Bredon Equivariant Homology of Representation Spheres

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Abstract

In this paper we compute the Bredon equivariant homology of representation spheres corresponding to the orientable three dimensional representations of cyclic groups and dihedral groups, as well as the symmetric group on three letters equipped with a permutation representation. These computations are greatly simplified by the introduction of a splitting theorem for the Burnside ring Mackey functor A.

1 Introduction

In this paper we present computations of the Bredon equivariant homology of representation spheres. We first introduce the basic equivariant notions of G-CW complexes and we give a definition of representation spheres. Because we will be computing the Bredon equivariant homology with coefficients in a specific Mackey functor, we also give a brief introduction to Mackey functors and coefficient systems. To fully understand the definition of the Bredon equivariant homology it is necessary to study the tensor product of Mackey functors, so this precedes the discussion of homolgy. Once these topics are introduced we discuss a splitting theorem for the Burnside ring Mackey functor. This decomposition, while interesting in its own right, makes the homology computations that follow significantly easier. Finally we will compute the Bredon equivariant homology of three representation spheres corresponding to the representations of cyclic groups, dihedral groups, and the permutation representation on the symmetric group on three letters.

Briefly, the homotopy theory of representation spheres has appeared in recent work on the Kervaire invariant problem [1], and these objects are generally of use in topics that employ $RO(G)$ grading.

2 G-CW Complexes and Representation Spheres

These are some of beginnings of equivariant algebraic topology, from chapter 1 of [2].

Let X, Y, Z be compactly generated spaces and let G, H be topological groups. We equip our spaces with a continuous G action and call these *G-spaces*. A map $f : X \to Y$ is *equivariant* if $f(g \cdot x) = g \cdot f(x)$ and is referred to as a *G-map*. We denote the category of G-spaces (over compactly generated spaces) and G-maps by $G\mathcal{U}$ (as opposed to $\mathcal{U}$ for the category of compactly generated spaces). Much of the theory developed for the category of compactly generated spaces works equally well in $G\mathcal{U}$.

Remark 1. *G acts diagonally on Cartesian products of spaces and if $f \in \mathrm{Map}(X, Y)$ then $g \cdot f$ is given by $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$.*

Lemma 2. *Just as the category of compactly generated spaces is Cartesian closed, $G\mathcal{U}$ is cartesian closed. That is, $\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$. The bijection above is not only natural, but it is a G-homeomorphism.*

We assume that subgroups of G are closed.

Definition 3. *For $H \subset G$, $X^H = \{x | hx = x \text{ for } h \in H\}$.*

Definition 4. *For $x \in X$, $G_x = \{h | hx = x\}$ is the isotropy group of x.*

Remark 5. *A homotopy between G-maps is defined in the usual way with the added constraint that the homotopy map is a G map. We can construct a homotopy category $hG\mathcal{U}$.*

A G-CW complex is essentially a CW complex with an action of G on the cells. Formally, a G-CW complex X is the union of G-spaces X^n formed inductively by attaching G-cells.

Definition 6. *A G-cell is the product of a cell D^n and an orbit: $G/H \times D^n$.*

In the case of G-CW complexes, to create X^{n+1} we attach a G-cell $G/H \times D^{n+1}$ to a G-space X^n via a G-map $G/H \times S^n \to X^n$. These attaching maps are determined by the restriction $S^n \to (X^n)^H$. We see this using the adjunctions from before:

$$G\mathcal{U}(G \times_H S^n, X^n) \cong H\mathcal{U}(S^n, X^n) \cong G\mathcal{U}(S^n, \mathrm{Map}_H(G, X^n)) \cong G\mathcal{U}(S^n, \mathrm{Map}(G/H, X^n)).$$

$\mathrm{Map}(G/H, X^n)$ contains G-maps, so if $f \in \mathrm{Map}(G/H, X^n)$ then $f(gH) = g \cdot f(H)$ so f is determined by where it sends the identity coset H. However, H can only be sent to an element of $(X^n)^H$, as $f(H) = h \cdot f(H)$. Therefore $\mathrm{Map}(G/H, X^n) = (X^n)^H$ as desired. In fact one can check that if a cell in a G-CW complex is preserved by an element of g, it must be fixed pointwise.

Representation Spheres

An orthogonal n-dimensional real representation of a group G preserves the unit sphere S^{n-1}. If we restrict the representation to S^{n-1} we have a G-space which we can write as a G-CW complex.

