CHAPTER 1

Stable Operations in Generalized Cohomology

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Contents

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1. Introduction

For any space X, the Steenrod algebra $\mathcal{A}$ of stable cohomology operations acts on the ordinary cohomology $H^*(X;\mathbb{F}_p)$ to make it an $\mathcal{A}$-algebra. Milnor discovered [22] that it is useful to treat $H^*(X;\mathbb{F}_p)$ as a comodule over the dual of $\mathcal{A}$, which becomes a Hopf algebra. Adams extended this program in [1, 3] to multiplicative generalized cohomology theories $E^*(-)$, under appropriate hypotheses. The coefficient ring E^* is now graded, and $E^*(X)$ is an E^*-algebra.

Our purpose is to describe the structure of the stable operations on $E^*(-)$ in a manner that will generalize in [9] to unstable operations. Unlike some treatments, we impose no finiteness or connectedness conditions whatever on the spaces and spectra involved, only a single freeness condition on E. We emphasize universal properties as the appropriate setting for many results. An early version of some of the ideas is presented in [8], which is limited to ordinary cohomology, MU, and BP.

For general E, the stable operations form the endomorphism ring $\mathcal{A} = E^*(E, o)$ of E (in our notation). For each $x \in E^*(X)$, we have the E^*-module homomorphism $x^*\colon \mathcal{A} \to E^*(X)$ given by $x^*r = \pm rx$. The key idea is (roughly) that given an E^*-module M, we define SM as the set of all E^*-module homomorphisms $\mathcal{A} \to M$; this is to be thought of as the set of candidates for the values of all operations on a typical element of M.

Generally, we encode the action of $\mathcal{A}$ on a stable module M as the function $\rho_M\colon M \to SM$ given by $(\rho_M x)r = \pm rx$. There is an E^*-module structure on SM (different from the obvious one) that makes ρ_M a homomorphism of E^*-modules. This is not yet enough; composition of operations makes the functor S what is known as a *comonad*, and we need (M, ρ_M) to be a *coalgebra* over this comonad. When M is an E^*-algebra, so is SM, and we can similarly define stable algebras.

This work serves as more than just a pattern for the promised unstable theory of [9]. To compare unstable structures with the analogous stable structures, we shall there construct suitable natural transformations; this is far easier to do when both theories are developed in the same manner. Much of the basic category theory is the same for either case; we keep it all here for convenience. Finally, we need specific stable results for later use.

Outline. In section 2, we introduce five assorted ring spectra E, which will serve throughout as our examples. We review some elementary category theory and set up notation.

In sections 3 and 4, we study E-(co)homology in enough detail to suggest what categories to use. In section 9, we consider (co)homology in the stable homotopy category of spectra. It is essential for us to work in the correct categories, in order to make our categorical machinery run smoothly; otherwise it does not run at all. We therefore take pains in section 6 to say precisely what our categories are.

In section 7, we discuss the various kinds of algebraic object, such as group, module, and ring, that we need in general categories. In section 8, we rework the definition of a module over a ring until we find a way that will generalize to the unstable context.

In section 10, we discuss stable modules from several points of view. We introduce the comonad S, and define a stable module as an S-coalgebra. Theorem 10.16 shows that $E^*(X)$ is (more or less) a stable module.

In section 11, we make the homology $E_*(E,o)$ a coalgebra (in a sense), provided only that it is a free E^*-module. A stable module then becomes a comodule over it; indeed, Thm. 11.13 shows that the theories of stable modules and stable comodules are entirely equivalent. Theorem 11.14 provides a useful universal property of $E_*(E,o)$. Theorem 11.35 shows that our structure on $E_*(E,o)$ agrees with that introduced by Adams [1].

Everything mentioned so far works for spectra X, too. In section 12, we take account of the multiplication present on $E^*(X)$ when X is a space by making SM an E^*-algebra whenever M is. This leads to the definition of a stable algebra. Again, there is an equivalent comodule version.

All our examples of E-cohomology come with a complex orientation. This has standard implications for the structure of $E^*(\mathbb{C}P^\infty)$ etc., which we review in section 5. In section 13, we study the consequences for operations.

In section 14, we present the structure on $E_*(E,o)$ in detail for each of our five examples E. We do not actually construct the operations, which are all well known. It is clear that many other examples are available.

In section 15, we study the special case of BP-cohomology in greater depth. For a general introduction, see Wilson [37]. Stable BP-operations are well established; a short early history would include Landweber [17], Novikov [28], Quillen [30], Adams [3], Zahler [41, 42], Miller-Ravenel-Wilson [21], and more recently, Ravenel's book [31]. We review Landweber's filtration theorem, for imitation in [9].

An index of symbols is included at the end.

Acknowledgements. We thank Dave Johnson and Steve Wilson for making this paper necessary. As noted, it serves chiefly as a platform for [9]. It incorporates several suggestions of Steve Wilson, especially the use of corepresented functors in section 8. We also thank Nigel Ray for pointing out some useful references.

2. Notation and five examples

Our five examples of commutative ring spectra E are:

- $H(\mathbb{F}_p)$ The Eilenberg-MacLane spectrum, for a fixed prime $p \geqslant 2$, which represents ordinary cohomology $H^*(-;\mathbb{F}_p)$ and is a ring spectrum (see e.g. Switzer [34, 13.88]);
- BP The Brown-Peterson spectrum, for a fixed prime $p \geqslant 2$ (which is suppressed from the notation), a ring spectrum by Quillen [29];
- MU The unitary (or complex) cobordism Thom spectrum, which is a ring spectrum (see e.g. Switzer [34, 13.89]);
- KU The complex Bott spectrum (often written K), which represents topological complex K-theory and is a ring spectrum [ibid., 13.90];

$K(n)$ The Morava K-theory spectrum, for a fixed prime $p > 2$ (again suppressed from the notation), and any $n \geqslant 0$. (We take $p > 2$ in order to ensure that the multiplication is commutative as well as associative; see Morava [26], and especially Shimada-Yagita [33, Cor. 6.7] or Würgler [38, Thm. 2.14]. See [16] for background information.)

In particular, $K(0) = H(\mathbb{Q})$ (for any p), and $K(1)$ is a summand of KU-theory mod p.

Indeed, all our ring spectra are understood to be commutative. Each E defines a multiplicative cohomology theory $E^*(X)$ and homology theory $E_*(X)$, which we discuss in sections 3 and 4. They have the same coefficient ring E^*.

Because we deal almost exclusively in cohomology, we assign the *degree* n to cohomology classes in $E^n(X)$ and elements of E^n; this forces homology classes in $E_n(X)$ to have degree $-n$. Note that under this convention, elements of BP^* and MU^* are given *negative* degrees.

For any space X, $E^*(X)$ and $E_*(X)$ are E^*-modules. We therefore adopt E^* as our ground ring throughout, and all tensor products and groups $\mathrm{Hom}(M,N)$ are taken over E^* unless otherwise specified. Except for (co)homology, we generally follow the practice of [25] in writing a graded group with components M^n as M rather than M^*. When we do write M^* (e.g. E^* as above), we mean the whole graded group, not a typical component.

All our rings and algebras are associative and are presumed to have a unit element 1, which is to be preserved by homomorphisms. Dually, coalgebras are assumed to be coassociative.

Summations are often understood as taken over all available values of the index.

We do not attempt to give each construct a unique symbol. For example, all multiplications are named ϕ, which we decorate as ϕ_S etc. only as needed to distinguish different multiplications. All actions are named λ and all coactions are named ρ. To compensate, we generally specify where each equation takes place.

Signs. We follow the convention that a minus sign should be introduced *whenever* two symbols of odd degree become transposed for any reason. As explained in [7], this is a purely lexical convention, which depends only on the order of appearance of the various symbols, not on their meanings. The principle is that consistency will be maintained provided one starts from equations that conform and performs only "reasonable" manipulations on them. The main requirement is that each symbol having a degree should appear exactly once in every term of an equation.

Category theory. Our basic reference is MacLane's book [20], which also provides most of our notation and terminology.

In any category $\mathcal{A}$, the set of morphisms from X to Y is denoted $\mathcal{A}(X,Y)$, or occasionally $\mathrm{Mor}(X,Y)$. If $\mathcal{A}$ is a *graded* category (always assumed additive), $\mathcal{A}^n(X,Y)$ denotes the abelian group of morphisms from X to Y of degree n. Unmarked arrows are intended to be the obvious morphisms. We write $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ for the projections from the product $X \times Y$ to its factors, and dually $i_1\colon X \to X \amalg Y$ and $i_2\colon Y \to X \amalg Y$ for coproducts. We also write $q\colon X \to T$

for the unique morphism to a terminal object T.

We denote by $I: \mathcal{A} \to \mathcal{A}$ the identity functor of $\mathcal{A}$. We sometimes find it useful to write a natural transformation α between functors $F, F': \mathcal{A} \to \mathcal{B}$ as

$$\alpha: F \to F': \mathcal{A} \to \mathcal{B} .$$

If $\mathcal{A}$ and $\mathcal{B}$ are graded, we can have $\deg(\alpha) = m \neq 0$; in this case, we require $\alpha Y \circ Ff = (-1)^{m \deg(f)} F'f \circ \alpha X$ for each morphism $f: X \to Y$. In contrast, our graded functors invariably preserve degree.

If $\alpha': F' \to F''$ is another natural transformation, we have the composite natural transformation $\alpha' \circ \alpha: F \to F''$. There is the identity natural transformation $1: F \to F$. Given also $G: \mathcal{B} \to \mathcal{C}$, we denote the composite functor as $GF: \mathcal{A} \to \mathcal{C}$ (never $G \circ F$), and define the natural transformation $G\alpha: GF \to GF': \mathcal{A} \to \mathcal{C}$ by $(G\alpha)X = G(\alpha X)$. Similarly, given $\beta: G \to G'$, we define $\beta F: GF \to G'F: \mathcal{A} \to \mathcal{C}$ by $(\beta F)X = \beta(FX)$. We also have $\beta\alpha = \beta F' \circ G\alpha = G'\alpha \circ \beta F: GF \to G'F': \mathcal{A} \to \mathcal{C}$ (or $\pm G'\alpha \circ \beta F$ in the graded case).

We make incessant use of Yoneda's Lemma [20, III.2].

Adjoint functors. It should be no surprise that we have numerous pairs of adjoint functors. Suppose given a functor $V: \mathcal{B} \to \mathcal{A}$ (which is usually, but not necessarily, some forgetful functor) and an object A in $\mathcal{A}$.

Definition 2.1. We call an object M in $\mathcal{B}$ *V-free* on A, with *basis* $i: A \to VM$, a morphism in $\mathcal{A}$, if for each B in $\mathcal{B}$, any morphism $f: A \to VB$ in $\mathcal{A}$ "extends" to a unique morphism $g: M \to B$ in $\mathcal{B}$, called the *left adjunct* of f, in the sense that $Vg \circ i = f: A \to VB$.

In the language of [20, III.1], i is a *universal arrow*, which induces the bijection $\mathcal{B}(M, B) \cong \mathcal{A}(A, VB)$. The free object M is unique up to canonical isomorphism, but there is no guarantee that one exists. In the favorable case when we have a free object FA for each A in $\mathcal{A}$, with basis $\eta A: A \to VFA$, there is a unique way to define Fh for each morphism h in $\mathcal{A}$ to make η natural; then F becomes a functor and the isomorphism

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, VB), \tag{2.2}$$

is natural in both A and B. Explicitly, we recover $f: A \to VB$ from $g: FA \to B$ as

$$f = Vg \circ \eta A: A \longrightarrow VFA \longrightarrow VB \qquad \text{in } \mathcal{A}. \tag{2.3}$$

For any B, we define $\epsilon B: FVB \to B$ in $\mathcal{B}$ as extending $1: VB \to VB$. Then $\epsilon: FV \to I$ is also natural, and we may construct the left adjunct g of f as

$$g = \epsilon B \circ Ff: FA \longrightarrow FVB \longrightarrow B \qquad \text{in } \mathcal{B}. \tag{2.4}$$

The fact that this is inverse to eq. (2.3) is neatly expressed by the pair of identities

$$\begin{aligned} &\text{(i)} \quad V\epsilon \circ \eta V = 1: V \longrightarrow V: \mathcal{B} \longrightarrow \mathcal{A} \\ &\text{(ii)} \quad \epsilon F \circ F\eta = 1: F \longrightarrow F: \mathcal{A} \longrightarrow \mathcal{B} \end{aligned} \tag{2.5}$$

We summarize the basic facts about adjoint functors from [20, Thm. IV.1.2].

Theorem 2.6. *The following conditions on a functor $V: \mathcal{B} \to \mathcal{A}$ are equivalent:*

(i) *V has a left adjoint $F: \mathcal{A} \to \mathcal{B}$;*

(ii) *V is a right adjoint to some functor $F: \mathcal{A} \to \mathcal{B}$;*

(iii) *There is a functor $F: \mathcal{A} \to \mathcal{B}$ and an isomorphism* (2.2)*, natural in A and B;*

(iv) *For all A in $\mathcal{A}$, we can choose a V-free object FA and a basis ηA of it;*

(v) *There is a functor $F: \mathcal{A} \to \mathcal{B}$ with natural transformations $\eta: I \to VF$ and $\epsilon: FV \to I$ that satisfy eqs.* (2.5). □

In view of the symmetry in (v), or between (i) and (ii), we have the dual result, which we do not state. Nevertheless, we do give the dual to Defn. 2.1.

Definition 2.7. The object N in $\mathcal{B}$ is *V-cofree* on A, with *cobasis* $p: VN \to A$, a morphism in $\mathcal{A}$, if for each B in $\mathcal{B}$, any $f: VB \to A$ in $\mathcal{A}$ "lifts" uniquely to a morphism $g: B \to N$ in $\mathcal{B}$, called the *right adjunct* of f, in the sense that $p \circ Vg = f$.

3. Generalized cohomology of spaces

In this section and the next, we review multiplicative cohomology theories $E^*(-)$ and their associated homology theories $E_*(-)$ in sufficient depth to decide what objects our categories should contain. We also establish much of our notation.

Spaces. We find we have to work mostly with *unbased* spaces. The most convenient spaces are CW-complexes. We denote by T the one-point space. It is sometimes useful to allow also spaces that are homotopy equivalent to CW-complexes, so that we can form products and loop spaces directly. A *pair* (X, A) of spaces is assumed to be a CW-pair (or homotopy equivalent, as a pair, to one).

Ungraded cohomology. For our purposes, an *ungraded cohomology theory* is a homotopy-invariant contravariant functor $h(-)$ that assigns to each space X an abelian group $h(X)$, and satisfies the usual two axioms:

(i) $h(-)$ is *half-exact:* If $X = A \cup B$, where A and B are well-behaved subspaces (e.g. subcomplexes of a CW-complex X), and $y \in h(A)$ and $z \in h(B)$ agree in $h(A \cap B)$, there exists $x \in h(X)$ (not in general unique) that lifts both y and z; (3.1)

(ii) $h(-)$ is *strongly additive:* For any topological disjoint union $X = \coprod_\alpha X_\alpha$, the inclusions $X_\alpha \subset X$ induce $h(X) \cong \prod_\alpha h(X_\alpha)$.

For a space X with basepoint $o \in X$, we may define the *reduced* cohomology $h(X, o)$ by the split short exact sequence

$$0 \longrightarrow h(X, o) \xrightarrow{\subset} h(X) \longrightarrow h(o) \longrightarrow 0. \tag{3.2}$$

We recover the *absolute* cohomology $h(X)$ by constructing the disjoint union X^+ of X with a (new) basepoint; by (ii), $h(X^+) \cong h(X) \oplus h(o)$ and the inclusion $X \subset X^+$ induces an isomorphism

$$h(X^+, o) \cong h(X) \ . \tag{3.3}$$

For a good pair (X, A) of spaces, we may define the *relative* cohomology as

$$h(X, A) = h(X/A, o), \tag{3.4}$$

and these groups behave as expected. We generalize eq. (3.2).

Lemma 3.5. *If A is a retract of X, we have the split short exact sequence*

$$0 \longrightarrow h(X, A) \longrightarrow h(X) \longrightarrow h(A) \longrightarrow 0.$$

If A has a basepoint o, we have also the split short exact sequence

$$0 \longrightarrow h(X, A) \longrightarrow h(X, o) \longrightarrow h(A, o) \longrightarrow 0.$$

□

With no basepoints, we have to be a little careful in representing $h(-)$. Let *Ho* be the homotopy category of spaces that are (homotopy equivalent to) CW-complexes.

Theorem 3.6. *Let $h(-)$ be an ungraded cohomology theory as above. Then:*

(a) *$h(-)$ is represented in Ho by an H-space H, with a universal class $\iota \in h(H, o) \subset h(H)$ that induces an isomorphism $Ho(X, H) \cong h(X)$ of abelian groups by $f \mapsto h(f)\iota$ for all X;*

(b) *For any cohomology theory $k(-)$, operations $\theta: h(-) \to k(-)$ correspond to elements $\theta\iota \in k(H)$.*

Proof. What Brown's representation theorem [10, Thm. 2.8, Ex. 3.1] actually provides is a based connected space (H', o), which represents $h(-, o)$ on *based connected* spaces (X, o) only. Then West [35] shows that $h(-, o)$ is represented on all *based* spaces by the product space

$$H = h(T) \times H', \tag{3.7}$$

where we treat the group $h(T)$ as a discrete space. By eq. (3.3), H also represents $h(-)$ in the unbased category *Ho*.

The map $\omega: T \to H$ that corresponds to $0 \in h(T)$ furnishes H with a (homotopically well-defined) basepoint, and the exact sequence (3.2) shows that $\iota \in h(H, o)$. Yoneda's Lemma represents the addition

$$Ho(X, H \times H) \cong h(X) \times h(X) \xrightarrow{+} h(X) \cong Ho(X, H)$$

by an *addition map* $\mu: H \times H \to H$ which makes H an H-space, and also gives (b). (Lemma 7.7(a) will tell us much more about H.) □

Example: KU. For finite-dimensional spaces X, the ungraded cohomology theory $KU(X)$ is defined (e.g. Husemoller [15]) as the Grothendieck group of complex vector bundles over X. The class of the vector bundle ξ is denoted $[\xi]$, and every element of $KU(X)$ has the form $[\xi] - [\eta]$. The trivial n-plane bundle is denoted simply n. Addition is defined from the Whitney sum of vector bundles, $[\xi] + [\eta] =$

$[\xi \oplus \eta]$, and multiplication from the tensor product, $[\xi][\eta] = [\xi \otimes \eta]$. In particular, $KU(T) \cong \mathbb{Z}$, as a ring.

