimes M' \longrightarrow M''$ of Mackey functors. By killing the higher homotopy groups of $HM \wedge HM'$, we obtain a canonical map

$$\iota : HM \wedge HM' \longrightarrow H(M \otimes M'),$$

and ι induces an isomorphism on $H_G^0(\cdot; M'') = [\cdot, HM'']_G$. It follows that pairings of G-spectra $HM \wedge HM' \longrightarrow HM''$ are in bijective correspondence with pairings $M \otimes M' \longrightarrow M''$. From here, it is clear how to define the notion of a ring in the category of Mackey functors — such objects are called Green functors — and to conclude that a ring structure on the G-spectrum HM determines and is determined by a structure of Green functor on the Mackey functor M. These observations come from work of Greenlees and myself on Tate cohomology.

There is a notion of a ring G-prespectrum; modulo $\lim^1$ problems, its associated G-spectrum (here constructed using the cylinder construction since one wishes to retain homotopical information) inherits a structure of ring G-spectrum. A good nonequivariant exposition that carries over to the equivariant context has been given by McClure.

J. F. Adams. Lectures on generalized cohomology. in Springer Lecture Notes in Mathematics, Vol. 99, 1-138.

J. P. C. Greenlees and J. P. May. Generalized Tate cohomology (§8). Memoirs Amer. Math. Soc. Number 543. 1995.

J. E. McClure. H_∞-ring spectra via space-level homotopy theory (§§1-2). In R. Bruner, et al, H_∞-ring spectra and their applications. Springer Lecture Notes in Mathematics, Vol. 1176. 1986.

B. Mitchell. Rings with several objects. Advances in Math 8(1972), 1-16.

CHAPTER XIV

An introduction to equivariant K-theory

by J. P. C. Greenlees

1. The definition and basic properties of K_G-theory

The aim of this chapter is to explain the basic facts about equivariant K-theory through the Atiyah-Segal completion theorem. Throughout, G is a compact Lie group and we focus on complex K-theory. Real K-theory works similarly.

We briefly outline the geometric roots of equivariant K-theory. A G-vector bundle over a G-space X is a G-map $\xi : E \longrightarrow X$ which is a vector bundle such that G acts linearly on the fibers, in the sense that $g : E_x \longrightarrow E_{gx}$ is a linear map. Since G is compact, all short exact sequences of G-vector bundles split. If X is a compact space, then $K_G(X)$ is defined to be the Grothendieck group of finite dimensional G-vector bundles over X. Tensor product of bundles makes $K_G(X)$ into a ring.

Many applications arise; for example, the equivariant K-groups are the homes for indices of G-manifolds and families of elliptic operators.

Any complex representation V of G defines a trivial bundle over X and, by the Peter-Weyl theorem, any G-vector bundle over a compact base space is a summand of such a trivial bundle. The cokernel of $K_G(*) \longrightarrow K_G(X)$ can therefore be described as the group of stable isomorphism classes of bundles over X, where two bundles are stably isomorphic if they become isomorphic upon adding an appropriate trivial bundle to each. When X has a G-fixed basepoint $*$, we write $\tilde{K}_G(X)$ for the isomorphic group $\ker(K_G(X) \longrightarrow K_G(*))$.

The definition of a G-vector bundle makes it clear that G-bundles over a free G-space correspond to vector bundles over the quotient under pullback. We deduce the basic reduction theorem:

(1.1) $$K_G(X) = K(X/G) \ \text{ if } \ X \ \text{ is } \ G-\text{free.}$$

This is essentially the statement that K-theory is split in the sense to be discussed in XVI§2. It provides the fundamental link between equivariant and nonequivariant K-theory.

Restriction and induction are the basic pieces of structure that link different ambient groups of equivariance.

If $i : H \longrightarrow G$ is the inclusion of a subgroup it is clear that a G-space or bundle can be viewed as an H-space or bundle; we thereby obtain a restriction map

$$i^* : K_G(X) \longrightarrow K_H(X).$$

There is another way of thinking about this map. For an H-space Y,

(1.2) $$K_G(G \times_H Y) \cong K_H(Y)$$

since a G-bundle over $G \times_H Y$ is determined by its underlying H-bundle over Y. For a G-space X, $G \times_H X \cong G/H \times X$, and the restriction map coincides with the map

$$K_G(X) \longrightarrow K_G(G/H \times X) \cong K_H(X)$$

induced by the projection $G/H \longrightarrow *$.

If H is of finite index in G, an H-bundle over a G-space may be made into a G-bundle by applying the functor $\mathrm{Hom}_H(G, \bullet)$. We thus obtain an induction map $i_* : K_H(X) \longrightarrow K_G(X)$. However if H is of infinite index this construction gives an infinite dimensional bundle. There are three other constructions one may hope to use. First, there is smooth induction, which Segal describes for the representation ring and which should apply to more general base manifolds than a point.

Second, there is the holomorphic transfer, which one only expects to exist when G/H admits the structure of a projective variety. The most important case is when H is the maximal torus in the unitary group $U(n)$, in which case a construction using elliptic operators is described by Atiyah. Its essential property is that it satisfies $i_*i^* = 1$. It is used in the proof of Bott periodicity.

Third, there is a transfer map

$$tr : \tilde{K}_H(\Sigma^W X) \cong \tilde{K}_G(G_+ \wedge_H \Sigma^W X) \longrightarrow \tilde{K}_G(\Sigma^V X)$$

induced by the Pontrjagin-Thom construction $t : S^V \longrightarrow G_+ \wedge_H S^W$ associated to an embedding of G/H in a representation V, where W is the complement of the image in V of the tangent H-representation $L = L(H)$ at the identity coset of G/H. Once we use Bott periodicity to set up $RO(G)$-graded K-theory, this may be interpreted as a dimension-shifting transfer $\tilde{K}_H^{q+L}(X) \longrightarrow \tilde{K}_G^q(X)$. Clearly this transfer is not special to K-theory: it is present in any $RO(G)$-graded theory.

M. F.Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. 19(1968), 113-140.

G. B.Segal. Equivariant K-theory. Pub. IHES 34(1968), 129-151.

2. Bundles over a point: the representation ring

Bundles over a point are representations and hence equivariant K-theory is module-valued over the complex representation ring $R(G)$. More generally, any G-vector bundle over a transitive G-space G/H is of the form $G\times_H V \longrightarrow G\times_H * = G/H$ for some representation V of H. Hence $K_G(G/H) = R(H)$. It follows that $K_G(X)$ takes values in the category of $R(G)$-modules, and thus it is important to understand the algebraic nature of $R(G)$.

Before turning to this, we observe that if G acts trivially on X, then

$$K_G(X) \cong R(G) \otimes K(X).$$

Indeed, the map $K(X) \longrightarrow K_G(X)$ obtained by regarding a vector bundle as a G-trivial G-vector bundle extends to a map $\mu : R(G) \otimes K(X)$ of $R(G)$-modules, and this map is the required isomorphism. An explicit inverse can be constructed as follows. For a representation V, let V denote the trivial G-vector bundle $X\times V \longrightarrow X$. The functor that sends a G-vector bundle ξ to the vector bundle $\mathrm{Hom}_G(V,\xi)$ induces a homomorphism $\varepsilon_V : K_G(X) \longrightarrow K(X)$. Let $\{V_i\}$ run through a set consisting of one representation V_i from each isomorphism class $[V_i]$ of irreducible representations. Then a G-vector bundle ξ over X breaks up as the Whitney sum of its subbundles $\boldsymbol{V_i} \otimes \mathrm{Hom}_G(\boldsymbol{V_i},\xi)$. Define $\nu : K_G(X) \longrightarrow R(G) \otimes K(X)$ by $\nu(\alpha) = \sum_i [V_i] \otimes \varepsilon_{V_i}(\alpha)$. It is then easy to check that μ and ν are inverse isomorphisms.

To understand the algebra of $R(G)$, one should concentrate on the so called "Cartan subgroups" of G. These are topologically cyclic subgroups H with finite Weyl groups $W_G(H) = N_G(H)/H$. Conjugacy classes of Cartan subgroups are in one-to-one correspondence with conjugacy classes of cyclic subgroups of the component group $\pi_0(G)$. Every element of G lies in some Cartan subgroup, and therefore the restriction maps give an injective ring homomorphism

$$R(G) \longrightarrow \prod_{(C)} R(C) \tag{2.1}$$

where the product is over conjugacy classes of Cartan subgroups.

The ring $R(G)$ is Noetherian. Indeed, by explicit calculation, $R(U(n))$ is Noetherian and the representation ring of a maximal torus T is finite over it. Any group G may be embedded in some $U(n)$, and it is enough to show that $R(G)$ is finitely generated as an $R(U(n))$-module. Now $R(G)$ is detected on finitely many topologically cyclic subgroups C, so it is enough to show each $R(C)$ is finitely generated over $R(U(n))$. But each such C is conjugate to a subgroup of T, and $R(C)$ is finite over $R(T)$.

The map (2.1) makes the codomain a finitely generated module over the domain and consequently the induced map of prime spectra is surjective and has finite fibers. By identifying the fibers it can then be shown that for any prime $\wp$ of $R(G)$ the set of minimal elements of

$$\{H \subseteq G \mid \wp \text{ is the restriction of a prime of } R(H)\}$$

constitutes a single conjugacy class (H) of subgroups, with H topologically cyclic. We say that (H) is the *support* of $\wp$. If $R(G)/\wp$ is of characteristic $p > 0$ then the component group of H has order prime to p.

The first easy consequence is that the Krull dimension of $R(G)$ is one more than the rank of G.

A more technical consequence which will become important to us later is that completion is compatible with restriction. Indeed restriction gives a ring homomorphism $\mathrm{res} : R(G) \longrightarrow R(H)$ by which we may regard an $R(H)$-module as an $R(G)$-module. Using supports, we see that if $I(G) = \ker\{\dim : R(G) \longrightarrow \mathbb{Z}\}$ is the augmentation ideal, the ideals $I(H)$ and $\mathrm{res}(I(G)).R(H)$ have the same radical. Consequently the $I(H)$-adic and $I(G)$-adic completions of an $R(H)$-module coincide.

Finally, using supports it is straightforward to understand localizations of equivariant K-theory at primes of $R(G)$. In fact if (H) is the support of $\wp$ the inclusion $X^{(H)} \longrightarrow X$ induces an isomorphism of $K_G(\ )_\wp$, where $X^{(H)}$ is the union of the fixed point spaces $X^{H'}$ with H' conjugate to H.

G. B.Segal. The representation ring of a compact Lie group. Pub. IHES 34(1968), 113-128.

3. Equivariant Bott periodicity

Equivariant Bott periodicity is the most important theorem in equivariant K-theory and is even more extraordinary than its nonequivariant counterpart. It underlies all of the amazing properties of equivariant K-theory. For a locally compact G-space X, define $K_G(X)$ to be the reduced K-theory of the one-point compactification $X_\#$ of X. That is, writing $*$ for the point at infinity,

$$K_G(X) = \ker(K_G(X_\#) \longrightarrow K_G(*).$$

When X is compact, $X_\#$ is the union X_+ of X and a disjoint G-fixed basepoint. We issue a warning: in general, for infinite G-CW complexes, $K_G(X)$ as just defined will not agree with the represented K_G-theory of X that will become available when we construct the K-theory G-spectrum in the next section.

Theorem 3.1 (Thom isomorphism). For vector bundles E over locally compact base spaces X, there is a natural Thom isomorphism

$$\phi : K_G(X) \xrightarrow{\cong} K_G(E).$$

There is a quick reduction to the case when X is compact, and in this case we can use that any G-bundle is a summand of the trivial bundle of some representation V to reduce to the case when $E = V \times X$. Here, with an appropriate description of the Thom isomorphism, one can reinterpret the statement as a convenient and explicit version of Bott periodicity. To see this, let $\lambda(V) \in R(G)$ denote the alternating sum of exterior powers

$$\lambda(V) = 1 - V + \lambda^2 V - \cdots + (-1)^{dim V}\lambda^{dim V} V,$$

let $e_V : S^0 \longrightarrow S^V$ be the based map that sends the non-basepoint to 0, and, taking X to be a point, let $b_V = \phi(1) \in \tilde{K}(S^V)$. Observe that e_V induces

$$e_V^* : \tilde{K}(S^V) \longrightarrow \tilde{K}(S^0) = R(G).$$

Theorem 3.2 (Bott periodicity). For a compact G-space X and a complex representation V of G, multiplication by b_V specifies an isomorphism

$$\phi : \tilde{K}_G(X_+) = K_G(X) \xrightarrow{\cong} K_G(V \times X) = \tilde{K}(S^V \wedge X_+).$$

Moreover, $e(V)^*(b_V) = \lambda(V)$.

The Thom isomorphism can be proven for line bundles, trivial or not, by arguing with clutching functions, as in the nonequivariant case. The essential point is to show that the K-theory of the projective bundle $P(E \oplus \mathbb{C})$ is the free $K_G(X)$-module generated by the unit element $\{1\}$ and the Hopf bundle H. This implies the case when E is a sum of trivial line bundles. If G is abelian, every V is a sum of one dimensional representations so the theorem is proved. This deals with the case of a torus T. The significantly new feature of the equivariant case is the use of holomorphic transfer to deduce the case of $U(n)$. Finally, by change of groups, the result follows for any subgroup of $U(n)$.

For real equivariant K-theory KO_G, the Bott periodicity theorem is true as stated provided that we restrict V to be a Spin representation of dimension divisible by eight. However, the proof is significantly more difficult, requiring the use of pseudo-differential operators.

Now we may extend $K_G(\bullet)$ to a cohomology theory. Following our usual conventions, we shall write K_G^* for the reduced theory on based G-spaces X. Since we need compactness, we consider based finite G-CW complexes, and we then have the notational conventions that in degree zero

$$K_G^0(X_+) = K_G(X) \text{ for finite } G\text{-CW complexes } X$$

and

$$K_G^0(X) = \tilde{K}_G(X) \text{ for based finite } G\text{-CW complexes } X.$$

Of course we could already have made the definition $K_G^{-q}(X) = K_G^0(\Sigma^q X)$ for positive q, but we now know that these are periodic with period 2 since $\mathbb{R}^2 = \mathbb{C}$. Thus we may take

$$K_G^{2n}(X) = K_G^0(X) \quad \text{and} \quad K_G^{2n+1}(X) = K_G^0(\Sigma^1 X) \text{ for all } n.$$

Note in particular that the coefficient ring is $R(G)$ in even degrees. It is zero in odd degrees because all bundles over S^1 are pullbacks of bundles over a point, $GL_n(\mathbb{C})$ being connected. We can extend this to an $RO(G)$-graded theory that is $R(G)$-periodic, but we let the construction of a representing G-spectrum in the next section take care of this for us.

M. F.Atiyah. Bott periodicity and the index of elliptic operators. Quart. J. Math. 19(1968), 113-140.

M. F.Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. Acta math. 112(1964), 229-247.

G. B.Segal. Equivariant K-theory. Pub. IHES 34(1968), 129-151.

4. Equivariant K-theory spectra

Following the procedures indicated in XII§9, we run through the construction of a G-spectrum that represents equivariant K-theory. Recall from VII.3.1 that the Grassmannian G-space $BU(n,V)$ of complex n-planes in a complex inner product G-space V classifies complex n-dimensional G-vector bundles if V is sufficiently large, for example if V contains a complete complex G-universe.

Diverging slightly from our usual notation, fix a complete G-universe $\mathscr{U}$. For each indexing space $V \subset \mathscr{U}$ and each $q \geq 0$, we have a classifying space

$$BU(q, V \oplus \mathscr{U})$$

for q-plane bundles. For $V \subseteq W$, we have an inclusion

$$BU(q, V \oplus \mathscr{U}) \longrightarrow BU(q + |W - V|, W \oplus \mathscr{U})$$

that sends a plane A to the plane $A + (W - V)$. Define

$$BU_G(V) = \coprod_{q \geq 0} BU(q, V \oplus \mathscr{U}).$$

We take the plane V in $BU(|V|, V \oplus \mathscr{U})$ as the canonical G-fixed basepoint of $BU_G(V)$. For $V \subset W$, we then have an inclusion $BU_G(V)$ in $BU_G(W)$ of based G-spaces. Define BU_G to be the colimit of the $BU_G(V)$.

For finite (unbased) G-CW complexes X, the definition of $K_G(X)$ as a Grothendieck group and the classification theorem for complex G-vector bundles lead to an isomorphism

$$[X_+, BU_G]_G \cong K_G(X) = K_G^0(X_+).$$

The finiteness ensures that our bundles embed in trivial bundles and thus have complements. In turn, this ensures that every element of the Grothendieck group is the difference of a bundle and a trivial bundle. For the proof, we may as well assume that X/G is connected. In this case, a G-map $\phi : X \longrightarrow BU_G$ factors through a map $f : BU_G(q, V \oplus \mathscr{U})$ for some q and V. If f classifies the G-bundle ξ, then the isomorphism sends ϕ to $\xi - V$.

The spaces $BU_G(V)$ and BU_G have the homotopy types of G-CW complexes. If we wish, we can replace them by actual G-CW complexes by use of the functor ? from G-spaces to G-CW complexes. For a complex representation V and based finite G-CW complexes X, Bott periodicity implies a natural isomorphism

$$[X, BU_G]_G \cong K_G^0(X) \cong K_G^0(\Sigma^V X) \cong [X, \Omega^V BU_G]_G.$$

By Adams' variant XIII.3.4 of Brown's representability theorem, this isomorphism is represented by a G-map $\tilde{\sigma} : BU_G \longrightarrow \Omega^V BU_G$, which must be an equivalence. However, we must check the vanishing of the appropriate $\lim^1$-term to see that the homotopy class of $\tilde{\sigma}$ is well-defined. Restricting to a cofinal sequence of representations so as to arrange transitivity (as in XIII.3.2), we have an Ω-G-prespectrum. It need not be Σ-cofibrant, but we can apply the cylinder construction K to make it so. Applying L, we then obtain a G-spectrum K_G. It is related to the Ω-G-prespectrum that we started with by a spacewise equivalence. Of course, the restriction to complex indexing spaces is no problem since we can extend to all real indexing spaces, as explained in XII§2.

Using real inner product spaces, we obtain an analogous G-space BO_G and an analogous isomorphism

$$[X, BO_G]_G \cong KO_G(X).$$

If we start with *Spin* representations of dimension $8n$, those being the ones for which we have real Bott periodicity, the same argument works to construct a G-spectrum KO_G that represents real K-theory.

5. The Atiyah-Segal completion theorem

It is especially important to understand bundles over the universal space EG, because of their role in the theory of characteristic classes. We have already mentioned one very simple construction of bundles. In fact for any representation V we may form the bundle $EG \times V \longrightarrow EG \times *$ and hence we obtain the homomorphism

$$\alpha : R(G) \longrightarrow K_G(EG).$$

Evidently α is induced by the projection map $\pi : EG \longrightarrow *$. The Atiyah-Segal completion theorem measures how near α is to being an isomorphism.

Of course, EG is a free G-CW complex. Any free G-CW complex is constructed from the G-spaces $G_+ \wedge S^n$ by means of wedges, cofibers, and passage to colimits. From the change of groups isomorphism $K_G^*(G_+ \wedge X) \cong K^*(X)$ we see that the augmentation ideal $I = I(G)$ acts as zero on the K-theory of any space $G_+ \wedge X$.

In particular the K-theory of a free sphere is complete as an $R(G)$-module for the topology defined by powers of I. Completeness is preserved by extensions of finitely generated modules, so we that $K_G^*(X)$ is I-complete for any finite free G-CW complex X. Completeness is also preserved by inverse limits so, *provided lim^1 error terms vanish*, the K-theory of EG is I-complete.

Remarkably the K-theory of EG is fully accounted for by the representation ring, in the simplest way allowed for by completeness. The Atiyah-Segal theorem can be seen as a comparison between the algebraic process of I-adic completion and the geometric process of "completion" by making a space free.

The map α has a counterpart in all degrees, and it is useful to allow a parameter space, which will be a based G-space X. Thus we consider the map

$$\pi^* : K_G^*(X) \longrightarrow K_G^*(EG_+ \wedge X).$$

Note that the target is isomorphic to the non-equivariant K-theory $K^*(EG_+ \wedge_G X)$, and the following theorem may be regarded as a calculation of this in terms of the more approachable group $K_G^*(X)$.

Theorem 5.1 (Atiyah-Segal). Provided that X is a finite G-CW-complex, the map π^* above is completion at the augmentation ideal, so that

$$K_G^*(EG_+ \wedge X) \cong K_G^*(X)_I^\wedge.$$

In particular,

$$K_G^0(EG_+) = R(G)_I^\wedge \quad \text{and} \quad K_G^1(EG_+) = 0.$$

We sketch the simplest proof, which is that of Adams, Haeberly, Jackowski, and May. We skate over two technical points and return to them at the end. For simplicity of notation, we omit the parameter space X. We do not yet know that $K_G^*(EG_+)$ is complete since we do not yet know that the relevant $\lim^1$-term vanishes. If we did know this, we would be reduced to proving that $\pi : EG_+ \longrightarrow S^0$ induces an isomorphism of I-completed K-theory.

If we also knew that "completed K-theory" was a cohomology theory it would then be enough to show that the cofiber of π was acyclic. It is standard to let $\tilde{E}G$ denote this cofiber, which is easily seen to be the unreduced suspension of EG with one of the cone points as base point. That is, it would be enough to prove that $K_G^*(\tilde{E}G) = 0$ after completion.

The next simplification is adapted from a step in Carlsson's proof of the Segal conjecture. If we argue by induction on the size of the group (which is possible since chains of subgroups of compact Lie groups satisfy the descending chain condition), we may suppose the result proved for all proper subgroups H of G. Accordingly, by change of groups, $K_G^*(G/H_+ \wedge Y) = 0$ after completion for any nonequivariantly contractible space Y and hence by wedges, cofibers, and colimits $K_G^*(E \wedge Y) = 0$

after completion for any G-CW complex E constructed using cells $G/H_+ \wedge S^n$ for various proper subgroups H.

Now if G is finite, let V denote the reduced regular representation and let $S^{\infty V}$ be the union of the representation spheres S^{kV}. For a general compact Lie group G, we let $S^{\infty V}$ denote the union of the representation spheres S^V as V runs over the indexing spaces V such that $V^G = 0$ in a complete G-universe U.

Evidently $S^{\infty V^H}$ is contractible if H is a proper subgroup and $S^{\infty V^G} = S^0$. Thus $S^{\infty V}/S^0$ has no G-fixed points and may be constructed using cells $G/H_+ \wedge S^n$ for proper subgroups H. Thus, by the inductive hypothesis, $K^*_G(S^{\infty V}/S^0 \wedge \tilde{E}G) = 0$ after completion, and hence

$$K^*_G(S^{\infty V} \wedge \tilde{E}G) \cong K^*_G(S^0 \wedge \tilde{E}G) = K^*_G(\tilde{E}G)$$

after completion. But evidently the inclusion

$$S^{\infty V} = S^{\infty V} \wedge S^0 \longrightarrow S^{\infty V} \wedge \tilde{E}G$$

is an equivariant homotopy equivalence (consider the various fixed point sets). This proves a most convenient reduction: it is enough to prove that $K^*_G(S^{\infty V}) = 0$ after completion.

In fact, it is easy to see that $K^*_G(S^{\infty V}) = 0$ after completion. When G is finite, one just notes that (ignoring $\lim^1$ problems again)

$$K^*_G(S^{\infty V}) = \lim_k K^*_G(S^{kV}) = \lim_k (K^*_G(S^0), \lambda(V)) = 0$$

because $\lambda(V) \in I$. Indeed the inverse limit has the effect of making the element $\lambda(V)$ invertible, and if $IM = M$ then $M^\wedge_I = 0$. The argument in the general compact Lie case is only a little more elaborate.

To make this proof honest, we must address the two important properties that we used without justification: (a) that completed K-theory takes cofiberings to exact sequences and (b) that the K-theories of certain infinite complexes are the inverse limits of the K-theories of their finite subcomplexes. In other words the points that we skated over were the linked problems of the inexactness of completion and the nonvanishing of $\lim^1$ terms.

Now, since $R(G)$ is Noetherian, completion *is* exact on finitely generated modules, and the K groups of finite complexes are finitely generated. Accordingly, one route is to arrange the formalities so as to only discuss finite complexes: this is the method of pro-groups, as in the original approach of Atiyah. It is elementary and widely useful. Instead of considering the single group $K_G^*(X)$ we consider the inverse system of groups $K_G^*(X_\alpha)$ as X_α runs over the finite subcomplexes of X.

We do not need to know much about pro-groups. A pro-group is just an inverse system of Abelian groups. There is a natural way to define morphisms, and the resulting category is Abelian. The fundamental technical advantage of working in the category of pro-groups is that, in this category, the inverse limit functor is exact. For any Abelian group valued functor h on G-CW complexes or spectra, we define the associated pro-group valued functor $\mathbf{h}$ by letting $\mathbf{h}(X)$ be the inverse system $\{h(X_\alpha)\}$, where X_α runs over the finite subcomplexes of X.

As long as all K-theory is interpreted as pro-group valued, the argument just given is honest. The conclusion of the argument is that, for a finite G-CW complex X, $\pi : EG_+ \wedge X \longrightarrow X$ induces an isomorphism of I-completed pro-group valued K-theory. Here the I-completion of a pro-$R(G)$-module $\mathbf{M} = \{M_\alpha\}$ is just the inverse system $\{M_\alpha/I^r M_\alpha\}$. When $\mathbf{M}$ is a constant system, such as $\mathbf{K}_G^*(S^0)$, this is just an inverse system of epimorphisms and has zero $\lim^1$. It follows from the isomorphism of pro-groups that $\lim^1$ is also zero for the progroup $K_G^*(EG_+ \wedge X)$, and hence the group $K_G^*(EG_+ \wedge X)$ is the inverse limit of the K-theories of the skeleta of $EG_+ \wedge X$. We may thus simply pass to inverse limits to obtain the conclusion of Theorem 3.1 as originally stated for ordinary rather than pro-$R(G)$-modules.

There is an alternative way to be honest: we could accept the inexactness and adapt the usual methods for discussing it by derived functors. In fact we shall later see how to realize the construction of left derived functors of completion geometrically. This approach leads compellingly to consideration of completions of K_G-module spectra and to the consideration of homology. We invite the interested reader to turn to Chapter XXIV (especially Section 7).

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6. The generalization to families

The above statements and proofs for the universal free G-space EG and the augmentation ideal I carry over with the given proofs to theorems about the universal $\mathscr{F}$-free space $E\mathscr{F}$ and the ideal

$$I\mathscr{F} = \bigcap_{H\in\mathscr{F}} ker\{res^G_H : R(G) \longrightarrow R(H)\}.$$

The only difference is that for most families $\mathscr{F}$ there is no reduction of $K_G(E\mathscr{F})$ to the nonequivariant K-theory of some other space. Note that, by the injectivity of (2.1), if $\mathscr{F}$ includes all cyclic subgroups then $I\mathscr{F} = 0$.

THEOREM 6.1. For any family $\mathscr{F}$ and any finite G-CW-complex X the projection map $E\mathscr{F} \longrightarrow *$ induces completion, so that

$$K^*_G(E\mathscr{F}_+ \wedge X) \cong K^*_G(X)^\wedge_{I\mathscr{F}}.$$

In particular

$$K^0_G(E\mathscr{F}_+) \cong R(G)^\wedge_{I\mathscr{F}} \quad \text{and} \quad K^1_G(E\mathscr{F}_+) = 0.$$

Two useful consequences of these generalizations are that K-theory is detected on finite subgroups and that isomorphisms are detected by cyclic groups.

THEOREM 6.2 (MCCLURE). (a) If X is a finite G-CW-complex and $x \in K_G(X)$ restricts to zero in $K_H(X)$ for all finite subgroups H of G then $x = 0$.
(b) If $f : X \longrightarrow Y$ is a map of finite G-CW-complexes that induces an isomorphism $K_C(Y) \longrightarrow K_C(X)$ for all finite cyclic subgroups C then $f^* : K_G(Y) \longrightarrow K_G(X)$ is also an isomorphism.

Thinking about characters, one might be tempted to believe that finite subgroups could be replaced by finite cyclic subgroups in (a), but that is false.

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CHAPTER XV

An introduction to equivariant cobordism

by S. R. Costenoble

1. A review of nonequivariant cobordism

We start with a brief summary of nonequivariant cobordism.

We define a sequence of groups $\mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{N}_2$, ... as follows: We say that two smooth closed k-dimensional manifolds M_1 and M_2 are *cobordant* if there is a smooth $(k+1)$-dimensional manifold W (the *cobordism*) such that $\partial W \cong M_1 \coprod M_2$; this is an equivalence relation, and $\mathcal{N}_k$ is the set of cobordism classes of k-dimensional manifolds. We make this into an abelian group with addition being disjoint union. The 0 element is the class of the empty manifold $\emptyset$; a manifold is cobordant to $\emptyset$ if it bounds. Every manifold is its own inverse, since $M \coprod M$ bounds $M \times I$. We can make the graded group $\mathcal{N}_*$ into a ring by using cartesian product as multiplication. This ring has been calculated: $\mathcal{N}_* \cong \mathbb{Z}/2[x_k \,|\, k \neq 2^i - 1]$. We'll say more about how we attack this calculation in a moment. This is the *unoriented bordism ring*, due to Thom.

Thom also considered the variant in which the manifolds are oriented. In this case, the cobordism is also required to be oriented, and the boundary ∂W is oriented so that its orientation, together with the inward normal into W, gives the restriction of the orientation of W to ∂W. The effect is that, if M is a closed oriented manifold, then $\partial(M \times I) = M \coprod (-M)$ where $-M$ denotes M with its orientation reversed. This makes $-M$ the negative of M in the resulting *oriented bordism ring* Ω_*. This ring is more complicated than $\mathcal{N}_*$, having both a torsion-free part (calculated by Thom) and a torsion part, consisting entirely of elements of order 2 (calculated by Milnor and Wall).

There are many other variants of these rings, including *unitary bordism*, $\mathscr{U}_*$, which uses "stably almost complex" manifolds; M is such a manifold if there is given an embedding $M \subset \mathbb{R}^n$ and a complex structure on the normal bundle to this embedding. The calculation is $\mathscr{U}_* \cong \mathbb{Z}[z_{2k}]$. This and other variants are discussed in Stong.

These rings are actually coefficient rings of certain homology theories, the *bordism theories* (there is a nice convention, due to Atiyah, that we use the name bordism for the homology theory, and the name cobordism for the related cohomology theory). If X is a space, we define the group $\mathscr{N}_k(X)$ to be the set of bordism classes of maps $M \longrightarrow X$, where M is a k-dimensional smooth closed manifold and the map is continuous. Cobordisms must also map into X, and the restriction of the map to the boundary must agree with the given maps on the k-manifolds. Defining the relative groups $\mathscr{N}_*(X,A)$ is a little trickier. We consider maps $(M,\partial M) \longrightarrow (X,A)$. Such a map is cobordant to $(N,\partial N) \longrightarrow (X,A)$ if there exists a triple $(W,\partial_0 W,\partial_1 W)$, where $\partial W = \partial_0 W \cup \partial_1 W$, the intersection $\partial_0 W \cap \partial_1 W$ is the common boundary $\partial(\partial_0 W) = \partial(\partial_1 W)$, and $\partial_0 W \cong M \coprod N$, together with a map $(W,\partial_1 W) \longrightarrow (X,A)$ that restricts to the given maps on $\partial_0 W$. (This makes the most sense if you draw a picture.) It's useful to think of W as having a "corner" at $\partial_0 W \cap \partial_1 W$; otherwise you have to use resmoothings to get an equivalence relation. It is now a pretty geometric exercise to show that there is a long exact sequence

$$\cdots \longrightarrow \mathscr{N}_k(A) \longrightarrow \mathscr{N}_k(X) \longrightarrow \mathscr{N}_k(X,A) \longrightarrow \mathscr{N}_{k-1}(A) \longrightarrow \cdots$$

where the "boundary map" is precisely taking the boundary. There are oriented, unitary, and other variants of this homology theory.

Calculation of these groups is possible largely because we know the representing spectra for these theories. Let TO (the *Thom prespectrum*) be the prespectrum whose kth space is $TO(k)$, the Thom space of the universal k-plane bundle over $BO(k)$. It is an inclusion prespectrum and, applying the spectrification functor L to it, we obtain the *Thom spectrum* MO. Its homotopy groups are given by

$$\pi_k(MO) = \operatorname{colim}_q \pi_{q+k}(TO(q)).$$

Then $\mathscr{N}_* \cong \pi_*(MO)$, and in fact MO represents unoriented bordism.

The proof goes like this: Given a k-dimensional manifold M, embed M in some $\mathbb{R}^{q+k}$ with normal bundle ν. The unit disk of this bundle is homeomorphic to a tubular neighborhood N of M in $\mathbb{R}^{q+k}$, and so there is a collapse map $c: S^{q+k} \longrightarrow$

$T\nu$ given by collapsing everything outside of N to the basepoint. There is also a classifying map $T\nu \longrightarrow TO(q)$, and the composite

$$S^{q+k} \longrightarrow T\nu \longrightarrow TO(q)$$

represents an element of $\pi_k(MO)$. Applying a similar construction to a cobordism gives a homotopy between the two maps obtained from cobordant manifolds. This construction, known as the *Pontrjagin-Thom construction*, describes the map $\mathscr{N}_k \longrightarrow \pi_k(MO)$.

The inverse map is constructed as follows: Given a map $f : S^{q+k} \longrightarrow TO(q)$, we may assume that f is transverse to the zero-section. The inverse image $M = f^{-1}(BO(q))$ is then a k-dimensional submanifold of S^{q+k} (provided that we use Grassmannian manifold approximations of classifying spaces), and the normal bundle to the embedding of M in S^{q+k} is the pullback of the universal bundle. Making a homotopy between two maps transverse provides a cobordism between the two manifolds obtained from the maps. One can now check that these two constructions are well-defined and inverse isomorphisms. The analysis of $\mathscr{N}_*(X, A)$ is almost identical.

In fact MO is a ring spectrum, and the Thom isomorphism just constructed is an isomorphism of rings. The product on MO is induced from the maps

$$TO(j) \wedge TO(k) \longrightarrow TO(j+k)$$

of Thom complexes arising from the classifying map of the external sum of the jth and kth universal bundle. This becomes clearer when one thinks in a coordinate-free way; in fact, it was inspection of Thom spectra that led to the description of the stable homotopy category that May gave in Chapter XII.

Now MO is a very tractable spectrum. To compute its homotopy we have available such tools as the Thom isomorphism, the Steenrod algebra (mod 2), and the Adams spectral sequence for the most sophisticated calculation. (Stong gives a calculation not using the spectral sequence.) The point is that we now have something concrete to work with, and adequate tools to do the job. For oriented bordism, we replace MO with MSO, which is constructed similarly except that we use the universal oriented bundles over the spaces $BSO(k)$. Here we use the fact that an orientation of a manifold is equivalent to an orientation of its normal bundle. Similarly, for unitary bordism we use the spectrum MU, constructed out of the universal unitary bundles.

The standard general reference is

R. E. Stong. Notes on Cobordism Theory. Princeton University Press. 1968.

2. Equivariant cobordism and Thom spectra

Now we take a compact Lie group G and try to generalize everything to the G-equivariant context. This generalization of nonequivariant bordism was first studied by Conner and Floyd. Using smooth G-manifolds throughout we can certainly copy the definition of cobordism to obtain the equivariant bordism groups $\mathscr{N}_*^G$ and, for pairs of G-spaces (X, A), the groups $\mathscr{N}_*^G(X, A)$. We shall concentrate on unoriented bordism. To define unitary bordism, we consider a unitary manifold to be a smooth G-manifold M together with an embedding of M in either V or $V \oplus \mathbb{R}$, where V is a complex representation of G, and a complex structure on the resulting normal bundle. The notion of an oriented G-manifold is complicated and still controversial, although for odd order groups it suffices to look at oriented manifolds with an action of G; the action of G automatically preserves the orientation.

It is also easy to generalize the Thom spectrum. Let $\mathscr{U}$ be a complete G-universe. In view of the description of the K-theory G-spectra in the previous chapter, it seems most natural to start with the universal n-plane bundles

$$\pi(V) : EO(|V|, V \oplus \mathscr{U}) \longrightarrow BO(|V|, V \oplus \mathscr{U})$$

for indexing spaces V in $\mathscr{U}$. Let $TO_G(V)$ be the Thom space of $\pi(V)$. For $V \subseteq W$, the pullback of $\pi(W)$ over the inclusion

$$BO(|V|, V \oplus \mathscr{U}) \longrightarrow BO(|W|, W \oplus \mathscr{U})$$

is the Whitney sum of $\pi(V)$ and the trivial bundle with fiber $W - V$. Its Thom space is $\Sigma^{W-V} TO_G(V)$, and the evident map of bundles induces an inclusion

$$\sigma : \Sigma^{W-V} TO_G(V) \longrightarrow TO_G(W).$$

This construction gives us an inclusion G-prespectrum TO_G. We define the real Thom G-spectrum to be its spectrification $MO_G = LTO_G$. Using complex representations throughout, we obtain the complex analogs TU_G and MU_G. This definition is essentially due to tom Dieck.

The interesting thing is that MO_G does not represent $\mathscr{N}_*^G$. It is easy to define a map $\mathscr{N}_*^G \longrightarrow \pi_*^G(MO_G) = MO_*^G$ using the Pontrjagin-Thom construction, but we cannot define an inverse. The problem is the failure of transversality in the equivariant context. As a simple example of this failure, consider the group $G = \mathbb{Z}/2$, let $M = *$ be a one-point G-set (a 0-dimensional manifold), let $N = \mathbb{R}$ with the nontrivial linear action of G, and let $Y = \{0\} \subset N$. Let $f : M \longrightarrow N$

be the only G-map that can be defined: it takes M to Y. Clearly f cannot be made transverse to Y, since it is homotopic only to itself. This simple example is paradigmatic. In general, given manifolds M and $Y \subset N$ and a map $f : M \longrightarrow N$, if f fails to be homotopic to a map transverse to Y it is because of the presence in the normal bundle to Y of a nontrivial representation of G that cannot be mapped onto by the representations available in the tangent bundle of M. Wasserman provided conditions under which we can get transversality. If G is a product of a torus and a finite group, he gives a sufficient condition for transversality that amounts to saying that, where needed, we will always have in M a nontrivial representation mapping onto the nontrivial representation we see in the normal bundle to Y. Others have given obstruction theories to transversality, for example Petrie and Waner and myself.

Using Wasserman's condition, it is possible (for one of his G) to construct the G-spectrum that *does* represent $\mathscr{N}_*^G$. Again, let $\mathscr{U}$ be a complete G-universe. We can construct a G-prespectrum to_G with associated G-spectrum mo_G by letting V run through the indexing spaces in our complete universe $\mathscr{U}$ as before, but replacing $\mathscr{U}$ by its G-fixed point space $\mathscr{U}^G \cong \mathbb{R}^\infty$ in the bundles we start with. That is, we start with the G-bundles

$$EO(|V|, V \oplus \mathscr{U}^G) \longrightarrow BO(|V|, V \oplus \mathscr{U}^G)$$

for indexing spaces V in $\mathscr{U}$. Again, restricting attention to complex representations, we obtain the complex analogs tu_G and mu_G. The fact that there are so few nontrivial representations present in the bundle $EO(|V|, V \oplus \mathscr{U}^G)$ allows us to use Wasserman's transversality results to show that mo_G represents $\mathscr{N}_*^G$. The inclusion $\mathscr{U}^G \longrightarrow \mathscr{U}$ induces a map

$$mo_G \longrightarrow MO_G$$

that represents the map $\mathscr{N}_*^G \longrightarrow MO_*^G$ that we originally hoped was an isomorphism.

On the other hand, there is also a geometric interpretation of MO_*^G. Using either transversality arguments or a clever argument due to Bröcker and Hook that works for all compact Lie groups, one can show that

$$MO_k^G(X, A) \cong \operatorname{colim}_V \mathscr{N}_{k+|V|}^G((X, A) \times (D(V), S(V))).$$

Here the maps in the colimit are given by multiplying manifolds by disks of representations, smoothing corners as necessary. We interpret this in the simplest case as follows. A class in $MO_k^G \cong \operatorname{colim}_V \mathscr{N}_{k+|V|}^G(D(V), S(V))$ is represented by a manifold $(M, \partial M)$ together with a map $(M, \partial M) \longrightarrow (D(V), S(V))$. This map is equivalent in the colimit to $(M \times D(W), \partial(M \times D(W))) \longrightarrow (D(V \oplus W), S(V \oplus W))$ together with the original map crossed with the identity on $D(W)$. We call the equivalence class of such a manifold over the disk of a representation a *stable manifold*. Its (virtual) dimension is $\dim M - \dim V$. We can then interpret MO^G_k as the group of cobordism classes of stable manifolds of dimension k. A similar interpretation works for $MO^G_k(X, A)$.

With this interpretation we can see clearly one of the differences between $\mathscr{N}^G_*$ and MO^G_*. If V is a representation of G with no trivial summands, then there is a stable manifold represented by $* \longrightarrow D(V)$, the inclusion of the origin. This represents a nontrivial element $\chi(V) \in MO^G_{-n}$ where $n = |V|$. This element is called the *Euler class* of V. Tom Dieck showed the nontriviality of these elements and we'll give a version of the argument below; note that if V had a trivial summand, then $* \longrightarrow D(V)$ would be homotopic to a map into $S(V)$, so that $\chi(V) = 0$. On the other hand, $\mathscr{N}^G_*$ has no nontrivial elements in negative dimensions, by definition.

Here is another, related difference: Stable bordism is periodic in a sense. If V is any representation of G, then, by the definition of MO_G, $MO_G(V) \cong MO_G(|V|)$; the point is that $MO_G(V)$ really depends only on $|V|$. This gives an equivalence $\Sigma^V MO_G \simeq \Sigma^n MO_G$ if $n = |V|$, or

$$MO_G \simeq \Sigma^{V-n} MO_G.$$

One way of defining an explicit equivalence is to start by classifying the bundle $V \longrightarrow *$ and so obtain an associated map of Thom complexes (a *Thom class*)

$$S^V \longrightarrow TO_G(\mathbb{R}^n) \subset MO_G(\mathbb{R}^n).$$

This is adjoint to a map $\mu(V) : S^{V-n} = \Sigma^\infty_n S^V \longrightarrow MO_G$. Reversing the roles of V and $\mathbb{R}^n$, we obtain an analogous map $S^{n-V} \longrightarrow MO_G$. It is not hard to check that these are inverse units in the $RO(G)$-graded ring MO^G_*. The required equivalence is the evident composite

$$S^{V-n} \wedge MO_G \longrightarrow MO_G \wedge MO_G \longrightarrow MO_G.$$

In homology, this gives isomorphisms of MO^G_*-modules

$$MO^G_*(\Sigma^{|V|} X) \cong MO^G_*(\Sigma^V X)$$

and

$$MO^G_k(X) \cong MO^G_{k+n}(\Sigma^V X)$$

for all k. This is really a special case of a Thom isomorphism that holds for every bundle. The *Thom class* of a bundle ξ is the element in cobordism represented by the map of Thom complexes $T\xi \longrightarrow TO_G(|\xi|) \subset MO_G(|\xi|)$ induced by the classifying map of ξ. Another consequence of the isomorphisms above is that $MO^G_V(X) \cong MO^G_n(X)$, so that the $RO(G)$-graded groups that we get are no different from the groups in integer grading. We can think of this as a periodicity given by multiplication by the unit $\mu(V)$. It should also be clear that, if $|V| = m$ and $|W| = n$, then the composite isomorphism

$$MO^G_k(X) \cong MO^G_{k+m}(\Sigma^V X) \cong MO^G_{k+m+n}(\Sigma^{V\oplus W} X)$$

agrees with the isomorphism $MO^G_k(X) \cong MO^G_{k+m+n}(\Sigma^{V\oplus W} X)$ associated with the representation $V \oplus W$.

We record one further consequence of all this. Consider the inclusion $e : S^0 \longrightarrow S^V$, where $|V| = n$. This induces a map

$$MO^G_{k+n}(X) \longrightarrow MO^G_{k+n}(\Sigma^V X) \cong MO^G_k(X).$$

It is easy to see geometrically that this is given by multiplication by the stable manifold $* \longrightarrow D(V)$, the inclusion of the origin, which represents $\chi(V) \in MO^G_{-n}$. The similar map in cobordism,

$$MO_G^k(X) \cong MO_G^{k+n}(\Sigma^V X) \longrightarrow MO_G^{k+n}(X)$$

is also given by multiplication by $\chi(V) \in MO^n_G$, as we can see by representing $\chi(V)$ by the stable map

$$S^0 \longrightarrow S^V \longrightarrow \Sigma^V MO_G \simeq \Sigma^n MO_G.$$

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3. Computations: the use of families

For computations, we start with the fact that $\mathcal{N}^G_*(X)$ is a module over $\mathcal{N}_*$ (the nonequivariant bordism ring, which we know) by cartesian product. The question is then its structure as a module. We'll take a look at the main computational techniques and at some of the simpler known results.

The main computational technique was introduced by Conner and Floyd. Recall that a *family of subgroups* of G is a collection of subgroups closed under conjugacy and taking of subgroups (in short, under subconjugacy). If $\mathscr{F}$ is such a family, we define an *$\mathscr{F}$-manifold* to be a smooth G-manifold all of whose isotropy groups are in $\mathscr{F}$. If we restrict our attention to closed $\mathscr{F}$-manifolds and cobordisms that are also $\mathscr{F}$-manifolds, we get the groups $\mathscr{N}_*^G[\mathscr{F}]$ of cobordism classes of manifolds with restricted isotropy. Similarly, we can consider the bordism theory $\mathscr{N}_*^G[\mathscr{F}](X, A)$. Now there is a relative version of this as well. Suppose that $\mathscr{F}' \subset \mathscr{F}$. An *$(\mathscr{F}, \mathscr{F}')$-manifold* is a manifold $(M, \partial M)$ where M is an $\mathscr{F}$-manifold and ∂M is an $\mathscr{F}'$-manifold (possibly empty, of course). To define cobordism of such manifolds, we must resort to manifolds with multipart boundaries, or manifolds with corners. Precisely, $(M, \partial M)$ is cobordant to $(N, \partial N)$ if there is a manifold $(W, \partial_0 W, \partial_1 W)$ such that W is an $\mathscr{F}$-manifold, $\partial_1 W$ is an $\mathscr{F}'$-manifold, and $\partial_0 W = M \coprod N$, where as usual $\partial W = \partial_0 W \cup \partial_1 W$ and $\partial_0 W \cap \partial_1 W$ is the common boundary of $\partial_0 W$ and $\partial_1 W$. With this definition we can form the relative bordism groups $\mathscr{N}_*^G[\mathscr{F}, \mathscr{F}']$. Of course, there is also an associated bordism theory, although to describe the relative groups of that theory requires manifolds with 2-part boundaries, and cobordisms with 3-part boundaries!

From a homotopy theoretic point of view it's interesting to notice that $\mathscr{N}_*^G[\mathscr{F}] \cong \mathscr{N}_*^G(E\mathscr{F})$, since a manifold over $E\mathscr{F}$ must be an $\mathscr{F}$-manifold, and any $\mathscr{F}$-manifold has a unique homotopy class of maps into $E\mathscr{F}$. Similarly, $\mathscr{N}_*^G[\mathscr{F}](X) \cong \mathscr{N}_*^G(X \times E\mathscr{F})$, and so on. For the purposes of computation, it is usually more fruitful to think in terms of manifolds with restricted isotropy, however. Notice that this gives us an easy way to define $MO_*^G[\mathscr{F}]$: it is $MO_*^G(E\mathscr{F})$. We can also interpret this in terms of stable manifolds with restricted isotropy.

As a first illustration of the use of families, we give the promised proof of the nontriviality of Euler classes.

LEMMA 3.1. Let G be a compact Lie group and V be a representation of G without trivial summands. Then $\chi(V) \neq 0$ in MO_{-n}^G, where $n = |V|$.

PROOF. Let $\mathscr{A}$ be the family of all subgroups, and let $\mathscr{P}$ be the family of proper subgroups. Consider the map $MO_*^G \longrightarrow MO_*^G[\mathscr{A}, \mathscr{P}]$. We claim that the image of $\chi(V)$ is invertible in $MO_*^G[\mathscr{A}, \mathscr{P}]$ (which is nonzero), so that $\chi(V) \neq 0$. Thinking in terms of stable manifolds, $\chi(V) = [* \longrightarrow D(V)]$. Its inverse is

$D(V) \longrightarrow *$, which lives in the group $MO_*^G[\mathscr{A}, \mathscr{P}]$ because $\partial D(V) = S(V)$ has no fixed points. It's slightly tricky to show that the product, which is represented by $D(V) \longrightarrow * \longrightarrow D(V)$, is cobordant to the identity $D(V) \longrightarrow D(V)$, as we have to change the interpretation of the boundary $S(V)$ of the source from being the "$\mathscr{P}$-manifold part" to being the "maps into $S(V)$ part". However, a little cleverness with $D(V) \times I$ does the trick. $\square$

Returning to our general discussion of the use of families, note that, for a pair of families $(\mathscr{F}, \mathscr{F}')$, there is a long exact sequence

$$\cdots \longrightarrow \mathscr{N}_k^G[\mathscr{F}'] \longrightarrow \mathscr{N}_k^G[\mathscr{F}] \longrightarrow \mathscr{N}_k^G[\mathscr{F}, \mathscr{F}'] \longrightarrow \mathscr{N}_{k-1}^G[\mathscr{F}'] \longrightarrow \cdots,$$

where the boundary map is given by taking boundaries. (This is of course the same as the long exact sequence associated with the pair of spaces $(E\mathscr{F}, E\mathscr{F}')$.) We would like to use this exact sequence to calculate $\mathscr{N}_*^G$ inductively. To set this up a little more systematically, suppose that we have a sequence $\mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots$ of families of subgroups whose union is the family of all subgroups. If we can calculate $\mathscr{N}_k^G[\mathscr{F}_0]$ and each relative term $\mathscr{N}_k^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$, we may be able to calculate every $\mathscr{N}_k^G[\mathscr{F}_p]$ and ultimately $\mathscr{N}_*^G$. We can also introduce the machinery of spectral sequences here: The long exact sequences give us an exact couple

$$\begin{tikzcd}
\mathscr{N}_*^G[\mathscr{F}_{p-1}] \arrow[rr] & & \mathscr{N}_*^G[\mathscr{F}_p] \arrow[dl] \\
& \mathscr{N}_*^G[\mathscr{F}_p, \mathscr{F}_{p-1}] \arrow[ul] &
\end{tikzcd}$$

and hence a spectral sequence with $E^1_{p,q} = \mathscr{N}_q^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$ that converges to $\mathscr{N}_*^G$.

This would all be academic if not for the fact that $\mathscr{N}_*^G[\mathscr{F}_p, \mathscr{F}_{p-1}]$ is often computable. Let us start off with the base of the induction: $\mathscr{N}_*^G[\{e\}, \emptyset] = \mathscr{N}_k^G[\{e\}]$. This is the bordism group of free closed G-manifolds. Now, if M is a free G-manifold, then M/G is also a manifold, of dimension $\dim M - \dim G$. There is a unique homotopy class of G-maps $M \longrightarrow EG$, which passes to quotients to give a map $M/G \longrightarrow BG$. Moreover, given the map $M/G \longrightarrow BG$ we can recover the original manifold M, since it is the pullback in the following diagram:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & EG \arrow[d] \\
M/G \arrow[r] & BG.
\end{tikzcd}$$

This applies equally well to manifolds with or without boundary, so it applies to cobordisms as well. This establishes the isomorphism

$$\mathscr{N}_k^G[\{e\}] \cong \mathscr{N}_{k-\dim G}(BG).$$

Now the bordism of a classifying space may or may not be easy to compute, but at least this is a nonequivariant problem.

The inductive step can also be reduced to a nonequivariant calculation. Suppose that G is finite or Abelian for convenience. We say that $\mathscr{F}$ and $\mathscr{F}'$ are adjacent if $\mathscr{F} = \mathscr{F}' \cup (H)$ for a single conjugacy class of subgroups (H), and it suffices to restrict attention to such an adjacent pair. Suppose that $(M, \partial M)$ is an $(\mathscr{F}, \mathscr{F}')$-manifold. Let $M^{(H)}$ denote the set of points in M with isotropy groups in (H); $M^{(H)}$ lies in the interior of M, since ∂M is an $\mathscr{F}'$-manifold, and $M^{(H)} = \cup_{K \in (H)} M^K$ is a union of closed submanifolds of M. Moreover, these submanifolds are pairwise disjoint, since (H) is maximal in $\mathscr{F}$. Therefore $M^{(H)}$ is a closed G-invariant submanifold in the interior of M, isomorphic to $G \times_{NH} M^H$. (Here is where it is convenient to have G finite or Abelian.) Thus $M^{(H)}$ has a G-invariant closed tubular neighborhood in M, call it N. Here is the key step: *$(M, \partial M)$ is cobordant to $(N, \partial N)$ as an $(\mathscr{F}, \mathscr{F}')$-manifold.* The cobordism is provided by $M \times I$ with corners smoothed (this is easiest to see in a picture).

As usual, let $WH = NH/H$. Now $(N, \partial N)$ is determined by the free WH-manifold M^H and the NH-vector bundle over it which is its normal bundle. Since WH acts freely on the base, each fiber is a representation of H with no trivial summands and decomposes into a sum of multiples of irreducible representations. This also decomposes the whole bundle: Suppose that the nontrivial irreducible representations of H are V_1, V_2, $\ldots$. Then $\nu = \oplus \nu_i$, where each fiber of each ν_i is a sum of copies of V_i. Clearly ν_i is completely determined by the free WH-bundle $\mathrm{Hom}_G(V_i, \nu_i)$, which has fibers $\mathbb{F}^n$ where $\mathbb{F}$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, depending on V_i. Notice, however, that the NH-action on ν induces certain isomorphisms among the ν_i: If V_i and V_j are conjugate representations under the action of NH, then ν_i and ν_j must be isomorphic.

The upshot of all of this is that $\mathscr{N}_k^G[\mathscr{F}, \mathscr{F}']$ is isomorphic to the group obtained in the following way. Suppose that the dimension of V_i is d_i and that $\mathrm{Hom}_G(V_i, V_i) = \mathbb{F}_i$, where $\mathbb{F}_i = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Consider free WH-manifolds M, together with a sequence of WH-bundles ξ_1, ξ_2, $\cdots$ over M, one for each V_i, the group of ξ_i being $O(\mathbb{F}_i, n_i)$ (i.e., $O(n_i)$, $U(n_i)$, or $Sp(n_i)$). If V_i and V_j are conjugate under the action of NH, then we insist that ξ_i and ξ_j be isomorphic.

The dimension of $(M;\xi_1,\xi_2,\cdots)$ is $\dim M+\sum n_i d_i$; that is, this should equal k. Now define $(M;\xi_1,\xi_2,\cdots)$ to be cobordant to $(N;\zeta_1,\zeta_2,\cdots)$ if there exists some $(W;\theta_1,\theta_2,\cdots)$ such that $\partial W = M\coprod N$ and the restriction of θ_i to ∂W is $\xi_i\coprod\zeta_i$. It should be reasonably clear from this description that we have an isomorphism

$$\mathscr{N}_k^G[\mathscr{F},\mathscr{F}'] \cong \sum_{j+\sum n_i d_i = k} \mathscr{N}_j^{WH}(EWH\times(\times_i BO(\mathbb{F}_i,n_i)))$$

where WH acts on $\times_i BO(\mathbb{F}_i,n_i)$ via its permutation of the representations of H. One more step and this becomes a nonequivariant problem: We take the quotient by WH, which we can do because the argument $EWH\times(\times_i BO(\mathbb{F}_i,n_i))$ is free (this being just like the case $\mathscr{N}_*^G[\{e\}]$ above). This gives

$$(3.2)\qquad \mathscr{N}_k^G[\mathscr{F},\mathscr{F}'] \cong \sum_{\dim WH+j+\sum n_i d_i = k} \mathscr{N}_j(EWH\times_{WH}(\times_i BO(\mathbb{F}_i,n_i))).$$

Notice that, if G is Abelian or if WH acts trivially on the representations of H for some other reason, then the argument is $BWH\times(\times_i BO(\mathbb{F}_i,n_i)))$.

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4. Special cases: odd order groups and $\mathbb{Z}/2$

If G is a finite group of odd order, then the differentials in the spectral sequence for $\mathscr{N}_*^G$ all vanish, and $\mathscr{N}_*^G$ is the direct sum over (H) of the groups displayed in (3.2). This is actually a consequence of a very general splitting result that will be explained in XVII§6. The point is that $\mathscr{N}_*^G$ is a $\mathbb{Z}/2$-vector space and, away from the order of the group, the Burnside ring $A(G)$ splits as a direct sum of copies of $\mathbb{Z}[1/|G|]$, one for each conjugacy class of subgroups of G. This induces splittings in all modules over the Burnside ring, including all $RO(G)$-graded homology theories (that is, those homology theories represented by spectra indexed on complete universes). The moral of the story is that, away from the order of the group, equivariant topology generally reduces to nonequivariant topology.

This observation can also be used to show that the spectra mo_G and MO_G split as products of Eilenberg-MacLane spectra, just as in the nonequivariant case. Remember that this depends on G having odd order.

Conner and Floyd computed the additive structure of $\mathscr{N}_*^{\mathbb{Z}/2}$, and Alexander computed its multiplicative structure. There is a split short exact sequence

$$0\longrightarrow \mathscr{N}_k^{\mathbb{Z}/2}\longrightarrow \oplus_{0\le n\le k}\mathscr{N}_{k-n}(BO(n))\longrightarrow \mathscr{N}_{k-1}(B\mathbb{Z}/2)\longrightarrow 0,$$

which is part of the long exact sequence of the pair $(\{\mathbb{Z}/2, e\}, \{e\})$. The first map is given by restriction to $\mathbb{Z}/2$-fixed points and the normal bundles to these. The second map is given by taking the unit sphere of a bundle, then taking the quotient by the antipodal map (a free $\mathbb{Z}/2$-action) and classifying the resulting $\mathbb{Z}/2$-bundle. This map is the only nontrivial differential in the spectral sequence. Now

$$\oplus_{0\leq n\leq k}\mathscr{N}_{k-n}(BO(n)) \cong \mathscr{N}_*[x_1, x_2, \cdots],$$

where $x_k \in \mathscr{N}_{k-1}(BO(1))$ is the class of the canonical line bundle over $\mathbb{R}P^{k-1}$. On the other hand,

$$\mathscr{N}_*(B\mathbb{Z}/2) \cong \mathscr{N}_*\{r_0, r_1, r_2, \cdots\}$$

is the free $\mathscr{N}_*$-module generated by $\{r_k\}$, where r_k is the class of $\mathbb{R}P^k \longrightarrow B\mathbb{Z}/2$. The splitting is the obvious one: it sends r_k to x_{k+1}. In fact, the x_k all live in the summand $\mathscr{N}_*(B\mathbb{Z}/2) = \mathscr{N}_*(BO(1))$, and the splitting is simply the inclusion of this summand. It follows that $\mathscr{N}_*^{\mathbb{Z}/2}$ is a free module over $\mathscr{N}_*$, and one can write down explicit generators. Alexander writes down explicit multiplicative generators.

A similar calculation can be done for $MO_*^{\mathbb{Z}/2}$. The short exact sequence is then

$$0 \longrightarrow MO_k^{\mathbb{Z}/2} \longrightarrow \oplus_n\mathscr{N}_{k-n}(BO) \longrightarrow \mathscr{N}_{k-1}(B\mathbb{Z}/2) \longrightarrow 0,$$

where now k and n range over the integers, positive and negative, and the sum in the middle is infinite. In fact,

$$\oplus_n\mathscr{N}_{*-n}(BO) \cong \mathscr{N}_*[x_1^{-1}, x_1, x_2, \cdots],$$

where the x_i are the images of the elements of the same name from the geometric case. Here x_1^{-1} is the image of χ_L, where L is the nontrivial irreducible representation of $\mathbb{Z}/2$.

It is natural to ask whether or not $mo_{\mathbb{Z}/2}$ and $MO_{\mathbb{Z}/2}$ are products of Eilenberg-MacLane $\mathbb{Z}/2$-spectra, as in the case of odd order groups. I showed that the answer turns out to be no.

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CHAPTER XVI

Spectra and G-spectra; change of groups; duality

In this and the following three chapters, we return to the development of features of the equivariant stable homotopy category. The basic reference is [LMS], and specific citations are given at the ends of sections.

1. Fixed point spectra and orbit spectra

Much of the most interesting work in equivariant algebraic topology involves the connection between equivariant constructions and nonequivariant topics of current interest. We here explain the basic facts concerning the relationships between G-spectra and spectra and between equivariant and nonequivariant cohomology theories.

We restrict attention to a complete G-universe U and we write $RO(G)$ for $RO(G;U)$. Given the details of the previous chapter, we shall be more informal about the $RO(G)$-grading from now on. In particular, we shall allow ourselves to write $E^{\alpha}_{G}(X)$ for $\alpha \in RO(G)$, ignoring the fact that, for rigor, we must first fix a presentation of α as a formal difference $V \ominus W$. We write S^{α} instead of $S^{V\ominus W}$ and, for G-spectra X and E, we write

$$E^{G}_{\alpha}(X) = [S^{\alpha}, E \wedge X]_{G} \tag{1.1}$$

and

$$E^{\alpha}_{G}(X) = [S^{-\alpha} \wedge X, E]_{G} = [S^{-\alpha}, F(X,E)]_{G}. \tag{1.2}$$

To relate this to nonequivariant theories, let $i : U^{G} \longrightarrow U$ be the inclusion of the fixed point universe. Recall that we have the forgetful functor

$$i^{*} : G\mathscr{S}U \longrightarrow G\mathscr{S}U^{G}$$

obtained by forgetting the indexing G-spaces with non-trivial G-action. The "underlying nonequivariant spectrum" of E is i^*E with its action by G ignored. Recall too that i^* has a left adjoint

$$i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$$

that builds in non-trivial representations. Explicitly, for a naive G-prespectrum D and an indexing G-space V,

$$(i_*D)(V) = D(V^G) \wedge S^{V-V^G}.$$

For a naive G-spectrum D, $i_*D = Li_*\ell D$, as usual. These change of universe functors play a subtle and critical role in relating equivariant and nonequivariant phenomena. Since, with G-actions ignored, the universes are isomorphic, the following result is intuitively obvious.

Lemma 1.3. For $D \in G\mathscr{S}U^G$, the unit G-map $\eta : D \longrightarrow i^*i_*D$ of the (i_*, i^*) adjunction is a nonequivariant equivalence. For $E \in G\mathscr{S}U$, the counit G-map $\varepsilon : i_*i^*E \longrightarrow E$ is a nonequivariant equivalence.

We define the fixed point spectrum D^G of a naive G-spectrum D by passing to fixed points spacewise, $D^G(V) = (DV)^G$. This functor is right adjoint to the forgetful functor from naive G-spectra to spectra:

$$(1.4)\quad G\mathscr{S}U^G(C, D) \cong \mathscr{S}U^G(C, D^G) \quad \text{for } C \in \mathscr{S}U^G \text{ and } D \in G\mathscr{S}U^G.$$

It is essential that G act trivially on the universe to obtain well-defined structural homeomorphisms on D^G. For $E \in G\mathscr{S}U$, we define $E^G = (i^*E)^G$. Composing the (i_*, i^*)-adjunction with (1.4), we obtain

$$(1.5)\quad G\mathscr{S}U(i_*C, E) \cong \mathscr{S}U^G(C, E^G) \quad \text{for } C \in \mathscr{S}U^G \text{ and } D \in G\mathscr{S}U^G.$$

The sphere G-spectra $G/H_+ \wedge S^n$ in $G\mathscr{S}U$ are obtained by applying i_* to the corresponding sphere G-spectra in $G\mathscr{S}U^G$. When we restrict (1.1) and (1.2) to integer gradings and take $H = G$, we see that (1.5) implies

$$(1.6)\qquad E_n^G(X) \cong \pi_n((E \wedge X)^G)$$

and

$$(1.7)\qquad E^n_G(X) \cong \pi_{-n}(F(X, E)^G).$$

As in the second isomorphism, naive G-spectra D represent $\mathbb{Z}$-graded cohomology theories on naive G-spectra or on G-spaces. In contrast, as we have already noted in XIII§3, we cannot represent interesting homology theories on G-spaces

X in the form $\pi_*((D \wedge X)^G)$ for a naive G-spectrum D: here smash products commute with fixed points, hence such theories vanish on X/X^G. For genuine G-spectra, there is a well-behaved natural map

$$E^G \wedge (E')^G \longrightarrow (E \wedge E')^G, \tag{1.8}$$

but, even when E' is replaced by a G-space, it is not an equivalence. In Section 3, we shall define a different G-fixed point functor that does commute with smash products.

Orbit spectra D/G of naive G-spectra are constructed by first passing to orbits spacewise on the prespectrum level and then applying the functor L from prespectra to spectra. Here $(\Sigma^\infty X)/G \cong \Sigma^\infty(X/G)$. The orbit functor is left adjoint to the forgetful functor to spectra:

$$\mathscr{S}U^G(D/G, C) \cong G\mathscr{S}U^G(D, C) \quad \text{for} \quad C \in \mathscr{S}U^G \quad \text{and} \quad D \in G\mathscr{S}U^G. \tag{1.9}$$

For a genuine G-spectrum E, it is tempting to define E/G to be $L((i^*E)/G)$, but this appears to be an entirely useless construction. For free actions, we will shortly give a substitute.

[LMS, especially I§3]

2. Split G-spectra and free G-spectra

The calculation of the equivariant cohomology of free G-spectra in terms of the nonequivariant cohomology of orbit spectra is fundamental to the passage back and forth between equivariant and nonequivariant phenomena. This requires the subtle and important notion of a "split G-spectrum".

Definition 2.1. A naive G-spectrum D is said to be split if there is a nonequivariant map of spectra $\zeta : D \longrightarrow D^G$ whose composite with the inclusion of D^G in D is homotopic to the identity map. A genuine G-spectrum E is said to be split if i^*E is split.

The K-theory G-spectra K_G and KO_G are split. Intuitively, the splitting is obtained by giving nonequivariant bundles trivial G-action. The cobordism spectra MO_G and MU_G are also split. The Eilenberg-MacLane G-spectrum HM associated to a Mackey functor M is split if and only if the canonical map $M(G/G) \longrightarrow M(G/e)$ is a split epimorphism; this implies that G acts trivially on $M(G/e)$, which is usually not the case. The suspension G-spectrum $\Sigma^\infty X$ of a G-space X is split if and only if X is stably a retract up to homotopy of X^G, which again is usually not the case. In particular, however, the sphere G-spectrum $S = \Sigma^\infty S^0$ is split. The following consequence of Lemma 1.3 gives more examples.

LEMMA 2.2. If $D \in G\mathscr{S}U^G$ is split, then $i_*D \in G\mathscr{S}U$ is also split.

The notion of a split G-spectrum is defined in nonequivariant terms, but it admits the following equivariant interpretation.

LEMMA 2.3. If E is a G-spectrum with underlying nonequivariant spectrum D, then E is split if and only if there is a map of G-spectra $i_*D \longrightarrow E$ that is a nonequivariant equivalence.

Recall that a based G-space is said to be free if it is free away from its G-fixed basepoint. A G-spectrum, either naive or genuine, is said to be free if it is equivalent to a G-CW spectrum built up out of free cells $G_+ \wedge CS^n$. The functors $\Sigma^\infty : \mathscr{T} \longrightarrow G\mathscr{S}U^G$ and $i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$ carry free G-spaces to free naive G-spectra and free naive G-spectra to free G-spectra. In all three categories, X is homotopy equivalent to a free object if and only if the canonical G-map $EG_+ \wedge X \longrightarrow X$ is an equivalence. A free G-spectrum E is equivalent to i_*D for a free naive G-spectrum D, unique up to equivalence; the orbit spectrum D/G is the substitute for E/G that we alluded to above. A useful mnemonic slogan is that "free G-spectra live in the trivial universe". Note, however, that we cannot take D = i*E: this is not a free G-spectrum. For example, $\Sigma^\infty G_+ \in G\mathscr{S}U^G$ clearly satisfies $(\Sigma^\infty G_+)^G = *$, but we shall see later that $i_*\Sigma^\infty G_+$, which is the genuine suspension G-spectrum $\Sigma^\infty G_+ \in G\mathscr{S}U$, satisfies $(i^*\Sigma^\infty G_+)^G = S$.

THEOREM 2.4. If E is a split G-spectrum and X is a free naive G-spectrum, then there are natural isomorphisms

$$E_n^G(i_*X) \cong E_n((\Sigma^{Ad(G)}X)/G) \quad \text{and} \quad E^n_G(i_*X) \cong E^n(X/G),$$

where $Ad(G)$ is the adjoint representation of G and E_* and E^* denote the theories represented by the underlying nonequivariant spectrum of E.

The cohomology isomorphism holds by inductive reduction to the case $X = G_+$ and use of Lemma 2.3. The homology isomorphism is quite subtle and depends on a dimension-shifting transfer isomorphism that we shall say more about later. This result is an essential starting point for the approach to generalized Tate cohomology theory that we shall present later.

In analogy with (1.8), there is a well-behaved natural map

$$\Sigma^\infty(X^G) \longrightarrow (\Sigma^\infty X)^G, \tag{2.5}$$

but it is not an equivalence.

[LMS, especially II.1.8, II.2.8, II.2.12, II.8.4]

3. Geometric fixed point spectra

There is a "geometric fixed-point functor"

$$\Phi^G : G\mathscr{S}U \longrightarrow \mathscr{S}U^G$$

that enjoys the properties

$$\Sigma^\infty(X^G) \simeq \Phi^G(\Sigma^\infty X) \tag{3.1}$$

and

$$\Phi^G(E) \wedge \Phi^G(E') \simeq \Phi^G(E \wedge E'). \tag{3.2}$$

To construct it, recall the definition of $\tilde{E}\mathscr{F}$ for a family $\mathscr{F}$ from V.2.8 and set

$$\Phi^G E = (E \wedge \tilde{E}\mathscr{P})^G, \tag{3.3}$$

where $\mathscr{P}$ is the family of all proper subgroups of G. Here $E \wedge \tilde{E}\mathscr{P}$ is H-trivial for all $H \in \mathscr{P}$.