3 Mackey Functors and Coefficient Systems

Mackey Functors

We begin with definitions of the Burnside category and Mackey functors. For a finite group G we shall denote the Burnside category as $\mathcal{B}_G$. Before defining $\mathcal{B}_G$ it is convenient to begin with an auxiliary category $\mathcal{B}'_G$. Given a finite group G, $\mathcal{B}'_G$ is the category whose objects are finite G-sets and morphisms from T to S are isomorphism classes of diagrams

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] & \\ T & & S \end{tikzcd}$$

We define composition of morphisms to be the pullback of two diagrams:

$$\begin{tikzcd} & U_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ T & & V \end{tikzcd} \circ \begin{tikzcd} & U_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\ S & & T \end{tikzcd} = \begin{tikzcd} & & U_1 \times_T U_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\ & U_1 \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & U_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ S & & T & & V \end{tikzcd}$$

If we define addition of morphisms as

$$\begin{tikzcd} & U_1 \arrow[dl] \arrow[dr] & \\ T & & S \end{tikzcd} + \begin{tikzcd} & U_2 \arrow[dl] \arrow[dr] & \\ T & & S \end{tikzcd} = \begin{tikzcd} & U_1 \coprod U_2 \arrow[dl] \arrow[dr] & \\ T & & S \end{tikzcd}$$

it follows that $Hom(S,T)$ is a commutative monoid. This construction of $\mathcal{B}'_G$ allows us to define $\mathcal{B}_G$:

Definition 7. *Given a finite group G, the Burnside category $\mathcal{B}_G$ is the category whose objects are the same as the objects of $\mathcal{B}'_G$ and for two G-sets S and T, $Hom_{\mathcal{B}_G}(S,T)$ is the Grothendieck group of $Hom_{\mathcal{B}'_G}(S,T)$.*

Given a map of G-sets $f : S \to T$, there are two corresponding morphisms in $\mathcal{B}_G$. The first is depicted on the left in the figure below and is referred to as a 'forward arrow' (denoted by f) and the second is on the right and is referred to as a 'backward arrow' (denoted by $\hat{f}$).

$$\begin{tikzcd} & S \arrow[dl, "="'] \arrow[dr, "f"] & \\ S & & T \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "="] & \\ T & & S \end{tikzcd}$$

Definition 8. *A Mackey functor for a group is an additive contravariant functor from the Burnside category $\mathcal{B}_G$ to the category of Abelian groups. A morphism of Mackey functors is a natural transformation.*

Recall that as a consequence of additivity and functoriality we have the following:

Proposition 9. *It is sufficient to define Mackey functors on G-sets of the form G/H for any subgroup H, quotient maps between these orbits, and conjugation maps from an orbit to an isomorphic orbit.*

Definition 10. *Denote by π^H_K, the quotient map from G/K to G/H where $K \subset H$ and by $\gamma_{x,H}$ the conjugation map that maps $H \to xHx^{-1}$. Given a Mackey functor M, we refer to $M(\pi^H_K)$ as a restriction map, $M(\hat{\pi}^H_K)$ as a transfer map, and $M(\gamma_{x,H})$ as a conjugation map. We shall denote restriction maps as r^H_K, transfer maps as t^H_K, and conjugation maps as τ.*

We present three examples of Mackey functors.

Example 11. *We denote by R the representation functor defined by $R(G/H) = RU(H)$ where $RU(H)$ is the complex representation ring of H. The restriction maps are restriction of representations, the transfer maps are induction of representations, and the conjugations $R(\gamma_{x,H})$ send a vector space with an action of H to the same vector space where the action is redefined as $h \cdot v = xhx^{-1}v$.*

Example 12. *Let M be an abelian group with G-action. Denote the fixed point functor by FP_M. Then $FP_M(G/H) = M^H$. Restriction maps are inclusions of M^H into M^K and transfer maps send $m \to \sum_{h\in H/K} hm$. The conjugations are maps from $M^H \to M^{^xH}$ which send $m \to xm$.*

Example 13. *The representable functor $[-, X] = Hom_{\mathcal{B}_G}(-, X)$ is a Mackey functor. The representable functor $A = [-, e]$ is called the Burnside ring Mackey functor. The Burnside ring is the Grothendieck group of finite G-sets up to isomorphism, and we specify the restriction and transfer maps under this identification. The restriction maps π^G_H send G-sets to H-sets, thereby restricting the group action. The transfer maps $\hat{\pi}^G_H$ send an H-set X to $X \times_H G = X \times G/(x, y) \sim (xh^{-1}, hy)$.*

Coefficient System

A coefficient system is a functor from the orbit category of a group to the category of abelian groups. Because the orbit category embeds in the Burnside category, and we can define a Mackey functor on orbits, we see that a Mackey functor determines two coefficient systems: one contravariant and one covariant [3].

4 Tensor Products

Definition 14. *We shall denote by $\hat{\otimes}$ the 'internal' tensor product of Mackey functors. Let M and N be Mackey functors, and let L, H, and K be subgroups of a group G.*

$$(M\hat{\otimes}N)(H) = \left(\bigoplus\nolimits_{K\subseteq H} M(K) \otimes_{\mathbb{Z}} N(K)\right)/\mathcal{J}$$

$\mathcal{J}$ is the submodule generated by:

$$t^L_K(m) \otimes n - m \otimes r^L_K(n) \text{ for } K \subseteq L \subseteq H, m \in M(K), n \in N(L)$$
$$m \otimes t^K_L(n) - r^K_L(m) \otimes n \text{ for } L \subseteq K \subseteq H, m \in M(K), n \in N(L)$$
$$hm \otimes hn - m \otimes n \text{ for } K \subseteq H, m \in M(K), n \in N(K), h \in H.$$

Proposition 15. *The Burnside ring Mackey functor A serves as the identity for $\hat{\otimes}$ [4].*

A generalization of the same tensor product to an arbitrary G-set X is given below.