Let (X, o) be a based connected space, still finite-dimensional. Because any vector bundle ξ over X has a stable inverse η such that $\xi \oplus \eta$ is trivial, every element of $KU(X, o)$ can be written in the form $[\xi] - n$ for some n-plane vector bundle ξ, provided n is large enough. The bundle ξ has a classifying map $X \to BU(n) \subset BU$, which leads to the representation $\mathit{Ho}(X, BU) \cong KU(X, o)$. As in the proof of Thm. 3.6, this extends to an isomorphism $\mathit{Ho}(X, \mathbb{Z} \times BU) \cong KU(X)$, valid for all finite-dimensional spaces X.

To extend $KU(-)$ to all spaces as an ungraded cohomology theory, we must *define* $KU(X) = \mathit{Ho}(X, \mathbb{Z} \times BU)$. It remains true that any vector bundle ξ over X defines an element $[\xi] \in KU(X)$, but in general, not all elements of $KU(X)$ have the form $[\xi] - [\eta]$.

Splittings. All our splittings depend on the following elementary result.

Lemma 3.8. *Assume that $\theta\colon h(-) \to h(-)$ is an idempotent cohomology operation, $\theta \circ \theta = \theta$. Then the image $\theta h(-)$ also satisfies the axioms* (3.1).

Proof. For (i), given $y \in \theta h(A)$ and $z \in \theta h(B)$ that agree in $h(A \cap B)$, the half-exactness of h yields an element $x \in h(X)$ that lifts y and z. Because θ is idempotent, $\theta x \in \theta h(X)$ also lifts y and z, to show that (i) holds.

For (ii), we need only the naturality of θ. Given elements $x_\alpha = \theta x'_\alpha \in \theta h(X_\alpha)$, axiom (ii) for h provides $x' \in h(X)$ that lifts each x'_α. Then $x = \theta x' \in \theta h(X)$ lifts each x_α, and is unique because h satisfies (ii). □

We immediately deduce the standard tool for constructing splittings. Theorem 3.6(b) allows us to write the identity operation as ι.

Lemma 3.9. *Let θ be an additive idempotent operation on the ungraded cohomology theory $h(-)$. Then:*

(a) *$\iota - \theta$ is also idempotent;*

(b) *We can define ungraded cohomology theories*

$$h'(X) = \mathrm{Ker}[\theta\colon h(X) \longrightarrow h(X)] = \mathrm{Im}[\iota - \theta\colon h(X) \longrightarrow h(X)]$$

and

$$h''(X) = \mathrm{Ker}[\iota - \theta\colon h(X) \longrightarrow h(X)] = \mathrm{Im}[\theta\colon h(X) \longrightarrow h(X)];$$

(c) *We have the natural direct sum decompositon $h(X) = h'(X) \oplus h''(X)$;*

(d) *If the H-spaces H' and H'' represent $h'(-)$ and $h''(-)$ as in Thm. 3.6(a), then $H' \times H''$ represents $h(-)$.* □

For future use in [9], we extend this result to certain *nonadditive* idempotent operations. To emphasize the nonadditivity, we retain the parentheses in $\theta(-)$.

Lemma 3.10. *Assume the* nonadditive *operation θ on the ungraded cohomology theory $h(-)$ satisfies the axioms:*

$$\begin{aligned}&\text{(i) } \theta(0)=0;\\ &\text{(ii) } \theta(x+y-\theta(y))=\theta(x) \text{ for any } x,y\in h(X).\end{aligned} \tag{3.11}$$

Then:

(a) *θ and $\iota-\theta$ are idempotent;*

(b) *We can define the kernel cohomology theory $h'(-)=\operatorname{Ker}\theta=\operatorname{Im}(\iota-\theta)$ as a subgroup of $h(-)$;*

(c) *We can define the coimage cohomology theory $h''(X) = \operatorname{Coim}\theta = h(X)/h'(X)$ as a quotient of $h(X)$, with projection $\pi\colon h(X)\to h''(X)$;*

(d) *We have the natural short exact sequence of ungraded cohomology theories*

$$0 \longrightarrow h'(X) \xrightarrow{\subset} h(X) \xrightarrow{\pi} h''(X) \longrightarrow 0; \tag{3.12}$$

(e) *θ induces a nonadditive operation $\overline{\theta}\colon h''(X)\to h(X)$ which splits* (3.12) *and induces the bijection of sets $h''(X)=\operatorname{Coim}\theta\cong\operatorname{Im}[\theta\colon h(X)\to h(X)]$;*

(f) *The short exact sequence* (3.12) *is represented by a fibration of H-spaces and H-maps*

$$H' \longrightarrow H \longrightarrow H''$$

in which $H\to H''$ admits a section (not an H-map) and $H\simeq H'\times H''$ as spaces.

Remark. Note the asymmetry of the situation. It is necessary to distinguish (cf. [20, VIII.3]) between the *coimage* of θ, which is a quotient group of $h(X)$, and the *image* of θ, which in interesting cases is only a *subset* of $h(X)$, not a subgroup (otherwise Lemma 3.9 would be available).

Proof. For (a), we put $x=\theta(y)$ in (ii) to see that θ is idempotent. If we put $x=0$ instead, we see that $\theta(y-\theta(y))=0$, which implies that $\iota-\theta$ is idempotent.

For (b), we have just seen that $\operatorname{Im}(\iota-\theta)\subset\operatorname{Ker}\theta$. The opposite inclusion is trivial, because if $\theta(x)=0$, we can write $x=(\iota-\theta)(x)$.

To see that $h'(X)$ is a subgroup, we first note that $0\in h'(X)$ by (i). Take any $z\in h'(X)$, which we may write as $z=y-\theta(y)$. Then by (ii), $x+z\in h'(X)$ if and only if $x\in h'(X)$. Therefore by Lemma 3.8 (which did not require θ to be additive), $h'(-)$ is a cohomology theory.

This allows us to define the coimage $h''(X)$ in (c) as an abelian group. By (ii) and (b), $\overline{\theta}$ in (e) is well defined and provides the inverse bijection to $\operatorname{Im}\theta\subset h(X)\to h''(X)$. By Lemma 3.8, $\operatorname{Im}\theta$ and hence $h''(-)$ satisfy the axioms (3.1), and h'' is a cohomology theory. Then (d) combines (b) and (c).

For (f), we represent π by a fibration $H\to H''$, which is an H-map of H-spaces. Then $\overline{\theta}$ is represented by a section. It follows from the short exact sequence (3.12) that the fibre of π represents h'. □

Graded cohomology. A *graded cohomology theory* $E^*(-)$ consists of an ungraded cohomology theory $E^n(-)$ for each integer n, connected by natural *suspension isomorphisms*

$$\Sigma\colon E^n(X)\cong E^{n+1}(S^1\times X, o\times X) \tag{3.13}$$

of abelian groups, much as in Conner-Floyd [12, §4]. By Lemma 3.5, there is a split short exact sequence

$$0 \longrightarrow E^{n+1}(S^1 \times X, o \times X) \longrightarrow E^{n+1}(S^1 \times X) \longrightarrow E^{n+1}(o \times X) \longrightarrow 0. \quad (3.14)$$

For a based space (X, o), Σ induces, with the help of eq. (3.4), the commutative diagram of split exact sequences

$$\begin{array}{ccccc} E^n(X,o) & \longrightarrow & E^n(X) & \longrightarrow & E^n(o) \\ \cong \vdots \Sigma & & \cong \downarrow \Sigma & & \cong \downarrow \Sigma \\ E^{n+1}(\Sigma X, o) & \longrightarrow & E^{n+1}\left(\dfrac{S^1 \times X}{o \times X}, o\right) & \longrightarrow & E^{n+1}(S^1 \times o, o) \end{array} \quad (3.15)$$

whose bottom row comes from Lemma 3.5, where the *suspension* of X is

$$\Sigma X = S^1 \wedge X = \frac{S^1 \times X}{S^1 \vee X} \cong \frac{S^1 \times X}{o \times X} \Big/ S^1 \times o \ .$$

We deduce the more commonly used *reduced* suspension isomorphism $\Sigma\colon E^n(X, o) \cong E^{n+1}(\Sigma X, o)$. In view of eq. (3.3), we recover eq. (3.13) as a special case.

By iteration of eq. (3.13), or analogy, there are k-fold suspension isomorphisms for all $k > 0$

$$\Sigma^k\colon E^n(X) \cong E^{n+k}(S^k \times X, o \times X) \ . \quad (3.16)$$

Theorem 3.17. *Any graded cohomology theory $E^*(-)$ is represented in Ho by an Ω-spectrum $n \mapsto \underline{E}_n$, consisting of H-spaces $\underline{E}_n$ equipped with universal elements $\iota_n \in E^n(\underline{E}_n, o) \subset E^n(\underline{E}_n)$ and isomorphisms (in Ho) of H-spaces $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$.*

Proof. Theorem 3.6 provides the *H*-spaces $\underline{E}_n$ and elements ι_n. Then as a functor of X, the sequence (3.14) is represented by the fibration of *H*-spaces

$$\Omega \underline{E}_{n+1} \longrightarrow \underline{E}_{n+1}^{S^1} \longrightarrow \underline{E}_{n+1}^{\{o\}}$$

(which is not to be confused with the path space fibration). In particular,

$$E^{n+1}(S^1 \times X, o \times X) \cong \mathsf{Ho}(X, \Omega \underline{E}_{n+1}), \quad (3.18)$$

and eq. (3.13) is represented by the desired isomorphism $\underline{E}_n \simeq \Omega \underline{E}_{n+1}$. □

Similarly, Σ^k in eq. (3.16) is represented by the iterated homotopy equivalence $\underline{E}_n \simeq \Omega^k \underline{E}_{n+k}$.

We find it more convenient to work with the left adjunct $\Sigma \underline{E}_n \to \underline{E}_{n+1}$ of the isomorphism. We introduce a sign, which is suggested by section 9.

Definition 3.19. For each n, we define the based structure map $f_n\colon \Sigma \underline{E}_n \to \underline{E}_{n+1}$ by

$$f_n^* \iota_{n+1} = (-1)^n \Sigma \iota_n \qquad \text{in } E^{n+1}(\Sigma \underline{E}_n, o). \quad (3.20)$$

Theorem 3.17 gives a 1–1 correspondence between cohomology classes and maps. We suspend in both senses and compare.

Lemma 3.21. *Given a based space X, suppose that the class $x \in E^n(X,o)$ corresponds to the based map $x_U\colon X \to \underline{E}_n$. Then the map $f_n \circ \Sigma x_U\colon \Sigma X \to \Sigma \underline{E}_n \to \underline{E}_{n+1}$ corresponds to the class $(-1)^n \Sigma x \in E^{n+1}(\Sigma X, o)$* (see diag. (3.15)).

Proof. In $E^*(\Sigma X, o)$, we have $(\Sigma x_U)^* f_n^* \iota_{n+1} = (-1)^n (\Sigma x_U)^* \Sigma \iota_n = (-1)^n \Sigma x$. □

Multiplicative graded cohomology. The cohomology theory $E^*(-)$ is *multiplicative* if $E^*(X)$ is naturally a commutative graded ring (with unit element 1_X and the customary signs) and eq. (3.13) is an isomorphism of $E^*(X)$-modules of degree 1, where we use the projection $p_2\colon S^1 \times X \to X$ to make (3.14) a short exact sequence of $E^*(X)$-modules. Explicitly, $\Sigma(xy) = (-1)^i (p_2^* x)\Sigma y$ for $x \in E^i(X)$ and $y \in E^*(X)$. The *coefficient ring* is defined as $E^* = E^*(T)$.

The natural ring structure on $E^*(X)$ is equivalent to having natural *cross product* pairings

$$\times\colon E^k(X) \times E^m(Y) \longrightarrow E^{k+m}(X \times Y)$$

that are biadditive, commutative, associative, and have $1_T \in E^*(T)$ as the unit. They may be defined in terms of the ring structure as $x \times y = (p_1^* x)(p_2^* y)$; conversely, given $x, y \in E^*(X)$, we recover $xy = \Delta^*(x \times y)$, using the diagonal map $\Delta\colon X \to X \times X$.

By means of $X \cong T \times X$, $E^*(X)$ becomes a module over $E^* = E^*(T)$, and we may rewrite the $\times$-product more usefully as

$$\times\colon E^*(X) \otimes E^*(Y) \longrightarrow E^*(X \times Y), \tag{3.22}$$

where the tensor product is taken over E^*. On the rare occasion that this is an isomorphism, it is called the *cohomology Künneth isomorphism*.

Definition 3.23. We define the *canonical generator* $u_1 \in E^1(S^1, o) \subset E^1(S^1)$ as corresponding to $\Sigma 1_T \in E^1(S^1 \times T, o \times T) \cong E^1(S^1, o)$, by taking $X = T$ in eq. (3.13).

Then by naturality, for any $x \in E^n(X)$ we have

$$\Sigma x = u_1 \times x \qquad \text{in } E^{n+1}(S^1 \times X, o \times X). \tag{3.24}$$

Similarly, $\Sigma^k x = u_k \times x$ in eq. (3.16), where the canonical generator $u_k \in E^k(S^k, o)$ corresponds to $\Sigma^k 1_T$.

Theorem 3.25. *A multiplicative structure on the graded cohomology theory $E^*(-)$ is represented by multiplication maps $\phi\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_k \wedge \underline{E}_m \to \underline{E}_{k+m}$ and a unit map $\eta\colon T \to \underline{E}_0$, such that:*

(a) *The cross product of $x \in E^k(X)$ and $y \in E^m(Y)$ is*

$$x \times y\colon X \times Y \xrightarrow{x \times y} \underline{E}_k \times \underline{E}_m \xrightarrow{\phi} \underline{E}_{k+m}; \tag{3.26}$$

(b) *The unit element of $E^*(X)$ is $1_X = \eta \circ q\colon X \to T \to \underline{E}_0$;*

(c) *Given $v \in E^h$, the module action $v: E^k(-) \to E^{k+h}(-)$ is represented by the map*

$$\xi v: \underline{E}_k \cong T \times \underline{E}_k \xrightarrow{v \times 1} \underline{E}_h \times \underline{E}_k \xrightarrow{\phi} \underline{E}_{k+h}; \tag{3.27}$$

(d) *The structure map $\Sigma \underline{E}_n \to \underline{E}_{n+1}$ of Defn. 3.19 is*

$$f_n: \Sigma \underline{E}_n = S^1 \wedge \underline{E}_n \xrightarrow{(-1)^n u_1 \wedge 1} \underline{E}_1 \wedge \underline{E}_n \xrightarrow{\phi} \underline{E}_{n+1}. \tag{3.28}$$

Proof. We take $\iota_k \times \iota_m \in E^{k+m}(\underline{E}_k \times \underline{E}_m)$ as ϕ and $1_T \in E^0(T)$ as η; then (a) and (b) follow by naturality. By definition, vx corresponds to $v \times x \in E^{k+h}(T \times X)$. Thus by eq. (3.26), scalar multiplication by v in $E^*(X)$ is represented by eq. (3.27); equivalently, we use the identity $vx = (v1)x$ in $E^*(X)$. By eq. (3.24), the map (3.28) takes ι_{n+1} to $(-1)^n \Sigma \iota_n$ and is therefore f_n. □

From now on, we shall assume that $E^*(-)$ is multiplicative. We shall have much more to say (in Cor. 7.8) about the spaces $\underline{E}_n$, once we have the language.

Example: KU. The key to making a graded cohomology theory out of $KU(-)$ is Bott periodicity, in the following form. (See Atiyah-Bott [6] or Husemoller [15, Ch. 10] for an elegant proof that is close to our point of view.) It gives us everything we need to build a periodic graded cohomology theory.

Theorem 3.29. (Bott) *The Hopf line bundle ξ over $\mathbb{C}P^1 \cong S^2$ induces an isomorphism*

$$([\xi] - 1) \times -: KU(X) \cong KU(S^2 \times X, o \times X)$$

for any space X.

Definition 3.30. We define the *graded* cohomology theory $KU^*(-)$ as having the representing spaces $\underline{KU}_{2n} = \mathbb{Z} \times BU$ and $\underline{KU}_{2n+1} = U$ for all integers n, so that $KU^{2n}(X) = \mathit{Ho}(X, \mathbb{Z} \times BU) = KU(X)$ and $KU^{2n+1}(X) = \mathit{Ho}(X, U)$.

In odd degrees, we use the suspension isomorphism

$$KU^{2n+1}(X) \cong KU^{2n+2}(S^1 \times X, o \times X) \cong \mathit{Ho}(X, \Omega(\mathbb{Z} \times BU)) \tag{3.31}$$

represented by $U \simeq \Omega BU = \Omega(\mathbb{Z} \times BU)$. In even degrees, rather than specify $\Sigma: KU^{2n}(X) \cong KU^{2n+1}(S^1 \times X, o \times X)$ directly, we use the double suspension isomorphism $\Sigma^2: KU^{2n}(X) \cong KU^{2n+2}(S^2 \times X, o \times X)$ provided by Thm. 3.29.

The ring structure on $KU(X)$ makes $KU^*(X)$ multiplicative, with the help of eq. (3.31). (The only case that presents any difficulty is

$$KU^{2m+1}(X) \times KU^{2n+1}(X) \longrightarrow KU^{2(m+n+1)}(X),$$

which requires another appeal to Thm. 3.29.)

The coefficient ring is clearly $\mathbb{Z}[u, u^{-1}]$, where we *define* $u \in KU^{-2} = KU(T) = \mathbb{Z}$ as the copy of 1. To keep the degrees straight, all we have to do is insert appropriate powers u^n everywhere. (It is traditional to simplify matters by setting $u = 1$, thus making $KU^*(-)$ a $\mathbb{Z}/2$-graded cohomology theory; however, this strategy is not available to us, as it would allow only operations that preserve this identification.) For example, Thm. 3.29 provides the canonical element

$$u_2 = u^{-1}([\xi] - 1) \qquad \text{in } KU^2(S^2, o) \subset KU^2(S^2). \tag{3.32}$$

The skeleton filtration. The cohomology $E^*(X)$ is usually uncountable for infinite X, which makes Künneth isomorphisms (3.22) unlikely without some kind of completion. This suggests that it ought to be given a topology.