The name "geometric fixed point spectrum" comes from an equivalent description of the functor Φ^G. There is an intuitive "spacewise G-fixed point functor" Φ^G from G-prespectra indexed on U to prespectra indexed on U^G. To be precise about this, we index G-prespectra on an indexing sequence $\{V_i\}$, so that $V_i \subset V_{i+1}$ and $U = \cup V_i$, and index prespectra on the indexing sequence $\left\{V_i^G\right\}$. Here we use indexing sequences to avoid ambiguities resulting from the fact that different indexing spaces in U can have the same G-fixed point space. For a G-prespectrum $D = \{DV_i\}$, the prespectrum $\Phi^G D$ is given by $(\Phi^G D)(V_i) = (DV_i)^G$, with structural maps $\Sigma^{V_{i+1}^G - V_i^G}(DV_i)^G \longrightarrow (DV_{i+1})^G$ obtained from those of D by passage to G-fixed points. We are interested in homotopical properties of this construction, and when applying it to spectra regarded as prespectra, we must first apply the cylinder functor K and CW approximation functor Γ discussed in XII§9. The relationship between the resulting construction and the spectrum-level construction (3.3) is as follows. Remember that ℓ denotes the forgetful functor from spectra to prespectra and L denotes its left adjoint.

THEOREM 3.4. For Σ-cofibrant G-prespectra D, there is a natural weak equivalence of spectra

$$\Phi^G LD \longrightarrow L\Phi^G D.$$

For G-CW spectra E, there is a natural weak equivalence of spectra

$$\Phi^G E \longrightarrow L\Phi^G K\Gamma\ell E.$$

It is not hard to deduce the isomorphisms (3.1) and (3.2) from this prespectrum level description of Φ^G.

[LMS, II§9]

4. Change of groups and the Wirthmüller isomorphism

In the previous sections, we discussed the relationship between G-spectra and e-spectra, where we write e both for the identity element and the trivial subgroup of G. We must consider other subgroups and quotient groups of G. First, consider a subgroup H. Since any representation of NH extends to a representation of G and since a WH-representation is just an H-fixed NH-representation, the H-fixed point space U^H of our given complete G-universe U is a complete WH-universe. We define

$$E^H = (i^*E)^H, \quad i : U^H \subset U. \tag{4.1}$$

This gives a functor $G\mathscr{S}U \longrightarrow (WH)\mathscr{S}U^H$. Of course, we can also define E^H as a spectrum in $\mathscr{S}U^G$. The forgetful functor associated to the inclusion $U^G \longrightarrow U^H$ carries the first version of E^H to the second, and we use the same notation for both. For $D \in (NH)\mathscr{S}U^H$, the orbit spectrum D/H is also a WH-spectrum.

Exactly as on the space level in I§1, we have induced and coinduced G-spectra generated by an H-spectrum $D \in H\mathscr{S}U$. These are denoted by

$$G \ltimes_H D \quad \text{and} \quad F_H[G, D).$$

The "twisted" notation $\ltimes$ is used because there is a little twist in the definitions to take account of the action of G on indexing spaces. As on the space level, these functors are left and right adjoint to the forgetful functor $G\mathscr{S}U \longrightarrow H\mathscr{S}U$: for $D \in H\mathscr{S}U$ and $E \in G\mathscr{S}U$, we have

$$G\mathscr{S}U(G \ltimes_H D, E) \cong H\mathscr{S}U(D, E) \tag{4.2}$$

and

$$H\mathscr{S}U(E, D) \cong G\mathscr{S}U(E, F_H[G, D)). \tag{4.3}$$

Again, as on the space level, for $E \in G\mathscr{S}U$ we have

$$G \ltimes_H E \cong (G/H)_+ \wedge E \tag{4.4}$$

and

$$F_H[G,E) \cong F(G/H_+, E). \tag{4.5}$$

As promised earlier, we can now deduce as in (1.6) that

$$\pi_n^H(E) \equiv [G/H_+ \wedge S^n, E]_G \cong [S^n, E]_H \cong \pi_n(E^H). \tag{4.6}$$

In cohomology, the isomorphism (4.2) gives

$$E_G^*(G \ltimes_H D) \cong E_H^*(D). \tag{4.7}$$

We shall not go into detail, but we can interpret this in terms of $RO(G)$ and $RO(H)$ graded theories via the evident functor $\mathscr{R}O(G) \longrightarrow \mathscr{R}O(H)$. The isomorphism (4.3) does not have such a convenient interpretation as it stands. However, there is a fundamental change of groups result — called the Wirthmüller isomorphism — which in its most conceptual form is given by a calculation of the functor $F_H[G,D)$. It leads to the following homological complement of (4.7). Let $L(H)$ be the tangent H-representation at the identity coset of G/H. Then

$$E_*^G(G \ltimes_H D) \cong E_*^H(\Sigma^{L(H)} D). \tag{4.8}$$

THEOREM 4.9 (GENERALIZED WIRTHMÜLLER ISOMORPHISM). For H-spectra D, there is a natural equivalence of G-spectra

$$F_H[G, \Sigma^{L(H)} D) \longrightarrow G \ltimes_H D.$$

Therefore, for G-spectra E,

$$[E, \Sigma^{L(H)} D]_H \cong [E, G \ltimes_H D]_G.$$

The last isomorphism complements the isomorphism from (4.2):

$$[G \ltimes_H D, E]_G \cong [D, E]_H. \tag{4.10}$$

We deduce (4.8) by replacing E in (4.9) by a sphere, replacing D by $E \wedge D$, and using the generalization

$$G \ltimes_H (D \wedge E) \cong (G \ltimes_H D) \wedge E$$

of (4.4).

[LMS, II§§2-4]
K. Wirthmüller. Equivariant homology and duality. Manuscripta Math. 11(1974), 373-390.

5. Quotient groups and the Adams isomorphism

Let N be a normal subgroup of G with quotient group J. In practice, one is often thinking of a quotient map $NH \longrightarrow WH$ rather than $G \longrightarrow J$. There is an analog of the Wirthmüller isomorphism — called the Adams isomorphism — that compares orbit and fixed-point spectra. It involves the change of universe functors associated to the inclusion $i : U^N \longrightarrow U$ and requires restriction to N-free G-spectra. We note first that the fixed point and orbit functors $G\mathscr{S}U^N \longrightarrow J\mathscr{S}U^N$ are right and left adjoint to the evident pullback functor from J-spectra to G-spectra: for $D \in J\mathscr{S}U^N$ and $E \in G\mathscr{S}U^N$,

$$G\mathscr{S}U^N(D,E) \cong J\mathscr{S}U^N(D,E^N) \tag{5.1}$$

and

$$J\mathscr{S}U^N(E/N,D) \cong G\mathscr{S}U^N(E,D). \tag{5.2}$$

Here we suppress notation for the pullback functor $J\mathscr{S}U^N \longrightarrow G\mathscr{S}U^N$. An N-free G-spectrum E indexed on U is equivalent to i_*D for an N-free G-spectrum D indexed on U^N, and D is unique up to equivalence. Thus our slogan that "free G-spectra live in the trivial universe" generalizes to the slogan that "N-free G-spectra live in the N-fixed universe". This gives force to the following version of (5.2). It compares maps of J-spectra indexed on U^N with maps of G-spectra indexed on U.

THEOREM 5.3. Let $J = G/N$. For N-free G-spectra E indexed on U^N and J-spectra D indexed on U^N,

$$[E/N, D]_J \cong [i_*E, i_*D]_G.$$

The conjugation action of G on N gives rise to an action of G on the tangent space of N at e; we call this representation $Ad(N)$, or $Ad(N;G)$. The following result complements the previous one, but is very much deeper. When $N = G$, it is the heart of the proof of the homology isomorphism of Theorem 2.4. We shall later describe the dimension-shifting transfer that is the basic ingredient in its proof.

THEOREM 5.4 (GENERALIZED ADAMS ISOMORPHISM). Let $J = G/N$. For N-free G-spectra $E \in G\mathscr{S}U^N$, there is a natural equivalence of J-spectra

$$E/N \longrightarrow (\Sigma^{-Ad(N)} i_*E)^N.$$

Therefore, for $D \in J\mathscr{S}U^N$,

$$[D, E/N]_J \cong [i_*D, \Sigma^{-Ad(N)} i_*E]_G.$$

This result is another of the essential starting points for the approach to generalized Tate cohomology that we will present later. The last two results cry out for general homological and cohomological interpretations, like those of Theorem 2.4. Looking back at Lemma 2.3, we see that what is needed for this are analogs of the underlying nonequivariant spectrum and of the characterization of split G-spectra that make sense for quotient groups J. What is so special about the trivial group is just that it is naturally both a subgroup and a quotient group of G.

The language of families is helpful here. Let $\mathscr{F}$ be a family. We say that a G-spectrum E is $\mathscr{F}$-free, or is an $\mathscr{F}$-spectrum, if E is equivalent to a G-CW spectrum all of whose cells are of orbit type in $\mathscr{F}$. Thus free G-spectra are $\{e\}$-free. We say that a map $f : D \longrightarrow E$ is an $\mathscr{F}$-equivalence if $f^H : D^H \longrightarrow E^H$ is an equivalence for all $H \in \mathscr{F}$ or, equivalently by the Whitehead theorem, if f is an H-equivalence for all $H \in \mathscr{F}$.

Returning to our normal subgroup N, let $\mathscr{F}(N) = \mathscr{F}(N;G)$ be the family of subgroups of G that intersect N in the trivial group. Thus an $\mathscr{F}(N)$-spectrum is an N-free G-spectrum. We have seen these families before, in our study of equivariant bundles. We can now state precise generalizations of Lemma 2.3 and Theorem 2.4. Fix spectra

$$D \in J\mathscr{S}U^N \quad \text{and} \quad E \in G\mathscr{S}U.$$

LEMMA 5.5. A G-map $\xi : i_*D \longrightarrow E$ is an $\mathscr{F}(N)$-equivalence if and only if the composite of the adjoint $D \longrightarrow (i^*E)^N$ of ξ and the inclusion $(i^*E)^N \longrightarrow i^*E$ is an $\mathscr{F}(N)$-equivalence.

THEOREM 5.6. Assume given an $\mathscr{F}(N)$-equivalence $i_*D \longrightarrow E$. For any N-free G-spectrum $X \in G\mathscr{S}U^N$,

$$E^G_*(\Sigma^{-Ad(N)}(i_*X)) \cong D^J_*(X/N) \quad \text{and} \quad E^*_G(i_*X) \cong D^*_J(X/N).$$

Given E, when do we have an appropriate D? We often have theories that are defined on the category of all compact Lie groups, or on a suitable sub-category. When such theories satisfy appropriate naturality axioms, the theory E_J associated to J will necessarily bear the appropriate relationship to the theory E_G associated to G. We shall not go into detail here. One assumes that the homomorphisms $\alpha : H \longrightarrow G$ in one's category induce maps of H-spectra $\xi_\alpha : \alpha^*E_G \longrightarrow E_H$ in a functorial way, where some bookkeeping with universes is needed to make sense of α^*, and one assumes that ξ_α is an H-equivalence if α is an inclusion. For each

$H \in \mathscr{F}(N)$, the quotient map $q : G \longrightarrow J$ restricts to an isomorphism from H to its image K. If the five visible maps,

$$H \subset G,\ K \subset J,\ q : G \longrightarrow J,\ q : H \longrightarrow K, \ \text{ and } \ q^{-1} : K \longrightarrow H,$$

are in one's category, one can deduce that $\xi_q : q^*E_J = i_*E_J \longrightarrow E_G$ is an $\mathscr{F}(N)$-equivalence. This is not too surprising in view of Lemma 2.3, but it is a bit subtle: there are examples where all axioms are satisfied, except that q^{-1} is not in the category, and the conclusion fails because ξ_q is not an H-equivalence. However, this does work for equivariant K-theory and the stable forms of equivariant cobordism, generalizing the arguments used to prove that these theories split. For K-theory, the Bott isomorphisms are suitably natural, by the specification of the Bott elements in terms of exterior powers. For cobordism, we shall explain in XXV§5 that MO_G and MU_G arise from functors, called "global $\mathscr{I}_*$ functors with smash product", that are defined on all compact Lie groups and their representations and take values in spaces with group actions. All theories with such a concrete geometric source are defined with suitable naturality on all compact Lie groups G.

J. F. Adams. Prerequisites (on equivariant theory) for Carlsson's lecture. Springer Lecture Notes in Mathematics Vol. 1051, 1984, 483-532.
[LMS, II§§8-9]

6. The construction of G/N-spectra from G-spectra

A different line of thought leads to a construction of J-spectra from G-spectra, $J = G/N$, that is a direct generalization of the geometric fixed point construction $\Phi^G E$. The appropriate analog of $\mathscr{P}$ is the family $\mathscr{F}[N]$ of those subgroups of G that do not contain N. Note that this is a family since N is normal. For a spectrum E in $G\mathscr{S}U$, we define

$$\Phi^N E \equiv (E \wedge \tilde{E}\mathscr{F}[N])^N. \tag{6.1}$$

We have the expected generalizations of (3.1) and (3.2): for a G-space X,

$$\Sigma^\infty(X^N) \simeq \Phi^N(\Sigma^\infty X) \tag{6.2}$$

and, for G-spectra E and E',

$$\Phi^N(E) \wedge \Phi^N(E') \simeq \Phi^N(E \wedge E'). \tag{6.3}$$

We can define $\Phi^H E$ for a not necessarily normal subgroup H by regarding E as an NH-spectrum. Although the Whitehead theorem appears naturally as a statement about homotopy groups and thus about the genuine fixed point functors characterized by the standard adjunctions, it is worth observing that it implies a version in terms of these Φ-fixed point spectra.

THEOREM 6.4. A map $f : E \longrightarrow E'$ of G-spectra is an equivalence if and only if each $\Phi^H f : \Phi^H E \longrightarrow \Phi^H E'$ is a nonequivariant equivalence.

Note that, for any family $\mathscr{F}$ and any G-spectra E and E',

$$[E \wedge E\mathscr{F}_+, E' \wedge \tilde{E}\mathscr{F}]_G = 0$$

since $E\mathscr{F}$ only has cells of orbit type G/H and $\tilde{E}\mathscr{F}$ is H-contractible for such H. Therefore the canonical G-map $E \longrightarrow E \wedge \tilde{E}\mathscr{F}$ induces an isomorphism

$$[E \wedge \tilde{E}\mathscr{F}, E' \wedge \tilde{E}\mathscr{F}]_G \cong [E, E' \wedge \tilde{E}\mathscr{F}]_G. \tag{6.5}$$

In the case of $\mathscr{F}[N]$, $E \longrightarrow E \wedge \tilde{E}\mathscr{F}[N]$ is an equivalence if and only if E is concentrated over N, in the sense that E is H-contractible if H does not contain N. Maps into such G-spectra determine and are determined by the J-maps obtained by passage to Φ^N-fixed point spectra. In fact, the stable category of J-spectra is equivalent to the full subcategory of the stable category of G-spectra consisting of the G-spectra concentrated over N.

THEOREM 6.6. For J-spectra $D \in J\mathscr{S}U^N$ and G-spectra $E \in G\mathscr{S}U$ concentrated over N, there is a natural isomorphism

$$[D, E^N]_J \cong [i_*D \wedge \tilde{E}\mathscr{F}[N], E]_G.$$

For J-spectra D and D', the functor $i_*(\cdot) \wedge \tilde{E}\mathscr{F}[N]$ induces a natural isomorphism

$$[D, D']_J \cong [i_*D \wedge \tilde{E}\mathscr{F}[N], i_*D \wedge \tilde{E}\mathscr{F}[N]]_G.$$

For general G-spectra E and E', the functor $\Phi^N(\cdot)$ induces a natural isomorphism

$$[\Phi^N E, \Phi^N E']_J \cong [E, E' \wedge \tilde{E}\mathscr{F}[N]]_G.$$

PROOF. The first isomorphism is a consequence of (5.1) and (6.5). The other two isomorphisms follow once one shows that the unit

$$D \longrightarrow (i_*D \wedge \tilde{E}\mathscr{F}[N])^N = \Phi^N(i_*D)$$

and counit

$$(i_*E^N) \wedge \tilde{E}\mathscr{F}[N] \longrightarrow E$$

of the adjunction are equivalences. One proves this by use of a spacewise N-fixed point functor, also denoted Φ^N, from G-prespectra to J-prespectra. This functor is defined exactly as was the spacewise G-fixed point functor in Section 3. It satisfies $\Phi^N(i_*D) = D$, and it commutes with smash products. The following generalization of Theorem 3.4, which shows that the prespectrum level functor Φ^N induces a functor equivalent to Φ^N on the spectrum level, leads to the conclusion. $\square$

THEOREM 6.7. For Σ-cofibrant G-prespectra D, there is a natural weak equivalence of J-spectra

$$\Phi^N LD \longrightarrow L\Phi^N D.$$

For G-CW spectra E, there is a natural weak equivalence of J-spectra

$$\Phi^N E \longrightarrow L\Phi^N K\Gamma\ell E.$$

As an illuminating example of the use of $RO(G)$-grading to allow calculational descriptions invisible to the $\mathbb{Z}$-graded part of a theory, we record how to compute the cohomology theory represented by $\Phi^N(E)$ in terms of the cohomology theory represented by E. This uses the Euler classes of representations, which appear ubiquitously in equivariant theory. For a representation V, we define $e(V) \in E^V_G(S^0)$ to be the image of $1 \in E^0_G(S^0) \cong E^V_G(S^V)$ under e^*, where $e : S^0 \longrightarrow S^V$ sends the basepoint to the point at ∞ and the non-basepoint to 0.

PROPOSITION 6.8. Let E be a ring G-spectrum. For a finite J-CW spectrum X, $(\Phi^N E)^*_J(X)$ is the localization of $E^*_G(X)$ obtained by inverting the Euler classes of all representations V such that $V^N = \{0\}$.

PROOF. By (6.3), $\Phi^N(E)$ inherits a ring structure from E. In interpreting the grading, we regard representations of J as representations of G by pullback. A check of fixed points, using the cofibrations $S(V) \longrightarrow B(V) \longrightarrow S^V$, shows that we obtain a model for $\tilde{E}\mathscr{F}[N]$ by taking the colimit of the spaces S^V as V ranges over the representations of G such that $V^N \cong \{0\}$. This leads to a colimit description of $(\Phi^N E)^*_J(X)$ that coincides algebraically with the cited localization. $\square$

With motivation from the last few results, the unfortunate alternative notation $E_J = \Phi^N(E_G)$ was used in [LMS] and elsewhere. This is a red herring from the point of view of Theorem 5.6, and it is ambiguous on two accounts. First, the J-spectrum $\Phi^N(E_G)$ depends vitally on the extension $J = G/N$ and not just on the group J. Second, in classical examples, the spectrum "E_J" will generally not agree with the preassigned spectrum with the same notation. For example, the subquotient J-spectrum "K_J" associated to the K-theory G-spectrum K_G is not the K-theory J-spectrum K_J. However, if S_G is the sphere G-spectrum, then the subquotient J-spectrum S_J is the sphere J-spectrum. We shall see that this easy fact plays a key conceptual role in Carlsson's proof of the Segal conjecture.

[LMS, II§9]

7. Spanier-Whitehead duality

We can develop abstract duality theory in any symmetric monoidal category, such as $\bar{h}G\mathscr{S}$ for our fixed complete G-universe U. While the elegant approach is to start from the abstract context, we shall specialize to $\bar{h}G\mathscr{S}$ from the start since we wish to emphasize equivariant phenomena. Define the dual of a G-spectrum X to be $DX = F(X,S)$. There is a natural map

$$\nu : F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z). \tag{7.1}$$

Using the unit isomorphism, it specializes to

$$\nu : (DX) \wedge X \longrightarrow F(X,X). \tag{7.2}$$

The adjoint of the unit isomorphism $S \wedge X \longrightarrow X$ gives a natural map $\eta : S \longrightarrow F(X,X)$. We say that X is "strongly dualizable" if there is a coevaluation map $\eta : S \longrightarrow X \wedge (DX)$ such that the following diagram commutes, where γ is the commutativity isomorphism.

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\eta"'] & X \wedge (DX) \arrow[d, "\gamma"] \\ F(X,X) & (DX) \wedge X \arrow[l, "\nu"] \end{tikzcd} \tag{7.3}$$

It is a categorical implication of the definition that the map ν of (7.1) is an isomorphism if either X or Z is strongly dualizable, and there are various other such formal consequences, such as $X \cong DD(X)$ when X is strongly dualizable. In particular, if X is strongly dualizable, then the map ν of (3.2) is an isomorphism. Conversely, if the map ν of (7.2) is an isomorphism, then X is strongly dualizable since the coevaluation map η can and must be defined to be the composite $\gamma\nu^{-1}\eta$ in (7.3). Note that we have an evaluation map $\varepsilon : DX \wedge X \longrightarrow S$ for any X.

THEOREM 7.4. A G-CW spectrum is strongly dualizable if and only if it is equivalent to a wedge summand of a finite G-CW spectrum.

PROOF. The evaluation map of X induces a natural map

$$\varepsilon_{\#} : [Y, Z \wedge DX]_G \longrightarrow [Y \wedge X, Z]_G \tag{$*$}$$

via $\varepsilon_{\#}(f) = (\mathrm{Id} \wedge \varepsilon)(f \wedge \mathrm{Id})$, and X is strongly dualizable if and only if $\varepsilon_{\#}$ is an isomorphism for all Y and Z. The Wirthmüller isomorphism implies that $D(\Sigma^\infty G/H_+)$ is equivalent to $G \ltimes_H S^{-L(H)}$, and diagram chases show that it also implies that $\varepsilon_{\#}$ is an isomorphism. Actually, this duality on orbits is the heart of the Wirthmüller isomorphism, and we shall explain it in direct geometric terms in the next section. If X is strongly dualizable, then so is ΣX. The cofiber of a map between strongly dualizable G-spectra is strongly dualizable since both sides of (*) turn cofibrations in X into long exact sequences. By induction on the number of cells, a finite G-CW spectrum is strongly dualizable, and it is formal that a wedge summand of a strongly dualizable G-spectrum is strongly dualizable. For the converse, which was conjectured in [LMS] and proven by Greenlees (unpublished), let X be a strongly dualizable G-CW spectrum with coevaluation map η. Then η factors through $A \wedge DX$ for some finite subcomplex A of X, the following diagram commutes, and its bottom composite is the identity:

$$\begin{tikzcd}
 & A \wedge (DX) \wedge X \arrow[r, "\mathrm{Id} \wedge \varepsilon"] \arrow[d] & A \wedge S \cong A \arrow[d] \\
X \cong S \wedge X \arrow[ur] \arrow[r, "\eta \wedge \mathrm{Id}"'] & X \wedge (DX) \wedge X \arrow[r, "\mathrm{Id} \wedge \varepsilon"'] & X \wedge S \cong X.
\end{tikzcd}$$

Therefore X is a retract up to homotopy and thus a wedge summand up to homotopy of A. $\square$

In contrast to the nonequivariant case, wedge summands of finite G-CW spectra need not be equivalent to finite G-CW spectra.

COROLLARY 7.5 (SPANIER-WHITEHEAD DUALITY). If X is a wedge summand of a finite G-CW spectrum and E is any G-spectrum, then

$$\nu : DX \wedge E \longrightarrow F(X, E)$$

is an isomorphism in $\bar{h}G\mathscr{S}$. Therefore, for any representation α,

$$E^G_\alpha(DX) \cong E^{-\alpha}_G(X).$$

So far, we have concentrated on the naturally given dual DX. However, it is important to identify the homotopy types of duals concretely, as we did in the case of orbits. There are a number of equivalent criteria. The most basic one goes as follows. Suppose given G-spectra X and Y and maps

$$\varepsilon : Y \wedge X \longrightarrow S \quad \text{and} \quad \eta : S \longrightarrow X \wedge Y$$

such that the composites

$$X \cong S \wedge X \xrightarrow{\eta \wedge \mathrm{Id}} X \wedge Y \wedge X \xrightarrow{\mathrm{Id} \wedge \varepsilon} X \wedge X \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{\mathrm{Id} \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge \mathrm{Id}} Y \wedge S \cong Y$$

are the respective identity maps. Then the adjoint $\tilde{\varepsilon} : Y \longrightarrow DX$ of ε is an equivalence and X is strongly dualizable with coevaluation map $(\mathrm{Id} \wedge \tilde{\varepsilon})\eta$. It is important to note that the maps η and ε that display the duality are not unique — much of the literature on duality is quite sloppy.

This criterion admits a homological interpretation, but we will not go into that here. It entails a reinterpretation in terms of the slant products relating homology and cohomology that we defined in XIII§5, and it works in the same way equivariantly as nonequivariantly.

[LMS, III§§1-3]

8. V-duality of G-spaces and Atiyah duality

There is a concrete space level version of the duality criterion just given. To describe it, let X and Y be G-spaces and let V be a representation of G. Suppose given G-maps

$$\varepsilon : Y \wedge X \longrightarrow S^V \quad \text{and} \quad \eta : S^V \longrightarrow X \wedge Y$$

such that the following diagrams are stably homotopy commutative, where $\sigma : S^V \longrightarrow S^V$ is the sign map, $\sigma(v) = -v$, and the γ are transpositions.

$$\begin{tikzcd} S^V \wedge X \arrow[r, "\eta \wedge \mathrm{Id}"] \arrow[dr, "\gamma"'] & X \wedge Y \wedge X \arrow[d, "\mathrm{Id} \wedge \varepsilon"] \\ & X \wedge S^V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \wedge S^V \arrow[r, "\mathrm{Id} \wedge \eta"] \arrow[d, "\gamma"'] & Y \wedge X \wedge Y \arrow[d, "\varepsilon \wedge \mathrm{Id}"] \\ S^V \wedge Y \arrow[r, "\sigma \wedge \mathrm{Id}"'] & S^V \wedge Y. \end{tikzcd}$$

On application of the functor Σ^∞_V, we find that $\Sigma^\infty X$ and $\Sigma^\infty_V Y$ are strongly dualizable and dual to one another by our spectrum level criterion.

For reasonable X and Y, say finite G-CW complexes, or, more generally, compact G-ENR's (ENR = Euclidean neighborhood retract), we can use the space level equivariant suspension and Whitehead theorems to prove that a pair of G-maps (ε, η) displays a V-duality between X and Y, as above, if and only if the fixed point pair (ε^H, η^H) displays an $n(H)$-duality between X^H and Y^H for each $H \subset G$, where $n(H) = dim(V^H)$.

If X is a compact G-ENR, then X embeds as a retract of an open set of a G-representation V. One can use elementary space level methods to construct an explicit V-duality between X_+ and the unreduced mapping cone $V \cup C(V - X)$. For a G-cofibration $A \longrightarrow X$, there is a relative version that constructs a V-duality between $X \cup CA$ and $(V - A) \cup C(V - X)$. The argument specializes to give an equivariant version of the Atiyah duality theorem, via precise duality maps. Recall that the Thom complex of a vector bundle is obtained by fiberwise one-point compactification followed by identification of the points at infinity. When the base space is compact, this is just the one-point compactification of the total space.

THEOREM 8.1 (ATIYAH DUALITY). If M is a smooth closed G-manifold embedded in a representation V with normal bundle ν, then M_+ is V-dual to the Thom complex $T\nu$. If M is a smooth compact G-manifold with boundary ∂M, $V = V' \oplus \mathbb{R}$, and $(M, \partial M)$ is properly embedded in $(V' \times [0, \infty), V' \times \{0\})$ with normal bundles ν' of ∂M in V' and ν of M in V, then $M/\partial M$ is V-dual to $T\nu$, M_+ is V-dual to $T\nu/T\nu'$, and the cofibration sequence

$$T\nu' \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'$$

is V-dual to the cofibration sequence

$$\Sigma(\partial M)_+ \longleftarrow M/\partial M \longleftarrow M_+ \longleftarrow (\partial M)_+.$$

We display the duality maps explicitly in the closed case. By the equivariant tubular neighborhood theorem, we may extend the embedding of M in V to an embedding of the normal bundle ν and apply the Pontrjagin-Thom construction to obtain a map $t : S^V \longrightarrow T\nu$. The diagonal map of the total space of ν induces the Thom diagonal $\Delta : T\nu \longrightarrow M_+ \wedge T\nu$. The map η is just $\Delta \circ t$. The map ε is equally explicit but a bit more complicated to describe. Let $s : M \longrightarrow \nu$ be the zero section. The composite of $\Delta : M \longrightarrow M \times M$ and $s \times \mathrm{Id} : M \times M \longrightarrow \nu \times M$ is an embedding with trivial normal bundle. The Pontrjagin-Thom construction gives a map $t : T\nu \wedge M_+ \longrightarrow M_+ \wedge S^V$. Let $\xi : M_+ \longrightarrow S^0$ collapse all of M to the non-basepoint. The map ε is just $(\xi \wedge \mathrm{Id}) \circ t$. This explicit construction implies that the maps $\xi : M_+ \longrightarrow S^0$ and $t : S^V \longrightarrow T\nu$ are dual to one another.

Let us specialize this discussion to orbits G/H (compare IX.3.4). Recall that $L = L(H)$ is the tangent H-representation at the identity coset of G/H. We have

$$\tau = G \times_H L(H) \quad \text{and} \quad T\tau = G_+ \wedge_H S^{L(H)}.$$

If G/H is embedded in V with normal bundle ν, then $\nu \oplus \tau$ is the trivial bundle $G/H \times V$. Let W be the orthogonal complement to $L(H)$ in the fiber over the identity coset, so that $V = L \oplus W$ as an H-space. Since G/H_+ is V-dual to $T\nu$, $\Sigma^\infty G/H_+$ is dual to $\Sigma_V^\infty T\nu$. Since $S^W \wedge S^{-V} \simeq S^{-L}$ as H-spectra, we find that $\Sigma_V^\infty T\nu \simeq G \ltimes_H S^{-L}$.

[LMS, III§§3-5]

9. Poincaré duality

Returning to general smooth G-manifolds, we can deduce an equivariant version of the Poincaré duality theorem by combining Spanier-Whitehead duality, Atiyah duality, and the Thom isomorphism.

DEFINITION 9.1. Let E be a ring G-spectrum and let ξ be an n-plane G-bundle over a G-space X. An E-orientation of ξ is an element $\mu \in E_G^\alpha(T\xi)$ for some $\alpha \in RO(G)$ of virtual dimension n such that, for each inclusion $i : G/H \longrightarrow X$, the restriction of μ to the Thom complex of the pullback $i^*\xi$ is a generator of the free $E_H^*(S^0)$-module $E_G^*(Ti^*\xi)$.

Here $i^*\xi$ has the form $G \times_H W$ for some representation W of H and $Ti^*\xi = G_+ \wedge S^W$ has cohomology $E_G^*(Ti^*\xi) \cong E_H^*(S^W) \cong E_H^{*-w}(S^0)$. Thus the definition makes sense, but it is limited in scope. If X is G-connected, then there is an obvious preferred choice for α, namely the fiber representation V at any fixed point of X: each W will then be isomorphic to V regarded as a representation of H. In general, however, there is no preferred choice for α and the existence of an orientation implies restrictions on the coefficients $E_H^*(S^0)$: there must be units in degree $\alpha - w \in RO(H)$. If $\alpha \neq w$, this forces a certain amount of periodicity in the theory. There is a great deal of further work, largely unpublished, by Costenoble, Waner, Kriz, and myself in the area of orientation theory and Poincaré duality, but the full story is not yet in place. Where it applies, the present definition does have the expected consequences.

THEOREM 9.2 (THOM ISOMORPHISM). Let $\mu \in E_G^\alpha(T\xi)$ be an orientation of the G-vector bundle ξ over X. Then

$$\cup\mu : E_G^\beta(X_+) \longrightarrow E_G^{\alpha+\beta}(T\xi)$$

is an isomorphism for all β.

There is also a relative version. Specializing to oriented manifolds, we obtain the Poincaré duality theorem as an immediate consequence. Observe first that, for bundles ξ and η over X, the diagonal map of X induces a canonical map

$$T(\xi \oplus \eta) \longrightarrow T(\xi \times \eta) \cong T\xi \wedge T\eta.$$

There results a pairing

$$(*) \qquad E_G^\alpha(T\xi) \otimes E_G^\beta(T\eta) \longrightarrow E_G^{\alpha+\beta}(T(\xi \oplus \eta)).$$

We say that a smooth compact G-manifold M is E-oriented if its tangent bundle τ is oriented, say via $\mu \in E_G^\alpha(T\tau)$. In view of our discussion above, this makes most sense when M is a V-manifold and we take α to be V. If M has boundary, the smooth boundary collar theorem shows that the normal bundle of ∂M in M is trivial, and we deduce that an orientation of M determines an orientation $\partial\mu$ of ∂M in degree $\alpha - 1$ such that, under the pairing (*), the product of $\partial\mu$ and the canonical orientation $\iota \in E_G^1(\Sigma(\partial M)_+)$ of the normal bundle is the restriction of μ to $T(\tau|\partial M)$. Similarly, if M is embedded in V, then μ determines an orientation ω of the normal bundle ν such that the product of μ and ω is the canonical orientation of the trivial bundle in $E_G^v(\Sigma^V M_+)$.

Definition 9.3 (Poincaré duality). If M is a closed E-oriented smooth G-manifold with orientation $\mu \in E_G^\alpha(T\tau)$, then the composite

$$D : E_G^\beta(M_+) \longrightarrow E_G^{V-\alpha+\beta}(T\nu) \longrightarrow E^G_{\alpha-\beta}(M)$$

of the Thom and Spanier-Whitehead duality isomorphisms is the Poincaré duality isomorphism; the element $[M] = D(1)$ in $E^G_\alpha(M)$ is called the fundamental class associated to the orientation. If M is a compact E-oriented smooth G-manifold with boundary, then the analogous composites

$$D : E_G^\beta(M_+) \longrightarrow E_G^{V-\alpha+\beta}(T\nu) \longrightarrow E^G_{\alpha-\beta}(M, \partial M)$$

and

$$D : E_G^\beta(M, \partial M) \longrightarrow E_G^{V-\alpha+\beta}(T\nu, T(\nu|\partial M)) \longrightarrow E^G_{\alpha-\beta}(M)$$

are called the relative Poincaré duality isomorphisms. With the Poincaré duality isomorphism for ∂M, they specify an isomorphism from the cohomology long exact sequence to the homology long exact sequence of $(M, \partial M)$. Here the element $[M] = D(1)$ in $E^G_\alpha(M, \partial M)$ is called the fundamental class associated to the orientation.

One can check that these isomorphisms are given by capping with the fundamental class, as one would expect.

S. R. Costenoble, J. P. May, and S. Waner. Equivariant orientation theory. Preprint.
S. R. Costenoble and S. Waner. Equivariant Poincaré duality. Michigan Math. J. 39(1992).
[LMS, III§6]

CHAPTER XVII

The Burnside ring

The basic references are tom Dieck and [LMS]; some specific citations will be given.

[tD] T. tom Dieck. Transformation groups and representation theory. Springer Lecture Notes in Mathematics. Vol. 766. 1979.

1. Generalized Euler characteristics and transfer maps

There are general categorical notions of Euler characteristic and trace maps that encompass a variety of phenomena in both algebra and topology. We again specialize directly to the stable category $\bar{h}G\mathscr{S}$. Let X be a strongly dualizable G-spectrum with coevaluation map $\eta : S \longrightarrow X \wedge DX$ and define the "Euler characteristic" $\chi(X)$ to be the composite

$$(1.1) \qquad \chi(X) : S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\varepsilon} S.$$

For a G-space X, we write $\chi(X) = \chi(\Sigma^\infty X_+)$; for a based G-space X, we write $\tilde{\chi}(X) = \chi(\Sigma^\infty X)$. We shall shortly define the Burnside ring $A(G)$ in terms of these Euler characteristics, and we shall see that it is isomorphic to $\pi_0^G(S)$, the zeroth stable homotopy group of G-spheres. Thus, via the unit isomorphism $S \wedge E \simeq E$, $A(G)$ acts on all G-spectra E and thus on all homotopy, homology, and cohomology groups of all G-spectra. Its algebraic analysis is central to a variety of calculations in equivariant stable homotopy theory.

Before getting to this, we give a closely related conceptual version of transfer maps. Assume given a diagonal map $\Delta : X \longrightarrow X \wedge X$. We are thinking of X as $\Sigma^\infty F_+$ for, say, a compact G-ENR F. We define the "transfer map" $\tau = \tau(X)$:

$S \longrightarrow X$ to be the following composite:

(1.2)

$$\tau : S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{\mathrm{Id} \wedge \Delta} DX \wedge X \wedge X \xrightarrow{\varepsilon \wedge \mathrm{Id}} S \wedge X \simeq X.$$

We shall later call these "pretransfer maps". When applied fiberwise in a suitable fashion, they will give rise to the transfer maps of bundles, which provide a crucial calculational device in both nonequivariant and equivariant cohomology theory.

These simple conceptual definitions lead to easy proofs of the basic properties of these fundamentally important maps. For example, to specify the relation between them, assume given a map $\xi = \xi(X) : X \longrightarrow S$ such that $(\mathrm{Id} \wedge \xi) \circ \Delta : X \longrightarrow X \wedge S$ is the unit isomophism. We are thinking of $\Sigma^\infty \xi$, where $\xi : F_+ \longrightarrow S^0$ is the evident collapse map. In the bundle context, the following immediate consequence of the definitions will determine the behavior of the composite of projection and transfer.

(1.3) The composite $\xi(X) \circ \tau(X) : S \longrightarrow S$ is equal to $\chi(X)$.

There are many other obvious properties with useful consequences.

Before getting to more of these, we assure the reader that if M is a smooth closed G-manifold embedded in a representation V, then application of the functor Σ_V^∞ to the explicit geometric transfer map

$$\tau(M) : S^V \longrightarrow \Sigma^V M_+$$

constructed in IX.3.1 does in fact give the same map as the transfer $\tau : S \longrightarrow S \wedge M_+$ of (1.2). By (1.3), it follows that the Euler characteristic $\chi(M)$ above is obtained by applying Σ_V^∞ to the Euler characteristic $\chi(M) : S^V \longrightarrow S^V$ of IX.3.2. One way to see this is to work out the description of the transfer map τ of (1.2) in the homotopical context of duality for G-ENR's and then specialize to manifolds as in XVI§8.

We shall return later to transfer maps, but we restrict attention to Euler characteristics here. We note first that, via a little Lie group theory, (1.5) leads to a calculation of the nonequivariant Euler characteristics $\chi((G/H)^K)$ for subgroups H and K. The key point is that, since $L(H)^H$ is the tangent space at the identity element of WH, WH is infinite if and only if $L(H)$ contains a trivial representation, in which case $e : S^0 \longrightarrow S^{L(H)}$ is null homotopic as an H-map.

Lemma 1.4. If WH is infinite, then $\chi(G/H) = 0$ and $\chi((G/H)^K) = 0$ for all K. If WH is finite and G/H embeds in V, then the degree of $f^K : S^{V^K} \longrightarrow S^{V^K}$ is the cardinality of the finite set $(G/H)^K$ for each K such that WK is finite.

This gains force from the next few results, which show how to compute $\chi(X)$ in terms of the $\chi(G/H)$ for any strongly dualizable X.

LEMMA 1.5. Let X and Y be strongly dualizable G-spectra.

(i) $\chi(X) = \chi(Y)$ if X is G-equivalent to Y.
(ii) $\chi(*)$ is the trivial map and $\chi(S)$ is the identity map.
(iii) $\chi(X \vee Y) = \chi(X) + \chi(Y)$ and $\chi(X \wedge Y) = \chi(X)\chi(Y)$.
(iv) $\chi(\Sigma^n X) = (-1)^n \chi(X)$.

A direct cofibration sequence argument from the definition of $\chi(X)$ gives the following much more substantial additivity relation.

THEOREM 1.6. For a G-map $f : X \longrightarrow Y$, $\chi(Cf) = \chi(Y) - \chi(X)$.

By induction on the number of cells, this gives the promised calculation of $\chi(X)$ in terms of the $\chi(G/H)$.

THEOREM 1.7. Let X be a finite G-CW spectrum, and let $\nu(H, n)$ be the number of n-cells of orbit type G/H in X. Then

$$\chi(X) = \sum_n \sum_{(H)} (-1)^n \nu(H, n) \chi(G/H).$$

Taking G to be the trivial group, we see from this formula that the Euler characteristic defined by (1.1) specializes to the classical nonequivariant Euler characteristic. The formula is written in terms of a chosen cell decomposition. On the space level, there is a canonical formula for $\chi(X)$ for any compact G-ENR X, namely

$$\chi(X) = \sum_{(H)} \chi(X_{(H)}/G)\chi(G/H). \tag{1.8}$$

Here $X_{(H)} = \{x | (G_x) = (H)\}$ and $\chi(X_{(H)}/G)$ is the sum of the "internal Euler characteristics" $\chi(M) = \chi(\bar{M}) - \chi(\partial M)$ of the path components M of $X_{(H)}$; $\bar{M}$ is the closure of M in X/G and $\partial M = \bar{M} - M$.

Define a homomorphism $d_H : \pi_0^G(S) \longrightarrow \mathbb{Z}$ by letting

$$d_H(x) = deg(f^H), \text{ where } f : S^V \longrightarrow S^V \text{ represents } x. \tag{1.9}$$

In view of XVI.6.2, $\Phi^H S$ is a nonequivariant sphere spectrum, and we can write this more conceptually as

$$d_H(x) = deg(\Phi^H(x)). \tag{1.10}$$

For a compact G-ENR X, we can deduce from (1.10) and standard properties of nonequivariant Euler characteristics that

$$d_H(\chi(X)) = \chi(X^H). \tag{1.11}$$

Similarly, for a finite G-CW spectrum X, we can deduce that

$$d_H(\chi(X)) = \chi(\Phi^H X). \tag{1.12}$$

Note that nothing like this can be true for the genuine fixed points of G-spectra: X^H is virtually never a finite CW-spectrum.

Formula (1.11) shows how the equivariant Euler characteristics of compact G-ENR's determine the nonequivariant Euler characteristics of their fixed point spaces. Conversely, by the following obstruction theoretic observation, the equivariant Euler characteristic is determined by nonequivariant Euler characteristics on fixed point spaces.

Proposition 1.13. Let V be a complex representation of G and let f and f' be G-maps $S^V \longrightarrow S^V$. Then $f \simeq f'$ if and only if $deg(f^H) = deg(f'^H)$ for all H such that WH is finite. Therefore, for compact G-ENR's X and Y, $\chi(X) = \chi(Y)$ if and only if $\chi(X^H) = \chi(Y^H)$ for all such H.

The integers $\chi(X^H)$ as H varies are restricted by congruences. For example, for a finite p-group, we saw in our study of Smith theory that $\chi(X^G) \equiv \chi(X) \bmod p$. More general congruences can be derived by use of the Bott isomorphism in equivariant K-theory.

Proposition 1.14. Let V be a complex representation of G and let f be a G-map $S^V \longrightarrow S^V$. If WH is finite, then

$$\sum [NH : NH \cap NK]\mu(K/H)deg(f^K) \equiv 0 \bmod \ |WH|,$$

where the sum runs over the H-conjugacy classes of groups K such that $H \subset K \subset NH$ and K/H is cyclic and where $\mu(K/H)$ is the number of generators of K/H. Therefore, for a compact G-ENR X,

$$\sum [NH : NH \cap NK]\mu(K/H)\chi(X^K) \equiv 0 \bmod \ |WH|.$$

Observe that this is really a result about the WH-maps f^H and is thus a result about finite group actions.

[tD, 5.1–5.4]
[LMS, III§§7-8 and V§1]

2. The Burnside ring $A(G)$ and the zero stem $\pi_0^G(S)$

For a finite group G, the Burnside ring $A(G)$ is the Grothendieck ring associated to the set of isomorphism classes of finite G-sets, with sum and product given by the disjoint union and Cartesian product of G-sets. There are ring homomorphisms $\phi_H : A(G) \longrightarrow \mathbb{Z}$ that send a finite G-set S to the cardinality of S^H. The product over conjugacy classes (H) gives a monomorphism $\phi : A(G) \longrightarrow C(G)$, where $C(G)$ is the product of a copy of $\mathbb{Z}$ for each (H). The image of ϕ is precisely the subring of tuples (n_H) of integers that satisfy the congruences

$$\sum [NH : NH \cap NK]\mu(K/H)n_K \equiv 0 \bmod |WH|.$$

It is an insight of Segal that $A(G)$ is isomorphic to $\pi_0^G(S)$.

The generalization of this insight to compact Lie groups is due to tom Dieck. We define $A(G)$ to be the set of equivalence classes of compact G-ENR's under the equivalence relation $X \approx Y$ if $\chi(X) = \chi(Y)$ in $\pi_0^G(S)$. Disjoint union and Cartesian product give a sum and product that make $A(G)$ into a ring; Cartesian product with a compact ENR K with trivial action and $\chi(K) = -1$ gives additive inverses. We can define $A(G)$ equally well in terms of finite G-CW complexes or finite G-CW spectra. However defined, the results of the previous section imply that, additively, $A(G)$ is the free Abelian group with a basis element $[G/H]$ for each conjugacy class (H) such that WH is finite. It is immediate that taking Euler characteristics specifies a monomorphism of rings $\chi : A(G) \longrightarrow \pi_0^G(S)$. We define

$$\phi_H = d_H \circ \chi : A(G) \longrightarrow \mathbb{Z}.$$

Then, by (1.11), $\phi_H([X]) = \chi(X^H)$ for a compact G-ENR X.

To define the appropriate version of $C(G)$ for compact Lie groups G we need a little topological algebra. We let $\mathscr{C}G$ be the set of closed subgroups of G and $\mathscr{F}G$ be the subset of those H such that WH is finite. Let ΓG and ΦG be the sets of conjugacy classes of subgroups in $\mathscr{C}G$ and $\mathscr{F}G$, respectively. The set ΓG is countable. The set ΦG is finite if and only if WT acts trivially on the maximal torus T. The set of orders of the finite groups $|WG/W_0G|$ has a finite bound.

There is a Hausdorff metric on $\mathscr{C}G$ that measures the distance between subgroups, and $\mathscr{F}G$ is a closed subspace of $\mathscr{C}G$. The conjugation action of G is continuous. With the orbit space topology, ΓG and ΦG are totally disconnected compact metric spaces. Recall that "totally disconnected" means that every singleton set $\{x\}$ is a component: the non-empty connected subspaces are points. It follows that ΦG has a neighborhood basis consisting of open and closed subsets S. Such a set is specified by a characteristic map $\zeta : \Phi G \longrightarrow S^0$ that send points in S to 1 and points not in S to -1. The proofs of many statements about $A(G)$ combine use of characteristic functions with compactness arguments.

Give $\mathbb{Z}$ the discrete topology and define $C(G)$ to be the ring of continuous (= locally constant) functions $\Phi G \longrightarrow \mathbb{Z}$. Since ΦG is compact, such a function takes finitely many values. The degree function $d(f) : \Phi G \longrightarrow \mathbb{Z}$ specified by $d(f)(H) = deg(f^H)$ for a G-map $f : S^V \longrightarrow S^V$ is continuous, hence there results a ring homomorphism $d : \pi_0^G(S) \longrightarrow C(G)$, and we define $\phi = d\chi : A(G) \longrightarrow C(G)$. Thus we have the following commutative diagram of rings:

$$\begin{tikzcd}
A(G) \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & \pi_0^G(S) \arrow[dl, "d"] \\
& C(G). &
\end{tikzcd}$$

Theorem 2.1. The homomorphism χ is an isomorphism. The homomorphisms ϕ and d are monomorphisms. For $H \in \Phi G$, there is a unique element $\gamma_H \in C(G)$ such that $|WH|\gamma_H = \phi([G/H])$, and $C(G)$ is the free Abelian group generated by these elements γ_H. A map $\nu : \Phi G \longrightarrow \mathbb{Z}$ is in the image of ϕ if and only if, for each $H \in \Phi G$,

$$\sum [NH : NH \cap NK]\mu(K/H)\nu_K \equiv 0 \bmod \ |WH|.$$

Moreover, there is an integer q such that $q(C(G)/A(G)) = 0$, and $q = |G|$ if G is finite.

The index of summation is that specified in Proposition 1.14, which shows that only maps ν that satisfy the congruences can be in the image of ϕ. We know by Proposition 1.13 that d and therefore ϕ is a monomorphism. It is not hard to prove the rest by inductive integrality arguments starting from rational linear combinations, provided that one knows a priori that the rationalization of ϕ is an isomorphism; we shall say something about why this is true shortly.

[tD, 5.5-5.6]
[LMS, V§2]

3. Prime ideals of the Burnside ring

Calculational understanding of the equivariant stable category depends on understanding of the algebraic properties of $A(G)$. For example, suppose given an idempotent $e \in A(G)$. Then $eA(G)$ is the localization of $A(G)$ at the ideal generated by e. For a G-spectrum X, define eX to be the telescope of iterates of

$e : X \longrightarrow X$. Then

$$\underline{\pi}_*(eX) = e\underline{\pi}_*(X).$$

Visibly, the canonical map $X \longrightarrow eX \vee (1-e)X$ induces an isomorphism of homotopy groups and is thus an equivalence. Therefore splittings of $A(G)$ in terms of sums of orthogonal idempotents determine splittings of the entire stable category $\bar{h}G\mathscr{S}$.

The first thing to say about $A(G)$ is that it is Noetherian if and only if the set ΦG is finite. For this reason, $A(G)$ is a much less familiar kind of ring for general compact Lie groups than it is for finite groups.

To understand the structure of any commutative ring A, one must understand its spectrum $\mathrm{Spec}(A)$ of prime ideals. In the case of $A(G)$, it is clear that every prime ideal pulls back from a prime ideal of $C(G)$. We define

$$q(H,p) = \{\alpha | \phi_H(\alpha) \equiv 0 \bmod p\}, \tag{3.1}$$

where p is a prime or $p = 0$. Although these are defined for all H, they are redundant when WH is infinite. There are further redundancies. We shall be precise about this since the basic sources — [tD] and [LMS] — require supplementation from a later note by Bauer and myself. The only proper inclusions of prime ideals are of the form $q(H,0) \subset q(H,p)$, hence $A(G)$ has Krull dimension one. For a given prime ideal q, we wish to describe $\{H | q = q(H,p)\}$. This is easy if $p = 0$.

PROPOSITION 3.2. Let $q = q(H,0)$ for a subgroup H of G.

(i) If $H \lhd J$ and J/T is a torus, then $q = q(J,0)$.
(ii) There is a unique conjugacy class (K) in ΦG such that $q = q(K,0)$; up to conjugation, $H \lhd K$ and K/H is a torus.
(iii) If $H \in \Phi G$ and $J \in \Phi G$, then $q(H,0) = q(J,0)$ if and only if $(H) = (J)$.

Fix a prime p. We say that a group G is "p-perfect" if it has no non-trivial quotient p-groups. For $H \subset G$, let H'_p be the maximal p-perfect subgroup of H; explicitly, H'_p is the inverse image in H of the maximal p-perfect subgroup of the finite group H/H_0. Then define $H_p \subset NH'_p$ to be the inverse image of a maximal torus in WH'_p; H_p is again p-perfect, but now WH_p is finite. This last fact is crucial; it will lead to some interesting new results further on.

THEOREM 3.3. Let $q = q(H,p)$ for a subgroup H of G and a prime p.

(i) If $H \lhd J$ and J/T is an extension of a torus by a finite p-group, then $q = q(J,p)$; if $H \in \Phi G$ and $|WH| \equiv 0 \bmod p$, then there exists $J \in \Phi G$ such that $H \lhd J$ and J/H is a finite p-group.

(ii) There is a unique conjugacy class (K) in ΦG such that $q = q(K,p)$ and $|WK|$ is prime to p; if $H \in \Phi G$ and H is p-perfect, then, up to conjugation, $H \lhd K$ and K/H is a finite p-group.

(iii) $K_p = K'_p$, and K_p is the unique normal p-perfect subgroup of K whose quotient is a finite p-group.

(iv) K_p is maximal in $\{J | q(J,p) = q \text{ and } J \text{ is } p-\text{perfect}\}$, and this property characterizes K_p up to conjugacy.

(v) $(H_p) = (K_p)$, hence $q(H,p) = q(J,p)$ if and only if $(H_p) = (J_p)$.

(vi) If $H \subset K_p$ and H is p-perfect, then $HT = K_p$, where T is the identity component of the center of K_p.

It is natural to let H^p denote the subgroup K of part (ii). If G is finite, we conclude that $q(J,p) = q$ if and only if $(H_p) \le (J) \le (H^p)$. For general compact Lie groups, the situation is more complicated and the following seemingly innocuous, but non-trivial, corollary of the theorem was left as an open question in [LMS].

COROLLARY 3.4. If $H \subset J \subset K$ and $q(H,p) = q(K,p)$, then $q(J,p) = q(K,p)$.

S. Bauer and J. P. May. Maximal ideals in the Burnside ring of a compact Lie group. Proc. Amer. Math. Soc. 102(1988), 684-686.
[tD, 5.7]
[LMS, V§3]

4. Idempotent elements of the Burnside ring

One reason that understanding the prime ideal spectrum of a commutative ring A is so important is the close relationship that it bears to idempotents. A decomposition of the identity element of A as a sum of othogonal idempotents determines and is determined by a partition of $\mathrm{Spec}(A)$ as a disjoint union of non-empty open subsets. In particular, $\mathrm{Spec}(A)$ is connected if and only if 0 and 1 are the only idempotents of A. This motivates us to compute the set $\pi\,\mathrm{Spec}(A(G))$ of components of $A(G)$; we topologize this set as a quotient space of $\mathrm{Spec}(A(G))$. However, there is a key subtlety here that was missed in [LMS]: while the components of any space are closed, they need not be open (unless the space is locally connected). In particular, since $\pi\,\mathrm{Spec}(A(G))$ is not discrete, the components of $\mathrm{Spec}(A(G))$ need not be open, and they therefore do not determine idempotents in general.

A compact Lie group G is perfect if it is equal to the closure of its commutator subgroup. It is solvable if it is an extension of a torus by a finite solvable group. Let $\mathscr{P}G$ denote the subspace of $\mathscr{C}G$ consisting of the perfect subgroups and let

ΠG be its orbit space of conjugacy classes; ΠG is countable, but it is usually not finite unless G is finite.

Any compact Lie group G has a minimal normal subgroup G_a such that G/G_a is solvable, and G_a is perfect. Passage from G to G_a is a continuous function $\mathscr{C}G \longrightarrow \mathscr{C}G$, $\mathscr{P}G$ is a closed subspace of $\mathscr{C}G$, and ΠG is a closed subspace of ΓG and is thus a totally disconnected compact metric space. There is a finite normal sequence connecting G_a to G each of whose subquotients is either a torus or a cyclic group of prime order. Via the results above, this implies that, for a given H, all prime ideals $q(H,p)$ are in the same component of $\mathrm{Spec}(A(G))$ as H_a. This leads to the following result.

Proposition 4.1. Define $\beta : \Pi G \longrightarrow \pi\,\mathrm{Spec}(A(G))$ by letting $\beta(L)$ be the component that contains $q(L,0)$. Then β is a homeomorphism.

In particular, G is solvable if and only if $A(G)$ contains no non-trivial idempotents. For example, the Feit-Thompson theorem that an odd order finite group G is solvable is equivalent to the statement that $A(G)$ has no non-trivial idempotents. (Several people have tried to use this fact as the starting point of a topological proof of the Feit-Thompson theorem, but without success.)

A key point in the proof, and in the proofs of the rest of the results of this section, is that, for a subring R of $\mathbb{Q}$, the function

$$q : \Phi G \times \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(A(G) \otimes R)$$

is a continuous closed surjection. This is deduced from the fact that

$$q : \Phi G \times \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(C(G) \otimes R)$$

is a homeomorphism. In turn, the latter holds by an argument that depends solely on the fact that ΦG is a totally disconnected compact Hausdorff space.

If L is a perfect subgroup of G that is not a limit of perfect subgroups, then the component of $\beta(L)$ in $\mathrm{Spec}(A(G)$ is open and L determines an idempotent e_L in $A(G)$. Even when G is finite, it is non-trivial to write e_L in the standard basis $\{[G/H] | (H) \in \Phi G\}$, and such a formula has not yet been worked out for general compact Lie groups. Nevertheless one can prove the following theorem. Observe that the trivial subgroup of G is perfect; we here denote it by 1.

Theorem 4.2. Let L be a perfect subgroup of G that is not a limit of perfect subgroups. Then there is an idempotent $e_L = e_L^G$ in $A(G)$ that is characterized by

$$\phi_H(e_L) = 1 \text{ if } (H_a) = (L) \quad \text{and} \quad \phi_H(e_L) = 0 \text{ if } (H_a) \neq (L).$$

Restriction from G to NL and passage to L-fixed points induce ring isomorphisms

$$e_L^G A(G) \longrightarrow e_L^{NL} A(NL) \longrightarrow e_1^{WL} A(WL).$$

[tD, 5.11]
[LMS, V§4]

5. Localizations of the Burnside ring

Let $A(G)_p$ denote the localization of $A(G)$ at a prime p and let $A(G)_0$ denote the rationalization of $A(G)$. We shall describe these localizations and the localizations of $A(G)$ at its prime ideals $q(H,p)$. We shall also explain the analysis of idempotents in $A(G)_p$, which is parallel to the analysis of idempotents in $A(G)$ just given but, in the full generality of compact Lie groups, is less well understood.

We begin with $A(G)_0$. Let $\mathbb{Z}_H$ denote $\mathbb{Z}$ regarded as an $A(G)$-module via $\phi_H : A(G) \longrightarrow \mathbb{Z}$.

Proposition 5.1. Let $(H) \in \Phi G$.

(i) The localization of $A(G)$ at $q(H,0)$ is the canonical homomorphism

$$A(G) \longrightarrow (A(G)/q(H,0))_0 \cong \mathbb{Q}.$$

(ii) $\phi_H : A(G) \longrightarrow \mathbb{Z}_H$ induces an isomorphism of localizations at $q(H,0)$.
(iii) $\phi : A(G) \longrightarrow C(G)$ induces an isomorphism of rationalizations.

Corollary 5.2. Rationalization $A(G) \longrightarrow A(G)_0 \cong C(G)_0$ is the inclusion of $A(G)$ in its total quotient ring, and $\phi : A(G) \longrightarrow C(G)$ is the inclusion of $A(G)$ in its integral closure in $C(G)_0$.

Here (i) makes essential use of the compactness of ΦG, and (i) implies (ii). To prove (iii) — which we needed to prove Theorem 2.1 — we can now exploit the fact that a map of rings is an isomorphism if it induces a homeomorphism on passage to Spec and an isomorphism upon localization at corresponding prime ideals. If G is finite, then $A(G)_0$ is just a finite product of copies of $\mathbb{Q}$. For general compact Lie groups G, $A(G)_0$ is a type of ring unfamiliar to topologists but familiar in other branches of mathematics under the name of an "absolutely flat" or "von Neumann regular" ring. One characterization of such a commutative ring is that all of its modules are flat; another, obviously satisfied by $A(G)_0$, is that the localization of A at any maximal ideal P is A/P. For any such ring A, Spec(A) is a totally disconnected compact Hausdorff space, and an ideal is finitely generated if and only if it is generated by a single idempotent element.

Proposition 5.3. Let p be a prime and let $(H) \in \Phi G$.

(i) The localization of $A(G)$ at $q(H,p)$ is the canonical homomorphism

$$A(G) \longrightarrow (A(G)/I(H,p))_p;$$

here $I(H,p) = \cap q(J,0)$, where the intersection runs over

$$\Phi(G;H,p) \equiv \{(J) | (J) \in \Phi G \text{ and } q(J,p) = q(H,p)\}.$$

(ii) The ring homomorphism

$$\prod \phi_J : A(G) \longrightarrow \prod \mathbb{Z}_J$$

is a monomorphism, where the product runs over $(J) \in \Phi(G;H,p)$.

The following statement only appears in the literature for finite groups. The general case relies on the full strength of Theorem 3.3, and the line of proof is the same as that of Theorem 3.6. The essential point is the analog of Proposition 3.5, and the essential point for this is the following assertion, which is trivially true for finite groups but has not yet been investigated for general compact Lie groups.

Conjecture 5.4. The function $\mathscr{C}G \longrightarrow \mathscr{C}G$ that sends H to H_p is continuous.

Theorem 5.5. Let L be a p-perfect subgroup of G that is maximal in the set of p-perfect subgroups H such that $q(H,p) = q(L,p)$ and is not a limit of such p-perfect subgroups. If Conjecture 3.10 holds, then there is an idempotent $e_L = e_L^G$ in $A(G)_p$ that is characterized by

$$\phi_H(e_L) = 1 \text{ if } (H_p) = (L) \text{ and } \phi_H(e_L) = 0 \text{ if } (H_p) \neq (L).$$

Restriction from G to NL and passage to L-fixed points induce ring isomorphisms

$$e_L^G A(G)_p \longrightarrow e_L^{NL} A(NL)_p \longrightarrow e_1^{WL} A(WL)_p.$$

Moreover, $e_L^G A(G)_p$ is isomorphic to the localization of $A(G)$ at $q(L,p)$. If G is finite, then

$$A(G)_p \cong \prod_{(L)} e_L^G A(G)_p.$$

Taking L to be any group in ΦG that is not a limit of groups in ΦL and taking H_0 to be H, we see that the statement is true when $p = 0$. Of course, in the general compact Lie case, $A(G)_p$ is no longer the product of the $e_L^G A(G)_p$. However, it seems possible that, by suitable arguments to handle limit groups L, $A(G)_p$ can be described sheaf theoretically in terms of these localizations. The point is that $A(G)_p$ has the unusual property that it is isomorphic to the ring of global sections of its structural sheaf over its maximal ideal spectrum. (Any commutative ring A is isomorphic to the ring of global sections of its structural sheaf over Spec(A).)

[tD, 7.8]
[LMS, V§5]

6. Localization of equivariant homology and cohomology

The results of the previous section imply algebraic decomposition and reduction theorems for the calculation of equivariant homology and cohomology theories. We shall go into some detail since, in the compact Lie case, the results of [LMS] require clarification. When G is finite, we shall obtain a natural reduction of the computation of homology and cohomology theories localized at a prime p to their calculation in terms of appropriate associated theories for subquotient p-groups of G. It is interesting that although the proof of this reduction makes heavy use of idempotents of $A(G)_p$, there is no reference to $A(G)$ in the description that one finally ends up with. We shall use this reduction in our proof of the generalized Segal conjecture.

Recall the geometric fixed point functors Φ^H from XVI§§3, 6. In view of (1.12), it should seem natural that this and not the genuine fixed point functor on G-spectra appears in the following results.

Theorem 6.1. *Let L be a perfect subgroup of G that is not a limit of perfect subgroups. For G-spectra X and Y, there are natural isomorphisms*

$$[X, e_L^G Y]_G \longrightarrow [X, e_L^{NL} Y]_{NL} \longrightarrow [\Phi^H X, e_1^{WL}\Phi^H Y]_{WL}.$$

We prefer to state the homological consequences in terms of G-spaces, but it applies just as well to Φ-fixed points of G-spectra.

Corollary 6.2. *Let E be a G-spectrum and X be a G-space. For $\alpha \in RO(G)$, let $\beta = r^G_{NL}\alpha \in RO(NL)$ and $\gamma = \beta^L \in RO(WL)$. Then there are natural isomorphisms*

$$e_L^G E_\alpha^G(X) \longrightarrow e_L^{NL} E_\beta^{NL}(X) \longrightarrow e_1^{WL} E_\gamma^{WL}(XL)$$

and

$$e_L^G E^\alpha_G(X) \longrightarrow e_L^{NL} E^\beta_{NL}(X) \longrightarrow e_1^{WL} E^\gamma_{WL}(XL),$$

where E_*^{NL} and E^*_{NL} denote the theories that are represented by E regarded as an NL-spectrum and E_*^{WL} and E^*_{WL} denote the theories that are represented by $\Phi^L E$.

Write X_p for the localization of a G-spectrum at a prime p. It can be constructed as the telescope of countably many iterates of $p : X \longrightarrow X$, and its properties are as one would expect from the G-space level.

THEOREM 6.3. Let L be a p-perfect subgroup of G that is maximal in the set of p-perfect subgroups H of G such that $q(H,p) = q(L,p)$ and is not a limit of such p-perfect subgroups. If G is finite, or if Conjecture 3.10 holds, then, for G-spectra X and Y, there are natural isomorphisms

$$[X, e_L^G Y_p]_G \longrightarrow [X, e_L^{NL} Y_p]_{NL} \longrightarrow [\Phi^H X, e_1^{WL}\Phi^H Y_p]_{WL}.$$

When $p = 0$, the statement holds for $L \in \Phi G$ if L is not a limit of groups in ΦG.

Here $\Phi^H(Y_p) \simeq (\Phi^H Y)_p$. We again state the homological version only for G-spaces, although it also applies to G-spectra and Φ-fixed points. There is a further isomorphism here that does not come from Theorem 4.3. We shall discuss it after stating the corollary.

COROLLARY 6.4. Let E be a G-spectrum and X be a G-space. With L as in Theorem 4.3, let VL be a p-Sylow subgroup of the finite group WL. For $\alpha \in RO(G)$, let $\beta = r_{NL}^G\alpha \in RO(NL)$, $\gamma = \beta^L \in RO(WL)$, and $\delta = r_{VL}^{WL}\gamma \in RO(VL)$. Then there are natural isomorphisms

$$e_L^G E_\alpha^G(X)_p \longrightarrow e_L^{NL} E_\beta^{NL}(X)_p \longrightarrow e_1^{WL} E_\gamma^{WL}(XL)_p \longrightarrow E_\delta^{VL}(X^L)_p^{inv}$$

and, assuming that X is a finite G-CW complex,

$$e_L^G E^\alpha_G(X)_p \longrightarrow e_L^{NL} E^\beta_{NL}(X)_p \longrightarrow e_1^{WL} E^\gamma_{WL}(XL)_p \longrightarrow E^\delta_{VL}(X^L)_p^{inv},$$

where E_*^{NL} and E^*_{NL} denote the theories represented by E regarded as an NL-spectrum, E_*^{WL} and E^*_{WL} denote the theories represented by $\Phi^L E$, and E_*^{VL} and E^*_{VL} denote the theories represented by $\Phi^L E$ regarded as a VL-spectrum. Therefore, if G is finite, then

$$E_*^G(X)_p \cong \prod_{(L)} E_*^{VL}(X^L)_p^{inv}$$

and, if X is a finite G-CW complex,

$$E^*_G(X)_p \cong \prod_{(L)} E^*_{VL}(X^L)_p^{inv}.$$

When $p = 0$, the statement holds with VL taken as the trivial group.

The ideas in XIII§1 are needed to be precise about the grading. Of course, there is no problem of interpretation for the $\mathbb{Z}$-graded part of the theories. For finite groups, this gives the promised calculation of the localization of equivariant homology and cohomology theories at p in terms of homology and cohomology theories that are associated to subquotient p-groups; in the case of rationalization, a better result will be described later. For general compact Lie groups, such a calculation may follow from the fact that one can reconstruct any module over

$A(G)_p$ as the module of global sections of its structural sheaf over the maximal ideal spectrum of $A(G)_p$. Intuitively, the idea is that the space of maximal ideals should carry the relevant Lie group theory; theories associated to subquotient p-groups should carry the algebraic topology.

We must still explain the "inv" notation and the final isomorphisms that appear in the corollary. These come from a typical application of the general concept of induction in the context of Mackey functors. We shall say more about this later, but we prefer to explain the idea without formalism here.

Let G be a finite group with p-Sylow subgroup K. We are thinking of WL and VL. For G-spectra X and Y, we define $([X,Y]_p^K)^{inv}$ to be the equalizer (= difference kernel) of the maps

$$[G/K_+ \wedge X, Y]_p^G \longrightarrow [G/K_+ \wedge G/K_+ \wedge X, Y]_p^G$$

induced by the two projections $G/K_+ \wedge G/K_+ \longrightarrow G/K_+$. Here we are using the notational convention

$$[X,Y]^G = [X,Y]_G.$$

For a G-spectrum E, we define $E_*^K(X)_p^{inv}$ by replacing X by sphere spectra and replacing Y by $E \wedge X$. We define $E_K^*(X)_p^{inv}$ by replacing X by its smash product with sphere spectra and replacing Y by E. The final isomorphisms of Corollary 3.4 are special cases of the following result; there we must restrict to finite X in cohomology because it is only for finite X that localized spectra represent algebraic localizations of cohomology groups.

Proposition 6.5. If G is a finite group with p-Sylow subgroup K, then, for any G-spectra X and Y, the projection $G/K_+ \wedge X \longrightarrow X$ induces an isomorphism

$$[X,Y]_p^G \longrightarrow ([X,Y]_p^K)^{inv}.$$

Actually, the relevant induction argument works to prove more generally that the analogous map

$$[X,Y]_{q(K,p)}^G \longrightarrow ([X,Y]_{q(K,p)}^K)^{inv}$$

is an isomorphism, where G is a compact Lie group and $(K) \in \Phi G$. The idea is that we have a complex

$$0 \longrightarrow [X,Y]^G \xrightarrow{d^0} [G/K_+ \wedge X, Y]^G \xrightarrow{d^1} [G/K_+ \wedge G/K_+ \wedge X, Y]^G \xrightarrow{d^2} \cdots,$$

where d^n is the alternating sum of the evident projection maps. When localized at $q(K,p)$, this complex acquires the contracting homotopy that is specified by $s^n = [G/K]^{-1}\tau^*$. Here, for any X, τ means

$$\tau \wedge \mathrm{Id} : X \cong S \wedge X \longrightarrow (G/K)_+ \wedge X,$$

where $\tau : S \longrightarrow (G/K)_+$ is the transfer map discussed in Section 1. It is immediate from (1.3) that the composite of τ and the projection $\xi : G/K_+ \longrightarrow S$ is the Euler characteristic $\chi(G/K) : S \longrightarrow S$. This implies that $\tau^*\xi^*$ is multiplication by $[G/K]$. The essential point is that $[G/K]$ becomes a unit in $A(G)_{q(K,p)}$. In the context of the proposition, the localization of $[X,Y]^K$ at $q(K,p)$ is the same as its localization at p.