Definition 16. *For any G-set X and G-maps $Y \xrightarrow{\phi} X$,*

$$(M\hat{\otimes}N)(X) = \left(\bigoplus\nolimits_{Y\xrightarrow{\phi}X} M(Y) \otimes_{\mathbb{Z}} N(Y)\right)/\mathcal{L}.$$

$\mathcal{L}$ is the submodule generated by:

$$M^*(f)(m') \otimes n - m' \otimes N_*(f)(n)$$
$$M_*(f)m \otimes n' - m \otimes N^*(f)(n')$$

for all $f : (Y, \phi) \to (Y', \phi')$, a morphism of G-sets Y, Y' such that $\phi' f = \phi$.

Lemma 1.6.1 in [5] proves that these definitions are equivalent. Briefly, notice that when we restrict the second definition to G-sets of the form G/H, $\mathcal{L} = \mathcal{J}$. In this case the difference between the two definitions is simply that in the latter, f varies over all possible morphisms $G/K \to G/H$, not just the canonical projection map. In the first definition this is accounted for by the third identity.

Because a Mackey functor determines two coefficient systems this discussion motivates an analogous definition for the tensor product of two coefficient systems.

Definition 17. *Let M be a contravariant coefficient system, let N be a covariant coefficient system, and let L, H, and K be subgroups of a group G. Let p_K^L be the projection map $G/K \to G/L$.*

$$(M \hat{\otimes} N)(H) = \left(\bigoplus_{K \subseteq H} M(K) \otimes_{\mathbb{Z}} N(K) \right) / \mathcal{J}$$

$\mathcal{J}$ is the submodule generated by:

$$M^*(p_K^L)(m) \otimes n - m \otimes N_*(p_K^L)(n) \text{ for } K \subseteq L \subseteq H, m \in M(K), n \in N(L)$$

$$hm \otimes hn - m \otimes n \text{ for } K \subseteq H, m \in M(K), n \in N(K), h \in H.$$

Note that this definition allows us to consider the tensor product of a Mackey functor and a coefficient system, simply by discarding one of the coefficient systems determined by the Mackey functor. A modification of the proof given by Buoc in [5] will show that this definition is equivalent to the definition of the tensor product of coefficient systems given by May in [2].

For all homology computations that follow we shall use definition 17.

5 Bredon Equivariant Homology

Let X be a G-CW complex. Then we define a chain complex of coefficient systems $\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H)$. Given a covariant coefficient system N, our cellular chains are given by $C_n^G(X; N) = (\underline{C}_n(X) \hat{\otimes} N)(G)$ with boundary maps $\partial = d \otimes 1$. Because Mackey functors provide both a covariant and contravariant coefficient system, this definition allows us to compute Bredon equivariant homology with coefficients in a Mackey functor.

The homology of the resulting chain complex is the Bredon equivariant homology of X.

6 $\mathbb{Z}$ Coefficients

To compute Bredon equivariant homology with coefficients in $\mathbb{Z}$ we let N be the trivial coefficient system which maps every orbit to $\mathbb{Z}$ and all morphisms to the identity morphism.

Proposition 18. $H_*^G(X; \mathbb{Z}) = H_*(X/G)$.

Proof. We must compute the homology of the coefficient system $C_n^G(X; N) = (\underline{C}_n(X) \hat{\otimes} N)(G)$. To determine the coefficient system we examine the module $\mathcal{J}$ in definition 17. In the first relation, $N_*(p_K^L)$ is trivial. $M^*(p_K^L) : X^L \to X^K$ is induced by the restriction of the L-action to K. If $x \in X^L$ then $x \in X^K$, so $M^*(p_K^L)(x) = x \in M(L)$. Therefore a quotient by the first relation reduces each $C_n^G(X; N)$ to the ordinary coefficient system $C_n(X)(G/e) = C_n(X; \mathbb{Z})$. The second relation allows us to quotient by the action of G on the elements of $C_n(X; \mathbb{Z})$. The resulting chain complex is $C_*(X/G)$. The result follows.

7 Coefficients in the Burnside Mackey Functor

In the discussion that follows we will give a decomposition of the Burnside Mackey functor A, and use this result to decompose the Bredon equivariant homology with coefficients in A.

We prove that the Mackey functor A splits as the sum of functors S_H defined as $S_H(X) = \mathbb{Z}\left[X^H/W(H)\right]$. With appropriate choices of transfer and restriction, we can verify that S_H is a Mackey functor.