Given any space X (which we take as a CW-complex), the *skeleton filtration* of $E^*(X)$ is defined by

$$F^s E^*(X) = \operatorname{Ker}[E^*(X) \longrightarrow E^*(X^{s-1})] = \operatorname{Im}[E^*(X, X^{s-1}) \longrightarrow E^*(X)] \tag{3.33}$$

for $s \geqslant 0$, where X^n denotes the n-skeleton of X, and this filtration is natural. It is a decreasing filtration by ideals,

$$E^*(X) = F^0 E^*(X) \supset F^1 E^*(X) \supset F^2 E^*(X) \supset \dots$$

Moreover, it is multiplicative,

$$(F^s E^*(X))(F^t E^*(X)) \subset F^{s+t} E^*(X) \qquad \text{(for all } s,t), \tag{3.34}$$

because $X^{s-1} \times X \cup X \times X^{t-1}$ contains the $(s+t-1)$-skeleton of $X \times X$, as in [34, Prop. 13.67].

When X is connected, with basepoint o, we recognize $F^1 E^*(X)$ from the exact sequence (3.2) as the augmentation ideal

$$F^1 E^*(X) = E^*(X, o) = \operatorname{Ker}[E^*(X) \longrightarrow E^*(o) = E^*]. \tag{3.35}$$

Filtered modules. We need to be somewhat more general.

Definition 3.36. Given any E^*-module M filtered by submodules $F^a M$, the associated *filtration topology* on M has a basis consisting of the cosets $x + F^a M$, for all $x \in M$ and all indices a.

For this to be a topology, we need the *directedness* condition, that given $F^a M$ and $F^b M$, there exists c such that $F^c M \subset F^a M \cap F^b M$.

We consider the projections $M \to M/F^a M$. We observe that M is Hausdorff if and only if the induced homomorphism $M \to \lim_a M/F^a M$ is monic, and that M is complete (in the sense that all Cauchy sequences $n \mapsto x_n \in M$ converge) if and only if it is epic. (A Cauchy sequence is one that satisfies $x_m - x_n \to 0$. However, its limit is unique only if M is Hausdorff.)

Definition 3.37. We define the *completion* of the filtered module M as $\widehat{M} = \lim_a M/F^a M$. The projections $M \to M/F^a M$ lift to define the *completion map* $M \to \widehat{M}$.

We shall observe in section 6 that $\widehat{M}$ has a canonical filtration that makes it complete Hausdorff.

In particular, we have the *skeleton topology* on $E^*(X)$. It is of course discrete when X is finite-dimensional. Since $E^*(X)/F^sE^*(X) \subset E^*(X^{s-1})$, Milnor's short exact sequence [24, Lemma 2]

$$0 \longrightarrow \lim_s{}^1 E^{k-1}(X^s) \longrightarrow E^k(X) \longrightarrow \lim_s E^k(X^s) \longrightarrow 0 \tag{3.38}$$

may be written in the form

$$0 \longrightarrow F^\infty E^k(X) \longrightarrow E^k(X) \longrightarrow \lim_s E^k(X)/F^sE^k(X) \longrightarrow 0, \tag{3.39}$$

where $F^\infty E^k(X) = \bigcap_s F^sE^k(X)$ and we recognize the limit term as the completion of $E^k(X)$. Thus the skeleton filtration is always complete, but examples show that it need not be Hausdorff. The elements of $F^\infty E^k(X)$ are called *phantom* classes. In this case, the completion is simply the *quotient* of $E^k(X)$ by the phantom classes.

Remark. The terminology is unfortunate, but standard. The word "complete" is sometimes understood to include "Hausdorff", which would leave us with no word to describe our situation. Here, completion is really Hausdorffification.

4. Generalized homology and duality

Associated to each of our multiplicative cohomology theories $E^*(-)$ is a multiplicative homology theory $E_*(-)$, whose coefficient ring $E_*(T)$ we can identify with $E^*(T) = E^*$. In this section, we study the relationship between them. We shall see in section 9 that the situation is quite general. In line with a suggestion of Adams [1], we have two main tools: a Künneth isomorphism, Thm. 4.2, and a universal coefficient isomorphism, Thm. 4.14. (With our emphasis on cohomology, we *never* write E_* for E^* or E_{-n} for E^n, as is often done.)

Homology too has external cross products

$$\times: E_*(X) \otimes E_*(Y) \longrightarrow E_*(X \times Y), \tag{4.1}$$

that make $E_*(X)$ an E^*-module. This is more often than (3.22) an isomorphism.

Theorem 4.2. *Assume that $E_*(X)$ or $E_*(Y)$ is a free or flat E^*-module. Then the pairing* (4.1) *induces the* Künneth isomorphism $E_*(X \times Y) \cong E_*(X) \otimes E_*(Y)$ *in homology.*

Proof. See Switzer [34, Thm. 13.75]. Assume that $E_*(Y)$ is flat. The idea is that as X varies, (4.1) is then a natural transformation of homology theories, which is an isomorphism for $X = T$ and therefore generally. □

This is particularly useful for $E = K(n)$ or $H(\mathbb{F}_p)$, for then all E^*-modules are free. When $E_*(X)$ is free (or flat), we can define the comultiplication

$$\psi: E_*(X) \xrightarrow{\Delta_*} E_*(X \times X) \xleftarrow{\cong} E_*(X) \otimes E_*(X), \tag{4.3}$$

which, together with the counit $\epsilon = q_*: E_*(X) \to E_*(T) = E^*$ induced by $q: X \to T$, makes $E_*(X)$ an E^*-coalgebra.

The homology analogue of Milnor's exact sequence (3.38) is simply [24, Lemma 1]

$$E_n(X) = \operatorname*{colim}_s E_n(X^s) \ . \tag{4.4}$$

Duality. Our only real use of homology is the Kronecker pairing

$$\langle -, - \rangle : E^*(X) \otimes E_*(X) \longrightarrow E^*,$$

which is E^*-bilinear in the sense that $\langle vx, z\rangle = v\langle x, z\rangle = (-1)^{hi}\langle x, vz\rangle$ for $x \in E^i(X)$, $z \in E_*(X)$, and $v \in E^h$. We convert it to the right adjunct form

$$d: E^*(X) \longrightarrow DE_*(X) \tag{4.5}$$

by defining $(dx)z = \langle x, z\rangle$. Here, DM denotes the *dual module* $\operatorname{Hom}^*(M, E^*)$ of any E^*-module M, defined by $(DM)^n = \operatorname{Hom}^n(M, E^*)$. (But we still like to write the evaluation as $\langle -, - \rangle : DM \otimes M \to E^*$.) This is the correct indexing to make DM an E^*-module and d a homomorphism of E^*-modules. It is reasonable to ask whether d is an isomorphism. We shall give a useful answer in Thm. 4.14.

There is an obvious natural pairing $\zeta_D: DM \otimes DN \to D(M \otimes N)$, defined by

$$\langle \zeta_D(f \otimes g), x \otimes y\rangle = (-1)^{\deg(x)\deg(g)}\langle f, x\rangle\langle g, y\rangle \qquad \text{in } E^*. \tag{4.6}$$

All these structure maps fit together in the commutative diagram

$$\begin{array}{ccccc} E^*(X) \otimes E^*(Y) & \xrightarrow{d\otimes d} & DE_*(X) \otimes DE_*(Y) & \xrightarrow{\zeta_D} & D(E_*(X) \otimes E_*(Y)) \\ \big\downarrow{\scriptstyle\times} & & & & \big\uparrow{\scriptstyle D\times} \\ E^*(X \times Y) & & \xrightarrow{\qquad\qquad d \qquad\qquad} & & DE_*(X \times Y) \end{array} \tag{4.7}$$

which, algebraically, states that $\langle x \times y, a \times b\rangle = \pm\langle x, a\rangle\langle y, b\rangle$. Its significance is that if any four of the maps are isomorphisms, so is the fifth.

We need more. We need a topology on $DE_*(X)$ to match the topology on $E^*(X)$. There is an obvious candidate. (We stress that the homology $E_*(X)$ invariably has the discrete topology.)

Definition 4.8. Given any E^*-module M, we define the *dual-finite filtration* on $DM = \operatorname{Hom}^*(M, E^*)$ as consisting of the submodules $F^L DM = \operatorname{Ker}[DM \to DL]$, where L runs through all finitely generated submodules of M. It gives rise by Defn. 3.36 to the *dual-finite topology* on DM.

This filtration is obviously Hausdorff, and we see it is complete by writing $DM = \lim_L DL$, the inverse limit of discrete E^*-modules. It certainly makes d continuous, because any finitely generated $L \subset E_*(X)$ lifts to $E_*(X^s)$ for some s, by eq. (4.4).

The profinite filtration. The skeleton filtration is adequate for discussing spaces *of finite type* (those having finite skeletons), but not all our spaces have finite type. We need a somewhat coarser topology that has better properties and a better chance of making d in (4.5) a homeomorphism.

Definition 4.9. Given a CW-complex X, we define the *profinite filtration* of $E^*(X)$ as consisting of all the ideals

$$F^a E^*(X) = \operatorname{Ker}[E^*(X) \longrightarrow E^*(X_a)] = \operatorname{Im}[E^*(X, X_a) \longrightarrow E^*(X)],$$

where X_a runs through all finite subcomplexes of X. We call the resulting filtration topology (see Defn. 3.36) the *profinite topology*.

The particular indexing set is not important and we rarely specify it. The ideals $F^a E^*(X)$ do form a directed system: given F^a and F^b, there exists X_c such that $F^c \subset F^a \cap F^b$, namely $X_c = X_a \cup X_b$.

This is our preferred topology on $E^*(X)$, for all spaces X. It is natural in X: given a map $f: X \to Y$, $f^*: E^*(Y) \to E^*(X)$ is continuous, because for each finite $X_a \subset X$, there is a finite $Y_b \subset Y$ for which $f X_a \subset Y_b$, so that $f^*(F^b) \subset F^a$. Indeed, it is the coarsest natural topology that makes $E^*(X)$ discrete for all finite X.

Of course, it coincides with the skeleton topology when X has finite type. However, it has one elementary property that the skeleton topology lacks.

Lemma 4.10. *For any disjoint union $X = \coprod_\alpha X_\alpha$, the profinite topology makes $E^k(X) \cong \prod_\alpha E^k(X_\alpha)$ a homeomorphism.* □

Definition 4.11. For any space X, we define its *completed E-cohomology* $E^*(X)\hat{}$ as the completion of $E^*(X)$ with respect to the profinite filtration.

A result of Adams [2, Thm. 1.8] shows that the profinite topology is always complete, that

$$E^*(X) \longrightarrow \lim_a E^*(X)/F^a E^*(X) \subset \lim_a E^*(X_a)$$

is surjective, which allows us to identify canonically

$$E^*(X)\hat{} = E^*(X) / \bigcap_a F^a E^*(X) \cong \lim_a E^*(X_a) \tag{4.12}$$

for all spaces X. This completed cohomology is not at all new; it was discussed at some length by Adams [ibid.].

As before, the topology on $E^*(X)$ need not be Hausdorff. The intersection $\bigcap_a F^a E^*(X)$ (which contains $F^\infty E^*(X)$) need not vanish, and its elements are called *weakly phantom* classes. In practice, one hopes there are none, so that $E^*(X)\hat{} = E^*(X)$.

Strong duality. We note that the morphism d in eq. (4.5) remains continuous with the profinite topology on $E^*(X)$.

Definition 4.13. We say the space X *has strong duality* if $d: E^*(X) \to DE_*(X)$ in (4.5) is a homeomorphism between the profinite topology on $E^*(X)$ and the dual-finite topology on $DE_*(X)$ (see Defn. 4.8).

Theorem 4.14. *Assume that $E_*(X)$ is a free E^*-module. Then X has strong duality, i. e. $d: E^*(X) \cong DE_*(X)$ is a homeomorphism between the profinite topology on $E^*(X)$ and the dual-finite topology on $DE_*(X)$. In particular, $E^*(X)$ is complete Hausdorff.*

This is best viewed as a stable result, and will be included in Thm. 9.25.

Künneth homeomorphisms. The cohomology Künneth homomorphism (3.22) is rarely an isomorphism, but our chances improve if we complete it. Generally, given E^*-modules M and N filtered by submodules F^aM and F^bN, we filter $M \otimes N$ by the submodules

$$\begin{aligned} F^{a,b}(M \otimes N) &= \operatorname{Im}[(F^aM \otimes N) \oplus (M \otimes F^bN) \longrightarrow M \otimes N] \\ &= \operatorname{Ker}[M \otimes N \longrightarrow (M/F^aM) \otimes (N/F^bN)] \end{aligned} \tag{4.15}$$

where the second form follows from the right exactness of $\otimes$. (Often, but not always, $F^aM \otimes N$ and $M \otimes F^bN$ are submodules of $M \otimes N$.) We construct the *completed tensor product* $M \mathbin{\widehat{\otimes}} N$ as the completion of $M \otimes N$ with respect to this filtration.

The filtration makes $\times$-multiplication (3.22) continuous, because given $Z_c \subset Z = X \times Y$, the inverse image of $F^cE^*(Z)$ contains $F^{a,b}(E^*(X) \otimes E^*(Y))$, provided $Z_c \subset X_a \times Y_b$. We may therefore complete it to

$$\times \colon E^*(X) \mathbin{\widehat{\otimes}} E^*(Y) \longrightarrow E^*(X \times Y)\hat{} \tag{4.16}$$

and ask whether this is an isomorphism. Again, we need more than a bijection.

Definition 4.17. If the pairing (4.16) is a homeomorphism and $E^*(X \times Y)\hat{} = E^*(X \times Y)$, we call the resulting homeomorphism $E^*(X \times Y) \cong E^*(X) \mathbin{\widehat{\otimes}} E^*(Y)$ a *Künneth homeomorphism.* (Note that we require $E^*(X \times Y)$ to be already Hausdorff.)

Similarly, $\zeta_D \colon DM \otimes DN \to D(M \otimes N)$ is continuous. We therefore complete diag. (4.7) to

$$\begin{array}{ccccc} E^*(X) \mathbin{\widehat{\otimes}} E^*(Y) & \xrightarrow{d \otimes d} & DE_*(X) \mathbin{\widehat{\otimes}} DE_*(Y) & \xrightarrow{\zeta_D} & D(E_*(X) \otimes E_*(Y)) \\ \Big\downarrow \times & & & & \Big\uparrow D\times \\ E^*(X \times Y) & & \xrightarrow{\qquad d \qquad} & & DE_*(X \times Y) \end{array} \tag{4.18}$$

Theorem 4.19. *Assume that $E_*(X)$ and $E_*(Y)$ are free E^*-modules. Then we have the Künneth homeomorphism $E^*(X \times Y) \cong E^*(X) \mathbin{\widehat{\otimes}} E^*(Y)$ in cohomology.*

Proof. The hypotheses, with the help of Thms. 4.2 and 4.14, make (4.18) a diagram of homeomorphisms. (For ζ_D, we may appeal to Lemma 6.15(e).) □

5. Complex orientation

All five of our examples of cohomology theories $E^*(-)$ are equipped with a complex orientation. This will provide Chern classes and a good supply of spaces with free E-homology.

The Chern class of a line bundle. Denote by $M(\xi)$ the Thom space of a vector bundle ξ. A *complex orientation* (for line bundles) assigns to each complex line bundle θ over any space X a natural Thom class $t(\theta) \in E^2(M(\theta))$, such that for the line bundle 1 over a point, $t(1) = u_2 \in E^2(S^2)$.

Remark. We assume here a specific homeomorphism between S^2 and the one-point compactification of $\mathbb{C}$, as determined by some orientation convention. In some contexts, it is useful to allow the slightly more general normalization $t(1) = \lambda u_2$, where $\lambda \in E^*$ may be any invertible element; but then $\lambda^{-1}t(\theta)$ is a Thom class in the stricter sense. We have no need here of this extra flexibility.

For our purposes, a closely related concept is more useful.

Definition 5.1. Given E, a *line bundle Chern class* assigns to each complex line bundle θ over any space X a class $x(\theta) \in E^2(X)$, called the *(first) E-Chern class* of θ, that satisfies the axioms:

(i) It is *natural*: Given a map $f: X' \to X$ and a line bundle θ over X, for the induced line bundle $f^*\theta$ over X' we have $x(f^*\theta) = f^*x(\theta)$ in $E^2(X')$;

(ii) It is *normalized*: For the Hopf line bundle ξ over $\mathbb{C}P^1 \cong S^2$, we have $x(\xi) = u_2 \in E^2(S^2)$, the canonical generator of $E^*(S^2)$.

It is easy to see that $x(\theta) = i^*t(\theta)$ satisfies the axioms, where $i: X \subset M(\theta)$ denotes the inclusion of the zero section. (Conversely, Connell [11, Thms. 4.1, 4.5] shows that every line bundle Chern class arises in this way, from a unique complex orientation.)

For $E = KU$, it is obvious from eq. (3.32) that

$$x(\theta) = u^{-1}([\theta] - 1) \in KU^2(X) \tag{5.2}$$

is a line bundle Chern class.

Complex projective spaces. Of course, Chern classes need not exist for general E. As the Hopf line bundle ξ over $\mathbb{C}P^\infty = BU(1)$ is universal, it is enough to have $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$. We start with $\mathbb{C}P^n$.