[tD, Ch 7]
[LMS, V§6]

CHAPTER XVIII

Transfer maps in equivariant bundle theory

The basic reference is [LMS]; specific citations are given at the ends of sections.

1. The transfer and a dimension-shifting variant

Transfer maps provide one of the main calculational tools in equivariant stable homotopy theory. We have given a first definition in XVII§1. We shall here refer to the "transfer map" there as a pretransfer. It will provide the map of fibers for the transfer maps of bundles, in a sense that we now make precise. We place ourselves in the context of VII§1, where we considered equivariant bundle theory. Thus we assume given an extension of compact Lie groups

$$1 \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Fix a complete Γ-universe U and note that U^Π is a complete G-universe. Let Y be a Π-free Γ-spectrum indexed on U^Π and let $B = Y/\Pi$. We are thinking of Y as $\Sigma^\infty X_+$ for a Π-free Γ-space X, but it changes nothing to work with spectra. In fact, this has some advantages. For example, relative bundles can be treated in terms of spectrum level cofibers, obviating complications that would arise if we restricted to spaces. Fix a compact Γ-ENR F. We could take F to be a spectrum as well, but we desist.

We have the orbit spectrum $E = Y \wedge_\Pi F_+$, which we think of as the total G-spectrum of a G-bundle with base G-spectrum B. Write $\pi : E \longrightarrow B$ for the map induced by the projection $F_+ \longrightarrow S^0$. Since F is a compact G-ENR, we have the stable pretransfer Γ-map $\tau(F) : S^0 \longrightarrow F_+$ of XVII§1; we have omitted notation for the suspension Γ-spectrum functor, and we shall continue to do so, but it is essential to remember that $\tau(F)$ is a map of genuine Γ-spectra indexed on U. As we discussed in XVI§5, Π-free Γ-spectra live in the Π-trivial Γ-universe U^{Π}. On maps, this gives that the inclusion $i : U^{\Pi} \longrightarrow U$ induces an isomorphism

$$i_* : [Y, Y \wedge F_+]_{\Gamma} \longrightarrow [i_*Y, i_*(Y \wedge F_+)]_{\Gamma} \cong [i_*Y, i_*Y \wedge F_+]_{\Gamma}.$$

Definition 1.1. Let $\tilde{\tau} : Y \longrightarrow Y \wedge F_+$ be the Γ-map indexed on U^{Π} such that

$$i_*(\tilde{\tau}) = \operatorname{Id} \wedge \tau(F) : i_*Y \longrightarrow i_*Y \wedge F_+.$$

Define the transfer

$$\tau = \tau(\pi) : B = Y/\Pi \longrightarrow Y \wedge_{\Pi} F_+ = E$$

to be the map of G-spectra indexed on U^{Π} that is obtained from $\tilde{\tau}$ by passage to orbits over Π.

When $G = e$, this gives the nonequivariant transfer; specialization to this case results in no significant simplification. Note that there is no finiteness condition on the base spectrum B.

The definition admits many variants. When we describe its properties, we shall often use implicitly that it does not require a complete Γ-universe, only a universe into which F can be embedded, so that duality applies.

We can apply the same construction to maps other than $\tau(F)$. We illustrate this by constructing the map that gives the generalized Adams isomorphism of XVI.5.4. Since the construction is a little intricate and will not be used in the rest of the chapter, the reader may prefer to skip ahead. The cited Adams isomorphism is a natural equivalence of G/N-spectra

$$E/N \longrightarrow (\Sigma^{-Ad(N)} i_*E)^N,$$

where N is a normal subgroup of G and E is an N-free G-spectrum indexed on the fixed points of a complete G-universe. By adjunction, such a map is determined by a "dimension-shifting transfer G-map"

$$i_*(E/N) \longrightarrow \Sigma^{-Ad(N)} i_*E.$$

We proceed to construct this map.

Construction 1.2. Let N be a normal subgroup of G and write Π for N considered together with its conjugation action c by G. Let Γ be the semi-direct product $G \times_c \Pi$. We then have the quotient map $\varepsilon : \Gamma \longrightarrow G$. We also have a twisted quotient map $\theta : \Gamma \longrightarrow G$, $\theta(g, n) = gn$, that restricts to the identity

$\Pi \longrightarrow N$. Let X be an N-free G-space and let θ^*X denote X regarded as a Γ-space via θ; then θ^*X is Π-free. It is easy to check that we have G-homeomorphisms

$$X \cong \theta^*X \times_\Pi N \quad \text{and} \quad X/G \cong \theta^*X \times_\Pi pt.$$

This tells us how to view X as a Π-free Γ-space, placing us in the context of Definition 1.1. Here, however, we really need the spectrum level generalization. Let E be an N-free G-spectrum indexed on $(U^\Pi)^N$, where U is a complete Γ-universe. Let $i : (U^\Pi)^N \longrightarrow U^\Pi$ be the inclusion and let $Y = i_*\theta^*E$. Then Y is a Π-free Γ-spectrum indexed on U^Π, and there are natural isomorphisms of G-spectra

$$i_*E \cong Y \wedge_\Pi N_+ \quad \text{and} \quad i_*(E/N) \cong Y/\Pi.$$

The relevant "pretransfer" in the present context is a map

$$t : S \longrightarrow \Sigma^{-Ad(N)} N_+$$

of Γ-spectra indexed on U. The tangent bundle of $N = \Gamma/G$ is the trivial bundle $N \times Ad(N)$, where Γ acts on $Ad(N)$ by pullback along ε. Embed N in a Γ-representation V and let W be the resulting representation $V - Ad(N)$ of Γ. Embedding a normal tube and taking the Pontrjagin-Thom construction, we obtain a Γ-map

$$S^V \longrightarrow \Gamma_+ \wedge_G S^W \cong N_+ \wedge S^W.$$

We obtain the pretransfer t by applying the suspension spectrum functor and then desuspending by V. We are now in a position to apply the construction of Definition 1.1. Letting j denote the inclusion of U^Π in U to avoid confusion with i, observe that

$$j_*(Y \wedge \Sigma^{-Ad(N)} N_+) \cong j_*(\Sigma^{-Ad(N)}(Y \wedge N_+)).$$

Thus, smashing Y with t, pulling back to the universe U^Π, and passing to orbits over Π, we obtain the desired transfer map

$$i_*(E/N) \cong Y/\Pi \longrightarrow \Sigma^{-Ad(N)}(Y \wedge_\Pi N_+) \cong \Sigma^{-Ad(N)} i_*E.$$

[LMS, II§7 and IV§3]

2. Basic properties of transfer maps

Now return to the context of Definition 1.1. While we shall not go into detail, the transfer can be axiomatized by the basic properties that we list in the following omnibus theorem. They are all derived from corresponding statements about pretransfer maps. By far the most substantial of these properties is (v), which is proven by a fairly elaborate exercise in diagram chasing of cofiber sequences in the context of Spanier-Whitehead duality.

THEOREM 2.1. The transfer satisfies the following properties.

(i) *Naturality.* The transfer is natural with respect to maps $f : Y \longrightarrow Y'$ of Π-free Γ-spectra.

(ii) *Stability.* For a representation V of G regarded by pullback as a representation of Γ, $\Sigma^V\tau$ coincides with the transfer

$$\tau : \Sigma^V(Y/\Pi) \cong (\Sigma^V Y)/\Pi \longrightarrow (\Sigma^V Y) \wedge_\Pi F_+ \cong \Sigma^V(Y \wedge_\Pi F_+).$$

(iii) *Normalization.* With $F = pt$, the transfer associated to the identity map is the identity map.

(iv) *Fiber invariance.* The following diagram commutes for an equivalence $\phi : F \longrightarrow F'$ of compact Γ-ENR's:

$$\begin{tikzcd} & Y/\Pi \arrow[dl, "\tau"'] \arrow[dr, "\tau"] & \\ Y \wedge_\Pi F_+ \arrow[rr, "\mathrm{Id} \wedge \phi"'] & & Y \wedge_\Pi F'_+. \end{tikzcd}$$

(v) *Additivity on fibers.* Let F be the pushout of a Γ-cofibration $F_0 \longrightarrow F_1$ and a Γ-map $F_0 \longrightarrow F_2$, where the F_k are compact Γ-ENR's. Let τ_k be the transfer associated to $Y \wedge_\Pi (F_k)_+ \longrightarrow Y/\Pi$ and let $j_k : Y \wedge_\Pi (F_k)_+ \longrightarrow Y \wedge_\Pi F_+$ be induced by the canonical map $F_k \longrightarrow F$. Then

$$\tau = j_1\tau_1 + j_2\tau_2 - j_0\tau_0.$$

(vi) *Change of groups.* Assume given an inclusion of extensions

$$\begin{tikzcd} 1 \arrow[r] & \Theta \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \Pi \arrow[r] & \Gamma \arrow[r] & G \arrow[r] & 1. \end{tikzcd}$$

Then the following diagram commutes for a Θ-free Λ-spectrum Y indexed on U^{Π} regarded as a Λ-universe:

$$\begin{tikzcd}
G \ltimes_H (Y/\Theta) \arrow[rr, "\mathrm{Id} \ltimes \tau"] \arrow[d, "\cong"'] & & G \ltimes_H (Y \wedge_\Theta F_+) \arrow[d, "\cong"] \\
(\Gamma \ltimes_\Lambda Y)/\Pi \arrow[r, "\tau"'] & (\Gamma \ltimes_\Lambda Y) \wedge_\Pi F_+ \arrow[r, "\cong"'] & (\Gamma \ltimes_\Lambda (Y \wedge F_+))/\Pi.
\end{tikzcd}$$

Modulo a fair amount of extra bookkeeping to make sense of it, part (vi) remains true if we require only the homomorphism $H \longrightarrow G$ in our given map of extensions to be an inclusion. There is also a change of groups property that holds for a map of extensions in which $\Theta \longrightarrow \Pi$ is the identity but the other two maps are unrestricted. Such properties are useful and important, but we shall not go into more detail here. Rather, we single out a particular example of the kind of information that they imply. Let $H \subset G$ and consider the bundles

$$G/H \longrightarrow pt \quad \text{and} \quad BH = EG \times_H (G/H) \longrightarrow BG$$

and the collapse maps $\varepsilon : EG_+ \longrightarrow S^0$ and $\varepsilon : EH_+ \longrightarrow S^0$.

Proposition 2.2. Let E be a split G-spectrum. Then the following diagram commutes:

$$\begin{tikzcd}
E_H^*(S^0) \arrow[r, "\varepsilon^*"] \arrow[d, "\tau^*"'] & E_H^*(EH_+) \arrow[r, "\cong"] \arrow[d, "\tau^*"] & E^*(BH_+) \arrow[d, "\tau^*"] \\
E_G^*(S^0) \arrow[r, "\varepsilon^*"'] & E_G^*(EG_+) \arrow[r, "\cong"'] & E^*(BG_+).
\end{tikzcd}$$

Here E^* is the theory represented by the underlying nonequivariant spectrum of E. For example, if E represents complex equivariant K-theory, then the transfer map on the left is induction $R(H) \longrightarrow R(G)$ and the transfer map on the right is the nonequivariant one. The horizontal maps become isomorphisms upon completion at augmentation ideals, by the Atiyah-Segal completion theorem.

[LMS, IV§§3–4]

3. Smash products and Euler characteristics

The transfer commutes with smash products, and a special case of this implies a basic formula in terms of Euler characteristics for the evaluation of the composite

$\xi \circ \tau$ for a G-bundle ξ. The commutation with smash products takes several forms. For an external form, we assume given extensions,

$$1 \longrightarrow \Pi_i \longrightarrow \Gamma_i \longrightarrow G_i \longrightarrow 1$$

and complete Γ_i-universes U_i for $i = 1$ and $i = 2$.

THEOREM 3.1. The following diagram of $(G_1 \times G_2)$-spectra indexed on the universe $(U_1)^{\Pi_1} \oplus (U_2)^{\Pi_2}$ commutes for Π_i-free Γ_i-spectra Y_i and finite Γ_i-spaces F_i:

$$\begin{tikzcd}
(Y_1/\Pi_1) \wedge (Y_2/\Pi_2) \arrow[r, "\tau\wedge\tau"] \arrow[d, "\cong"'] & (Y_1 \wedge_{\Pi_1} F_{1+}) \wedge (Y_2 \wedge_{\Pi_2} F_{2+}) \arrow[d, "\cong"] \\
(Y_1 \wedge Y_2)/(\Pi_1 \times \Pi_2) \arrow[r, "\tau"'] & (Y_1 \wedge Y_2) \wedge_{\Pi_1 \times \Pi_2} (F_1 \times F_2)_+ .
\end{tikzcd}$$

When $G = G_1 = G_2$, we can use change of groups to internalize this result. Modulo a certain amount of detail to make sense of things, we see in this case that the diagram of the previous theorem can be interpreted as a commutative diagram of G-spectra. Either specializing this result or just inspecting definitions, we obtain the following useful observation. We revert to the notations of Definition 1.1, so that U is a Γ-complete universe.

COROLLARY 3.2. Let Y be a Π-free Γ-spectrum indexed on U^{Π}, F be a compact Γ-ENR, and E be a G-spectrum indexed on U^{Π}. Then the following diagram commutes:

$$\begin{tikzcd}
(Y \wedge E)/\Pi \arrow[r, "\tau"] \arrow[d, "\cong"'] & (Y \wedge E) \wedge_{\Pi} F_+ \arrow[d, "\cong"] \\
(Y/\Pi) \wedge E \arrow[r, "\tau\wedge\mathrm{id}"'] & (Y \wedge_{\Pi} F_+) \wedge E .
\end{tikzcd}$$

In the presence of suitable diagonal maps, this leads to homological formulas involving cup and cap products. While more general results are valid and useful, we shall restrict attention to the case of a given space-level bundle. Here the previous corollary and diagram chases give the following result.

COROLLARY 3.3. Let X be a Π-free Γ-space and F be a compact Γ-ENR. Then the following diagram commutes, where we have written Δ for various maps induced from the diagonal maps of X and F.

$$\begin{tikzcd}
(X/\Pi)_+ \arrow[r, "\tau"] \arrow[d, "\Delta"'] & (X \times_\Pi F)_+ \arrow[dd, "\Delta"] & (X/\Pi)_+ \arrow[l, "\tau"'] \arrow[d, "\Delta"] \\
(X/\Pi)_+ \wedge (X/\Pi)_+ \arrow[d, "\tau \wedge \mathrm{id}"'] & & (X/\Pi)_+ \wedge (X/\Pi)_+ \arrow[d, "\mathrm{id} \wedge \tau"] \\
(X \times_\Pi F)_+ \wedge (X/\Pi)_+ & (X \times_\Pi F)_+ \wedge (X \times_\Pi F)_+ \arrow[l, "\mathrm{id} \wedge \xi"] \arrow[r, "\xi \wedge \mathrm{id}"'] & (X/\Pi)_+ \wedge (X \times_\Pi F)_+.
\end{tikzcd}$$

Retaining the hypotheses of the corollary and constructing cup and cap products as in XIII§5, we easily deduce the following formulas relating the maps induced on homology and cohomology by the maps Δ, τ, and ξ displayed in its diagram.

PROPOSITION 3.4. The following formulas hold, where E is a ring G-spectrum.

(i) $\tau^*(w) \cup y = \tau^*(w \cup \xi^*(y))$ for $w \in E_G^*(X \times_\Pi F)$ and $y \in E_G^*(Y/\Pi)$
(ii) $x \cup \tau^*(z) = \tau^*(\xi^*(x) \cup z)$ for $x \in E_G^*(X/\Pi)$ and $z \in E_G^*(Y \wedge_\Pi F_+)$
(iii) $y \cap \tau^*(x) = \xi_*(\tau_*(y) \cap w)$ for $y \in E_G^*(Y/\Pi)$ and $w \in E_G^*(X \times_\Pi F)$
(iv) $\tau_*(y) \cap \xi^*(x) = \tau_*(y \cap x)$ for $y \in E_G^*(Y/\Pi)$ and $x \in E_G^*(X/\Pi)$

Define the Euler characteristic of the bundle $\xi : X \times_\Pi F \longrightarrow X$ to be

$$\chi(\xi) = \tau^*(1) \in E_G^0(X/\Pi). \tag{3.5}$$

Taking $w = 1$ in the first equation above, we obtain the following conclusion.

COROLLARY 3.6. The composite

$$E_G^*(X/\Pi_+) \xrightarrow{\xi^*} E_G^*(X \times_\Pi F)_+ \xrightarrow{\tau^*} E_G^*(X/\Pi_+)$$

is multiplication by $\chi(\xi)$.

In many applications of the transfer, one wants to use this by proving that $\chi(\xi)$ is a unit and deducing that $E_G^*(X/\Pi_+)$ is a direct summand of $E_G^*(X \times_\Pi F)_+$. When $\chi(\xi)$ is or is not a unit is not thoroughly understood. The strategy for studying the problem is to relate $\chi(\xi)$ to the Euler characteristic

$$\chi(F) = \xi^*(\tau(F)) \in \pi_\Gamma^0(S).$$

We need a bit of language in order to state the basic result along these lines.

If $X/\Pi = G/H$, then $X = \Gamma/\Lambda$ for some Λ such that $\Lambda \cap \Pi = e$. The composite $\Lambda \subset \Gamma \longrightarrow G$ maps Λ isomorphically onto H. Inverting this isomorphism, we obtain a homomorphism $\alpha : H \cong \Lambda \subset \Gamma$. For a general Π-free Γ-space X and an orbit $G/H \subset X/\Pi$, the pullback bundle over G/H gives rise to such a homomorphism $\alpha : H \longrightarrow \Gamma$, which we call the fiber representation of X at G/H. Write α^*F for F regarded as an H-space by pullback along α.

Theorem 3.7. Let X be a Π-free Γ-space and F be a Γ-space. Let $B = X/\Pi$ and consider the bundle $\xi : X \times_\Pi F \longrightarrow B$. For a ring G-spectrum E, the Euler characteristic $\chi(\xi) \in E^0_G(B_+)$ is a unit if any of the following conditions hold.

(i) $\chi(\alpha^*F) \in E^0_H(S)$ is a unit for each fiber representation $\alpha : H \longrightarrow \Gamma$ of X.
(ii) B is G-connected with basepoint $*$ and $\chi(\alpha^*F) \in E^0_G(S)$ is a unit, where $\alpha : G \longrightarrow F$ is the fiber representation of X at $*$.
(iii) B is G-free and the nonequivariant Euler characteristic $\chi(F) \in E^0_e(S)$ is a unit.

Nonequivariantly, with $G = e$, the connectivity hypothesis of (ii) is inconsequential, but it is a serious limitation in the equivariant case and one must in general fall back on (i). The following implication is frequently used.

Theorem 3.8. If G is a finite p-group and $\xi : Y \longrightarrow B$ is a finite G-cover whose fiber F has cardinality prime to p, then the composite map

$$\Sigma^\infty B_+ \xrightarrow{\tau} \Sigma^\infty Y_+ \xrightarrow{\xi} \Sigma^\infty B_+$$

become an equivalence upon localization at p.

[LMS, IV§5]

4. The double coset formula and its applications

This section summarizes results of Feshback that are generalized and given simpler proofs in [LMS]. Basically, they are consequences of the additivity on fibers of transfer maps. That result leads to decomposition theorems for the computation of the transfer associated to any stable bundle $\xi : Y \wedge_\Pi F_+ \longrightarrow Y/\Pi$, and we state these first. Since we must keep track of varying orbits, we write

$$\xi(\Lambda, \Gamma) : Y \wedge_\Pi (\Gamma/\Lambda)_+ \longrightarrow Y/\Pi$$

for the stable bundle associated to a Π-free Γ-spectrum Y and the Γ-space Γ/Λ, and we write $\tau(\Lambda, \Gamma)$ for the associated transfer map.

THEOREM 4.1. Let F be a finite Γ-CW complex and let

$$j_i : \Gamma/\Lambda_i \subset \Gamma/\Lambda_i \times D^{n_i} \longrightarrow F$$

be the composite of the inclusion of an orbit and the ith characteristic map for some enumeration of the cells of F. Then, for any Π-free Γ-spectrum Y,

$$\tau = \sum_i (-1)^{n_i} j_i \tau(\Lambda_i, \Gamma) : Y/\Pi \longrightarrow Y \wedge_\Pi F_+.$$

There is a more invariant decomposition that applies to a general compact Γ-ENR F. For $\Lambda \subset \Gamma$, we let $F^{(\Lambda)}$ be the subspace of points whose isotropy groups are conjugate to Λ. A path component M of the orbit space $F^{(\Lambda)}/\Gamma$ is called an orbit type component of F/Γ. If $\bar{M}$ is the closure of M in F/Γ and $\partial M = \bar{M} - M$, we defined the (nonequivariant) internal Euler characteristic $\chi(M)$ to be the reduced Euler characteristic of the based space $\bar{M}/\partial M$.

THEOREM 4.2. Let F be a compact Γ-ENR and let

$$j_M : \Gamma/\Lambda \subset M \subset F$$

be the inclusion of an orbit in the orbit type component M. Then, for any Π-free Γ-spectrum Y,

$$\tau = \sum_M \chi(M) j_M \tau(\Lambda, \Gamma) : Y/\Pi \longrightarrow Y \wedge_\Pi F_+.$$

While it is possible to deduce a double coset formula in something close to our full generality, we shall simplify the bookkeeping by restricting to the case when $\Gamma = G \times \Pi$, which is the case of greatest importance in the applications. Recall that a principal (G, Π)-bundle is the same thing as a Π-free $(G \times \Pi)$-space and let Y be a Π-free $(G \times \Pi)$-spectrum indexed on U^Π, where U is a complete $(G \times \Pi)$-universe. For a subgroup Λ of Π, we have the stable (G, Π)-bundle

$$\xi(\Lambda, \Pi) : Y/\Lambda \cong Y \wedge_\Pi (\Pi/\Lambda)_+ \longrightarrow Y/\Pi$$

with associated transfer map $\tau(\Lambda, \Pi)$.

THEOREM 4.3 (DOUBLE COSET FORMULA). Let Λ and Φ be subgroups of Π and let $\Lambda\backslash\Pi/\Phi$ be the double coset space regarded as the space of orbits under Λ of Π/Φ. Let $\{m\}$ be a set of representatives in Π for the orbit type component manifolds M of $\Lambda\backslash\Pi/\Phi$ and let $\chi(M)$ be the internal Euler characteristic of M in $\Lambda\backslash\Pi/\Phi$. Then, for any Π-free $(G\times\Pi)$-spectrum Y, the composite

$$Y/\Lambda \xrightarrow{\xi} Y/\Pi \xrightarrow{\tau} Y/\Phi$$

is the sum over M of $\chi(M)$ times the composite

$$Y/\Lambda \xrightarrow{\tau} Y/\Phi^m\cap\Lambda \xrightarrow{\xi} Y/\Phi^m \xrightarrow{c_m} Y/\Phi.$$

Here $\Phi^m = m\Phi m^{-1}$ and c_m is induced by the left Π-map $\Pi/\Phi^m \longrightarrow \Pi/\Phi$ given by right multiplication by m. In symbols,

$$\tau(\Phi,\Pi)\xi(\Lambda,\Pi) = \sum_M \chi(M)\, c_m \circ \xi(\Phi^m\cap\Lambda, \Phi^M)\circ\tau(\Phi^m\cap\Lambda,\Lambda).$$

Proof. The composite

$$\Lambda/\Phi^m\cap\Lambda \xrightarrow{\subset} \Pi/\Phi^m \xrightarrow{c_m} \Pi/\Phi$$

is a homeomorphism onto the double coset $\Lambda\, m\,\Phi$. Modulo a little diagram chasing and the use of change of groups, the conclusion follows directly from the previous theorem applied to $\xi(\Lambda,\Pi)$. □

If Φ has finite index in Π, then M is the point $\Lambda\, m\,\Phi$ and $\chi(M)=1$. Here the formula is of the same form as the classical double coset formula in the cohomology of groups. Observe that the formula depends only on the structure of the fibers and has the same form equivariantly as in the nonequivariant case $G=e$ (which is the case originally proven by Feshback, at least over compact base spaces).

The theorem is most commonly used for the study of classifying spaces, with $Y=\Sigma^\infty E(G,\Pi)_+$. Here $E(G,\Pi)/\Phi$ is a classifying G-space for principal (G,Φ)-bundles and the result takes the following form.

Corollary 4.4. The composite

$$\Sigma^\infty B(G,\Lambda)_+ \xrightarrow{\xi} \Sigma^\infty B(G,\Pi)_+ \xrightarrow{\tau} \Sigma^\infty B(G,\Phi)_+$$

is the sum over M of $\chi(M)$ times the composite

$$\Sigma^\infty B(G,\Lambda)_+ \xrightarrow{\tau} \Sigma^\infty B(G,\Phi^m\cap\Lambda)_+ \xrightarrow{\xi} \Sigma^\infty B(G,\Phi^m)_+ \xrightarrow{c_m} \Sigma^\infty B(G,\Phi)_+.$$

Of course, the formula is very complicated in general. However, many terms simplify or disappear in special cases. For example, if the group $W\Phi = N\Phi/\Pi$ is infinite, then the transfer $\tau(\Phi, \Pi)$ is trivial. This observation and a little book-keeping, lead to the following examples where the formula reduces to something manageable.

COROLLARY 4.5. Let Y be any Π-free $(G \times \Pi)$-spectrum.

(i) If N is the normalizer of a maximal torus T in Π, then

$$\tau(N, \Pi)\xi(T, \Pi) = \xi(T, N) : Y/T \longrightarrow Y/N.$$

(ii) If T is a maximal torus in Π, then

$$\tau(T, \Pi)\xi(T, \Pi) = \sum c_m : Y/T \longrightarrow Y/T,$$

where the sum ranges over a set of coset representatives for the Weyl group $W = WT$ of Π.

(iii) If Λ is normal and of finite index in Π, then

$$\tau(\Lambda, \Pi)\xi(\Lambda, \Pi) = \sum c_m : Y/\Lambda \longrightarrow Y/\Lambda,$$

where the sum runs over a set of coset representatives for Π/Λ.

Typically, the double coset formula is applied to the computation of $E^*_G(Y/\Pi)$ in terms of $E^*_G(Y/\Phi)$ for a subgroup Φ. Here it is used in combination with the Euler characteristic formula of Corollary 3.6 and the unit criteria of Theorem 3.7. We need a definition to state the conclusions.

DEFINITION 4.6. An element $x \in E^*_G(Y/\Phi)$ is said to be stable if

$$\xi(\Phi \cap \Phi^m, \Phi)^*(x) = \xi(\Phi \cap \Phi^m, \Phi^m)^* c^*_m(x)$$

for all $m \in \Pi$. Let $E^*_G(Y/\Phi)^S$ denote the set of stable elements and observe that $\operatorname{Im} \xi(\Phi, \Pi)^* \subset E^*_G(Y/\Phi)^S$ since $\xi(\Phi, \Pi) \circ c_m = \xi(\Phi^m, \Pi)$.

The double coset and Euler characteristic formulas have the following direct implication.

THEOREM 4.7. Let X be a Π-free $(G\times\Pi)$-space and let E be a ring G-spectrum. Let $\Phi \subset \Pi$ and consider $\xi = \xi(\Phi, \Pi)$. If $\chi(\xi) \in E^0_G(X/\Pi_+)$ is a unit, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/\Phi_+)^S$$

is an isomorphism.

Unfortunately, only the first criterion of Theorem 3.7 applies to equivariant classifying spaces, and more work needs to be done on this. However, we have the following application of its last two criteria, and the nonequivariant case $G = e$ gives considerable information about nonequivariant characteristic classes.

THEOREM 4.8. Let X be a Π-free $(G\times\Pi)$-space and let E be a ring G-spectrum. Assume further that X/Π is either G-connected with trivial fiber representation $G \longrightarrow \Pi$ at any fixed point or G-free.

(i) If N is the normalizer of a maximal torus in Π, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/N_+)^S$$

is an isomorphism.

(ii) If $N(p)$ is the inverse image in the normalizer of a maximal torus T of a p-Sylow subgroup of the Weyl group $W = WT$ and E is p-local, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/N(p)_+)^S$$

is an isomorphism.

(iii) If T is a maximal torus in Π and E is local away from the order of the Weyl group $W = WT$, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/T_+)^W$$

is an isomorphism.

(iv) If Φ is normal and of finite index in Π and E is local away from $|\Pi/\Phi|$, then

$$\xi^* : E^*_G(X/\Pi_+) \longrightarrow E^*_G(X/\Phi_+)^{\Pi/\Phi}$$

is an isomorphism.

It is essential here that we are looking at theories represented by local spectra and not at theories obtained by algebraically localizing theories represented by general spectra. The point is that if F is the localization of a spectrum E at a set of primes T, then $F^*_G(X)$ is usually not isomorphic to $E^*_G(X)\otimes\mathbb{Z}_T$ unless X is a finite G-CW complex. The proof of the unit criteria makes use of the wedge axiom, which is not satisfied by the algebraically localized theories.

M. Feshbach. The transfer and compact Lie groups. Trans. Amer. Math. Soc. 251(1979), 139-169.
[LMS, IV§6]

5. Transitivity of the transfer

While a transitivity relation can be formulated and proven in our original general context of extensions of compact Lie groups, we shall content ourselves with its statement in the classical context of products $G \times \Pi$. We suppose given compact Lie groups G, Π, and Φ and a complete $(G \times \Pi \times \Phi)$-universe U'. Then $U = (U')^{\Phi}$ is a complete $(G \times \Pi)$-universe and $U^{\Pi} = (U')^{\Pi \times \Phi}$ is a complete G-universe.

We shall consider transitivity for stable bundles that are built up from bundles of fibers. Let P be a Φ-free finite $(\Pi \times \Phi)$-CW complex with orbit space $K = P/\Phi$ and let J be any finite Φ-CW complex. Let $F = P \times_{\Phi} J$. The resulting bundle $\zeta : F \longrightarrow K$ is to be our bundle of fibers. Here F and K are finite Π-CW complexes and ζ is a (Π, Φ)-bundle with fiber J. By pullback, we may regard ζ as a $(G \times \Pi, \Phi)$-bundle. With these hypotheses, we have a transitivity relation for pretransfers that leads to a transitivity relation for stable G-bundles. It is proven by using additivity and naturality to reduce to the case when P is an orbit and then using a change of groups argument.

THEOREM 5.1. The following diagram of $(G \times \Pi \times \Phi)$-spectra commutes:

$$\begin{tikzcd} & S \arrow[dl, "\tau(K)"'] \arrow[dr, "\tau(F)"] & \\ \Sigma^\infty K_+ \arrow[rr, "\tau(\zeta)"'] & & \Sigma^\infty F_+. \end{tikzcd}$$

THEOREM 5.2. Let Y be a Π-free $(G \times \Pi)$-spectrum indexed on U^{Π}. Observe that the G-map $\mathrm{id} \wedge_{\Pi} \zeta : D \wedge_{\Pi} F_+ \longrightarrow D \wedge_{\Pi} K_+$ is a stable $(G, \Pi \times \Phi)$-bundle with fiber J and consider the following commutative diagram of stable G-bundles:

$$\begin{tikzcd} Y \wedge_\Pi F_+ \arrow[rr, "\mathrm{id} \wedge_\Pi \zeta"] \arrow[dr, "\xi"'] & & Y \wedge_\Pi K_+ \arrow[dl, "\xi'"] \\ & Y/\Pi. & \end{tikzcd}$$

The following diagram of G-spectra commutes:

$$\begin{tikzcd} & Y/\Pi \arrow[dl, "\tau(\xi')"'] \arrow[dr, "\tau(\xi)"] & \\ Y \wedge_\Pi K_+ \arrow[rr, "\tau(\mathrm{id} \wedge_\Pi \zeta)"'] & & Y \wedge_\Pi F_+. \end{tikzcd}$$

The special case $P = \Pi$ is of particular interest. It gives transitivity for the diagram of transfers associated to the commutative diagram

$$\begin{tikzcd}
Y \wedge_{\Pi} (\Pi \times_{\Phi} J)_{+} \arrow[r, "\cong"] \arrow[d] & Y \wedge_{\Phi} J_{+} \arrow[d] \\
Y/\Pi & Y/\Phi. \arrow[l, "\xi(\Phi,\Pi)"']
\end{tikzcd}$$

[LMS, IV§7]

CHAPTER XIX

Stable homotopy and Mackey functors

1. The splitting of equivariant stable homotopy groups

One can reprove the isomorphism $A(G) \cong \pi_0^G(S)$ by means of the following important splitting theorem for the stable homotopy groups of G-spaces in terms of nonequivariant stable homotopy groups. When G is finite, we shall see that this result provides a bridge connecting the equivariant and non-equivariant versions of the Segal conjecture. Recall that $Ad(G)$ denotes the adjoint representation of G. Remember that our homology theories, including π_*, are understood to be reduced.

THEOREM 1.1. For based G-spaces Y, there is a natural isomorphism

$$\pi_*^G(Y) \cong \sum_{(H)\in\Gamma G} \pi_*(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H).$$

Observe that the sum ranges over all conjugacy classes, not just the conjugacy classes $(H) \in \Phi G$. However, WH is finite if and only if $Ad(WH) = 0$, and $EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H$ is connected if $Ad(WH) \neq 0$.

COROLLARY 1.2. For based G-spaces Y, there is a natural isomorphism

$$\pi_0^G(Y) \cong \sum_{(H)\in\Phi G} H_0(WH; \pi_0(Y^H)).$$

With $Y = S^0$, this is consistent with the statement that $A(G)$ is $\mathbb{Z}$-free on the basis $\{[G/H] \,|\, (H) \in \Phi G\}$. We shall come back to this point in the discussion of Mackey functors in Section 3. Theorem 1.1 implies a description of the G-fixed point spectra of equivariant suspension spectra.

THEOREM 1.3. For based G-spaces Y, there is a natural equivalence

$$(\Sigma^\infty Y)^G \simeq \bigvee_{(H)\in\Gamma G} \Sigma^\infty(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H).$$

Here the suspension spectrum functors are $\Sigma^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U$ on the left and $\Sigma^\infty : \mathscr{T} \longrightarrow \mathscr{S}U^G$ on the right, where U is a fixed complete G-universe. Actually, the most efficient proof seems to be to first write down an explicit map

$$\theta = \sum \theta_H : \sum \pi_*(EWH_+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H) \longrightarrow \pi_*^G(Y)$$

of homology theories in Y and use it to prove Theorem 1.1 and then write down an explicit map

$$\xi = \sum \xi_H : \bigvee \Sigma^\infty(EWH_+ \wedge_W H\Sigma^{Ad(WH)} Y^H) \longrightarrow (\Sigma^\infty Y)^G$$

of spectra and prove by a diagram chase that the map induced on homotopy groups by the wedge summand ξ_H is the same as the map induced by the summand θ_H. We shall first write down these maps and then say a little about the proofs.

Since the definitions of our maps proceed one H at a time, we abbreviate notation by writing:

$$N = NH, \quad W = WH, \quad E = EWH, \quad \text{and} \quad A = Ad(WH).$$

We let L be the tangent N-representation at the identity coset of G/N. A Lie theoretic argument shows that $(G/N)^H$ is a single point, and this implies that $L^H = \{0\}$. Now θ_H is defined by the following commutative diagram:

$$\begin{tikzcd}
\pi_*(E_+ \wedge_W \Sigma^A Y^H) \arrow[r, "\alpha"] \arrow[d, "\theta_H"'] & \pi_*^W(E_+ \wedge Y^H) \arrow[r, "\lambda"] & \pi_*^N(\Sigma^L(E_+ \wedge Y)) \arrow[d, "\omega"] \\
\pi_*^G(Y) & \pi_*^G((G \times_N E)_+ \wedge Y)) \arrow[l, "(\rho\wedge\mathrm{Id})_*"] & \pi_*^G(G_+ \wedge_N (E_+ \wedge Y)). \arrow[l, "\zeta_*"]
\end{tikzcd}$$

Here α is an instance of the Adams isomorphism of XVI.5.4, ω is an instance of the Wirthmüller isomorphism of XVI.4.9, ζ_* is induced by a canonical isomorphism of G-spectra, $\rho : (G \times_N E)_+ \longrightarrow S^0$ is the collapse map, and λ is the composite of the map $\pi_*^W \longrightarrow \pi_*^N$ obtained by regarding W-maps as H-fixed N-maps and the map induced by the inclusion of fixed point spaces

$$E_+ \wedge Y^H = (\Sigma^L(E + \wedge Y))^H \longrightarrow \Sigma^L(E_+ \wedge Y).$$

Why is the sum θ of the θ_H an isomorphismΓ Clearly θ is a map of homology theories in Y. Recall the spaces $E(\mathscr{F}', \mathscr{F})$ defined in V.4.6 for inclusions of families

$\mathscr{F} \subset \mathscr{F}'$. For a homology theory E_* on G-spaces (or G-spectra), we define the associated homology theory concentrated between $\mathscr{F}$ and $\mathscr{F}'$ by

$$E[\mathscr{F}', \mathscr{F}]_*(X) = E_*(X \wedge E(\mathscr{F}', \mathscr{F})).$$

We say that $(\mathscr{F}', \mathscr{F})$ is an adjacent pair if $\mathscr{F}' - \mathscr{F}$ consists of a single conjugacy class of subgroups. One can check, using an easy transfinite induction argument in the compact Lie case, that a map of homology theories is an isomorphism if the associated maps of homology theories concentrated between adjacent families are all isomorphisms.

Returning to θ, consider an adjacent pair of families with $\mathscr{F}' - \mathscr{F} = (H)$. We find easily that $EWJ_+ \wedge E(\mathscr{F}', \mathscr{F})$ is WJ-contractible unless $(H) = (J)$. Therefore, when we concentrate our theories between $\mathscr{F}$ and $\mathscr{F}'$, all of the summands of the domain of θ vanish except the domain of θ_H. It remains to prove that θ_H is an isomorphism when Y is replaced by $Y \wedge E(\mathscr{F}', \mathscr{F})$. We claim that each of the maps in the diagram defining θ_H is then an isomorphism, and three of the five are always isomorphisms. It is easy to see that $(G \times_N E)^H = E^H$, which is a contractible space. Since $E(\mathscr{F}', \mathscr{F})^J$ is contractible unless $(J) = (H)$, the Whitehead theorem implies that $\rho \wedge \mathrm{Id}$ is a G-homotopy equivalence.

It only remains to consider λ. Passage to H-fixed points on representative maps gives a homomorphism

$$\phi : \pi_*^N(\Sigma^L(E_+ \wedge Y \wedge E(\mathscr{F}', \mathscr{F})) \longrightarrow \pi_*^W(E_+ \wedge Y^H \wedge E(\mathscr{F}', \mathscr{F})^H)$$

such that $\phi \circ \lambda = \mathrm{Id}$. It suffices to show that ϕ is an isomorphism. As an N-space, $E(\mathscr{F}', \mathscr{F})$ is $E(\mathscr{F}'|N, \mathscr{F}|N)$. While $(\mathscr{F}'|N, \mathscr{F}|N)$ need not be an adjacent pair, $\mathscr{F}'|N - \mathscr{F}|N$ is the disjoint union of N-conjugacy classes (K), where the K are G-conjugate to H. It follows that $E(\mathscr{F}'|N, \mathscr{F}|N)$ is N-equivalent to a wedge of spaces $E(\mathscr{E}', \mathscr{E})$, where each $(\mathscr{E}', \mathscr{E})$ is an adjacent pair with $\mathscr{E}' - \mathscr{E} = (K)$ for some such K. However, it is easy to see that $E_+ \wedge E(\mathscr{E}', \mathscr{E})$ is N-contractible unless the N-conjugacy classes (H) and (K) are equal. Thus only the wedge summand $E(\mathscr{E}', \mathscr{E})$ with $\mathscr{E}' - \mathscr{E} = (H)$ contributes to the source and target of ϕ. Here $(H) = \{H\}$ since H is normal in N. A check of fixed points shows that $E(\mathscr{E}', \mathscr{E})^H$ is W-equivalent to E_+.

We now claim more generally that

$$\phi : \pi_*^N(Y \wedge E(\mathscr{E}', \mathscr{E})) \longrightarrow \pi_*^W(Y^H \wedge E(\mathscr{E}', \mathscr{E})^H) = \pi_*^W(Y^H \wedge E_+)$$

is an isomorphism for any N-CW complex Y. Writing out both sides as colimits of space level homotopy classes of maps, we see that it suffices to check that

$$\phi : [X, Y \wedge E(\mathscr{E}', \mathscr{E})]_N \longrightarrow [X^H, Y^H \wedge E_+]_W$$

is a bijection for any N-CW complex X. By easy cofibration sequence arguments, we may assume that all isotropy groups of X (except at its basepoint) are in $\mathscr{E}' - \mathscr{E} = \{H\}$. This uses the fact that the set $X_{\mathscr{E}}$ of points of X with isotropy group not in $\mathscr{E}$ is a subcomplex: we first show that X can be replaced by $X/X_{\mathscr{E}'}$, which has isotropy groups in $\mathscr{E}'$, and we then show that this new X can be replaced by $X_{\mathscr{E}}$, which has isotropy groups in $\mathscr{E}' - \mathscr{E}$. Under this assumption, $X = X^H$ and the conclusion is obvious.

Retaining our abbreviated notations, we next describe the map

$$\xi_H : \Sigma^\infty(E_+ \wedge_W \Sigma^A Y^H) \longrightarrow (\Sigma^\infty Y)^G.$$

This is a map of spectra indexed on U^G, and it suffices to describe its adjoint map of G-spectra indexed on U:

$$\tilde{\xi}_H : \Sigma^\infty(E_+ \wedge_W \Sigma^A Y^H) \longrightarrow \Sigma^\infty Y.$$

Here we regard $E_+ \wedge_W \Sigma^A Y^H$ as a G-trivial G-space, and the relevant suspension spectrum functor is $\Sigma^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U$ on both left and right. Suppressing notation for Σ^∞, implicitly using certain commutation relations between Σ^∞ and other functors, and abbreviating notation by setting $Z = E_+ \wedge_W \Sigma^A Y^H$, we define $\tilde{\xi}_H$ to be the composite displayed in the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rr, "\tau\wedge \mathrm{Id}"] \arrow[d, "\xi_H"'] & & G/N_+ \wedge Z \arrow[r, "\cong"] & G_+ \wedge_N Z \arrow[d, "\mathrm{Id}\wedge\tau"] \\
Y & (G \times NE)_+ \wedge Y \arrow[l, "\rho\wedge \mathrm{Id}"] & G_+ \wedge_N (E_+ \wedge Y) \arrow[l, "\zeta"] & G_+ \wedge_N (E_+ \wedge Y^H) \arrow[l, "\supset"]
\end{tikzcd}$$

On the top line, τ is the transfer stable G-map $S^0 \longrightarrow G/N_+$ of IX.3.4 (or XVII.1.2). At the right, $\tau : E_+ \wedge_W \Sigma^A Y^H \longrightarrow E_+ \wedge Y^H$ is the stable N-map obtained by applying $i_* : W\mathscr{S}U^H \longrightarrow N\mathscr{S}U$, $i : U^H \longrightarrow U$, to the dimension-shifting transfer W-map of XVIII.1.2 that is at the heart of the Adams isomorphism that appears in the definition of θ_H. A diagram chase shows that the map on homotopy groups induced by ξ_H coincides with θ_H, and the wedge sum of the ξ_H is therefore an equivalence.

T. tom Dieck. Orbittypen und äquivariante Homologie. I. Arch. Math. 23(1972), 307-317.

T. tom Dieck. Orbittypen und äquivariante Homologie. II. Arch. Math. 26(1975), 650-662.
[LMS, V§§10-11]

2. Generalizations of the splitting theorems

We here formulate generalizations of Theorems 1.1 and 1.3 that are important in the study of generalized versions of the Segal conjecture. The essential ideas are the same as those just sketched, but transfer maps of bundles enter into the picture and the bookkeeping needed to define the relevant maps and prove that the relevant diagrams commute is quite complicated. We place ourselves in the context in which we studied generalized equivariant bundles in VII§1. Thus let Π be a normal subgroup of a compact Lie group Γ with quotient group G. Let $E(\Pi;\Gamma)$ be the universal $(\Pi;\Gamma)$-bundle of VII.2.1. Let $Ad(\Pi;\Gamma)$ denote the adjoint representation of Γ derived from Π; it is the tangent space of Π at e with the action of Γ induced by the conjugation action of Γ on Π. We regard G-spaces as Γ-spaces by pullback. For based Γ-spaces X and Y, we write

$$\{X,Y\}_n^\Gamma = [\Sigma^n\Sigma^\infty X, \Sigma^\infty Y]_\Gamma$$

for integers n. With these notations, we have the following results.

THEOREM 2.1. Let X be a based G-space and Y be a based Γ-space. Assume either that X is a finite G-CW complex or that Π is finite. Then $\{X,Y\}_*^\Gamma$ is naturally isomorphic to the direct sum over the Γ-conjugacy classes of subgroups Λ of Π of the groups

$$\{X, E(W_\Pi\Lambda; W_\Gamma\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda;W_\Gamma\Lambda)}Y^\Lambda\}_*^{W_\Gamma\Lambda/W_\Pi\Lambda}.$$

Here the quotient homomorphism $\Gamma \longrightarrow G$ induces an inclusion of $W_\Gamma\Lambda/W_\Pi\Lambda$ in G and so fixes an action of this group on X. Of course, when G is finite, the adjoint representations in the theorem are all zero. If we set $\Pi = \Gamma$ (and rename it G), then the theorem reduces to a mild generalization of Theorem 1.1. When Π is finite, the specified sum satisfies the wedge axiom. In general, the sum is infinite and we must restrict to finite G-CW complexes X.

THEOREM 2.2. For based Γ-spaces Y, there is a natural equivalence of G-spectra from $(\Sigma^\infty Y)^\Pi$ to the wedge over the Γ-conjugacy classes of subgroups Λ of Π of the suspension spectra of the G-spaces

$$G_+ \wedge_{W_\Gamma\Lambda/W_\Pi\Lambda} (E(W_\Pi\Lambda; W_\Gamma\Lambda)_+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda;W_\Gamma\Lambda)}Y^\Lambda).$$

Here the suspension spectrum functor applied to Y is $\Sigma^\infty : \Gamma\mathscr{T} \longrightarrow \Gamma\mathscr{S}U$ and that applied to the wedge summands is $\Sigma^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U^\Pi$, where U is a complete Γ-universe.

[LMS, V§§10-11]

3. Equivalent definitions of Mackey functors

In IX§4, we defined a Mackey functor to be an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$, and we have observed that the Burnside category $\mathscr{B}_G$ is just the full subcategory of the stable category whose objects are the orbit spectra $\Sigma^\infty G/H_+$, but with objects denoted G/H. This is the appropriate definition of a Mackey functor for general compact Lie groups, but we show here that it is equivalent to an older, and purely algebraic, definition when G is finite. We first describe the maps in $\mathscr{B}_G$. As observed in IX§4, their composition is hard to describe in general. However, for finite groups G, there is a conceptual algebraic description. In fact, in this case there is an extensive literature on the algebraic theory of Mackey functors, and we shall say just enough to be able to explain the important idea of induction theorems in the next section.

When we specialize the diagram-chasing needed for the proofs in Section 1 to the calculation of $\pi_0^G(Y)$, we arrive at the following simple conclusion. Recall Corollary 1.2.

Proposition 3.1. For any based G-space Y, $\pi_0^G(Y)$ is the free Abelian group generated by the following composites, where (H) runs over ΦG and y runs over a representative point in Y^H of each non-basepoint component of Y^H/WH:

$$S \xrightarrow{\tau} \Sigma^\infty G/H_+ \xrightarrow{\Sigma^\infty \tilde{y}} \Sigma^\infty Y;$$

here τ is the transfer and $\tilde{y} : G/H_+ \longrightarrow Y$ is the based G-map such that $\tilde{y}(eH) = y$.

There is a useful conceptual reformulation of this calculation. Since we are interested in orbits G/H, we switch to unbased G-spaces.

Corollary 3.2. Let X be an unbased G-space. For $H \subset G$, the group

$$\pi_0^H(X_+) = [\Sigma^\infty G/H_+, \Sigma^\infty X_+]_G$$

is isomorphic to the free Abelian group generated by the equivalence classes of diagrams of space level G-maps

$$G/H \xleftarrow{\phi} G/K \xrightarrow{\chi} X,$$

where $K \subset H$ and $W_H K$ is finite. Here (ϕ, χ) is equivalent to (ϕ', χ') if there is a G-homeomorphism $\xi : G/K \longrightarrow G/K'$ such that the following diagram is G-homotopy commutative:

$$\begin{tikzcd} & G/K \arrow[dl, "\phi"'] \arrow[dd, "\xi"] \arrow[dr, "\chi"] & \\ G/H & & X \\ & G/K' \arrow[ul, "\phi'"] \arrow[ur, "\chi'"'] & \end{tikzcd}$$

We are thinking of ϕ as the corresponding transfer map $\Sigma^\infty G/H_+ \longrightarrow \Sigma^\infty G/K_+$, namely $G \ltimes_H (\tau)$, where $\tau : S^0 \longrightarrow \Sigma^\infty H/K_+$ is the transfer H-map.

This result specializes to give a good description of the maps of $\mathscr{B}_G$. In principle, their composition can be described in terms of a double coset formula, but this is quite hard to compute with in general. However, when G is finite, it admits an attractive conceptual reformulation.

To see this, let $\hat{\mathscr{B}}_G$ be the category whose objects are the finite G-sets and whose morphisms are the stable G-maps $X_+ \longrightarrow Y_+$. That is, up to an abbreviated notation for objects, $\hat{\mathscr{B}}_G$ is the full subcategory of the stable category whose objects are the $\Sigma^\infty X_+$ for finite G-sets X. Clearly $\mathscr{B}_G$ embeds as a full subcategory of $\hat{\mathscr{B}}_G$, and every object of $\hat{\mathscr{B}}_G$ is a disjoint union of objects of $\mathscr{B}_G$. We easily find that maps in $\hat{\mathscr{B}}_G$ can be described as equivalence classes $[\phi, \chi]$ of pairs (ϕ, χ), exactly as in the previous corollary, but now the composite of maps

$$V \xleftarrow{\phi} W \xrightarrow{\chi} X \quad \text{and} \quad X \xleftarrow{\psi} Y \xrightarrow{\omega} Z$$

can be specified as the equivalence class of the diagram

$$\begin{tikzcd} & & P \arrow[dl] \arrow[dr] & & \\ & W \arrow[dl, "\phi"'] \arrow[dr, "\xi"] & & Y \arrow[dl, "\psi"'] \arrow[dr, "\omega"] & \\ V & & X & & Z, \end{tikzcd}$$

where the top square is a pullback. This gives a complete description of $\hat{\mathscr{B}}_G$ in purely algebraic terms, with disjoint unions thought of as direct sums. It is important, and obvious, that this category is abstractly self-dual. Moreover, the duality isomorphism is given topologically by Spanier-Whitehead duality on orbits.

Since an additive functor necessarily preserves any finite direct sums in its domain, it is clear that an additive contravariant functor $\mathscr{B}_G \longrightarrow \mathscr{A}b$ determines and is determined by an additive contravariant functor $\hat{\mathscr{B}}_G \longrightarrow \mathscr{A}b$. In turn, as a matter of algebra, an additive contravariant functor $\hat{\mathscr{B}}_G \longrightarrow \mathscr{A}b$ determines and is determined by a Mackey functor in the classical algebraic sense. Precisely, such a Mackey functor M consists of a contravariant functor M^* and a covariant functor M_* from finite G-sets to Abelian groups. These functors have the same object function, denoted M, and M converts disjoint unions to direct sums. Writing $M^*\alpha = \alpha^*$ and $M_*\alpha = \alpha_*$, it is required that $\alpha^* \circ \beta_* = \delta_* \circ \gamma^*$ for pullback diagrams of finite G-sets

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & Z. \end{tikzcd}$$

For an additive contravariant functor $M : \hat{\mathscr{B}}_G \longrightarrow \mathscr{A}b$, the maps $M[\phi, 1]$ and $M[1, \chi]$ specify the covariant and contravariant parts ϕ^* and χ_* of the corresponding algebraic Mackey functor, and conversely.

[LMS, V§9]

4. Induction theorems

Assuming that G is finite, and working with the algebraic notion of a Mackey functor just defined, we now consider the problem of computing $M(*)$, where $* = G/G$, in terms of the $M(G/H)$ for proper subgroups H. For a finite G-set X, let X^n be the product of n copies of X and let $\pi_i : X^{n+1} \longrightarrow X^n$ be the projection that omits the ith variable. We then have the chain complex

$$(*) \qquad 0 \longrightarrow M(*) \longrightarrow M(X) \longrightarrow M(X^2) \longrightarrow \cdots,$$

where the differential $d^n : M(X^n) \longrightarrow M(X^{n+1})$ is the alternating sum of the maps $(\pi_i)^*$, $0 \leq i \leq n$. Let $M(X)^{inv}$ be the kernel of d^1, namely the equalizer of $(\pi_0)^*$ and $(\pi_1)^*$. We are interested in determining when the resulting map $M(*) \longrightarrow M(X)^{inv}$ is an isomorphism. Of course, this will surely hold if the sequence $(*)$ is exact. We have already seen an instance of this kind of argument in XVII§6.

When is (*) exactΓ Let M_X be the Mackey functor that sends a finite G-set Y to $M(X \times Y)$, and similarly for maps. The projections $\pi : X \times Y \longrightarrow Y$ induce a map of Mackey functors $\theta^X : M \longrightarrow M_X$. We say that M is "X-injective" if θ^X is a split monomorphism. If θ^X is split by $\psi : M_X \longrightarrow M$, so that $\psi \circ \theta_X = \mathrm{Id}$, then the homomorphisms

$$\psi(X^n) : M(X \times X^n) \longrightarrow M(X^{n+1})$$

specify a contracting homotopy for (*). Therefore (*) is exact if M is X-injective.

When is M X-injectiveΓ To obtain a good criterion, we must first specify the notion of a pairing $\mu : L \times M \longrightarrow N$ of Mackey functors. This consists of maps $\mu : L(X) \otimes M(X) \longrightarrow N(X)$ for finite G-sets X such that the evident covariant and contravariant functoriality diagrams and the following Frobenius diagram commute for a G-map $f : X \longrightarrow Y$.

$$\begin{tikzcd}
L(X) \otimes M(Y) \arrow[rr, "f_* \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes f^*"'] & & L(Y) \otimes M(Y) \arrow[d, "\mu"] \\
L(X) \otimes M(X) \arrow[r, "\mu"'] & N(X) \arrow[r, "f_*"'] & N(Y).
\end{tikzcd}$$

A Green functor is a Mackey functor R together with a pairing μ that makes each $R(X)$ a commutative and associative unital ring, the maps f^* being required to preserve units and thus to be ring homomorphisms. The notion of a module M over a Green functor R is defined in the evident way. With these notions, one can prove the following very useful general fact.

Proposition 4.1. If R is a Green functor and the projection $X \longrightarrow *$ induces an epimorphism $R(X) \longrightarrow R(*)$, then every R-module M is X-injective. Therefore $M(*) \cong M(X)^{inv}$ for every R-module M.

For a Green functor R, there is a unique minimal set $\{(H)\}$ of conjugacy classes of subgroups of G such that $R(\coprod G/H) \longrightarrow R(*)$ is an epimorphism; this set is called the "defect set" of R. By an "induction theorem", we understand an identification of the defect set of a Green functor. For example, the complex representation rings $R(H)$ are the values on G/H of a Green functor $\underline{R}$, and the "Brauer induction theorem" states that the set of products $P \times C$ of a p-group P and a cyclic group C in G contains a defect set of $\underline{R}$. We will shortly give another example, one that we will use later to reduce the generalized Segal conjecture to the case of finite p-groups.

We must first explain the relationship of Burnside rings to Mackey functors. For a finite G-set X, we have a Grothendieck ring $\underline{A}(X)$ of isomorphism classes of G-sets over X. The multiplication is obtained by pulling Cartesian products back along the diagonal of the base G-set X. When $X = *$, this is the Burnside ring $A(G)$. More generally, a G-set $\alpha : T \longrightarrow G/H$ over G/H determines and is determined by the H-set $\alpha^{-1}(eH)$, and it follows that $\underline{A}(G/H) \cong A(H)$. A G-map $f : X \longrightarrow Y$ determines $f^* : \underline{A}(Y) \longrightarrow \underline{A}(X)$ by pullback along f, and it determines $f_* : \underline{A}(X) \longrightarrow \underline{A}(Y)$ by composition with f. In more down to earth terms, if $f : G/H \longrightarrow G/K$ is the G-map induced by an inclusion $H \subset K$, then $f^* : A(K) \longrightarrow A(H)$ sends a K-set to the same set regarded as an H-set and $f_* : A(H) \longrightarrow A(K)$ sends an H-set X to the K-set $K \times_H X$; we call f_* induction. This gives the Burnside Green functor $\underline{A}$.

Any Mackey functor M is an $\underline{A}$-module via the pairings

$$\underline{A}(X) \otimes M(X) \longrightarrow M(X)$$

that send $\alpha \otimes m$, $\alpha : Y \longrightarrow X$, to $\alpha_*\alpha^*(m)$. Therefore, by pullback along the ring map $A(G) = \underline{A}(*) \longrightarrow \underline{A}(X)$, each $M(X)$ is an $A(G)$-module. Any Green functor R has a unit map of Green functors $\eta : \underline{A} \longrightarrow R$ that sends $\alpha : Y \longrightarrow X$ to $\alpha_*\alpha^*(1)$. Thus we see that the Burnside Green functor plays a universal role.

Observe that we can localize Mackey functors termwise at any multiplicative subset S of $A(G)$. We can complete Mackey functors that are termwise finitely $A(G)$-generated at any ideal $I \subset A(G)$. We wish to establish an induction theorem applicable to such localized and completed Mackey functors. This amounts to determination of the defect set of $S^{-1}\underline{A}^{\wedge}_I$.

It is useful to use a little commutative algebra. The following observation is standard algebra, but its relevance to the present question was noticed in work of Adams, Haeberly, Jackowski, and myself and its extension by Haeberly. We shall state it for pro-modules — which are just inverse systems of modules — but only actual modules need be considered at the moment. Its pro-module version will be used in the proof of the generalized Segal conjecture in XX§§2, 3. Localizations of completions of pro-modules $\mathbf{M} = \{M_\alpha\}$ are understood to be inverse systems

$$S^{-1}\mathbf{M}^{\wedge}_I = \{S^{-1}M_\alpha / I^r M_\alpha\}.$$

LEMMA 4.2. Let $\mathbf{M}$ be a pro-finitely generated module over a commutative ring A, let S be a multiplicative subset of A, and let I be an ideal of A. Then $S^{-1}\mathbf{M}^{\wedge}_I$

is pro-zero if and only if $(S_P)^{-1}\mathbf{M}_{\hat{P}}$ is pro-zero for every prime ideal P such that $P \cap S = \emptyset$ and $P \supset I$, where S_P is the multiplicative subset $A - P$.

For a prime ideal P of $A(G)$, we let $K(P)$ be a maximal element of the set $\{H | P = q(H,p)\}$. We have discussed these subgroups in XVII§3.

LEMMA 4.3. $\{(K(P))\}$ is the defect set of the Green functor $(S_P)^{-1}\underline{A}$.

PROOF. Essentially this result was observed, in less fancy language, at the end of XVII§6. The subgroup $K = K(P)$ is characterized by $P = q(K,p)$ and $|WK| \not\equiv 0 \bmod p$. (We allow $p = 0$.) The composite

$$A(G) \longrightarrow A(K) \longrightarrow A(G)$$

of restriction and induction is multiplication by $[G/K]$. Since this element of $A(G)$ maps to a unit in $A(G)_{q(K,p)}$, the displayed composite becomes an isomorphism upon localization at $q(K,p)$. □

PROPOSITION 4.4. Let S be a multiplicative subset of $A(G)$ and let I be an ideal of $A(G)$. Then the defect set of the Green functor $S^{-1}\underline{A}_{\hat{I}}$ is

$$\{(K(P)) | P \cap S = \emptyset \ \text{ and } \ P \supset I\}.$$

PROOF. The statement means that the sum of transfer maps

$$\sum S^{-1}A(K(P))_{\hat{I}} \longrightarrow S^{-1}A(G)_{\hat{I}}$$

is an epimorphism, and Lemmas 4.2 and 4.3 imply that its cokernel is zero. □

The starting point for arguments like this was the following result of McClure and myself, which is the special case when $S = \{1\}$ and I is the augmentation ideal (alias $q(e,0)$). If $P = q(e,p)$, then $K(P)$ is a p-Sylow subgroup of G.

COROLLARY 4.5. If I is the augmentation ideal of $A(G)$, then the defect set of the Green functor $\underline{A}_{\hat{I}}$ is the set of p-Sylow subgroups of G.

This will be applied in conjunction with the following observation.

LEMMA 4.6. Let M be a Mackey functor over a finite p-group G and let $\pi^* : M(*) \longrightarrow M(G)$ be induced by the projection $G \longrightarrow *$. Then the p-adic and I-adic topologies coincide on $\mathrm{Ker}(\pi^*)$.

PROOF. Since multiplication by $[G]$ is the composite $\pi_*\pi^*$, $[G]\operatorname{Ker}(\pi^*) = 0$. Since $[G] - |G| \in I$, $|G|\operatorname{Ker}(\pi^*) \subset I\operatorname{Ker}(\pi^*)$. If $H \neq e$, then $\phi_H([G/K] - |G/K|)$ is divisible by p because $G/K - (G/K)^H$ is a disjoint union of non-trivial H-orbits. Therefore $\phi(I) \subset pC(G)$. Let $|G| = p^n$. Since $|G|C(G) \subset \phi(A(G))$, we see that $\phi(I^{n+1}) \subset p\phi(I)$ and thus $I^{n+1} \subset pI$. The conclusion follows. $\square$

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5. Splittings of rational G-spectra for finite groups G

We here analyze the rational equivariant stable category for finite groups G. The essential point is that any rational G-spectrum splits as a product of Eilenberg-MacLane G-spectra $K(M,n) = \Sigma^n HM$.

THEOREM 5.1. Let G be finite. Then, for rational G-spectra X, there is a natural equivalence $X \longrightarrow \prod K(\underline{\pi}_n(X), n)$.

There is something to prove here since the counterexamples of Triantafillou discussed in III§3 show that, unless G is cyclic of prime power order, the conclusion is false for naive G-spectra. A counterexample of Haeberly shows that the conclusion is also false for genuine G-spectra when G is the circle group, the rationalization of KU_G furnishing a counterexample. Greenlees has recently studied what does happen for general compact Lie groups.

The proof of Theorem 5.1 depends on two facts, one algebraic and one topological. We assume that G is finite in the rest of this section.

PROPOSITION 5.2. In the Abelian category of rational Mackey functors, all objects are projective and injective.

The analog for coefficient systems is false, and so is the analog for compact Lie groups. The following result is easy for finite groups and false for compact Lie groups.

PROPOSITION 5.3. For $H \subset G$ and $n \neq 0$, $\underline{\pi}_n(G/H_+) \otimes \mathbb{Q} = 0$.

Let $\mathscr{M} = \mathscr{M}_G$ denote the Abelian category of Mackey functors over G. For G-spectra X and Y, there is an evident natural map

$$\theta : [X, Y]_G \longrightarrow \prod \mathrm{Hom}_{\mathscr{M}}(\underline{\pi}_n(X), \underline{\pi}_n(Y)).$$

Let Y be rational. By the previous result and the Yoneda lemma, θ is an isomorphism when $X = \Sigma^\infty G/H_+$ for any H. Throwing in suspensions, we can extend θ to a graded map

$$\theta : Y_G^q(X) = [X, Y]_G^q = [\Sigma^{-q} X, Y]_G \longrightarrow \prod \mathrm{Hom}_{\mathscr{M}}(\underline{\pi}_n(\Sigma^{-q} X), \underline{\pi}_n(Y)).$$

It is still an isomorphism when X is an orbit. Of course, we obtain the same groups if we replace X and the Mackey functors $\underline{\pi}_n(\Sigma^{-q} X)$ by their rationalizations. Since the Mackey functors $\underline{\pi}_n(Y)$ are injective, the right hand side is a cohomology theory on G-spectra X. Clearly θ is a map of cohomology theories and this already proves the following result. With $Y = \prod K(\underline{\pi}_n(X), n)$, Theorem 5.1 is an easy consequence.

THEOREM 5.4. If Y is rational, then θ is a natural isomorphism.

This classifies rational G-spectra, and we next classify maps between them. Recall that $\phi \otimes \mathbb{Q} : A(G) \otimes \mathbb{Q} \longrightarrow C(G) \otimes \mathbb{Q}$ is an isomorphism and that $C(G) \otimes \mathbb{Q}$ is the product of a copy of $\mathbb{Q}$ for each conjugacy class (H). There results a complete set of orthogonal idempotents $e_H = e_H^G$ in $A(G) \otimes \mathbb{Q}$. Multiplication by the e_H induces splittings of $A(G) \otimes \mathbb{Q}$-modules, rational Mackey functors, and rational G-spectra, and we have the commutation relation

$$\underline{\pi}_n(e_H X) \cong e_H \underline{\pi}_n(X).$$

In all three settings, there are no non-zero maps $e_H X \longrightarrow e_J Y$ unless H is conjugate to J. This gives refinements of Theorems 5.1 and 5.4.

THEOREM 5.5. For rational G-spectra X, there are natural equivalences

$$X \simeq \bigvee e_H X \simeq \prod K(e_H \underline{\pi}_n(X), n).$$

THEOREM 5.6. For rational G-spectra X and Y, there are natural isomorphisms

$$[X, Y]_G \cong \sum [e_H X, e_H Y]_G \cong \sum \prod \mathrm{Hom}_{\mathscr{M}}(e_H \underline{\pi}_n(X), e_H \underline{\pi}_n(Y)).$$

Moreover, if $V_{n,H}(X) = (e_H \underline{\pi}_n(X))(G/H) \subset \pi_n(X^H)$, then

$$\mathrm{Hom}_{\mathscr{M}}(e_H \underline{\pi}_n(X), e_H \underline{\pi}_n(Y)) \cong \mathrm{Hom}_{WH}(V_{n,H}(X), V_{n,H}(Y)).$$

Thus the computation of maps between rational G-spectra reduces to the computation of maps between functorially associated modules over subquotient groups. The last statement of the theorem is a special case of the following algebraic result.

THEOREM 5.7. For rational Mackey functors M and N, there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{M}}(e_H M, e_H N) \cong \mathrm{Hom}_{WH}(V_H(M), V_H(N)),$$

where $V_H(M)$ is the $\mathbb{Q}[WH]$-module $(e_H M)(G/H) \subset M(G/H)$.

The proof of Proposition 5.2 is based on the following consequence of the fact that $V_H(N)$ is a projective and injective $\mathbb{Q}[WH]$-module.

LEMMA 5.8. If the conclusion of Theorem 5.7 holds for all N and for a given M and H, then $e_H M$ is projective; if the conclusion holds for all M and for a given N and H, then $e_H N$ is injective.

Now let $\mathscr{M}_{\mathbb{Q}}$ be the category of rational Mackey functors over G. Let $\mathscr{Q}[G]$ be the category of $\mathbb{Q}[G]$-modules. Fix $H \subset G$. Then there are functors

$$U_H : \mathscr{M}_{\mathbb{Q}} \longrightarrow \mathscr{Q}[WH] \quad \text{and} \quad F_H : \mathscr{Q}[WH] \longrightarrow \mathscr{M}_{\mathbb{Q}}.$$

Explicitly,

$$U_H M = M(G/H) \quad \text{and} \quad (F_H V)(G/K) = (\mathbb{Q}[(G/K)^H] \otimes V)^{WH}.$$

These functors are both left and right adjoint to each other if we replace $\mathscr{M}_{\mathbb{Q}}$ by its full subcategory $\mathscr{M}_{\mathbb{Q}}/H$ of those Mackey functors M such that $M(G/J) = 0$ for all proper subconjugates J of H. Since $(F_H V)(G/K) = 0$ unless H is subconjugate to K, $F_H V$ is in $\mathscr{M}_{\mathbb{Q}}/H$.

PROOFS OF PROPOSITION 5.2 AND THEOREM 5.7. One easily proves both of these results when $M = F_H V$ by use of the adjunctions and idempotents. Even integrally, every Mackey functor M is built up by successive extensions from Mackey functors of the form $F_H V$. Rationally, the extensions split by the projectivity of the $F_H V$. Therefore any rational Mackey functor M is a direct sum of Mackey functors of the form $F_H V$ for varying H and V. $\square$

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CHAPTER XX

The Segal conjecture

1. The statement in terms of completions of G-spectra

There are many ways to think about the Segal conjecture and its generalizations. Historically, the original source of the conjecture was just the obvious analogy between K-theory and stable cohomotopy. According to the Atiyah-Segal completion theorem, the K-theory of the classifying space of a compact Lie group G is isomorphic to the completion of the representation ring $R(G)$ at its augmentation ideal. Here $R(G)$ is $K_G^0(S^0)$, and $K_G^1(S^0) = K^1(BG_+) = 0$. The Burnside ring $A(G)$ is $\pi_G^0(S^0)$, and it is natural to guess that the stable cohomotopy of BG is isomorphic to the completion of $\pi_G^*(S^0)$ at the augmentation ideal I of $A(G)$. This guess is the Segal conjecture. It is not true for compact Lie groups in general, but it turns out to be correct for finite groups G. We shall restrict ourselves to finite groups throughout our discussion. A survey of what is known about the Segal conjecture for compact Lie groups has been given by Lee and Minami.