Recall that it suffices to define a Mackey functor on transitive G-sets. Let $L \subseteq K$. Then we have a quotient map $\pi_L^K : G/L \to G/K$. We wish to define transfer maps $t_L^K = S_H(\hat{\pi}_L^K) : S_H(G/L) \to S_H(G/K)$ and restriction maps $r_L^K = S_H(\pi_L^K) : S_H(G/K) \to S_H(G/L)$. First identify

$$(G/L)^H/W(H) \simeq \mathrm{Map}_G(G/H, G/L)/Aut_G(G/H)$$

$$(G/K)^H/W(H) \simeq \mathrm{Map}_G(G/H, G/K)/Aut_G(G/H).$$

Recall that $\mathrm{Map}_G(G/H, G/L)$ is trivial unless H is conjugate to a subgroup of L. There are three cases to consider, up to composing with the appropriate conjugation.

Case 1: H is not a subgroup of L or K. In this case both $S_H(G/L)$ and $S_H(G/K)$ are trivial. So the transfer and restriction maps are also trivial: the transfer map composes the trivial element of $\mathrm{Map}_G(G/H, G/L)$ with π_L^K to give the trivial element of $\mathrm{Map}_G(G/H, G/K)$. The restriction map is the inverse of the transfer map.

Case 2: H is a subgroup of K but not L. In this case $S_H(G/L)$ is trivial, so the transfer and restriction maps must be trivial once again. The transfer map composes a trivial map with a quotient map, which gives the trivial element of $\mathrm{Map}_G(G/H, G/K)$. The restriction map sends all elements of $\mathrm{Map}_G(G/H, G/K)$ to the identity in $\mathrm{Map}_G(G/H, G/L)$.

Case 3: $H \subseteq L \subseteq K$. In this case neither of $S_H(G/L)$ and $S_H(G/K)$ are trivial. The transfer is defined as $t_L^k(f) = \pi_L^K \circ f$. Since any element of $\mathrm{Map}_G(G/H, G/K)$ factors through G/L, $r_L^K(g)$ is the element of $\mathrm{Map}_G(G/H, G/L)$ obtained by taking the pullback of the diagram $G/L \xrightarrow{\pi_L^K} G/K \xleftarrow{\pi_H^K} G/H$. In fact, just as in the previous cases, the restriction and transfer maps are inverses of each other.

Now that the transfers and restrictions of S_H have been articulated we can verify that S_H is a Mackey functor.

Proposition 19. *S_H is a Mackey functor.*

Proof. We prove that S_H satisfies the two conditions required of a Mackey functor in [6]. First we show that if X_1, X_2, X_3, X_4 are G-sets then the commutativity of the following pullback diagram with G-maps $\alpha, \beta, \gamma, \delta$ implies the commutativity of the second diagram.

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha"] \arrow[d, "\beta"] & X_2 \arrow[d, "\gamma"] \\
X_3 \arrow[r, "\delta"] & X_4
\end{tikzcd}$$

$$\begin{tikzcd}
S_H(X_1) \arrow[d, "S_H(\hat{\beta})"] & S_H(X_2) \arrow[l, "S_H(\alpha)"'] \arrow[d, "S_H(\hat{\gamma})"] \\
S_H(X_3) & S_H(X_4) \arrow[l, "S_H(\delta)"']
\end{tikzcd}$$

Notice that if we apply $\mathrm{Map}_G(G/H, -)$ to the first diagram the result still commutes, and taking a quotient by $W(H)$ respects the diagram. The morphisms that result are all transfer maps for S_H. Since the restriction and transfer maps are inverse to each other, the second diagram above must commute.

Next we must verify that $S_H(X \sqcup Y) \cong S_H(X) \oplus S_H(Y)$ is an isomorphism. Well,

$$(X \sqcup Y)^H/W(H) \cong (X^H \sqcup Y^H)/W(H) \cong X^H/W(H) \oplus Y^H/W(H).$$

It follows that S_H is indeed a Mackey functor.

We will now verify the splitting of A on objects in $\mathcal{B}_G$. We introduce the notation $[H] \subseteq G$ to refer to a conjugacy class of subgroups of G with representative element H.

Proposition 20. $\bigoplus_{[H]\subseteq G} \mathbb{Z}[X^H/W(H)] = A(X)$

Proof. We identify the fixed point set X^H with $\mathrm{Map}_G(G/H, X)$ and note that this provides a natural action of the Weyl group $W(H) = N(H)/H = \mathrm{Aut}_G(G/H)$ on X^H. Next, recall that the Burnside ring functor A is the representable functor $[-, e]$. The ring $A(X)$ contains all isomorphism classes of spans $X \leftarrow G/H \to e$. Therefore we have a map $X^H \to A(X)$. In fact, the Weyl group acts as the identity on equivalence classes of spans, so there is an embedding $\varphi_H : X^H/W(H) \to A(X)$. Let $\Phi = \bigoplus_{[H]\subseteq G} \varphi_H$. Φ is certainly an injection since φ_H is an embedding and the images of φ_H and φ_K are disjoint when $H \neq K$. Viewing $A(X)$ as isomorphisms classes of spans, for any span in $A(X)$ the backwards span gives an element of $X^H/W(H)$ for some H, so Φ is surjective.