Lemma 5.3. (Dold) *Assume that the Hopf line bundle ξ over $\mathbb{C}P^n$ has the Chern class $x = x(\xi) \in E^2(\mathbb{C}P^n)$, where $n \geqslant 0$. Then:*

(a) *$E^*(\mathbb{C}P^n) = E^*[x : x^{n+1} = 0]$, a truncated polynomial algebra over E^*;*

(b) *We have the duality isomorphism $d: E^*(\mathbb{C}P^n) \cong DE_*(\mathbb{C}P^n)$;*

(c) *$E_*(\mathbb{C}P^n)$ is the free E^*-module with basis $\{\beta_0, \beta_1, \beta_2, \dots, \beta_n\}$, where $\beta_i \in E_{2i}(\mathbb{C}P^n)$ is defined as dual to x^i.*

Proof. See Adams [3, Lemmas II.2.5, II.2.14] or Switzer [34, Props. 16.29, 16.30]. The idea is that the presence of x forces the Atiyah-Hirzebruch spectral sequences for both $E^*(\mathbb{C}P^n)$ and $E_*(\mathbb{C}P^n)$ to collapse. (There is of course no topology on $E^*(\mathbb{C}P^n)$ to check.) One has to verify that $x^{n+1} = 0$ exactly. In terms of the skeleton filtration, $x \in F^2E^*(\mathbb{C}P^n)$. Then by eq. (3.34), $x^{n+1} \in F^{2n+2}E^*(\mathbb{C}P^n) = 0$. □

The result for $\mathbb{C}P^\infty$ follows immediately, by eq. (4.4) and Thm. 4.14, and also clarifies exactly how non-unique a complex orientation is. Similarly named elements correspond under inclusion.

Lemma 5.4. (Dold) *Assume that we have the Chern class* $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$. *Then:*

(a) $E^*(\mathbb{C}P^\infty) = E^*[[x]]$, *the algebra of formal power series in* x *over* E^*, *filtered by powers of the ideal* (x);

(b) *We have strong duality* $d: E^*(\mathbb{C}P^\infty) \cong DE_*(\mathbb{C}P^\infty)$;

(c) $E_*(\mathbb{C}P^\infty)$ *is the free* E^**-module with basis* $\{\beta_0, \beta_1, \beta_2, \beta_3, \ldots\}$, *where* $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ *is dual to* x^i *for* $i \geqslant 0$. □

Chern classes of a vector bundle. We proceed to BU by way of $\mathbb{C}P^\infty = BU(1) \subset BU$. A useful intermediate step is the torus group $T(n) = U(1) \times \ldots \times U(1)$, for which $BT(n) = BU(1) \times \ldots \times BU(1)$. We have Künneth isomorphisms

$$E_*(BT(n)) \cong E_*(\mathbb{C}P^\infty) \otimes E_*(\mathbb{C}P^\infty) \otimes \ldots \otimes E_*(\mathbb{C}P^\infty)$$

in homology by Thm. 4.2, and

$$E^*(BT(n)) = E^*[[x_1, x_2, \ldots, x_n]] \cong E^*(\mathbb{C}P^\infty) \widehat{\otimes} \ldots \widehat{\otimes} E^*(\mathbb{C}P^\infty) \tag{5.5}$$

in cohomology by Thm. 4.19, where $x_i = p_i^* x(\xi) = x(p_i^*\xi)$.

Lemma 5.6. *Assume* E *has a line bundle Chern class. Then:*

(a) $E^*(BU) = E^*[[c_1, c_2, c_3, \ldots]]$, *where* $c_i \in E^{2i}(BU)$ *restricts to the* i*th elementary symmetric function of the* $x_j \in E^*(BT(n))$ *for any* $n \geqslant i$, *and* $E^*(BU(n)) = E^*[[c_1, c_2, \ldots, c_n]]$ *is the quotient of this with* $c_i = 0$ *for all* $i > n$;

(b) *We have strong duality* $d: E^*(BU) \cong DE_*(BU)$ *and* $d: E^*(BU(n)) \cong DE_*(BU(n))$, *and in particular,* $E^*(BU)$ *and* $E^*(BU(n))$ *are Hausdorff;*

(c) $E_*(BU) = E^*[\beta_1, \beta_2, \beta_3, \ldots]$, *where* β_i *is inherited from* $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ *by* $\mathbb{C}P^\infty = BU(1) \subset BU$ *and* $\beta_0 \mapsto 1$, *and* $E_*(BU(n)) \subset E_*(BU)$ *is the* E^**-free submodule spanned by all monomials of polynomial degree* $\leqslant n$ *in the* β_i.

Proof. See Adams [3, Lemma II.4.1] or Switzer [34, Thms. 16.31, 16.32]. □

From this it is immediate, as in Conner-Floyd [12, Thm. 7.6], Adams [3, Lemma II.4.3], or Switzer [34, Thm. 16.2], that general Chern classes exist. The axioms determine them uniquely on $BT(n)$, and this is enough.

Theorem 5.7. *Assume* E *has a complex orientation. Then there exist uniquely* E*-*Chern classes $c_i(\xi) \in E^{2i}(X)$, *for* $i > 0$ *and any complex vector bundle* ξ *over any space* X, *that satisfy the axioms:*

(i) *Naturality:* $c_i(f^*\xi) = f^*c_i(\xi) \in E^{2i}(X')$ *for any vector bundle* ξ *over* X *and any map* $f: X' \to X$;

(ii) *For any* n*-plane bundle* ξ, $c_i(\xi) = 0$ *for all* $i > n$;

(iii) *For any line bundle* ξ, $c_1(\xi) = x(\xi)$;

(iv) *For any vector bundles ξ and η over X, we have the* Cartan formula

$$c_k(\xi \oplus \eta) = c_k(\xi) + \sum_{i=1}^{k-1} c_{k-i}(\xi) c_i(\eta) + c_k(\eta) \qquad \text{in } E^*(X). \qquad \square$$

The unitary groups. We study the unitary group U by means of the Bott map $b\colon \Sigma(\mathbb{Z} \times BU) \to U$, one of the structure maps of the Ω-spectrum KU. The Hopf line bundle θ over $\mathbb{C}P^{n-1}$ defines the *unbased* inclusion

$$\mathbb{C}P^{n-1} \subset \mathbb{C}P^\infty = BU(1) \subset BU \cong 1 \times BU \subset \mathbb{Z} \times BU. \qquad (5.8)$$

Its fibre over the point $A \in \mathbb{C}P^{n-1}$ is $\mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$, where we also regard A as a line in $\mathbb{C}^n$.

When we apply Bott periodicity as in Thm. 3.29, we obtain the element

$$([\xi]-1) \times [\theta] = [(\xi \otimes \theta) \oplus \theta^\perp] - n \qquad \text{in } KU(S^2 \times \mathbb{C}P^{n-1}),$$

where $\theta^\perp$ denotes the orthogonal complement bundle having the fibre $\mathrm{Hom}_{\mathbb{C}}(A^\perp, \mathbb{C})$ over $A \in \mathbb{C}P^{n-1}$. The n-plane bundle $(\xi \otimes \theta) \oplus \theta^\perp$ is, by design, trivial over $D^2 \times \mathbb{C}P^{n-1}$ for any 2-disk $D^2 \subset S^2$, and its clutching function

$$h\colon S^1 \times \mathbb{C}P^{n-1} \longrightarrow U(n) \qquad (5.9)$$

induces the Bott map b, restricted as in (5.8). Here, $S^1 \subset \mathbb{C}$ is to be regarded as the circle group. We read off that (for suitable choices of orientation) $h(z, A)\colon \mathbb{C}^n \to \mathbb{C}^n$ is the well-known map that preserves $A^\perp$ and on A is multiplication by z; explicitly, on any vector $Y \in \mathbb{C}^n$, it is

$$h(z, A)Y = Y + (z-1)\, \langle Y, X \rangle\, X \qquad \text{in } \mathbb{C}^n, \qquad (5.10)$$

where X is any unit vector in A. (From the group-theoretic point of view, the image of h is the union of all the conjugates of $U(1) \subset U(n)$.)

In [40], Yokota used (essentially) this map h and the multiplication in $U(n)$ to construct explicit cell decompositions of $SU(n)$ and hence $U(n)$, and deduce their ordinary (co)homology. The method works equally well for E-(co)homology.

Lemma 5.11. *Assume that E has a line bundle Chern class. Then $E_*(U(n))$ is a free E^*-module with a basis consisting of all the Pontryagin products $\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_k}$, where $n > i_1 > i_2 > \ldots i_k \geqslant 0$, $k \geqslant 0$ (we allow the empty product 1), $\gamma_i = h_*(z \times \beta_i) \in E_{2i+1}(U(n))$ with h as in eq. (5.9), and $z \in E_1(S^1)$ is dual to u_1.*

Proof. Because we are decomposing $U(n)$ rather than $SU(n)$, we use a slightly different (and simpler) decomposition. We regard $U(n)$ as a principal right $U(n-1)$-bundle over S^{2n-1}, with projection map $\pi\colon U(n) \to S^{2n-1}$ given by $\pi g = g e_n$, where $e_n = (0, 0, \ldots, 0, 1) \in \mathbb{C}^n$ and we recognize $U(n-1)$ as the subgroup of $U(n)$ that fixes e_n. Given $g \in U(n) - U(n-1)$, so that $\pi g \neq e_n$, it is easy to solve eq. (5.10), as in [40], for a unique pair (z, A) such that $h(z, A)e_n = \pi g$, which allows us to write $g = h(z, A)g'$ for some $g' \in U(n-1)$. Moreover, $z \neq 1$ and $A \notin \mathbb{C}P^{n-2}$; in other words, $\pi \circ h$ identifies the top cell of $S^1 \times \mathbb{C}P^{n-1}$ with $S^{2n-1} - e_n$.

It follows that the map

$$S^1 \times \mathbb{C}P^{n-1} \times U(n-1) \xrightarrow{h \times 1} U(n) \times U(n-1) \xrightarrow{\mu} U(n)$$

induces the isomorphism in the commutative square

$$\begin{array}{ccc} E_*(S^1 \times \mathbb{C}P^{n-1}) \otimes E_*(U(n-1)) & \longrightarrow & E_*(U(n)) \\ \downarrow & & \downarrow \\ E_*(S^1 \times \mathbb{C}P^{n-1}, K) \otimes E_*(U(n-1)) & \xrightarrow{\cong} & E_*(U(n), U(n-1)) \end{array}$$

where $K = S^1 \times \mathbb{C}P^{n-2} \cup 1 \times \mathbb{C}P^{n-1}$. From Lemma 5.3, we deduce that both vertical arrows are split epic and obtain the decomposition

$$E_*(U(n)) \cong E_*(U(n-1)) \oplus \gamma_{n-1} E_*(U(n-1))$$

of $E_*(U(n))$ as the direct sum (with a shift) of two copies of $E_*(U(n-1))$, as the multiplication by γ_{n-1} is an embedding. The result now follows by induction on n, starting from $U(1) = S^1$.

Alternatively, we apply the Atiyah-Hirzebruch homology spectral sequence to the map h, to deduce that the spectral sequence for $E_*(U(n))$ collapses whenever that for $E_*(\mathbb{C}P^{n-1})$ does. □

Corollary 5.12. *Assume that E has a line bundle Chern class, and that E^* has no 2-torsion. Then $E_*(U) = \Lambda(\gamma_0, \gamma_1, \gamma_2, \ldots)$, an exterior algebra on the generators $\gamma_i = b_*(z \times \beta_i)$, where $b\colon \Sigma(\mathbb{Z} \times BU) \to U$ denotes the Bott map and $\beta_i \in E_{2i}(\mathbb{Z} \times BU)$ is inherited from $\mathbb{C}P^\infty$ by the inclusion* (5.8).

Proof. We let $n \to \infty$ in the Lemma and use eq. (4.4). The homotopy commutativity of U gives $\gamma_j \gamma_i = -\gamma_i \gamma_j$ and hence $\gamma_i^2 = 0$. □

The formal group law. Conspicuous by its absence is any formula for $c_i(\xi \otimes \eta)$. For line bundles, the universal example is $p_1^* \xi \otimes p_2^* \xi$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, where ξ denotes the Hopf line bundle. In view of eq. (5.5), there must be some formula

$$x(\xi \otimes \eta) = x(\xi) + x(\eta) + \sum_{i,j} a_{i,j}\, x(\xi)^i x(\eta)^j = F(x(\xi), x(\eta)) \tag{5.13}$$

that is valid in the universal case, and therefore generally, where

$$F(x, y) = x + y + \sum_{i,j} a_{i,j}\, x^i y^j \qquad \text{in } E^*[[x, y]] \tag{5.14}$$

is a well-defined formal power series with coefficients $a_{i,j} \in E^{-2i-2j+2}$ for $i > 0$ and $j > 0$. (In the common case that the series is infinite, it may be necessary to interpret eq. (5.13) in the completion $E^*(X)\hat{}$ of $E^*(X)$.) By use of the splitting principle (working in $BT(n)$) and heavy algebra, one can in principle determine formulae for $c_i(\xi \otimes \eta)$ for general complex vector bundles.

The series $F(x, y)$ is known as the *formal group law* of E (or more accurately, of its Chern class $x(-)$). It satisfies the three identities:

$$\begin{aligned} &\text{(i)} \ \ F(x, y) = F(y, x); \\ &\text{(ii)} \ \ F(F(x, y), z) = F(x, F(y, z)); \\ &\text{(iii)} \ \ F(x, 0) = x\ . \end{aligned} \tag{5.15}$$

The first two reflect the commutativity and associativity of $\otimes$. The last comes from $\xi \otimes \epsilon \cong \xi$ for a trivial line bundle ϵ, and shows that $F(x,y)$ has no terms of the form $a_{i,0}x^i$ other than x.

In the case $E = KU$, we can write down

$$x(\xi \otimes \eta) = x(\xi) + x(\eta) + ux(\xi)x(\eta) \qquad \text{in } KU^*(X) \tag{5.16}$$

directly from eq. (5.2), since $x(\xi \otimes \eta) = u^{-1}([\xi][\eta] - 1)$; in other words, the formal group law for KU is $F(x,y) = x + y + uxy$.

6. The categories

In this section we introduce the major categories we need, based on the discussion in section 3. We also fix some terminology and notation. Our basic reference for category theory is MacLane [20]. The ground ring throughout is our coefficient ring E^*, a commutative graded ring.

$\mathcal{A}^{\mathrm{op}}$ denotes the *dual category* of any category $\mathcal{A}$. It has a morphism $f^{\mathrm{op}}: Y \to X$ for each morphism $f: X \to Y$ in $\mathcal{A}$. If $\mathcal{A}$ is graded (and therefore additive), $\deg(f^{\mathrm{op}}) = \deg(f)$ and composition in $\mathcal{A}^{\mathrm{op}}$ is given by $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (-1)^{\deg(f)\deg(g)}(g \circ f)^{\mathrm{op}}$.

Set denotes the category of sets. Cartesian products serve as products and disjoint unions as coproducts. The one-point set T is a terminal object, and the empty set is an initial object.

Ho denotes the homotopy category of *unbased* spaces that are homotopy equivalent to a CW-complex. *This will be our main category of spaces.* Milnor proved [23, Prop. 3] that it admits products $X \times Y$, with never any need to retopologize. The one-point space T is a terminal object. Arbitrary disjoint unions serve as coproducts; in particular, any space is the disjoint union of connected spaces. We identify $E^k(X) = \mathit{Ho}(X, \underline{E}_k)$ according to Thm. 3.17.

Of course, any equivalent category will serve as well. We reserve the option of taking any specific space to be a CW-complex, extending constructions to the rest of *Ho* by naturality.

Ho′ denotes the homotopy category of *based* spaces as in *Ho*, where the basepoint o is assumed to be non-degenerate; all maps and homotopies are to preserve the basepoint. Although this category is more common, we use it only rarely. Milnor proved [23, Cor. 3] that the loop space ΩX of such a space X again lies in the category. Finite cartesian products remain products, but the one-point space T becomes a zero object and arbitrary wedges (one-point unions) serve as coproducts. The exact sequence (3.2) identifies $E^k(X, o) = \mathit{Ho}'(X, \underline{E}_k)$.

Stab denotes the stable homotopy category (in any of various equivalent versions, e.g. [3]). It is an additive category, and has the point spectrum as a zero object. Arbitrary wedges of spectra serve as coproducts. It is equipped with a *stabilization* functor $\mathit{Ho}' \to \mathit{Stab}$, which we suppress from our notation. There is a biadditive smash product functor $\wedge$: *Stab* $\times$ *Stab* $\to$ *Stab*, which (up to coherent isomorphisms) is commutative and associative, has the sphere spectrum T^+ as a unit, and is compatible with the smash product in *Ho*$'$. We define the *suspension* $\Sigma X = S^1 \wedge X$, which is therefore compatible with Σ: *Ho*$' \to$ *Ho*$'$.

Stab* denotes the *graded* stable homotopy category; it has the same objects as *Stab*, with maps of any degree as morphisms. It is a graded additive category. We write *Stab*$^n(X,Y) = \{X,Y\}^n$ for the group of maps of degree n (in the conventions of section 2). Given a fixed choice of one of the two isomorphisms $S^1 \simeq T^+$ in *Stab** of degree 1, we define the canonical natural *desuspension isomorphism*

$$\Sigma X = S^1 \wedge X \simeq T^+ \wedge X \simeq X \tag{6.1}$$

of degree 1 for any spectrum X. (We do not give it a symbol.) Composition with it yields isomorphisms, for any $n \geqslant 0$:

$$\{X, \Sigma^n Y\} \cong \{X,Y\}^n; \qquad \{X,Y\}^{-n} \cong \{\Sigma^n X, Y\};$$

which express *Stab** in terms of *Stab* and Σ.

However, there is a difficulty with smash products. Given maps $f: X \to X'$ and $g: Y \to Y'$ of degrees m and n, the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{f \wedge Y} & X' \wedge Y \\ {\scriptstyle X \wedge g}\downarrow & (-1)^{mn} & \downarrow{\scriptstyle X' \wedge g} \\ X \wedge Y' & \xrightarrow{f \wedge Y'} & X' \wedge Y' \end{array}$$

commutes only up to the indicated sign $(-1)^{mn}$, owing to the necessity of shuffling suspension factors. Consequently, the graded smash product is a functor defined *not* on *Stab*$^* \times$ *Stab**, but on a new graded category (which might be called *Stab*$^* \otimes$ *Stab**) with the biadditivity and signs built in. All we need to know is how to compose: given also $f': X' \to X''$ of degree m' and $g': Y' \to Y''$, we have

$$(g' \wedge f') \circ (g \wedge f) = (-1)^{m'n} (g' \circ g) \wedge (f' \circ f) : X \wedge Y \longrightarrow X'' \wedge Y''. \tag{6.2}$$

From the topological point of view, this is the source of the principle of signs (see section 2). For example, a map $f: X \to Y$ of degree n induces, for any W and Z, the homomorphisms of graded groups of degree n:

$$\begin{aligned} f_*&: \textit{Stab}^*(W,X) \to \textit{Stab}^*(W,Y) \quad \text{given by } f_* g = f \circ g; \\ f^*&: \textit{Stab}^*(Y,Z) \to \textit{Stab}^*(X,Z) \quad \text{given by } f^* g = (-1)^{n \deg(g)} g \circ f. \end{aligned} \tag{6.3}$$

Ab denotes the category of abelian groups. It is the prototypical abelian category and needs no review here.