Here we are thinking about $\mathbb{Z}$-graded theories, and that is the right way to think about the proof. However, one can also think about the result in purely equivariant terms, and the conclusion then improves to a result about G-spectra and thus about $RO(G)$-graded cohomology theories. To see this, let's at first generalize and consider any G-spectrum k_G. We have the projection $EG_+ \longrightarrow S^0$, and it induces a G-map

$$\varepsilon : k_G = F(S^0, k_G) \longrightarrow F(EG_+, k_G). \tag{1.1}$$

We think of ε as a kind of geometric completion of k_G.

It is natural to think about such completions more generally. Let $\mathscr{F}$ be a family of subgroups of G. We have the projection $E\mathscr{F}_+ \longrightarrow S^0$, and we have the induced

G-map

$$\varepsilon : k_G = F(S^0, k_G) \longrightarrow F(E\mathscr{F}_+, k_G). \tag{1.2}$$

We think of ε as the geometric completion of k_G at $\mathscr{F}$.

We want to compare this with an algebraic completion. The family $\mathscr{F}$ determines an ideal $I\mathscr{F}$ of $A(G)$, namely

$$I\mathscr{F} = \bigcap_{H\in\mathscr{F}} \mathrm{Ker}(A(G) \longrightarrow A(H)). \tag{1.3}$$

Just as $I = I\{e\} = q(e,0)$, by definition, it turns out algebraically that

$$I\mathscr{F} = \bigcap_{H\in\mathscr{F}} q(H,0). \tag{1.4}$$

Since $A(G)$ plays the same role in equivariant theory that $\mathbb{Z}$ plays in nonequivariant theory, it is natural to introduce completions of G-spectra at ideals of the Burnside ring. This is quite easy to do. For an element α of $A(G)$, define $S_G[\alpha^{-1}]$, the localization of the sphere G-spectrum S_G at α, to be the telescope of countably many iterates of $\alpha : S_G \longrightarrow S_G$. Then let $K(\alpha)$ be the fiber of the canonical map $S_G \longrightarrow S_G[\alpha^{-1}]$. For an ideal I generated by a set $\{\alpha_1, \cdots, \alpha_n\}$, define

$$K(I) = K(\alpha_1) \wedge \cdots \wedge K(\alpha_n). \tag{1.5}$$

It turns out that, up to equivalence, $K(I)$ is independent of the choice of generators of I. Now define

$$(k_G)\hat{}_I = F(K(I), k_G). \tag{1.6}$$

By construction, $K(I)$ comes with a canonical map $\zeta : K(I) \longrightarrow S_G$, and there results a map

$$\gamma : k_G \longrightarrow (k_G)\hat{}_I. \tag{1.7}$$

We call γ the completion of k_G at the ideal I. For those who know about such things, we remark that completion at I is just Bousfield localization at $K(I)$. We shall later use "brave new algebra" to generalize this construction.

Now specialize to $I = I\mathscr{F}$ for a family $\mathscr{F}$. For $\alpha \in I\mathscr{F}$, $\alpha : S_G \longrightarrow S_G$ is null homotopic as an H-map for any $H \in \mathscr{F}$. Therefore $S_G[\alpha^{-1}]$ is H-contractible, $K(I\mathscr{F})$ is H-equivalent to S_G, and the cofiber of ζ is H-contractible. This implies that there is a unique G-map

$$\xi : \Sigma^\infty E\mathscr{F}_+ \longrightarrow K(I\mathscr{F}) \tag{1.8}$$

over S_G. There results a canonical map of G-spectra

$$\xi^* : F(K(I\mathscr{F}), kG) \longrightarrow F(E\mathscr{F}_+, k_G). \tag{1.9}$$

We view this as a comparison map relating the algebraic to the geometric completion of k_G at $\mathscr{F}$.

One can ask for which G-spectra k_G the map ξ^* is an equivalence. We can now state what I find to be the most beautiful version of the Segal conjecture. Recall that $D(E) = F(E, S_G)$.

THEOREM 1.10. For every family $\mathscr{F}$, the map

$$\xi^* : (S_G)\hat{}_{I\mathscr{F}} = D(K(I\mathscr{F})) \longrightarrow D(E\mathscr{F}_+)$$

is an equivalence of G-spectra.

Parenthetically, one can also pass to smash products rather than function spectra from the map ξ, obtaining

$$\xi_* : k_G \wedge E\mathscr{F}_+ \longrightarrow k_G \wedge K(I\mathscr{F}_+). \tag{1.11}$$

One can ask for which G-spectra k_G this map is an equivalence. A standard argument shows that ξ^* is an equivalence if k_G is a ring spectrum and ξ_* is an equivalence. Once we introduce Tate theory, we will be able to give a remarkable partial converse. The point to make here is that ξ_* is an equivalence for K_G, as we shall explain in XXIV§7, but is certainly not an equivalence for S_G. That would be incompatible with the splitting of $(S_G)^G$ in XIX§1. Our original analogy will only take us so far.

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2. A calculational reformulation

What does Theorem 1.10 say calculationallyΓ To give an answer, we go back to our algebraic completions. The I-adic completion functor is neither left nor right exact in general, and it has left derived functors L_i^I. Because $A(G)$ has Krull dimension one, these vanish for $i > 1$. In precise analogy with the calculation of the homotopy groups of p-adic completions of spaces, we find that, for any G-spectrum X, there is a natural short exact sequence

$$0 \longrightarrow L_1^I(\underline{\pi}_{q-1}(X)) \longrightarrow \underline{\pi}_q(X\hat{_I}) \longrightarrow L_0^I(\underline{\pi}_q(X)) \longrightarrow 0, \tag{2.1}$$

where we apply our derived functors to Mackey functors termwise. Thinking cohomologically, for any G-spectra X and k_G, there are natural short exact seqences

$$0 \longrightarrow L_1^I((k_G^{\nu+1}(X)) \longrightarrow ((k_G)\hat{_I})_G^{\nu}(X) \longrightarrow L_0^I(k_G^{\nu}(X)) \longrightarrow 0. \tag{2.2}$$

As a matter of algebra, the L_i^I admit the following descriptions, which closely parallels the algebra we summarized when we discussed completions at p in II§4. Abbreviate $A = A(G)$ and consider an A-module M. Then we have the following natural short exact sequences.

$$0 \longrightarrow \lim{}^1 \operatorname{Tor}_1^A(A/I^r, M) \longrightarrow L_0^I(M) \longrightarrow M\hat{_I} \longrightarrow 0. \tag{2.3}$$

$$0 \longrightarrow \lim{}^1 \operatorname{Tor}_2^A(A/I^r, M) \longrightarrow L_1^I(M) \longrightarrow \lim \operatorname{Tor}_1^A(A/I^r, M) \longrightarrow 0. \tag{2.4}$$

There is interesting algebra in the passage from the topological definition of completion to the algebraic interpretation (2.1). Briefly, there are "local homology groups" $H_i^I(M)$ analogous to Grothendieck's local cohomology groups. Our topological construction mimics the algebraic definition of the $H_i^I(M)$, and, as a matter of algebra, $L_i^I(M) \cong H_i^I(M)$. This leads to alternative procedures for calculation, but begins to take us far from the Segal conjecture. We shall return to the relevant algebra in Chapter XXIV.

The last two formulas show that, if M is finitely generated, then $L_0^I(M) \cong M\hat{_I}$ and $L_1^I(M) = 0$. When a G-spectrum k_G is bounded below and of finite type, in the sense that each of its homotopy groups is finitely generated, we can construct a model for $(k_G)\hat{_I}$ and study its properties by induction up a Postnikov tower, exactly as we studied p-completion in II§5. As there, we find that a map from k_G to an "I-complete spectrum" that induces I-adic completion on all homotopy groups is equivalent to the I-completion of k_G. Moreover, a sufficient condition for a bounded below spectrum to be I-complete is that its homotopy groups are finitely generated modules over $A(G)\hat{_I}$.

We deduce from XIX.1.1 that S_G is of finite type. Thus the I-adic completions of its homotopy groups are bounded below and of finite type over $A(G)\hat{_I}$. A little diagram chase now shows that the following theorem will imply Theorem 1.10.

THEOREM 2.5. The map $\varepsilon : S_G \longrightarrow D(E\mathscr{F}_+)$ induces an isomorphism

$$\underline{\pi}_*(S_G)\hat{_{I\mathscr{F}}} \longrightarrow \underline{\pi}_*(D(E\mathscr{F}_+)).$$

There is an immediate problem here. A priori, we do not know anything about the homotopy groups of $D(E\mathscr{F}_+)$, which, on the face of it, need be neither bounded below nor of finite type. There is a $\lim^1$ exact sequence for their calculation in terms of the duals of the skeleta of $E\mathscr{F}_+$. To prove that the $\lim^1$ terms vanish, and to make sure that we are always working with finitely generated $A(G)$-modules, we work with pro-groups and only pass to actual inverse limits at the very end. We have already said nearly all that we need to say about this in XIV§5. Recall that, for any Abelian group valued functor h on G-CW complexes or spectra, we define the associated pro-group valued functor $\mathbf{h}$ by letting $\mathbf{h}(X)$ be the inverse system $\{h(X_\alpha)\}$, where X_α runs over the finite subcomplexes of X. Our functors take values in finitely generated $A(G)$-modules. For an ideal I in $A(G)$ and such a pro-module $\mathbf{M} = \{M_\alpha\}$, $\mathbf{M}\hat{_I}$ is the inverse system $\{M_\alpha/I^r M_\alpha\}$. For a multiplicative subset S, $S^{-1}M = \{S^{-1}M_\alpha\}$.

We define pro-Mackey functors just as we defined Mackey functors, but changing the target category from groups to pro-groups. Now Theorem 2.5 will follow from its pro-Mackey functor version.

THEOREM 2.6. The map $\varepsilon : S_G \longrightarrow D(E\mathscr{F}_+)$ induces an isomorphism

$$\underline{\boldsymbol{\pi}}_*(S_G)\hat{_{I\mathscr{F}}} \longrightarrow \underline{\boldsymbol{\pi}}_*(D(E\mathscr{F}_+)).$$

The point is that the pro-groups on the left certainly satisfy the Mittag-Leffler condition guaranteeing the vanishing of $\lim^1$ terms, hence the $\lim^1$ terms for the calculation of $\underline{\pi}_*(D(E\mathscr{F}_+))$ vanish and we obtain the isomorphism of Theorem 2.5 on passage to limits. We now go back to something we omitted: making sense of the induced map in Theorem 2.6. For a finite G-CW complex X such that X^H is empty for $H \notin \mathscr{F}$, we find by induction on the number of cells and the very definition of $I\mathscr{F}$ that $\underline{\pi}_*(D(X_+))$ is annihilated by some power of $I\mathscr{F}$. This implies that the canonical pro-map

$$\underline{\boldsymbol{\pi}}_*(D(X_+)) \longrightarrow \underline{\boldsymbol{\pi}}_*(D(X_+))\hat{_{I\mathscr{F}}}$$

is an isomorphism. Applying this to the finite subcomplexes of $E\mathscr{F}$, we see that the right side in Theorem 2.6 is $I\mathscr{F}$-adically complete. Thus the displayed map makes sense.

J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. Topology 27(1988), 7-21.
J. P. C. Greenlees and J. P. May. Derived functors of I-adic completion and local homology. J. Algebra 149(1992), 438-453.

3. A generalization and the reduction to p-groups

Now we change our point of view once more, thinking about individual pro-homotopy groups rather than Mackey functors. Using a little algebra to check that the ideal in $A(H)$ generated by the image of $I\mathscr{F}$ under restriction has the same radical as $I(\mathscr{F}|H)$, we see that the Hth term of the map in Theorem 2.6 is

$$\boldsymbol{\pi}_H^*(S^0)\hat{_{I(\mathscr{F}|H)}} \longrightarrow \boldsymbol{\pi}_H^*(E(\mathscr{F}|H)_+).$$

We may as well proceed by induction on the order of G, so that we may assume this map to be an isomorphism for all proper subgroups. In any case, Theorem 2.6 can be restated as follows.

THEOREM 3.1. The map $E\mathscr{F} \longrightarrow *$ induces an isomorphism

$$\boldsymbol{\pi}_G^*(S^0)\hat{_{I\mathscr{F}}} \longrightarrow \boldsymbol{\pi}_G^*(E\mathscr{F}_+).$$

Now $E\mathscr{F} \longrightarrow *$ is obviously an example of an $\mathscr{F}$-equivalence, that is, a map that induces an equivalence on H-fixed points for $H \in \mathscr{F}$. We are really proving an invariance theorem:

An $\mathscr{F}$-equivalence $f : X \longrightarrow Y$ induces an isomorphism $\boldsymbol{\pi}_G^*(f)\hat{_{I\mathscr{F}}}$.

We can place this in a more general framework. Given a set $\mathscr{H}$ of subgroups of G, closed under conjugacy, we say that a cohomology theory is $\mathscr{H}$-invariant if it carries $\mathscr{H}$-equivalences to isomorphisms. We say that a G-space X is $\mathscr{H}$-contractible if X^H is contractible for $H \in \mathscr{H}$. By an immediate cofiber sequence argument, a theory is $\mathscr{H}$-invariant if and only if it vanishes on $\mathscr{H}$-contractible spaces. It is not difficult to show that, for any cohomology theory h^*, there is a unique minimal class $\mathscr{H}$ such that h^* is $\mathscr{H}$-invariant: determination of this class gives a best possible invariance theorem for h^*. Given an ideal I and a collection $\mathscr{H}$, we can try to obtain such a theorem for the theory $\boldsymbol{\pi}_G^*(\cdot)\hat{_I}$.

Answers to such questions in the context of localizations rather than completions have a long history and demonstrated value, but there one usually assumes that $\mathscr{H}$ is closed under passage to larger rather than smaller subgroups. For such a "cofamily" $\mathscr{H}$, we have the $\mathscr{H}$-fixed point subcomplex $X^{\mathscr{H}} = \{x | G_x \in \mathscr{H}\}$;

the inclusion $i : X^{\mathscr{H}} \longrightarrow X$ is an $\mathscr{H}$-equivalence. A cohomology theory is $\mathscr{H}$-invariant if and only it carries all such inclusions i to isomorphisms.

It seems eminently reasonable to ask about localizations and completions together. We can now state the following generalization of Theorem 3.1. Define the support of a prime ideal P in $A(G)$ to be the conjugacy class (L) such that P is in the image of $\mathrm{Spec}(A(L))$ but is not in the image of $\mathrm{Spec}(A(K))$ for any subgroup K of L. We know what the supports are: (H) for $q(H,0)$ and (H_p) for $q(H,p)$.

Theorem 3.2. For any multiplicative subset S and ideal I, the cohomology theory $S^{-1}\boldsymbol{\pi}_G^*(\cdot)^{\hat{}}_I$ is $\mathscr{H}$-invariant, where

$$\mathscr{H} = \bigcup\{\mathrm{Supp}(P) | P \cap S = \emptyset \text{ and } P \supset I\}.$$

With $S = \emptyset$ and $I = I\mathscr{F}$, Theorem 3.1 follows once one checks that the resulting $\mathscr{H}$ is contained in $\mathscr{F}$. In fact it equals $\mathscr{F}$ since the primes that contain $I\mathscr{F}$ are all of the $q(H,p)$ with $H \in \mathscr{F}$, and this allows $p = 0$. It looks as if we have made our work harder with this generalization but in fact, precisely because we have introduced localization, which we have already studied in some detail, the general theorem quickly reduces to a very special case.

In fact, by XIX.4.2, it is enough to show that $(S_P)^{-1}\boldsymbol{\pi}_G^*(X)^{\hat{}}_P = 0$ if X^L is contractible for $L \in \mathrm{Supp}(P)$, where $S_P = A - P$. By XVII.5.5, there is an idempotent $e_L^G \in A(G)_p$ such that $(S_P)^{-1}A(G) = e_L^G A(G)_p$. Remembering that the Φ-fixed point functor satisfies $\Phi^H S_G = S_H$, we see that, for any finite G-CW complex X, XVII.6.4 specializes to give the chain of isomorphisms

$$e_L^G \pi_G^n(X)_p \longrightarrow e_L^{NL}\pi_{NL}^n(X)_p \longrightarrow e_1^{WL}\pi_{WL}^n(X^L)_p \longrightarrow \pi_{VL}^n(X^L)_p^{inv}$$

where VL is a p-Sylow subgroup of WL. The transfer argument used to prove the last isomorphism gives further that $\pi_{VL}^n(X^L)_p^{inv}$ is naturally a direct summand in $\pi_{VL}^n(X^L)_p$. Passing to pro-modules, we conclude that $(S_P)^{-1}\boldsymbol{\pi}_G^*(X)^{\hat{}}_P$ is a direct summand in $\boldsymbol{\pi}_{VL}^*(X^L)^{\hat{}}_p$. Therefore Theorem 3.2 is implied by the following special case.

Theorem 3.3. The theory $\boldsymbol{\pi}_G^*(\cdot)^{\hat{}}_p$ is e-invariant for any finite p-group G. That is, it vanishes on nonequivariantly contractible G-spaces.

This is Carlsson's theorem, and we will discuss its proof in the next section. In the case of the augmentation ideal there is a shortcut to the reduction to p-groups and p-adic completion: it is immediate from XIX.4.5 and XIX.4.6. Let us say a word about the nonequivariant interpretation of the Segal conjecture in this case. Since S_G is a split G-spectrum, we can conclude that

$$\pi_G^*(S^0)\hat{_I} \cong \pi_G^*(EG_+) \cong \pi^*(BG_+). \tag{3.4}$$

Of course, the cohomotopy groups on the left lie in non-positive degrees and are just the homotopy groups reindexed. By XIX.1.1,

$$\pi_*^G(S^0) = \sum_{(H)} \pi_*(BWH_+). \tag{3.5}$$

The left side is a ring, but virtually nothing seems to be known about the multiplicative structure on the right. Nor is much known about the $A(G)$-module structure. Of course, the last problem disappears upon completion in the case of p-groups, by XIX.4.6.

J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. Topology 27(1988), 7-21.

4. The proof of the Segal conjecture for finite p-groups

There are two basic strategies. One is to use (3.5) and a nonequivariant interpretation of the completion map to reduce to a nonequivariant problem. For elementary p-groups, the ideas that we discussed in the context of the Sullivan conjecture can equally well be used to prove the Segal conjecture, and Lannes has an unpublished nonequivariant argument that handles general p-groups.

The other is to use equivariant techniques, which is the method used by Carlsson. Historically, Lin first proved the Segal conjecture for $\mathbb{Z}/2$, Gunawardena for $\mathbb{Z}/p$, p odd, and Adams, Gunawardena, and Miller for general elementary Abelian p-groups, all using nonequivariant techniques and the Adams spectral sequence. Carlsson's theorem reduced the case of general finite p-groups to the case of elementary Abelian p-groups. His ideas also led to a substantial simplification of the proof in the elementary Abelian case, as was first observed by Caruso, Priddy, and myself. For this reason, the full original proof of Adams, Gunawardena, and Miller was never published. Since I have nothing to add to the exposition that Caruso, Priddy, and I gave, which includes complete details of a variant of Carlsson's proof of the reduction to elementary Abelian p-groups, I will give an outline that may gain clarity by the subtraction of most of the technical details.

We assume throughout that G is a finite p-group. We begin with a general G-spectrum k_G, and we will work with the bitheory

$$k_G^q(X;Y) = k_{-q}^G(X;Y)$$

on spaces X and Y. It can be defined as the cohomology of X with coefficients in the spectrum $Y \wedge k_G$. The following easy first reduction of Carlsson is a key step. It holds for both represented and pro-group valued theories. Let $\mathscr{P}$ be the family of proper subgroups of G.

LEMMA 4.1. Assume that k_H^* is e-invariant for all $H \in \mathscr{P}$. Then k_G^* is e-invariant if and only if $k_G^*(\tilde{E}\mathscr{P}) = 0$.

PROOF. Let X be e-contractible. We must show that $k_G^*(X) = 0$ if $k_G^*(\tilde{E}\mathscr{P}) = 0$. Write $Y = \tilde{E}\mathscr{P}$. Then $Y^G = S^0$ and Y is H-contractible for $H \in \mathscr{P}$. Let $Z = Y/S^0$. We have the cofiber sequence

$$X \longrightarrow X \wedge Y \longrightarrow X \wedge Z.$$

We claim that $k_G^*(W \wedge Y) = 0$ for any G-CW complex W and that $k_G^*(X \wedge Z) = 0$ for any G-CW complex Z such that $Z^G = *$. The first claim holds by hypothesis on orbit types G/G and holds trivially on orbit types G/H with $H \in \mathscr{P}$. The second claim holds on orbits by the induction hypothesis. The general cases of both claims follow. $\square$

The cofiber sequence $EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$ gives rise to a long exact sequence

$$(4.2)\quad \longrightarrow k_G^q(Y;EG_+) \longrightarrow k_G^q(Y) \longrightarrow k_G^q(Y;\tilde{E}G) \xrightarrow{\delta} k_G^{q+1}(Y;EG_+) \longrightarrow .$$

The $\tilde{E}G$ terms carry the singular part of the problem; the EG_+ terms carry the free part.

Let us agree once and for all that all of our theories are to be understood as pro-group valued and completed at p, since that is the form of the theorem we need to prove. We must show that $\pi_G^*(Y) = 0$. However, studying more general theories allows a punch line in the elementary Abelian case: there the map δ in (4.2) is proven to be an isomorphism by comparison with a theory for which the analogue of the Segal conjecture holds trivially.

For a normal subgroup K of H with quotient group J write $k_{H/K}^* = k_J^*$ for the theory represented by $\Phi^K(k_H)$, where k_H denotes k_G regarded as an H-spectrum. We pointed out the ambiguity of the notation k_J^* at the end of XVI§6, but we also observed there that the notation π_J^* is correct and unambiguous. As we shall explain in the next section, we can analyze the singular terms in (4.2) in terms of these subquotient theories.

THEOREM 4.3. Assume that k_J^* is e-invariant for every proper subquotient J of G and let $Y = \tilde{E}\mathscr{P}$.

(i) If G is not elementary Abelian, then $k_G^*(Y; \tilde{E}G) = 0$.
(ii) If $G = (\mathbb{Z}/p)^r$, then $k_G^*(Y; \tilde{E}G)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1} k_{G/G}^*(S^0)$.

Warning: the nonequivariant theory $k_{G/G}^*$ is usually quite different from the underlying nonequivariant theory $k^* = k_e^*$.

As we shall explain in Section 6, we can use Adams spectral sequences to analyze the free terms in (4.2).

THEOREM 4.4. Assume that k_G is split and k is bounded below and let $Y = \tilde{E}\mathscr{P}$.

(i) If G is not elementary Abelian, then $k_G^*(Y; EG_+) = 0$.
(ii) If $G = (\mathbb{Z}/p)^r$ and $H^*(k)$ is finite dimensional, then $k_G^*(Y; EG_+)$ is the direct sum of $p^{r(r-1)/2}$ copies of $\Sigma^r k^*(S^0)$.

The hypothesis that $H^*(k)$ be finite dimensional in (ii) is extremely restrictive, although it is satisfied trivially when k is the sphere spectrum. The hypothesis is actually necessary. We shall see in Section 7 that the theories $\pi_G^*(\cdot; B_G\Pi_+)$ are e-invariant for finite groups Π. They satisfy all other hypotheses of our theorems, but here k^* and $k_{G/G}^*$ are different. In such cases, the calculation of $k_G^*(Y; EG_+)$ falls out from the e-invariance, which must be proven differently, and (4.2).

Carlsson's reduction is now the case π_G of the following immediate inductive consequence of the first parts of Theorems 4.3 and 4.4.

THEOREM 4.5. Suppose that G is not elementary Abelian. Assume

(i) k_J^* is e-invariant for all elementary Abelian subquotients J;
(ii) k_J is split and $k_{K/K}$ is bounded below for all non-elementary Abelian subquotients $J = H/K$.

Then k_J^* is e-invariant for all subquotients J, including $J = G$.

Returning to cohomotopy and the proof of the Segal conjecture, it only remains to prove that the map δ in (4.2) is an isomorphism when $G = (\mathbb{Z}/p)^r$. We assume that the result has been proven for $1 \leq q < r$. Comparing Theorems 4.3 and 4.4, we see that the map δ in (4.2) is a map between free π^*-modules on the same number of generators. It suffices to show that δ is a bijection on generators, which means that it is an isomorphism in degree $r-1$. Here δ is a map between free modules on the same number of generators over the p-adic integers $\mathbb{Z}_p^\wedge$, so that it will be an isomorphism if it is a monomorphism when reduced mod p.

To prove this, let $k_G = F(EG_+, H\underline{\mathbb{F}}_p)$, where $H\mathbb{F}_p$ is the Eilenberg-MacLane G-spectrum associated to the "constant Mackey functor" at $\mathbb{F}_p$ that we obtain from IX.4.3. This theory, like any other theory represented by a function spectrum $F(EG_+, \cdot)$, is e-invariant. Since $\pi_0^G(H\underline{\mathbb{F}}_p) = \mathbb{F}_p$, we have a unit map $S_G \longrightarrow H\underline{\mathbb{F}}_p$, and we compose with $\varepsilon : H\underline{\mathbb{F}}_p \longrightarrow k_G$ to obtain $\eta : S_G \longrightarrow k_G$. There is an induced map $S = S_{G/G} \longrightarrow k_{G/G}$, and a little calculation shows that it sends the unit in $\pi^0(S)$ to an element that is non-zero mod p. We can also check that the subquotient theories k_J^* are all e-invariant. By the naturality of (4.2), we have the commutative diagram

$$\begin{array}{ccc}
\pi_G^{r-1}(Y;\tilde{E}G) & \xrightarrow{\delta} & \pi_G^r(Y;EG_+) \\
\Big\downarrow{\scriptstyle \eta_*} & & \Big\downarrow{\scriptstyle \eta_*} \\
k_G^{r-1}(Y;\tilde{E}G) & \xrightarrow[\delta]{} & k_G^r(Y;EG_+).
\end{array}$$

The bottom map δ is an isomorphism since $k_G^*(Y) = 0$. The left map η_* is the sum of $p^{r(r-1)/2}$ copies of $\Sigma^{r-1}\eta_*$, $\eta_* : \pi^0(S) \longrightarrow \pi^0(k_{G/G})$, and is therefore a monomorphism mod p. Thus the top map δ is a monomorphism mod p, and this concludes the proof.

J. F. Adams, J. H. Gunawardena, and H. Miller. The Segal conjecture for elementary Abelian p-groups-I. Topology 24(1985), 435-460.

G. Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. Annals Math. 120(1984), 189-224.

J. Caruso, J. P. May, and S. B. Priddy. The Segal conjecture for elementary Abelian p-groups-II. Topology 26(1987), 413-433.

5. Approximations of singular subspaces of G-spaces

Let SX denote the singular set of a G-space X, namely the set of points with non-trivial isotropy group. The starting point of the proof of Theorem 4.3 is the space level observation that the inclusions

$$SX \longrightarrow X \quad \text{and} \quad S^0 \longrightarrow \tilde{E}G$$

induce bijections

$$[X, \tilde{E}G \wedge X']_G \longrightarrow [SX, \tilde{E}G \wedge X']_G \longleftarrow [SX, X']_G.$$

We may represent theories on finite G-CW complexes via colimits of space level homotopy classes of maps. The precise formula is not so important. What is important is that, when calculating $k_G^*(X; \tilde{E}G)$, we get a colimit of terms of the general form $[SW, Z]_G$. We can replace S here by other functors T on spaces that satisfy appropriate axioms and still get a cohomology theory in X, called $k_G^*(X; T)$. Such functors are called "S-functors". Natural transformations $T \longrightarrow T'$ induce maps of theories, contravariantly. We have a notion of a cofibration of S-functors, and cofibrations give rise to long exact sequences. In sum, we have something like a cohomology theory on S-functors T.

We construct a filtered S-functor A that approximates the singular functor S. Let $\mathscr{A} = \mathscr{A}(G)$ be the partially ordered set of non-trivial elementary Abelian subgroups of G, thought of as a G-category with a map $A \longrightarrow B$ when $B \subset A$, with G acting by conjugation. If $G \neq e$, the classifying space $B\mathscr{A}$ is G-contractible. In fact, if C is a central subgroup of order p, then the diagram $A \longleftarrow AC \longrightarrow C$ displays the values on an object A of three G-equivariant functors on $\mathscr{A}$ together with two equivariant natural transformations between them; these induce a G-homotopy from the identity to the constant G-map at the vertex C.

We can parametrize $\mathscr{A}$ by points of SX. Precisely, we construct a topological G-category $\mathscr{A}[X]$ whose objects are pairs (A, x) such that $x \in X^A$; there is a morphism $(A, x) \longrightarrow (B, y)$ if $B \subset A$ and $y = x$, and G acts by $g(A, x) = (gAg^{-1}, gx)$. Projection on the X-coordinate gives a functor $\mathscr{A}[X] \longrightarrow SX$, where SX is a category in the trivial way, and $B\mathscr{A}[X] \longrightarrow BSX = SX$ is a G-homotopy equivalence. The subspace $B\mathscr{A}[*]$ of $B\mathscr{A}[X]$ is G-contractible. Let $AX = B\mathscr{A}[X]/B\mathscr{A}[*]$. We still have a G-homotopy equivalence $AX \longrightarrow SX$, but now A is an S-functor and our equivalences give a map of S-functors. For any space Y, we have

$$k_G^*(Y; \tilde{E}G) \cong k_G^*(Y; S) \cong k_G^*(Y; A).$$

The functor A arises from geometric realizations of simplicial spaces and carries the simplicial filtration $F_q A$; here $F_{-1}A = *$ and $F_{r-1}A = A$, where $r = \operatorname{rank}(G)$. Inspection of definitions shows that the successive subquotients satisfy

$$(F_q A / F_{q-1} A)(X) = \bigvee \Sigma^q (G_+ \wedge_{H(\omega)} X^{A(\omega)}).$$

Here ω runs over the G-conjugacy classes of strictly ascending chains $(A_0, \cdots, A_q)$ of non-trivial elementary Abelian subgroups of G, $H(\omega)$ is the isotropy group of ω, namely $\{g|gA_ig^{-1} = A_i,\ 0 \le i \le q\}$, and $A(\omega) = A_q$. For each normal subgroup K of a subgroup H of G, there is an S-functor $C(K,H)$ whose value on X is $G_+ \wedge_H X^K$, and, as S-functors,

$$(F_qA/F_{q-1}A) = \bigvee \Sigma^q C(A(\omega), H(\omega)). \tag{5.1}$$

By direct inspection of definitions, we find that, for any space Y,

$$k_G^*(Y; C(K,H)) \cong k_{H/K}^*(Y^K). \tag{5.2}$$

This is why the Φ-fixed point functors enter into the picture.

To prove Theorem 4.3, we restrict attention to $Y = \tilde{E}\mathscr{P}$. If G is not elementary Abelian, then Y^K is contractible and the subquotients H/K are proper for all pairs (K,H) that appear in (5.1). If $G = (\mathbb{Z}/p)^r$, and $q \le r-2$, this is still true. All these terms vanish by hypothesis. If $G = (\mathbb{Z}/p)^r$, we are left with the case $q = r-1$. Here $A(\omega) = H(\omega) = G$ for all chains ω, there are $p(p-1)/2$ chains ω, and $Y^G = S^0$. Using (5.2), Theorem 4.3 follows.

G. Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. Annals Math. 120(1984), 189-224.

J. Caruso, J. P. May, and S. B. Priddy. The Segal conjecture for elementary Abelian p-groups-II. Topology 26(1987), 413-433.

6. An inverse limit of Adams spectral sequences

We turn to the proof of Theorem 4.4. Its hypothesis that k_G is split allows us to reduce the problem to a nonequivariant one, and the hypothesis that the underlying nonequivariant spectrum k is bounded below ensures the convergence of the relevant Adams spectral sequences. We prove Theorem 4.4 by use of a particularly convenient model Y for $\tilde{E}\mathscr{P}$, namely the union of the G-spheres S^{nV}, where V is the reduced regular complex representation of G. It is a model since $V^G = \{0\}$ and $V^H \neq 0$ for $H \in \mathscr{P}$.

In general, for any representation V, there is a Thom spectrum BG^{-V}. Here we may think of $-V$ as the negative of the representation bundle $EG \times_G V \longrightarrow BG$, regarded as a map $-V : BG \longrightarrow BO \times \mathbb{Z}$. If V is suitably oriented, for example if V is complex, there is a Thom isomorphism showing that $H^*(BG^{-V})$ is a free $H^*(BG)$-module on one generator ι_v of degree $-n$, where n is the (real) dimension of V. We take cohomology with mod p coefficients. For $V \subset W$, there is a map $f : BG^{-W} \longrightarrow BG^{-V}$ such that $f^* : H^*(BG^{-V}) \longrightarrow H^*(BG^{-W})$ carries

ι_v to $\chi(W-V)\iota_w$. Here $\chi(V)\in H^*(BG)$ is the Euler class of V, which is the Euler class of its representation bundle. For a split G-spectrum k_G we have an isomorphism

$$k_*^G(S^V;EG_+)\cong k_*(BG^{-V}).$$

For $V\subset W$, the map $f_*: k_*(BG^{-W})\longrightarrow k_*(BG^{-V})$ corresponds under the isomorphisms to the map induced by $e: S^V\longrightarrow S^W$. (The paper of mine cited at the end gives details on all of this.) With our model Y for $\tilde{E}\mathscr{P}$, we now see that

$$k_G^{-q}(Y;EG_+)=k_q^G(Y;EG_+)\cong \lim k_q(BG^{-nV}).$$

Remember that we are working p-adically; we complete spectra at p without change of notation. The inverse limit E_r of Adams spectral sequences of an inverse sequence $\{X_n\}$ of bounded below spectra of finite type over the p-adic integers $\mathbb{Z}_p$ converges from

$$E_2=\operatorname{Ext}_A(\operatorname{colim} H^*(X_n),\mathbb{F}_p)$$

to $\lim\pi_*(X_n)$. With $X_n=k\wedge BG^{-nV}$, this gives an inverse limit of Adams spectral sequences that converges from

$$E_2=\operatorname{Ext}_A(H^*(k)\otimes \operatorname{colim} H^*(BG^{-nV}),\mathbb{F}_p)$$

to $k_G^*(Y;EG_+)$. The colimit is taken with respect to the maps

$$\chi(V): H^*(BG^{-nV})\longrightarrow H^*(BG^{-(n+1)V}).$$

Since $V^H\neq\{0\}$, $\chi(V)$ restricts to zero in $H^*(BH)$ for all $H\in\mathscr{P}$. A theorem of Quillen implies that $\chi(V)$ must be nilpotent if G is not elementary Abelian, and this implies that $E_2=0$. This proves part (i) of Theorem 4.4.

Now assume that $G=(\mathbb{Z}/p)^r$. Let $L=\chi(V)\in H^{2(p^r-1)}(BG)$. Then

$$\operatorname{colim} H^*(BG^{-nV})=H^*(BG)[L^{-1}].$$

It is easy to write L down explicitly, and the heart of part (ii) is the following purely algebraic calculation of Adams, Gunawardena, and Miller, which gives the E_2 term of our spectral sequence.

Theorem 6.1. Let $St=H^*(BG)[L^{-1}]\otimes_A\mathbb{F}_p$, and regard St as a trivial A-module. Then St is concentrated in degree $-r$ and has dimension $p^{r(r-1)/2}$. The quotient homomorphism $\varepsilon: H^*(BG)[L^{-1}]\longrightarrow St$ induces an isomorphism

$$\operatorname{Ext}_A(K\otimes St,\mathbb{F}_p)\longrightarrow \operatorname{Ext}_A(K\otimes H^*(BG)[L^{-1}],\mathbb{F}_p)$$

for any finite dimensional A-module K.

The notation "St" stands for Steinberg: $GL(r, \mathbb{F}_p)$ acts naturally on everything in sight, and St is the classical Steinberg representation.

Let W be the wedge of $p^{r(r-1)/2}$ copies of S^{-r}. It follows by convergence that there is a compatible system of maps $W \longrightarrow BG^{-nV}$ that induces an isomorphism

$$k_*(W) = \pi_*(k \wedge W) \longrightarrow \lim \pi_*(k \wedge BG^{-nV}) \cong k_G^{-*}(Y; EG_+).$$

This gives Theorem 4.4(ii). It also implies the following remarkable corollary, which has had many applications.

Corollary 6.2. The wedge of spheres W is equivalent to the homotopy limit, $BG^{-\infty V}$, of the Thom spectra BG^{-nV}. In particular, with $G = \mathbb{Z}/2$, S^{-1} is equivalent to the spectrum $\operatorname{holim} \mathbb{R}P^{\infty}_{-i}$.

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7. Further generalizations; maps between classifying spaces

Even before the Segal conjecture was proven, Lewis, McClure, and I showed that it would have the following implication. Let G and Π be finite groups and let $A(G, \Pi)$ be the Grothendieck group of Π-free finite $(G \times \Pi)$-sets. Observe that $A(G, \Pi)$ is an $A(G)$-module and let I be the augmentation ideal of $A(G)$.

Theorem 7.1. There is a canonical isomorphism

$$\alpha_I^\wedge : A(G, \Pi)_I^\wedge \longrightarrow [\Sigma^\infty BG_+, \Sigma^\infty B\Pi_+].$$

The map $\alpha : A(G, \Pi) \longrightarrow [\Sigma^\infty BG_+, \Sigma^\infty B\Pi_+]$ can be described explicitly in terms of transfer maps and classifying maps (and the paper of mine cited at the end gives more about the relationship between the algebra on the left and the topology on the right). A Π-free $(G \times \Pi)$-set T determines a principal Π-bundle

$$EG \times_G T \longrightarrow EG \times_G T/\Pi,$$

which is classified by a map $\xi(T) : EG \times_G T/\Pi \longrightarrow B\Pi$. It also determines a (not necessarily connected) finite cover

$$EG \times_G T/\Pi \longrightarrow EG \times_G \{*\} = BG,$$

which has a stable transfer map $\tau(T) : BG_+ \longrightarrow (EG \times_G T/\Pi)_+$. Both ξ and τ are additive in T, and α is the unique homomorphism such that

$$\alpha(T) = \xi(T) \circ \tau(T).$$

In principle, this reduces to pure algebra the problem of computing stable maps between the classifying spaces of finite groups. Many authors have studied the relevant algebra — Nishida, Martino and Priddy, Harris and Kuhn, Benson and Feshback, and Webb, among others — and have obtained a rather good understanding of such maps. We shall not go into these calculations. Rather, we shall place the result in a larger context and describe some substantial generalizations.

Recall that we interpreted the consequences of the Sullivan conjecture for maps between classifying spaces as statements about equivariant classifying spaces. Analogously, Theorem 7.1 is a consequence of a result about the suspension G-spectra of equivariant classifying spaces.

Theorem 7.2. The cohomology theory $\pi_G^*(\cdot; \Sigma^\infty(B_G\Pi)_+)\hat{_I}$ is e-invariant. Therefore the map $EG \longrightarrow *$ induces an isomorphism

$$\pi_G^*(S^0; \Sigma^\infty(B_G\Pi)_+)\hat{_I} \longrightarrow \pi_G^*(EG_+; \Sigma^\infty(B_G\Pi)_+) \cong \pi^*(BG_+; \Sigma^\infty B\Pi_+).$$

The isomorphism on the right comes from XVI.2.4. In degree zero, this is Theorem 7.1. The description of the map α of that result is obtained by describing the map of Theorem 7.2 in nonequivariant terms, using the splitting theorem for $(B_G\Pi)^G$ of VII.2.7, the splitting theorem for the homotopy groups of suspension spectra of XIX.1.2, and some diagram chasing.

We next point out a related consequence of the generalization of the Segal conjecture to families. In it, we let Π be a normal subgroup of a finite group Γ.

Theorem 7.3. The projection $E(\Pi;\Gamma) \longrightarrow *$ induces an isomorphism

$$A(\Gamma)\hat{_{I\mathscr{F}(\Pi;\Gamma)}} \longrightarrow \pi_\Gamma^0(E(\Pi;\Gamma)_+) \cong \pi_G^0(B(\Pi;\Gamma)_+).$$

This is just the degree zero part of Theorem 2.5 for the family $\mathscr{F}(\Pi;\Gamma)$ in the group Γ; the last isomorphism is a consequence of XVI.5.4. With the Burnside ring replaced by the representation ring, a precisely analogous result holds in K-theory, but in that context the result generalizes to an arbitrary extension of compact Lie groups. Of course, these may be viewed as calculations of equivariant characteristic classes. It is natural to ask if Theorems 7.1 and 7.3 admit a common generalization or, better, if the completion theorems of which they are special cases admit a common generalization.

A result along these lines was proven by Snaith, Zelewski, and myself. Here, for the first time in our discussion, we let compact Lie groups enter into the picture. We consider finite groups G and J and a compact Lie group Π. Let $A(G\times J,\Pi)$ be the Grothendieck group of principal $(G\times J,\Pi)$-bundles over finite $(G\times J)$-sets. This is an $A(G\times J)$-module, and we can complete it at the ideal $I\mathscr{F}_G(J)$. As in VII§1, $\mathscr{F}_G(J)$ is the family of subgroups H of $G\times J$ such that $H\cap J=e$.

Theorem 7.4. There is a canonical isomorphism

$$\alpha\hat{_{I\mathscr{F}_G(J)}} : A(G\times J,\Pi)\hat{_{I\mathscr{F}_G(J)}} \longrightarrow [\Sigma^\infty B_G J_+, \Sigma^\infty B_G\Pi_+]_G.$$

Again, the map $\alpha : A(G\times J,\Pi) \longrightarrow [\Sigma^\infty B_G J_+, \Sigma^\infty B_G\Pi_+]_G$ is given on principal $(G\times J,\Pi)$-bundles as composites of equivariant classifying maps and equivariant transfer maps. Although the derivation is not quite immediate, this result is a consequence of an invariance result exactly analogous to the version of the Segal conjecture given in Theorem 3.2.

Theorem 7.5. Let Π be a normal subgroup of a compact Lie group Γ with finite quotient group G. Let S be a multiplicatively closed subset of $A(G)$ and let I be an ideal in $A(G)$. Then the cohomology theory $S^{-1}\boldsymbol{\pi}_G^*(\cdot; B(\Pi;\Gamma)_+)\hat{_I}$ is $\mathscr{H}$-invariant, where

$$\mathscr{H} = \bigcup\{\mathrm{Supp}(P) | P\cap S=\emptyset \ \text{ and } \ P\supset I\}.$$

The statement is identical with that of Theorem 3.2, except that we have substituted $B(\Pi;\Gamma)_+$ for S^0 as the second variable of our bitheory. We could generalize a bit further by substituting $E(\Pi;\Gamma)_+\wedge_\Pi X$ for any finite Γ-CW complex X. What other G-spaces can be substitutedΓ The elementary p-group case of the proof of the Segal conjecture makes it clear that one cannot substitute an arbitrary G-space. In fact, very little more than what we have already stated is known.

Theorem 7.5 specializes to give the analog of Theorem 3.1.

Theorem 7.6. Let $\mathscr{F}$ be a family in G, where $G=\Gamma/\Pi$. The map $E\mathscr{F}\longrightarrow *$ induces an isomorphism

$$\boldsymbol{\pi}_G^*(S^0; B(\Pi;\Gamma)_+)\hat{_{I\mathscr{F}}} \longrightarrow \boldsymbol{\pi}_G^*(E\mathscr{F}_+; \Sigma^\infty B(\Pi;\Gamma)_+).$$

We can restate this in Mackey functor form, as in Theorem 2.5, and then deduce a conceptual formulation generalizing Theorem 1.10.

THEOREM 7.7. For every family $\mathscr{F}$ in G, the map

$$\xi^* : F(K(I\mathscr{F}), \Sigma^\infty B(\Pi;\Gamma)_+) \longrightarrow F(E\mathscr{F}_+, \Sigma^\infty B(\Pi;\Gamma)_+)$$

is an equivalence of G-spectra.

This extends the calculational consequences to the $RO(G)$-graded represented theories. Exactly as in Sections 1–3, all of these theorems reduce to the following special case.

THEOREM 7.8. Let Π be a normal subgroup of a compact Lie group Γ such that the quotient group G is a finite p-group. Then the theory $\boldsymbol{\pi}_G^*(\cdot; B(\Pi;\Gamma)_+)^\wedge_p$ is e-invariant.

The proof is a bootstrap argument starting from the Segal conjecture. When Γ is finite, the result can be deduced from the generalized splitting theorem of XIX.2.1 and the case of the Segal conjecture for Γ that deals with the family of subgroups of Γ that are contained in Π. When Γ is a finite extension of a torus, the result is then deduced by approximating Γ by an expanding sequence of finite groups; this part of the argument entails rather rather elaborate duality and colimit arguments, together with several uses of the generalized Adams isomorphisms XVI.5.4. Finally, the general case is deduced by a transfer argument.

As is discussed in my paper with Snaith and Zelewski, and more extensively in the survey of Lee and Minami, these results connect up with and expands what is known about the Segal conjecture for compact Lie groups.

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CHAPTER XXI

Generalized Tate cohomology

by J. P. C. Greenlees and J. P. May

In this chapter, we will describe some joint work on the generalization of the Tate cohomology of a finite group G with coefficients in a G-module V to the Tate cohomology of a compact Lie group G with coefficients in a G-spectrum k_G. There has been a great deal of more recent work in this area, with many calculations and applications. We shall briefly indicate some of the main directions.

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1. Definitions and basic properties

Tate cohomology has long played a prominent role in finite group theory and its applications. For a finite group G and a G-module V, the Tate cohomology $\hat{H}^*_G(V)$ is obtained as follows. One starts with a free resolution

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}[G]$-modules, dualizes it to obtain a resolution

$$0 \longrightarrow \mathbb{Z} \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots ,$$

renames $P_i^* = P_{-i-1}$, and splices the two sequences together to obtain a $\mathbb{Z}$-graded exact complex P of finitely generated free $\mathbb{Z}[G]$-modules with a factorization $P_0 \longrightarrow \mathbb{Z} \longrightarrow P_{-1}$ of d_0. The complex P is called a "complete resolution of $\mathbb{Z}$", and $\hat{H}^*_G(V)$ is defined to be the cohomology of the cochain complex $\mathrm{Hom}_G(P, V)$. There results a "norm exact sequence" that relates $\hat{H}^*_G(V)$, $H^G_*(V)$, and $H^*_G(V)$.

In connection with Smith theory, Swan generalized this algebraic theory to a cohomology theory $\hat{H}^*_G(X;V)$ on G-spaces X, using $\mathrm{Hom}(P\otimes C_*(X),V)$. (Swan took X to be a G-simplicial complex, but singular chains could be used.) When $G=S^1$ or $G=S^3$ and X is a CW-complex with a cellular action by G, there is a closely analogous theory that is obtained by replacing P by $\mathbb{Z}[u,u^{-1}]$, where u has degree -2 or -4. Here $\mathrm{Hom}(P\otimes C_*(X),V)$ has differential

$$d(p\otimes x)=p\otimes d(x)+pu\otimes i\cdot x,$$

where $i\in C_1(S^1)$ or $i\in C_3(S^3)$ is the fundamental class. For S^1, this is periodic cyclic cohomology theory.

We shall give a very simple definition of a common generalization of these variants of Tate theory. In fact, as part of a general "norm cofibration sequence", we shall associate a Tate G-spectrum $t(k_G)$ to any G-spectrum k_G, where G is any compact Lie group. The construction is closely related to the "stable homotopy limit problem" and to nonequivariant stable homotopy theory.

We have the cofiber sequence

$$EG_+\longrightarrow S^0\longrightarrow \tilde{E}G, \tag{1.1}$$

and the projection $EG_+\longrightarrow S^0$ induces the canonical map of G-spectra

$$\varepsilon : k_G=F(S^0,k_G)\longrightarrow F(EG_+,k_G). \tag{1.2}$$

Taking the smash product of the cofibering (1.1) with the map (1.2), we obtain the following map of cofiberings of G-spectra:

$$\begin{array}{ccccc} k_G\wedge EG_+ & \longrightarrow & k_G & \longrightarrow & k_G\wedge \tilde{E}G \\ \downarrow{\scriptstyle \varepsilon\wedge\mathrm{id}} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon\wedge\mathrm{id}} \\ F(EG_+,k_G)\wedge EG_+ & \longrightarrow & F(EG_+,kG) & \longrightarrow & F(EG_+,kG)\wedge \tilde{E}G. \end{array} \tag{1.3}$$

We have seen most of the ingredients of this diagram in our discussion of the Segal conjecture. We introduce abbreviated notations for these spectra. Define

$$f(k_G)=k_G\wedge EG_+. \tag{1.4}$$

We call $f(k_G)$ the free G-spectrum associated to k_G. It represents the appropriate generalized version of the Borel homology theory $H_*(EG\times_G X)$. Precisely, if

k_G is split with underlying nonequivariant spectrum k, then, by XVI.2.4,

$$f(k_G)_*(X) \cong k_*(EG_+ \wedge_G \Sigma^{Ad(G)} X). \tag{1.5}$$

We refer to the homology theories represented by G-spectra of the form $f(k_G)$ as Borel homology theories. We refer to the cohomology theories represented by the $f(k_G)$ simply as f-cohomology theories. Define

$$f'(k_G) = F(EG_+, k_G) \wedge EG_+. \tag{1.6}$$

It is clear that the map $\varepsilon \wedge \mathrm{Id} : f(k_G) \longrightarrow f'(k_G)$ is always an equivalence, so that the G-spectra $f(k_G)$ and $f'(k_G)$ can be used interchangeably. We usually drop the notation f', preferring to just use f. Define

$$f^-(k_G) = k_G \wedge \tilde{E}G. \tag{1.7}$$

We call $f^-(k_G)$ the singular G-spectrum associated to k_G.

Define

$$c(k_G) = F(EG_+, k_G). \tag{1.8}$$

We call $c(k_G)$ the geometric completion of k_G. The problem of determining the behavior of $\varepsilon : k_G \longrightarrow c(k_G)$ on G-fixed point spectra is the "stable homotopy limit problem". We have already discussed this problem in several cases, and we have seen that it is best viewed as the equivariant problem of comparing the geometric completion $c(k_G)$ with the algebraic completion $(k_G)\hat{_I}$ of k_G at the augmentation ideal of the Burnside ring or of some other ring more closely related to k_G. As one would expect, $c(k_G)$ represents the appropriate generalized version of Borel cohomology $H^*(EG \times_G X)$. Precisely, if k_G is a split G-spectrum with underlying nonequivariant spectrum k, then, by XVI.2.4,

$$c(k_G)^*(X) \cong k^*(EG_+ \wedge_G X). \tag{1.9}$$

We therefore refer to the cohomology theories represented by G-spectra $c(k_G)$ as Borel cohomology theories. We refer to the homology theories represented by the $c(k_G)$ as c-homology theories.

Finally, define

$$t(k_G) = F(EG_+, k_G) \wedge \tilde{E}G = f^- c(k_G). \tag{1.10}$$

We call $t(k_G)$ the Tate G-spectrum associated to k_G. It is the singular part of the geometric completion of k_G. Our primary focus will be on the theories represented by the $t(k_G)$. These are our generalized Tate homology and cohomology theories.

With this cast of characters, and with the abbreviation of $\varepsilon \wedge \mathrm{id}$ to ε, the diagram (1.3) can be rewritten in the form

(1.11)
$$\begin{tikzcd} f(k_G) \arrow[r] \arrow[d, "\varepsilon"', "\simeq"] & k_G \arrow[r] \arrow[d, "\varepsilon"] & f^-(k_G) \arrow[d, "\varepsilon"] \\ f'(k_G) \arrow[r] & c(k_G) \arrow[r] & t(k_G). \end{tikzcd}$$

The bottom row is the promised "norm cofibration sequence". The theories represented by the spectra on this row are all e-invariant.

The definition implies that if X is a free G-spectrum, then

$$t(k_G)_*(X) = 0 \quad \text{and} \quad t(k_G)^*(X) = 0.$$

Similarly, if X is a nonequivariantly contractible G-spectrum, then

$$c(k_G)^*(X) = 0 \quad \text{and} \quad f(k_G)_*(X) = 0.$$

By definition, Tate homology is a special case of c-homology,

(1.12)
$$t(k_G)_n(X) = c(k_G)_n(\tilde{E}G \wedge X).$$

The two vanishing statements imply that Tate cohomology is a special case of f-cohomology,

(1.13)
$$t(k_G)^n(X) \cong f(k_G)^{n+1}(\tilde{E}G \wedge X).$$

In fact, on the spectrum level, the vanishing statements imply the remarkable equivalence

(1.14)
$$t(k_G) \equiv F(EG_+, k_G) \wedge \tilde{E}G \simeq F(\tilde{E}G, \Sigma EG_+ \wedge k_G) \equiv F(\tilde{E}G, \Sigma f(k_G)).$$

It is a consequence of the definition that $t(k_G)$ is a ring G-spectrum if k_G is a ring G-spectrum, and then $t(k_G)^G$ is a ring spectrum.

Much of the force of our definitional framework comes from the fact that (1.11) is a diagram of genuine and conveniently explicit G-spectra indexed on representations, so that all of the $\mathbb{Z}$-graded cohomology theories in sight are $RO(G)$-gradable. The $RO(G)$-grading is essential to the proofs of many of the results discussed below. Nevertheless, it is interesting to give a naive reinterpretation of the fixed point cofibration sequence associated to the norm sequence.

With our definitions, the Tate homology of X is

$$t(k_G)_*(X) = \pi_*((t(k_G) \wedge X)^G).$$

Since any k_G is e-equivalent to j_G for a naive G-spectrum j_G and Tate theory is e-invariant, we may as well assume that $k_G = i_* j_G$. Provided that X is a finite G-CW complex, the spectrum $(t(k_G) \wedge X)^G$ is then equivalent to the cofiber of an appropriate transfer map

$$\begin{array}{c} (j_G \wedge \Sigma^{Ad(G)} X)_{hG} \equiv (j_G \wedge EG_+ \wedge \Sigma^{Ad(G)} X)/G \\ \big\downarrow \\ (j_G \wedge X)^{hG} \equiv F(EG_+, j_G \wedge X)^G. \end{array}$$

A description like this was first written down by Adem, Cohen, and Dwyer. When G is finite, $X = S^0$, and j_G is a nonequivariant spectrum k given trivial action by G, this reduces to

$$k \wedge BG_+ \longrightarrow F(BG_+, k).$$

The interpretation of Tate theory as the third term in a long sequence whose other terms are Borel k-homology and Borel k-cohomology is then transparent.

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2. Ordinary theories; Atiyah-Hirzebruch spectral sequences

Let M be a Mackey functor and V be the $\pi_0(G)$-module $M(G/e)$. The norm sequence of HM depends only on V: if M and M' are Mackey functors for which $M(G/e) \cong M'(G/e)$ as $\pi_0(G)$-modules, then the norm cofibration sequences of HM and HM' are equivalent. We therefore write

$$(2.1) \qquad \hat{H}^G_*(X; V) = t(HM)_*(X) \quad \text{and} \quad \hat{H}^*_G(X; V) = t(HM)^*(X).$$

For finite groups G, this recovers the Tate-Swan cohomology groups, as the notation anticipates. We sketch the proof. The simple objects to the eyes of ordinary cohomology are cells, and the calculation depends on an analogue of the skeletal filtration of a CW complex that mimics the construction of a complete resolution. The idea is to splice the skeletal filtration of EG_+ with its Spanier-Whitehead dual. More precisely, we define an integer graded filtration on $\tilde{E}G$, or rather on its suspension spectrum, by letting

$$F^i\tilde{E}G = \begin{cases} \tilde{E}G^{(i)} = S^0 \cup C(EG_+^{(i-1)}) & \text{for } i \geq 1 \\ S^0 & \text{for } i = 0 \\ D(\tilde{E}G^{(-i)}) & \text{for } i \leq -1. \end{cases}$$

The ith subquotient of this filtration is a finite wedge of spectra $S^i \wedge G_+$, and the E^1 term of the spectral sequence that is obtained by applying ordinary nonequivariant integral homology is a complete resolution of $\mathbb{Z}$. Therefore, if one takes the smash product of this filtration with the skeletal filtration of X and applies an equivariant cohomology theory $k_G^*(\cdot)$, one obtains the "Atiyah-Hirzebruch-Tate" spectral sequence

$$E_2^{p,q} = \hat{H}_G^p(X; k^q) \Longrightarrow t(k)_G^{p+q}(X). \tag{2.2}$$

Here k is the underlying nonequivariant spectrum of k_G, and $k^q = \pi_{-q}(k)$ regarded as a G-module. To see that the target is Tate cohomology as claimed, note that the "cohomological" description (1.14) of the Tate spectrum gives

$$t(k)_G^*(X) = [\tilde{E}G \wedge X, k \wedge \Sigma EG_+]_G^*.$$

There are compensating shifts of grading in the identifications of the E_2 terms and of the target, so that the grading works out as indicated in (2.2).

When $k_G = HM$, the spectral sequence collapses at the E_2-term by the dimension axiom, and this proves that $t(HM)_G^*(X)$ is the Tate-Swan cohomology of X. In general, we have a whole plane spectral sequence, but it converges strongly to $t(k_G)^*(X)$ provided that there are not too many non-zero higher differentials. When k_G is a ring spectrum, it is a spectral sequence of differential algebras.

With a little care about the splice point and the model of EG used, we can apply part of this construction to compact Lie groups G of dimension $d > 0$. In this case, there is a "gap" in the appropriate filtration of $\tilde{E}G$:

$$F^i\tilde{E}G = \begin{cases} \tilde{E}G^{(i)} = S^0 \cup C(EG_+^{(i-1)}) & \text{for } i \geq 1 \\ S^0 & \text{for } -d \leq i \leq 0 \\ D(\tilde{E}G^{(-i)}) & \text{for } i < -d. \end{cases}$$

The gap is dictated by the fact that the Spanier-Whitehead dual of G_+ is $G_+ \wedge S^{-d}$.

In the case of Eilenberg-MacLane spectra, this gives an explicit chain level calculation of the coefficient groups $\hat{H}_*^G(V) \equiv \hat{H}_*^G(S^0; V)$ in terms of the ordinary (unreduced) homology and cohomology groups of the classifying space BG:

$$\hat{H}^n_G(V) = t(HM)^n \cong \begin{cases} H^n(BG;V) & \text{if } 0 \le n \\ 0 & \text{if } -d \le n < 0 \\ H_{-n-1-d}(BG;V) & \text{if } n \le -d-1. \end{cases} \tag{2.3}$$

However, we would really like a chain complex for calculating the ordinary Tate cohomology of G-CW complexes X, and for groups of positive dimension it is not obvious how to make one. At present, we only have such descriptions for $G = S^1$ and $G = S^3$. In these cases, we can exploit the obvious cell structure on G and the standard models $S(\mathbb{C}^\infty)$ and $S(\mathbb{H}^\infty)$ for EG to put a cunning G-CW structure on $EG_+ \wedge X$ and to derive an appropriate filtration of $\tilde{E}G \wedge X$ when G acts cellularly on X. In the case of S^1, the resulting chain complex is a cellular version of Jones' complex for cyclic cohomology, and this proves that $t(H\underline{\mathbb{Z}})^*_{S^1}(X)$ is the periodic cyclic cohomology $\hat{H}^*_{S^1}(X)$, as defined by Jones in terms of the singular complex of X. There is a precisely analogous identification in the case of S^3. In general, the problem of giving $\tilde{E}G \wedge X$ an appropriate filtration appears to be intractable, although a few other small groups are under investigation.

Despite this difficulty, we still have spectral sequences of the form (2.2) for general compact Lie groups G, where $k^q = \pi_{-q}(k)$ is now regarded as a $\pi_0(G)$-module. However, in the absence of a good filtration of $\tilde{E}G \wedge X$, we construct the spectral sequences by using a Postnikov filtration of k_G. In this generality, the ordinary Tate groups $\hat{H}^*_G(X;V)$ used to describe the E_2 terms are not familiar ones, and systematic techniques for their calculation do not appear in the literature. One approach to their calculation is to use the skeletal filtration of X together with (2.3) and change of groups. More systematic approaches involve the construction of spectral sequences that converge to $\hat{H}^*_G(X;V)$, and there are several sensible candidates. This is an area that needs further investigation, and we shall say no more about it here.

We have similar and compatible spectral sequences for Borel and f-cohomology, and in these cases too the E_2-terms depend only on the graded $\pi_0(G)$-module k^*, as one would expect from the e-invariance of the bottom row of Diagram (1.11). This very weak dependence on k_G makes the bottom row much more calculationally accessible than the top row.

3. Cohomotopy, periodicity, and root invariants

For finite groups G, the Segal conjecture directly implies the determination of the Tate spectrum associated to the sphere spectrum S_G. Indeed, we have

$$t(S_G) = F(EG_+, S^0) \wedge \tilde{E}G \simeq (S_G)^\wedge_I \wedge \tilde{E}G \simeq (\Sigma^\infty \tilde{E}G)^\wedge_I. \tag{3.1}$$

For instance, if G is a p-group, then

$$t(S_G) \simeq (\Sigma^\infty \tilde{E}G)^\wedge_p, \tag{3.2}$$

and we may calculate from the splitting theorem XIX.1.1 that, after completion,

$$t(S_G)^G_*(X) = \bigoplus_{(H)\neq(1)} \pi_*(EW_G(H)_+ \wedge_{W_G(H)} X^H). \tag{3.3}$$

With $X = S^0$, the summand for $H = G$ is $\pi_*(S^0)$, and it follows that, for each G, the Atiyah-Hirzebruch-Tate spectral sequence defines a "root invariant" on the stable stems. Its values are cosets in the Tate cohomology group $\hat{H}^*(G; \pi_*(S^0))$. Essentially, the root invariant assigns to an element $\alpha \in \pi_*(S^0)$ all elements of E^2 of the appropriate filtration that project to the image of α in the E^∞ term of the spectral sequence.

These invariants have not been much investigated beyond the classical case of $G = C_p$, the cyclic group of order p. In this case, our construction agrees with earlier constructions of the root invariant. Indeed, this is a consequence of the observations that, if $G = C_2$ and $k_G = i_*k$ is the G-spectrum associated to a non-equivariant spectrum k, then

$$t(k_G)^G \simeq \operatorname{holim}(\mathbb{R}P^\infty_{-i} \wedge \Sigma k) \tag{3.4}$$

and, if $G = C_p$ for an odd prime p and $k_G = i_*k$, then

$$t(k_G)^G \simeq \operatorname{holim}(L^\infty_{-i} \wedge \Sigma k), \tag{3.5}$$

where L^∞_{-i} is the lens space analog of $\mathbb{R}P^\infty_{-i}$. Taking $k = S$, there results a spectral sequence that agrees with our Atiyah-Hirzebruch-Tate spectral sequence and was used in the classical definition of the root invariant.

Similarly, if G is the circle group and $k_G = i_*k$, then

$$t(k_G)^G \simeq \operatorname{holim}(\mathbb{C}P^\infty_{-i} \wedge \Sigma^2 k). \tag{3.6}$$

These are all special cases of a phenomenon that occurs whenever G acts freely on the unit sphere of a representation V, and this phenomenon is the source of periodic behavior in Tate theory. The point is that the union of the S^{nV} is then a model for $\tilde{E}G$, and we can use this model to evaluate the right side as a homotopy limit in the equivalence (1.14). This immediately gives (3.4)–(3.6). These equivalences allow us to apply nonequivariant calculations of Davis, Mahowald, and others of spectra on the right sides to study equivariant theories. We will say a little more about this in Section 6. It also gives new insight into the nonequivariant theories. In particular, if k is a ring spectrum, then $t(k_G)^G$ is a ring spectrum. Looking nonequivariantly at the right sides, this is far from clear.

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D. M. Davis, D. C. Johnson, J. Klippenstein, M. Mahowald and S. Wegmann. The spectrum $(P \wedge BP\langle 2\rangle)_{-\infty}$. Trans. American Math. Soc. 296(1986) 95-110.

4. The generalization to families

The theory described above is only part of the story: it admits a generalization in which the universal free G-space EG is replaced by the universal $\mathscr{F}$-space $E\mathscr{F}$ for any family $\mathscr{F}$ of subgroups of G. The definitions above deal with the case $\mathscr{F} = \{e\}$, and there is a precisely analogous sequence of definitions for any other family. We have the cofibering

$$E\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{F}, \tag{4.1}$$

and the projection $E\mathscr{F}_+ \longrightarrow S^0$ induces a G-map

$$\varepsilon : k_G = F(S^0, k_G) \longrightarrow F(E\mathscr{F}_+, k_G). \tag{4.2}$$

Taking the smash product of the cofibering (4.1) with the map (4.2), we obtain the following map of cofiberings of G-spectra:

$$\begin{array}{ccccc} k_G \wedge E\mathscr{F}_+ & \longrightarrow & k_G & \longrightarrow & k_G \wedge \tilde{E} \\ \downarrow{\scriptstyle \varepsilon\wedge\mathrm{id}} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon\wedge\mathrm{id}} \\ F(E\mathscr{F}_+, kG) \wedge E\mathscr{F}_+ & \longrightarrow & F(E\mathscr{F}_+, kG) & \longrightarrow & F(E\mathscr{F}_+, kG) \wedge \tilde{E}\mathscr{F}. \end{array} \tag{4.3}$$

Define the $\mathscr{F}$-free G-spectrum associated to k_G to be

$$f_{\mathscr{F}}(k_G) = k_G \wedge E\mathscr{F}_+. \tag{4.4}$$

We refer to the homology theories represented by G-spectra $f_{\mathscr{F}}(k_G)$ as $\mathscr{F}$-Borel homology theories. Define

$$f'_{\mathscr{F}}(k_G) = F(E\mathscr{F}_+, k_G) \wedge E\mathscr{F}_+. \tag{4.5}$$

Again, $\varepsilon \wedge \mathrm{Id} : f_{\mathscr{F}}(k_G) \longrightarrow f'_{\mathscr{F}}(k_G)$ is an equivalence, hence we usually use the notation $f_{\mathscr{F}}$. Define the $\mathscr{F}$-singular G-spectrum associated to k_G to be

$$f_{\mathscr{F}}^{-}(k_G) = k_G \wedge \tilde{E}\mathscr{F}. \tag{4.6}$$

Define the geometric $\mathscr{F}$-completion of kG to be

$$c_{\mathscr{F}}(k_G) = F(E\mathscr{F}_+, k_G). \tag{4.7}$$

We refer to the cohomology theories represented by G-spectra $c_{\mathscr{F}}(k_G)$ as $\mathscr{F}$-Borel cohomology theories. The map $\varepsilon : k_G \longrightarrow c_{\mathscr{F}}(k_G)$ of (4.2) is the object of study of such results as the generalized Atiyah-Segal completion theorem and the generalized Segal conjecture of Adams-Haeberly-Jackowski-May. As in these results, one version of the $\mathscr{F}$-homotopy limit problem is the equivariant problem of comparing the geometric $\mathscr{F}$-completion $c_{\mathscr{F}}(k_G)$ with the algebraic completion $(k_G)\hat{}_{I\mathscr{F}}$ of k_G at the ideal $I\mathscr{F}$ of the Burnside ring or at an analogous ideal in a ring more closely related to k_G. Observe that we usually do not have analogs of (1.5) and (1.9) for general families $\mathscr{F}$; the Adams isomorphism XVI.5.4 and the discussion around it are relevant at this point.

Define

$$t_{\mathscr{F}}(k_G) = F(E\mathscr{F}_+, k_G) \wedge \tilde{E}\mathscr{F} = f_{\mathscr{F}}^{-} c_{\mathscr{F}}(k_G). \tag{4.8}$$

We call $t_{\mathscr{F}}(k_G)$ the $\mathscr{F}$-Tate G-spectrum associated to k_G. These G-spectra represent $\mathscr{F}$-Tate homology and cohomology theories. With this cast, and with the abbreviation of $\varepsilon \wedge \mathrm{id}$ to ε, the diagram (4.3) can be rewritten in the form

$$\begin{tikzcd} f_{\mathscr{F}}(k_G) \arrow[r] \arrow[d, "\varepsilon"', "\simeq"] & k_G \arrow[r] \arrow[d, "\varepsilon"] & f_{\mathscr{F}}^{-}(k_G) \arrow[d, "\varepsilon"] \\ f'_{\mathscr{F}}(k_G) \arrow[r] & c_{\mathscr{F}}(k_G) \arrow[r] & t_{\mathscr{F}}(k_G). \end{tikzcd} \tag{4.9}$$

We call the bottom row the "$\mathscr{F}$-norm cofibration sequence". The theories represented by the spectra on this row are all $\mathscr{F}$-invariant.

The diagram leads to a remarkable and illuminating relationship between the Tate theories and the $\mathscr{F}$-homotopy limit problem. Recall that $I\mathscr{F} \subset A(G)$ is the intersection of the kernels of the restrictions $A(G) \longrightarrow A(H)$ for $H \in \mathscr{F}$.

THEOREM 4.10. *The spectra $c_{\mathscr{F}}(k_G)$ are $I\mathscr{F}$-complete. The spectra $f_{\mathscr{F}}(kG)$ and $t_{\mathscr{F}}(k_G)$ are $I\mathscr{F}$-complete if k_G is bounded below.*

We promised in XX§1 to relate the questions of when

$$\xi^* : (k_G)\hat{_{I\mathscr{F}}} = F(K(I\mathscr{F}), k_G) \longrightarrow F(E\mathscr{F}_+, k_G) = c_{\mathscr{F}}(k_G)$$

and

$$\xi_* : k_G \wedge E\mathscr{F}_+ \longrightarrow k_G \wedge K(I\mathscr{F})$$

are equivalences. The answer is rather surprising.

THEOREM 4.11. Let k_G be a ring G-spectrum, where G is finite. Then ξ_* is an equivalence if and only if ξ^* is an equivalence and $t_{\mathscr{F}}(k_G)$ is rational.

The proof is due to the first author and will be discussed in XXIV§8. We shall turn to relevant examples in the next section.