Theorem 21. *The morphisms $\varphi_H : X^H/W(H) \to A(X)$ are natural transformations of Mackey functors, yielding a decomposition of the Burnside ring Mackey functor A.*

$$A \cong \bigoplus_{[H]\subseteq G} S_H.$$

Proof. Let $L \subseteq K$. We must verify that the two diagrams below commute:

$$\begin{tikzcd} S_H(K) \arrow[r, "\varphi_K"] \arrow[d, "S_H(\pi_L^K)"] & A(K) \arrow[d, "A(\pi_L^K)"] \\ S_H(L) \arrow[r, "\varphi_H"] & A(L) \end{tikzcd}$$

$$\begin{tikzcd} S_H(K) \arrow[r, "\varphi_K"] & A(K) \\ S_H(L) \arrow[r, "\varphi_H"] \arrow[u, "S_H(\hat{\pi}_L^K)"] & A(L) \arrow[u, "A(\hat{\pi}_L^K)"] \end{tikzcd}$$

The diagram on top corresponds to commutativity for restrictions, and the diagram on the bottom corresponds to transfers. We only check the restriction and transfers in the case where $H \subseteq L \subseteq K$.

Given $f \in S_H(K)$, the restriction map for S_H gives a map $f' \in S_H(L)$. The isomorphism class of spans $G/K \xleftarrow{f} G/H \to e$ will be sent to $G/L \xleftarrow{f'} G/H \to e$ by $A(\pi_L^K)$, because the morphism on the left of the span is obtained by the same pullback that determines the restriction on S_H. So the diagram corresponding to the restrictions will commute.

Let $f \in S_H(L)$. Then $\varphi_H(f)$ equals the isomorphism classes of spans $G/L \xleftarrow{f} G/H \to e$. Moving upwards $A(\pi_L^K)$ applied to this span gives the span$G/L \xleftarrow{\pi_L^K \circ f} G/H \to e$. On the top left, $t_L^K(f) = \pi_L^K \circ f$. Finally we apply φ_K to see that the bottom diagram does indeed commute for the transfer maps.

Proposition 22. $H_*^G(X; A) = \bigoplus_{[H]\subseteq G} H_*(X^H/W(H))$

Proof. The result will follow if we prove that the corresponding chain complexes are isomorphic with compatible boundary maps. So we show that $C_n^G(X; A) = (\underline{C}_n(X) \hat{\otimes} A)(G)$ is isomorphic to $\bigoplus_{[H]\subseteq G} C_n(X^H/W(H))$.

Since we are working in the case where N is the Mackey functor A, we examine $\mathcal{J}$ carefully, referring to definition 17. Let $M = \underline{C}_*(X)$. First, $M^*(p_K^L) : X^L \to X^K$ is induced by the restriction of the L-action to K. If $x \in X^L$ then $x \in X^K$, so $M^*(p_K^L)(x) = x \in X^K$. Next, $N_*(p_K^L)$ is the transfer map of A, t_K^L. So the first relation in definition 17 can be rewritten as

$$m \otimes n - m \otimes t_K^L(n) \text{ for } K \subseteq L \subseteq H, m \otimes n \in M(K) \otimes N(K).$$

What is the transfer map of A? Well, $t_K^L(K/N) = L/N$. Therefore for a given $H \subseteq G$, the only elements of $C_n(G/H) \otimes A(G/H)$ that are not in the kernel of the quotient by the relation above are of the form $m \otimes [H/H]$. So each term in the summand is one dimensional.

For the second relation we first check the G-action on $\underline{C}_*(X)(G/K) = C_*(X^K)$. This action will be induced by the map $G/K \to G/gKg^{-1}$, so we get a map $X^K \to X^{gKg^{-1}}$ which sends $x \mapsto gx$. In the event that $g \in N(K)$ the normalizer of K, we have a map $X^K \to X^K$ which sends $x \mapsto gx$. So this relation performs a quotient by the action of the Weyl group $W(K) = N(K)/K$, and it identifies $C_n(G/H) \otimes A(G/H)$ and $C_n(G/H') \otimes A(G/H')$ if H and H' are conjugate. Therefore the two chain complexes are isomorphic with compatible chain maps (since the chain maps are $\delta \otimes 1$), so the result follows for homology.