Ab* denotes the graded category of graded abelian groups, graded by all integers (positive and negative).

Mod denotes the additive category of (necessarily graded) E^*-modules, in which the morphisms are E^*-module homomorphisms of degree 0. Degreewise direct products $\prod_\alpha M_\alpha$ and sums $\bigoplus_\alpha M_\alpha$ serve as products and coproducts. It is equipped with the biadditive functor $\otimes\colon \mathit{Mod} \times \mathit{Mod} \to \mathit{Mod}$ (taken over E^*), which is associative, commutative, and has E^* as unit (up to coherent isomorphisms).

We note that the homology functor $E_*(-)\colon \mathit{Ho} \to \mathit{Mod}$ preserves arbitrary coproducts, i. e. is strongly additive.

Mod* denotes the graded category of E^*-modules, in which homomorphisms of any degree are allowed. That is, $\mathit{Mod}^*(M,N)$ is the graded group whose component $\mathit{Mod}^n(M,N)$ in degree n is the group of E^*-module homomorphisms $f\colon M \to N$ of degree n, with components $f^i\colon M^i \to N^{i+n}$ that satisfy $f^{i+h}(vx) = (-1)^{nh}v(f^i x)$ for $x \in M^i$ and $v \in E^h$. The sign must be present if the algebra is to imitate the topology.

Moreover, $\mathit{Mod}^*(M,N)$ is an E^*-module in the obvious way, with vf defined by $(vf)x = v(fx) = \pm f(vx)$ for $v \in E^*$. Given E^*-module homomorphisms $g\colon L' \to L$ and $h\colon M \to M'$, we define $\operatorname{Hom}(g,h)\colon \mathit{Mod}^*(L,M) \to \mathit{Mod}^*(L',M')$ by

$$\operatorname{Hom}(g,h)f = \mathit{Mod}^*(g,h)f = (-1)^{\deg(g)(\deg(f)+\deg(h))} h \circ f \circ g\colon L' \longrightarrow M', \quad (6.4)$$

to make it a homomorphism of E^*-modules. Similarly for tensor products: given morphisms $f\colon L \to L'$ and $g\colon M \to M'$, we define the morphism $f \otimes g\colon L \otimes M \to L' \otimes M'$ in *Mod** by

$$(f \otimes g)(x \otimes y) = (-1)^{\deg(g)\deg(x)} fx \otimes gy\ .$$

If also $f'\colon L' \to L''$ and $g'\colon M' \to M''$, composition is given, like eq. (6.2), by

$$(g' \otimes f') \circ (g \otimes f) = (-1)^{\deg(f')\deg(g)} (g' \circ g) \otimes (f' \circ f)\colon L \otimes M \longrightarrow L'' \otimes M''\ . \quad (6.5)$$

We imitate the suspension isomorphisms (3.13) and (3.16) algebraically by introducing suspension functors into *Mod* and *Mod**.

Definition 6.6. Given an E^*-module M and any integer k, we define the *k-fold suspension* $\Sigma^k M$ of M by shifting everything up in degree by k: $(\Sigma^k M)^i$ is a formal copy of M^{i-k}, consisting of the elements $\Sigma^k x$ for $x \in M^{i-k}$.

To make the function $\Sigma^k\colon M \to \Sigma^k M$ an isomorphism of E^*-modules of degree k, we must define the action of $v \in E^h$ on $\Sigma^k M$ by

$$v(\Sigma^k x) = (-1)^{hk}\Sigma^k(vx) \qquad \text{in } \Sigma^k M. \quad (6.7)$$

Further, $\Sigma^k\colon M \cong \Sigma^k M$ becomes a natural isomorphism $I \cong \Sigma^k$ of degree k of functors on *Mod** if we define $\Sigma^k f\colon \Sigma^k M \to \Sigma^k N$ by $(\Sigma^k f)(\Sigma^k x) = (-1)^{kn}\Sigma^k(fx)$ on a morphism $f\colon M \to N$ of any degree n. (Here, Σ denotes both a natural isomorphism and a functor.)

Alg denotes the category of commutative E^*-algebras. It admits arbitrary degreewise cartesian products $\prod_\alpha A_\alpha$ as products. The tensor product $A \otimes B$ of algebras serves as the coproduct of A and B, and E^* is the initial object.

Categories of filtered objects. The discussion in sections 3 and 4 strongly suggests that for cohomology, we need filtered versions of *Mod*, *Mod**, and *Alg*.

FMod denotes the category of complete Hausdorff *filtered* E^*-modules and *continuous* E^*-module homomorphisms of degree 0. An object M is an E^*-module M, equipped with a directed system of E^*-submodules F^aM, and hence a topology as in Defn. 3.36. (We do not require the indexing set to be the integers, or even countable.) These are required to satisfy $M = \lim_a M/F^aM$, to make the topology complete Hausdorff. The category remains an additive category.

The forgetful functor $V\colon$ *FMod* $\to$ *Mod* simply discards the filtration. Conversely, any E^*-module M may be treated as a *discrete* filtered module by taking 0 as the only submodule F^aM; this defines an inclusion *Mod* $\subset$ *FMod*. Generally, a filtered module M is discrete if and only if some F^aM is zero.

We frequently encounter filtered E^*-modules M that are not complete Hausdorff. We defined the completion $\widehat{M} = \lim_a M/F^aM$ of M in Defn. 3.37. The completion map $M \to \widehat{M}$ is monic if and only if M is Hausdorff, and epic if and only if M is complete. Each $\widehat{M} \to M/F^aM$ is epic, because $M \to M/F^aM$ is.

We filter $\widehat{M}$ in the obvious way, by $F^a\widehat{M} = \operatorname{Ker}[\widehat{M} \to M/F^aM]$. This filters the completion map and induces isomorphisms $M/F^aM \cong \widehat{M}/F^a\widehat{M}$; it is now obvious that $\widehat{M}$ is indeed complete Hausdorff (as the terminology demands) and so an object of *FMod*. If M happens to be already complete Hausdorff, $M \to \widehat{M}$ is an isomorphism in *FMod*. We make frequent use of the expected universal property: given an object N of *FMod*, any continuous E^*-module homomorphism $M \to N$ factors uniquely through a morphism $\widehat{M} \to N$ in *FMod*. In the language of Defn. 2.1, $\widehat{M}$ is V-free on M, with the completion map $M \to \widehat{M}$ as a basis.

If $F^aM \subset F^bM$, we can write $F^bM/F^aM = \operatorname{Ker}[M/F^aM \to M/F^bM]$. If we now fix F^bM and apply the left exact functor $\lim_a$, we see that the completion of F^bM, filtered by those F^aM contained in it, is just $\operatorname{Ker}[\widehat{M} \to M/F^bM] = F^b\widehat{M}$, as expected.

None of the above facts requires the filtration to be countable.

The obvious filtration (4.15) on the tensor product $M \otimes N$ is rarely complete, even when M and N are. We therefore complete it to define the *completed tensor product* $M \mathbin{\widehat{\otimes}} N$ in *FMod*. In view of the second form of (4.15), it may usefully be written

$$M \mathbin{\widehat{\otimes}} N = \lim_{a,b}\,[(M/F^aM) \otimes (N/F^bN)]. \tag{6.8}$$

This makes it clear that $\widehat{M} \mathbin{\widehat{\otimes}} \widehat{N} = M \mathbin{\widehat{\otimes}} N$, that it does not matter whether we complete M and N first or not. (We continue to write $f \otimes g$ rather than $f \mathbin{\widehat{\otimes}} g$ for the completed morphisms, leaving it to the context to indicate that completion is assumed.)

FMod* denotes the graded category of complete Hausdorff filtered E^*-modules, in which continuous E^*-module homomorphisms of any degree are allowed.

We give the E-cohomology $E^*(X)$ of a space X the profinite topology from Defn. 4.9, and complete it to $E^*(X)\hat{}$ as in Defn. 4.11 if necessary; by Lemma 4.10, the functor $E^*(-)\hat{}: \mathit{Ho}^{\mathrm{op}} \to \mathit{FMod}$ takes arbitrary coproducts in *Ho* to products in *FMod*. Thus cohomology remains strongly additive in this enriched sense.

As noted in section 4, the profinite topology on E-cohomology makes cup and cross products continuous,which suggests our other main category.

FAlg denotes the category of complete Hausdorff commutative filtered E^*-algebras A, with multiplication $\phi: A \otimes A \to A$ and unit $\eta: E^* \to A$. We filter objects as in *FMod*, except that the filtration is now by *ideals* $F^a A$. Then ϕ is automatically continuous, and it is sometimes useful to complete it to $A \mathbin{\widehat{\otimes}} A \to A$. We have the forgetful functor $\mathit{FAlg} \to \mathit{FMod}$.

Degreewise cartesian products serve as products, and we note that the cohomology functor $E^*(-)\hat{}: \mathit{Ho}^{\mathrm{op}} \to \mathit{FAlg}$ takes coproducts in *Ho* to products in *FAlg*. The initial object is just E^* itself. Coproducts in *FAlg* are less obvious.

Lemma 6.9. *The completed tensor product $A \mathbin{\widehat{\otimes}} B$ of algebras serves as the coproduct in the category FAlg.*

Proof. We first consider the uncompleted tensor product $A \otimes B$, made into an E^*-algebra in the standard way, filtered as in (4.15) by the ideals

$$F^{a,b}(A \otimes B) = \mathrm{Im}[(F^a A \otimes B) \oplus (A \otimes F^b B) \longrightarrow A \otimes B].$$

We define continuous injections $i: A \to A \otimes B$ and $j: B \to A \otimes B$ by $ix = x \otimes 1$ and $jy = 1 \otimes y$. Given continuous homomorphisms $f: A \to C$ and $g: B \to C$, where C is any object in *FAlg*, there is a unique homomorphism of algebras $h: A \otimes B \to C$ satisfying $h \circ i = f$ and $h \circ j = g$, defined by $h(x \otimes y) = (fx)(gy)$, thanks to the commutativity of C. It is also continuous: given $F^c C \subset C$, choose $F^a A$ and $F^b B$ such that $f(F^a A) \subset F^c C$ and $g(F^b B) \subset F^c C$; then $h(F^{a,b}(A \otimes B)) \subset F^c C$. Because $A \otimes B$ is rarely complete, we complete it, and the homomorphism h, to obtain the desired unique algebra homomorphism $\widehat{h}: A \mathbin{\widehat{\otimes}} B \to C$ in *FAlg*. □

Although $E^*(-)\hat{}$ does not in general take products in *Ho* to coproducts in *FAlg*, it does in the favorable cases when we have the Künneth homeomorphism $E^*(X \times Y) \cong E^*(X) \mathbin{\widehat{\otimes}} E^*(Y)$ as in Defn. 4.17.

The module of indecomposables. If $(A, \phi, \eta, \epsilon)$ is a (completed) algebra with counit (or augmentation) $\epsilon: A \to E^*$ (which is required to be a morphism of algebras as in e.g. a Hopf algebra), the augmentation ideal $\overline{A} = \mathrm{Ker}\,\epsilon$ splits off as an E^*-module, $A \cong E^* \oplus \overline{A}$. One can define the *module of indecomposables* $QA = \overline{A}/\overline{A}\,\overline{A}$, i.e. $\mathrm{Coker}[\phi: \overline{A} \otimes \overline{A} \to \overline{A}]$ (or $\mathrm{Coker}[\phi: \overline{A} \mathbin{\widehat{\otimes}} \overline{A} \to \overline{A}]$ in the completed case). A cleaner way to write this categorically is

$$QA = \mathrm{Coker}[\phi - A \otimes \epsilon - \epsilon \otimes A: A \otimes A \longrightarrow A] \qquad \text{in } \mathit{Mod}, \tag{6.10}$$

as we see by using the splitting of A; the homomorphism here is zero on $\overline{A} \otimes 1$ and $1 \otimes \overline{A}$ and -1 on $E^* = E^* \otimes E^*$.

Lemma 6.11. *The functor Q, defined on (completed) E^*-algebras with counit, preserves finite coproducts: $Q(A \otimes B) \cong QA \oplus QB$ (or $Q(A \mathbin{\widehat{\otimes}} B) \cong QA \oplus QB$) and $QE^* = 0$.*

Proof. For $C = A\otimes B$ (and similarly $A\mathbin{\widehat{\otimes}} B$) we have the direct sum decomposition

$$\overline{C} = (\overline{A}\otimes 1) \oplus (1\otimes\overline{B}) \oplus (\overline{A}\otimes\overline{B}).$$

Then $\phi(\overline{C}\otimes\overline{C})$ contains $\overline{A}\,\overline{A}\otimes 1$ from $(\overline{A}\otimes 1)\otimes(\overline{A}\otimes 1)$, $1\otimes\overline{B}\,\overline{B}$ similarly, and $\overline{A}\otimes\overline{B}$ from $(\overline{A}\otimes 1)\otimes(1\otimes\overline{B})$. The image is the direct sum of these, because the other six pieces of $\overline{C}\otimes\overline{C}$ give nothing new. This allows us to read off the cokernel. □

Coalg denotes the category of cocommutative E^*-coalgebras, with comultiplication $\psi\colon A\to A\otimes A$ and counit $\epsilon\colon A\to E^*$.

When $E_*(X)$ is a free E^*-module, eq. (4.3) and $q_*\colon E_*(X)\to E^*$ make it an object in *Coalg*.

Lemma 6.12. *In the category Coalg:*

(a) *The tensor product $A\otimes B$ of two coalgebras is again a coalgebra* (see e. g. [25, §2])*, and serves as the product;*

(b) *E^* is the terminal object;*

(c) *Arbitrary direct sums $\bigoplus_\alpha A_\alpha$ of coalgebras serve as coproducts.* □

There are also the slightly more general *completed coalgebras* A, where A is filtered as above and we have instead $\psi\colon A\to A\mathbin{\widehat{\otimes}} A$. If A and B are completed coalgebras, so is $A\mathbin{\widehat{\otimes}} B$.

The module of primitives. If (A,ψ,ϵ,η) is a (completed) coalgebra with unit (e. g. a Hopf algebra), where $\eta\colon E^*\to A$ is required to be a morphism of coalgebras, we can define, dually to eq. (6.10), the *module of coalgebra primitives*

$$PA = \operatorname{Ker}[\psi - A\otimes\eta - \eta\otimes A\colon A\longrightarrow A\otimes A]\subset A \tag{6.13}$$

in *Mod* (or *FMod*, with $A\mathbin{\widehat{\otimes}} A$ in place of $A\otimes A$), a submodule of A. The dual of Lemma 6.11 holds.

Lemma 6.14. *The functor P, defined on (completed) coalgebras with unit, preserves finite products: $P(A\otimes B)\cong PA\oplus PB$ (or $P(A\mathbin{\widehat{\otimes}} B)\cong PA\oplus PB$) and $PE^* = 0$.* □

Dual modules. We warn that the completed tensor product $\widehat{\otimes}$ does *not* make *FMod* a closed category (as $-\mathbin{\widehat{\otimes}} M$ admits no right adjoint). Nor do we attempt to topologize $\mathit{FMod}(M,N)$ in general.

Nevertheless, we found it useful in Defn. 4.8 to filter the dual $DM = \mathit{Mod}^*(M,E^*)$ of a *discrete* E^*-module M by the submodules $F^LDM = \operatorname{Ker}[DM\to DL]$, where L runs through all finitely generated submodules of M. Then $DM = \lim_L DL$ in *FMod*, where each DL is discrete; in particular, DM is automatically complete Hausdorff.

The dual $Df\colon DN\to DM$ of any homomorphism $f\colon M\to N$ is continuous, because $(Df)^{-1}(F^LDM) = F^{fL}DN$. In the important case when M is free, we obtain topologically equivalent filtrations by taking only those L that are (i) free of finite rank, or (ii) free of finite rank, and a summand of M, or (iii) generated by finite subsets of a given basis of M.

Lemma 6.15. *Let M, M_α, and N be discrete E^*-modules. Then:*

(a) *The canonical isomorphism $D(M \oplus N) \cong DM \oplus DN = DM \times DN$ is a homeomorphism;*

(b) *The canonical isomorphism $D(\bigoplus_\alpha M_\alpha) \cong \prod_\alpha DM_\alpha$ is a homeomorphism;*

(c) *If $f: M \to N$ is epic, then the dual $Df: DN \to DM$ is a topological embedding;*

(d) *The functor D takes colimits in Mod to limits in FMod;*

(e) *$\zeta_D: DM \widehat{\otimes} DN \cong D(M \otimes N)$ in FMod, if M or N is a free E^*-module.*

Proof. In (a), $D(M\oplus N) \to DM \times DN$ is continuous because D is a functor. Given a basic open set $F^L D(M\oplus N) \subset D(M\oplus N)$, where $L \subset M\oplus N$ is finitely generated, there are finitely generated submodules $P \subset M$ and $Q \subset N$ such that $L \subset P \oplus Q$; then $F^P DM \oplus F^Q DN \subset F^L D(M \oplus N)$ shows that we have a homeomorphism. More generally, we get (b).

In (c), we can lift any finitely generated submodule $L \subset N$ to a finitely generated submodule $K \subset M$ such that $fK = L$. Then $F^L DN = DN \cap F^K DM$ in DM.

If $C = \mathrm{Coker}[f: M \to N]$, we have $DC = \mathrm{Ker}[Df: DN \to DM]$ as an E^*-module. By (c), the topology on DC is correct, so that D sends cokernels to kernels. This, with (b), gives (d).