When G is finite and k_G is an Eilenberg-MacLane G-spectrum HM, the $\mathscr{F}$-Tate G-spectrum $t_{\mathscr{F}}(HM)$ represents the generalization to homology and cohomology theories on G-spaces and G-spectra of certain "Amitsur-Dress-Tate cohomology theories" $\hat{H}^*_{\mathscr{F}}(M)$ that figure prominently in induction theory. We again obtain generalized Atiyah-Hirzebruch-Tate spectral sequences in the context of families. These vastly extend the web of symmetry relations relating equivariant theory with the stable homotopy groups of spheres. In particular, for a finite p-group G, if we use the family $\mathscr{P}$ of all proper subgroups of G, we obtain a spectral sequence whose E_2-term is $\hat{H}^{\mathscr{P}}_*(\underline{\pi}^G_*)$ and which converges to $(\pi_*)\hat{_p}$. We have moved the groups $\pi_*(BWH_+)$ from the target to ingredients in the calculation of E_2. In this spectral sequence the "root invariant" of an element $\alpha \in \pi_q$ lies in degree at least $q(|G| - 1)$. The root invariant measures where elements are detected in E^2 of the spectral sequence, and the dependence on the order of G indicates an increasing dependence of lower degree homotopy groups of spheres on higher degree homotopy groups of classifying spaces.

More generally, if G is any finite group, we use the family $\mathscr{P}$ to obtain two related spectral sequences, both of which converge to the completion of the nonequivariant stable homotopy groups of spheres at $n(\mathscr{P})$, where $n(\mathscr{P})$ is the product of those primes p such that $\mathbb{Z}/p\mathbb{Z}$ is a quotient of G. For example, if G is a nonabelian group of order pq, $p < q$, then $n(\mathscr{P}) = p$ and the spectral sequences provide a mechanism for the prime q to affect stable homotopy groups at the prime p. One of the spectral sequences is the Atiyah-Hirzebruch-Tate spectral sequence whose E_2-term is the Amitsur-Dress-Tate homology $\hat{H}^{\mathscr{P}}_*(\underline{\pi}^G_*)$. The other comes from a filtration of $\tilde{E}G$ in terms of the regular representation of G. These spectral sequences lead to new equivariant root invariants, and the basic Bredon-Jones-Miller root invariant theorem generalizes to the spectral sequence constructed by use of the regular representation.

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5. Equivariant K-theory

Our most interesting calculation shows that, for any finite group G, $t(K_G)$ is a rational G-spectrum, namely

$$t(K_G) \simeq \bigvee K(\hat{J} \otimes \mathbb{Q}, 2i), \tag{5.1}$$

where $\hat{J}$ is the Mackey functor of completed augmentation ideals of representation rings and i ranges over the integers. In this case, the relevant Atiyah-Hirzebruch-Tate spectral sequence is rather amazing. Its E_2-term is torsion, being annihilated by multiplication by the order of G. If G is cyclic, then $E_2 = E_\infty$ and the spectral sequence certainly converges strongly. In general, the E_2-term depends solely on the classical Tate cohomology of G and not at all on its representation ring, whereas $t(K_G)^*$ depends solely on the representation ring and not at all on the Tate cohomology. Needless to say, the proof of (5.1) is not based on use of the spectral sequence.

In fact, and the generalization is easier to prove than the special case, $t_{\mathscr{F}}(K_G)$ turns out to be rational for every family $\mathscr{F}$. Again, there results an explicit calculation of $t_{\mathscr{F}}(K_G)$ as a wedge of Eilenberg-MacLane spectra. Let $J\mathscr{F}$ be the intersection of the kernels of the restrictions $R(G) \longrightarrow R(H)$ for $H \in \mathscr{F}$. It is clear by character theory that

$$J\mathscr{F} = \{\chi | \chi(g) = 0 \text{ if the group generated by } g \text{ is in } \mathscr{F}\},$$

and we define a rationally complementary ideal $J'\mathscr{F}$ by

$$J'\mathscr{F} = \{\chi | \chi(g) = 0 \text{ if the group generated by } g \text{ is not in } \mathscr{F}\}.$$

Then (5.1) generalizes to

$$t_{\mathscr{F}}(K_G) \simeq \bigvee K((R/J'\mathscr{F})\hat{_{J\mathscr{F}}} \otimes \mathbb{Q}, 2i), \tag{5.2}$$

where $(R/J'\mathscr{F})_{J\mathscr{F}}^{\wedge}$ denotes the Mackey functor whose value at G/H is the completion at the ideal $J(\mathscr{F}|_H)$ of the quotient $R(H)/J'(\mathscr{F}|H)$. This is consistent with (5.1) since, when $\mathscr{F} = \{e\}$, $J'(\mathscr{F}|H)$ is a copy of $\mathbb{Z}$ generated by the regular representation of H and JH maps isomorphically onto $R(H)/\mathbb{Z}$. It follows in all cases that the completions $t_{\mathscr{F}}(K_G)_{I\mathscr{F}}^{\wedge}$ are contractible.

The following folklore result is proven in our paper on completions at ideals of the Burnside ring. On passage to π_0^G, the unit $S_G \longrightarrow K_G$ induces the homomorphism $A(G) \longrightarrow R(G)$ that sends a finite set X to the permutation representation $\mathbb{C}[X]$. We regard $R(G)$-modules as $A(G)$-modules by pullback.

THEOREM 5.3. The completion of an $R(G)$-module M at the ideal $J\mathscr{F}$ of $R(G)$ is isomorphic to the completion of M at the ideal $I\mathscr{F}$ of the Burnside ring $A(G)$.

In fact, the proof shows that the ideals $I\mathscr{F}R(G)$ and $J\mathscr{F}$ of $R(G)$ have the same radical. Therefore the generalized completion theorem of Adams-Haeberly-Jackowski-May discussed in XIV.6.1 implies that

$$\xi^* : (K_G)_{I\mathscr{F}}^{\wedge} \longrightarrow F(E\mathscr{F}_+, K_G)$$

is an equivalence. By (5.2) and Theorem 4.11, this in turn implies that

$$\xi_* : k_G \wedge E\mathscr{F}_+ \longrightarrow k_G \wedge K(I\mathscr{F})$$

is an equivalence. In fact, the latter result was proven by the first author before the implication was known; we shall explain his argument and discuss the algebra behind it in Chapter XXIV.

As a corollary of the calculation of $t(K_G)$, we obtain a surprisingly explicit calculation of the nonequivariant K-homology of the classifying space BG:

$$K_0(BG) \cong \mathbb{Z} \quad \text{and} \quad K_1(BG) \cong J(G)_{J(G)}^{\wedge} \otimes (\mathbb{Q}/\mathbb{Z}). \tag{5.4}$$

In fact, (5.1) and (5.4) both follow easily once we know that $t(K_G)$ is rational. Given that, we have the exact sequence

$$\cdots \to K_*^G(EG_+) \otimes \mathbb{Q} \to K_G^*(EG_+) \otimes \mathbb{Q} \to t(K)_G^* \to \cdots ,$$

which turns out to be short exact. The Atiyah-Segal theorem shows that

$$K_G^*(EG_+) \otimes \mathbb{Q} \cong R(G)_J^{\wedge}[\beta, \beta^{-1}] \otimes \mathbb{Q},$$

where β is the Bott element. Rationally, the K-homology of EG_+ is a summand of K_*^G, and in fact $K_*^G(EG_+) \otimes \mathbb{Q} \cong \mathbb{Q}[\beta, \beta^{-1}]$. It is not hard to identify the maps and conclude that

$$t(K)_G^* = \{R(G)/\mathbb{Z}\}_J^\wedge[\beta, \beta^{-1}] \otimes \mathbb{Q}.$$

Since, as explained in XIX§5, all rational G-spectra split, this gives the exact equivariant homotopy type claimed in (5.1). Now we can deduce (5.4) by analysis of the *integral* norm sequence, using the Atiyah-Segal completion theorem to identify $K_G^*(EG_+)$.

We must still say something about why $t(K_G)$ and all other $t_{\mathscr{F}}(K_G)$ are rational. An inductive scheme reduces the proof to showing that $t_{\mathscr{F}}(K_G) \wedge \tilde{E}\mathscr{P}$ is rational, where $\mathscr{P}$ is the family of proper subgroups of G. If V is the reduced regular complex representation of V, then $S^{\infty V}$ is a model for $\tilde{E}\mathscr{P}$. It follows that, for any K_G-module spectrum M and any spectrum X, $(M \wedge \tilde{E}\mathscr{P})_*^G(X)$ is the localization of $M_*^G(X)$ away from the Euler class (which is the total exterior power) $\lambda(V) \in R(G)$. Since $\lambda(V)$ is in $J\mathscr{P}$, it restricts to zero in all proper subgroups. Since the product over the cyclic subgroups C of G of the restrictions $R(G) \longrightarrow R(C)$ is an injection, $\lambda(V) = 0$ and the conclusion holds trivially unless G is cyclic. In that case, the Atiyah-Hirzebruch-Tate spectral sequence for $t_{\mathscr{P}}(K_G)_*(X)$ gives that primes that do not divide the order n of G act invertibly since n annihilates the E^2-term. An easy calculational argument in representation rings handles the remaining primes.

The evident analogs of all of these statements for real K-theory are also valid.

In the case of connective K-theory, we do not have the same degree of periodicity to help, and the calculations are harder. Results of Davis and Mahowald give the following result.

THEOREM 5.5. If $G = C_p$ for a prime p, then

$$t(ku_G) \simeq \prod_{n \in \mathbb{Z}} \Sigma^{2n} H(\hat{J}),$$

and similarly for connective real K-theory.

This result led us to the overoptimistic conjecture that its conclusion would generalize to arbitrary finite groups. However, Bayen and Bruner have shown that the conjecture fails for both real and complex connective K-theory.

Finally, we must point out that the restriction to finite groups in the discussion above is essential; even for $G = S^1$ something more complicated happens since in that case $t(K_G)^G$ is a homotopy inverse limit of wedges of even suspensions of

K and each even degree homotopy group of $t(K_G)^G$ is isomorphic to $\mathbb{Z}[[\chi]][\chi^{-1}]$, where $1-\chi$ is the canonical irreducible one-dimensional representation of G. In particular, $t(K_G)$ is certainly not rational. Similarly, still taking $G = S^1$, each even degree homotopy group of $t(k_G)^G$ is isomorphic to $\mathbb{Z}[[\chi]]$. In this case, we can identify the homotopy type of the fixed point spectrum:

$$t(ku_{S^1})^{S^1} \simeq \prod_{n\in\mathbb{Z}} \Sigma^{2n} ku_{S^1}. \tag{5.6}$$

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6. Further calculations and applications

Philosophically, one of the main differences between the calculation of the Tate K-theory for finite groups and for the circle group is that the Krull dimension of $R(G)$ is one in the case of finite groups and two in the case of the circle group. Quite generally, the complexity of the calculations increases with the Krull dimension of the coefficient ring. It is relevant that the Krull dimension of $R(G)$ for a compact connected Lie group G is one greater than its rank.

For finite groups, most calculations that have been carried out to date concern ring G-spectra k_G, like those that represent K-theory, that are so related to cobordism as to have Thom isomorphisms of the general form

$$k_*^G(\Sigma^V X) \cong k_*^G(\Sigma^{|V|} X) \tag{6.1}$$

for all complex representations V. Let $e(V) : S^0 \longrightarrow S^V$ be the inclusion. Applying $e(V)^*$ to the element $1 \in k_G^0(S^0) \cong k_G^V(S^V)$, we obtain an element of $k_G^V(S^0) = k_{-V}^G(S^0)$. The Thom isomorphism yields an isomorphism between this group and the integer coefficient group $k_{-|V|}^G$, and there results an Euler class $\chi(V) \in k_{-|V|}^G$. As in our indication of the rationality of $t(K_G)$, localizations and other algebraic constructions in terms of such Euler classes can often lead to explicit calculations.

This works particularly well in cases, such as p-groups, where G acts freely on a product of unit spheres $S(V_1) \times \cdots \times S(V_n)$ for some representations $V_1, \ldots, V_n$. This implies that the smash product $S(\infty V_1)_+ \wedge \cdots \wedge S(\infty V_n)_+$ is a model for EG_+, and there results a filtration of $\tilde{E}G$ that has subquotients given by wedges of smash products of spheres. This gives rise to a different spectral sequence for the computation of $t(k_G)^*_G(X)$. When $X = S^0$, the E_2-term can be identified as the "Čech cohomology $\check{H}^*_{J'}(k^*(BG))$ of the k^*_G-module $k^*(BG_+)$ with respect to the ideal $J' = (\chi(V_1), \cdots, \chi(V_n)) \subset k^*_G$". The relevant algebraic definitions will be given in Chapter XXIV. These groups depend only on the radical of J', and, when k^*_G is Noetherian, it turns out that J' has the same radical as the augmentation ideal $J = \mathrm{Ker}(k^*_G \longrightarrow k^*)$.

The interesting mathematics begins with the calculation of the E_2-term, where the nature of the Euler classes for the particular theory becomes important. In fact, this spectral sequence collapses unusually often because the complexity is controlled by the Krull dimension of the coefficients. In cases where one can calculate the coefficients $t(k)^*_G$, one can often also deduce the homotopy type of the fixed point spectrum $t(k_G)^G$ because $t(k_G)^G$ is a module spectrum over k. However, the periodic and connective cases have rather different flavors. In the periodic case the algebra of the coefficients has a field-like appearance and is more often enough to determine the homotopy type of the fixed point spectrum $t(k_G)^G$. In the connective case the algebra of the coefficients in the answer has the appearance of a complete local ring and some sort of Adams spectral sequence argument seems to be necessary to deduce the topology from the algebra. In very exceptional circumstances, such as the use of rationality in the case of K_G, one can go on to deduce the equivariant homotopy type of $t(k_G)$.

In the discussion that follows, we consider equivariant forms k_G of some familiar nonequivariant theories k. We may take k_G to be i_*k, but any split G-spectrum with underlying nonequivariant spectrum k could be used instead. Technically, it is often best to use $F(EG_+, i_*k)$. This has the advantage that its coefficients can often be calculated, and it can be thought of as a geometric completion of any other candidate (and an algebraic completion of any candidate for which a completion theorem holds).

The most visible feature of the calculations to date is that the Tate construction tends to decrease chromatic periodicity. We saw this in the case of K_G, where the periodicity reduced from one to zero. This appears in especially simple form in a theorem of Greenlees and Sadofsky: if $K(n)$ is the nth Morava K-theory spectrum, whose coefficient ring is the graded field

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}], \quad \deg v_n = 2p^n - 2,$$

then

$$t(K(n)_G) \simeq *. \tag{6.2}$$

In fact, this is a quite easy consequence of Ravenel's result that $K(n)^*(BG_+)$ is finitely generated over $K(n)^*$. Another example of this nature is a calculation of Fajstrup, which shows that if the spectrum KR that represents K-theory with reality is regarded as a C_2-spectrum, then the associated Tate spectrum is trivial.

These calculations illustrate another phenomenon that appears to be general: it seems that the Tate construction reduces the Krull dimension of periodic theories. More precisely, the Krull dimension of $t(k_G)^0_G$ is usually less than that of k^0_G. In the case of Morava K-theory, one deduces from Ravenel's result that $K(n)^0_G$ is finite over $K(n)^0$ and thus has dimension 0. The contractibility of $t(K(n)_G)$ can then be thought of as a degenerate form of dimension reduction. More convincingly, work of Greenlees and Sadofsky shows that for many periodic theories for which k^0_G is one dimensional, $t(k_G)^0_G$ is finite dimensional over a field. The higher dimensional case is under consideration by Greenlees and Strickland.

This reduction of Krull dimension is reflected in the E_2-term of the spectral sequence cited above. When k is v_n-periodic for some n, one typically first proves that some v_i, $i < n$ is invertible on $t(k_G)$ and then uses the localisation of the norm sequence

$$\cdots \to k^G_*(EG_+)\left[v_i^{-1}\right] \to k^*_G(EG_+)\left[v_i^{-1}\right] \to t(k_G)^*_G \to \cdots$$

to assist calculations. For example, consider the spectra $E(n)$ with coefficient rings

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}].$$

Since there is a cofiber sequence $E(2)/p \xrightarrow{v_1} E(2)/p \to K(2)$, we deduce from (6.2) that v_1 is invertible on $t((E(2)/p)_G)$. More generally v_{n-1} is invertible on a suitable completion of $t(E(n)_G)$.

The intuition that the Tate construction lowers Krull dimension is reflected in the following conjecture about the spectra $BP\langle n\rangle$ with coefficient rings

$$BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n].$$

Conjecture 6.3 (Davis-Johnson-Klippenstein-Mahowald-Wegmann).

$$t(BP\langle n\rangle_{C_p})^{C_p} \simeq \prod_{n\in\mathbb{Z}} \Sigma^{2n} BP\langle n-1\rangle_p^\wedge.$$

The cited authors proved the case $n = 2$; the case $n = 1$ was due to Davis and Mahowald. Since $BP\langle n\rangle_*$ has Krull dimension $n+1$, the depth of the conjecture increases with n.

We end by pointing the reader to what is by far the most striking application of generalized Tate cohomology. In a series of papers, Madsen, Bökstedt, Hesselholt, and Tsalidis have used the case of S^1 and its subgroups to carry out fundamentally important calculations of the topological cyclic homology and thus of the algebraic K-theory of number rings. It would take us too far afield to say much about this. Madsen has given two excellent surveys. In another direction, Hesselholt and Madsen have calculated the coefficient groups of the S^1-tate spectrum associated to the periodic J-theory spectrum at an odd prime. The calculation is consistent with the following conjecture.

Conjecture 6.4 (Hesselholt-Madsen).

$$t(J_G)^{S^1} \simeq K'(1) \vee \Sigma K'(1) \vee (\prod_{n\in\mathbb{Z}} \Sigma^{2n+1} K)/(\bigvee_{n\in\mathbb{Z}} \Sigma^{2n+1} K),$$

where $K'(1)$ is the Adams summand of p-complete K-theory with homotopy groups concentrated in degrees $\equiv 0 \bmod 2(p-1)$.

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CHAPTER XXII

Brave new algebra

1. The category of S-modules

Let us return to the introductory overview of the stable homotopy category given in XII§1. As said there, Elmendorf, Kriz, Mandell, and I have gone beyond the foundations of Chapter XII to the construction of a new category of spectra, the category of "S-modules", that has a smash product that is symmetric monoidal (associative, commutative, and unital up to coherent natural isomorphisms) on the point-set level. The complete treatment is given in [EKMM], and an exposition has been given in [EKMM′]. The latter emphasizes the logical development of the foundations. Here, instead, we will focus more on the structure and applications of the theory. Working nonequivariantly in this chapter, we will describe the new categories of rings, modules, and algebras and summarize some of their more important applications. All of the basic theory generalizes to the equivariant context and, working equivariantly, we will return to the foundations and outline the construction of the category of S-modules in the next chapter. We begin work here by summarizing its properties.

An S-module is a spectrum (indexed on some fixed universe U) with additional structure, and a map of S-modules is a map of spectra that preserves the additional structure. The sphere spectrum S and, more generally, any suspension spectrum $\Sigma^\infty X$ has a canonical structure of S-module. The category of S-modules is denoted $\mathscr{M}_S$. It is symmetric monoidal with unit object S under a suitable smash product, which is denoted $\wedge_S$, and it also has a function S-module functor, which is denoted F_S. The expected adjunction holds:

$$\mathscr{M}_S(M \wedge_S N, P) \cong \mathscr{M}_S(M, F_S(N, P)).$$

Moreover, for based spaces X and Y, there is a natural isomorphism of S-modules

$$\Sigma^\infty X \wedge_S \Sigma^\infty Y \cong \Sigma^\infty(X \wedge Y).$$

When regarded as a functor from spaces to S-modules, rather than as a functor from spaces to spectra, Σ^∞ is *not* left adjoint to the zeroth space functor Ω^∞; rather, we have an adjunction

$$\mathscr{M}_S(\Sigma^\infty X, M) \cong \mathscr{T}(X, \mathscr{M}_S(S, M)).$$

Here the space of maps $\mathscr{M}_S(S, M)$ is not even equivalent to $\Omega^\infty M$. As observed by Hastings and Lewis, this is intrinsic to the mathematics: since $\mathscr{M}_S$ is symmetric monoidal, $\mathscr{M}_S(S,S)$ is a commutative topological monoid, and it therefore cannot be equivalent to the space $QS^0 = \Omega^\infty S$.

For an S-module M and a based space X, the smash product $M \wedge X$ is an S-module and

$$M \wedge X \cong M \wedge_S \Sigma^\infty X.$$

Cylinders, cones, and suspensions of S-modules are defined by smashing with I_+, I, and S^1. A homotopy between maps $f, g : M \longrightarrow N$ of S-modules is a map $M \wedge I_+ \longrightarrow N$ that restricts to f and g on the ends of the cylinder. The function spectrum $F(X, M)$ is not an S-module; $F_S(\Sigma^{\Sigma ty} X, M)$ is the appropriate substitute and must be used when defining cocylinder, path, and loop S-modules.

The category $\mathscr{M}_S$ is cocomplete (has all colimits), its colimits being created in $\mathscr{S}$. That is, the colimit in $\mathscr{S}$ of a diagram of S-modules is an S-module that is the colimit of the given diagram in $\mathscr{M}_S$. It is also complete (has all limits). The limit in $\mathscr{S}$ of a diagram of S-modules is not quite an S-module, but it takes values in a category $\mathscr{S}[\mathbb{L}]$ of "$\mathbb{L}$-spectra" that lies intermediate between spectra and S-modules. Limits in $\mathscr{S}[\mathbb{L}]$ are created in $\mathscr{S}$, and the forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$ has a right adjoint that creates the limits in $\mathscr{M}_S$. We shall explain this scaffolding in XXIII§2. For pragmatic purposes, what matters is that limits exist and have the same weak homotopy types as if they were created in $\mathscr{S}$.

There is a "free S-module functor" $\mathbb{F}_S : \mathscr{S} \longrightarrow \mathscr{M}_S$. It is not quite free in the usual sense since its right adjoint $\mathbb{U}_S : \mathscr{M}_S \longrightarrow \mathscr{S}$ is not quite the evident forgetful functor. This technicality reflects the fact that the forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}[\mathbb{L}]$ is a left rather than a right adjoint. Again, for pragmatic purposes, what matters is that $\mathbb{U}_S$ is naturally weakly equivalent to the evident forgetful functor.

We define sphere S-modules by

$$S^n_S = \mathbb{F}_S S^n.$$

We define the homotopy groups of an S-module to be the homotopy groups of the underlying spectrum and find by the adjunction cited in the previous paragraph that they can be computed as

$$\pi_n(M) = h\mathscr{M}_S(S^n_S, M).$$

From here, we develop the theory of cell and CW S-modules precisely as we developed the theory of cell and CW spectra, taking the spheres S^n_S as the domains of attaching maps of cells CS^n_S. We construct the "derived category of S-modules", denoted $\mathscr{D}_S$, by adjoining formal inverses to the weak equivalences and find that $\mathscr{D}_S$ is equivalent to the homotopy category of CW S-modules. The following fundamental theorem then shows that no homotopical information is lost if we replace the stable homotopy category $\bar{h}\mathscr{S}$ by the derived category $\mathscr{D}_S$.

THEOREM 1.1. The following conclusions hold.

(i) The free functor $\mathbb{F}_S : \mathscr{S} \longrightarrow \mathscr{M}_S$ carries CW spectra to CW S-modules.
(ii) The forgetful functor $\mathscr{M}_S \longrightarrow \mathscr{S}$ carries S-modules of the homotopy types of CW S-modules to spectra of the homotopy types of CW spectra.
(iii) Every CW S-module M is homotopy equivalent as an S-module to $\mathbb{F}_S E$ for some CW spectrum E.

The free functor and forgetful functors establish an adjoint equivalence between the stable homotopy category $\bar{h}\mathscr{S}$ and the derived category $\mathscr{D}_S$. This equivalence of categories preserves smash products and function objects. Thus

$$\mathscr{D}_S(\mathbb{F}_S E, M) \cong \bar{h}\mathscr{S}(E, M),$$

$$\mathbb{F}_S : \bar{h}\mathscr{S}(E, E') \xrightarrow{\cong} \mathscr{D}_S(\mathbb{F}_S E, \mathbb{F}_S E'),$$

$$\mathbb{F}_S(E \wedge E') \simeq (\mathbb{F}_S E) \wedge_S (\mathbb{F}_S E'),$$

and

$$\mathbb{F}_S(F(E, E')) \simeq F_S(\mathbb{F}_S E, \mathbb{F}_S E').$$

We can describe the equivalence in the language of (closed) model categories in the sense of Quillen, but we shall say little about this. Both $\mathscr{S}$ and $\mathscr{M}_S$ are model categories whose weak equivalences are the maps that induce isomorphisms of homotopy groups. The q-cofibrations (or Quillen cofibrations) are the retracts of inclusions of relative cell complexes (that is, cell spectra or cell S-modules). The q-fibrations in $\mathscr{S}$ are the Serre fibrations, namely the maps that satisfy the covering homotopy property with respect to maps defined on the cone spectra

$\Sigma_q^\infty CS^n$, where $q \geq 0$ and $n \geq 0$. The q-fibrations in $\mathscr{M}_S$ are the maps $M \longrightarrow N$ of S-modules whose induced maps $\mathbb{U}_S M \longrightarrow \mathbb{U}_S N$ are Serre fibrations of spectra.

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2. Categories of R-modules

Let us think about S-modules algebraically. There is a perhaps silly analogy that I find illuminating. Algebraically, it is of course a triviality that Abelian groups are essentially the same things as $\mathbb{Z}$-modules. Nevertheless, these notions are conceptually different. Thinking of brave new algebra in stable homotopy theory as analogous to classical algebra, I like to think of spectra as analogues of Abelian groups and S-modules as analogues of $\mathbb{Z}$-modules. While it required some thought and work to figure out how to pass from spectra to S-modules, now that we have done so we can follow our noses and mimic algebraic definitions word for word in the category of S-modules, thinking of $\wedge_S$ as analogous to $\otimes_{\mathbb{Z}}$ and F_S as analogous to $\mathrm{Hom}_{\mathbb{Z}}$.

We think of rings as $\mathbb{Z}$-algebras, and we define an S-algebra R by requiring a unit $S \longrightarrow R$ and product $R \wedge_S R \longrightarrow R$ such that the evident unit and associativity diagrams commute. We say that R is a commutative S-algebra if the evident commutativity diagram also commutes. We define a left R-module similarly, requiring a map $R \wedge_S M \longrightarrow M$ such that the evident unit and associativity diagrams commute.

For a right R-module M and left R-module N, we define an S-module $M \wedge_R N$ by the coequalizer diagram

$$M \wedge_S R \wedge_S N \underset{\mathrm{Id} \wedge_S \nu}{\overset{\mu \wedge_S \mathrm{Id}}{\rightrightarrows}} M \wedge_S N \longrightarrow M \wedge_R N,$$

where μ and ν are the given actions of R on M and N. Similarly, for left R-modules M and N, we define an S-module $F_R(M, N)$ by an appropriate equalizer diagram. We then have adjunctions exactly like those relating $\otimes_R$ and Hom_R in algebra.

If R is commutative, then $M \wedge_R N$ and $F_R(M,N)$ are R-modules, the category $\mathscr{M}_R$ of R-modules is symmetric monoidal with unit R, and we have the expected adjunction relating $\wedge_R$ and F_R. We can go on to define (R,R')-bimodules and to derive a host of formal relations involving smash products and function modules over varying rings, all of which are exactly like their algebraic counterparts.

For a left R-module M and a based space X, $M \wedge X \cong M \wedge_S \Sigma^\infty X$ and $F_S(\Sigma^\infty X, M)$ are left R-modules. If K is an S-module, then $M \wedge_S K$ is a left and $F_S(M,K)$ is a right R-module. We have theories of cofiber and fiber sequences of R-modules exactly as for spectra. We define the free R-module generated by a spectrum X to be

$$\mathbb{F}_R X = R \wedge_S \mathbb{F}_S X.$$

Again the right adjoint $\mathbb{U}_R$ of this functor is naturally weakly equivalent to the forgetful functor from R-modules to spectra. We define sphere R-modules by

$$S^n_R = \mathbb{F}_R S^n = R \wedge_S S^n_S$$

and find that

$$\pi_n(M) = h\mathscr{M}_R(S^n_R, M).$$

There is also a natural weak equivalence of R-modules $\mathbb{F}_R S \longrightarrow R$.

We develop the theory of cell and CW R-modules exactly as we developed the theory of cell and CW spectra, using the spheres S^n_R as the domains of attaching maps. However, the CW theory is only of interest when R is connective ($\pi_n(R) = 0$ for $n < 0$) since otherwise the cellular approximation theorem fails. We construct the derived category $\mathscr{D}_R$ from the category $\mathscr{M}_R$ of R-modules by adjoining formal inverses to the weak equivalences and find that $\mathscr{D}_R$ is equivalent to the homotopy category of cell R-modules.

Brown's representability theorem holds in the category $\mathscr{D}_R$: a contravariant set-valued functor k on $\mathscr{D}_R$ is representable in the form $kM \cong \mathscr{D}_R(M,N)$ if and only if k converts wedges to products and converts homotopy pushouts to weak pullbacks. However, as recently observed by Neeman in an algebraic context, Adams' variant for functors defined on finite cell R-modules only holds under a countability hypothesis on $\pi_*(R)$.

The category $\mathscr{M}_R$ is a model category. The weak equivalences and q-fibrations are the maps of R-modules that are weak equivalences and q-fibrations when regarded as maps of S-modules. The q-cofibrations are the retracts of relative cell R-modules. It is also a tensored and cotensored topological category. That is, its

Hom sets are based topological spaces, composition is continuous, and we have adjunction homeomorphisms

$$\mathscr{M}_R(M \wedge X, N) \cong \mathscr{T}(X, \mathscr{M}_R(M, N)) \cong \mathscr{M}_R(M, F_S(\Sigma^\infty X, N)).$$

Recently, Hovey, Palmieri, and Strickland have axiomatized the formal properties that a category ought to have in order to be called a "stable homotopy category". The idea is to abstract those properties that are independent of any underlying point-set level foundations and see what can be derived from that starting point. Our derived categories $\mathscr{D}_R$ provide a wealth of examples.

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3. The algebraic theory of R-modules

The categories $\mathscr{D}_R$ are both tools for the the study of classical algebraic topology, and interesting new subjects of study in their own right. In particular, they subsume much of classical algebra. The Eilenberg-MacLane spectrum HR associated to a (commutative) discrete ring R is a (commutative) S-algebra, and the Eilenberg-MacLane spectrum HM associated to an R-module is an HR-module. Moreover, the derived category $\mathscr{D}_{HR}$ is equivalent to the algebraic derived category $\mathscr{D}_R$ of chain complexes over R, and the equivalence converts derived smash products and function modules in topology to derived tensor products and Hom functors in algebra. In algebra, the homotopy groups of derived tensor product and Hom functors compute Tor and Ext, and we have natural isomorphisms

$$\pi_n(HM \wedge_{HR} HN) \cong Tor_n^R(M, N)$$

for a right R-module M and left R-module N and

$$\pi_{-n}(F_{HR}(HM, HN)) \cong Ext_R^n(M, N)$$

for left R-modules M and N, where HM is taken to be a CW HR-module.

Now return to the convention that R is an S-algebra. By the equivalence of $\bar{h}\mathscr{S}$ and $\mathscr{D}_S$, we see that homology and cohomology theories on spectra are subsumed as homotopy groups of smash products and function modules over S. Precisely, for a CW S-module M and an S-module N,

$$\pi_n(M \wedge_S N) = M_n(N)$$

and

$$\pi_{-n}(F_S(M, N)) = N^n(M).$$

These facts suggest that we should think of the homotopy groups of smash product and function R-modules ambiguously as generalizations of both Tor and Ext groups and homology and cohomology groups. Thus, for a right cell R-module M and a left R-module N, we define

$$Tor_n^R(M,N) = \pi_n(M \wedge_R N) = M_n^R(N) \tag{3.1}$$

and, for a left cell R-module M and a left R-module N, we define

$$Ext_R^n(M,N) = \pi_{-n}(F_R(M,N)) = N_R^n(M). \tag{3.2}$$

We assume that M is a cell module to ensure that these are well-defined derived category invariants.

These functors enjoy many properties familiar from both the algebraic and topological settings. For example, assuming that R is commutative, we have a natural, associative, and unital system of pairings of R^*-modules ($R^n = \pi_{-n}(R)$)

$$\mathrm{Ext}_R^*(M,N) \otimes_{R^*} \mathrm{Ext}_R^*(L,M) \longrightarrow \mathrm{Ext}_R^*(L,N).$$

Similarly, setting $D_RM = F_R(M,R)$, a formal argument in duality theory implies a natural isomorphism

$$Tor_n^R(D_RM,N) \cong \mathrm{Ext}_R^{-n}(M,N)$$

for finite cell R-modules M and arbitrary R-modules N. Thought of homologically, this isomorphism can be interpreted as Spanier-Whitehead duality: for a finite cell R-module M and any R-module N,

$$N_n^R(D_RM) \cong N_R^{-n}(M).$$

There are spectral sequences for the computation of these invariants. As usual, for a spectrum E, we write $E_n = \pi_n(E) = E^{-n}$.

Theorem 3.3. For right and left R-modules M and N, there is a spectral sequence

$$E_{p,q}^2 = \mathrm{Tor}_{p,q}^{R_*}(M_*,N_*) \Longrightarrow \mathrm{Tor}_{p+q}^R(M,N);$$

For left R-modules M and N, there is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_{R^*}^{p,q}(M^*,N^*) \Longrightarrow \mathrm{Ext}_R^{p+q}(M,N).$$

If R is commutative, these are spectral sequences of differential R_*-modules, and the second admits pairings converging from the evident Yoneda pairings on the E_2 terms to the natural pairings on the limit terms.

Setting $M = \mathbb{F}_R X$ in these two spectral sequences, we obtain universal coefficient spectral sequences.

THEOREM 3.4 (UNIVERSAL COEFFICIENT). For an R-module N and any spectrum X, there are spectral sequences of the form

$$\mathrm{Tor}^{R_*}_{*,*}(R_*(X), N_*) \Longrightarrow N_*(X)$$

and

$$\mathrm{Ext}^{*,*}_{R^*}(R_{-*}(X), N^*) \Longrightarrow N^*(X).$$

Replacing R and N by Eilenberg-Mac Lane spectra HR and HN for a discrete ring R and R-module N, we obtain the classical universal coefficient theorems. Replacing N by $\mathbb{F}_R Y$ and by $F_R(\mathbb{F}_R Y, R)$ in the two universal coefficient spectral sequences, we obtain Künneth spectral sequences.

THEOREM 3.5 (KÜNNETH). For any spectra X and Y, there are spectral sequences of the form

$$\mathrm{Tor}^{R_*}_{*,*}(R_*(X), R_*(Y)) \Longrightarrow R_*(X \wedge Y)$$

and

$$\mathrm{Ext}^{*,*}_{R^*}(R_{-*}(X), R^*(Y)) \Longrightarrow R^*(X \wedge Y).$$

Under varying hypotheses, the Künneth theorem in homology generalizes to an Eilenberg-Moore type spectral sequence. Here is one example.

THEOREM 3.6. Let E and R be commutative S-algebras and M and N be R-modules. Then there is a spectral sequence of differential $E_*(R)$-modules of the form

$$\mathrm{Tor}^{E_*(R)}_{p,q}(E_*(M), E_*(N)) \Longrightarrow E_{p+q}(M \wedge_R N).$$

4. The homotopical theory of R-modules

Thinking of the derived category of R-modules as an analog of the stable homotopy category, we have the notion of an R-ring spectrum, which is just like the classical notion of a ring spectrum in the stable homotopy category.

DEFINITION 4.1. An R-ring spectrum A is an R-module A with unit $\eta : R \longrightarrow A$ and product $\phi : A \wedge_R A \longrightarrow A$ in $\mathscr{D}_R$ such that the following left and right unit diagram commutes in $\mathscr{D}_R$:

$$\begin{tikzcd}
R \wedge_R A \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[dr, "\lambda"'] & A \wedge_R A \arrow[d, "\phi"] & A \wedge_R R \arrow[l, "\mathrm{id}\wedge\eta"'] \arrow[dl, "\lambda\tau"] \\
& A. &
\end{tikzcd}$$

A is associative or commutative if the appropriate diagram commutes in $\mathscr{D}_R$. If A is associative, then an A-module spectrum M is an R-module M with an action $\mu : A \wedge_R M \longrightarrow M$ such that the evident unit and associativity diagrams commute in $\mathscr{D}_R$.

LEMMA 4.2. If A and B are R-ring spectra, then so is $A \wedge_R B$. If A and B are associative or commutative, then so is $A \wedge_R B$.

When $R = S$, S-ring spectra and their module spectra are equivalent to classical ring spectra and their module spectra. By neglect of structure, an R-ring spectrum A is an S-ring spectrum and thus a ring spectrum in the classical sense; its unit is the composite of the unit of R and the unit of A and its product is the composite of the product of A and the canonical map

$$A \wedge A \simeq A \wedge_S A \longrightarrow A \wedge_R A.$$

If A is commutative or associative as an R-ring spectrum, then it is commutative and associative as an S-ring spectrum and thus as a classical ring spectrum. The R-ring spectra and their module spectra play a role in the study of $\mathscr{D}_R$ analogous to the role played by ring and module spectra in classical stable homotopy theory. Moreover, the new theory of R-ring and module spectra provides a powerful constructive tool for the study of the classical notions. The point is that, in $\mathscr{D}_R$, we have all of the internal structure, such as cofiber sequences, that we have in the stable homotopy category.

This can make it easy to construct R-ring spectra and modules in cases when a direct proof that they are merely classical ring spectra and modules is far more difficult, if it can be done at all. We assume that R is a commutative S-algebra and illustrate by indicating how to construct M/IM and $M[Y^{-1}]$ for an R-module M, where I is the ideal generated by a sequence $\{x_i\}$ of elements of R_* and Y is a countable multiplicatively closed set of elements of R_*. We shall also state some results about when these modules have R-ring structures and when such structures are commutative or associative.

We have isomorphisms

$$M_n \cong h\mathscr{M}_R(S^n_R, M).$$

The suspension $\Sigma^n M$ is equivalent to $S^n_R \wedge_R M$ and, for $x \in R_n$, the composite map of R-modules

$$S^n_R \wedge_R M \xrightarrow{x\wedge \mathrm{id}} R \wedge_R M \xrightarrow{\lambda} M \tag{4.3}$$

is a module theoretic version of the map $x\cdot : \Sigma^n M \longrightarrow M$.

Definition 4.4. Define M/xM to be the cofiber of the map (4.3) and let $\rho : M \longrightarrow M/xM$ be the canonical map. Inductively, for a finite sequence $\{x_1, \dots, x_n\}$ of elements of R_*, define

$$M/(x_1, \dots, x_n)M = N/x_nN, \quad \text{where} \quad N = M/(x_1, \dots, x_{n-1})M.$$

For a sequence $X = \{x_i\}$, define $M/XM = \operatorname{tel} M/(x_1, \dots, x_n)M$, where the telescope is taken with respect to the successive canonical maps ρ.

Clearly we have a long exact sequence

$$\cdots \longrightarrow \pi_{q-n}(M) \xrightarrow{x\cdot} \pi_q(M) \xrightarrow{\rho_*} \pi_q(M/xM) \longrightarrow \pi_{q-n-1}(M) \longrightarrow \cdots .$$

If x is regular for $\pi_*(M)$ ($xm = 0$ implies $m = 0$), then ρ_* induces an isomorphism of R_*-modules

$$\pi_*(M)/x \cdot \pi_*(M) \cong \pi_*(M/xM).$$

If $\{x_1, \dots, x_n\}$ is a regular sequence for $\pi_*(M)$, in the sense that x_i is regular for $\pi_*(M)/(x_1, \dots, x_{i-1})\pi_*(M)$ for $1 \le i \le n$, then

$$\pi_*(M)/(x_1, \dots, x_n)\pi_*(M) \cong \pi_*(M/(x_1, \dots, x_n)M),$$

and similarly for a possibly infinite regular sequence $X = \{x_i\}$. The following result implies that M/XM is independent of the ordering of the elements of the set X. We write R/X instead of R/XR.

Lemma 4.5. For a set X of elements of R_*, there is a natural weak equivalence

$$(R/X) \wedge_R M \longrightarrow M/XM.$$

In particular, for a finite set $X = \{x_1, \dots, x_n\}$,

$$R/(x_1, \dots, x_n) \simeq (R/x_1) \wedge_R \cdots \wedge_R (R/x_n).$$

If I denotes the ideal generated by X, then it is reasonable to define

$$M/IM = M/XM.$$

However, this notation must be used with caution since, if we fail to restrict attention to regular sequences X, the homotopy type of M/XM will depend on the set X and not just on the ideal it generates. For example, quite different modules are obtained if we repeat a generator x_i of I in our construction.

To construct localizations, let $\{y_i\}$ be any sequence of elements of Y that is cofinal in the sense that every $y \in Y$ divides some y_i. If $y_i \in R_{n_i}$, we may represent y_i by an R-map $S^0_R \longrightarrow S^{-n_i}_R$, which we also denote by y_i. Let $q_0 = 0$ and, inductively, $q_i = q_{i-1} + n_i$. Then the R-map

$$y_i \wedge \mathrm{id} : S^0_R \wedge_R M \longrightarrow S^{-n_i}_R \wedge_R M$$

represents multiplication by y_i. Smashing over R with $S^{-q_{i-1}}_R$, we obtain a sequence of R-maps

$$S^{-q_{i-1}}_R \wedge_R M \longrightarrow S^{-q_i}_R \wedge_R M. \tag{4.6}$$

Definition 4.7. Define the localization of M at Y, denoted $M[Y^{-1}]$, to be the telescope of the sequence of maps (4.6). Since $M \cong S^0_R \wedge_R M$ in $\mathscr{D}_R$, we may regard the inclusion of the initial stage $S^0_R \wedge_R M$ of the telescope as a natural map $\lambda : M \longrightarrow M[Y^{-1}]$.

Since homotopy groups commute with localization, we see immediately that λ induces an isomorphism of R_*-modules

$$\pi_*(M[Y^{-1}]) \cong \pi_*(M)[Y^{-1}].$$

As in Lemma 4.5, the localization of M is the smash product of M with the localization of R.

Lemma 4.8. For a multiplicatively closed set Y of elements of R_*, there is a natural equivalence

$$R[Y^{-1}] \wedge_R M \longrightarrow M[Y^{-1}].$$

Moreover, $R[Y^{-1}]$ is independent of the ordering of the elements of Y. For sets X and Y, $R[(X \cup Y)^{-1}]$ is equivalent to the composite localization $R[X^{-1}][Y^{-1}]$.

The behavior of localizations with respect to R-ring structures is now immediate.

PROPOSITION 4.9. Let Y be a multiplicatively closed set of elements of R_*. If A is an R-ring spectrum, then so is $A[Y^{-1}]$. If A is associative or commutative, then so is $A[Y^{-1}]$.

PROOF. It suffices to observe that $R[Y^{-1}]$ is an associative and commutative R-ring spectrum with unit λ and product the equivalence

$$R[Y^{-1}] \wedge_R R[Y^{-1}] \simeq R[Y^{-1}][Y^{-1}] \simeq R[Y^{-1}]. \quad \square$$

This doesn't work for quotients since $(R/X)/X$ is not equivalent to R/X. However, we can analyze the problem by analyzing the deviation, and, by Lemma 4.5, we may as well work one element at a time. We have a necessary condition for R/x to be an R-ring spectrum that is familiar from classical stable homotopy theory.

LEMMA 4.10. Let A be an R-ring spectrum. If A/xA admits a structure of R-ring spectrum such that $\rho : A \longrightarrow A/xA$ is a map of R-ring spectra, then $x : A/xA \longrightarrow A/xA$ is null homotopic as a map of R-modules.

Thus, for example, the Moore spectrum $S/2$ is not an S-ring spectrum since the map $2 : S/2 \longrightarrow S/2$ is not null homotopic. We have the following sufficient condition for when R/x does have an R-ring spectrum structure.

THEOREM 4.11. Let $x \in R_m$, where $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$. Then R/x admits a structure of R-ring spectrum with unit $\rho : R \longrightarrow R/x$. Therefore, for every R-ring spectrum A and every sequence X of elements of R_* such that $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$ if $x \in X$ has degree m, A/XA admits a structure of R-ring spectrum such that $\rho : A \longrightarrow A/XA$ is a map of R-ring spectra.

For an R-ring spectrum A and an element x as in the theorem, we give $A/xA \simeq (R/x) \wedge_R A$ the product induced by one of our constructed products on R/x and the given product on A. We refer to any such product as a "canonical" product on A/xA. We also have sufficient conditions for when the canonical product is unique and when a canonical product is commutative or associative.

THEOREM 4.12. Let $x \in R_m$, where $\pi_{m+1}(R/x) = 0$ and $\pi_{2m+1}(R/x) = 0$. Let A be an R-ring spectrum and assume that $\pi_{2m+2}(A/xA) = 0$. Then there is a unique canonical product on A/xA. If A is commutative, then A/xA is commutative. If A is associative and $\pi_{3m+3}(A/xA) = 0$, then A/xA is associative.

This leads to the following conclusion.

THEOREM 4.13. Assume that $R_i = 0$ if i is odd. Let X be a sequence of non zero divisors in R_* such that $\pi_*(R/X)$ is concentrated in degrees congruent to zero mod 4. Then R/X has a unique canonical structure of R-ring spectrum, and it is commutative and associative.

This is particularly valuable when applied with $R = MU$. The classical Thom spectra arise in nature as E_∞ ring spectra and give rise to equivalent commutative S-algebras. In fact, inspection of the prespectrum level definition of Thom spectra in terms of Grassmannians first led to the theory of E_∞ ring spectra and therefore of S-algebras. Of course,

$$MU_* = \mathbb{Z}[x_i | deg\, x_i = 2i]$$

Thus the results above have the following immediate corollary.

THEOREM 4.14. Let X be a regular sequence in MU_*, let I be the ideal generated by X, and let Y be any sequence in MU_*. Then there is an MU-ring spectrum $(MU/X)[Y^{-1}]$ and a natural map of MU-ring spectra (the unit map)

$$\eta : MU \longrightarrow (MU/X)[Y^{-1}]$$

such that

$$\eta_* : MU_* \longrightarrow \pi_*((MU/X)[Y^{-1}])$$

realizes the natural homomorphism of MU_*-algebras

$$MU_* \longrightarrow (MU_*/I)[Y^{-1}].$$

If MU_*/I is concentrated in degrees congruent to zero mod 4, then there is a unique canonical product on $(MU/X)[Y^{-1}]$, and this product is commutative and associative.

In comparison with earlier constructions of this sort based on the Baas-Sullivan theory of manifolds with singularities or on Landweber's exact functor theorem (where it applies), we have obtained a simpler proof of a substantially stronger result since an MU-ring spectrum is a much richer structure than just a ring spectrum and commutativity and associativity in the MU-ring spectrum sense are much more stringent conditions than mere commutativity and associativity of the underlying ring spectrum.

5. Categories of R-algebras

In the previous section, we considered R-ring spectra, which are homotopical versions of R-algebras. We also have a pointwise definition of R-algebras that is just like the definition of S-algebras. That is, R-algebras and commutative R-algebras A are defined via unit and product maps $R \longrightarrow A$ and $A \wedge_R A \longrightarrow A$ such that the appropriate diagrams commute in the symmetric monoidal category $\mathscr{M}_R$. All of the standard formal properties of algebras in classical algebra carry over directly to these brave new algebras. For example, a commutative R-algebra A is the same thing as a commutative S-algebra together with a map of S-algebras $R \longrightarrow A$ (the unit map), and the smash product $A \wedge_R A'$ of commutative R-algebras A and A' is their coproduct in the category of commutative R-algebras.

Some of the most subtantive work in [EKMM] concerns the understanding of the categories $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ of R-algebras and commutative R-algebras. The crucial point is to be able to compute the homotopical behavior of formal constructions in these categories. Technically, what is involved is the homotopical understanding of the forgetful functors from $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ to $\mathscr{M}_R$. Although not in itself enough to answer these questions, the context of enriched model categories is essential to give a framework in which they can be addressed. We shall indicate some of the main features here, but this material is addressed to the relatively sophisticated reader who has some familiarity with enriched category and model category theory. It provides the essential technical underpinning for the applications to Bousfield localization and topological Hochschild homology that are summarized in the following two sections.

Both $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ are tensored and cotensored topological categories. In fact, they are topologically complete and cocomplete, which means that they have not only the usual limits and colimits but also "indexed" limits and colimits. Limits are created in the category of R-modules, but colimits are less obvious constructions. In the absence of basepoints in their Hom sets, these categories are enriched over the category $\mathscr{U}$ of unbased spaces. The cotensors in both cases are the function S-algebras $F_S(\Sigma^\infty X_+, A)$ with the R-algebra structure induced from the diagonal on X and the product on A. The tensors are less familiar. They are denoted $A \otimes_{\mathscr{A}_R} X$ and $A \otimes_{\mathscr{C}\mathscr{A}_R} X$. These are different constructions in the two cases, but we write $A \otimes X$ when the context is understood. We have adjunctions

$$(5.1) \qquad \mathscr{A}_R(A \otimes X, B) \cong \mathscr{U}(X, \mathscr{A}_R(A, B)) \cong \mathscr{A}_R(A, F_S(\Sigma^\infty X_+, B)),$$

and similarly in the commutative case. Some idea of the structure and meaning of tensors is given by the following result. For R-algebras A and B and a space X, we say that a map $f : A \wedge X_+ \longrightarrow B$ of R-modules is a pointwise map of R-algebras if each composite $f \circ i_x : A \longrightarrow B$ is a map of R-algebras, where, for $x \in X$, $i_x : A \longrightarrow A \wedge X_+$ is the map induced by the evident inclusion $\{x\}_+ \longrightarrow X_+$.

Proposition 5.2. For R-algebras A and spaces X there is a natural map of R-modules

$$\omega : A \wedge X_+ \longrightarrow A \otimes X$$

such that a pointwise map $f : A \wedge X_+ \longrightarrow B$ of R-algebras uniquely determines a map $\tilde{f} : A \otimes X \longrightarrow B$ of R-algebras such that $f = \tilde{f} \circ \omega$. The same statement holds for commutative R-algebras.

More substantial results tell how to compute tensors when X is the geometric realization of a simplicial set or simplicial space. These results are at the heart of the development and understanding of model category structures on the categories $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$. In both categories, the weak equivalences and q-fibrations are the maps of R-algebras that are weak equivalences or q-fibrations of underlying R-modules. It follows that the q-cofibrations are the maps of R-algebras that satisfy the left lifting property with respect to the acyclic q-fibrations. (The LLP is recalled in VI§5.) However, the q-cofibrations admit a more explicit description as retracts of relative "cell R-algebras" or "cell commutative R-algebras". Such cell algebras are constructed by using free algebras generated by sphere spectra as the domains of attaching maps and mimicking the construction of cell R-modules, using coproducts, pushouts, and colimits in the relevant category of R-algebras.

The question of understanding the homotopical behavior of the forgetful functors from $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ to $\mathscr{M}_R$ now takes the form of understanding the homotopical behavior of q-cofibrant algebras (retracts of cell algebras) with respect to these forgetful functors. However, the formal properties of model categories have nothing to say about this homotopical question.

In what follows, let R be a fixed q-cofibrant commutative R-algebra. Since R is the initial object of $\mathscr{A}_R$ and of $\mathscr{C}\mathscr{A}_R$, it is q-cofibrant both as an R-algebra and as a commutative R-algebra. However, it is not q-cofibrant as an R-module. Therefore the most that one could hope of the underlying R-module of a q-cofibrant R-algebra is the conclusion of the following result.

Theorem 5.3. If A is a q-cofibrant R-algebra, then A is a retract of a cell R-module relative to R. That is, the unit $R \longrightarrow A$ is a q-cofibration of R-modules.

The conclusion fails in the deeper commutive case. The essential reason is that the free commutative R-algebra generated by an R-module M is the wedge of the symmetric powers M^j/Σ_j, and passage to orbits obscures the homotopy type of the underlying R-module. The following technically important result at least gives the homotopy type of the underlying spectrum.

THEOREM 5.4. Let R be a q-cofibrant commutative S-algebra. If M is a cell R-module, then the projection

$$\pi : (E\Sigma_j)_+ \wedge_{\Sigma_j} M^j \longrightarrow M^j/\Sigma_j$$

is a homotopy equivalence of spectra.

The following theorem provides a workable substitute for Theorem 5.3. It shows that the derived smash product is represented by the point-set level smash product on a large class $\bar{\mathscr{E}}_R$ of R-modules, one that in particular includes the underlying R-modules of all q-cofibrant R-algebras and commutative R-algebras.

THEOREM 5.5. There is a collection $\bar{\mathscr{E}}_R$ of R-modules of the underlying homotopy types of CW spectra that is closed under wedges, pushouts, colimits of countable sequences of cofibrations, homotopy equivalences, and finite smash products over R and that contains all q-cofibrant R-modules and the underlying R-modules of all q-cofibrant R-algebras and all q-cofibrant commutative R-algebras. Moreover, if $M_1, \cdots, M_n$ are R-modules in $\bar{\mathscr{E}}_R$ and $\gamma_i : N_i \longrightarrow M_i$ are weak equivalences, where the N_i are cell R-modules, then

$$\gamma_1 \wedge_R \cdots \wedge_R \gamma_n : N_1 \wedge_R \cdots \wedge_R N_n \longrightarrow M_1 \wedge_R \cdots \wedge_R M_n$$

is a weak equivalence. Therefore the cell R-module $N_1 \wedge_R \cdots \wedge_R N_n$ represents $M_1 \wedge_R \cdots \wedge_R M_n$ in the derived category $\mathscr{D}_R$.

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6. Bousfield localizations of R-modules and algebras

Bousfield localization is a basic tool in the study of classical stable homotopy theory, and the construction generalizes readily to the context of brave new algebra. In fact, using our model category structures, this context leads to a smoother treatment than can be found in the classical literature. More important, as we shall sketch, any brave new algebraic structure is preserved by Bousfield localization.

Let R be an S-algebra and E be a cell R-module. A map $f : M \longrightarrow N$ of R-modules is said to be an E-equivalence if

$$\mathrm{id} \wedge_R f : E \wedge_R M \longrightarrow E \wedge_R N$$

is a weak equivalence. An R-module W is said to be E-acyclic if $E \wedge_R W \simeq *$, and a map f is an E-equivalence if and only if its cofiber is E-acyclic. We say that an R-module L is E-local if $f^* : \mathscr{D}_R(N, L) \longrightarrow \mathscr{D}_R(M, L)$ is an isomorphism for any E-equivalence f or, equivalently, if $\mathscr{D}_R(W, L) = 0$ for any E-acyclic R-module W. Since this is a derived category criterion, it suffices to test it when W is a cell R-module. A localization of M at E is a map $\lambda : M \longrightarrow M_E$ such that λ is an E-equivalence and M_E is E-local. The formal properties of such localizations discussed by Bousfield carry over verbatim to the present context. There is a model structure on $\mathscr{M}_R$ that implies the existence of E-localizations of R-modules.

THEOREM 6.1. The category $\mathscr{M}_R$ admits a new structure as a topological model category in which the weak equivalences are the E-equivalences and the cofibrations are the q-cofibrations in the standard model structure, that is, the retracts of the inclusions of relative cell R-modules.

We call the fibrations in the new model structure E-fibrations. They are determined formally as maps that satisfy the right lifting property with respect to the E-acyclic q-cofibrations, namely the q-cofibrations that are E-equivalences. (The RLP is recalled in VI§5.) One can characterize the E-fibrations more explicitly, but the following result gives all the relevant information. Say that an R-module L is E-fibrant if the trivial map $L \longrightarrow *$ is an E-fibration.

THEOREM 6.2. An R-module is E-fibrant if and only if it is E-local. Any R-module M admits a localization $\lambda : M \longrightarrow M_E$ at E.

In fact, one of the standard properties of a model category shows that we can factor the trivial map $M \longrightarrow *$ as the composite of an E-acyclic q-cofibration $\lambda : M \longrightarrow M_E$ and an E-fibration $M_E \longrightarrow *$, so that the first statement implies the second. The following complement shows that the localization of an R-module at a spectrum (not necessarily an R-module) can be constructed as a map of R-modules.

PROPOSITION 6.3. Let K be a CW-spectrum and let E be the R-module $\mathbb{F}_R K$. Regarded as a map of spectra, a localization $\lambda : M \longrightarrow M_E$ of an R-module M at E is a localization of M at K.

The result generalizes to show that, for an R-algebra A, the localization of an A-module at an R-module E can be constructed as a map of A-modules.

PROPOSITION 6.4. Let A be a q-cofibrant R-algebra, let E be a cell R-module, and let F be the A-module $A \wedge_R E$. Regarded as a map of R-modules, a localization $\lambda : M \longrightarrow M_F$ of an A-module M at F is a localization of M at E.

Restrict R to be a q-cofibrant commutative S-algebra in the rest of this section. We then have the following fundamental theorem about localizations of R-algebras.

THEOREM 6.5. For a cell R-algebra A, the localization $\lambda : A \longrightarrow A_E$ can be constructed as the inclusion of a subcomplex in a cell R-algebra A_E. Moreover, if $f : A \longrightarrow B$ is a map of R-algebras into an E-local R-algebra B, then f lifts to a map of R-algebras $\tilde{f} : A_E \longrightarrow B$ such that $\tilde{f} \circ \lambda = f$; if f is an E-equivalence, then $\tilde{f}$ is a weak equivalence. The same statements hold for commutative R-algebras.

The idea is to replace the category $\mathscr{M}_R$ by either the category $\mathscr{A}_R$ or the category $\mathscr{C}\mathscr{A}_R$ in the development just sketched. That is, we attempt to construct new model category structures on $\mathscr{A}_R$ and $\mathscr{C}\mathscr{A}_R$ in such a fashion that a factorization of the trivial map $A \longrightarrow *$ as the composite of an E-acyclic q-cofibration and a q-fibration in the appropriate category of R-algebras gives a localization of the underlying R-module of A. The argument doesn't quite work to give a model structure because the module level argument uses vitally that a pushout of an E-acyclic q-cofibration of R-modules is an E-equivalence. There is no reason to believe that this holds for q-cofibrations of R-algebras. However, we can use Theorems 5.3–5.5 to prove that it does hold for pushouts of inclusions of subcomplexes in cell R-algebras along maps to cell R-algebras. This gives enough information to prove the theorem.

The theorem implies in particular that we can construct the localization of R at E as the unit $R \longrightarrow R_E$ of a q-cofibrant commutative R-algebra. This leads to a new perspective on localizations in classical stable homotopy theory. To see this, we compare the derived category $\mathscr{D}_{R_E}$ to the stable homotopy category $\mathscr{D}_R[E^{-1}]$ associated to the model structure on $\mathscr{M}_R$ that is determined by E. Thus $\mathscr{D}_R[E^{-1}]$ is obtained from $\mathscr{D}_R$ by inverting the E-equivalences and is equivalent to the full subcategory of $\mathscr{D}_R$ whose objects are the E-local R-modules. Observe that, for a cell R-module M, we have the canonical E-equivalence

$$\xi = \eta \wedge \mathrm{id} : M \cong R \wedge_R M \longrightarrow R_E \wedge_R M.$$

The following observation is the same as in the classical case.

LEMMA 6.6. If M is a finite cell R-module, then $R_E \wedge_R M$ is E-local and therefore ξ is the localization of M at E.

We say that localization at E is smashing if, for all cell R-modules M, $R_E \wedge_R M$ is E-local and therefore ξ is the localization of M at E. The following observation is due to Wolbert.

PROPOSITION 6.7 (WOLBERT). If localization at E is smashing, then the categories $\mathscr{D}_R[E^{-1}]$ and $\mathscr{D}_{R_E}$ are equivalent.

These categories are closely related even when localization at E is not smashing, as the following elaboration of Wolbert's result shows. Remember that R is assumed to be commutative.

THEOREM 6.8. The following three categories are equivalent.

(i) The category $\mathscr{D}_R[E^{-1}]$ of E-local R-modules.
(ii) The full subcategory $\mathscr{D}_{R_E}[E^{-1}]$ of $\mathscr{D}_{R_E}$ whose objects are the R_E-modules that are E-local as R-modules.
(iii) The category $\mathscr{D}_{R_E}[F^{-1}]$ of F-local R_E-modules, where $F = R_E \wedge_R E$.

This implies that the question of whether or not localization at E is smashing is a question about the category of R_E-modules, and it leads to the following factorization of the localization functor. In the case $R = S$, this shows that the commutative S-algebras S_E and their categories of modules are intrinsic to the classical theory of Bousfield localization.

THEOREM 6.9. Let $F = R_E \wedge_R E$. The E-localization functor

$$(\cdot)_E : \mathscr{D}_R \longrightarrow \mathscr{D}_R[E^{-1}]$$

is equivalent to the composite of the extension of scalars functor

$$R_E \wedge_R (\cdot) : \mathscr{D}_R \longrightarrow \mathscr{D}_{R_E}$$

and the F-localization functor

$$(\cdot)_F : \mathscr{D}_{R_E} \longrightarrow \mathscr{D}_{R_E}[F^{-1}].$$

COROLLARY 6.10. Localization at E is smashing if and only if all R_E-modules are E-local as R-modules, so that

$$\mathscr{D}_R[E^{-1}] \approx \mathscr{D}_{R_E} \approx \mathscr{D}_{R_E}[F^{-1}].$$

We illustrate the constructive power of Theorem 6.5 by showing that the algebraic localizations of R considered in Section 4 actually take R to commutative R-algebras on the point set level and not just on the homotopical level (as given by Proposition 4.9). Thus let Y be a countable multiplicatively closed set of elements of R_*. Using Lemma 4.8, we see that localization of R-modules at $R[Y^{-1}]$ is smashing and is given by the canonical maps

$$\lambda = \lambda \wedge_R \mathrm{id} : M \cong R \wedge_R M \longrightarrow R[Y^{-1}] \wedge_R M.$$

THEOREM 6.11. The localization $R \longrightarrow R[Y^{-1}]$ can be constructed as the unit of a cell R-algebra.

By multiplicative infinite loop space theory and our model category structure on the category of S-algebras, the spectra ko and ku that represent real and complex connective K-theory can be taken to be q-cofibrant commutative S-algebras. The spectra that represent periodic K-theory can be reconstructed up to homotopy by inverting the Bott element $\beta_O \in \pi_8(ko)$ or $\beta_U \in \pi_2(ku)$. That is,

$$KO \simeq ko[\beta_O^{-1}] \qquad \text{and} \qquad KU \simeq ku[\beta_U^{-1}].$$

We are entitled to the following result as a special case of the previous one.

THEOREM 6.12. The spectra KO and KU can be constructed as commutative ko and ku-algebras.

In particular, KO and KU are commutative S-algebras, but it seems very hard to prove this directly. Wolbert has studied the algebraic structure of the derived categories of modules over the connective and periodic versions of the real and complex K-theory S-algebras.

REMARK 6.13. For finite groups G, Theorem 6.12 applies with the same proof to construct the periodic spectra KO_G and KU_G of equivariant K-theory as commutative ko_G and ku_G-algebras. As we shall discuss in Chapter XXIV, this leads to an elegant proof of the Atiyah-Segal completion theorem in equivariant K-cohomology and of its analogue for equivariant K-homology.

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7. Topological Hochschild homology and cohomology

As another application of brave new algebra, we describe the topological Hochschild homology of R-algebras with coefficients in bimodules. We assume familiarity with the classical Hochschild homology of algebras (as in Cartan and Eilenberg, for example). The study of this topic and of topological cyclic homology, which takes topological Hochschild homology as its starting point and involves equivariant considerations, is under active investigation by many people. We shall just give a brief introduction.

We assume given a q-cofibrant commutative S-algebra R and a q-cofibrant R-algebra A. If A is commutative, we require it to be q-cofibrant as a commutative R-algebra. We define the enveloping R-algebra of A by

$$A^e = A \wedge_R A^{op},$$

where A^{op} is defined by twisting the product on A, as in algebra. If A is commutative, then $A^E \cong A \wedge_R A$ and the product $A^e \longrightarrow A$ is a map of R-algebras. We also assume given an (A, A)-bimodule M; it can be viewed as either a left or a right A^e-module.

Definition 7.1. Working in derived categories, define topological Hochschild homology and cohomology with values in $\mathscr{D}_R$ by

$$THH^R(A; M) = M \wedge_{A^e} A \qquad \text{and} \qquad THH_R(A; M) = F_{A^e}(A, M).$$

If A is commutative, then these functors take values in the derived category $\mathscr{D}_{A^e}$. On passage to homotopy groups, define

$$THH^R_*(A; M) = \mathrm{Tor}^{A^e}_*(M, A) \qquad \text{and} \qquad THH^*_R(A; M) = \mathrm{Ext}^*_{A^e}(A, M).$$

When $M = A$, we delete it from the notations.

Since we are working in derived categories, we are implicitly taking M to be a cell A^e-module in the definition of $THH^R(A; M)$ and approximating A by a weakly equivalent cell A^e-module in the definition of $THH_R(A; M)$.

Proposition 7.2. If A is a commutative R-algebra, then $THH^R(A)$ is isomorphic in $\mathscr{D}_{A^e}$ to a commutative A^e-algebra.

The module structures on $THH^R(A;M)$ have the following implication.

PROPOSITION 7.3. If either R or A is the Eilenberg-Mac Lane spectrum of a commutative ring, then $THH^R(A;M)$ is a product of Eilenberg-Mac Lane spectra.

We have spectral sequences that relate algebraic and topological Hochschild homology. For a commutative graded ring R_*, a graded R_*-algebra A_* that is flat as an R_*-module, and a graded (A_*,A_*)-bimodule M_*, we define

$$HH^{R_*}_{p,q}(A_*;M_*) = \mathrm{Tor}^{(A_*)^e}_{p,q}(M_*,A_*) \quad \text{and} \quad HH^{p,q}_{R^*}(A^*;M^*) = \mathrm{Ext}^{p,q}_{(A^*)^e}(A^*,M^*),$$

where p is the homological degree and q is the internal degree. (This algebraic definition would not be correct in the absence of the flatness hypothesis.) When $M_* = A_*$, we delete it from the notation. If A_* is commutative, then $HH^{R_*}_{*,*}(A_*)$ is a graded A_*-algebra. Observe that $(A^{op})_* = (A_*)^{op}$.

In view of Theorem 5.5, the spectral sequence of Theorem 3.2 specializes to give the following spectral sequences relating algebraic and topological Hochschild homology.

THEOREM 7.4. There are spectral sequences of the form

$$E^2_{p,q} = \mathrm{Tor}^{R_*}_{p,q}(A_*,A^{op}_*) \Longrightarrow (A^e)_{p+q},$$

$$E^2_{p,q} = \mathrm{Tor}^{(A^e)_*}_{p,q}(M_*,A_*) \Longrightarrow THH^R_{p+q}(A;M),$$

and

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{(A^e)^*}(A^*,M^*) \Longrightarrow THH_R^{p+q}(A;M).$$

If A_* is a flat R_*-module, so that the first spectral sequence collapses, then the initial terms of the second and third spectral sequences are, respectively,

$$HH^{R_*}_{*,*}(A_*;M_*) \quad \text{and} \quad HH^{*,*}_{R^*}(A^*;M^*).$$

This is of negligible use in the absolute case $R = S$, where the flatness hypothesis on A_* is unrealistic. However, in the relative case, it implies that algebraic Hochschild homology and cohomology are special cases of topological Hochschild homology and cohomology.

THEOREM 7.5. Let R be a (discrete, ungraded) commutative ring, let A be an R-flat R-algebra, and let M be an (A,A)-bimodule. Then

$$HH^R_*(A;M) \cong THH^{HR}_*(HA;HM)$$

and

$$HH_R^*(A;M) \cong THH^*_{HR}(HA;HM).$$

If A is commutative, then $HH_*^R(A) \cong THH_*^{HR}(HA)$ as A-algebras.

We concentrate on homology henceforward. In the absolute case $R = S$, it is natural to approach $THH_*^S(A;M)$ by first determining the ordinary homology of $THH^S(A;M)$, using the case $E = H\mathbb{F}_p$ of the following spectral sequence, and then using the Adams spectral sequence. A spectral sequence like the following one was first obtained by Bökstedt. Under flatness hypotheses, there are variants in which E need only be a commutative ring spectrum, e.g. Theorem 7.12 below.

THEOREM 7.6. Let E be a commutative S-algebra. There are spectral sequence of differential $E_*(R)$-modules of the forms

$$E^2_{p,q} = \mathrm{Tor}^{E_*R}_{p,q}(E_*A, E_*(A^{op})) \Longrightarrow E_{p+q}(A^e)$$

and

$$E^2_{p,q} = \mathrm{Tor}^{E_*(A^e)}_{p,q}(E_*(M), E_*(A)) \Longrightarrow E_{p+q}(THH^R(A;M)).$$

There is an alternative description of topological Hochschild homology in terms of the brave new algebra version of the standard complex for the computation of Hochshild homology. Write A^p for the p-fold $\wedge_R$-power of A, and let

$$\phi : A \wedge_R A \longrightarrow A \quad \text{and} \quad \eta : R \longrightarrow A$$

be the product and unit of A. Let

$$\xi_\ell : A \wedge_R M \longrightarrow M \quad \text{and} \quad \xi_r : M \wedge_R A \longrightarrow M$$

be the left and right action of A on M. We have cyclic permutation isomorphisms

$$\tau : M \wedge_R A^p \wedge_R A \longrightarrow A \wedge_R M \wedge_R A^p.$$

The topological analogue of passage from a simplicial k-module to a chain complex of k-modules is passage from a simplicial spectrum E_* to its spectrum level geometric realization $|E_*|$; this construction is studied in [EKMM].