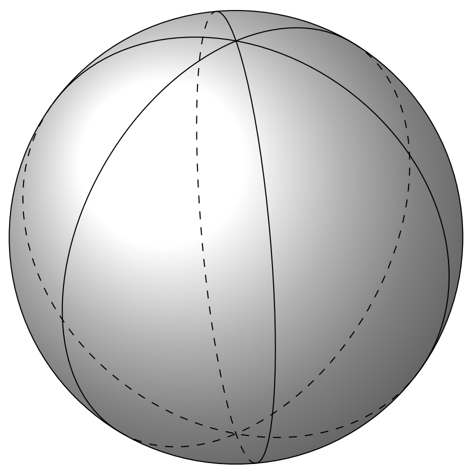
Alternatively, this result can be obtained by applying proposition 21 to the chain complex of the G-CW complex and verifying that the boundary maps are compatible. In the proof above the compatibility of the boundary maps comes immediately since the boundary maps of $C_n^G(X; A)$ are $\partial \otimes 1$.

Corollary 23. $H_*^G(X; A) = \bigoplus_{[H] \subseteq G} H_*^{W(H)}(X^H; \mathbb{Z})$

Proof. This folows immediately from proposition 18 and proposition 22.

8 Computations for $G = C_n$

The G-CW complex of the representation sphere of the cyclic representation of C_n is depicted below.



Given proposition 22, computations of the Bredon equivariant homology with coefficients in A becomes algorithmic.

Proposition 24. *When X is the representation sphere of $G = C_n$ acting on S^2 by rotation around a fixed axis, $H_0^G(X; A) = \bigoplus_{H \subseteq G} \mathbb{Z}$, $H_1^G(X; A) = 0$, $H_2^G(X; A) = \mathbb{Z}$, and $H_n^G(X; A) = 0$ for $n > 2$.*

Proof. Since C_n is abelian, we must iterate over every subgroup of G. The fixed point set of any rotation is the two antipodal points at the intersection of S^2 and the axis of rotation. Therefore for every $H \subseteq G$, X^H is the union of two points, and for $H = e$ $X^H = S^2$. Again, since C_n is abelian, $W(H) = N(H)/H = G/H$. When $H \neq e$, the Weyl group acts trivially on X^H. When $H = e$ we must quotient S^2 by C_n. This quotient is homeomorphic to S^2. Therefore $H_0(X^H/W(H)) = \mathbb{Z}$ for every $H \neq e$. When $H = e$, $H_0(X^H/W(H)) = 0$. $H_1(X^H/W(H)) = 0$ for all $H \subseteq G$. $H_2(X^H/W(H)) = 0$ for all $H \subseteq G$ except $H = e$ in which case $H_2(X^H/W(H)) = \mathbb{Z}$. We can immediately deduce $H_*^G(X; A)$ using proposition 22.

$$H_0^G(X; A) = \bigoplus_{H \subseteq G} \mathbb{Z}$$

$$H_1^G(X; A) = 0$$

$$H_2^G(X; A) = \mathbb{Z}$$

9 Computations for $G = S_3$

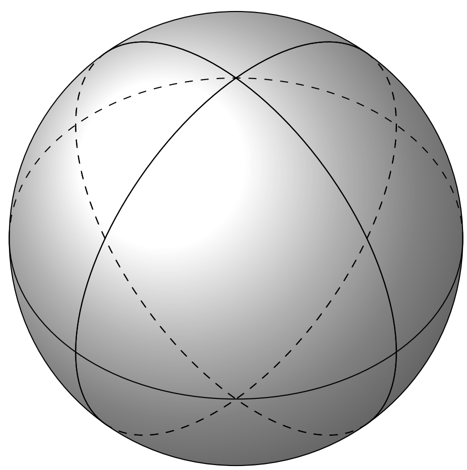
Proposition 25. *When X is the representation sphere of $G = S_3$ acting on S^2 by permutation of the axes, $H_0^G(X; A) = \mathbb{Z} \oplus \mathbb{Z}$, $H_1^G(X; A) = \mathbb{Z}$, and $H_n^G(X; A) = 0$ for $n \geq 2$.*

Proof. The Burnside category for $G = S_3$ takes the following form (up to conjugations):