In (e), we may assume M is free. Equality is obvious for $M = E^*$ and therefore, by additivity, for M free of finite rank. By (d) and eq. (6.8), the general case is the limit in FMod of the isomorphisms $DL \otimes DN \cong D(L \otimes N)$ as L runs through the free submodules of M of finite rank that are summands of M. □

The evaluation $e: DL \otimes L \to E^*$, which we write as $e(r \otimes c) = \langle r, c\rangle$, is standard. The dual concept, of a homomorphism $E^* \to DL \otimes L$ for suitable L, is far less known, even for finite-dimensional vector spaces.

Lemma 6.16. *Let L be a discrete free E^*-module. We can define the* universal element *$u = u_L \in DL \widehat{\otimes} L$ by the property that for any $r \in DL = \mathsf{Mod}^*(L, E^*)$, the homomorphism*

$$DL \widehat{\otimes} \langle r, -\rangle : DL \widehat{\otimes} L \longrightarrow DL \otimes E^* \cong DL$$

takes u to r. It induces the following isomorphisms of E^*-modules:

(a) *$\mathsf{Mod}^*(L, M) \cong DL \widehat{\otimes} M$ for any discrete E^*-module M, by $f \mapsto (DL \otimes f)u$, with inverse $r \otimes x \mapsto [c \mapsto (-1)^{\deg(c)\deg(x)} \langle r, c\rangle\, x]$;*

(b) *$\mathsf{FMod}^*(DL, N) \cong N \widehat{\otimes} L$ for any object N of FMod, by $g \mapsto (g \otimes L)u$, with inverse $y \otimes c \mapsto [r \mapsto (-1)^e \langle r, c\rangle\, y]$, where $e = \deg(r)\deg(c) + \deg(r)\deg(y) + \deg(c)\deg(y)$;*

(c) *$\mathsf{FMod}^*(DL, E^*) \cong E^* \otimes L \cong L$, by $g \leftrightarrow c$, where $c = (g \otimes L)u$ and $gr = (-1)^{\deg(r)\deg(c)} \langle r, c\rangle$.*

Remark. We are not claiming to have isomorphisms in FMod. Indeed, for reasons already mentioned, we do not even topologize $\mathsf{FMod}^*(DL, N)$ etc. In any case, the obvious E^*-module structures are the wrong ones for our applications.

Proof. In terms of an E^*-basis $\{c_\alpha : \alpha \in \Lambda\}$ of L, u is given by

$$u = u_L = \sum_\alpha (-1)^{\deg(c_\alpha)} c_\alpha^* \otimes c_\alpha \in DL \mathbin{\widehat{\otimes}} L,$$

where c_α^* denotes the linear functional dual to c_α, given by $\langle c_\alpha^*, c_\alpha \rangle = 1$ and $\langle c_\alpha^*, c_\beta \rangle = 0$ for $\beta \neq \alpha$. In effect, (a) generalizes the definition of u, and is clearly an isomorphism when L has finite rank, with inverse as stated.

For general L, we let K run through all the free submodules of L of finite rank. The functor $\mathit{Mod}^*(-, M)$ automatically takes the colimit $L = \operatorname{colim}_K K$ to a limit. On the right, the functor $- \mathbin{\widehat{\otimes}} M$ preserves the limit $DL = \lim_K DK$ by eq. (6.8).

Similarly, (b) is obvious when L has finite rank and N is discrete. For general L and discrete N, any continuous homomorphism $DL \to N$ must factor through some DK, so that on the left, we have the colimit $\operatorname{colim}_K \mathit{Mod}^*(DK, N)$. On the right, we also have a colimit, $N \otimes L = \operatorname{colim}_K N \otimes K$ (as no completion is needed). This gives (b) for discrete N and general L. For general N, we observe that both sides preserve the limit $N = \lim_b N/F^b N$, with the help of eq. (6.8).

In the special case (c) of (b), the defining property of u implies by naturality that $gr = \pm\langle r, (g \otimes L)u \rangle$ for any $r \in DL$ and any $g: DL \to E^*$. □

It will be convenient to rearrange the signs in (b).

Corollary 6.17. *The general element* $\sum_\alpha (-1)^{\deg(y_\alpha)\deg(c_\alpha)} y_\alpha \otimes c_\alpha \in N \mathbin{\widehat{\otimes}} L$ *of degree* k *corresponds to the general morphism* $DL \to N$ *of degree* k *given by*

$$r \mapsto (-1)^{k \deg(r)} \sum_\alpha \langle r, c_\alpha \rangle \, y_\alpha \ . \qquad \square$$

7. Algebraic objects in categories

It has been known for a long time (e. g. Lawvere [19]) how to define algebraic objects in general categories. We are primarily interested in abelian group objects and generalizations, especially E^*-module and E^*-algebra objects, where E^* is a fixed commutative graded ring. We review the material on categories we need from MacLane's book [20, Chs. VI, VII].

Group objects. Let $\mathcal{C}$ be any category having a terminal object T and (enough) finite products. (Recall that T is the empty product.)

A *group object* in $\mathcal{C}$ is an object G equipped with a *multiplication* morphism $\mu: G \times G \to G$, a *unit* morphism $\omega: T \to G$, and an *inversion* morphism $\nu: G \to G$, that satisfy the usual axioms, expressed as well-known commutative diagrams (which may be viewed in [32, §1]). Then for any object X, $\mathcal{C}(X, G)$ becomes a group (as we see generally in Lemma 7.7), whose unit element is $\omega \circ q: X \to T \to G$. In the group $\mathcal{C}(G, G)$, ν is the inverse of 1_G.

An *abelian* group object G has μ commutative (another diagram); in this case, we call μ the *addition* and ω the *zero* morphism. Then the group $\mathcal{C}(X, G)$ is abelian.

If H is another group object in $\mathcal{C}$, a morphism $f: G \to H$ is a *morphism of group objects* if it commutes with the three structure morphisms; as is standard for sets and true generally (again by Lemma 7.7), it is enough to check μ. Thus we form the category $\mathit{Gp}(\mathcal{C})$ of all group objects in $\mathcal{C}$; one important example is $\mathit{Gp}(\mathit{Ho})$.

Example. In the category *Set*, one writes the structure maps of an abelian group object as $\mu(x,y) = x+y$, $\omega(a) = 0$, and $\nu(x) = -x$, where $T = \{a\}$. Then the axioms take the form $(x+y)+z = x+(y+z)$, $x+0 = x$, $x+(-x) = 0$, and $x+y = y+x$, the usual axioms for an abelian group.

Example. An (abelian) group object A in *Coalg* is a cocommutative Hopf algebra over E^*, with (commutative) multiplication $\phi: A \otimes A \to A$ and unit $\eta: E^* \to A$; the canonical antiautomorphism $\chi: A \to A$ is by [25, Defn. 8.4] the inversion ν. (Recall from Lemma 6.12(a) that $A \otimes A$ is the product in *Coalg*.)

Dually, a *cogroup object* in $\mathcal{C}$ is simply a group object G in the dual category $\mathcal{C}^{\mathrm{op}}$. That is, we use coproducts instead of products, an initial object I instead of T, and reverse all the arrows; so that G is equipped with a comultiplication $G \to G \amalg G$, counit $G \to I$, and inversion $G \to G$, satisfying the evident rules.

Example. A commutative Hopf algebra A over E^* may be regarded as a cogroup object in *Alg* with comultiplication $\psi: A \to A \otimes A$, counit $\epsilon: A \to E^*$, and inversion $\chi: A \to A$. (As in Lemma 6.9, $A \otimes A$ is the coproduct.)

Example. In the based homotopy category Ho', the circle S^1, and hence the suspension ΣX, are well-known cogroup objects.

In any additive category, we have abelian group objects for free.

Lemma 7.1. *In a (graded) additive category $\mathcal{C}$:*

(a) *Every object admits a unique structure as abelian group object and as abelian cogroup object;*

(b) *Every morphism is a morphism of abelian (co)group objects;*

(c) *The (graded) abelian group structure on $\mathcal{C}(X,Y)$ resulting from the group object Y or the cogroup object X is the given one.*

Proof. The zero object is terminal, which forces $\omega = 0$. The sum $G \oplus G$ serves as both product and coproduct. The axioms force $\mu = p_1 + p_2$ and $\nu = -1_G: G \to G$, and these choices work. The dual of an additive category is again additive. □

The *product* $G \times H$ of two group objects is another group object, with the obvious multiplication

$$\mu: G \times H \times G \times H \cong G \times G \times H \times H \xrightarrow{\mu\times\mu} G \times H, \tag{7.2}$$

unit $\omega \times \omega: T \cong T \times T \to G \times H$, and inversion $\nu \times \nu: G \times H \to G \times H$. This serves as the product in the category $\mathit{Gp}(\mathcal{C})$. The trivial group object T, with the unique structure morphisms, serves as the terminal object.

This allows one to define group objects in $\mathit{Gp}(\mathcal{C})$, as follows. To say that G is an *object* of $\mathit{Gp}(\mathcal{C})$ means that it is equipped with a multiplication μ_G, unit ω_G, and inversion ν_G that make it a group object in $\mathcal{C}$. In diag. (7.2) we made $G \times G$ an object of $\mathit{Gp}(\mathcal{C})$. Then G is a *group object* in $\mathit{Gp}(\mathcal{C})$ if it is equipped also with morphisms $\mu: G \times G \to G$, $\omega: T \to G$, and $\nu: G \to G$ in $\mathit{Gp}(\mathcal{C})$ that satisfy the axioms. The following useful result is well known.

Proposition 7.3. *Let G be a group object in the category $\mathit{Gp}(\mathcal{C})$. Then the two group structures on G coincide and are abelian.*

Proof. Lemma 7.7 will show that it is sufficient to consider the case $\mathcal{C} = \mathit{Set}$, where the result is a standard exercise (e. g. [20, Ex. III.6.4]). □

Module objects. A *graded group object* M in $\mathcal{C}$ is a function $n \mapsto M^n$ that assigns to each integer n (positive or negative) an abelian group object M^n in $\mathcal{C}$. (Note that the infinite product $\prod_n M^n$ and coproduct are irrelevant.)

An E^**-module object* in a (graded) category $\mathcal{C}$ is a graded group object $n \mapsto M^n$ that is equipped with morphisms $\xi v: M^n \to M^{n+h}$ of abelian group objects (of degree h) for all $v \in E^*$ and all n, where $h = \deg(v)$, subject to the axioms:

$$\begin{aligned} &\text{(i) } \xi(v+v') = \xi v + \xi v' \text{ in the group } \mathcal{C}(M^n, M^{n+h}), \text{ for } v, v' \in E^h; \\ &\text{(ii) } \xi(vv') = \xi v \circ \xi v' \text{ for all } v, v' \in E^*; \\ &\text{(iii) } \xi 1 = 1: M^n \to M^n. \end{aligned} \tag{7.4}$$

It follows that the inversion $\nu = \xi(-1) = -1$ in $\mathcal{C}(M^n, M^n)$.

In an additive category, Lemma 7.1 shows that all we need is a graded object $n \mapsto M^n$ equipped with morphisms $\xi v: M^n \to M^{n+h}$ that satisfy the axioms (7.4). If $\mathcal{C}$ is graded, we often (but not always) have only a single object M, with $M^n = M$ for all n; then the definition reduces to a graded ring homomorphism $\xi: E^* \to \operatorname{End}^*_{\mathcal{C}}(M)$.

In a *graded* category, the concept of E^*-module object is self-dual, thanks to the commutativity of E^* (provided we watch the signs and indexing): $n \mapsto M^n$ is an E^*-module object in $\mathcal{C}$, with v acting by $\xi v: M^n \to M^{n+h}$, if and only if $n \mapsto M^{-n}$ is an E^*-module object in $\mathcal{C}^{\mathrm{op}}$, with v acting by $(\xi v)^{\mathrm{op}}: M^{n+h} \to M^n$ in $\mathcal{C}^{\mathrm{op}}$. (But we note that this observation fails in general in *ungraded* additive categories, because the required signs are absent.)

Algebra objects. A *(commutative) monoid object* in $\mathcal{C}$ is an object G equipped with a multiplication morphism $\phi: G \times G \to G$ and a unit morphism $\eta: T \to G$ that satisfy the axioms for associativity, (commutativity,) and unit. Apart from the lack of inverses and a change in notation, this is like a group object.

A *graded monoid object* is a graded object $n \mapsto M^n$, equipped with multiplications $\phi: M^k \times M^m \to M^{k+m}$ and a unit $\eta: T \to M^0$, that satisfy the axioms for associativity and unit. (There is a problem in defining commutativity for graded monoid objects, because extra structure is needed to handle the signs.)

An E^**-algebra object* in $\mathcal{C}$ is an E^*-module object that is also a graded monoid object, with the two structures related by three commutative diagrams that interpret the two distributive laws and $(vx)y = v(xy) = \pm\, x(vy)$. It is commutative if $yx = \pm\, xy$, interpreted as another diagram. Here, the sign $(-1)^n$ becomes $\xi((-1)^n)$.

It is often useful to replace the v-action $\xi v\colon M^n \to M^{n+h}$ in an E^*-algebra object by the simpler morphism $\eta_v = \xi v \circ \eta\colon T \to M^h$, so that $\eta_1 = \eta$; the diagram

$$\begin{array}{ccccc} T \times M^n & \xrightarrow{\eta \times M^n} & M^0 \times M^n & \xrightarrow{\xi v \times M^n} & M^h \times M^n \\ & \searrow^{\cong} & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi} \\ & & M^n & \xrightarrow{\xi v} & M^{n+h} \end{array}$$

shows that we can recover ξv from η_v as the composite

$$\xi v\colon M^n \cong T \times M^n \xrightarrow{\eta_v \times M^n} M^h \times M^n \xrightarrow{\phi} M^{n+h}. \tag{7.5}$$

Equivalently, we have interpreted the identity $vx = (v1)x$.

General algebraic objects. Other kinds of algebraic object can be defined similarly, provided they are (or can be) described in terms of operations $\alpha\colon G^{\times n(\alpha)} \to G$ subject to *universal* laws, where $G^{\times n} = G \times G \times \ldots \times G$, with n factors. Frequently, our algebraic object lies in the dual category $\mathcal{C}^{\mathrm{op}}$ and is the corresponding *coalgebraic* object in $\mathcal{C}$. Our general results extend without difficulty (except notationally) to the dual and graded variants, and we omit details.

The following observation is quite elementary but extremely useful.

Lemma 7.6. *Let G be an algebraic object in $\mathcal{C}$ that is equipped with operations $\alpha\colon G^{\times n(\alpha)} \to G$, and $V\colon \mathcal{C} \to \mathcal{D}$ be a functor.*

(a) *If V preserves (enough) finite powers of G, then VG is an algebraic object in $\mathcal{D}$ of the same kind, equipped with the operations*

$$\alpha\colon (VG)^{\times n(\alpha)} \cong V(G^{\times n(\alpha)}) \xrightarrow{V\alpha} VG;$$

(b) *If $f\colon G \to H$ is a morphism of algebraic objects in $\mathcal{C}$, where H is another algebraic object of the same kind, and V preserves (enough) powers of G and H, then $Vf\colon VG \to VH$ is a morphism of algebraic objects in $\mathcal{D}$;*

(c) *If $\theta\colon V \to W$ is a natural transformation, where $W\colon \mathcal{C} \to \mathcal{D}$ is another functor that preserves (enough) powers of G, then $\theta G\colon VG \to WG$ is a morphism of algebraic objects in $\mathcal{D}$.* □

More precisely, V and W do not need to preserve *all* finite powers, only the powers of G and H that actually appear in the operations and laws (including the terminal object T, if used).

Example. As S^1 is a cogroup object in Ho', (a) shows that the loop space ΩX on any based space X becomes a group object in Ho', and hence in Ho. If X is already a group object in Ho', (a) provides a second group object structure on ΩX; but by Prop. 7.3, these two group structures coincide and are abelian.

One common case where this lemma applies trivially is when V is an additive functor between additive categories. There are other functors of interest that automatically preserve products: for any object X in $\mathcal{C}$, the corepresented functor $\mathcal{C}(X,-):\mathcal{C} \to \mathit{Set}$ preserves products by definition, and dually, $\mathcal{C}(-,X) = \mathcal{C}^{\mathrm{op}}(X,-):\mathcal{C}^{\mathrm{op}} \to \mathit{Set}$ takes coproducts in $\mathcal{C}$ to products in Set. Then Lemma 7.6 gives parts (a), (b), and (c) of the following.

Lemma 7.7. *Let G and H be fixed objects in the category $\mathcal{C}$, and V and W be the contravariant represented functors $\mathcal{C}(-,G), \mathcal{C}(-,H):\mathcal{C}^{\mathrm{op}} \to \mathit{Set}$ (or dually, covariant corepresented functors $\mathcal{C}(G,-), \mathcal{C}(H,-):\mathcal{C} \to \mathit{Set}$).*

(a) *If G is a (co)algebraic object in $\mathcal{C}$, then for any object X in $\mathcal{C}$, VX is naturally an algebraic object in Set of the same kind;*

(b) *With G as in (a), then for any morphism $f:X \to Y$ in $\mathcal{C}$, $Vf:VY \to VX$ (or $Vf:VX \to VY$) is a morphism of algebraic objects in Set;*

(c) *Any morphism $f:G \to H$ of (co)algebraic objects in $\mathcal{C}$ induces a natural morphism $\mathcal{C}(X,f):VX \to WX$ (or $\mathcal{C}(f,X):WX \to VX$) of algebraic objects in Set;*

(d) *Conversely, if VX has a natural algebraic structure, it is induced as in (a) by a unique (co)algebraic structure on G of the same kind, provided the necessary (co)powers of G exist in $\mathcal{C}$;*

(e) *Any natural transformation of algebraic objects $VX \to WX$ (or $WX \to VX$) in Set is induced as in (c) by a unique morphism $f:G \to H$ of (co)algebraic objects in $\mathcal{C}$.*

Proof. In (d), we may identify $\mathcal{C}(X,G)^{\times n}$ with $\mathcal{C}(X,G^{\times n})$. Then by Yoneda's Lemma, each natural transformation $\alpha:\mathcal{C}(-,G)^{\times n} \to \mathcal{C}(-,G)$ is induced by a unique morphism, which we also call $\alpha:G^{\times n} \to G$; the uniqueness shows that the same laws apply, thus making G an algebraic object. Part (e) is similar. □

This allows us to clarify Thm. 3.17.