DEFINITION 7.7. Define a simplicial R-module $thh^R(A;M)_*$ as follows. Its R-module of p-simplices is $M \wedge_R A^p$. Its face and degeneracy operators are

$$d_i = \begin{cases} \xi_r \wedge (\mathrm{id})^{p-1} & \text{if } i = 0 \\ \mathrm{id} \wedge (\mathrm{id})^{i-1} \wedge \phi \wedge (\mathrm{id})^{p-i-1} & \text{if } 1 \le i < p \\ (\xi_\ell \wedge (\mathrm{id})^{p-1}) \circ \tau & \text{if } i = p \end{cases}$$

and $s_i = \mathrm{id} \wedge (\mathrm{id})^i \wedge \eta \wedge (\mathrm{id})^{p-i}$. Define

$$thh^R(A;M) = |thh^R(A;M)_*|;$$

When $M = A$, we delete it from the notation, writing $thh^R(A)_*$ and $|thh^R(A)_*|$.

PROPOSITION 7.8. Let A be a commutative R-algebra. Then $thh^R(A)$ is a commutative A-algebra and $thh^R(A;M)$ is a $thh^R(A)$-module.

As in algebra, the starting point for a comparison of definitions is the relative two-sided bar construction $B^R(M,A,N)$. It is defined for a commutative S-algebra R, an R-algebra A, and right and left A-modules M and N. Its R-module of p-simplices is $M \wedge_R A^p \wedge N$. There is a natural map

$$\psi : B^R(A,A,N) \longrightarrow N$$

of A-modules that is a homotopy equivalence of R-modules. More generally, there is a natural map of R-modules

$$\psi : B^R(M,A,N) \longrightarrow M \wedge_A N$$

that is a weak equivalence of R-modules when M is a cell A-module. The relevance of the bar construction to thh is shown by the following observation, which is the same as in algebra. We write

$$B^R(A) = B^R(A,A,A);$$

$B^R(A)$ is an (A,A)-bimodule; on the simplicial level, $B^R_0(A) = A^e$.

PROPOSITION 7.9. For (A,A)-bimodules M, there is a natural isomorphism

$$thh^R(A;M) \cong M \wedge_{A^e} B^R(A).$$

Therefore, for cell A^e-modules M, the natural map

$$thh^R(A;M) \cong M \wedge_{A^e} B^R(A) \xrightarrow{\mathrm{id}\wedge\psi} M \wedge_{A^e} A = THH^R(A;M)$$

is a weak equivalences of R-modules, or of A^e-modules if A is commutative.

While we assumed that M is a cell A^e-module in our derived category level definition of THH, we are mainly interested in the case $M = A$ of our point-set level construction thh, and A is not of the A^e-homotopy type of a cell A^e-module except in trivial cases. However, Theorem 5.5 leads to the following result.

THEOREM 7.10. Let $\gamma : M \longrightarrow A$ be a weak equivalence of A^e-modules, where M is a cell A^e-module. Then the map

$$thh^R(\mathrm{id};\gamma) : thh^R(A;M) \longrightarrow thh^R(A;A) = thh^R(A)$$

is a weak equivalence of R-modules, or of A^e-modules if A is commutative. Therefore $THH^R(A;M)$ is weakly equivalent to $thh^R(A)$.

COROLLARY 7.11. In the derived category $\mathscr{D}_R$, $THH^R(A) \cong thh^R(A)$.

Use of the standard simplicial filtration of the standard complex gives us the promised variant of the spectral sequence of Theorem 7.6. For simplicity, we restrict attention to the absolute case $R = S$.

THEOREM 7.12. Let E be a commutative ring spectrum, A be an S-algebra, and M be a cell A^e-module. If $E_*(A)$ is E_*-flat, there is a spectral sequence of the form

$$E^2_{p,q} = HH^{E_*}_{p,q}(E_*(A); E_*(M)) \Longrightarrow E_{p+q}(thh^S(A; M)).$$

If A is commutative and $M = A$, this is a spectral sequence of differential $E_*(A)$-algebras, the product on E^2 being the standard product on Hochschild homology.

McClure, Schwänzl, and Vogt observed that, when A is commutative, as we assume in the rest of the section, there is an attractive conceptual reinterpretation of the definition of $thh^R(A)$. Recall that the category $\mathscr{C}\mathscr{A}_R$ of commutative R-algebras is tensored over the category of unbased spaces. By writing out the standard simplicial set S^1_* whose realization is the circle and comparing faces and degeneracies, it is easy to check that there is an identification of simplicial commutative R-algebras

$$thh^R(A)_* \cong A \otimes S^1_*. \tag{7.13}$$

Passing to geometric realization and identifying S^1 with the unit complex numbers, we obtain the following consequence.

THEOREM 7.14 (MCCLURE, SCHWÄNZL, VOGT). For commutative R-algebras A, there is a natural isomorphism of commutative R-algebras

$$thh^R(A) \cong A \otimes S^1.$$

The product of $thh^R(A)$ is induced by the codiagonal $S^1 \amalg S^1 \longrightarrow S^1$. The unit $\zeta : A \longrightarrow thh^R(A)$ is induced by the inclusion $\{1\} \to S^1$.

The adjunction (5.1) that defines tensors implies that the functor $thh^R(A)$ preserves colimits in A, something that is not at all obvious from the original definition. The theorem and the adjunction (5.1) imply much further structure on $thh^R(A)$.

Corollary 7.15. The pinch map $S^1 \longrightarrow S^1 \vee S^1$ and trivial map $S^1 \longrightarrow *$ induce a (homotopy) coassociative and counital coproduct and counit

$$\psi : thh^R(A) \longrightarrow thh^R(A) \wedge_A thh^R(A) \quad \text{and} \quad \varepsilon : thh^R(A) \longrightarrow A$$

that make $thh^R(A)$ a homotopical Hopf A-algebra.

The product on S^1 gives rise to a map

$$\alpha : (A \otimes S^1) \otimes S^1 \cong A \otimes (S^1 \times S^1) \longrightarrow A \otimes S^1.$$

Corollary 7.16. For an integer r, define $\phi^r : S^1 \longrightarrow S^1$ by $\phi^r(e^{2\pi it}) = e^{2\pi irt}$. The ϕ^r induce power operations

$$\Phi^r : thh^R(A) \longrightarrow thh^R(A).$$

These are maps of R-algebras such that

$$\Phi^0 = \zeta\varepsilon, \quad \Phi^1 = \mathrm{id}, \quad \Phi^r \circ \Phi^s = \Phi^{rs},$$

and the following diagrams commute:

$$\begin{tikzcd}
thh^R(A) \otimes S^1 \arrow[r, "\alpha"] \arrow[d, "\Phi^r \otimes \phi^s"'] & thh^R(A) \arrow[d, "\Phi^{r+s}"] \\
thh^R(A) \otimes S^1 \arrow[r, "\alpha"'] & thh^R(A).
\end{tikzcd}$$

Consider naive S^1-spectra and let S^1 act trivially on R and A. Via the adjunction (5.1), the map α gives rise to an action of S^1 on $thh^R(A)$.

Corollary 7.17. $thh^R(A)$ is a naive commutative S^1-R-algebra. If B is a naive commutative S^1-R-algebra and $f : A \longrightarrow B$ is a map of commutative R-algebras, then there is a unique map $\tilde{f} : thh^R(A) \longrightarrow B$ of naive commutative S^1-R-algebras such that $\tilde{f} \circ \zeta = f$.

Finally, the description of tensors in Proposition 5.2 leads to the following result.

Corollary 7.18. There is a natural S^1-equivariant map of R-modules

$$\omega : A \wedge S^1_+ \longrightarrow thh^R(A)$$

such that if B is a commutative R-algebra and $f : A \wedge S^1_+ \longrightarrow B$ is a map of R-modules that is a pointwise map of R-algebras, then f uniquely determines a map of R-algebras $\tilde{f} : thh^R(A) \longrightarrow B$ such that $f = \tilde{f} \circ \omega$.

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CHAPTER XXIII

Brave new equivariant foundations

by A. D. Elmendorf, L. G. Lewis, Jr., and J. P. May

1. Twisted half-smash products

We here give a quick sketch of the basic constructions behind the work of the last chapter. Although the basic source, [EKMM], is written nonequivariantly, it applies verbatim to the equivariant context in which we shall work in this chapter. We shall take the opportunity to describe some unpublished perspectives on the role of equivariance in the new theory.

The essential starting point is the twisted half-smash product construction from [LMS]. Although we have come this far without mentioning this construction, it is in fact central to equivariant stable homotopy theory. Before describing it, we shall motivate it in terms of the main theme of this chapter, which is the construction of the category of $\mathbb{L}$-spectra. As we shall see, this is the main step in the construction of the category of S-modules.

Fix a compact Lie group G and a G-universe U and consider the category $G\mathscr{S}U$ of G-spectra indexed on U. Write U^j for the direct sum of j copies of U. Recall that we have an external smash product $\wedge : G\mathscr{S}U \times G\mathscr{S}U \longrightarrow G\mathscr{S}U^2$ and an internal smash product $f_* \circ \wedge : G\mathscr{S}U^2 \longrightarrow G\mathscr{S}U$ for each G-linear isometry $f : U^2 \longrightarrow U$. The external smash product is suitably associative, commutative, and unital on the point set level, hence we may iterate and form an external smash product $\wedge : (G\mathscr{S}U)^j \longrightarrow G\mathscr{S}U^j$ for each $j \geq 1$, the first external smash power being the identity functor. For each G-linear isometry $f : U^j \longrightarrow U$, we have an associated internal smash product $f_* \circ \wedge : G\mathscr{S}U^j \longrightarrow G\mathscr{S}U$. We allow the case $j = 0$; here $G\mathscr{S}\{0\} = G\mathscr{T}$, the only linear isometry $\{0\} \longrightarrow U$ is the inclusion i, and i_* is the suspension G-spectrum functor. At least if we restrict attention to tame G-spectra, the functors induced by varying f are all equivalent (see Theorem 1.5 below). Thus varying G-linear isometries $f : U^j \longrightarrow U$ parametrize equivalent internal smash products.

There is a language for the discussion of such parametrized products in various mathematical contexts, namely the language of "operads" that was introduced for the study of iterated loop space theory in 1972. Let $\mathscr{L}(j)$ denote the space $\mathscr{I}(U^j, U)$ of linear isometries $U^j \longrightarrow U$. Here we allow all linear isometries, not just the G-linear ones, and G acts on $\mathscr{L}(j)$ by conjugation. Thus the fixed point space $\mathscr{L}(j)^G$ is the space of G-linear isometries $U^j \longrightarrow U$. The symmetric group Σ_j acts freely from the right on $\mathscr{L}(j)$, and the actions of G and Σ_j commute. The equivariant homotopy type of $\mathscr{L}(j)$ depends on U. If U is complete, then, for $\Lambda \subset G \times \Sigma_j$, $\mathscr{L}(j)^\Lambda$ is empty unless $\Lambda \cap \Sigma_j = e$ and contractible otherwise. That is, $\mathscr{L}(j)$ is a universal (G, Σ_j)-bundle. We have maps

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times \cdots \times \mathscr{L}(j_k) \longrightarrow \mathscr{L}(j_1 + \cdots + j_k)$$

defined by

$$\gamma(g; f_1, \ldots, f_k) = g \circ (f_1 \oplus \cdots \oplus f_k).$$

These data are interrelated in a manner codified in the definition of an operad, and $\mathscr{L}$ is called the "linear isometries G-operad" of the universe U. When U is complete, $\mathscr{L}$ is an E_∞ G-operad.

There is a "twisted half-smash product"

$$\mathscr{L}(j) \ltimes (E_1 \wedge \cdots \wedge E_j) \tag{1.1}$$

into which we can map each of the j-fold internal smash products $f_*(E_1 \wedge \cdots \wedge E_j)$. Moreover, if we restrict attention to tame G-spectra, then each of these maps into the twisted half-smash product (1.1) is an equivalence. The twisted half-smash products $\mathscr{L}(1) \ltimes E$ and $\mathscr{L}(2) \ltimes E \wedge E'$ are the starting points for the construction of the category of $\mathbb{L}$-spectra and the definition of its smash product. We shall return to this point in the next section, after saying a little more about twisted smash products of G-spectra.

Suppose given G-universes U and U', and let $\mathscr{I}(U, U')$ be the G-space of linear isometries $U \longrightarrow U'$, with G acting by conjugation. Let A be an (unbased) G-space together with a given G-map $\alpha : A \longrightarrow \mathscr{I}(U, U')$. We then have a twisted half-smash product functor

$$\alpha \ltimes (\cdot) : G\mathscr{S}U \longrightarrow G\mathscr{S}U'.$$

When A has the homotopy type of a G-CW complex and $E \in G\mathscr{S}U$ is tame, different choices of α give homotopy equivalent G-spectra $\alpha \ltimes E$. For this reason, and because we often have a canonical choice of α in mind, we usually abuse notation by writing $A \ltimes E$ instead of $\alpha \ltimes E$. Thus we think of A as a space over $\mathscr{I}(U, U')$.

When A is a point, α is a choice of a G-linear isometry $f : U \longrightarrow U'$. In this case, the twisted half-smash functor is just the change of universe functor $f_* : G\mathscr{S}U \longrightarrow G\mathscr{S}U'$ (see XII.3.1–3.2). Intuitively, one may think of $\alpha \ltimes E$ as obtained by suitably topologizing and giving a G-action to the union of the nonequivariant spectra $\alpha(a)_*(E)$ as a runs through A. Another intuition is that the twisted half-smash product is a generalization to spectra of the "untwisted" functor $A_+ \wedge X$ on based G-spaces X. This intuition is made precise by the following "untwisting formula" that relates twisted half-smash products and shift desuspensions.

Proposition 1.2. For a G-space A over $\mathscr{I}(U, U')$ and an isomorphism $V \cong V'$ of indexing G-spaces, where $V \subset U$ and $V' \subset U'$, there is an isomorphism of G-spectra

$$A \ltimes \Sigma^\infty_V X \cong A_+ \wedge \Sigma^\infty_{V'} X$$

that is natural in G-spaces A over $\mathscr{I}(U, U')$ and based G-spaces X.

The twisted-half smash product functor enjoys essentially the same formal properties as the space level functor $A_+ \wedge X$. For example, we have the following properties, whose space level analogues are trivial to verify.

Proposition 1.3. The following statements hold.

(i) There is a canonical isomorphism $\{\mathrm{id}_U\} \ltimes E \cong E$.

(ii) Let $A \to \mathscr{I}(U, U')$ and $B \to \mathscr{I}(U', U'')$ be given and give $B \times A$ the composite structure map

$$B \times A \longrightarrow \mathscr{I}(U', U'') \times \mathscr{I}(U, U') \xrightarrow{\circ} \mathscr{I}(U, U'').$$

Then there is a canonical isomorphism

$$(B \times A) \ltimes E \cong B \ltimes (A \ltimes E).$$

(iii) Let $A \to \mathscr{I}(U_1, U_1')$ and $B \to \mathscr{I}(U_2, U_2')$ be given and give $A \times B$ the composite structure map

$$A \times B \longrightarrow \mathscr{I}(U_1, U_1') \times \mathscr{I}(U_2, U_2') \xrightarrow{\oplus} \mathscr{I}(U_1 \oplus U_2, U_1' \oplus U_2').$$

Let E_1 and E_2 be G-spectra indexed on U_1 and U_2 respectively. Then there is a canonical isomorphism

$$(A \times B) \ltimes (E_1 \wedge E_2) \cong (A \ltimes E_1) \wedge (B \ltimes E_2).$$

(iv) For $A \to \mathscr{I}(U, U')$, $E \in G\mathscr{S}U$, and a based G-space X, there is a canonical isomorphism

$$A \ltimes (E \wedge X) \cong (A \ltimes E) \wedge X.$$

The functor $A \ltimes (\bullet)$ has a right adjoint twisted function spectrum functor

$$F[A, \cdot) : G\mathscr{S}U' \longrightarrow G\mathscr{S}U,$$

which is the spectrum level analog of the function G-space $F(A_+, X)$. Thus

$$G\mathscr{S}U'(A \ltimes E, E') \cong G\mathscr{S}U(E, F[A, E')). \tag{1.4}$$

The functor $A \ltimes E$ is homotopy-preserving in E, and it therefore preserves homotopy equivalences in the variable E. However, it only preserves homotopies over $\mathscr{I}(U, U')$ in A. Nevertheless, it very often preserves homotopy equivalences in the variable A. The following central technical result is an easy consequence of Proposition 1.2 and XII.9.2. It explains why all j-fold internal smash products are equivalent to the twisted half-smash product (1.1).

Theorem 1.5. *Let $E \in G\mathscr{S}U$ be tame and let A be a G-space over $\mathscr{I}(U, U')$. If $\phi : A' \longrightarrow A$ is a homotopy equivalence of G-spaces, then $\phi \ltimes \mathrm{id} : A' \ltimes E \longrightarrow A \ltimes E$ is a homotopy equivalence of G-spectra.*

Since $A \ltimes E$ is a G-CW spectrum if A is a G-CW complex and E is a G-CW spectrum, this has the following consequence.

Corollary 1.6. *Let $E \in G\mathscr{S}U$ have the homotopy type of a G-CW spectrum and let A be a G-space over $\mathscr{I}(U, U')$ that has the homotopy type of a G-CW complex. Then $A \ltimes E$ has the homotopy type of a G-CW spectrum.*

[LMS, Chapter VI]

J. P. May. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Volume 271. 1972.

2. The category of $\mathbb{L}$-spectra

Return to the twisted half-smash product of (1.1). We think of it as a canonical j-fold internal smash product. However, if we are to take this point of view seriously, we must take note of the difference between E and its "1-fold smash product" $\mathscr{L}(1) \ltimes E$. The space $\mathscr{L}(1)$ is a monoid under composition, and the formal properties of twisted half-smash products imply a natural isomorphism

$$\mathscr{L}(1) \ltimes (\mathscr{L}(1) \ltimes E) \cong (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E,$$

where, on the right, $\mathscr{L}(1) \times \mathscr{L}(1)$ is regarded as a G-space over $\mathscr{L}(1)$ via the composition product. This product induces a map

$$\mu : (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E \longrightarrow \mathscr{L}(1) \ltimes E,$$

and the inclusion $\{1\} \longrightarrow \mathscr{L}(1)$ induces a map $\eta : E \longrightarrow \mathscr{L}(1) \ltimes E$. The functor $\mathbb{L}$ given by $\mathbb{L}E = \mathscr{L}(1) \ltimes E$ is a monad under the product μ and unit η. We therefore have the notion of a G-spectrum E with an action $\xi : \mathbb{L}E \longrightarrow E$ of $\mathbb{L}$; the evident associativity and unit diagrams are required to commute.

DEFINITION 2.1. An $\mathbb{L}$-spectrum is a G-spectrum M together with an action of the monad $\mathbb{L}$. Let $G\mathscr{S}[\mathbb{L}]$ denote the category of $\mathbb{L}$-spectra.

The formal properties of $G\mathscr{S}[\mathbb{L}]$ are virtually the same as those of $G\mathscr{S}$; since $\mathscr{L}(1)$ is a contractible G-space, so are the homotopical properties. For tame G-spectra E, we have a natural equivalence $E = \mathrm{id}_* E \longrightarrow \mathbb{L}E$. For $\mathbb{L}$-spectra M that are tame as G-spectra, the action $\xi : \mathbb{L}M \longrightarrow M$ is a weak equivalence. Taking the $\mathbb{L}S^n$ as sphere $\mathbb{L}$-modules, we obtain a theory of G-CW $\mathbb{L}$-spectra exactly like the theory of G-CW spectra. The functor $\mathbb{L}$ preserves G-CW spectra. We let $\bar{h}G\mathscr{S}[\mathbb{L}]$ be the category that is obtained from the homotopy category $hG\mathscr{S}[\mathbb{L}]$ by formally inverting the weak equivalences and find that it is equivalent to the homotopy category of G-CW $\mathbb{L}$-spectra. The functor $\mathbb{L} : G\mathscr{S} \longrightarrow G\mathscr{S}[\mathbb{L}]$ and the forgetful functor $G\mathscr{S}[\mathbb{L}] \longrightarrow G\mathscr{S}$ induce an adjoint equivalence between the stable homotopy category $\bar{h}G\mathscr{S}$ and the category $\bar{h}G\mathscr{S}[\mathbb{L}]$.

Via the untwisting isomorphism $\mathscr{L}(1) \ltimes \Sigma^\infty X \cong \mathscr{L}(1)_+ \wedge \Sigma^\infty X$ and the obvious projection $\mathscr{L}(1)_+ \longrightarrow S^0$, we obtain a natural action of $\mathbb{L}$ on suspension spectra. However, even when X is a G-CW complex, $\Sigma^\infty X$ is not of the homotopy type of a G-CW $\mathbb{L}$-spectrum, and it is the functor $\mathbb{L} \circ \Sigma^\infty$ and not the functor Σ^∞ that is left adjoint to the zeroth space functor $G\mathscr{S}[\mathbb{L}] \longrightarrow \mathscr{T}$.

The reason for introducing the category of $\mathbb{L}$-spectra is that it has a well-behaved "operadic smash product", which we define next. Via instances of the structural maps γ of the operad $\mathscr{L}$, we have both a left action of the monoid $\mathscr{L}(1)$ and a right action of the monoid $\mathscr{L}(1) \times \mathscr{L}(1)$ on $\mathscr{L}(2)$. These actions commute with each other. If M and N are $\mathbb{L}$-spectra, then $\mathscr{L}(1) \times \mathscr{L}(1)$ acts from the left on the external smash product $M \wedge N$ via the map

$$\xi : (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \cong (\mathscr{L}(1) \ltimes M) \wedge (\mathscr{L}(1) \ltimes N) \xrightarrow{\xi \wedge \xi} M \wedge N.$$

To form the twisted half smash product on the left, we think of $\mathscr{L}(1) \times \mathscr{L}(1)$ as mapping to $\mathscr{I}(U^2, U^2)$ via direct sum of linear isometries. The smash product over $\mathscr{L}$ of M and N is simply the balanced product of the two $\mathscr{L}(1) \times \mathscr{L}(1)$-actions.

Definition 2.2. Let M and N be $\mathbb{L}$-spectra. Define the operadic smash product $M \wedge_{\mathscr{L}} N$ to be the coequalizer displayed in the diagram

$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \underset{\mathrm{id} \ltimes \xi}{\overset{\gamma \ltimes \mathrm{id}}{\rightrightarrows}} \mathscr{L}(2) \ltimes (M \wedge N) \longrightarrow M \wedge_{\mathscr{L}} N.$$

Here we have implicitly used the isomorphism

$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N) \cong \mathscr{L}(2) \ltimes [(\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (M \wedge N)]$$

given by Proposition 1.4(ii). The left action of $\mathscr{L}(1)$ on $\mathscr{L}(2)$ induces a left action of $\mathscr{L}(1)$ on $M \wedge_{\mathscr{L}} N$ that gives it a structure of $\mathbb{L}$-spectrum.

We may mimic tensor product notation and write

$$M \wedge_{\mathscr{L}} N = \mathscr{L}(2) \ltimes_{\mathscr{L}(1) \times \mathscr{L}(1)} (M \wedge N).$$

This smash product is commutative, and a special property of the linear isometries operad, first noticed by Hopkins, implies that it is also associative. There is a function $\mathbb{L}$-spectrum functor $F_{\mathscr{L}}$ to go with $\wedge_{\mathscr{L}}$; it is constructed from the external and twisted function spectra functors, and we have the adjunction

$$G\mathscr{S}[\mathbb{L}](M \wedge_{\mathscr{L}} M', M'') \cong G\mathscr{S}[\mathbb{L}](M, F_{\mathscr{L}}(M, M')). \tag{2.3}$$

The smash product $\wedge_{\mathscr{L}}$ is not unital. However, there is a natural map

$$\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$$

of $\mathbb{L}$-spectra that is always a weak equivalence of spectra. It is not usually an isomorphism, but another special property of the linear isometries operad implies that it is an isomorphism if $M = S$ or if $M = S \wedge_{\mathscr{L}} N$ for any $\mathbb{L}$-spectrum N. Thus any $\mathbb{L}$-spectrum is weakly equivalent to one whose unit map is an isomorphism. This makes sense of the following definition, in which we understand S to mean the sphere G-spectrum indexed on our fixed chosen G-universe U.

DEFINITION 2.4. An S-module is an $\mathbb{L}$-spectrum M such that $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is an isomorphism. The category $G\mathscr{M}_S$ of S-modules is the full subcategory of $G\mathscr{S}[\mathbb{L}]$ whose objects are the S-modules. For S-modules M and M', define

$$M \wedge_S M' = M \wedge_{\mathscr{L}} M' \quad \text{and} \quad F_S(M, M') = S \wedge_{\mathscr{L}} F_{\mathscr{L}}(M, M').$$

Although easy to prove, one surprising formal feature of the theory is that the functor $S \wedge_{\mathscr{L}} (\cdot) : G\mathscr{S}[\mathbb{L}] \longrightarrow G\mathscr{M}_S$ is right and not left adjoint to the forgetful functor; it is left adjoint to the functor $F_{\mathscr{L}}(S, \cdot)$. This categorical situation dictates our definition of function S-modules. It also dictates that we construct limits of S-modules by constructing limits of their underlying $\mathbb{L}$-spectra and then applying the functor $S \wedge_{\mathscr{L}} (\bullet)$, as indicated in XXII§1. The free S-module functor $\mathbb{F}_S : G\mathscr{S} \longrightarrow G\mathscr{M}_S$ is defined by

$$\mathbb{F}_S(E) = S \wedge_{\mathscr{L}} \mathbb{L}E.$$

It is left adjoint to the functor $F_{\mathscr{L}}(S, \cdot) : G\mathscr{M}_S \longrightarrow G\mathscr{S}$, and this is the functor that we denoted by $\mathbb{U}_S$ in XXII§1. From this point, the properties of the category of S-modules that we described in XXII§1 are inherited directly from the good properties of the category of $\mathbb{L}$-spectra.

3. A_∞ and E_∞ ring spectra and S-algebras

We defined S-algebras and their modules in terms of structure maps that make the evident diagrams commute in the symmetric monoidal category of S-modules. There are older notions of A_∞ and E_∞ ring spectra and their modules that May, Quinn, and Ray introduced nonequivariantly in 1972; the equivariant generalization was given in [LMS]. Working equivariantly, an A_∞ ring spectrum is a spectrum R together with an action by the linear isometries G-operad $\mathscr{L}$. Such an action is given by G-maps

$$\theta_j : \mathscr{L}(j) \ltimes R^j \longrightarrow R, \ j \geq 0,$$

such that appropriate associativity and unity diagrams commute. If the θ_j are Σ_j-equivariant, then R is said to be an E_∞ ring spectrum. Similarly a left module M over an A_∞ ring spectrum R is defined in terms of maps

$$\mu_j : \mathscr{L}(j) \ltimes R^{j-1} \wedge M \longrightarrow M, \ j \geq 1;$$

in the E_∞ case, we require these maps to be Σ_{j-1}-equivariant. It turns out that the higher θ_j and μ_j are determined by the θ_j and μ_j for $j \leq 2$. That is, we have the following result, which might instead be taken as a definition.

THEOREM 3.1. An A_∞ ring spectrum is an $\mathbb{L}$-spectrum R with a unit map $\eta : S \longrightarrow R$ and a product $\phi : R \wedge_{\mathscr{L}} R \to R$ such that the following diagrams commute:

$$\begin{tikzcd} S \wedge_{\mathscr{L}} R \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[dr, "\lambda"'] & R \wedge_{\mathscr{L}} R \arrow[d, "\phi"] & R \wedge_{\mathscr{L}} S \arrow[l, "\mathrm{id}\wedge\eta"'] \arrow[dl, "\lambda\tau"] \\ & R & \end{tikzcd}$$

and

$$\begin{tikzcd} R \wedge_{\mathscr{L}} R \wedge_{\mathscr{L}} R \arrow[r, "\mathrm{id}\wedge\phi"] \arrow[d, "\phi\wedge\mathrm{id}"'] & R \wedge_{\mathscr{L}} R \arrow[d, "\phi"] \\ R \wedge_{\mathscr{L}} R \arrow[r, "\phi"'] & R; \end{tikzcd}$$

R is an E_∞ ring spectrum if the following diagram also commutes:

$$\begin{tikzcd} R \wedge_{\mathscr{L}} R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R \wedge_{\mathscr{L}} R \arrow[dl, "\phi"] \\ & R. & \end{tikzcd}$$

A module over an A_∞ or E_∞ ring spectrum R is an $\mathbb{L}$-spectrum M with a map $\mu : R \wedge_{\mathscr{L}} M \to M$ such that the following diagrams commute:

$$\begin{tikzcd} S \wedge_{\mathscr{L}} M \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[dr, "\lambda"'] & R \wedge_{\mathscr{L}} M \arrow[d, "\mu"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \wedge_{\mathscr{L}} R \wedge_{\mathscr{L}} M \arrow[r, "\mathrm{id}\wedge\mu"] \arrow[d, "\phi\wedge\mathrm{id}"'] & R \wedge_{\mathscr{L}} M \arrow[d, "\mu"] \\ R \wedge_{\mathscr{L}} M \arrow[r, "\mu"] & M. \end{tikzcd}$$

This leads to the following description of S-algebras.

COROLLARY 3.2. An S-algebra or commutative S-algebra is an A_∞ or E_∞ ring spectrum that is also an S-module. A module over an S-algebra or commutative S-algebra R is a module over the underlying A_∞ or E_∞ ring spectrum that is also an S-module.

In particular, we have a functorial way to replace A_∞ and E_∞ ring spectra and their modules by weakly equivalent S-algebras and commutative S-algebras and their modules.

COROLLARY 3.3. For an A_∞ ring spectrum R, $S \wedge_{\mathscr{L}} R$ is an S-algebra and $\lambda : S \wedge_{\mathscr{L}} R \longrightarrow R$ is a weak equivalence of A_∞ ring spectra, and similarly in the E_∞ case. If M is an R-module, then $S \wedge_{\mathscr{L}} M$ is an $S \wedge_{\mathscr{L}} R$-module and

$\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is a weak equivalence of R-modules and of modules over $S \wedge_{\mathscr{L}} R$ regarded as an A_∞ ring spectrum.

Thus the earlier definitions are essentially equivalent to the new ones, and earlier work gives a plenitude of examples. Thom G-spectra occur in nature as E_∞ ring G-spectra. For finite groups G, multiplicative infinite loop space theory works as it does nonequivariantly; however, the details have yet to be fully worked out and written up: that is planned for a later work. This theory gives that the Eilenberg-Mac Lane G-spectra of Green functors, the G-spectra of connective real and complex K-theory, and the G-spectra of equivariant algebraic K-theory are E_∞ ring spectra. As observed in XXII.6.13, it follows that the G-spectra of periodic real and complex K-theory are also E_∞ ring G-spectra. Nonequivariantly, many more examples are known due to recent work, mostly unpublished, of such people as Hopkins, Miller, and Kriz.

J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Volume 577. 1977.
J. P. May. Multiplicative infinite loop space theory. J. Pure and Applied Algebra, 26(1982), 1–69.

4. Alternative perspectives on equivariance

We have developed the theory of $\mathbb{L}$-spectra and S-modules starting from a fixed given G-universe U. However, there are alternative perspectives on the role of the universe and of equivariance that shed considerable light on the theory. Much of this material does not appear in the literature, and we give proofs in Section 6 after explaining the ideas here. Let S_U denote the sphere G-spectrum indexed on a G-universe U. The essential point is that while the categories $G\mathscr{S}U$ of G-spectra indexed on U vary as U varies, the categories $G\mathscr{M}_{S_U}$ of S_U-modules do not: all such categories are actually isomorphic. These isomorphisms preserve homotopies and thus pass to ordinary homotopy categories. However, they do not preserve weak equivalences and therefore do not pass to derived categories, which do vary with U. This observation first appeared in a paper of Elmendorf and May, but we shall begin with a different explanation than the one we gave there.

We shall explain matters by describing the categories of G-spectra and of $\mathbb{L}$-G-spectra indexed on varying universes U in terms of algebras over monads defined on the ground category $\mathscr{S} = \mathscr{S}\mathbb{R}^\infty$ of nonequivariant spectra indexed on $\mathbb{R}^\infty$. Abbreviate notation by writing L for the monoid $\mathscr{L}(1) = \mathscr{I}(\mathbb{R}^\infty, \mathbb{R}^\infty)$. Any G-universe U is isomorphic to $\mathbb{R}^\infty$ with an action by G through linear isometries. The action may be written in the form $gx = f(g)(x)$ for $x \in \mathbb{R}^\infty$, where $f : G \longrightarrow L$ is a homomorphism of monoids. To fix ideas, we shall write $\mathbb{R}^\infty_f$ for the G-universe determined by such a homomorphism f. For a spectrum E, we then define

$$\mathbb{G}_f E = G \ltimes E,$$

where the twisted half-smash product is determined by the map f. The multiplication and unit of G determine maps $\mu : \mathbb{G}_f\mathbb{G}_f E \longrightarrow \mathbb{G}_f E$ and $\eta : E \longrightarrow \mathbb{G}_f E$ that give $\mathbb{G}_f$ a structure of monad in $\mathscr{S}$. As was observed in [LMS], the category $G\mathscr{S}\mathbb{R}^\infty_f$ of G-spectra indexed on $\mathbb{R}^\infty_f$ is canonically isomorphic to the category $\mathscr{S}[\mathbb{G}_f]$ of algebras over the monad $\mathbb{G}_f$. Of course, we also have the monad $\mathbb{L}$ in $\mathscr{S}$ with $\mathbb{L}E = L \ltimes E$; by definition, a nonequivariant $\mathbb{L}$-spectrum is an algebra over this monad.

Proposition 4.1. The following statements about the monads $\mathbb{L}$ and $\mathbb{G}_f$ hold for any homomorphism of monoids $f : G \longrightarrow L = \mathscr{I}(\mathbb{R}^\infty, \mathbb{R}^\infty)$.

(i) $\mathbb{L}$ restricts to a monad in the category $\mathscr{S}[\mathbb{G}_f]$ of G-spectra indexed on $\mathbb{R}^\infty_f$.
(ii) $\mathbb{G}_f$ restricts to a monad in the category $\mathscr{S}[\mathbb{L}]$ of $\mathbb{L}$-spectra indexed on $\mathbb{R}^\infty$.
(iii) The composite monads $\mathbb{L}\mathbb{G}_f$ and $\mathbb{G}_f\mathbb{L}$ in $\mathscr{S}$ are isomorphic.

Moreover, up to isomorphism, the composite monad $\mathbb{L}\mathbb{G}_f$ is independent of f.

Corollary 4.2. The category $G\mathscr{S}\mathbb{R}^\infty_f[\mathbb{L}] = \mathscr{S}[G_f][\mathbb{L}]$ of $\mathbb{L}$-G-spectra indexed on $\mathbb{R}^\infty_f$ is isomorphic to the category $\mathscr{S}[\mathbb{L}][\mathbb{G}_f]$ of G-$\mathbb{L}$-spectra indexed on $\mathbb{R}^\infty_f$. Up to isomorphism, this category is independent of f.

The isomorphisms that we shall obtain preserve spheres and operadic smash products and so restrict to give isomorphisms between categories of S-modules.

Corollary 4.3. Up to isomorphism, the category $G\mathscr{M}_{S_U}$ of S_U-modules is independent of the G-universe U.

Thus a structure of $S_{\mathbb{R}^\infty}$-module on a naive G-spectrum is so rich that it encompasses an S_U-action on a G-spectrum indexed on U for any universe U. This richness is possible because the action of G on U can itself be expressed in terms of the monoid L.

There is another way to think about these isomorphisms, which is given in Elmendorf and May and which we now summarize. It is motivated by the definition of the operadic smash product.

DEFINITION 4.4. Fix universes U and U', write $\mathbb{L}$ and $\mathbb{L}'$ for the respective monads in $G\mathscr{S}U$ and $G\mathscr{S}U'$ and write $\mathscr{L}$ and $\mathscr{L}'$ for the respective G-operads. For an $\mathbb{L}$-spectrum M, define an $\mathbb{L}'$-spectrum $I_U^{U'}M$ by

$$I_U^{U'}M = \mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} M.$$

That is, $I_U^{U'}M$ is the coequalizer displayed in the diagram

$$\mathscr{I}(U,U') \ltimes (\mathscr{I}(U,U) \ltimes M) \underset{\mathrm{id}\ltimes\xi}{\overset{\gamma\ltimes\mathrm{id}}{\rightrightarrows}} \mathscr{I}(U,U') \ltimes M \longrightarrow I_U^{U'}M.$$

Here $\xi : \mathscr{I}(U,U) \ltimes M \longrightarrow M$ is the given action of $\mathbb{L}$ on M. We regard the product $\mathscr{I}(U,U') \times \mathscr{I}(U,U)$ as a space over $\mathscr{I}(U,U')$ via the composition map

$$\gamma : \mathscr{I}(U,U') \times \mathscr{I}(U,U) \longrightarrow \mathscr{I}(U,U');$$

Proposition 1.3(ii) gives a natural isomorphism

$$\mathscr{I}(U,U') \ltimes (\mathscr{I}(U,U) \ltimes M) \cong (\mathscr{I}(U,U') \times \mathscr{I}(U,U)) \ltimes M.$$

This makes sense of the map $\gamma \ltimes \mathrm{id}$ in the diagram. The required left action of $\mathscr{I}(U',U')$ on $I_U^{U'}M$ is induced by the composition product

$$\gamma : \mathscr{I}(U',U') \times \mathscr{I}(U,U') \longrightarrow \mathscr{I}(U,U'),$$

which induces a natural map of coequalizer diagrams on passage to twisted half-smash products.

PROPOSITION 4.5. Let U, U', and U'' be G-universes. Consider the functors

$$I_U^{U'} : G\mathscr{S}U[\mathbb{L}] \longrightarrow G\mathscr{S}U'[\mathbb{L}'] \quad \text{and} \quad \Sigma_U^\infty : G\mathscr{T} \longrightarrow G\mathscr{S}U[\mathbb{L}].$$

(i) $I_U^{U'} \circ \Sigma_U^\infty$ is naturally isomorphic to $\Sigma_{U'}^\infty$.
(ii) $I_{U'}^{U''} \circ I_U^{U'}$ is naturally isomorphic to $I_U^{U''}$.
(iii) I_U^U is naturally isomorphic to the identity functor.

Therefore the functor $I_U^{U'}$ is an equivalence of categories with inverse $I_{U'}^U$. Moreover, the functor $I_U^{U'}$ is continuous and satisfies $I_U^{U'}(M \wedge X) \cong (I_U^{U'}M) \wedge X$ for $\mathbb{L}$-spectra M and based G-spaces X. In particular, it is homotopy preserving, and $I_U^{U'}$ and $I_{U'}^U$ induce inverse equivalences of homotopy categories.

Now suppose that $U = \mathbb{R}_f^\infty$ and $U' = \mathbb{R}_{f'}^\infty$. Since the coequalizer defining $I_U^{U'}$ is the underlying nonequivariant coequalizer with a suitable action of G, we see that, with all group actions ignored, the functor $I_U^{U'}$ is naturally isomorphic to the identity functor on $\mathscr{S}[\mathbb{L}]$. In this case, the equivalences of categories of the previous result are natural isomorphisms and, tracing through the definitions, one can check that they agree with the equivalences given by the last statement of Corollary 4.2. Therefore the following result, which applies to any pair of universes U and U', is an elaboration of Corollary 4.3.

PROPOSITION 4.6. The following statements hold.

(i) $I_U^{U'} S_U$ is canonically isomorphic to $S_{U'}$.

(ii) For $\mathbb{L}$-spectra M and N, there is a natural isomorphism

$$\omega : I_U^{U'}(M \wedge_{\mathscr{L}} N) \cong (I_U^{U'} M) \wedge_{\mathscr{L}'} (I_U^{U'} N).$$

(iii) The following diagram commutes for all $\mathbb{L}$-spectra M:

$$\begin{tikzcd} I_U^{U'}(S_U \wedge_{\mathscr{L}} M) \arrow[rr, "\omega"] \arrow[dr, "I_U^{U'}\lambda"'] & & S_{U'} \wedge_{\mathscr{L}'} (I_U^{U'} M) \arrow[dl, "\lambda"] \\ & I_U^{U'} M. & \end{tikzcd}$$

(iv) M is an S_U-module if and only if $I_U^{U'} M$ is an $S_{U'}$-module.

Therefore the functors $I_U^{U'}$ and $I_{U'}^{U}$ restrict to inverse equivalences of categories between $G\mathscr{M}_{S_U}$ and $G\mathscr{M}_{S_{U'}}$ that induce inverse equivalences of categories between $hG\mathscr{M}_{S_U}$ and $hG\mathscr{M}_{S_{U'}}$.

This has the following consequence, which shows that, on the point-set level, our brave new equivariant algebraic structures are independent of the universe in which they are defined.

THEOREM 4.7. The functor $I_U^{U'} : G\mathscr{M}_{S_U} \longrightarrow G\mathscr{M}_{S_{U'}}$ is monoidal. If R is an S_U-algebra and M is an R-module, then $I_U^{U'} R$ is an $S_{U'}$-algebra and $I_U^{U'} M$ is an $I_U^{U'} R$-module.

The ideas of this section are illuminated by thinking model theoretically. We focus attention on the category $G\mathscr{M}_{\mathbb{R}^\infty}$, where G acts trivially on $\mathbb{R}^\infty$. We can reinterpret our results as saying that the model categories of S_U-modules for varying universes U are all isomorphic to the category $G\mathscr{M}_{\mathbb{R}^\infty}$, but given a model structure that depends on U. Indeed, for any $U = \mathbb{R}^\infty_f$, we have the isomorphism of categories $I_U^{\mathbb{R}^\infty} : G\mathscr{M}_U \longrightarrow G\mathscr{M}_{\mathbb{R}^\infty}$, and we can transport the model category structure of $G\mathscr{M}_U$ to a new model category structure on $G\mathscr{M}_{\mathbb{R}^\infty}$, which we call the U-model structure on $G\mathscr{M}_{\mathbb{R}^\infty}$.

The essential point is that $I_U^{\mathbb{R}^\infty}$ does not carry the cofibrant sphere S_U-modules $S^n_{S_U} = S_U \wedge_{\mathscr{L}} \mathbb{L}S^n$ to the corresponding cofibrant sphere $S_{\mathbb{R}^\infty}$-modules. The weak equivalences in the U-model structure are the maps that induce isomorphisms on homotopy classes of $S_{\mathbb{R}^\infty}$-module maps out of the "U-spheres" $G/H_+ \wedge I_U^{\mathbb{R}^\infty} S^n_{S_U}$. We define U-cell and relative U-cell $S_{\mathbb{R}^\infty}$-modules by using these U-spheres as the domains of their attaching maps. The U-cofibrations are the retracts of the relative U-cell $S_{\mathbb{R}^\infty}$-modules, and the U-fibrations are then determined as the maps that satisfy the right lifting property with respect to the acyclic U-cofibrations.

A. D. Elmendorf and J. P. May. Algebras over equivariant sphere spectra. Preprint, 1995.

5. The construction of equivariant algebras and modules

The results of the previous section are not mere esoterica. They lead to homotopically well-behaved constructions of brave new equivariant algebraic structures from brave new nonequivariant algebraic structures. The essential point is to understand the homotopical behavior of point-set level constructions that have desirable formal properties. We shall explain the solutions to two natural problems in this direction.

First, suppose given a nonequivariant S-algebra R and an R-module M; for definiteness, we suppose that these spectra are indexed on the fixed point universe U^G of a complete G-universe U. Is there an S_G-algebra R_G and an R_G-module M_G whose underlying nonequivariant spectra are equivalent to R and M in a way that preserves the brave new algebraic structuresΓ In this generality, the only obvious candidates for R_G and M_G are i_*R and i_*M, where $i : U^G \longrightarrow U$ is the inclusion. In any case, we want R_G and M_G to be equivalent to i_*R and i_*M. However, the change of universe functor i_* does not preserve brave new algebraic structures. Thus the problem is to find a functor that does preserve such structures and yet is equivalent to i_*. A very special case of the solution of this problem has been used by Benson and Greenlees to obtain calculational information about the ordinary cohomology of classifying spaces of compact Lie groups.

Second, suppose given an S_G-algebra R_G with underlying nonequivariant S-algebra R and suppose given an R-module M. Can we construct an R_G-module M_G whose underlying nonequivariant R-module is MΓ Note in particular that the problem presupposes that, up to equivalence, the underlying nonequivariant spectrum of R_G is an S-algebra, and similarly for modules. We are thinking of MU_G and MU, and the solution of this problem gives equivariant versions as MU_G-modules of all of the spectra, such as the Brown-Peterson and Morava K-theory spectra, that can be constructed from MU by killing some generators and inverting others.

The following homotopical result of Elmendorf and May combines with Theorem 4.7 to solve the first problem. In fact, it shows more generally that, up to isomorphism in derived categories, any change of universe functor preserves brave new algebraic structures. Observe that, for a linear isometry $f : U \longrightarrow U'$ and S_U-modules $M \in G\mathscr{M}_{S_U}$, we have a composite natural map

$$\alpha : f_*M \longrightarrow \mathscr{I}(U,U') \ltimes M \longrightarrow I^U_{U'}M$$

of G-spectra indexed on U', where the first arrow is induced by the inclusion $\{f\} \longrightarrow \mathscr{I}(U,U')$ and the second is the evident quotient map.

THEOREM 5.1. Let $f : U \longrightarrow U'$ be a G-linear isometry. Then for sufficiently well-behaved S_U-modules $M \in G\mathscr{M}_{S_U}$ (those in the collection $\bar{\mathscr{E}}_{S_U}$ of XXII.5.5), the natural map $\alpha : f_*M \longrightarrow I^{U'}_U M$ is a homotopy equivalence of G-spectra indexed on U'

Remember that $\bar{\mathscr{E}}_{S_U}$ includes the q-cofibrant objects in all of our categories of brave new algebras and modules. We are entitled to conclude that, up to equivalence, the change of universe functor f_* preserves brave new algebras and modules. The most important case is the inclusion $i : U^G \longrightarrow U$. If we start from any nonequivariant q-cofibrant brave new algebraic structure, then, up to equivalence, the change of universe functor i_* constructs from it a corresponding equivariant brave new algebraic structure.

Turning to the second problem that we posed, we give a result (due to May) that interrelates brave new algebraic structures in $G\mathscr{M}_U$ and $\mathscr{M}_{U^G}$. Its starting point is the idea of combining the operadic smash product with the functors $I^U_{U'}$. We think of U as the basic universe of interest in what follows.

DEFINITION 5.2. Let U, U', and U'' be G-universes. For an $\mathbb{L}'$-spectrum M and an $\mathbb{L}''$-spectrum N, define an $\mathbb{L}$-spectrum $M \wedge_{\mathscr{L}} N$ by

$$M \wedge_{\mathscr{L}} N = I^U_{U'}M \wedge_{\mathscr{L}} I^U_{U''}N.$$

The formal properties of this product can be deduced from those of the functors $I^U_{U'}$ together with those of the operadic smash product for the fixed universe U. In particular, since the functor $I^U_{U'}$ takes $S_{U'}$-modules to S_U-modules and the smash product over S_U is the restriction to S_U-modules of the smash product over $\mathscr{L}$, we have the following observation.

LEMMA 5.3. The functor $\wedge_{\mathscr{L}} : G\mathscr{S}U'[\mathbb{L}'] \times G\mathscr{S}U''[\mathbb{L}''] \longrightarrow G\mathscr{S}U[\mathbb{L}]$ restricts to a functor

$$\wedge_{S_U} : G\mathscr{M}_{S_{U'}} \times G\mathscr{M}_{S_{U''}} \longrightarrow G\mathscr{M}_{S_U}.$$

This allows us to define modules indexed on one universe over algebras indexed on another.

DEFINITION 5.4. Let $R \in G\mathscr{M}_{S_{U''}}$ be an $S_{U''}$-algebra and let $M \in G\mathscr{M}_{S_{U'}}$. Say that M is a right R-module if it is a right $I_{U''}^{U'}R$-module, and similarly for left modules.

To define smash products over R in this context, we use the functors $I_{U'}^{U}$ to index everything on our preferred universe U and then take smash products there.

DEFINITION 5.5. Let $R \in G\mathscr{M}_{S_{U''}}$ be an $S_{U''}$-algebra, let $M \in G\mathscr{M}_{S_{U'}}$ be a right R-module and let $N \in G\mathscr{M}_{U'''}$ be a left R-module. Define

$$M \wedge_R N = I_{U'}^{U} M \wedge_{I_{U''}^{U} R} I_{U'''}^{U} N.$$

These smash products inherit good formal properties from those of the smash products of R-modules, and their homotopical properties can be deduced from the homotopical properties of the smash product of R-modules and the homotopical properties of the functors $I_{U'}^{U}$, as given by Theorem 5.1.

Now specialize to consideration of $U^G \subseteq U$. Write S_G for the sphere G-spectrum indexed on U and S for the nonequivariant sphere spectrum indexed on U^G. We take S_G-modules to be in $G\mathscr{M}_U$ and S-modules to be in $\mathscr{M}_{U^G}$ in what follows.

THEOREM 5.6. Let R_G be a commutative S_G-algebra and assume that R_G is split as an algebra with underlying nonequivariant S-algebra R . Then there is a monoidal functor $R_G \wedge_R (\cdot) : \mathscr{M}_R \longrightarrow G\mathscr{M}_{R_G}$. If M is a cell R-module, then $R_G \wedge_R M$ is split as a module with underlying nonequivariant R-module M. The functor $R_G \wedge_R (\cdot)$ induces a derived monoidal functor $\mathscr{D}_R \longrightarrow G\mathscr{D}_{R_G}$. Therefore, if M is an R-ring spectrum (in the homotopical sense), then $R_G \wedge_R M$ is an R_G-ring G-spectrum.

The terms "split as an algebra" and "split as a module" are a bit technical, and we will explain them in a moment. However, we have the following important example; see XV§2 for the definition of MU_G.

PROPOSITION 5.7. The G-spectrum MU_G that represents stable complex cobordism is a commutative S_G-algebra, and it is split as an algebra with underlying nonequivariant S-algebra MU.

We shall return to this point and say something about the proof of the proposition in XXV§7. We conclude that, for any compact Lie group G and any MU-module M, we have a corresponding split MU_G-module $M_G \equiv MU_G \wedge_{MU} M$. This allows us to transport the nonequivariant constructions of XXII§4 into the equivariant world. For example, taking $M = BP$ or $M = K(n)$, we obtain equivariant Brown-Peterson and Morava K-theory MU_G-modules BP_G and $K(n)_G$. Moreover, if M is an MU-ring spectrum, then M_G is an MU_G-ring G-spectrum, and M_G is associative or commutative if M is so.

We must still explain our terms and sketch the proof of Theorem 5.6. The notion of a split G-spectrum was a homotopical one involving the change of universe functor i_*, and neither that functor nor its right adjoint i^* preserves brave new algebraic structures. We are led to the following definitions.

DEFINITION 5.8. A commutative S_G-algebra R_G is split as an algebra if there is a commutative S-algebra R and a map $\eta : I_{U^G}^U R \longrightarrow R_G$ of S_G-algebras such that η is a (nonequivariant) equivalence of spectra and the natural map $\alpha : i_*R \longrightarrow I_{U^G}^U R$ is an (equivariant) equivalence of G-spectra. We call R the (or, more accurately, an) underlying nonequivariant S-algebra of R_G.

Since the composite $\eta \circ \alpha$ is a nonequivariant equivalence and the natural map $R \longrightarrow i^*i_*R$ is a weak equivalence (provided that R is tame), R is weakly equivalent to i^*R_G with G-action ignored. Thus R is a highly structured version of the underlying nonequivariant spectrum of R_G. Clearly R_G is split as a G-spectrum with splitting map $\eta \circ \alpha$.

We have a parallel definition for modules.

DEFINITION 5.9. Let R_G be a commutative S_G-algebra that is split as an algebra with underlying S-algebra R and let M_G be an R_G-module. Regard M_G as an $I_{U^G}^U R$-module by pullback along η. Then M_G is split as a module if there is an R-module M and a map $\chi : I_{U^G}^U M \longrightarrow M_G$ of $I_{U^G}^U R$-modules such that χ is a (nonequivariant) equivalence of spectra and the natural map $\alpha : i_*M \longrightarrow I_{U^G}^U M$ is an (equivariant) equivalence of G-spectra. We call M the (or, more accurately, an) underlying nonequivariant R-module of M_G.

Again, M is a highly structured version of the underlying nonequivariant spectrum of M_G, and M_G is split as a G-spectrum with splitting map $\chi \circ \alpha$. The ambiguity that we allow in the notion of an underlying object is quite useful: it allows us to use Theorem 5.1 and q-cofibrant approximation (of S-algebras and of R-modules) to arrange the condition on α in the definitions if we have succeeded in arranging the other conditions.

For the proof of Theorem 5.6, Definition 5.5 specializes to give the required functor $R_G \wedge_R (\cdot)$, and it is clearly monoidal. We may as well assume that our given underlying nonequivariant S-algebra R is q-cofibrant as an S-algebra. Let

M be a cell R-module. By Theorem 5.1, the condition on α in the definition of an underlying R-module is satisfied. Define

$$\chi = \eta \wedge \mathrm{id} : I^U_{U^G} M \cong I^U_{U^G} R \wedge_{I^U_{U^G} R} I^U_{U^G} M \longrightarrow R_G \wedge_{I^U_{U^G} R} I^U_{U^G} M = M_G.$$

Clearly χ is a map of $I^U_{U^G} R$-modules, and it is not hard to prove that it is an equivalence of spectra. Thus M_G is split as a module with underlying nonequivariant R-module M. That is the main point, and the rest follows without difficulty.

D. J. Benson and J. P. C. Greenlees. Commutative algebra for cohomology rings of classifying spaces of compact Lie groups. Preprint. 1995.
A. D. Elmendorf and J. P. May. Algebras over equivariant sphere spectra. Preprint, 1995.
J. P. May. Equivariant and nonequivariant module spectra. Preprint, 1995.

6. Comparisons of categories of $\mathbb{L}$-G-spectra

We prove Proposition 4.1 and Corollary 4.2 here. The proof of Proposition 4.1 is based on the comparison of certain monoids constructed from the monoids G and L and the homomorphism $f : G \longrightarrow L$. Thus let $G \ltimes_f L$ and $L \rtimes_f G$ be the left and right semidirect products of G and L determined by f. As spaces,

$$G \ltimes_f L = G \times L \quad \text{and} \quad L \rtimes_f G = L \times G,$$

and their multiplications are specified by

$$(g,m)(g',m') = (gg', f({g'}^{-1}) m f(g') m')$$

and

$$(m,g)(m',g') = (m f(g) m' f(g^{-1}), gg').$$

There is an isomorphism of monoids

$$\tau : G \ltimes_f L \longrightarrow L \rtimes_f G$$

specified by

$$\tau(g,m) = (f(g) m f(g^{-1}), g);$$

there is also an isomorphism of monoids

$$\zeta : G \ltimes_f L \longrightarrow G \times L$$

specified by

$$\zeta(g,m) = (g, f(g)m);$$

its inverse takes (g,m) to $(g, f(g^{-1})m)$. Let

$$\pi : G \times L \longrightarrow L$$

be the projection. We regard $G \times L$ as a monoid over L via π and we regard $G \ltimes_f L$ and $L \times_f G$ as monoids over L via the composites $\pi \circ \zeta$ and $\pi \circ \zeta \circ \tau^{-1}$, so that ζ and τ are isomorphisms over L. Using Proposition 1.3, we see that, for spectra $E \in \mathscr{S}$, the map τ induces a natural isomorphism

$$(6.1) \qquad \tau : \mathbb{G}_f\mathbb{L}E \cong (G \ltimes_f L) \ltimes E \longrightarrow (L \rtimes_f G) \ltimes E \cong \mathbb{L}\mathbb{G}_f E$$

and the map ζ induces a natural isomorphism

$$(6.2) \qquad \zeta : \mathbb{G}_f\mathbb{L}E \cong (G \ltimes_f L) \ltimes E \longrightarrow (G \times L) \ltimes E \cong G_+ \wedge \mathbb{L}E.$$

In the domains and targets here, the units and products of the given monoids determine natural transformations η and μ that give the specified composite monad structures to the displayed functors $\mathscr{S} \longrightarrow \mathscr{S}$. Elementary diagram chases on the level of monoids imply that the displayed natural transformations are well-defined isomorphisms of monads. If f is the trivial homomorphism that sends all of G to $1 \in L$, then $G \ltimes_f L = G \times L$. Thus in (6.2) we are comparing the monad for the G-universe $\mathbb{R}^\infty_f$ to the monad determined by $\mathbb{R}^\infty$ regarded as a trivial G-universe. The conclusions of Proposition 4.1 follow, and Corollary 4.2 follows as a matter of category theory.

The following two lemmas in category theory may or may not illuminate what is going on. The first is proven in [EKMM] and shows why Corollary 4.2 follows from Proposition 4.1. The second dictates exactly what "elementary diagram chases" are needed to complete the proof of Proposition 4.1.

Lemma 6.3. Let $\mathbb{S}$ be a monad in a category $\mathscr{C}$ and let $\mathbb{T}$ be a monad in the category $\mathscr{C}[\mathbb{S}]$ of $\mathbb{S}$-algebras. Then the category $\mathscr{C}[\mathbb{S}][\mathbb{T}]$ of $\mathbb{T}$-algebras in $\mathscr{C}[\mathbb{S}]$ is isomorphic to the category $\mathscr{C}[\mathbb{T}\mathbb{S}]$ of algebras over the composite monad $\mathbb{T}\mathbb{S}$ in $\mathscr{C}$.

Here the unit of $\mathbb{T}\mathbb{S}$ is the composite $\mathrm{id} \longrightarrow \mathbb{S} \longrightarrow \mathbb{T}\mathbb{S}$ given by the units of $\mathbb{S}$ and $\mathbb{T}$ and the product on $\mathbb{T}\mathbb{S}$ is the composite $\mathbb{T}\mathbb{S}\mathbb{T}\mathbb{S} \longrightarrow \mathbb{T}\mathbb{T}\mathbb{S} \longrightarrow \mathbb{T}\mathbb{S}$, where the second map is given by the product of $\mathbb{T}$ and the first is obtained by applying $\mathbb{T}$ to the action $\mathbb{S}\mathbb{T}\mathbb{S} \longrightarrow \mathbb{T}\mathbb{S}$ given by the fact that $\mathbb{T}$ is a monad in $\mathscr{C}[\mathbb{S}]$. In our applications, we are taking $\mathbb{T}$ to be the restriction to $\mathscr{C}[\mathbb{S}]$ of a monad in $\mathscr{C}$. This requires us to start with monads $\mathbb{S}$ and $\mathbb{T}$ that commute with one another.

Lemma 6.4. Let $\mathbb{S}$ and $\mathbb{T}$ be monads in $\mathscr{C}$. Suppose there is a natural isomorphism $\tau : \mathbb{ST} \longrightarrow \mathbb{TS}$ such that the following diagrams commute:

$$\begin{tikzcd} \mathbb{SST} \arrow[rr, "\mu"] \arrow[d, "\mathbb{S}\tau"'] & & \mathbb{ST} \arrow[d, "\tau"] \\ \mathbb{STS} \arrow[r, "\tau"'] & \mathbb{TSS} \arrow[r, "\mathbb{T}\mu"'] & \mathbb{TS} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathbb{T} \arrow[dl, "\eta"'] \arrow[dr, "\mathbb{T}\eta"] & \\ \mathbb{ST} \arrow[rr, "\tau"'] & & \mathbb{TS}. \end{tikzcd}$$

Then $\mathbb{T}$ restricts to a monad in $\mathscr{C}[\mathbb{S}]$ to which the previous lemma applies. Suppose further that these diagrams with the roles of $\mathbb{S}$ and $\mathbb{T}$ reversed also commute, as do the following diagrams:

$$\begin{tikzcd} \mathbb{STST} \arrow[r, "\mathbb{S}\tau^{-1}"] \arrow[d, "\tau\circ\mathbb{ST}\tau"'] & \mathbb{SSTT} \arrow[r, "\mathbb{SS}\mu"] & \mathbb{SST} \arrow[r, "\mu"] & \mathbb{ST} \arrow[d, "\tau"] \\ \mathbb{TSTS} \arrow[r, "\mathbb{T}\tau"'] & \mathbb{TTSS} \arrow[r, "\mathbb{TT}\mu"'] & \mathbb{TTS} \arrow[r, "\mu"'] & \mathbb{TS} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{id} \arrow[r, "\eta"] \arrow[d, equal] & \mathbb{T} \arrow[r, "\eta"] & \mathbb{ST} \arrow[d, "\tau"] \\ \mathrm{id} \arrow[r, "\eta"'] & \mathbb{S} \arrow[r, "\eta"'] & \mathbb{TS}. \end{tikzcd}$$

Then $\tau : \mathbb{ST} \longrightarrow \mathbb{TS}$ is an isomorphism of monads. Therefore the categories $\mathscr{C}[\mathbb{S}][\mathbb{T}]$ and $\mathscr{C}[\mathbb{T}][\mathbb{S}]$ are both isomorphic to the category $\mathscr{C}[\mathbb{ST}] \cong \mathscr{C}[\mathbb{TS}]$.

Here, for the first statement, if X is an $\mathbb{S}$-algebra with action ξ, then the required action of $\mathbb{S}$ on $\mathbb{T}X$ is the composite $\mathbb{ST}X \xrightarrow{\tau} \mathbb{TS}X \xrightarrow{\mathbb{T}\xi} \mathbb{T}X$.

CHAPTER XXIV

Brave New Equivariant Algebra

by J. P. C. Greenlees and J. P. May

1. Introduction

We shall explain how useful it is to be able to mimic commutative algebra in equivariant topology. Actually, the nonequivariant specializations of the constructions that we shall describe are also of considerable interest, especially in connection with the chromatic filtration of stable homotopy theory. We have discussed this in an expository paper [GM1], and that paper also says more about the relevant algebraic constructions than we shall say here. We shall give a connected sequence of examples of brave new analogues of constructions in commutative algebra. The general pattern of how the theory works is this. We first give an algebraic definition. We next give its brave new analogue. The homotopy groups of the brave new analogue will be computable in terms of a spectral sequence that starts with the relevant algebraic construction computed on coefficient rings and modules. The usefulness of the constructions is that they are often related by a natural map to or from an analogous geometric construction that one wishes to compute. Localization and completion theorems say when such maps are equivalences.

The Atiyah-Segal completion theorem and the Segal conjecture are examples of this paradigm that we have already discussed. However, very special features of those cases allowed them to be handled without explicit use of brave new algebra: the force of Bott periodicity in the case of K-theory and the fact that the sphere G-spectrum acts naturally on the stable homotopy category in the case of cohomotopy. We shall explain how brave new algebra gives a coherent general framework for the study of such completion phenomena in cohomology and analogous localization phenomena in homology. We have given another exposition of these matters in [GM2], which says more about the basic philosophy. We shall describe the results in a little greater generality here and so clarify the application to K-theory. We shall also explain the relationship between localization theorems and Tate theory, which we find quite illuminating.

[GM1] J. P. C. Greenlees and J. P. May. Completions in algebra and topology. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 255-276.
[GM2] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. In "Handbook of Algebraic Topology", edited by I.M. James. North Holland, 1995, pp 277-324.

2. Local and Čech cohomology in algebra

Suppose given a ring R, which may be graded and which need not be Noetherian, and suppose given a finitely generated ideal $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. If R is graded the α_i are required to be homogeneous.

For any element α, we may consider the stable Koszul cochain complex

$$K^\bullet(\alpha) = \big(R \to R[\alpha^{-1}]\big)$$

concentrated in codegrees 0 and 1. Notice that we have a fiber sequence

$$K^\bullet(\alpha) \longrightarrow R \longrightarrow R[\alpha^{-1}]$$

of cochain complexes.

We may now form the tensor product

$$K^\bullet(\alpha_1, \ldots, \alpha_n) = K^\bullet(\alpha_1) \otimes \ldots \otimes K^\bullet(\alpha_n).$$

It is clear that this complex is unchanged if we replace some α_i by a power, and it is not hard to check the following result.

Lemma 2.1. If $\beta \in I$, then $K^\bullet(\alpha_1, \ldots, \alpha_n)[\beta^{-1}]$ is exact. Up to quasi-isomorphism, the complex $K^\bullet(\alpha_1, \ldots, \alpha_n)$ depends only on the ideal I.

Therefore, up to quasi-isomorphism, $K^\bullet(\alpha_1, \ldots, \alpha_n)$ depends only on the radical of the ideal I, and we henceforth write $K^\bullet(I)$ for it.

Following Grothendieck, we define the local cohomology groups of an R-module M by

$$H^*_I(R; M) = H^*(K^\bullet(I) \otimes M). \tag{2.2}$$

It is easy to see that $H^0_I(R;M)$ is the submodule

$$\Gamma_I(M) = \{m \in M | I^k m = 0 \text{ for some positive integer } k\}$$

of I-power torsion elements of M. If R is Noetherian it is not hard to prove that $H^*_I(R;\cdot)$ is effaceable and hence that local cohomology calculates the right derived functors of $\Gamma_I(\cdot)$. It is clear that the local cohomology groups vanish above codegree n; in the Noetherian case Grothendieck's vanishing theorem shows that they are actually zero above the Krull dimension of R. Observe that if $\beta \in I$ then $H^*_I(R;M)[\beta^{-1}] = 0$; this is a restatement of the exactness of $K^\bullet(I)[\beta^{-1}]$.

The Koszul complex $K^\bullet(\alpha)$ comes with a natural map $\varepsilon : K^\bullet(\alpha) \longrightarrow R$; the tensor product of such maps gives an augmentation $\varepsilon : K^\bullet(I) \longrightarrow R$. Define the Čech complex $\check{C}^\bullet(I)$ to be $\Sigma(\text{Ker}\,\varepsilon)$. (The name is justified in [GM1].) By inspection, or as an alternative definition, we then have the fiber sequence of cochain complexes

$$K^\bullet(I) \longrightarrow R \longrightarrow \check{C}^\bullet(I). \tag{2.3}$$

We define the Čech cohomology groups of an R-module M by

$$\check{C}H^*_I(R;M) = H^*(\check{C}^\bullet(I) \otimes M). \tag{2.4}$$

We often delete R from the notation for these functors. The fiber sequence (2.3) gives rise to long exact sequences relating local and Čech cohomology, and these reduce to exact sequences

$$0 \longrightarrow H^0_I(M) \longrightarrow M \longrightarrow \check{C}H^0_I(M) \longrightarrow H^1_I(M) \longrightarrow 0$$

together with isomorphisms

$$H^i_I(M) \cong \check{C}H^{i-1}_I(M).$$

A. Grothendieck (notes by R.Hartshorne). Local cohomology. Springer Lecture notes in mathematics, Vol. 42. 1967.

3. Brave new versions of local and Čech cohomology

Turning to topology, we fix a compact Lie group G and consider G-spectra indexed on a complete G-universe U. We let S_G be the sphere G-spectrum, and we work in the category of S_G-modules. Fix a commutative S_G-algebra R and consider R-modules M. We write

$$M^G_n = \pi^G_n(M) = M^{-n}_G.$$

Thus R^G_* is a ring and M^G_* is an R^G_*-module.

Mimicking the algebra, for $\alpha \in R^G_*$ we define the Koszul spectrum $K(\alpha)$ by the fiber sequence

$$K(\alpha) \longrightarrow R \longrightarrow R[\alpha^{-1}].$$

Here, suppressing notation for suspensions, $R[\alpha^{-1}] = \mathrm{hocolim}(R \xrightarrow{\alpha} R \xrightarrow{\alpha} \dots)$; it is an R-module and the inclusion of R is a module map; therefore $K(\alpha)$ is an R-module. Analogous to the filtration at the chain level, we obtain a filtration of $K(\alpha)$ by viewing it as $\Sigma^{-1}(R[1/\alpha] \cup CR)$.

Next we define the Koszul spectrum of a sequence $\alpha_1, \dots, \alpha_n$ by

$$K(\alpha_1, \dots, \alpha_n) = K(\alpha_1) \wedge_R \dots \wedge_R K(\alpha_n).$$

Using the same proof as in the algebraic case we conclude that, up to equivalence, $K(\alpha_1, \dots, \alpha_n)$ depends only on the radical of $I = (\alpha_1, \dots, \alpha_n)$; we therefore denote it $K(I)$. We then define the homotopy I-power torsion (or local cohomology) module of an R-module M by

$$\Gamma_I(M) = K(I) \wedge_R M. \tag{3.1}$$

In particular, $\Gamma_I(R) = K(I)$.

To calculate the homotopy groups of $\Gamma_I(M)$ we use the product of the filtrations of the $K(\alpha_i)$ given above. Since the filtration models the algebra precisely, there results a spectral sequence of the form

$$E^2_{s,t} = H^{-s}_I(R^G_*; M^G_*)_t \Rightarrow \pi^G_{s+t}(\Gamma_I(M)) \tag{3.2}$$

with differentials $d^r : E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$.

REMARK 3.3. In practice it is often useful to use the fact that the algebraic local cohomology $H^*_I(R; M)$ is essentially independent of R. Indeed if the generators of I come from a ring R_0 (in which they generate an ideal I_0) via a ring homomorphism $\theta : R_0 \to R$, then $H^*_{I_0}(R_0; M) = H^*_I(R; M)$. In practice we often use this if the ideal I of R^G_* may be radically generated by elements of degree 0. This holds for any ideal of S^G_* since the elements of positive degree in S^G_* are nilpotent.

Similarly, we define the Čech spectrum of I by the cofiber sequence of R-modules

$$K(I) \longrightarrow R \longrightarrow \check{C}(I). \tag{3.4}$$

We think of $\check{C}(I)$ as analogous to $\tilde{E}G$. We then define the homotopical localization (or Čech cohomology) module associated to an R-module M by

$$M[I^{-1}] = \check{C}(I) \wedge_R M. \tag{3.5}$$

In particular, $R[I^{-1}] = \check{C}(I)$. Again, we have a spectral sequence of the form

$$E^2_{s,t} = \check{C}H_I^{-s}(R_*^G; M_*^G)_t \Rightarrow \pi^G_{s+t}(M[I^{-1}]) \tag{3.6}$$

with differentials $d^r : E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$.

The "localization" $M[I^{-1}]$ is generally not a localization of M at a multiplicatively closed subset of R_*. However, the term is justified by the following theorem from [GM1, §5]. Recall the discussion of Bousfield localization from XXII§6.