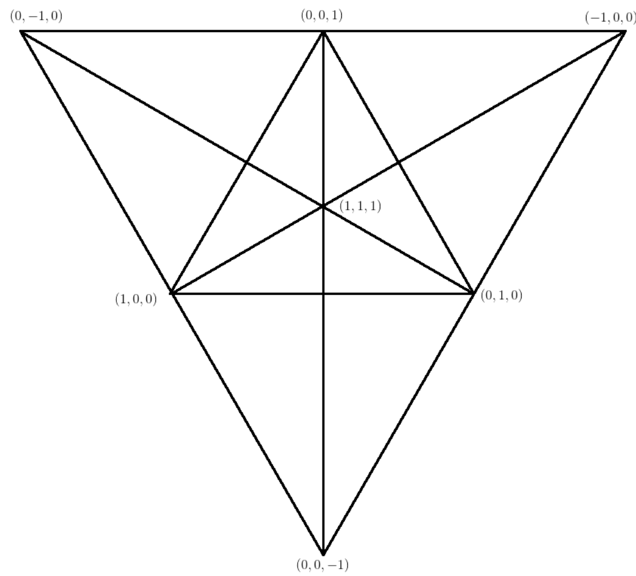
$$\begin{tikzcd}
 & G/G \arrow[dl, bend right=20, "\hat{\pi}^G_{C_2}"'] \arrow[dr, bend left=20, "\hat{\pi}^G_{C_3}"] & \\
G/C_2 \arrow[ur, bend left=40, "\pi^G_{C_2}"] \arrow[dr, bend left=20, "\hat{\pi}^{C_3}_e"] & & G/C_3 \arrow[ul, bend right=40, "\pi^G_{C_3}"'] \arrow[dl, bend right=20, "\hat{\pi}^{C_2}_e"'] \\
 & G/e \arrow[ul, bend left=40, "\pi^{C_2}_e"] \arrow[ur, bend right=40, "\pi^{C_3}_e"'] &
\end{tikzcd}$$

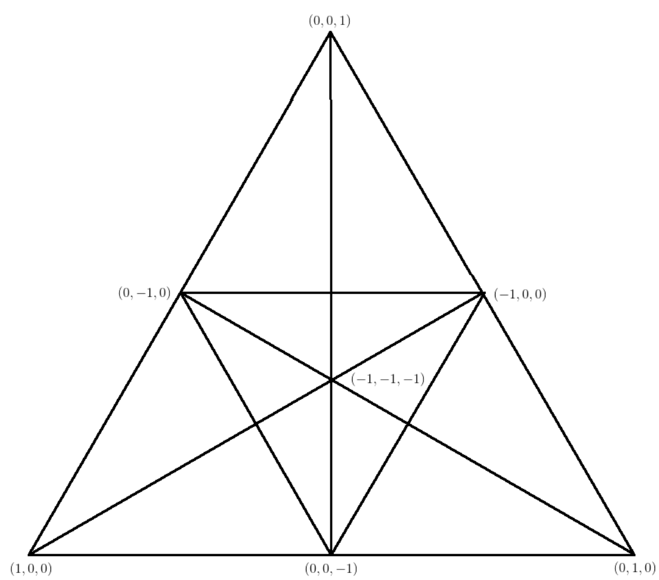
We must identify the G-CW complex of the sphere of this permutation representation. A general construction for any permutation representation can be found using barycentric subdivision. We start with an octahedral approximation of S^2, and we perform barycentric subdivision on each of the octants.

The G-CW complex of the representation sphere of the cyclic representation of C_n is depicted below.



The first diagram below is a view of the simplicial approximation of the G-CW complex of S^2 from the positive side of the line $x = y = z$ (the same view point for the sphere above). The three octants adjacent to the octant with positive coordinates on each axis are depicted in this diagram. The second diagram is a view from the negative side of the line $x = y = z$. The necessary sides are identified.





The stabilizers of each 1-cell are shown in the diagram. We can deduce from the diagrams above that $X^e = S^2$, $X^{C_2} = S^1$, and X^{C_3} and X^G are each the union of two points (the antipodal points $(1,1,1)$ and $(-1,-1,-1)$. When $H = e$, $W(H) = G$, and the quotient S^2/G is contractible. When $H = C_2$, $W(H)$ acts trivially on X^H, so $X^{C_2}/W(C_2) = S^1$. When $H = C_3$, $W(H)$ acts trivially on the two point set, so $X^{C_3}/W(C_3)$ is still the two point set. Similarly, X^{S_3} is a two point set. Therefore $H_0^G(X;A) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_1^G(X;A) = \mathbb{Z}$ and all other homology groups are zero.

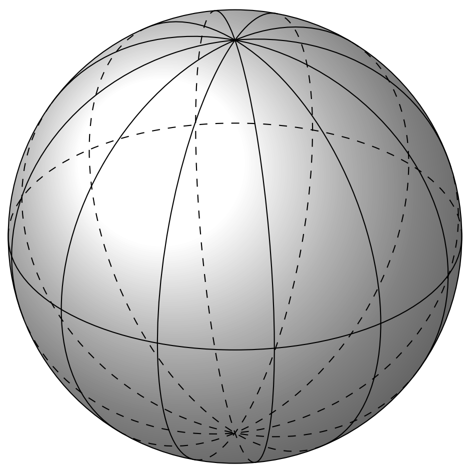
10 Computations for D_n

The dihedral group D_n has the following presentation:

$$D_n = \langle r, s | r^n = s^2 = 1, srs = r^{-1} \rangle.$$

Let us begin by introducing the orthogonal representation of D_n in R^3. Identify D_n as the symmetry group of the n-gon where r is a rotation by $2\pi/n$ and s is a reflection across a line of symmetry.