Corollary 7.8. *We have the E^*-algebra object $n \mapsto \underline{E}_n$ in the category Ho; in particular, each $\underline{E}_n$ is an abelian group object in Ho. Moreover, each equivalence $\underline{E}_n \simeq \Omega\underline{E}_{n+1}$ is an isomorphism of group objects.*

Proof. We apply (d) and (e) to the cohomology functors $E^n(-):\mathit{Ho}^{\mathrm{op}} \to \mathit{Set}$, represented according to Thm. 3.17 by the spaces $\underline{E}_n$. Part (e) also gives the last assertion; by Prop. 7.3, the group structure on $\Omega\underline{E}_{n+1}$ is well defined. □

Symmetric monoidal categories. The theory presented so far is not general enough. In order to express the multiplicative structures, we need symmetric monoidal categories. We review the few basic facts we need from MacLane [20, Ch. VII].

A *(symmetric) monoidal category* $(\mathcal{C},\otimes,K)$ is a category $\mathcal{C}$ equipped with a bifunctor $\otimes:\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object $K = K_{\mathcal{C}}$. (But if $\mathcal{C}$ is graded, we need a more general kind of bifunctor $\otimes$ that is biadditive and includes signs, with composition as in eq. (6.5).) It is understood (but suppressed from our notation) that the specification includes [ibid.] coherent natural isomorphisms for associativity, (commutativity, with signs if $\mathcal{C}$ is graded) and $K \otimes X \cong X \cong X \otimes K$.

As examples, we have $(\mathit{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$, $(\mathit{Mod}, \otimes, E^*)$, $(\mathit{FMod}, \widehat{\otimes}, E^*)$, $(\mathit{Stab}, \wedge, T^+)$, the graded versions of all these, and the dual $(\mathcal{C}^{\mathrm{op}}, \otimes, K)$ of any symmetric monoidal category. The original example was $(\mathcal{C}, \times, T)$, for any category $\mathcal{C}$ that admits finite products (including the empty product T).

Example. We define the symmetric monoidal category $(\mathit{Set}^{\mathbb{Z}}, \times, T)$ of graded sets. For this purpose, the graded set $n \mapsto A^n$ is best treated as the disjoint union $A = \coprod_n A^n$, equipped with the degree function $A \to \mathbb{Z}$ given by $\deg(A^n) = n$. The product $A \times B$ is given the degree function $\deg((x,y)) = \deg(x) + \deg(y)$. The unit object is the set T consisting of one point in degree zero.

The purpose (for us) of a (symmetric) monoidal category is to extend the definition of monoid object. A *(commutative) monoid object* in $(\mathcal{C}, \otimes, K)$ is an object M of $\mathcal{C}$ that is equipped with a *multiplication* morphism $\phi: M \otimes M \to M$ and a *unit* morphism $\eta: K \to M$ (both of degree 0 if $\mathcal{C}$ is graded) that satisfy the usual axioms for associativity, (commutativity,) and left and right unit. In $(\mathit{Set}, \times, T)$, this reduces to the usual concept of (commutative) monoid; more generally, in $(\mathcal{C}, \times, T)$, it reduces to the concept of (commutative) monoid object as before.

A *graded monoid object* in $(\mathcal{C}, \otimes, K)$ is a graded object $n \mapsto M^n$ in $\mathcal{C}$ equipped with multiplications $\phi: M^k \otimes M^m \to M^{k+m}$ and unit $\eta: K \to M^0$ (with degree 0) that satisfy the axioms for associativity and two-sided unit. (Again, we defer the discussion of commutativity.) Morphisms of monoids are defined in the obvious way.

A *(symmetric) monoidal functor* $(F, \zeta_F, z_F): (\mathcal{C}, \otimes, K_{\mathcal{C}}) \to (\mathcal{D}, \otimes, K_{\mathcal{D}})$ between (symmetric) monoidal categories consists of a functor $F: \mathcal{C} \to \mathcal{D}$, together with a natural transformation $\zeta_F: FX \otimes FY \to F(X \otimes Y)$ and a morphism $z_F: K_{\mathcal{D}} \to FK_{\mathcal{C}}$ in $\mathcal{D}$. Of course, ζ_F and z_F are required to respect the isomorphisms for associativity, (commutativity,) and unit. If M is a (commutative) monoid object in $\mathcal{C}$, FM will be one in $\mathcal{D}$, equipped with the obvious multiplication

$$\phi: FM \otimes FM \xrightarrow{\zeta_F(M,M)} F(M \otimes M) \xrightarrow{F\phi} FM$$

and unit $F\eta \circ z_F: K_{\mathcal{D}} \to FK_{\mathcal{C}} \to FM$.

We do *not* require ζ_F and z_F to be isomorphisms (but if they are, so much the better). One example is the duality functor

$$(D, \zeta_D, z_D): (\mathit{Mod}^{\mathrm{op}}, \otimes, E^*) \longrightarrow (\mathit{FMod}, \widehat{\otimes}, E^*)$$

defined by $DM = \mathit{Mod}^*(M, E^*)$ and filtered in Defn. 4.8, where $z_D: E^* \cong DE^*$ is obvious and ζ_D was originally defined in eq. (4.6) and completed later for diag. (4.18). By Lemma 6.15(e), ζ_D is sometimes an isomorphism. Another example is the symmetric monoidal functor

$$(\mathcal{C}(X, -), \zeta, z): (\mathcal{C}, \times, T) \longrightarrow (\mathit{Set}, \times, T)$$

used in Lemma 7.7 to map an algebraic object in $\mathcal{C}$ to the corresponding algebraic object in *Set*; in this case, ζ and z are automatically isomorphisms.

Monoidal functors compose in the obvious way. Given another (symmetric) monoidal functor $(G, \zeta_G, z_G): (\mathcal{D}, \otimes, K_{\mathcal{D}}) \to (\mathcal{E}, \otimes, K_{\mathcal{E}})$, the composite (symmetric) monoidal functor $(GF, \zeta_{GF}, z_{GF})\colon (\mathcal{C}, \otimes, K_{\mathcal{C}}) \to (\mathcal{E}, \otimes, K_{\mathcal{E}})$ uses the natural transformation

$$\zeta_{GF}\colon GFX \otimes GFY \xrightarrow{\zeta_G} G(FX \otimes FY) \xrightarrow{G\zeta_F} GF(X \otimes Y)$$

and morphism

$$z_{GF}\colon K_{\mathcal{E}} \xrightarrow{z_G} GK_{\mathcal{D}} \xrightarrow{Gz_F} GFK_{\mathcal{C}}\ .$$

Given two (symmetric) monoidal functors

$$(F, \zeta_F, z_F), (G, \zeta_G, z_G)\colon (\mathcal{C}, \otimes, K_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \otimes, K_{\mathcal{D}}),$$

a natural transformation $\theta\colon F \to G$ is called *monoidal* if there are commutative diagrams

$$\begin{array}{ccc} FX \otimes FY & \xrightarrow{\theta X \otimes \theta Y} & GX \otimes GY \\ \downarrow{\scriptstyle \zeta_F(X,Y)} & & \downarrow{\scriptstyle \zeta_G(X,Y)} \\ F(X \otimes Y) & \xrightarrow{\theta(X \otimes Y)} & G(X \otimes Y) \end{array} \qquad \begin{array}{ccc} K_{\mathcal{D}} & & \\ \downarrow{\scriptstyle z_F} & \searrow{\scriptstyle z_G} & \\ FK_{\mathcal{C}} & \xrightarrow{\theta K_{\mathcal{C}}} & GK_{\mathcal{C}} \end{array}$$

Thus if X is a monoid object in $\mathcal{C}$, $\theta X\colon FX \to GX$ will be a morphism of monoid objects in $\mathcal{D}$.

We adapt Lemma 7.7 to monoidal functors.

Lemma 7.9. *Given a graded monoid object $n \mapsto C^n$ in the (graded) monoidal category $(\mathcal{C}^{\mathrm{op}}, \otimes, K)$, write $(FM)^n = \mathcal{C}(C^n, M)$ for any object M in $\mathcal{C}$. Then:*

(a) *We can make F a monoidal functor*

$$(F, \zeta_F, z_F)\colon (\mathcal{C}, \otimes, K) \longrightarrow (\mathsf{Set}^{\mathbb{Z}}, \times, T); \tag{7.10}$$

(b) *If the graded monoid object $n \mapsto D^n$ defines similarly the monoidal functor G, then a morphism $h\colon C \to D$ in $\mathcal{C}^{\mathrm{op}}$ of graded monoid objects induces a monoidal natural transformation $\theta\colon F \to G$.*

Proof. Let the multiplications and unit of C be $\phi\colon C^k \otimes C^m \to C^{k+m}$ and $\eta\colon K \to C^0$ (in $\mathcal{C}^{\mathrm{op}}$). We defined FM as a graded set. Given $f \in (FM)^k$ and $g \in (FN)^m$, we define $\zeta_F(f, g) \in F(M \otimes N)^{k+m} = \mathcal{C}(C^{k+m}, M \otimes N)$ as the composite

$$C^{k+m} \xrightarrow{\phi^{\mathrm{op}}} C^k \otimes C^m \xrightarrow{f \otimes g} M \otimes N \qquad \text{in } \mathcal{C}. \tag{7.11}$$

The morphism $z_F\colon T \to (FK)^0 = \mathcal{C}(C^0, K)$ has $\eta^{\mathrm{op}}\colon C^0 \to K$ as its image. In (b), we define $(\theta M)^n\colon (FM)^n = \mathcal{C}(C^n, M) \to \mathcal{C}(D^n, M) = (GM)^n$ as composition in $\mathcal{C}$ with $h^{\mathrm{op}}\colon D^n \to C^n$. The necessary verification is routine. □

Additive symmetric monoidal categories. We need a slightly more general categorical structure, arranged in two layers. If the category $\mathcal{C}$ is both monoidal and additive, it will be appropriate to use the monoidal structure $(\mathcal{C}, \otimes, K)$ to define multiplication, but to return to the additive structure of $\mathcal{C}$ to define addition. In this situation, we require the bifunctor $\otimes$ to be biadditive. Rather than strive for great generality, we limit attention to the cases we actually need. (We do not attempt to define the tensor product of E^*-module objects.)

Because $\mathcal{C}$ is additive, an E^*-module object reduces simply to a graded object $n \mapsto M^n$ equipped with morphisms $\xi v: M^n \to M^{n+h}$ for all $v \in E^*$ and all n (where $h = \deg(v)$) that satisfy the axioms (7.4). Further, we can now define *commutative* graded monoid objects $n \mapsto M^n$, including the expected sign.

Definition 7.12. A *(commutative) E^*-algebra object* in the (possibly graded) additive (symmetric) monoidal category $(\mathcal{C}, \otimes, K)$ is a graded object $n \mapsto M^n$ equipped with:

(i) morphisms $\xi v: M^n \to M^{n+h}$, for all n, h, and $v \in E^h$, that make it an E^*-module object in $\mathcal{C}$;

(ii) morphisms (ϕ, η) that make it a graded (commutative) monoid object;

in such a way that the diagrams commute up to the indicated sign:

$$\begin{array}{ccc} M^k \otimes M^m & \xrightarrow{\phi} & M^{k+m} \\ \downarrow{\scriptstyle \xi v \otimes 1} & & \downarrow{\scriptstyle \xi v} \\ M^{k+h} \otimes M^m & \xrightarrow{\phi} & M^{k+m+h} \end{array} \qquad \begin{array}{ccc} M^k \otimes M^m & \xrightarrow{\phi} & M^{k+m} \\ \downarrow{\scriptstyle 1 \otimes \xi v} & (-1)^{kh} & \downarrow{\scriptstyle \xi v} \\ M^k \otimes M^{m+h} & \xrightarrow{\phi} & M^{k+m+h} \end{array} \tag{7.13}$$

In the commutative case, the two diagrams are equivalent.

Example. An E^*-algebra object in $(\mathit{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is just an E^*-algebra.

We can again simplify the structure by replacing the v-actions ξv by the single morphism $\eta_v = \xi v \circ \eta: K \to M^h$ for each $v \in E^h$; as in eq. (7.5), we recover ξv from η_v as the composite

$$\xi v: M^n \cong K \otimes M^n \xrightarrow{\eta_v \otimes M^n} M^h \otimes M^n \xrightarrow{\phi} M^{n+h}.$$

Lemma 7.14. *Let $n \mapsto C^n$ be a (commutative) E^*-algebra object in the (graded) additive (symmetric) monoidal category $(\mathcal{C}^{\mathrm{op}}, \otimes, K)$. Then the functor* (7.10) *becomes a (symmetric) monoidal functor*

$$(F, \zeta_F, z_F): (\mathcal{C}, \otimes, K) \longrightarrow (\mathit{Mod}, \otimes, E^*) \qquad (\text{or } (\mathit{Mod}^*, \otimes, E^*)).$$

Proof. For fixed L, the functor $\mathcal{C}(-, L): \mathcal{C}^{\mathrm{op}} \to \mathit{Ab}$ (or Ab^*) takes the E^*-module object C in $\mathcal{C}^{\mathrm{op}}$ to the E^*-module FL, by Lemma 7.7(a). The action of $v \in E^h$ on FL is the composition $\mathrm{Mor}((\xi v)^{\mathrm{op}}, L): FL \to FL$ with $(\xi v)^{\mathrm{op}}: C^{n+h} \to C^n$ (including signs as in eq. (6.4) if $\mathcal{C}$ is graded). As L varies, F takes values in Mod by Lemma 7.7(b); diags. (7.13) show that $\zeta_F: FL \times FN \to F(L \otimes N)$ is E^*-bilinear, allowing us to write $\zeta_F: FL \otimes FN \to F(L \otimes N)$. We define $z_F: E^* \to FK$ on $v \in E^h$ as

$$z_F v: C^h \xrightarrow{(\xi v)^{\mathrm{op}}} C^0 \xrightarrow{\eta^{\mathrm{op}}} K \qquad \text{in } \mathcal{C}, \tag{7.15}$$

to make it an E^*-module homomorphism. □

8. What is a module?

In this section, we study the relationship between the category R-*Mod* of left R-modules and the category *Ab* of abelian groups from several points of view, in order to abstract and generalize it to cover all our main objects of interest in a uniform manner. The central theme is the classical construction by Eilenberg and Moore [13] (or see MacLane [20, Ch. VI]) of a pair of adjoint functors by means of algebras in categories, except that the less familiar (but equivalent) dual formulation, in terms of comonads, turns out to be appropriate.

This will serve as a pattern for our definitions. There are of course variants for graded categories and graded objects. Graded categories can be handled by replacing the graded group $\mathcal{A}^*(X,Y)$ by the group $\bigoplus_n \mathcal{A}^n(X,Y)$, or sometimes even the disjoint union of the sets $\mathcal{A}^n(X,Y)$. Graded objects can be handled by working in the category $\mathcal{A}^{\mathbb{Z}}$ of graded objects $n \mapsto X_n$ in $\mathcal{A}$. We omit details.

The ring R is usually not commutative. Like all our rings, it is understood to have a multiplication ϕ and a unit element 1_R; we define the unit homomorphism $\eta\colon \mathbb{Z} \to R$ by $\eta 1 = 1_R$. The associativity and unit axioms on R take the form of three commutative diagrams in *Ab*:

$$
\text{(i)}\quad \begin{array}{ccc}
R\otimes R\otimes R & \xrightarrow{\phi\otimes R} & R\otimes R \\
\downarrow{\scriptstyle R\otimes\phi} & & \downarrow{\scriptstyle \phi} \\
R\otimes R & \xrightarrow{\phi} & R
\end{array}
\qquad
\text{(ii)}\quad \begin{array}{ccc}
\mathbb{Z}\otimes R & \xrightarrow{\eta\otimes R} & R\otimes R \\
& \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \phi} \\
& & R
\end{array}
\qquad
\text{(iii)}\quad \begin{array}{ccc}
R\otimes \mathbb{Z} & \xrightarrow{R\otimes\eta} & R\otimes R \\
& \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \phi} \\
& & R
\end{array}
\tag{8.1}
$$

In this section (only), all tensor products $\otimes$ and Hom groups are taken over the integers $\mathbb{Z}$.

First Answer. The standard definition of a left R-module (e. g. [25, Defn. 1.2]) equips an abelian group M with a *left action* $\lambda_M\colon R\otimes M \to M$ in *Ab*. It is required to satisfy the usual two axioms, which we express as commutative diagrams:

$$
\text{(i)}\quad \begin{array}{ccc}
R\otimes R\otimes M & \xrightarrow{\phi\otimes M} & R\otimes M \\
\downarrow{\scriptstyle R\otimes\lambda_M} & & \downarrow{\scriptstyle \lambda_M} \\
R\otimes M & \xrightarrow{\lambda_M} & M
\end{array}
\qquad
\text{(ii)}\quad \begin{array}{ccc}
\mathbb{Z}\otimes M & \xrightarrow{\eta\otimes M} & R\otimes M \\
& \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \lambda_M} \\
& & M
\end{array}
\tag{8.2}
$$

Second Answer. We make our First Answer more functorial by introducing the functor $T = R\otimes -\colon Ab \to Ab$. We define natural transformations $\phi\colon TT \to T$ and $\eta\colon I \to T$ on A by $\phi A = \phi_R \otimes A\colon R\otimes R\otimes A \to R\otimes A$ and $(\eta A)x = 1\otimes x \in R\otimes A$. The action on M is now a morphism $\lambda_M\colon TM \to M$, and the axioms (8.2) take the cleaner form

$$\text{(i)}\quad \begin{array}{ccc} TTM & \xrightarrow{\phi M} & TM \\ \downarrow{\scriptstyle T\lambda_M} & & \downarrow{\scriptstyle \lambda_M} \\ TM & \xrightarrow{\lambda_M} & M \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} M & \xrightarrow{\eta M} & TM \\ & \searrow^{=} & \downarrow{\scriptstyle \lambda_M} \\ & & M \end{array} \tag{8.3}$$

Third Answer. We have so far attempted to describe a module structure over a ring without first properly defining a ring structure. In particular, we have not yet mentioned the fact that R is itself an R-module, as is evident by comparing axioms (8.2) with two axioms of (8.1). The function of the other axiom (8.1)(iii) is to ensure that R is a *free* module on one generator 1_R: given $x \in M$, there is a unique module homomorphism $f\colon R \to M$ that satisfies $f1_R = x$, since $fr = f(r1_R) = rf1_R = rx$.