Theorem 3.7. For any finitely generated ideal $I = (\alpha_1, \ldots, \alpha_n)$ of R_*^G, the map $M \longrightarrow M[I^{-1}]$ is Bousfield localization with respect to the R-module $R[I^{-1}]$ or, equivalently, with respect to the wedge of the R-modules $R[\alpha_i^{-1}]$.

Observe that we have a natural cofiber sequence

$$\Gamma_I(M) \longrightarrow M \longrightarrow M[I^{-1}] \tag{3.8}$$

relating our I-power torsion and localization functors.

4. Localization theorems in equivariant homology

For an R-module M, we have the fundamental cofiber sequence of R-modules

$$EG_+ \wedge M \longrightarrow M \longrightarrow \tilde{E}G \wedge M. \tag{4.1}$$

Such sequences played a central role in our study of the Segal conjecture and Tate cohomology, for example, and we would like to understand their homotopical behavior. In favorable cases, the cofiber sequence (3.8) models this sequence and so allows computations via the spectral sequences of the previous section. The relevant ideal is the augmentation ideal

$$I = \mathrm{Ker}(\mathrm{res}^G_1 : R^G_* \to R_*).$$

In order to apply the constructions of the previous section, we need an assumption. It will be satisfied automatically when R^G_* is Noetherian.

Assumption 4.2. Up to taking radicals, the ideal I is finitely generated. That is, there are elements $\alpha_1, \ldots, \alpha_n \in I$ such that

$$\sqrt{(\alpha_1, \ldots, \alpha_n)} = \sqrt{I}.$$

Under Assumption (4.2), it is reasonable to let $K(I)$ denote $K(\alpha_1, \ldots, \alpha_n)$. The canonical map $\varepsilon : K(I) \longrightarrow R$ is then a nonequivariant equivalence. Indeed, this is a special case of the following observation, which is evident from our constructions.

LEMMA 4.3. Let $H \subseteq G$, let $\beta_i \in R_*^G$, and let $\gamma_i = \mathrm{res}_H^G(\beta_i) \in R_*^H$. Then, regarded as a module over the S_H-algebra $R|_H$,

$$K(\beta_1, \cdots, \beta_n)|_H = K(\gamma_1, \cdots, \gamma_n).$$

Therefore, if $\beta_i \in \mathrm{Ker}\,\mathrm{res}_H^G$, then the natural map $K(\beta_1, \cdots, \beta_n) \longrightarrow R$ is an H-equivalence.

Here the last statement holds since $K(0) = R$. If we take the smash product of ε with the identity map of EG_+, we obtain a G-equivalence of R-modules $EG_+ \wedge K(I) \longrightarrow EG_+ \wedge R$. Working in the derived category $G\mathscr{D}_R$, we may invert this map and compose with the map

$$EG_+ \wedge K(I) \longrightarrow S^0 \wedge K(I) = K(I)$$

induced by the projection $EG_+ \longrightarrow S^0$ to obtain a map of R-modules over R

$$\kappa : EG_+ \wedge R \longrightarrow K(I). \tag{4.4}$$

Passing to cofibers we obtain a compatible map

$$\tilde{\kappa} : \tilde{E}G \wedge R \longrightarrow \check{C}(I). \tag{4.5}$$

Finally, taking the smash product over R with an R-module M, there results a natural map of cofiber sequences

$$\begin{tikzcd} EG_+ \wedge M \arrow[r] \arrow[d, "\kappa"'] & M \arrow[r] \arrow[d, equal] & \tilde{E}G \wedge M \arrow[d, "\tilde{\kappa}"] \\ \Gamma_I(M) \arrow[r] & M \arrow[r] & M[I^{-1}]. \end{tikzcd} \tag{4.6}$$

Clearly κ is an equivalence if and only if $\tilde{\kappa}$ is an equivalence. When the latter holds, it should be interpreted as stating that the 'topological' localization of M away from its free part is equivalent to the 'algebraic' localization of M away from I. We adopt this idea in a definition. Recall the homotopical notions of an R-ring spectrum A and of an A-module spectrum from XXII.4.1; we tacitly assume throughout the chapter that all given R-ring spectra are associative and commutative.

DEFINITION 4.7. The 'localization theorem' holds for an R-ring spectrum A if

$$\tilde{\kappa}_A = \tilde{\kappa} \wedge \mathrm{id} : \tilde{E}G \wedge A = \tilde{E}G \wedge R \wedge_R A \longrightarrow \check{C}(I) \wedge_R A$$

is a weak equivalence of R-modules, that is, if it is an isomorphism in $G\mathscr{D}_R$. It is equivalent that

$$\kappa_A = \kappa \wedge \mathrm{id} : EG_+ \wedge A = EG_+ \wedge R \wedge_R A \longrightarrow K(I) \wedge_R A$$

be an isomorphism in $G\mathscr{D}_R$.

In our equivariant context, we define the A-homology of an R-module M by

$$A_n^{G,R}(M) = \pi_n^G(M \wedge_R A); \tag{4.8}$$

compare XXII.3.1. This must not be confused with $A_n^G(X) = \pi_n^G(X \wedge A)$, which is defined on all G-spectra X. When $A = R$, $A_*^{G,R}$ is the restriction of A_*^G to R-modules. When $R = S_G$, A_*^{G,S_G} is A_*^G thought of as a theory defined on S_G-modules. In general, for G-spectra X, we have the relation

$$A_*^G(X) \cong A_*^{G,R}(\mathbb{F}_R X), \tag{4.9}$$

where the free R-module $\mathbb{F}_R X$ is weakly equivalent to the spectrum $X \wedge R$. The localization theorem asserts that κ is an $A_*^{H,R}$-isomorphism for all subgroups H of G and thus that the cofiber $C\kappa$ is $A_*^{H,R}$-acyclic for all H. Observe that the definition of κ implies that $C\kappa$ is equivalent to $\tilde{E}G \wedge K(I)$. We are mainly interested in the case $A = R$, but we shall see in the next section that the localization theorem holds for K_G regarded as an S_G-ring spectrum, although it fails for S_G itself. The conclusion of the localization theorem is inherited by arbitrary A-modules.

LEMMA 4.10. If the localization theorem holds for the R-ring spectrum A, then the maps

$$EG_+ \wedge M \longrightarrow \Gamma_I(M) \quad \text{and} \quad \tilde{E}G \wedge M \longrightarrow M[I^{-1}]$$

of (4.6) are isomorphisms in $G\mathscr{D}_R$ for all A-modules M.

PROOF. $C\kappa \wedge_R M$ is trivial since it is a retract in $G\mathscr{D}_R$ of $C\kappa \wedge_R A \wedge_R M$. $\square$

When this holds, we obtain the isomorphism

$$M_*^G(EG_+) = \pi_*^G(EG_+ \wedge M) \cong \pi_*^G(\Gamma_I(M))$$

on passage to homotopy groups. Here, in favorable cases, the homotopy groups on the right can be calculated by the spectral sequence (3.2). When M is split and G is finite, the homology groups on the left are the (reduced) homology groups

$M_*(BG_+)$ defined with respect to the underlying nonequivariant spectrum of M; see XVI§2. We also obtain the isomorphism

$$M^G_*(\tilde{E}G) = \pi^G_*(\tilde{E}G \wedge M) \cong \pi^G_*(M[I^{-1}]);$$

the homotopy groups on the right can be calculated by the spectral sequence (3.5).

More generally, it is valuable to obtain a localization theorem about $EG_+ \wedge_G X$ for a general based G-space X, obtaining the result about BG_+ by taking X to be S^0. To obtain this, we simply replace M by $M \wedge X$ in the first equivalence of the previous lemma. If M is split, we conclude from XVI§2 that

$$\pi^G_*(\Sigma^{-Ad(G)}(EG_+ \wedge X \wedge M)) \cong M_*(EG_+ \wedge_G X),$$

where $Ad(G)$ is the adjoint representation of G. Thus we have the following implication.

Corollary 4.11. If the localization theorem holds for A and M is an A-module spectrum that is split as a G-spectrum, then

$$\Gamma_I(\Sigma^{-Ad(G)} M \wedge X)^G_* \cong M_*(EG_+ \wedge_G X)$$

for any based G-space X. Therefore there is a spectral sequence of the form

$$E^2_{s,t} = H^{-s}_I(R^G_*; M^G_*(\Sigma^{-Ad(G)}\Sigma^\infty X))_t \Rightarrow M_{s+t}(EG_+ \wedge_G X).$$

5. Completions, completion theorems, and local homology

The localization theorem also implies a completion theorem. In fact, applying the functor $F_R(\cdot, M)$ to the map κ, we obtain a cohomological analogue of Lemma 4.10. To give the appropriate context, we define the completion of an R-module M at a finitely generated ideal I by

$$M^\wedge_I = F_R(K(I), M). \tag{5.1}$$

We shall shortly return to algebra and define certain "local homology groups" $H^*_I(R; M)$ that are closely related to the I-adic completion functor. In the topological context, it will follow from the definitions that the filtration of $K(I)$ gives rise to a spectral sequence of the form

$$E^{s,t}_2 = H^I_{-s}(R^*_G; M^*_G)^t \Rightarrow \pi^G_{-s-t}(M^\wedge_I) \tag{5.2}$$

with differentials $d_r : E^{s,t}_r \to E^{s+r,t-r+1}_r$. Here, if R^*_G is Noetherian and M^*_G is finitely generated, then

$$\pi^G_{-*}(M^\wedge_I) = (M^*_G)^\wedge_I.$$

Again, a theorem from [GM1, §5] gives an interpretation of the completion functor as a Bousfield localization.

THEOREM 5.3. For any finitely generated ideal $I = (\alpha_1, \dots, \alpha_n)$ of R_*^G, the map $M \longrightarrow M_I^\wedge$ is Bousfield localization in the category of R-modules with respect to the R-module $K(I)$ or, equivalently, with respect to the smash product of the R-modules R/α_i.

Returning to the augmentation ideal I, we have the promised cohomological implication of the localization theorem; the case $M = A$ is called the 'completion theorem' for A.

LEMMA 5.4. If the localization theorem holds for the R-ring spectrum A, then the map

$$M_I^\wedge = F_R(K(I), M) \longrightarrow F_R(EG_+ \wedge R, M) \cong F(EG_+, M)$$

is an isomorphism in $G\mathscr{D}_R$ for all A-module spectra M.

PROOF. $F_R(C\kappa, M)$ is trivial since any map $C\kappa \longrightarrow M$ factors as a composite

$$C\kappa \longrightarrow C\kappa \wedge_R A \longrightarrow M \wedge_R A \longrightarrow M,$$

and similarly for suspensions of $C\kappa$. $\square$

When this holds, we obtain the isomorphism

$$\pi_{-*}^G(M_I^\wedge) \cong M_G^*(EG_+)$$

on passage to homotopy groups. If M is split, the cohomology groups on the right are the (reduced) cohomology groups $M^*(BG_+)$ defined with respect to the underlying nonequivariant spectrum of M; see XVI§2.

To obtain a completion theorem about $EG_+ \wedge_G X$ for a based G-space X, we replace M by $F(X, M)$ in the previous lemma. If M is split, then

$$\pi_*^G(F(EG_+ \wedge X, M)) \cong M^*(EG_+ \wedge_G X).$$

COROLLARY 5.5. If the localization theorem holds for A and M is an A-module spectrum that is split as a G-spectrum, then

$$(F(X, M)_I^\wedge)_G^* \cong M^*(EG_+ \wedge_G X)$$

for any based G-space X. Therefore there is a spectral sequence of the form

$$E_2^{s,t} = H_{-s}^I(R_G^*; M_G^*(X))^t \Rightarrow M^{s+t}(EG_+ \wedge_G X).$$

Thus, when it holds, the localization theorem for A implies a calculation of both $M_*(EG_+ \wedge_G X)$ and $M^*(EG_+ \wedge_G X)$ for all split A-modules M and all based G-spaces X.

We must still define the algebraic construction whose brave new counterpart is given by our completion functors. Returning to the algebraic context of Section 1, we want to define a suitable dual to local cohomology. Since local cohomology is obtained as $H^*(K \otimes M)$ for a suitable complex K, we expect to have to take $H_*(\mathrm{Hom}(K, M))$. However this will be badly behaved unless we first replace K by a complex of projective R-modules. Thus we choose an R-free complex $PK^\bullet(I)$ and a homology isomorphism $PK^\bullet(I) \longrightarrow K^\bullet(I)$. Since both complexes consist of flat modules we could equally well have used $PK^\bullet(I)$ in the definition of local cohomology. For finitely generated ideals $I = (\alpha_1, \cdots, \alpha_n)$, we take tensor products and define $PK^\bullet(I) = PK^\bullet(\alpha_1) \otimes \ldots \otimes PK^\bullet(\alpha_n)$; independence of generators follows from that of $K^\bullet(I)$.

We may then define local homology by

$$H_*^I(R; M) = H_*(\mathrm{Hom}(PK^\bullet(I), M)). \tag{5.6}$$

We often omit R from the notation. Because we chose a projective complex we obtain a third quadrant univeral coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s(H_I^{-t}(R), M) \Rightarrow H^I_{-t-s}(R; M)$$

with differentials $d_r : E_r^{s,t} \rightarrow E_r^{s+t,t-r+1}$ that relates local cohomology to local homology.

It is not hard to check from the definition that if R is Noetherian and M is either free or finitely generated, then $H_0^I(R; M) \cong M_I^\wedge$, and one may also prove that in these cases the higher local homology groups are zero. It follows that $H_*^I(R; M)$ calculates the left derived functors of the (not necessarily right exact) I-adic completion functor. In fact, this holds under weaker hypotheses on R than that it be Noetherian.

Returning to our topological context, it is now clear that if R is a commutative S_G-algebra and I is a finitely generated ideal in R_*^G, then the completion functor $M_I^\wedge$ on R-modules is the brave new analogue of local homology: we have the spectral sequence (5.2).

J.P.C. Greenlees and J.P. May. Derived functors of I-adic completion and local homology. J. Algebra **149** (1992), 438-453.

6. A proof and generalization of the localization theorem

To prove systematically that the map κ_A of (4.7) is a weak equivalence we need to know that when we restrict the map κ of (4.4) to a subgroup H, we obtain an analogous map of H-spectra. Write I_H for the augmentation ideal $\mathrm{Ker}(\mathrm{res}^H_1 \subset R^H_*)$. Even for cohomotopy it is not true that $\mathrm{res}(I_G) = I_H$, but in that case they do have the same radical. To give a general result, we must assume that this holds.

ASSUMPTION 6.1. For all subgroups $H \subseteq G$

$$\sqrt{\mathrm{res}(I_G)} = \sqrt{I_H}.$$

For theories such as cohomotopy and K-theory, where we understand all of the primes of R^G_*, this is easy to verify. Note that both (4.2) and (6.1) are assumptions on R that have nothing to do with A. We need an assumption that relates R^G_* to A^G_*. Let $J = J_G$ be the augmentation ideal in A^G_*. The unit $R \longrightarrow A$ induces a homomorphism of rings $R^G_* \longrightarrow A^G_*$ that is compatible with restrictions to subgroups, hence we have an inclusion of ideals $I \cdot A^G_* \subseteq J$.

ASSUMPTION 6.2. The augmentation ideals of R^*_G and A^*_G are related by

$$\sqrt{I \cdot A^G_*} = \sqrt{J}.$$

Recall from (4.8) that $A^{G,R}_*(M) = \pi^G_*(M \wedge_R A)$. The final ingredient of our proof will be the existence of Thom isomorphisms

$$A^{G,R}_*(S^V \wedge M) \cong A^{G,R}_*(S^{|V|} \wedge M) \tag{6.3}$$

of $A^{G,R}_*$-modules for all complex representations V and R-modules M. For example, with $A = R$, homotopical bordism and K-theory have such Thom isomorphisms. Cohomotopy does not, and that is why our proof (and the theorem) fail in that case.

THEOREM 6.4 (LOCALIZATION). If A is an R-ring spectrum such that, for all subgroups H of G, the theories $A^{H,R}_*(\cdot)$ admit Thom isomorphisms and if assumptions (4.2), (6.1), and (6.2) hold for G and for all of its subgroups, then the localization theorem holds for A.

PROOF. We have observed that the cofiber of κ is equivalent to $\tilde{E}G \wedge K(I)$. We must prove that $\tilde{E}G \wedge K(I) \wedge_R A \simeq *$. We proceed by induction on the size of the group. By Assumption (6.1) and Lemma 4.3, we see that

$$(\tilde{E}G \wedge K(I_G))|_H \simeq \tilde{E}H \wedge K(I_H).$$

Thus our inductive assumption implies that

$$G/H_+ \wedge \tilde{E}G \wedge K(I) \wedge_R A \simeq *$$

for all proper subgroups $H \subset G$. Arguing exactly as in Carlsson's first reduction, XX.4.1, of the Segal conjecture for finite p-groups, we find that it suffices to prove that $\tilde{E}\mathscr{P} \wedge K(I) \wedge_R A \simeq *$. Indeed, $(\tilde{E}\mathscr{P})^G = S^0$ and $\tilde{E}\mathscr{P}/S^0$ can be constructed from cells $G/H_+ \wedge S^n$ with H proper. Therefore

$$(\tilde{E}\mathscr{P}/S^0) \wedge \tilde{E}G \wedge K(I) \wedge_R A \simeq *$$

and thus

$$\tilde{E}G \wedge K(I) \wedge_R A \simeq \tilde{E}\mathscr{P} \wedge \tilde{E}G \wedge K(I) \wedge_R A.$$

However, the map $\tilde{E}\mathscr{P} \longrightarrow \tilde{E}\mathscr{P} \wedge \tilde{E}G$ induced by the map $S^0 \longrightarrow \tilde{E}G$ is a G-equivalence by a check of fixed point spaces.

Now, if G is finite, consider the reduced regular representation V. As we observed in the proof of the Segal conjecture, $S^{\infty V} = \operatorname{colim} S^{kV}$ is a model for $\tilde{E}\mathscr{P}$ since $V^H \neq 0$ if H is proper and $V^G = 0$. For a general compact Lie group G, we write $S^{\infty V}$ for the colimit of the spheres S^V, where V runs over a suitably large set of representations V such that $V^G = \{0\}$, for example all such V that are contained in a complete G-universe U. Again, $S^{\infty V}$ is a model for $\tilde{E}\mathscr{P}$.

At this point we must recall how Thom isomorphisms give rise to Euler classes $\chi(V) \in A^{G,R}_{-|V|}$. Indeed the inclusion $e : S^0 \longrightarrow S^V$ and the Thom isomorphism give a natural map of $A^{G,R}_*$-modules

$$A^{G,R}_*(X) \xrightarrow{e_*} A^{G,R}_*(S^V \wedge X) \cong A^{G,R}_*(S^{|V|} \wedge X) \cong A^{G,R}_{*-|V|}(X),$$

and this map is given by multiplication by $\chi(V)$. Thus, for finite G,

$$\begin{aligned} A^{G,R}_*(S^{\infty V} \wedge K(I)) &= \operatorname{colim}_k A^{G,R}_*(S^{kV} \wedge K(I)) \\ &= \operatorname{colim}_k (A^{G,R}_*(K(I)), \chi(V)) \\ &= A^{G,R}_*(K(I))[\chi(V)^{-1}]. \end{aligned}$$

Here $\chi(V)$ is in J since e is nonequivariantly null homotopic. Therefore, using Assumption 6.2 and Remark 3.3, we see that

$$H^*_I(R^G_*; N)[\chi(V)^{-1}] \cong H^*_J(A^G_*; N)\left[\chi(V)^{-1}\right] = 0$$

for any A_*^G-module N. From the spectral sequence (3.2), we deduce that

$$A_*^{G,R}(S^{\infty V} \wedge K(I)) = 0.$$

A little elaboration of the argument gives the same conclusion when G is a general compact Lie group. Since $S^{\infty V}$ is H-equivariantly contractible for all proper subgroups H, this shows that $S^{\infty V} \wedge K(I) \wedge_R A \simeq *$, as required. $\square$

There is a substantial generalization of the theorem that admits virtually the same proof. Recall from V.4.6 that, for a family $\mathscr{F}$, we have the cofiber sequence

$$E\mathscr{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathscr{F}.$$

We discussed family versions of the Atiyah-Segal completion theorem in XIV§6 and of the Segal conjecture in XX§§1-3. As in those cases, we define

$$I\mathscr{F} = \cap_{H\in\mathscr{F}} \operatorname{Ker}(\operatorname{res}_H^G : R_*^G \longrightarrow R_*^H).$$

Arguing exactly as above, we obtain a map

$$\kappa : E\mathscr{F}_+ \wedge R \longrightarrow K(I\mathscr{F}). \tag{6.5}$$

DEFINITION 6.6. The '$\mathscr{F}$-localization theorem' holds for an R-ring spectrum A if

$$\kappa_A = \kappa \wedge \mathrm{id} : E\mathscr{F}_+ \wedge A = E\mathscr{F}_+ \wedge R \wedge_R A \longrightarrow K(I\mathscr{F}) \wedge_R A$$

is a weak equivalence of R-modules, that is, if it is an isomorphism in $G\mathscr{D}_R$.

We combine and record the evident analogs of Lemmas 4.10 and 5.4.

LEMMA 6.7. *If the $\mathscr{F}$-localization theorem holds for the R-ring spectrum A, then the maps*

$$E\mathscr{F}_+ \wedge M \longrightarrow \Gamma_{I\mathscr{F}}(M), \qquad \tilde{E}\mathscr{F} \wedge M \longrightarrow M[I\mathscr{F}^{-1}],$$

and

$$M_{I\mathscr{F}}^{\wedge} = F_R(K(I\mathscr{F}), M) \longrightarrow F_R(E\mathscr{F}_+ \wedge R, M) \cong F(E\mathscr{F}_+, M)$$

are isomorphisms in $G\mathscr{D}_R$ for all A-modules M.

A family $\mathscr{F}$ in G restricts to a family $\mathscr{F}|_H = \{K | K \in \mathscr{F} \text{ and } K \subset H\}$, and Assumptions 4.2, 6.1, and 6.2 admit evident analogs for $I\mathscr{F}$.

THEOREM 6.8 (F-LOCALIZATION). *If A is an R-ring spectrum such that, for all subgroups H of G, the theories $A_*^{R|_H}(\cdot)$ admit Thom isomorphisms and if, for a given family $\mathscr{F}$, the $\mathscr{F}$ versions of assumptions (4.2), (6.1), and (6.2) hold for G and for all of its subgroups, then the $\mathscr{F}$-localization theorem holds for A.*

PROOF. Here we must prove that $\tilde{E}\mathscr{F} \wedge K(I\mathscr{F}) \wedge_R A \simeq *$, and we assume that $G \notin \mathscr{F}$ since otherwise $\tilde{E}\mathscr{F} \simeq *$. As in the proof of the localization theorem, since the evident map $\tilde{E}\mathscr{P} \longrightarrow \tilde{E}\mathscr{P} \wedge \tilde{E}\mathscr{F}$ is a G-equivalence, the problem reduces inductively to showing that $\tilde{E}\mathscr{P} \wedge K(I\mathscr{F}) \wedge_R A \simeq *$. We take $S^{\infty V}$ as our model for $\tilde{E}\mathscr{P}$ and see that, since $V^H = \{O\}$ for all $H \neq G$, $\chi(V) \in J\mathscr{F}$. The rest is the same as in the proof of the localization theorem. $\square$

REMARK 6.9. It is perhaps of philosophical interest to note that the localization theorem is true for all R that satisfy (4.2) and (6.1) *provided that we work with $RO(G)$-graded rings*. Indeed the proof is the same except that instead of using the integer graded element $\chi(V) \in R^G_{-|V|}$ we must use $e(V) = e_*(1) \in R^G_{-V}$. The conclusion is only that there are spectral sequences

$$H^*_I(R^G_*) \Rightarrow R^G_*(EG_+)$$

and so forth, where $RO(G)$-grading of R^G_* is understood. In practice this theorem is not useful because the $RO(G)$-graded coefficient ring is hard to compute and is usually of even greater Krull dimension than the integer graded coefficent ring R^G_*. The Thom isomorphisms allow us to translate the $RO(G)$-graded augmentation ideal into its integer graded counterpart.

7. The application to K-theory

We can apply the $\mathscr{F}$-localization theorem to complex and real periodic equivariant K-theory in two quite different ways. The essential point is that Bott periodicity clearly gives the Thom isomorphisms necessary for both applications (see XIV§3). Unfortunately, for entirely different reasons, both applications are at present limited to finite groups.

First, we recall from XXII.6.13 that, for finite groups G, complex and real equivariant K-theory are known to be represented by commutative S_G-algebras. In view of Bott periodicity, we may restrict attention to the (complex or real) representation ring of G regarded as the subring of degree zero elements of K^G_* or KO^G_* (compare Remark 3.3), and our complete understanding of these rings makes verification of the $\mathscr{F}$ versions of (4.2) and (6.1) straightforward. In fact, these verifications work for arbitrary compact Lie groups G. The following theorem would hold in that generality if only we knew that K_G and KO_G were represented by commutative S_G-algebras in general. For this reason, although the completion theorem is known for all compact Lie groups, the localization theorem is only known for finite groups. The problem is that, at this writing, equivariant infinite loop space theory has not yet been developed for compact Lie groups of equivariance.

THEOREM 7.1. Let G be finite. Then, for every family $\mathscr{F}$, the $\mathscr{F}$-localization theorem holds for K_G regarded as a K_G-algebra, and similarly for KO_G.

Second, we have the first author's original version of the $\mathscr{F}$-localization theorem for K-theory. For that version, we regard K_G and KO_G as S_G-ring spectra. Here we may restrict attention to the Burnside ring of G regarded as the subring of degree zero elements of $\pi_*^G(S_G)$. Again, when G is finite, our complete understanding of $A(G)$ makes verification of the $\mathscr{F}$ versions of (4.2) and (6.1) straightforward, and we observed in and after XXI.5.3 that the $\mathscr{F}$ version of (6.2) holds. Note, however, that $A(G)$ is not Noetherian for general compact Lie groups, so that (4.2) and (6.1) are not available to us in that generality. Moreover, $A(G)$ and $R(G)$ are not closely enough related for (6.2) to hold. For example, the augmentation ideal of $A(G)$ is zero when G is a torus.

THEOREM 7.2. Let G be finite. Then, for every family $\mathscr{F}$, the $\mathscr{F}$-localization theorem holds for K_G regarded as an S_G-ring spectrum, and similarly for KO_G.

In the standard case $\mathscr{F} = \{e\}$, we explained in XXI§5 how Tate theory allows us to process the conclusions of the theorems to give an explicit computation of $K_*(BG)$; see XXI.5.4. The following references give further computational information. A comment on the relative generality of the two theorems is in order. The first only gives information about K_G-modules of the brave new sort, whereas the second gives information about K_G-module spectra of the classical sort. However, a remarkable result of Wolbert shows that the nonequivariant implications are the same: every classical K-module spectrum is weakly equivalent to the underlying spectrum of a brave new K-module.

J. P. C. Greenlees. K homology of universal spaces and local cohomology of the representation ring. Topology 32(1993), 295-308.

J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoir American Math. Soc. No. 543. 1995.

J. Wolbert. Toward an algebraic classification of module spectra. Preprint, 1995. University of Chicago. (Part of 1996 PhD thesis in preparation.)

8. Local Tate cohomology

When the $\mathscr{F}$-localization theorem holds, it implies good algebraic behaviour of the $\mathscr{F}$-Tate spectrum. We here explain what such good behaviour is by defining the algebraic ideal to which the Tate spectrum aspires: the local Tate cohomology groups of a module. We proceed by strict analogy with the construction of the topological $\mathscr{F}$-Tate spectrum,

$$t_{\mathscr{F}}(k) = F(E\mathscr{F}_+, k) \wedge \tilde{E}\mathscr{F}.$$

Thus, again working in the algebraic context of Section 1, we define the local Tate cohomology groups to be

$$\hat{H}^*_I(R; M) = H^*(\mathrm{Hom}(PK^\bullet(I), M) \otimes P\check{C}^\bullet(I)). \tag{8.1}$$

Here $P\check{C}^\bullet(I)$ is the projective Čech complex, which is defined by the algebraic fiber sequence

$$PK^\bullet(I) \longrightarrow R \longrightarrow P\check{C}^\bullet(I) \tag{8.2}$$

of chain complexes. There results a local Tate spectral sequence of the form

$$E_2^{*,*} = \check{H}^*_I(H^I_*(R; M)) \Rightarrow \hat{H}^*_I(R; M).$$

In favorable cases this starts with the Čech cohomology of the derived functors of I-adic completion.

The usefulness of the definition becomes apparent from the form that periodicity takes in this manifestation of Tate theory. It turns out that unexpectedly many elements of R induce isomorphisms of the R-module $\hat{H}^*_I(R; M)$. It is simplest to state this formally when R has Krull dimension 1.

Theorem 8.3 (Rationality). If R is Noetherian and of Krull dimension 1, then multiplication by any non-zero divisor of R is an isomorphism on $\hat{H}^*_I(R; M)$.

The Burnside ring $A(G)$ and the representation ring $R(G)$ of a finite group G are one dimensional Noetherian rings of particular topological interest.

Corollary 8.4. Let G be finite. For any ideal I of $A(G)$ and any $A(G)$-module M, $\hat{H}^*_I(A(G); M)$ is a rational vector space.

Corollary 8.5. Let G be finite. For any ideal I of $R(G)$ and any $R(G)$-module M, $\hat{H}^*_I(R(G); M)$ is a rational vector space.

Returning to our S_G-algebra R and its modules M, we define the 'I-local Tate spectrum' of M for a finitely generated ideal $I \subset R^G_*$ by

$$t_I(M) = F_R(K(I), M) \wedge_R \check{C}(I). \tag{8.6}$$

It is then immediate that there is a spectral sequence

$$E_2^{s,t} = \hat{H}_I^s(R_G^*; M_G^*)^t \Rightarrow \pi_{-s-t}^G(t_I(M)). \tag{8.7}$$

In particular, we may draw topological corollaries from Corollaries 8.4 and 8.5.

Corollary 8.8. Let G be finite. For any ideal I in $A(G) = \pi_0^G(S_G)$ and any G-spectrum E, $t_I(E)$ is a rational G-spectrum.

Corollary 8.9. Let G be finite. For any ideal I in $R(G) = \pi_0^G(K_G)$ and any K_G-module M, $t_I(M)$ is a rational G-spectrum.

Now assume the $\mathscr{F}$ version of (4.2). Let A be an R-ring spectrum and consider the diagram

$$\begin{tikzcd}
E\mathscr{F}_+ \wedge A \arrow[r] \arrow[d, "\kappa_A"'] & S^0 \wedge A \arrow[r] \arrow[d, equal] & \tilde{E}\mathscr{F} \wedge A \arrow[d, "\tilde{\kappa}_A"] \\
K(I\mathscr{F}) \wedge_R A \arrow[r] & A \arrow[r] & \check{C}(I\mathscr{F}) \wedge_R A.
\end{tikzcd}$$

If the $\mathscr{F}$-localization theorem holds for A, then κ_A and $\tilde{\kappa}_A$ are weak equivalences of R-modules. We may read off remarkable implications for the Tate spectrum $t_{\mathscr{F}}(M)$ of any A-module spectrum M. If κ_A is a weak equivalence, this $\mathscr{F}$-Tate spectrum is equivalent to the $I\mathscr{F}$-local Tate spectrum: a manipulation of isotropy groups is equivalent to a manipulation of ideals in brave new commutative algebra.

Theorem 8.10. If the $\mathscr{F}$-localization theorem holds for the R-ring spectrum A, then the $\mathscr{F}$-Tate and $I\mathscr{F}$-local Tate spectra of any A-module spectrum M are equivalent:

$$t_{\mathscr{F}}(M) \simeq t_{I\mathscr{F}}(M).$$

Proof. Since $F_R(X, M)$ is an A-module for any R-module X, Lemma 6.7 implies that all maps in the following diagram are weak equivalences of R-modules:

$$
\begin{tikzcd}
 & t_{\mathscr{F}}(M) \arrow[d, equal] \\
F_R(K(I\mathscr{F}),M)\wedge_R R\wedge \tilde{E}\mathscr{F} \arrow[r] \arrow[d] & F_R(E\mathscr{F}_+\wedge R, M)\wedge_R R\wedge \tilde{E}\mathscr{F} \arrow[d] \\
F_R(K(I\mathscr{F}),M)\wedge_R \check{C}(I\mathscr{F}) \arrow[r] \arrow[d, equal] & F_R(E\mathscr{F}_+\wedge R, M)\wedge_R \check{C}(I\mathscr{F}). \\
t_{I\mathscr{F}}(M) & \square
\end{tikzcd}
$$

Theorem 8.3 gives a striking consequence.

COROLLARY 8.11. Assume that R_*^G is Noetherian of dimension 1 and $\mathbb{Z}$-torsion free. If the $\mathscr{F}$-localization theorem holds for an R-ring spectrum A, then the $\mathscr{F}$-Tate spectrum $t_{\mathscr{F}}(M)$ is rational for any A-module M.

REMARK 8.12. Upon restriction to the Burnside ring $A(G)=\pi_0^G(S_G)$, the corollary applies to $R=S_G$. In this case it has a converse: if the completion theorem holds for A and $t_{\mathscr{F}}(A)$ is rational, then the localization theorem holds for A. The proof (which is in our memoir on Tate cohomology) uses easy formal arguments and the fact that $\kappa: E\mathscr{F}_+\wedge S_G \longrightarrow K(I\mathscr{F})$ is a rational equivalence.

We should comment on analogues of Corollary 8.11 in the higher dimensional case. The essence of Theorem 8.10 is that if the localization theorem holds for A, then the Tate spectrum of an A-module M is algebraic and is therefore dominated by the behaviour of the local Tate cohomology groups $\hat{H}_I^*(R_G^*;M_G^*)$ via the spectral sequence (8.7). Now these groups are modules over the ring $\hat{H}_I^*(R_G^*)$, so an understanding of the prime ideal spectrum of this ring is fundamental. For example, the first author's proof of the Rationality Theorem shows that analogues of it hold under appropriate hypotheses on $\mathrm{spec}(R_G^*)$.

These comments are relevant to the discussion of XXI§6. As noted there, we know that applying the Tate construction to spectra of type $E(n)$, on which v_n is invertible, forces v_{n-1} to be invertible (in a suitable completion). One guesses that this can be explained in terms of the subvariety of $\mathrm{Spec}(E(n)_G^*)$ defined by v_{n-1} and its intersection with that of I. Unfortunately our ignorance of $E(n)_G^*$ prevents us from justifying this intuition.

J. P. C. Greenlees. Tate cohomology in commutative algebra. J. Pure and Applied Algebra 94(1994), 59-83.

J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoir American Math. Soc. No. 543. 1995.

CHAPTER XXV

Localization and completion in complex bordism

by J. P. C. Greenlees and J. P. May

1. The localization theorem for stable complex bordism

There is a large literature that is concerned with the calculation of homology and cohomology groups $M_*(BG)$ and $M^*(BG)$ for MU-module spectra M, such as MU itself, K, BP, $K(n)$, $E(n)$, and so forth. Here G is a compact Lie group, in practice a finite group or a finite extension of a torus. The results do not appear to fall into a common pattern.

Nevertheless, there is a localization and completion theorem for stable complex bordism, and this shows that all such calculations must fit into a single general pattern dominated by the structure of the equivariant bordism ring MU_*^G. Indeed, as we showed in XXIII§5, there is a general procedure for constructing an equivariant version M_G of any nonequivariant MU-module M. Since M_G is split with underlying nonequivariant MU-module M, the theorem applies to the calculation of $M_*(BG_+)$ and $M^*(BG_+)$ for all such M. This is not, at present, calculationally useful since rather little is known about MU_*^G. Nevertheless, the theorem gives an intriguing new relation between equivariant and nonequivariant algebraic topology.

While the basic philosophy behind the theorem is the same as for the localization theorem XXIV.6.4, that result does not apply because its basic algebraic assumptions, XXIV.4.2 and 6.1, do not hold. In particular, since the augmentation ideal of MU_*^G is certainly not finitely generated and presumably not radically finitely generated, it is not even clear what we mean by the localization theorem, and different techniques are needed for its proof. Let $J = J_G$ denote the augmentation ideal of MU^G_* (with integer grading understood). For finitely generated subideals I of J, we can perform all of the topological constructions discussed in the previous chapter.

THEOREM 1.1. Let G be finite or a finite extension of a torus. Then, for any sufficiently large finitely generated ideal $I \subset J$, $\kappa : EG_+ \wedge MU_G \to K(I)$ is an equivalence.

It is reasonable to define $K(J)$ to be $K(I)$ for any sufficiently large I and to define $\Gamma_J(M_G)$ and $(M_G)^\wedge_J$ similarly. The theorem implies that these MU_G-modules are independent of the choice of I.

Consequences are drawn exactly as they were for the localization theorem in Sections 4 and 5 of the previous chapter. In particular,

$$EG_+ \wedge M_G \to \Gamma_J(M_G) \quad \text{and} \quad (M_G)^\wedge_J \to F(EG_+, M_G)$$

are equivalences for any MU_G-module M_G.

The fact that the theorem holds for a finite extension of a torus and thus for the normalizer of a maximal torus in an arbitrary compact Lie group strongly suggests that the following generalization should be true, but we have not succeeded in finding a proof.

CONJECTURE 1.2. The theorem remains true for any compact Lie group G.

Most of this chapter is taken from the following paper, which gives full details. The last section discusses an earlier completion "theorem" for MU^G_* when G is a compact Abelian Lie group. While it may be true, we have only been able to obtain a complete proof in special cases.

J. P. C. Greenlees and J. P. May. Localization and completion theorems for MU-module spectra. Preprint, 1995.

2. An outline of the proof

We shall emphasize the general strategy. Let G be a compact Lie group and let S_G be the sphere G-spectrum. We assume given a commutative S_G-algebra R_G with underlying nonequivariant commutative S-algebra R. As in the localization theorem, we shall assume that the theory R^G_* has Thom isomorphisms

$$R^G_*(S^V \wedge X) \cong R^G_*(S^{|V|} \wedge X) \tag{2.1}$$

for complex representations V and G-spectra X. More precisely, we shall assume this for all subgroups $H \subseteq G$, and we shall later impose a certain naturality condition on these Thom isomorphisms. We have already seen in XV§2 that MU_G has such Thom isomorphisms. As in the proof of XXIV.6.4, the Thom isomorphism gives rise to an Euler class $\chi(V) \in R^G_{|-V|}$. Let J_H be the augmentation ideal $\mathrm{Ker}(\mathrm{res}^H_1 : R^H_* \longrightarrow R_*)$; remember that $J = J_G$.

DEFINITION 2.2. Assume that R^H_* has Thom isomorphisms for all $H \subseteq G$. Let I be a finitely generated subideal of J and, for $H \subseteq G$, let $r^G_H(I)$ denote the resulting subideal $\mathrm{res}^G_H(I) \cdot R^H_*$ of J_H. We say that I is sufficiently large at H if there is a non-zero complex representation V of H such that $V^H = 0$ and the Euler class $\chi(V) \in R^H_*$ is in the radical $\sqrt{r^G_H(I)}$. We say that the ideal I is sufficiently large if it is sufficiently large at all $H \subseteq G$.

We have the canonical map of R_G-modules

$$\kappa : EG_+ \wedge R_G \longrightarrow K(I),$$

and our goal is to prove that it is an equivalence. The essential point of our strategy is the following result, which reduces the problem to the construction of a sufficiently large finitely generated subideal I of J.

THEOREM 2.3. Assume that R^H_* has Thom isomorphisms for all $H \subseteq G$. If I is a sufficiently large finitely generated subideal of J, then

$$\kappa : EG_+ \wedge R_G \longrightarrow K(I)$$

is an equivalence.

PROOF. The cofiber of κ is equivalent to $\tilde{E}G \wedge K(I)$, and we must prove that this is contractible. Using the transitivity of restriction maps to see that $r^G_H(I)$ is a large enough subideal of R^H_*, we see that the hypotheses of the theorem are inherited by any subgroup. Therefore we may assume inductively that the theorem holds for $H \in \mathscr{P}$. Observing that

$$(\tilde{E}G \wedge K(I))|_H = \tilde{E}H \wedge K(r^G_H(I))$$

for $H \subseteq G$, we see that our definition of a sufficiently large ideal provides exactly what is needed to allow us to obtain the conclusion by parroting the proof the localization theorem XXIV.6.4. $\square$

Thus our problem is to prove that there is a large enough finitely generated ideal I. One's first instinct is to take I to be generated by finitely many well chosen Euler classes. While that does work in some cases, we usually need to add in other elements, and we shall do so by exploiting norm, or "multiplicative transfer", maps. We explain the strategy before stating what it means for a theory to have such norm maps.

We assume from now on that G is a toral group, namely an extension

$$1 \longrightarrow T \longrightarrow G \longrightarrow F \longrightarrow 1,$$

where T is a torus and F is a finite group.

THEOREM 2.4. If G is toral and the R_*^H for $H \subseteq G$ admit norm maps and Thom isomorphisms, then J contains a sufficiently large finitely generated subideal.

The proof of the theorem depends on two lemmas. As usual, we write

$$\mathrm{res}_H^G : R(G) \longrightarrow R(H)$$

for the restriction homomorphism. When H has finite index in G, we write

$$\mathrm{ind}_H^G : R(H) \longrightarrow R(G)$$

for the induction homomorphism. Recall that $\mathrm{ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

LEMMA 2.5. There are finitely many non-zero complex representations $V_1, \cdots, V_s$ of T such that T acts freely on the product of the unit spheres of the representations

$$\mathrm{res}_T^G \mathrm{ind}_T^G V_i.$$

While this is not obvious, its proof requires only elementary Lie theory and does not depend on the use of norm maps. We shall say no more about it since it is irrelevant when G is finite.

LEMMA 2.6. Let F' be a subgroup of F with inverse image G' in G. There is an element $\xi(F')$ of J such that

$$\mathrm{res}_{G'}^G(\xi(F')) = \chi(V')^{w'},$$

where V' is the reduced regular complex representation of F' regarded by pullback as a representation of G' and w' is the order of $WG' = NG'/G'$.

We shall turn to the proof of this in the next section, but we first show how these lemmas imply Theorem 2.4.

PROOF OF THEOREM 2.4. We claim that the ideal

$$I = (\chi(\operatorname{ind}_T^G V_1), \cdots, \chi(\operatorname{ind}_T^G V_s)) + (\xi(F')|F' \subseteq F)$$

is sufficiently large.

If H is a subgroup of G that intersects T non-trivially, then, by Lemma 2.5, $(\operatorname{res}_T^G \operatorname{ind}_T^G V_i)^{H \cap T} = \{0\}$ for some i and therefore $(\operatorname{ind}_T^G V_i)^H = \{0\}$. Since

$$\chi(\operatorname{res}_H^G \operatorname{ind}_T^G V_i) = \operatorname{res}_H^G(\chi(\operatorname{ind}_T^G V_i)) \in r_H^G(I),$$

this shows that I is sufficiently large at H in this case.

If H is a subgroup of G that intersects T trivially, as is always the case when G is finite, then H maps isomorphically to its image F' in F. If G' is the inverse image of F' in G and V' is the reduced regular complex representation of F' regarded as a representation of G', then $\operatorname{res}_H^{G'}(V')$ is the reduced regular complex representation of H and $(\operatorname{res}_H^{G'}(V'))^H = 0$. By Lemma 2.6, we have $\operatorname{res}_{G'}^G(\xi(F')) = \chi(V')^{w'}$ and therefore

$$\chi(\operatorname{res}_H^{G'}(V'))^{w'} = \operatorname{res}_H^{G'}(\chi(V')^{w'}) = \operatorname{res}_H^{G'} \operatorname{res}_{G'}^G(\xi(F')) = \operatorname{res}_H^G(\xi(F')) \in r_H^G(I).$$

This shows that I is sufficiently large at H in this case. $\square$

3. The norm map and its properties

We must still explain the proof of Lemma 2.6, and to do so we must explain our hypothesis that R_*^G has norm maps. We shall give a rather crude definition that prescribes exactly what we shall use in the proof. The crux of the matter is a double coset formula, and we need some notations in order to state it. For $g \in G$ and $H \subseteq G$, let ${}^gH = gHg^{-1}$ and let $c_g : {}^gH \longrightarrow H$ be the conjugation isomorphism. For a based H-space X, we have a natural isomorphism

$$c_g : R_*^H(X) \longrightarrow R_*^{{}^gH}({}^gX),$$

where ${}^gX$ denotes X regarded as a ${}^gH$-space by pullback along c_g. We also have a natural restriction homomorphism

$$\operatorname{res}_H^G : R_*^G(X) \longrightarrow R_*^H(X).$$

DEFINITION 3.1. We say that R_*^G has norm maps if, for a subgroup H of finite index n in G and an element $y \in R_{-r}^H$, where $r \geq 0$ is even, there is an element

$$\overline{\operatorname{norm}}_H^G(1 + y) \in \sum_{i=0}^{n} R_{-ri}^G$$

that satisfies the following properties; here $1 = 1_H \in R_0^H$ denotes the identity element.

(i) $\overline{\mathrm{norm}}_G^G(1+y) = 1+y$.
(ii) $\overline{\mathrm{norm}}_H^G(1) = 1$.
(iii) [The double coset formula]

$$res_K^G\,\overline{\mathrm{norm}}_H^G\,(1+y) = \prod_g \overline{\mathrm{norm}}_{{}^gH\cap K}^K\ \mathrm{res}_{{}^gH\cap K}^{{}^gH}\ c_g(1+y),$$

where K is any subgroup of G and $\{g\}$ runs through a set of double coset representatives for $K\backslash G/H$.

Proof of Lemma 2.6. Since the restriction of the reduced regular representation of F' to any proper subgroup contains a trivial representation, the restriction of $\chi(V') \in R_*^{G'}$ to a subgroup that maps to a proper subgroup of F' is zero. In $R_*^{G'}$, the double coset formula gives

$$\mathrm{res}_{G'}^G\ \overline{\mathrm{norm}}_{G'}^G\,(1+\chi(V')) = \prod_g \overline{\mathrm{norm}}_{{}^gG'\cap G'}^{G'}\ \mathrm{res}_{{}^gG'\cap G'}^{{}^gG'}\ c_g(1+\chi(V')), \tag{3.2}$$

where g runs through a set of double coset representatives for $G'\backslash G/G'$. We require that our Thom isomorphisms be natural with respect to conjugation in the sense that their Euler classes satisfy $c_g(\chi(V)) = \chi({}^gV)$, where ${}^gV$ is the pullback of V along c_g. In particular, this gives that

$$c_g(1+\chi(V')) = 1+\chi({}^gV').$$

Here ${}^gV'$ is the reduced regular representation of ${}^gG'$. Clearly ${}^gG'\cap G'$ is the inverse image in G of ${}^gF'\cap F'$. If ${}^gF'\cap F'$ is a proper subgroup of F', then the restriction of $\chi(V')$ to ${}^gG'\cap G'$ is zero. Therefore all terms in the product on the right side of (3.2) are 1 except for those that are indexed on elements $g \in NG'$. There is one such g for each element of $WG' = NG'/G'$, and the term in the product that is indexed by each such g is just $1+\chi(V')$. Therefore (3.2) reduces to

$$\mathrm{res}_{G'}^G\ \overline{\mathrm{norm}}_{G'}^G(1+\chi(V')) = (1+\chi(V'))^{w'}. \tag{3.3}$$

If V' has real dimension r, then the summand of $(1+\chi(V'))^{w'}$ in degree rw' is $\chi(V')^{w'}$. Since $\mathrm{res}_{G'}^G$ preserves the grading, we may take $\xi(F')$ to be the summand of degree rw' in $\overline{\mathrm{norm}}_{G'}^G(1+\chi(V'))$. $\square$

4. The idea behind the construction of norm maps

We give an intuitive idea of the construction here, but we need some preliminaries to establish the context. Let H be a subgroup of finite index n in a compact Lie group G. The norm map is intimately related to $\mathrm{ind}_H^G : RO(H) \longrightarrow RO(G)$, and we begin with a description of induction that suggests an action of G on the nth smash power X^n of any based H-space X. Recall that the wreath product $\Sigma_n \int H$ is the set $\Sigma_n \times H^n$ with the product

$$(\sigma, h_1, \dots, h_n)(\tau, h'_1, \dots, h'_n) = (\sigma\tau, h_{\tau 1}h'_1, \cdots, h_{\tau n}h'_n).$$

Choose coset representatives $t_1, \dots, t_n$ for H in G and define the "monomial representation"

$$\alpha : G \longrightarrow \Sigma_n \int H$$

by the formula

$$\alpha(\gamma) = (\sigma(\gamma), h_1(\gamma), \dots, h_n(\gamma)),$$

where $\sigma(\gamma)$ and $h_i(\gamma)$ are defined implicitly by the formula

$$\gamma t_i = t_{\sigma(\gamma)(i)} h_i(\gamma).$$

LEMMA 4.1. The map α is a homomorphism of groups. If α' is defined with respect to a second choice of coset representatives $\{t'_i\}$, then α and α' differ by a conjugation in $\Sigma_n \int H$.

The homomorphism α is implicitly central to induction as the following lemma explains. Write $\alpha^* W$ for a representation W of $\Sigma_n \int H$ regarded as a representation of G by pullback along α.

LEMMA 4.2. Let V be a representation of H. Then the sum nV of n copies of V is a representation of $\Sigma_n \int H$ with action given by

$$(\sigma, h_1, \dots, h_n)(v_1, \dots, v_n) = (h_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(n)} v_{\sigma^{-1}(n)}),$$

and $\alpha^*(nV)$ is isomorphic to the induced representation $\mathrm{ind}_H^G V = \mathbb{R}[G] \otimes_{\mathbb{R}[H]} V$.

LEMMA 4.3. If X is a based H-space, then the smash power X^n is a $(\Sigma_n \int H)$-space with action given by

$$(\sigma, h_1, \dots, h_n)(x_1 \wedge \dots \wedge x_n) = h_{\sigma^{-1}(1)} x_{\sigma^{-1}(1)} \wedge \dots \wedge h_{\sigma^{-1}(n)} x_{\sigma^{-1}(n)}.$$

For a based $\Sigma_n \int H$-space Y, such as $Y = X^n$ for a based H-space X, write $\alpha^* Y$ for Y regarded as a G-space by pullback along α. Note in particular that $\alpha^*((S^V)^n) \cong S^{\mathrm{ind}_H^G V}$ for an H-representation V.

To begin the construction of $\overline{norm}_H^G$, one constructs a natural function

$$\mathrm{norm}_H^G : R_0^H(X) \longrightarrow R_0^G(\alpha^* X^n). \tag{4.4}$$

The norm map $\overline{\mathrm{norm}}_H^G$ of Definition 3.1 is then obtained by taking X to be the wedge $S^0 \vee S^r$, studying the decomposition of X^n into wedge summands of G-spaces described in terms of smash powers of spheres and thus of representations, and using Thom isomorphisms to translate the result to integer gradings. We shall say no more about this step here. The properties of $\overline{\mathrm{norm}}_H^G$ are deduced from the following properties of norm_H^G.

$$\mathrm{norm}_G^G \quad \text{is the identity function.} \tag{4.5}$$

$$\mathrm{norm}_H^G(1_H) = 1_G, \quad \text{where } 1_H \in R_0^H(S^0) \text{ is the identity element.} \tag{4.6}$$

$$\mathrm{norm}_H^G(xy) = \mathrm{norm}_H^G(x)\,\mathrm{norm}_H^G(y) \ \text{ if } x \in R_0^H(X) \text{ and } y \in R_0^H(Y). \tag{4.7}$$

Here the product xy on the left is defined by use of the evident map

$$R_0^H(X) \otimes R_0^H(Y) \longrightarrow R_0^H(X \wedge Y) \tag{4.8}$$

and similarly on the right, where we must also use the isomorphism

$$R_0^G(X^n \wedge Y^n) \cong R_0^G((X \wedge Y)^n).$$

The most important property is the double coset formula

$$\mathrm{res}_K^G\ \mathrm{norm}_H^G(x) = \prod_g \mathrm{norm}_{{}^gH\cap K}^K\ \mathrm{res}_{{}^gH\cap K}^{{}^gH}\ c_g(x), \tag{4.9}$$

where K is any subgroup of G and $\{g\}$ runs through a set of double coset representatives for $K\backslash G/H$. Here, if ${}^gH \cap K$ has index $n(g)$ in ${}^gH$, then $n = \sum n(g)$ and the product on the right is defined by use of the evident map

$$\bigotimes_g R_0^K(X^{n(g)}) \longrightarrow R_0^K(X^n). \tag{4.10}$$

An element of $R_0^H(X)$ is represented by an H-map $x : S_G \longrightarrow R_G \wedge X$. There is no difficulty in using the product on R_G to produce an H-map

$$S_G \cong (S_G)^n \xrightarrow{x^n} (R_G \wedge X)^n \cong (R_G)^n \wedge X^n \longrightarrow R_G \wedge X^n. \tag{4.11}$$

The essential point of the construction is to do this in such a way as to produce a G-map: this will be $\mathrm{norm}_H^G(x)$. This is the basic idea, but carrying it out entails several difficulties. Of course, since our group actions involve permutations of smash powers, we must be working in the brave new world of associative and commutative smash products, with an associative and commutative multiplication on R_G. Our first instinct is to interpret the smash powers in (4.11) in terms of $\wedge_S$. Certainly the maps in (4.11) are then both H-maps and Σ_n-maps. However, the H-action on $(R_G)^n$ does not come by pullback along the diagonal of an H^n-action, so that $\Sigma_n \int H$ need not act on $(R_G)^n$. This is only to be expected since $(R_G)^n$ is indexed on the original complete G-universe U on which R_G is indexed, not on a complete $\Sigma_n \int H$-universe. Since our G-actions come by restriction of actions of wreath products $\Sigma_n \int H$, it is essential to bring $(\Sigma_n \int H)$-spectra into the picture. External smash products seem more reasonable than $\wedge_S$ for this purpose since the external smash power $(R_G)^n$ is indexed on the complete $\Sigma_n \int H$-universe U^n.

5. Global $\mathcal{I}_*$-functors with smash product

The solution to the difficulties that we have indicated is to work with a restricted kind of commutative S_G-algebra, namely one that arises from a global $\mathscr{I}_*$-functor with smash product, abbreviated $\mathscr{G}\mathscr{I}_*$-FSP. Unlike general commutative S_G-algebras, these have structure given directly in terms of external smash products, as is needed to make sense of (4.11).

The notion of an $\mathscr{I}_*$-FSP was introduced by May, Quinn, and Ray around 1973, under the ugly name of an $\mathscr{I}_*$-prefunctor. (The name "functor with smash product" was introduced much later by Bökstedt, who rediscovered essentially the same concept.) While $\mathscr{I}_*$-FSP's were originally defined nonequivariantly, the definition transcribes directly to one in which a given compact Lie group G acts on everything in sight. The adjective "global" means that we allow G to range through all compact Lie groups G, functorially with respect to homomorphisms of compact Lie groups. We let $\mathscr{G}$ denote the category of compact Lie groups and their homomorphisms.

Definition 5.1. Define the global category $\mathscr{G}\mathscr{T}$ of equivariant based spaces to have objects (G, X), where G is a compact Lie group and X is a based G-space. The morphisms are the pairs

$$(\alpha, f) : (G, X) \longrightarrow (G', X')$$

where $\alpha : G \longrightarrow G'$ is a homomorphism of Lie groups and $f : X \longrightarrow X'$ is an α-equivariant map, in the sense that $f(gx) = \alpha(g)f(x)$ for all $x \in X$ and $g \in G$. Topologize the set of maps $(G, X) \longrightarrow (G', X')$ as a subspace of the evident product of mapping spaces and observe that composition is continuous.

Definition 5.2. Define the global category $\mathscr{G}\mathscr{I}_*$ of finite dimensional equivariant complex inner product spaces to have objects (G, V), where G is a compact Lie group and V is a finite dimensional inner product space with an action of G through linear isometries. The morphisms are the pairs

$$(\alpha, f) : (G, V) \longrightarrow (G', V')$$

where $\alpha : G \longrightarrow G'$ is a homomorphism and $f : V \longrightarrow V'$ is an α-equivariant linear isomorphism.

The definitions work equally well with real inner product spaces; we restrict attention to complex inner product spaces for convenience in our present application. Each morphism (α, f) in $\mathscr{G}\mathscr{I}_*$ factors as a composite

$$(G, V) \xrightarrow{(\mathrm{id}, f)} (G, W) \xrightarrow{(\alpha, \mathrm{id})} (H, W),$$

where G acts through α on W. We have a similar factorization of morphisms in $\mathscr{G}\mathscr{T}$. We also have forgetful functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}$ and $\mathscr{G}\mathscr{T} \longrightarrow \mathscr{G}$. We shall be interested in functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over $\mathscr{G}$, that is, functors that preserve the group coordinate. For example, one-point compactification of inner product spaces gives such a functor, which we shall denote by $S^\bullet$. As in this example, the space coordinate of our functors will be the identity on morphisms of the form (α, id).

Definition 5.3. A $\mathscr{G}\mathscr{I}_*$-functor is a continuous functor $T : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over $\mathscr{G}$, written (G, TV) on objects (G, V), such that

$$T(\alpha, \mathrm{id}) = (\alpha, \mathrm{id}) : (G, TW) \longrightarrow (H, TW)$$

for a representation W of H and a homomorphism $\alpha : G \longrightarrow H$.

The following observation is the germ of the definition of the norm map.

Lemma 5.4. Let $A = \mathrm{Aut}(G, V)$ be the group of automorphisms of (G, V) in the category $\mathscr{G}\mathscr{I}_*$. For any $\mathscr{G}\mathscr{I}_*$-functor T, the group $A \ltimes G$ acts on the space TV.

Define the direct sum functor $\oplus : \mathscr{G}\mathscr{I}_* \times \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{I}_*$ by

$$(G,V) \oplus (H,W) = (G \times H, V \oplus W).$$

Define the smash product functor $\wedge : \mathscr{G}\mathscr{T} \times \mathscr{G}\mathscr{T} \longrightarrow \mathscr{G}\mathscr{T}$ by

$$(G,X) \wedge (H,Y) = (G \times H, X \wedge Y).$$

These functors lie over the functor $\times : \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$.

DEFINITION 5.5. A $\mathscr{G}\mathscr{I}_*$-FSP is a $\mathscr{G}\mathscr{I}_*$-functor together with a continuous natural unit transformation $\eta : S^\bullet \longrightarrow T$ of functors $\mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ and a continuous natural product transformation $\omega : T \wedge T \longrightarrow T \circ \oplus$ of functors $\mathscr{G}\mathscr{I}_* \times \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ such that the composite

$$TV \cong TV \wedge S^0 \xrightarrow{\mathrm{id} \wedge \eta} TV \wedge T(0) \xrightarrow{\omega} T(V \oplus 0) \cong TV$$

is the identity map and the following unity, associativity, and commutativity diagrams commute:

$$\begin{tikzcd}
S^V \wedge S^W \arrow[r, "\eta\wedge\eta"] \arrow[d, "\cong"'] & TV \wedge TW \arrow[d, "\omega"] \\
S^{V\oplus W} \arrow[r, "\eta"'] & T(V \oplus W),
\end{tikzcd}$$

$$\begin{tikzcd}
TV \wedge TW \wedge TZ \arrow[r, "\omega\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\omega"'] & T(V \oplus W) \wedge TZ \arrow[d, "\omega"] \\
TV \wedge T(W \oplus Z) \arrow[r, "\omega"'] & T(V \oplus W \oplus Z),
\end{tikzcd}$$

and

$$\begin{tikzcd}
TV \wedge TW \arrow[r, "\omega"] \arrow[d, "\tau"'] & T(V \oplus W) \arrow[d, "T(\tau)"] \\
TW \wedge TV \arrow[r, "\omega"'] & T(W \oplus V).
\end{tikzcd}$$

Actually, this is the notion of a commutative $\mathscr{G}\mathscr{I}_*$-FSP; for the more general non-commutative notion, the commutativity diagram must be replaced by a weaker centrality of unit diagram. Observe that the space coordinate of each map $T(\alpha, f)$ is necessarily a homeomorphism since $(\alpha, f) = (\alpha, \mathrm{id}) \circ (\mathrm{id}, f)$ and f is an isomorphism. Spheres and Thom complexes give naturally occurring examples.

EXAMPLE 5.6. The sphere functor $S^\bullet$ is a $\mathscr{G}\mathscr{I}_*$-FSP with unit given by the identity maps of the S^V and product given by the isomorphisms $S^V \wedge S^W \cong S^{V\oplus W}$. For any $\mathscr{G}\mathscr{I}_*$-FSP T, the unit $\eta : S^\bullet \longrightarrow T$ is a map of $\mathscr{G}\mathscr{I}_*$-FSP's.

EXAMPLE 5.7. Let $\dim V = n$ and, as in XV§2, define TV to be the one-point compactification of the canonical n-plane bundle EV over the Grassmann manifold $Gr_n(V \oplus V)$. An action of G on V induces an action of G that makes EV a G-bundle and TV a based G-space. Take $V = V \oplus \{0\}$ as a canonical basepoint in $Gr_n(V \oplus V)$. The inclusion of the fiber over the basepoint induces a map $\eta : S^V \longrightarrow TV$. The canonical bundle map $EV \oplus EW \longrightarrow E(V \oplus W)$ induces a map $\omega : TV \wedge TW \longrightarrow T(V \oplus W)$. With the evident definition of T on morphisms, T is a $\mathscr{G}\mathscr{I}_*$-functor.

It is useful to regard a $\mathscr{G}\mathscr{I}_*$-FSP as a $\mathscr{G}\mathscr{I}_*$-prespectrum with additional structure.

DEFINITION 5.8. A $\mathscr{G}\mathscr{I}_*$-prespectrum is a $\mathscr{G}\mathscr{I}_*$-functor $T : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ together with a continuous natural transformation $\sigma : T \wedge S^\bullet \longrightarrow T \circ \oplus$ of functors $\mathscr{G}\mathscr{I}_* \times \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{T}$ such that the composites

$$TV \cong TV \wedge S^0 \xrightarrow{\sigma} T(V \oplus 0) \cong TV$$

are identity maps and each of the following diagrams commutes:

$$\begin{tikzcd}
TV \wedge S^W \wedge S^Z \arrow[r, "\sigma\wedge\mathrm{id}"] \arrow[d, "\cong"'] & T(V \oplus W) \wedge S^Z \arrow[d, "\sigma"] \\
TV \wedge S^{W\oplus Z} \arrow[r, "\sigma"'] & T(V \oplus W \oplus Z).
\end{tikzcd}$$

LEMMA 5.9. If T is a $\mathscr{G}\mathscr{I}_*$-FSP, then T is a $\mathscr{G}\mathscr{I}_*$-prespectrum with respect to the composite maps

$$\sigma : TV \wedge S^W \xrightarrow{\mathrm{id}\wedge\eta} TV \wedge TW \xrightarrow{\omega} T(V \oplus W).$$

It is evident that a $\mathscr{G}\mathscr{I}_*$-prespectrum restricts to a G-prespectrum indexed on U for every G and U.

NOTATIONS 5.10. Let $T_{G,U}$ denote the G-prespectrum indexed on U associated to a $\mathscr{G}\mathscr{I}_*$-FSP T. Write $R_{G,U}$ for the spectrum $LT_{G,U}$ associated to $T_{G,U}$.

There is a notion of an $\mathscr{L}$-prespectrum, due to May, Quinn, and Ray, and $T_{G,U}$ is an example. The essential point is that if $f : U^j \longrightarrow U$ is a linear isometry and V_i are indexing spaces in U, then we have maps

$$(5.11) \qquad \xi_j(f) : TV_1 \wedge \cdots TV_j \xrightarrow{\omega} T(V_1 \oplus \cdots \oplus V_j) \xrightarrow{Tf} Tf(V_1 \oplus \cdots \oplus V_j).$$

The notion of an $\mathscr{L}$-prespectrum was first defined in terms of just such maps. It was later redefined more conceptually in [LMS] in terms of maps

$$(5.12) \qquad \xi_j : \mathscr{L}(j) \ltimes E^j \longrightarrow E$$

induced by the $\xi_j(f)$. It was shown in the cited sources that the spectrification functor L converts $\mathscr{L}$-prespectra to $\mathscr{L}$-spectra. We conclude that, for every G and every G-universe U, $R_{G,U}$ is an $\mathscr{L}$-spectrum and thus an E_∞ ring G-spectrum when U is complete. Of course, the $\mathscr{L}$-spectrum $R_{G,U}$ determines the weakly equivalent commutative $S_{G,U}$-algebra $S_{G,U} \ltimes_{\mathscr{L}} R_{G,U}$.

M. Bökstedt. Topological Hochschild homology. Preprint, 1990.
J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Volume 577. 1977.

6. The definition of the norm map

We have the following crucial observation about $\mathscr{G}\mathscr{I}_*$-FSP's.

PROPOSITION 6.1. Let T be a $\mathscr{G}\mathscr{I}_*$-FSP. For an H-representation V, $(TV)^n$ and $T(V^n)$ are $\Sigma_n \wr H$-spaces and the map

$$\omega : (TV)^n \longrightarrow T(V^n)$$

is $(\Sigma_n \wr H)$-equivariant. If U is an H-universe, then U^n is a $(\Sigma_n \wr H)$-universe and the maps ω define a map of $(\Sigma_n \wr H)$-prespectra indexed on U^n

$$\omega : (T_{H,U})^n \longrightarrow T_{\Sigma_n \wr H, U^n},$$

where $(T_{H,U})^n$ is the nth external smash power of $T_{H,U}$. If $T = S^\bullet$, then ω is an isomorphism of prespectra.

This allows us to define the norm maps we require. Recall Notations 5.10.

DEFINITION 6.2. Let T be a $\mathscr{G}\mathscr{I}_*$-FSP, let X be a based H-space, and let U be a complete H-universe. An element $x \in R_0^H(X)$ is given by a map of H-spectra $x : S_{H,U} \longrightarrow R_{H,U} \wedge X$. Let G act on U^n through $\alpha : G \longrightarrow \Sigma_n \wr H$, observe that the G-universe U^n is then complete, and define the norm of x to be the element of $R_0^G(\alpha^* X^n)$ given by the composite map of G-spectra indexed on U^n displayed in the commutative diagram:

$$\begin{tikzcd}
S_{G,U^n} \arrow[r, "\omega^{-1}"] \arrow[d, "\mathrm{norm}_H^G(x)"'] & (S_{H,U})^n \arrow[r, "x^n"] & (R_{H,U} \wedge X)^n \arrow[d, "\cong"] \\
R_{G,U^n} \wedge X^n & & (R_{H,U})^n \wedge X^n. \arrow[ll, "\omega \wedge \mathrm{id}"]
\end{tikzcd} \tag{6.3}$$

Strictly speaking, if we start with H-spectra defined in fixed complete H-universes U_H for all H, then we must choose an isomorphism $U_G \cong U_H^n$ to transfer the norm to a map of spectra indexed on U_G, but it is more convenient to derive formulas from the definition as given. From here, all of the properties of the norm except the double coset formula are easy consequences of the definition. The proof of the double coset formula is in principle straightforward diagram chasing from the definitions, but it requires precise combinatorial understanding of double cosets and some fairly elaborate bookkeeping. It is noteworthy that the formula is actually derived from a precise equality of the point set level maps that represent the two sides of the formula.

7. The splitting of MU_G as an algebra

In the context of $\mathscr{G}\mathscr{I}_*$-FSP's, we can complete an unfinished piece of business, namely an indication of the proof that MU_G is split as an algebra in the sense of XXIII.5.8. This was at the heart of our assertion that MU-modules M naturally give rise to split MU_G-modules M_G. In fact, the result we need applies to the S_G-algebra associated to any $\mathscr{G}\mathscr{I}_*$-FSP T, and we adopt Notations 5.10.

We need a preliminary observation. If $f : U \longrightarrow U'$ is a linear isometry, we have maps $Tf : TV \longrightarrow T(fV)$ for indexing spaces $V \subset U$. These specify a map of prespectra $T_{G,U} \longrightarrow f^* T_{G,U'}$ indexed on U and thus, by adjunction, a map $f_* T_{G,U} \longrightarrow T_{G,U'}$ of prespectra indexed on U'. On passage to spectra, these glue together to define a map

$$\xi : \mathscr{I}(U, U') \ltimes R_{G,U} \longrightarrow R_{G,U'}. \tag{7.1}$$

Moreover, this map factors over coequalizers to give a map of $\mathbb{L}'$-spectra

$$\xi : I_U^{U'} R_{G,U} = \mathscr{I}(U, U') \ltimes_{\mathscr{I}(U,U)} R_{G,U} \longrightarrow R_{G,U'}. \tag{7.2}$$

PROPOSITION 7.3. Consider

$$R' = S_{e,U^G} \wedge_{\mathscr{L}} R_{e,U^G} \quad \text{and} \quad R_G = S_{G,U} \wedge_{\mathscr{L}} R_{G,U}$$

(where the subscripts $\mathscr{L}$ refer respectively to U^G and to U) and let $\gamma : R \longrightarrow R'$ be a q-cofibrant approximation of the commutative S-algebra R'. Then the commutative S_G-algebra R_G is split as an algebra with underlying nonequivariant S-algebra R.