We embed this n-gon in the xy plane so that the center point of the n-gon is at the origin and the line of symmetry across which s reflects lies on the x-axis. Now we define the representation $\sigma : D_n \to GL_3(\mathbb{R})$. Let $\sigma(r)$ be a rotation of $2\pi/n$ about the z axis, and let $\sigma(s)$ be a rotation of π about the x axis. The G-CW complex of this representation sphere has a simplicial approximation that is the suspension of the n-gon with a 0-cell bisecting each side.



We shall take a brief aside to describe the subgroup structure of the dihedral group.

D_n

We cite the following two lemmas from [7]:

Lemma 26. *Subgroups of D_n take the one of the following two forms:*

- $\langle r^d \rangle$ *with $d|n$ and index $2d$*
- $\langle r^d, r^i s \rangle$ *with $d|n$, $0 \leq i < d$, and index d.*

Lemma 27. *If n is odd and $m|2n$:*

- *If m is odd then every subgroup of D_n of index m is conjugate to $\langle r^m, s \rangle$.*
- *If m is even the only subgroup of D_n with index m is $\langle r^{\frac{m}{2}} \rangle$.*

If n is even and $m|2n$:

- *If m is odd then every subgroup of D_n of index m is conjugate to $\langle r^m, s \rangle$.*
- *If m is even and $m \not| n$ then the only subgroup of D_n with index m is $\langle r^{\frac{m}{2}} \rangle$.*
- *If m is even and $m|n$ then any subgroup of D_n with index m is $\langle r^{\frac{m}{2}} \rangle$ or is conjugate to exactly one of $\langle r^m, s \rangle$ or $\langle r^m, rs \rangle$.*

Next we refer to the simplicial approximation of our G-CW complex and apply Proposition 14.

Suppose n is odd. Notice that if a subgroup H of $G = D_n$ contains both a rotation r^d and s the fixed point set of the sphere $X = S^2$ is empty because the axis of the rotation r is orthogonal to the axis of the rotation s. Therefore if m is odd the only subgroup H with a nontrivial fixed point set X^H is the group $\langle s \rangle$ which occurs when $m = n$. This fixed point set contains the two points where the axis of rotation for s

intersects S^2. If m is even the subgroup $\langle r^{\frac{m}{2}}\rangle$ yields a nontrivial fixed point set containing the two points where the axis of rotation of r intersects S^2.

Since $srs = r^{-1}$, $s \in N(\langle r^{\frac{m}{2}}\rangle)$. Therefore $N(\langle r^{\frac{m}{2}}\rangle) = D_n$. So $W(\langle r^{\frac{m}{2}}\rangle) = D_n/\langle r^{\frac{m}{2}}\rangle$. Thus the quotient $X^{\langle r^{\frac{m}{2}}\rangle}/W(\langle r^{\frac{m}{2}}\rangle)$ is a single point which contributes trivial homology to the Bredon equivariant homology.

The normalizer of $\langle s\rangle$ is itself, so $W(\langle s\rangle)$ is trivial. Therefore the quotient $X^{\langle s\rangle}/W(\langle s\rangle)$ is a two point space which contributes a copy of $\mathbb{Z}$ to the 0^{th} term of the Bredon equivariant homology. Thus, when n is odd, the Bredon equivariant homology is supported by a single copy of $\mathbb{Z}$ in dimensions 0 and 2 and is zero everywhere else.

Finally, if $H = e$, the quotient S^2/D_n is homeomorphic to S^2, so there is a $\mathbb{Z}$ in the second degree of the Bredon homology.

If n is even, subgroups of the form $\langle r^m, s\rangle$ and $\langle r^m, rs\rangle$ give empty fixed point sets unless $m = 0$, in which case $X^{\langle}s\rangle$ and $X^{\langle}rs\rangle$ yield distinct fixed point sets each containing two points. Subgroups of the form $\langle r^{\frac{m}{2}}\rangle$ still give fixed point sets with two points, however upon taking the quotient by the Weyl group the contribution to homology is trivial. Therefore the Bredon equivariant homology has two copies of $\mathbb{Z}$ in degree 0 and one copy of $\mathbb{Z}$ in degree 2.

The results of this section are summarized in the following theorem.

Theorem 28. *Let $G = D_n = \langle r, s | r^n = s^2 = 1, srs = r^{-1}\rangle$. Let $\sigma : G \to GL_3(\mathbb{R})$ be the representation such that $\sigma(r)$ is rotation about the z axis by $2\pi/n$ and $\sigma(s)$ is rotation about the x axis by π. Let X be the corresponding representation sphere. Then when n is odd,*

$$H_0^G(X; A) = \mathbb{Z}$$
$$H_1^G(X; A) = 0$$
$$H_2^G(X; A) = \mathbb{Z}.$$

When n is even,

$$H_0^G(X; A) = \mathbb{Z} \oplus \mathbb{Z}$$
$$H_1^G(X; A) = 0$$
$$H_2^G(X; A) = \mathbb{Z}.$$

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