PROOF. It suffices to to construct a map $\eta' : I_{U^G}^U R' \longrightarrow R_G$ of S_G-algebras that is a nonequivariant equivalence of spectra, since we can then precompose it with $I_{U^G}^U \gamma$ to obtain a map $\eta : I_{U^G}^U R \longrightarrow R_G$ of S_G-algebras that is a nonequivariant equivalence. In fact, we shall construct a map η' that is actually an isomorphism. Replace U and U' by U^G and U in (7.2). It is not hard to check from the definition of a $\mathscr{G}\mathscr{I}_*$-FSP that

$$R_{e,U^G} = R_{G,U^G} \quad \text{and} \quad R^{\#}_{G,U} = R_{e,U^{\#}}, \tag{7.4}$$

where the superscript $\#$ denotes that we are forgetting actions by G. That is, R_{G,U^G} is R_{e,U^G} regarded as a G-trivial G-spectrum indexed on the G-trivial universe U^G, and $R_{G,U}$ regarded as a nonequivariant spectrum indexed on $U^{\#}$ is $R_{e,U^{\#}}$. The first equality in (7.4) allows us to specialize the map ξ to obtain a map of E_∞ ring spectra

$$\xi : I_{U^G}^U R_{e,U^G} = \mathscr{I}(U^G, U) \ltimes_{I(U^G,U^G)} R_{G,U^G} \longrightarrow R_{G,U}. \tag{7.5}$$

The second equality allows us to identify the target of the underlying map $\xi^{\#}$ of nonequivariant spectra with $R_{e,U^{\#}}$, and it is not hard to check that $\xi^{\#}$ is actually an isomorphism of spectra. We obtain the required map η' on passage to S_G-algebras, using from XXIII.4.5 that we have an isomorphism of S_G-algebras

$$I_{U^G}^U R' \cong S_G \wedge_{\mathscr{L}} I_{U^G}^U R_{e,U^G}. \quad \square$$

J. P. May. Equivariant and nonequivariant module spectra. Preprint, 1995.

8. Löffler's completion conjecture

While computations of MU_*^G are in general out of reach, they are more manageable for compact Abelian Lie groups. Moreover, in this case $MU^*(BG_+)$ is well understood due to work of Landweber and others. Early on in the study of stable complex cobordism, Löffler stated the following assertion as a theorem, although details of proof never appeared.

CONJECTURE 8.1 (LÖFFLER). If G is a compact Abelian Lie group, then

$$(MU_G^*)^\wedge_J \cong MU^*(BG_+).$$

When this holds, it combines with our topological result to force the following algebraic conclusion. A direct proof would be out of reach.

COROLLARY 8.2. If G is a compact Abelian Lie group such that the conjecture holds and I is a sufficiently large ideal in MU_*^G, then

$$H_0^I(MU_G^*) \cong ((MU_G)\hat{}_I)_G^* \cong (MU_G^*)\hat{}_I$$

and

$$H_p^I(MU_G^*) = 0 \quad \text{if} \quad p \neq 0.$$

We do not know whether or not the conjecture holds in general, but it does hold in many cases, as we shall explain in the rest of this section. We also indicate the flaw in the argument sketched by Löffler. We are indebted to Comezaña for details, and our proofs rely on results that he will prove in the next chapter. In particular, the following result is XXVI.5.3; it is stated by Löffler, but no proof appears in the literature.

THEOREM 8.3. For a compact Abelian Lie group G, $MU_*^G$is a free MU^*-module on even degree generators.

Since MU_G is a split G-spectrum, the projection $EG \longrightarrow *$ induces a natural map

$$\alpha : MU_G^*(X) \longrightarrow MU_G^*(EG_+ \wedge X) \cong MU^*(EG_+ \wedge_G X).$$

We shall mainly concern ourselves with the case $X = S^0$ relevant to Conjecture 8.1. We may take EG to be a G-CW complex with finite skeleta, and there results a model for BG as a CW complex with finite skeleta BG^n. We shall need the following result of Landweber.

PROPOSITION 8.4 (LANDWEBER). For a compact Lie group G and a finite G-CW complex X, the natural map $MU^*(EG_+ \wedge_G X) \longrightarrow \lim MU^*(EG_+^n \wedge_G X)$ is an isomorphism.

The vanishing of $\lim^1$ terms here is analogous to part of the Atiyah-Segal completion theorem. In fact, in view of the Conner-Floyd isomorphism

$$K^*(X) \cong MU^*(X) \otimes_{MU^*} K^*$$

for finite X, the result for MU can be deduced from its counterpart for K. Some power J^q of the augmentation ideal of MU_*^G annihilates $MU_G^*(X)$ for any finite free G-CW complex X, by the usual induction on the number of cells, and we conclude that $MU_G^*(EG_+) \cong MU^*(BG_+)$ is J-adically complete. Therefore α gives rise to a natural map

$$MU_G^*(X)^\wedge_J \longrightarrow MU^*(EG_+ \wedge_G X)$$

on finite G-CW complexes X.

A basic tool in the study of this map is the Gysin sequence

(8.5)

$$\cdots \longrightarrow MU_G^{q-2d}(X) \xrightarrow{\chi(V)} MU_G^q(X) \longrightarrow MU_G^q(X \wedge SV_+) \rightarrow MU_G^{q-2d+1}(X) \longrightarrow \cdots,$$

where V is a complex G-module of complex dimension d and we write SV and DV for the unit sphere and unit disc of V. Noting that DV is G-contractible and DV/SV is equivalent to S^V, we can obtain this directly from the long exact sequence of the pair (DV, SV) by use of the Thom isomorphism

$$MU_G^{q-2d}(X) \longrightarrow MU^q(X \wedge S^V).$$

LEMMA 8.6. Conjecture 8.1 holds when $G = S^1$.

PROOF. Let $V = \mathbb{C}$ with the standard action of S^1. Since $SV = S^1$, we have $MU_{S^1}^*(SV_+) \cong MU^*$, which of course is concentrated in even degrees. Therefore the Gysin sequence for V, with $X = S^0$, breaks up into short exact sequences and multiplication by $\chi(V)$ is a monomorphism on $MU_{S^1}^*$. By the multiplicativity of Euler classes, $\chi(nV) = \chi(V)^n$. Thus multiplication by $\chi(nV)$ is also a monomorphism and the Gysin sequence of nV breaks up into short exact sequences

$$0 \longrightarrow MU_{S^1}^{2q-2nd} \xrightarrow{\chi(V)^n} MU_{S^1}^{2q} \longrightarrow MU_{S^1}^{2q}(S(nV)_+) \longrightarrow 0.$$

Since S^1 acts freely on SV, the union $S(\infty V)$ of the $S(nV)$ is a model for ES^1. On passage to limits, there results an isomorphism

$$(MU_{S^1}^*)^\wedge_{(\chi(V))} \cong MU_{S^1}^*(S(\infty V)_+) \cong MU^*(BS^1_+).$$

It is immediate from the Gysin sequence that $J_{S^1} = (\chi(V))$, and the result follows. $\square$

Clearly the proof implies the standard calculation $MU^*(BS^1) \cong MU^*[[c]]$, where $c \in MU^2(BS^1)$ is the image of the Euler class.

The steps of the argument generalize to give the following two results.

LEMMA 8.7. For any compact Abelian Lie group G,

$$(MU^*_{G\times S^1})^\wedge_{(\chi(V))} \cong MU^*_G(BS^1_+) \cong MU^*_G[[c]].$$

PROOF. Here we regard $V = \mathbb{C}$ as a representation of $G \times S^1$, with G acting trivially, and we note that $S(V) \cong (G \times S^1)/G$, so that $MU^*_{G\times S^1}(S(V)_+) \cong MU^*_G$. The rest of the proof is as in Lemma 8.6. $\square$

LEMMA 8.8. Let $T = T^r$ be a torus, let $V_q = \mathbb{C}$ with T acting through its projection to the qth factor, and let $\chi_q = \chi(V_q)$. Then $J_T = (\chi_1, \cdots, \chi_r)$.

PROOF. Clearly J_T annihilates $MU^*_T(S(V_1)_+ \wedge \cdots \wedge S(V_r)_+) \cong MU^*$. By an easy inductive use of Gysin sequences, we find that, for $1 \leq q \leq r$,

$$MU^*_T(S(V_1)_+ \wedge \cdots \wedge S(V_q)_+) \cong MU^*_T/(\chi_1, \cdots, \chi_q)MU^*_T.$$

The rest of the proof is as in Lemma 8.6. $\square$

We put the previous two lemmas together to obtain Conjecture 8.1 for tori.

PROPOSITION 8.9. Conjecture 8.1 holds when G is a torus.

PROOF. Write $G = T \times S^1$ and assume inductively that the conclusion holds for T. Letting c_q be the image of χ_q, we find that

$$\begin{aligned}(MU^*_G)^\wedge_{J_G} \cong (MU^*_G)^{\wedge\,\wedge}_{J_T\,J_{S^1}} &\cong (MU^*_G)^{\wedge\,\wedge}_{J_{S^1}\,J_T} \cong (MU^*_T[[c_r]])^\wedge_{J_T} \\ &\cong (MU^*_T)^\wedge_{J_T}[[c_r]] \cong MU^*[[c_1, \cdots, c_r]] \cong MU^*(BG_+),\end{aligned}$$

the first equality being an evident identification of a double limit with a single one. $\square$

We would like to deduce the general case of Conjecture 8.1 from the case of a torus. Thus, for the rest of the section, we consider a group $G = C_1 \times \cdots \times C_r$, where each C_q is either S^1 or a subgroup of S^1. This fixes an embedding of G in the torus $T = T^r$, and of course every compact Abelian Lie group can be written in this form. We have the following pair of lemmas, the first of which follows from the known calculation of $MU^*(BG_+)$; see for example the second paper of Landweber below.

LEMMA 8.10. The restriction map $MU^*(BT_+) \longrightarrow MU^*(BG_+)$ is an epimorphism. In particular, $MU^*(BG_+)$ is concentrated in even degrees.

LEMMA 8.11. The restriction map $MU^*_T \longrightarrow MU^*_G$ is an epimorphism. In particular, J_T maps epimorphically onto J_G and the completion of an MU^G_*-module at J_G is isomorphic to its completion at J_T.

PROOF. It suffices to prove that each restriction map

$$MU^*_{T^q\times C_{q+1}\times\cdots\times C_r} \longrightarrow MU^*_{T^{q-1}\times C_q\times C_{q+1}\times\cdots\times C_r}$$

is an epimorphism. Let C_q be cyclic of order $k(q)$. Let $V_q = \mathbb{C}$ regarded as a T-module with all factors of S^1 acting trivially except the qth, which acts via its $k(q)$th power map. Restricting V_q to a representation of $T^q \times C_{q+1} \times \cdots \times C_r$, we see that its unit sphere can be identified with the quotient group

$$(T^q \times C_{q+1} \times \cdots \times C_r)/(T^{q-1} \times C_q \times C_{q+1} \times \cdots \times C_r).$$

With $X = S^0$ and $G = T^q \times C_{q+1} \times \cdots \times C_r$, the Gysin sequence of $\chi(V_q)$ breaks up into short exact sequences that give the conclusion. $\square$

Now consider the following commutative diagram:

$$\begin{tikzcd} (MU^*_T)^\wedge_{J_T} \arrow[r] \arrow[d] & MU^*(BT_+) \arrow[d] \\ (MU^*_G)^\wedge_{J_G} \arrow[r] & MU^*(BG_+). \end{tikzcd} \tag{8.12}$$

The top horizontal arrow is an isomorphism and both vertical arrows are epimorphisms. Thus Conjecture 8.1 will hold if the following conjecture holds.

CONJECTURE 8.13. The map $(MU^*_G)^\wedge_{J_G} \longrightarrow MU^*(BG)$ is a monomorphism.

LEMMA 8.14. Conjecture 8.1 holds if G is a finite cyclic group.

PROOF. We embed G in S^1 and consider the standard representation $V = \mathbb{C}$ of S^1 as a representation of G. Again, $S(\infty V)$ is a model for EG. With $X = S^0$, the Gysin sequence (8.5) breaks up into four term exact sequences. Here we cannot conclude that multiplication by $\chi(V)$ is a monomorphism: its kernel is the image in MU_G^* of the odd degree elements of $MU_G^*(S(V)_+)$. However, in even degrees, the Gysin sequences of the representations nV give isomorphisms

$$MU_G^*/\chi(V)^n MU_G^* \cong MU_G^{2*}(S(nV)_+).$$

Therefore $(MU_G^*)^\wedge_{(\chi(V))}$ maps isomorphically onto $MU^{2*}(BG_+)$. This proves Conjecture 8.13; indeed, since $MU^*(BG_+)$ is concentrated in even degrees, it proves Conjecture 8.1 directly. $\square$

Löffler asserts without proof that the general case of Conjecture 8.13 follows by the methods above. However, although $MU^*(BG_+)$ is concentrated in even degrees, the intended inductive proof may founder over the presence of odd degree elements in Gysin sequences, and we do not know whether or not the conjecture is true in general.

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CHAPTER XXVI

Some calculations in complex equivariant bordism

by G. Comezaña

In this chapter we shall explain some basic results about the homology and cohomology theories represented by the spectrum MU_G. These theories arise from stabilized bordism groups of G-manifolds carrying a certain "complex structure"; exactly what this means is something we feel is not adequately discussed in the literature. Since the chapter includes a substantial amount of well-known information, as well as some new material and proofs of results claimed without proof elsewhere, we make no claims to originality except where noted. The author would like to thank Steven Costenoble for discussions and insights that have thrown a great deal of light on the subject matter.

1. Notations and terminology

G will stand throughout for a compact (and, in most cases, Abelian) Lie group, and subgroups of a such a group will be assumed to be closed. All manifolds considered will be compact and smooth, and all group actions smooth. If (X, A) and (Y, B) are pairs of G-spaces, we will use the notation $(X, A) \times (Y, B)$ for the pair $(X \times Y, (X \times B) \cup (Y \times A))$. Homology and cohomology theories on G-spaces will be reduced.

G-vector bundles over a G-space will be assumed to carry an inner product (which will be hermitian if the bundle is complex). Unless explicit mention to the contrary is made, representations will be understood to be finite-dimensional and $\mathbb{R}$-linear. Depending on the context, we shall sometimes think of V as a G-vector bundle over a point. If ξ is a G-bundle, $|\xi|$ will stand for its real dimension, $S(\xi)$

for its unit sphere, $D(\xi)$ for its unit disk, and $T(\xi)$ for its Thom space. If V is a representation of G, S^V will denote its one-point compactification. The trivial G-vector bundle over a G-space X with fiber V will be denoted ε_V.

We define the V-suspension $\Sigma^V X$ of a based G-space X to be $X \wedge S^V$; thus if ε_V is the trivial G-vector bundle over X with fiber V, then $T(\varepsilon_V) = \Sigma^V X$. We define the V-suspension $\Sigma^V(X, A)$ of a pair of spaces to be $(X, A) \times (DV, SV)$. In both cases, Σ^V is a functor; if V is a subrepresentation of W with orthogonal complement $W - V$, the inclusion induces a natural transformation $\sigma^{W-V} : \Sigma^V \longrightarrow \Sigma^W$.

2. Stably almost complex structures and bordism

When G is the trivial group, a *stably almost complex structure* on a compact smooth manifold M is an element $[\xi] \in \tilde{K}(M)$, which goes to the class $[\nu M]$ of the stable normal bundle under the map

$$\tilde{K}(M) \longrightarrow \widetilde{KO}(M).$$

It is, of course, essentially equivalent to define this with $[\tau M]$ replacing $[\nu M]$, since these classes are additive inverses in $\widetilde{KO}(M)$.

The following definition gives the obvious equivariant generalization of this.

DEFINITION 2.1. If $[\xi] \in \tilde{K}_G(M)$ is a lift of $[\nu M] \in \widetilde{KO}_G(M)$ under the natural map, we call the pair $(M, [\xi])$ a normally almost complex G-manifold.

We will use the notation $M_{[\xi]}$ when necessary, but we will drop $[\xi]$ whenever there is no risk of confusion.

The bordism theory of these objects, denoted mu^G_*, is the "complex analog" of the unoriented theory mo^G_* discussed in Chapter XV. If V is a *complex* G-module and $(M, \partial M)_{[\xi]}$ is a G-manifold with a stably almost complex structure, then its V-suspension becomes a G-manifold after "straightening the angles", and $[\xi] - [\varepsilon_V]$ is a complex structure on $\Sigma^V(M, \partial M)$. This gives rise to a *suspension homomorphism*

$$\sigma^V : mu^G_*(X, A) \longrightarrow mu^G_{*+|V|}(\Sigma^V(X, A)),$$

which sends the class of a map $(M, \partial M) \longrightarrow (X, A)$ to the class of its suspension. Due to the failure of G-transversality, both the suspension homomorphisms and the Pontrjagin-Thom map are generally not bijective.

We construct a stabilized version of this theory as follows. Let $\mathscr{U}$ be an infinite-dimensional complex G-module equipped with a hermitian inner product whose underlying $\mathbb{R}$-linear structure is that of a complete G-universe. Define

$$MU_*^G(X,A) = \operatorname{colim}_V mu_*^G(\Sigma^V(X,A)),$$

where V ranges over all finite-dimensional complex G-subspaces of $\mathscr{U}$ and the colimit is taken over all suspension maps induced by inclusions. We should perhaps point out that MU_*^G is not a connective theory unless G is trivial. The advantage of this new theory over mu_*^G is that the bad behavior of the Pontrjagin-Thom map is corrected, and the maps induced by suspension by complex G-modules are isomorphisms by construction. This should be interpreted as a form of periodicity. Homology or cohomology theories with this property are often referred to in the literature as *complex-stable.* Other examples of such theories include equivariant complex K-theory, its associated Borel construction, etc. Complex-stability isomorphisms should not be confused with suspension isomorphisms of the form

$$\Sigma^V : h_*^G(X,A) \longrightarrow h_{*+[V]}^G(\Sigma^V(X,A)),$$

which are part of the structure of all $RO(G)$-graded homology theories.

MU_*^G or, more precisely, its dual cohomology theory was first constructed by tom Dieck in terms of a G-prespectrum TU_G, bearing the same relationship to complex Grassmanians as the G-prespectrum TO_G discussed in XV§2, does to real ones. An argument of Bröcker and Hook for unoriented bordism readily adapts to the complex case to show the equivalence of the two approaches. In what follows, we shall focus on the spectrification MU_G of TU_G. As with any representable equivariant homology theory, MU_*^G can be extended to an $RO(G)$-graded homology theory, but we shall concern ourselves only with integer gradings. We point out, however, that complex-stable theories are always $RO(G)$-gradable.

A key feature of MU_G, proven in XXV§7, is the fact that it is a split G-spectrum; this may be seen geometrically as a consequence of the fact that the augmentation map $MU_*^G \longrightarrow MU_*$, given on representatives by neglect of structure, can be split by regarding non-equivariant stably almost complex manifolds as G-manifolds with trivial action. The splitting makes $MU_*^G = MU_*^G(S^0)$ a module over the ring MU_*.

The multiplicative structure of the ring G-spectrum MU_G can be interpreted geometrically as coming from the fact that the class of normally stably almost complex manifolds is closed under finite products. The complex-stability isomorphisms are well-behaved with respect to the multiplicative structure: in cohomology, we have a commutative diagram

$$\begin{tikzcd}
MU_G^*(X) \otimes MU_G^*(Y) \arrow[r] \arrow[d, "\sigma^V \otimes \sigma^W"'] & MU_G^*(X \wedge Y) \arrow[d, "\sigma^{V \oplus W}"] \\
MU_G^{*+|V|}(\Sigma^V X) \otimes MU_G^{*+|W|}(\Sigma^W Y) \arrow[r] & MU_G^{*+|V|+|W|}(\Sigma^{V \oplus W} X \wedge Y)
\end{tikzcd}$$

for all based G-spaces X and Y and complex G-modules V and W. In general, for a multiplicative cohomology theory, commutativity of a diagram of the form above is assumed as part of the definition of complex-stability. K_G^* is another example of a multiplicative complex-stable cohomology theory, as is the Borel construction on any such theory.

The role of MU_G in the equivariant world is analogous to that of MU in classical homotopy theory, for its associated cohomology theory has a privileged position among those which are multiplicative, complex-stable, and have natural Thom classes (for complex G-vector bundles). We record the axiomatic definition of such theories.

Definition 2.2. A G-equivariant multiplicative cohomology theory h_G^* is said to have natural Thom classes for complex G-vector bundles if for every such bundle ξ of complex dimension n over a pointed G-space X there exists a class $\tau_\xi \in h_G^{2n}(T(\xi))$, with the following three properties:

(1) *Naturality:* If $f : Y \longrightarrow X$ is a pointed G-map, then $\tau_{f^*\xi} = f^*(\tau_\xi)$.

(2) *Multiplicativity:* If ξ and η are complex G-vector bundles over X, then

$$\tau_{\xi\oplus\eta} = \tau_\xi \times \tau_\eta \in h_G^{|\xi|+|\eta|}(T(\xi \oplus \eta)).$$

(3) *Normalization:* If V is a complex G-module, then $\tau_V = \sigma^V(1)$.

The following result, which admits a quite formal proof (given for example by Okonek) explains the universal role played by MU_G.

Proposition 2.3. If h_G^* is a multiplicative, complex-stable, cohomology theory with natural Thom classes for complex G-bundles, then there is a unique natural transformation $MU_G^*(\bullet) \longrightarrow h_G^*(\bullet)$ of multiplicative cohomology theories that takes Thom classes to Thom classes.

Returning to homology, for a complex G-bundle ξ of complex dimension k, the Thom class of ξ gives rise to a Thom isomorphism

$$\tau : MU_*^G(T(\xi)) \longrightarrow MU_{*-2k}^G(B(\xi)_+),$$

and similarly in cohomology. This isomorphism is constructed in the same way as in the nonequivariant case (see e.g. [LMS]), without using any feature of MU_*^G other than the existence and formal properties of Thom classes. However, in this special case, its *inverse* has a rather pleasant geometric interpretation: if $f : M \longrightarrow B(\xi)$ represents an element in $mu_n^G(B(\xi)_+)$, the map $\overline{f}$ in the pullback diagram

$$\begin{tikzcd} E(f^*\xi) \arrow[r, "\overline{f}"] \arrow[d, "f^*\xi"'] & E(\xi) \arrow[d, "\xi"'] \\ M \arrow[r, "f"'] & B(\xi) \end{tikzcd}$$

represents an element in $mu_{n+2k}^G(T(\xi))$. This procedure allows the construction of a homomorphism which stabilizes to the inverse of the Thom isomorphism. See Bröcker and Hook for the details of a treatment of the Thom isomorphism (in the unoriented case) that uses this interpretation.

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3. Tangential structures

Unfortunately, both mu_*^G and MU_*^G are rather intractable from the computational point of view. In order to address this difficulty, we shall introduce a new bordism theory, much more amenable to calculation, whose stabilization is also MU_G^*.

Consider the following variant of reduced K-theory: if X is a G-space, instead of taking the quotient by the subgroup generated by all trivial complex G-bundles, take the quotient by the subgroup generated by those trivial bundles of the form $\mathbb{C}^n \times X$, where G acts trivially on $\mathbb{C}^n$. We denote the group so obtained as $\check{K}_G$; there is an analogous construction in the real case, which we denote $\check{KO}_G$.

Definition 3.1. A tangentially stably almost complex manifold is a smooth manifold equipped with a lift of the class $[\tau M] \in \check{KO}_G(M)$ to $\check{K}_G(M)$.

We shall refer to the bordism theory of these manifolds as *tangential complex bordism*, denoted $\Omega_*^{U,G}$.

We warn the reader that nowhere in the literature is the distinction between the complex bordism theories $\Omega_*^{U,G}$ and mu_*^G made clear. This is not mere pedantry on our part, as our next result will show. It was pointed out to the author by Costenoble that this result does not hold for normally stably almost complex G-manifolds.

PROPOSITION 3.2. If M is a tangentially stably almost complex G-manifold and $H \subseteq G$ is a closed normal subgroup, then the G-tubular neighborhood around M^H has a complex structure.

We stress the fact that no stabilization is necessary to get a complex structure on the tubular neighborhood; this lies at the heart of the calculations we shall carry out later in the chapter.

PROOF. The first thing to observe is that $\tau(M^H) = (\tau M|_{M^H})^H$ as real vector bundles. If ξ is the restriction to M^H of a complex G-vector bundle over M that represents its tangential stably almost complex structure, and the underlying real G-vector bundle of ξ is $\tau M|_{M^H} \oplus \varepsilon_{\mathbb{R}^n}$, then $(\xi^H)^-$ is a complex G-vector bundle. We have

$$\xi = \xi^H \oplus (\xi^H)^- = (\tau M|_{M^H})^H \oplus \varepsilon_{\mathbb{R}^n} \oplus \nu(M^H, M).$$

This gives the desired structure. $\square$

We next explore the relation between mu_*^G and $\Omega_*^{U,G}$. There is a commutative square

$$\begin{tikzcd} \check{K}_G(X) \arrow[r] \arrow[d] & \check{KO}_G(X) \arrow[d] \\ \widetilde{K}_G(X) \arrow[r] & \widetilde{KO}_G(X) \end{tikzcd}$$

that yields a natural transformation of homology theories $\phi : mu_*^G \longrightarrow \Omega_*^{U,G}$. Just as we did with mu_*^G, we may stabilize $\Omega_*^{U,G}$ with respect to suspensions by finite-dimensional complex subrepresentations of a complete G-universe, obtaining a new complex-stable homology theory which we shall provisionally denote $\check{MU}_*^G$. The map ϕ stabilizes to a natural transformation $\Phi : \check{MU}_*^G \longrightarrow MU_*^G$. The following result was first proved by the author and Costenoble by a different argument and is central to the results of this chapter.

THEOREM 3.3. Φ is an isomorphism of homology theories.

We shall need the following standard result.

LEMMA 3.4. *(Change of groups isomorphism)* If $H \subseteq G$ is a closed subgroup of codimension j, then for all H-spaces X there is an isomorphism

$$mu^H_*(X_+) \xrightarrow{\cong} mu^G_{*+j}((G \times_H X)_+)$$

induced by application of the functor $G \times_H (\bullet))$ to representatives of bordism classes of maps, and similarly for pairs. The analogous result holds for $\Omega^{U,G}_*$ and MU^G_*.

SKETCH PROOF. If we apply the functor $G \times_H (\bullet)$ to a map $f : M \longrightarrow X$ that represents an element of $mu^H_n(X_+)$, we obtain an element of $mu^G_{n+j}((G \times_H X)_+)$. Conversely, if $g : N \longrightarrow G \times_H X$ represents an element of $mu^G_{n+j}((G \times_H X)_+)$ and if $\pi : G \times_H X \longrightarrow X$ is the evident H-map, we set $M = (\pi g)^{-1}(eH)$ and see that M is an H-manifold such that $N = G \times_H M$ and the restriction of g to M represents an element of $mu^H_n(X_+)$. $\square$

PROOF OF THEOREM 3.3. We show first that the theorem is true for $G = SU(2k+1)$ and then extend the result to the general case by a change of groups argument.

We recall a few standard facts about representations of special unitary groups (e.g., from Bröcker and tom Dieck). Let M be the complex $SU(2k+1)$-module such that $M = \mathbb{C}^{2k+1}$ with the action of $SU(2k+1)$ given by matrix multiplication and let $\Lambda^i = \Lambda^i M$. Then $R(SU(2k+1))$ is the polynomial algebra over $\mathbb{Z}$ on the representations Λ^i, $1 \leq i \leq 2k$, all of which are irreducible and of complex type. Furthermore, $\Lambda^{2k-i+1} = \overline{\Lambda^i}$. This implies that any irreducible real representation of $SU(2k+1)$ is either trivial or admits a complex structure. To see this, let W be a non-trivial irreducible real $SU(2k+1)$-module. Suppose first that $W \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible. Since the restriction to $\mathbb{R}$ of an irreducible complex representation of quaternionic type is irreducible, our assumptions imply that $W \otimes_{\mathbb{R}} \mathbb{C}$ is of real type and of the form $V \otimes_{\mathbb{C}} \overline{V}$, where V is a monomial in the Λ^i, $1 \leq i \leq k$. We have

$$(\overline{V} \otimes_{\mathbb{C}} V) \otimes_{\mathbb{R}} \mathbb{C} \cong (2W) \otimes_{\mathbb{R}} \mathbb{C} \cong 2(V \otimes_{\mathbb{C}} \overline{V})$$

as complex representations. On the other hand, since $2W \cong V \otimes_{\mathbb{C}} \overline{V}$, we have isomorphisms of complex $SU(2k+1)$-modules

$$(2W) \otimes_{\mathbb{R}} \mathbb{C} \cong (V \otimes_{\mathbb{C}} \overline{V}) \otimes_{\mathbb{R}} \mathbb{C} \cong V \otimes_{\mathbb{C}} (\overline{V} \otimes_{\mathbb{R}} \mathbb{C})$$

and

$$V \otimes_{\mathbb{C}} (\overline{V} \otimes_{\mathbb{R}} \mathbb{C}) \cong (V \otimes_{\mathbb{C}} V) \oplus (V \otimes_{\mathbb{C}} \overline{V})$$

(because $\overline{V} \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus \overline{V}$). So it follows that

$$2(V \otimes_{\mathbb{C}} \overline{V}) \cong (V \otimes_{\mathbb{C}} V) \oplus (V \otimes_{\mathbb{C}} \overline{V}),$$

which is absurd in view of the structure of $RSU(2k+1)$. Thus W must be reducible and so it is either of the form $V_1 \oplus V_1$, for an irreducible complex V_1 of quaternionic type, or $V_1 \oplus \overline{V_1}$, for an irreducible complex V_1 of complex type. The first possibility is ruled out by the fact that all self-conjugate irreducible complex representations of $SU(2k+1)$ are of real type. So we must have

$$2W \cong V_1 \oplus \overline{V}_1 \cong 2V$$

as real representations, and therefore, using the uniqueness of isotypical decompositions, we may conclude that $W \cong V$ as real representations.

Now let X be a $SU(2k+1)$-space and consider a map representing an element in $MU_*^G(X)$. By complex-stability, there is no loss of generality in assuming that our map is of the form $f : M \longrightarrow X$, where $\tau M \oplus \epsilon_V \cong \xi$, V is a real representation, and ξ is a complex $SU(2k+1)$-vector bundle. By the remark above, $V = W \oplus \mathbb{R}^k$ for a complex representation W. Then $\Sigma^W(M, \partial M)$ is a tangentially stably almost complex manifold and the class of $\Sigma^W f$ is in the image of ϕ. It follows that Φ is surjective. A similar argument applied to bordisms shows that Φ is injective.

To obtain the general case, observe that any compact Lie group embeds in $U(2k)$, and $U(2k)$ embeds in $SU(2k+1)$ (via the map that sends $A \in U(2k)$ to $(det\ A)^{-1} \cdot 1_{\mathbb{R}} \oplus A$), and apply Lemma 3.4. $\square$

T. Bröcker and T. tom Dieck. representations of compact Lie groups. Springer. 1985.
C. Okonek. Der Conner-Floyd-Isomorphismus für Abelsche Gruppen. Mathematische Zeitschrift 179(1982), pp. 201–212.

4. Calculational tools

For the remainder of the chapter, all Lie groups we consider will be Abelian.

There is a long list of names associated to the calculation of $\Omega_*^{U,G}(S^0)$ for different classes of compact Lie groups: Landweber (cyclic groups), Stong (Abelian p-groups), Ossa (finite Abelian groups), Löffler (Abelian groups), Lazarov (groups of order pq for distinct primes p and q), and Rowlett (extensions of a cyclic group by a cyclic group of relatively prime order). All of these authors rely on the study of fixed point sets by various subgroups, together with their normal bundles, through the use of bordism theories with suitable restrictions on isotropy subgroups.

The main calculational tool is the use of families of subgroups, which works in exactly the same fashion as was discussed in the real case in XV§3. Recall that, for a family $\mathscr{F}$, an $\mathscr{F}$-space is a G-space all of whose isotropy subgroups are in $\mathscr{F}$ and that we write $E\mathscr{F}$ for the universal $\mathscr{F}$-space. Recall too that, for a G-homology theory h_*^G and a pair of families $(\mathscr{F}, \mathscr{F}')$, $\mathscr{F}' \subseteq \mathscr{F}$, there is an associated homology theory $h_*^G[\mathscr{F}, \mathscr{F}']$, defined on pairs of G-spaces as

$$h_*^G[\mathscr{F}, \mathscr{F}'](X, A) = h_*^G(X \times E\mathscr{F}, (X \times \mathscr{F}') \cup (A \times E\mathscr{F})).$$

When $\mathscr{F}' = \emptyset$, we use the notation $h_*^G[\mathscr{F}]$. The theories $h_*^G[\mathscr{F}]$, $h_*^G[\mathscr{F}']$, and $h_*^G[\mathscr{F}, \mathscr{F}']$ fit into a long exact sequence. Of course, there is an analogous construction in cohomology.

In the special case of $\Omega_*^{U,G}$ (and similarly for other bordism theories), it is easy to see that $\Omega_*^{U,G}[\mathscr{F}, \mathscr{F}']$ has an alternative interpretation: it is the bordism theory of $(\mathscr{F}, \mathscr{F}')$- tangentially almost-complex manifolds, that is, compact, tangentially almost complex $\mathscr{F}$-manifolds with boundary, whose boundary is an $\mathscr{F}'$-manifold.

DEFINITION 4.1. A pair of families $(\mathscr{F}, \mathscr{F}')$ of subgroups of G is called a neighboring pair differing by H if there is a subgroup H such that if $K \in \mathscr{F} - \mathscr{F}'$, then H is a subconjugate of K.

This notion was first used by Löffler, but the terminology is not standard. A special case is the more usual notion of an *adjacent* pair of families pair differing by H, which is a neighboring pair $(\mathscr{F}, \mathscr{F}')$ such that $\mathscr{F} - \mathscr{F}'$ consists of those subgroups conjugate to H.

The next proposition explains the importance of neighboring families. We introduce some terminology and notation to facilitate its discussion.

Given a subgroup H of an Abelian Lie group G, we choose a set $\mathscr{C}_{G,H}$ of finite dimensional complex G-modules whose restrictions to H form a non-redundant, complete set of irreducible, nontrivial complex H-modules. If $\mathbb{C}$ denotes the trivial irreducible representation, we let $\mathscr{C}_{G,H}^+ = \mathscr{C}_{G,H} \cup \{\mathbb{C}\}$. For a nonnegative even integer k, we shall call an array of nonnegative integers $P = (p_V)_{V \in \mathscr{C}_{G,H}}$ a (G, H)-*partition* of k if

$$k = \sum_{V \in \mathscr{C}_{G,H}} 2p_V.$$

For such a partition P, we let

$$BU(P,G) = \prod_{V \in \mathscr{C}_{G,H}} BU(p_V, G).$$

We let $\mathscr{P}(k,G,H)$ denote the set of all (G,H)-partitions of k.

Proposition 4.2. If $(\mathscr{F}, \mathscr{F}')$ is a neighboring pair of families of subgroups of a compact Abelian Lie group G differing by a subgroup H, then

$$\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X,A) \cong \bigoplus_{\substack{0 \le 2k \le n \\ P \in \mathscr{P}(2k,G,H)}} \Omega_{n-2k}^{U,G/H}[\mathscr{F}/H]((X^H, A^H) \times BU(P, G/H)),$$

where $\mathscr{F}/H$ denotes the family of subgroups of G/H that is obtained by taking the quotient of each element of $\mathscr{F} - \mathscr{F}'$ by H.

Sketch of proof. For simplicity, we concentrate on the absolute case. Let $f : M \longrightarrow X$ represent an element in $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X_+)$ and let T be a (closed) G-tubular neighborhood of M^H. We may view T as the total space of the unit disc bundle of the normal bundle to M^H. We may also view T as an n-dimensional $\mathscr{F}$-manifold whose boundary is an $\mathscr{F}'$-manifold. Thus T represents an element of $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](S^0)$, and we see that $[f] = [f|_T]$ in $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X_+)$. Furthermore, $[f] = 0$ if and only if there is an H-trivial G-nullbordism of $f|_T$, equipped with a complex G-vector bundle whose unit disc bundle restricts to T on M^H. Observe that M^H breaks up into various components of constant even codimension. In other words, $\Omega_n^{U,G}[\mathscr{F},\mathscr{F}'](X_+)$ can be identified with the direct sum, with $2k$ ranging between 0 and n, of bordism of H-trivial $\mathscr{F}$-manifolds of dimension $n-2k$ equipped with a complex G-vector bundle of dimension k, containing no H-trivial summands. Note the twofold importance of Proposition 3.2: not only are we using that M^H is tangentially almost complex, but also that its tubular neighborhood carries a complex structure.

Consider the bundle-theoretic analog of the isotypical decomposition of a linear representation. For complex G-vector bundles E and F over a space X we may construct the vector bundle $\mathrm{Hom}_{\mathbb{C}}(E,F)$ whose fiber over $x \in X$ is $\mathrm{Hom}_{\mathbb{C}}(E_x, F_x)$; G acts on $\mathrm{Hom}_{\mathbb{C}}(E,F)$ by conjugation. If X is H-trivial, then $\mathrm{Hom}_H(E,F) = (\mathrm{Hom}_{\mathbb{C}}(E,F))^H$ is an H-trivial G-subbundle; if one regards X as a (G/H)-space, then $\mathrm{Hom}_H(E,F)$ becomes a (G/H)-vector bundle over X.

We apply this to $F = T$ and $E = \varepsilon_V$, where V is a complex G-module whose restriction to H is irreducible, thus obtaining a (G/H)-vector bundle which we call the V-*multiplicity* of E. The evaluation map

$$\bigoplus_{V \in \mathscr{C}^+_{G,H}} \operatorname{Hom}_H(\varepsilon_V, T) \otimes_{\mathbb{C}} \varepsilon_V \longrightarrow T$$

is a G-vector bundle isomorphism, and this decomposition into isotypical summands is unique. Note that in the special case we are considering, the multiplicity associated to the trivial representation is 0, so the sum really does run over $\mathscr{C}_{G,H}$.

T can therefore be identified with a direct sum of (G/H)-vector bundles over M^H, each corresponding to an irreducible complex representation of H, and M^H breaks into a disjoint union of components on which the dimension of each multiplicity remains constant; each of these components has therefore an associated (G,H)-partition, accounting for the summation over $\mathscr{P}(2k, G, H)$ in our formula. Clearly the bundle on the component associated to a (G,H)-partition P is classified by $BU(P, G/H)$. $\square$

Similar methods allow us to prove the following standard result.

PROPOSITION 4.3. With the notation above, if H is a subgroup of an Abelian Lie group G, then

$$BU(n,G)^H \cong \coprod_{P \in \mathscr{P}(n,G,H)} \prod_{V \in \mathscr{C}^+_{G,H}} BU(p_V, G/H)$$

as H-trivial G-spaces.

PROOF. It suffices to observe that the right hand side classifies n-dimensional complex G-vector bundles over H-trivial G-spaces. $\square$

P. S. Landweber. Unitary bordism of cyclic group actions. Proceedings of the Amer. Math. Soc. 31(1972), pp. 564–570.

C. Lazarov. Actions of groups of order pq. Transactions of the Amer. Math. Soc. 173(1972), pp. 215–230.

P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. Manuscripta Mathematica 12(1974), 307–327.

E. Ossa, Unitary bordism of Abelian groups. Proceedings of the American Mathematical Society 33(1972), pp. 568–571.

R.J. Rowlett. Bordism of metacyclic group actions. Michigan Mathematical Journal 27(1980), pp. 223–233.

R. Stong. Complex and oriented equivariant bordism. in Topology of Manifolds (J.C. Cantrell and C.H. Edwards, editors). Markham, Chicago 1970.

5. Statements of the main results

We come now to a series of theorems, some old, some new, that are consequences of the previous results. In all of them, we consider a given compact Abelian Lie group G.

THEOREM 5.1 (LÖFFLER). If V is a complex G-module, and X is a disjoint union of pairs of G-spaces of the form

$$(DV, SV) \times \prod_{i=1}^{k} BU(n_i, G),$$

then $\Omega_*^{U,G}(X)$ is a free MU_*-module concentrated in even degrees.

THEOREM 5.2. With the same hypotheses on X, the map

$$\Omega_*^{U,G}(BU(n,G) \times X) \longrightarrow \Omega_*^{U,G}(BU(n+1,G) \times X)$$

induced by Whitney sum with the trivial bundle $\varepsilon_{\mathbb{C}}$ is a split monomorphism of MU_*-modules.

THEOREM 5.3. MU_*^G is a free MU_*-module concentrated in even degrees.

THEOREM 5.4. The stabilization map $\Omega_*^{U,G} \longrightarrow MU_*^G$ is a split monomorphism of MU_*-modules.

Theorem 5.3 is stated in the second paper of Löffler cited below, but there seems to be no proof in the literature. Ours is a refinement of the ideas in the proof of Theorem 5.1, which yields Theorem 5.4 as a by-product, and is entirely self-contained (that is, it does not depend on results on finite Abelian groups). Tom Dieck has used a completely different method to prove a weaker version of Theorem 5.4, for G cyclic of prime order, but to the best of our knowledge nothing of the sort has previously been claimed or proved at our level of generality. Theorem 5.2, which also seems to be new, is required in the course of the proof of Theorem 5.3 and is of independent interest.

In the light of these results, it is natural to conjecture, probably overoptimistically, that MU_*^G is free over MU_* and concentrated in even degrees for any compact Lie group G. We have succeeded in verifying this for a class of non-Abelian groups that includes $O(2)$ and the dihedral groups. The statement about the injectivity of the stabilization map also holds for these groups. We hope to extend these results to other classes of non-Abelian groups; details will appear elsewhere.

The results above should be proven in the given order, but, since the proofs have a large overlap, we shall deal with all of them simultaneously.

We shall proceed by induction on the number of "cyclic factors" of the group, where, for the purposes of this discussion, S^1 counts as a cyclic group. The argument in each case is as follows: the result is either trivial or well-known for the trivial group. Then, one shows that if the result is true for a compact Lie group G, it also holds for $G \times S^1$, and this in turn implies the same for $G \times \mathbb{Z}_n$.

T. tom Dieck. Bordism of G-manifolds and integrality theorems. Topology 9(1970), pp. 345–358.
P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. Manuscripta Mathematica 12(1974), 307–327.
P. Löffler. Equivariant unitary bordism and classifying spaces. Proceedings of the International Symposium on Topology and its Applications, Budva, Yugoslavia 1973, pp. 158–160.

6. Preliminary lemmas and families in $G \times S^1$

For brevity, the subgroups $\{1\} \times S^1 \subseteq G \times S^1$ and $\{1\} \times \mathbb{Z}_n \subseteq G \times \mathbb{Z}_n$ will be denoted S^1 and $\mathbb{Z}_n$, respectively.

We shall need to consider the following families of subgroups of $G \times S^1$:

$$\begin{aligned} \mathscr{F}_i &= \{H \subseteq G \times S^1 \mid |H \cap S^1| \leq i\} \\ \mathscr{F}_\infty &= \{H \subseteq G \times S^1 \mid H \cap S^1 \neq S^1\} \\ \mathscr{A} &= \{\text{all closed subgroups of } G \times S^1\} \end{aligned}$$

These give rise to the neighboring pairs $(\mathscr{F}_{i+1}, \mathscr{F}_i)$ (differing by $\mathbb{Z}_{i+1}$) and $(\mathscr{A}, \mathscr{F}_\infty)$ (differing by S^1). Observe that $\mathscr{F}_\infty$ is the union of its subfamilies $\mathscr{F}_i$.

Lemma 6.1. Let G be a compact Lie group and X be a pair of $(G \times S^1)$-spaces. Then

$$\Omega_*^{U,G\times S^1}(X \times S^1) \cong \Omega_{*-1}^{U,G}(X)$$

and

$$\Omega_*^{U,G\times S^1}((X \times S^1)/\mathbb{Z}_n) \cong \Omega_{*-1}^{U,G\times \mathbb{Z}_n}(X),$$

where $G \times S^1$ acts on S^1 and $S^1/\mathbb{Z}_n$ through the projection $G \times S^1 \longrightarrow S^1$; the same statement holds for the theories $mu_*^{G\times S^1}$ and $MU_*^{G\times S^1}$.

The proofs of these isomorphisms are easy verifications and will be omitted; see Löffler. We shall also need the following result of Conner and Smith.

Lemma 6.2. A graded, projective, bounded below MU_*-module is free.

Lemma 6.3. Consider a diagram of projective modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f_1"'] & B \arrow[r] \arrow[d, "f_2"'] & C \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0.
\end{tikzcd}$$

If f_1 and f_3 (resp. f_2 and f_3) are split monomorphisms, so is f_2 (resp. f_1).

Proof. Add a third row consisting of the cokernels of the f_i, which will be exact by the Snake Lemma. An easy diagram chase shows that the modules in the new row are projective, and therefore the conclusion follows. $\square$

Note that we make no assumptions about compatibility of the splittings.

Remark 6.4. If X is a pair of G-spaces of the kind appearing in the statement of Theorem 5.1 and H is a subgroup of G, then restricting the action to H yields an H-pair of the same kind. Moreover, by Proposition 4.3, X^H is a (G/H)-pair of the same type. This class of pairs of spaces is also closed under cartesian product with $BU(n,G)$ and with pairs of the form (DW, SW) for a complex G-module W.

P. E. Conner, L. Smith, *On the complex bordism of finite complexes*, Publications Mathématiques de l'IHES, no. 37 (1969), pp. 417–521.
P. Löffler. Bordismengruppen unitärer Torusmannigfaltigkeiten. Manuscripta Mathematica 12(1974), 307–327.

7. On the families $\mathscr{F}_i$ in $G \times S^1$

In what follows, for a G-pair X and a homology theory h_*, ψ will designate a map of the form

$$\psi : h_*(BU(n,G) \times X) \longrightarrow h_*(BU(n+1,G) \times X)$$

that is induced by taking the Whitney sum of the universal complex G-bundle over $BU(n,G)$ and the trivial G-bundle $\varepsilon_{\mathbb{C}}$.

Suppose that all four theorems stated above have been proved for G. We shall deduce the following result in the case $G \times S^1$.

Theorem 7.1. The following statements hold for each $i \geq 1$ and for $i = \infty$.

(1) $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$ is a free MU_*-module concentrated in *odd* degrees.
(2) The map

$$\psi : \Omega_*^{U,G\times S^1}[\mathscr{F}_i](BU(n,G\times S^1)\times X) \longrightarrow \Omega_*^{U,G\times S^1}[\mathscr{F}_i](BU(n+1,G\times S^1)\times X)$$

is a split monomorphism of MU_*-modules.

(3) If W is an irreducible complex $(G \times S^1)$-module, then

$$\sigma^W : \Omega_*^{U,G\times S^1}[\mathscr{F}_i](X) \longrightarrow \Omega_{*+2}^{U,G\times S^1}[\mathscr{F}_i]((DW, SW) \times X)$$

is a split monomorphism of MU_*-modules.

(4) The map $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X) \longrightarrow \Omega_*^{U,G\times S^1}(X)$ is zero.

PROOF. We first prove this for $i = 1$, making use of a suitable model for the space $E\mathscr{F}_1$. Let $(W_i)_{i\geq 1}$ be a sequence of irreducible complex $(G \times S^1)$-modules such that S^1 acts freely on their unit circles, and every isomorphism class of such $(G \times S^1)$-modules appears infinitely many times. Let $V_k = \bigoplus_{i=1}^k W_i$ and

$$SV_\infty = \mathrm{colim}_k SV_k;$$

SV_∞ is the required space. Note also that this space embeds into the equivariantly contractible space

$$DV_\infty = \mathrm{colim}_k DV_k.$$

Using Lemma 6.1 and our assumptions about G, we see that $\Omega_*^{U,G\times S^1}(SV_1 \times X)$ is a free MU_*-module concentrated in odd degrees, and that

$$\sigma^W : \Omega_*^{U,G\times S^1}(SV_1 \times X) \longrightarrow \Omega_*^{U,G\times S^1}((DW, SW) \times SV_1 \times X)$$

and

$$\Omega_*^{U,G\times S^1}(SV_1 \times BU(n, G \times S^1) \times X) \longrightarrow \Omega_*^{U,G\times S^1}(SV_1 \times BU(n+1, G \times S^1) \times X)$$

are split monomorphisms of MU_*-modules.

We calculate $\Omega_*^{U,G\times S^1}((SV_{k+1}, SV_k) \times X)$ using the homotopy equivalence

$$(SV_{k+1}, SV_k) \simeq (SW_{k+1} * SV_k, DW_{k+1} \times SV_k),$$

and the excisive inclusion

$$SW_{k+1} \times (DV_k, SV_k) \longrightarrow (SW_{k+1} * SV_k, DW_{k+1} \times SV_k).$$

The action of $G \times S^1$ on SW_{k+1} determines and is determined by a split group epimorphism $G \times S^1 \longrightarrow S^1$ with kernel $H \subseteq G \times S^1$, $H \cong G$. This implies that SW_{k+1} is $(G \times S^1)$-homeomorphic to $(G \times S^1)/H$. By a change of groups argument and the inductive hypothesis, we see that $\Omega_*^{G,U}((SV_{k+1}, SV_k) \times X)$ is free and concentrated in odd degrees and that the maps induced respectively by suspension by an irreducible complex G-module and by addition of the bundle $\varepsilon_{\mathbb{C}}$ are split monomorphisms of MU_*-modules.

The diagram with exact columns (in which j is odd)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega_j^{U,G\times S^1}(SV_k \times X) \arrow[r, "\sigma^W"] \arrow[d] & \Omega_{j+2}^{U,G\times S^1}((DW,SW)\times SV_k \times X) \arrow[d] \\
\Omega_j^{U,G\times S^1}(SV_{k+1} \times X) \arrow[r, "\sigma^W"] \arrow[d] & \Omega_{j+2}^{U,G\times S^1}((DW,SW)\times SV_{k+1} \times X) \arrow[d] \\
\Omega_j^{U,G\times S^1}((SV_{k+1}, SV_k) \times X) \arrow[r, "\sigma^W"] \arrow[d] & \Omega_{j+2}^{U,G\times S^1}((DW,SW)\times (SV_{k+1}, SV_k) \times X) \arrow[d] \\
0 & 0
\end{tikzcd}$$

and the results above show by induction that, for all $k \geq 1$, $\Omega_*^{U,G\times S^1}(SV_k \times X)$ is free and concentrated in odd degrees and that σ^W is a split monomorphism. An analogous diagram shows the same is true for the map ψ induced by adding $\varepsilon_{\mathbb{C}}$.

To complete the proofs of (1) – (3) when $i = 1$, it suffices to observe that each step in the colimit contributes a direct summand to SV_∞. To prove (4), let $f : M \longrightarrow X \times SV_\infty$ represent an element of $\Omega_*^{U,G\times S^1}[\mathscr{F}_1](X)$. Since S^1 acts freely on M and all actions on a circle are linear, $p : M \longrightarrow M/S^1$ is the unit circle bundle of a 1-dimensional complex G-bundle E (the complex structure is given by multiplication by $i \in S^1$). Obviously, the circle bundle bounds a disc bundle, whose total space is a complex $(G \times S^1)$-manifold W. Any point $x \in W$ can be written as ty, where $t \in [0,1]$ and $y \in M$, so f extends to an equivariant map $F : W \longrightarrow X \times DV_\infty$ defined as $F(ty) = tf(y)$, where the multiplication on the right hand side is given by the linear structure of DV_∞.

We prove the case $i \geq 1$ of Theorem 7.1 by induction on i. Observe first that the case $i = \infty$ will follow directly from the case of finite i since

$$E\mathscr{F}_\infty = \mathrm{colim}_i E\mathscr{F}_i.$$

Indeed, we shall see that each stage in the construction of $E\mathscr{F}_\infty$ as a colimit contributes a free direct summand to $\Omega_*^{U,G\times S^1}[\mathscr{F}_\infty](X)$ on which σ^W and ψ are split monomorphisms of MU_*-modules and the map to $\Omega_*^{U,G\times S^1}(X)$ is zero.

Applying Proposition 4.2 with (G, H) replaced by $(G \times S^1, \mathbb{Z}_{i+1})$ and noting that $(G \times S^1)/\mathbb{Z}_{i+1} \cong G \times S^1$ and that, under this isomorphism, the family $\mathscr{F}_{i+1}/\mathbb{Z}_{i+1}$ corresponds to the family $\mathscr{F}_1$, we find that

$$\Omega_n^{U,G\times S^1}[\mathscr{F}_{i+1}, \mathscr{F}_i](X) \cong \bigoplus_{\substack{0\le 2k\le n \\ P\in\mathscr{P}(2k, G\times S^1, \mathbb{Z}_{n+1})}} \Omega_{n-2k}^{U,G\times S^1}[\mathscr{F}_1](X^{\mathbb{Z}_{n+1}} \times BU(P, G \times S^1)).$$

Thus the case $i = 1$, combined with Remark 6.4, shows that the left-hand side is free and concentrated in *odd* degrees.

One then concludes, by using the long exact sequences of the pairs $[\mathscr{F}_{i+1}, \mathscr{F}_i]$, that for all i, $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$ is concentrated in *odd* degrees.

The diagrams with exact columns (in which j is odd)

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega_j^{U,G}[\mathscr{F}_i](BU(n, G\times S^1)\times X) \arrow[r] \arrow[d] & \Omega_j^{U,G}[\mathscr{F}_i](BU(n+1, G\times S^1)\times X) \arrow[d] \\
\Omega_j^{U,G}[\mathscr{F}_{i+1}](BU(n, G\times S^1)\times X) \arrow[r] \arrow[d] & \Omega_j^{U,G}[\mathscr{F}_{i+1}](BU(n+1, G\times S^1)\times X) \arrow[d] \\
\Omega_j^{U,G}[\mathscr{F}_{i+1}, \mathscr{F}_i](BU(n, G\times S^1)\times X) \arrow[r] \arrow[d] & \Omega_j^{U,G}[\mathscr{F}_{i+1}, \mathscr{F}_i](BU(n+1, G\times S^1)\times X) \arrow[d] \\
0 & 0
\end{tikzcd}$$

show that, for all i, $\Omega_*^{U,G}[\mathscr{F}_i](X)$ is a free MU_*-module and the map induced by addition of $\varepsilon_{\mathbb{C}}$ is a split monomorphism of MU_*-modules.

The study of the suspension map σ^W must be broken into two cases. Since W is an irreducible representation of $G \times S^1$, its fixed point space W^{S^1} is either W or $\{0\}$ and therefore either

(1) $W^{\mathbb{Z}_{i+1}} = W$ or
(2) $W^{\mathbb{Z}_{i+1}} = \{0\}$.

In the first case, the map

$$(7.2) \quad \sigma^W : \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_{i+1}, \mathscr{F}_i](X) \longrightarrow \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_{i+1}, \mathscr{F}_i]((DW, SW) \times X),$$

can be regarded via Proposition 4.2 as a direct sum of suspension maps

$$\Omega_{2l+1}^{U,G\times S^1}[\mathscr{F}_1](Y) \longrightarrow \Omega_{2l+3}^{U,G\times S^1}[\mathscr{F}_1]((DW,SW)\times Y),$$

where $Y = X^{\mathbb{Z}_{i+1}} \times BU(P, G\times S^1)$ for some partition P of $2(j-l)$ and we think of W as a representation of $G\times(S^1/\mathbb{Z}_{i+1}) \cong G\times S^1$. Thus it follows from the case $i=1$ that (7.2) is a split monomorphism of MU_*-modules in this case.

For the second case consider a $(G\times S^1, \mathbb{Z}_{i+1})$-partition $P = (p_V)_{V\in\mathscr{C}_{G\times S^1,\mathbb{Z}_{i+1}}}$ of an even integer k. Let $P' = (p'_V)_{V\in\mathscr{C}_{G\times S^1}}$ denote the $(G\times S^1,\mathbb{Z}_{i+1})$-partition of $k+2$ defined by

$$p'_V = \begin{cases} p_V + 1 & \text{if } V = W \\ p_V & \text{otherwise.} \end{cases}$$

Since $W^{\mathbb{Z}_{i+1}} = \{0\}$, Proposition 4.2 implies that the map (7.2) can be interpreted as a direct sum of maps of the form

$$\psi : \Omega_{2l+1}^{U,G\times S^1}[\mathscr{F}_1](X^{\mathbb{Z}_{i+1}}\times BU(P,G)) \longrightarrow \Omega_{2l+3}^{U,G\times S^1}[\mathscr{F}_1](X^{\mathbb{Z}_{i+1}}\times BU(P',G))$$

induced by addition of $\varepsilon_{\mathbb{C}}$ to the multiplicity bundle corresponding to the V in the decomposition. We know already that maps of this kind are split monomorphisms of MU_*-modules, and we conclude that (7.2) is always a split monomorphism of MU_*-modules.

Now the following diagram with exact columns implies inductively that, for all i, σ^W is a split monomorphism of MU_*-modules on $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_i](X) \arrow[r, "\sigma^W"] \arrow[d] & \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_i]((DW,SW)\times X) \arrow[d] \\
\Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_{i+1}](X) \arrow[r, "\sigma^W"] \arrow[d] & \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_{i+1}]((DW,SW)\times X) \arrow[d] \\
\Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_{i+1},\mathscr{F}_i](X) \arrow[r, "\sigma^W"] \arrow[d] & \Omega_{2j+3}^{U,G\times S^1}[\mathscr{F}_{i+1},\mathscr{F}_i]((DW,SW)\times X) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Finally, to prove (4) of Theorem 7.1, let $f : M \longrightarrow X$ represent an element of $\Omega_*^{U,G\times S^1}[\mathscr{F}_i](X)$, $i > 1$, and suppose that we have already proved that

$$\Omega_*^{U,G\times S^1}[\mathscr{F}_j](X) \longrightarrow \Omega_*^{U,G\times S^1}(X)$$

is zero for all $j < i$. We shall construct a bordism with no isotropy restrictions from f to a map $f' : M' \longrightarrow X$ where M' is an $\mathscr{F}_{i-1}$-manifold. By the induction hypothesis, this will complete the proof.

Let us pause for a moment to explain informally how the bordism will be constructed. The idea is based on a standard technique in geometric topology known as "attaching handles". Any sphere S^k is the boundary of a disc D^{k+1}; if $S^k \subset N^n$ is embedded with trivial normal bundle in a manifold N and has a tubular neighborhood T, we can obtain a bordism of N to a new manifold by crossing N with the unit interval and pasting $D^{k+1} \times D^{n-k-1}$ (a *handle* with *core* D^k) to $N \times I$ by identifying $T \times \{1\}$ with $S^k \times D^{n-k-1}$. Our construction will be basically "attaching a generalized handle" to our manifold M. Instead of an embedded sphere, we shall use $M^{\mathbb{Z}_i}$, which bounds a manifold W; this will be the "core" of our "handle". The "handle" itself will be the total space of a disc bundle over W. The total space of its restriction to $M^{\mathbb{Z}_i}$ will be equivariantly diffeomorphic to a tubular neighborhood of $M^{\mathbb{Z}_i}$ in M, so we may take $M \times I$ and glue the "handle" in the obvious way, thus obtaining the desired bordism. Of course, all the required properties of the bordism have to be checked, and an extension of f to the bordism has to be constructed. We give the details next.

Consider a tubular neighborhood T of $M^{\mathbb{Z}_i}$, regarded as the total space of a disc bundle over $M^{\mathbb{Z}_i}$. We shall use the notation ST for the corresponding unit circle bundle, and T° for $T - ST$. We remark that $M - T^\circ$ and ST are $\mathscr{F}_{i-1}$-manifolds. When there is no danger of confusion, we shall make no notational distinction between a bundle and its total space.

Let λ denote a generator of $\mathbb{Z}_i \subset S^1 \subset \mathbb{C}$, and let V_k, $0 < k < i$, be 1-dimensional representations of $\mathbb{Z}_i$ such that λ acts by multiplication by λ^k. These form a complete, non-redundant set of nontrivial irreducible representations, and each of the V_k's obviously extends to $G \times S^1$ (an element $(g, s) \in G \times S^1$ acts by multiplication by s^k). We use these to obtain an isotypical decomposition of T. Let T_k denote the bundle $\mathrm{Hom}_{\mathbb{Z}_i}(\varepsilon_{V_k}, T)$.

Since $M^{\mathbb{Z}_i}$ is $(S^1/\mathbb{Z}_i)$-free, our proof in the case $i = 1$ shows that $f|_{M^{\mathbb{Z}_i}}$ bounds a map $\tilde{f} : W \longrightarrow X$, where W is the total space of a $\mathbb{Z}_i$-trivial 1-dimensional

$(G \times S^1)$-disc bundle over $Z = M^{\mathbb{Z}_i}/(S^1/\mathbb{Z}_i)$ whose unit circle bundle is $M^{\mathbb{Z}_i}$.

Passage to orbits gives a pull-back diagram

$$\begin{tikzcd} T_k \arrow[r] \arrow[d] & T_k/(S^1/\mathbb{Z}_i) \arrow[d] \\ N \arrow[r] & N/(S^1/\mathbb{Z}_i), \end{tikzcd}$$

for each k, where the right vertical arrow is a G-disc bundle, which may also be thought of as a $(G \times (S^1/\mathbb{Z}_i))$-bundle with trivial $(S^1/\mathbb{Z}_i)$-action. This makes the diagram above a pull-back of $(G \times (S^1/\mathbb{Z}_i))$-vector bundles. Since the zero-section of this bundle can be identified with $Z = SW/(S^1/\mathbb{Z}_i)$, we have a diagram of $(G \times (S^1/\mathbb{Z}_i))$-bundles

$$\begin{tikzcd} T_k \arrow[rr] \arrow[dr] \arrow[dd] & & T_k/(S^1/\mathbb{Z}_i) \arrow[dd] \\ & p^*(T_k/(S^1/\mathbb{Z}_i)) \arrow[ur] \arrow[dd] & \\ N \arrow[rr] \arrow[dr, "i"'] & & N/(S^1/\mathbb{Z}_i). \\ & W \arrow[ur, "p"'] & \end{tikzcd}$$

Clearly the bundle $\hat{T} = \bigoplus_k p^*(T_k/(S^1/\mathbb{Z}_i)) \otimes \varepsilon_{V_k}$ extends T to W; we claim that its unit sphere bundle is an $\mathscr{F}_{i-1}$-manifold. To prove this, observe that

$$W - Z \cong M^{\mathbb{Z}_i} \times [0, 1),$$

where $[0, 1)$ has trivial action, and so $S\hat{T}|_{W-Z}$ is equivariantly homeomorphic to $S\hat{T}|_{W-Z} \times [0, 1)$. Therefore, S^1-stabilizers of points in $S\hat{T} - ST$ not already present in ST can only appear in $S\hat{T}|_Z$, but since there is no component associated to the trivial representation (recall our remark in the course of the proof of Proposition 4.2) all these are *proper* subgroups of $\mathbb{Z}_i$, so the claim follows.

Let

$$M' \cong (M - T^\circ) \cup_{ST} S\hat{T};$$

by construction, this is an $\mathscr{F}_{i-1}$-manifold. Since $T \cup W$ is a $(G \times S^1)$-deformation retract of $\hat{T}$, there is a map $\hat{f} : W \longrightarrow X$ with $\hat{f}|_T = f|_T$ and $\hat{f}|_W = \tilde{f}$. We obtain a bordism by crossing M with the closed unit interval, pasting $\hat{T}$ to $M \times \{1\}$ along

$T \times \{1\}$, and extending f in the obvious way to a map F from the bordism into X. The maps $f' = F|_{M'}$ and f represent the same element in the bordism of X with no isotropy restrictions, as required. $\square$

8. Passing from G to $G \times S^1$ and $G \times \mathbb{Z}_k$

To complete the proofs of our theorems, it suffices to prove the following result, in which we again assume that we have proven all of our theorems for G.

THEOREM 8.1. Let $C = S^1$ or $C = \mathbb{Z}_k$. The following statements hold.

(1) $\Omega_*^{U,G\times C}(X)$ is a free MU_*-module concentrated in *even* degrees.

(2) The map

$$\psi : \Omega_*^{U,G\times C}(BU(n, G \times S^1) \times X) \longrightarrow \Omega_*^{U,G\times C}(BU(n+1, G \times S^1) \times X)$$

is a split monomorphism of MU_*-modules.

(3) If W is an irreducible complex $(G \times C)$-module, then

$$\sigma^W : \Omega_*^{U,G\times C}(X) \longrightarrow \Omega_{*+2}^{U,G\times C}((DW, SW) \times X)$$

is a split monomorphism of MU_*-modules.

We first show that $\Omega_*^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$ is a free MU_*-module concentrated in even degrees and that σ^W and ψ here are split monomorphisms of MU_*-modules. By Proposition 4.2, we have

$$\Omega_n^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X) \cong \bigoplus_{\substack{0 \le 2k \le n \\ P \in \mathscr{P}(2k, G\times S^1, S^1)}} \Omega_{n-2k}^{U,G}(X^{S^1} \times BU(P, G)).$$

Thus, by the induction hypothesis, $\Omega_n^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$ is free over MU_* and concentrated in even degrees, and the maps ψ induced by addition of $\varepsilon_{\mathbb{C}}$ are split monomorphisms of MU_*-modules.

Theorem 7.1(4) implies that the long exact sequence of the pair $(\mathscr{A}, \mathscr{F}_\infty)$ breaks into short exact sequences. In particular, the map

$$\Omega_*^{U,G\times S^1}(X) \longrightarrow \Omega_*^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$$

is a monomorphism, hence $\Omega_*^{U,G\times S^1}(X)$ is concentrated in even degrees.

In order to study the effect of σ^W on $\Omega_n^{U,G\times S^1}[\mathscr{A}, \mathscr{F}_\infty](X)$, it is necessary to distinguish two cases:

(1) $W^{S^1} = W$ and

(2) $W^{S^1} = \{0\}$.

The analysis is similar to the one carried out in the previous section and will be omitted; it yields the expected conclusion: σ^W is a split monomorphism of MU_*-modules on $\Omega_n^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X)$.

The diagram with exact columns

$$\begin{CD}
0 @. 0\\
@VVV @VVV\\
\Omega_{2j}^{U,G\times S^1}(X) @>{\sigma^W}>> \Omega_{2j+2}^{U,G\times S^1}((DW,SW)\times X)\\
@VVV @VVV\\
\Omega_{2j}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X) @>{\sigma^W}>> \Omega_{2j+2}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty]((DW,SW)\times X)\\
@VVV @VVV\\
\Omega_{2j-1}^{U,G\times S^1}[\mathscr{F}_\infty](X) @>{\sigma^W}>> \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_\infty]((DW,SW)\times X)\\
@VVV @VVV\\
0 @. 0
\end{CD}$$

together with Lemmas 6.2 and 6.3 shows that $\Omega_*^{U,G\times S^1}(X)$ is projective, and therefore free, and that σ^W is a split monomorphism of MU_*-modules on $\Omega_*^{U,G\times S^1}(X)$. A similar diagram gives the corresponding conclusion for ψ.

This completes the proof of Theorem 8.1 for $C = S^1$, and it remains to deal with the case $C = \mathbb{Z}_k$. Let V denote the 1-dimensional complex representation of $G\times S^1$ on which G acts trivially and an element $e^{2\pi i t}\in S^1$ acts by multiplication by $e^{2\pi i t k}$. Since S^1 acts without fixed points on $SV\times X$, $\Omega_*^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](SV\times X) = 0$. Therefore, by the long exact sequence of the pair (DV,SV),

$$\Omega_*^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X) \longrightarrow \Omega_*^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty]((DV,SV)\times X)$$

is an isomorphism, and, by the long exact sequence of the pair $(\mathscr{A},\mathscr{F}_\infty)$,

$$\Omega_*^{U,G\times S^1}[\mathscr{F}_\infty](SV\times X) \longrightarrow \Omega_*^{U,G\times S^1}(SV\times X)$$

is an isomorphism.

By Theorem 7.1, we conclude that $\Omega_*^{U,G\times S^1}(SV\times X)$ is a free MU_*-module concentrated in odd degrees. This being so, the long exact sequence of the pair (DV,SV) breaks up into short exact sequences

$$0\longrightarrow \Omega_{2j}^{U,G\times S^1}(X) \xrightarrow{\alpha} \Omega_{2j}^{U,G\times S^1}((DV,SV)\times X) \longrightarrow \Omega_{2j-1}^{U,G\times S^1}(SV\times X)\longrightarrow 0.$$

Since SV can be identified with $S^1/\mathbb{Z}_k$, we conclude from Lemma 3.4 that

$$\Omega_*^{U,G\times\mathbb{Z}_k}(X) \cong \text{coker } \alpha.$$

Now apply the Snake Lemma to the diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\Omega_{2j}^{U,G\times S^1}(X) \arrow[rr, "\alpha"] \arrow[d] & & \Omega_{2j}^{U,G\times S^1}((DV,SV)\times X) \arrow[d] \\
\Omega_{2j}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty](X) \arrow[rr, "\cong"] \arrow[d] & & \Omega_{2j}^{U,G\times S^1}[\mathscr{A},\mathscr{F}_\infty]((DV,SV)\times X) \arrow[d] \\
\Omega_{2j-1}^{U,G\times S^1}[\mathscr{F}_\infty](X) \arrow[rr, two heads, "\beta"] \arrow[d] & & \Omega_{2j-1}^{U,G\times S^1}[\mathscr{F}_\infty]((DV,SV)\times X) \arrow[d] \\
0 & & 0.
\end{tikzcd}$$

Since α is a monomorphism and β is an epimorphism, we see that coker $\alpha \cong$ ker β. Since ker β is a free MU_*-module concentrated in odd degrees, $\Omega_*^{U,G\times\mathbb{Z}_k}(X)$ is free and concentrated in even degrees.

To show that σ^W is a split monomorphism, let $Y = (DW, SW)\times X$ and consider the maps

$$\alpha' : \Omega_{2j+2}^{U,G\times S^1}(Y) \longrightarrow \Omega_{2j+2}^{U,G\times S^1}((DV,SV)\times Y)$$

and

$$\beta' : \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_\infty](Y) \longrightarrow \Omega_{2j+1}^{U,G\times S^1}[\mathscr{F}_\infty]((DV,SV)\times Y)$$

that fit into the diagram obtained from the previous one by raising all degrees by two and replacing X by Y. Then σ^W induces a map from the original diagram to the new diagram, and there results a commutative square

$$\begin{tikzcd}
\text{coker } \alpha \arrow[r, "\sigma^W"] \arrow[d, "\cong"'] & \text{coker } \alpha' \arrow[d, "\cong"] \\
\text{ker } \beta \arrow[r, "\sigma^W"'] & \text{ker } \beta'.
\end{tikzcd}$$

By Lemma 6.2, the bottom arrow is a split monomorphism of MU_*-modules, hence so is the top arrow. The proof that ψ is a split monomorphism is similar.

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