The three axioms on R translate into commutative diagrams of natural transformations in Ab:

$$\text{(i)}\quad \begin{array}{ccc} TTT & \xrightarrow{\phi T} & TT \\ \downarrow{\scriptstyle T\phi} & & \downarrow{\scriptstyle \phi} \\ TT & \xrightarrow{\phi} & T \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} T & \xrightarrow{\eta T} & TT \\ & \searrow^{=} & \downarrow{\scriptstyle \phi} \\ & & T \end{array} \qquad \text{(iii)}\quad \begin{array}{ccc} T & \xrightarrow{T\eta} & TT \\ & \searrow^{=} & \downarrow{\scriptstyle \phi} \\ & & T \end{array} \tag{8.4}$$

Thus a ring structure on R is equivalent to what is known as a *monad* (or *triple*) structure (ϕ, η) on the functor T. By analogy, we call ϕ the *multiplication* and η the *unit* of the monad T. We recognize an R-module as being precisely what is known as a *T-algebra*, namely, an object M equipped with an action morphism $\lambda_M\colon TM \to M$ that satisfies the axioms (8.3).

Fourth Answer. More generally, the first two axioms of (8.4) show that for any abelian group A, the action $\phi A\colon TTA \to TA$ makes TA an R-module, which we call FA; this defines a functor $F\colon \mathit{Ab} \to R\text{-}\mathit{Mod}$. We thus have the factorization $T = VF$, where $V\colon R\text{-}\mathit{Mod} \to \mathit{Ab}$ denotes the forgetful functor. We similarly factor $\phi = V\epsilon F\colon TT = V(FV)F \to VF = T$, where $\epsilon\colon FV \to I$ is defined on the R-module M as $\epsilon M = \lambda_M\colon R \otimes M \to M$; by axiom (8.3)(i), ϵM lies in $R\text{-}\mathit{Mod}$. In this formulation, axiom (8.4)(i) simply defines the natural transformation $V\epsilon\epsilon F\colon TTT = V(FVFV)F \to VF = T$, while the other two reduce to the identities (2.5) relating η and ϵ.

All this works in any category $\mathcal{A}$, as an application of Thm. 2.6(v).

Theorem 8.5. (Eilenberg-Moore) *Given a monad (T, ϕ, η) in $\mathcal{A}$, let $\mathcal{B}$ be the category of T-algebras, $V\colon \mathcal{B} \to \mathcal{A}$ the forgetful functor, and $F\colon \mathcal{A} \to \mathcal{B}$ the functor that assigns to each A in $\mathcal{A}$ the T-algebra $FA = (TA, \phi A)$. Then F is left adjoint to V, $\mathcal{B}(FA, M) \cong \mathcal{A}(A, VM)$ for any M in $\mathcal{B}$, and FA is V-free on A with basis $\eta A\colon A \to TA = VFA$* (in the language of Defn. 2.1).

Proof. We have already outlined most of the proof in the special case when $\mathcal{A} = \mathit{Ab}$ and $T = R \otimes -$, and can apply Thm. 2.6. For further details, see Eilenberg-Moore [13, Thm. 2.2] or MacLane [20, Thm. VI.2.1]. □

The image of F is known as the Kleisli category of all V-free objects.

Fifth Answer. The problem with our answers so far is that they rely heavily on the tensor product, which really has little to do with modules. While tensor products are (as we shall see) convenient for computation, they are simply not available in the nonadditive context of [9].

We therefore replace the functor $T = R \otimes -$ by its equivalent right adjoint $H = \operatorname{Hom}(R, -)\colon \mathit{Ab} \to \mathit{Ab}$. The right adjoint of $\phi\colon TT \to T$ is the *comultiplication* $\psi\colon H \to HH$, which is given on A as the homomorphism

$$\psi A\colon \operatorname{Hom}(R, A) \longrightarrow \operatorname{Hom}(R, \operatorname{Hom}(R, A))$$

that sends $f\colon R \to A$ to $s \mapsto [r \mapsto f(rs)]$. The right adjoint of $\eta\colon I \to T$ is the *counit* $\epsilon\colon H \to I$, where $\epsilon A\colon \operatorname{Hom}(R, A) \to A$ is simply evaluation on 1_R. The axioms (8.4) dualize to

$$\text{(i)}\quad \begin{array}{ccc} H & \xrightarrow{\psi} & HH \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle H\psi} \\ HH & \xrightarrow{\psi H} & HHH \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} H & \xrightarrow{\psi} & HH \\ & \searrow^{=} & \downarrow{\scriptstyle H\epsilon} \\ & & H \end{array} \qquad \text{(iii)}\quad \begin{array}{ccc} H & \xrightarrow{\psi} & HH \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon H} \\ & & H \end{array} \qquad (8.6)$$

which state that (H, ψ, ϵ) is what is known as a *comonad* in Ab.

Similarly, we replace the action λ_M on a module M by the right adjunct *coaction* $\rho_M\colon M \to HM = \operatorname{Hom}(R, M)$. This is given explicitly by $(\rho_M x)r = rx$, which also shows us how to recover the action from ρ_M. The way to think of $\operatorname{Hom}(R, M)$ is as the set of all possible candidates for the R-action on a typical element of M; then ρ_M selects for each $x \in M$ the action $r \mapsto rx$. The action axioms (8.3) become

$$\text{(i)}\quad \begin{array}{ccc} M & \xrightarrow{\rho_M} & HM \\ {\scriptstyle\rho_M}\downarrow & & \downarrow{\scriptstyle \psi M} \\ HM & \xrightarrow{H\rho_M} & HHM \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} M & \xrightarrow{\rho_M} & HM \\ & \searrow^{=} & \downarrow{\scriptstyle \epsilon M} \\ & & M \end{array} \qquad (8.7)$$

which state that M is what is called a *coalgebra* over the comonad H. Occasionally, it is useful to evaluate the right side of (i) on a typical $r \in R$, to yield the commutative square

$$\begin{array}{ccc} M & \xrightarrow{\quad r_M \quad} & M \\ {\scriptstyle\rho_M}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ \operatorname{Hom}(R, M) & \xrightarrow{\operatorname{Hom}(r^*, M)} & \operatorname{Hom}(R, M) \end{array} \qquad (8.8)$$

where $r_M: M \to M$ denotes the action of r on M and $r^*: R \to R$ denotes right multiplication by r.

A homomorphism $f: M \to N$ of R-modules is now a morphism in $\mathcal{Ab}$ for which we have the commutative square

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & HM \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Hf} \\ N & \xrightarrow{\rho_N} & HN \end{array} \tag{8.9}$$

This description successfully avoids all tensor products. It too works quite generally.

Theorem 8.10. *Given a comonad H in $\mathcal{A}$, let $\mathcal{C}$ be the category of H-coalgebras, $V: \mathcal{C} \to \mathcal{A}$ the forgetful functor, and $C: \mathcal{A} \to \mathcal{C}$ the functor that assigns to each A in $\mathcal{A}$ the H-coalgebra HA with the coaction $\psi A: HA \to HHA$. Then C is right adjoint to V, $\mathcal{A}(VM, A) \cong \mathcal{C}(M, CA)$ for all M in $\mathcal{C}$, and $CA = (HA, \psi A)$ is V-cofree on A with cobasis $\epsilon A: HA = VCA \to A$* (in the language of Defn. 2.7).

Proof. This is just Thm. 8.5 in the dual category $\mathcal{A}^{\mathrm{op}}$. □

Sixth Answer. The previous answer is certainly elegant, but we shall need an alternate description of R-modules that does not use ψ and ϵ. The key to achieving this is not to take adjuncts of everything.

Given an element $x \in M$, we put $f = \rho_M x: R \to M$ (given by $fr = rx$). Then commutativity of the square

$$\begin{array}{ccc} R & \xrightarrow{\rho_R} & HR \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Hf} \\ M & \xrightarrow{\rho_M} & HM \end{array} \tag{8.11}$$

expresses the law $(sr)x = s(rx)$. In other words, $f: R \to M$ is a homomorphism of R-modules. The law $1_R x = x$ is expressed as $f 1_R = x$.

Seventh Answer. The first level of abstraction in category theory is to avoid dealing with the elements of a set. The next level is to avoid dealing with the objects in a category. We have not yet used the fact that H is a corepresented functor. Given any functor $F: \mathcal{Ab} \to \mathcal{Ab}$, Yoneda's Lemma (dualized) yields a 1–1 correspondence between natural transformations $\theta: H \to F$ and elements $(\theta R)\mathrm{id}_R \in FR$, where $\theta R: \mathrm{Hom}(R, R) = HR \to FR$ and $\mathrm{id}_R \in HR$ denotes the identity morphism of R. For example, $\psi: H \to HH$ corresponds to $\rho_R \in HHR$, the coaction on the R-module R, and $\epsilon: H \to I$ corresponds to $1_R \in R = IR$. We note that $\rho_R 1_R = \mathrm{id}_R$.

To this end, we replace the object M by the corepresented functor $F_M = \mathrm{Hom}(M, -): \mathcal{Ab} \to \mathcal{Ab}$. (We already did this for $M = R$, to get $F_R = H$.) We replace the coaction morphism $\rho_M: M \to HM$ by the equivalent natural transformation $\rho_M: F_M \to F_M H: \mathsf{Ab} \to \mathsf{Ab}$; explicitly, $\rho_M N: F_M N \to F_M HN$ is

$$\rho_M N: \mathrm{Hom}(M,N) \xrightarrow{H} \mathrm{Hom}(HM,HN) \xrightarrow{\mathrm{Hom}(\rho_M,1)} \mathrm{Hom}(M,HN) \ . \quad (8.12)$$

The axioms (8.7) translate into equivalent commutative diagrams of natural transformations

$$\text{(i)} \quad \begin{array}{ccc} F_M & \xrightarrow{\rho_M} & F_M H \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle F_M \psi} \\ F_M H & \xrightarrow{\rho_M H} & F_M HH \end{array} \qquad \text{(ii)} \quad \begin{array}{ccc} F_M & \xrightarrow{\rho_M} & F_M H \\ & \searrow^{=} & \downarrow{\scriptstyle F_M \epsilon} \\ & & F_M \end{array} \qquad (8.13)$$

We observe that if we take $M = R$, these reduce to axioms (8.6)(i) and (ii).

Eighth Answer. In our applications, we do not have the luxury of starting out with a comonad; we have to construct it. Consequently, we are not able to invoke Thm. 8.10 directly. Instead, we generalize our Sixth Answer. We have to treat modules and rings together.

We assume that $\mathcal{A}$ is a category of sets with structure in the sense that we are given a faithful forgetful functor $W: \mathcal{A} \to \mathsf{Set}$. We assume given:

(i) A functor $H: \mathcal{A} \to \mathcal{A}$;
(ii) An object R in $\mathcal{A}$ that corepresents H in the sense that $WHM = \mathcal{A}(R,M)$, naturally in M;
(iii) An element 1_R of the set WR; (8.14)
(iv) A morphism $\rho_R: R \to HR$ in $\mathcal{A}$, which we call the *pre-coaction* on R, such that $W\rho_R: WR \to WHR = \mathcal{A}(R,R)$ in Set carries $1_R \in WR$ to the identity morphism $\mathrm{id}_R: R \to R$ of R.

We impose no further axioms at this point. In fact, we call *any* morphism $\rho_M: M \to HM$ a *pre-coaction on* M, and a morphism $f: M \to N$ a *morphism of pre-coactions* if it makes diag. (8.9) commute. To see what it takes to make ρ_M a coaction, we consider the function

$$W\rho_M: WM \longrightarrow WHM = \mathcal{A}(R,M) \qquad \text{in } \mathsf{Set} \ .$$

Definition 8.15. Given an object M of $\mathcal{A}$, a *coaction* on M is a pre-coaction $\rho_M: M \to HM$ such that for any element $x \in WM$, the morphism $f = (W\rho_M)x: R \to M$ in $\mathcal{A}$ satisfies:

(i) f makes diag. (8.11) commute, i. e. is a morphism of pre-coactions;
(ii) $Wf: WR \to WM$ sends $1_R \in WR$ to $x \in WM$.

We do not assume yet that ρ_R is itself a coaction. Lemma 8.20 will show that in the presence of suitable additional structure, this definition does agree with previous notions of what a coaction should be.

Ninth Answer. We generalize our Seventh Answer to the category $\mathcal{A}$ as above. We convert everything to corepresented functors. We make no claims to elegance, only that the machinery does what we need.

We replace an object M by the corepresented functor $F_M = \mathcal{A}(M,-): \mathcal{A} \to \mathit{Set}$, and a pre-coaction $\rho_M: M \to HM$ by the equivalent natural transformation $\rho_M: F_M \to F_M H: \mathcal{A} \to \mathit{Set}$. Explicitly, $\rho_M N: F_M N \to F_M HN$ is (cf. eq. (8.12))

$$\rho_M N: \mathcal{A}(M,N) \xrightarrow{H} \mathcal{A}(HM,HN) \xrightarrow{\mathcal{A}(\rho_M,HN)} \mathcal{A}(M,HN) \ . \tag{8.16}$$

In particular, we convert the pre-coaction ρ_R to the natural transformation $\rho_R: WH \to WHH$, where $\rho_R N: WHN \to WHHN$ is

$$\rho_R N: \mathcal{A}(R,N) \xrightarrow{H} \mathcal{A}(HR,HN) \xrightarrow{\mathcal{A}(\rho_R,HN)} \mathcal{A}(R,HN) \ . \tag{8.17}$$

Similarly, if $g: M \to N$ is a morphism of pre-coactions, we obtain the natural transformation $F_g: F_N \to F_M$ and from diag. (8.9) the commutative square

$$\begin{array}{ccc} F_N & \xrightarrow{\rho_N} & F_N H \\ \big\downarrow{\scriptstyle F_g} & & \big\downarrow{\scriptstyle F_g H} \\ F_M & \xrightarrow{\rho_M} & F_M H \end{array} \tag{8.18}$$

We now assume that H is equipped with natural transformations:

(i) $\psi: H \to HH$ such that $W\psi: WH \to WHH$ is the natural transformation ρ_R of eq. (8.17);
(ii) $\epsilon: H \to I$ such that $W\epsilon R: WHR = \mathcal{A}(R,R) \to WR$ sends id_R to 1_R. (8.19)

We assume *no* further properties of ψ and ϵ. In particular, (i) implies (and by naturality is equivalent to) the statement that

$$\mathcal{A}(R,R) = WHR \xrightarrow{W\psi R} WHHR = \mathcal{A}(R,HR)$$

takes id_R to the morphism ρ_R.

Lemma 8.20. *Assume we have a category $\mathcal{A}$ equipped with W, H, R, ψ, and ϵ, satisfying the axioms* (8.14) *and* (8.19). *Then given an object M of $\mathcal{A}$, a pre-coaction $\rho_M: M \to HM$ is a coaction in the sense of Defn. 8.15 if and only if it makes diags.* (8.7) *commute.*

Proof. Since W is faithful, we may apply W to diags. (8.7) and work with diagrams of sets. Thus (i) becomes

$$\begin{array}{ccccccc}
 & & WM & \xrightarrow{W\rho_M} & WHM & \xleftarrow{=} & \mathcal{A}(R,M) \\
 & & \downarrow{\scriptstyle W\rho_M} & & \downarrow{\scriptstyle WH\rho_M} & & \downarrow{\scriptstyle \mathcal{A}(R,\rho_M)} \\
\mathcal{A}(R,M) & \xrightarrow{=} & WHM & \xrightarrow{W\psi M} & WHHM & \xleftarrow{=} & \mathcal{A}(R,HM)
\end{array}$$

We evaluate on any $x \in WM$ and put $f = (W\rho_M)x: R \to M$. The upper route gives $\rho_M \circ f: R \to HM$, while the lower route gives $Hf \circ \rho_R: R \to HM$ by axiom (8.19)(i). These agree if and only if f is a morphism of pre-coactions as in diag. (8.11).

For diag. (8.7)(ii) we consider

$$\begin{array}{ccccc}
WM & \xrightarrow{W\rho_M} & WHM & \xleftarrow{WHf} & WHR \\
 & \searrow{\scriptstyle =} & \downarrow{\scriptstyle W\epsilon M} & & \downarrow{\scriptstyle W\epsilon R} \\
 & & WM & \xleftarrow{Wf} & WR
\end{array}$$

The element $f \in WHM = \mathcal{A}(R,M)$ lifts to $\mathrm{id}_R \in WHR = \mathcal{A}(R,R)$, which by axiom (8.19)(ii) maps to $1_R \in WR$. Thus $(Wf)1_R = x$ is exactly what we need. □

As in our Seventh Answer, we convert the objects in diags. (8.7) to corepresented functors.

Corollary 8.21. *The pre-coaction $\rho_M: M \to HM$ is a coaction (in the sense of Defn. 8.15) if and only if the associated natural transformation $\rho_M: F_M \to F_M H: \mathcal{A} \to \mathsf{Set}$ makes diags.* (8.13) *commute.* □

Now we can recover the full strength of Thm. 8.10.

Lemma 8.22. *Assume that $\rho_R: R \to HR$ is a coaction in the sense of Defn. 8.15, and that ψ and ϵ satisfy axioms* (8.19)*. Then:*

(a) *ψ and ϵ make H a comonad in $\mathcal{A}$;*

(b) *A pre-coaction $\rho_M: M \to HM$ makes M an H-coalgebra if and only if it is a coaction in the sense of Defn. 8.15.*

Proof. The first two axioms of (8.6) are just axioms (8.13) for $M = R$, which we have by Cor. 8.21. For the third, we have to show that $W\epsilon HN \circ W\psi N: WHN \to WHN$ is the identity. We evaluate on $g \in WHN = \mathcal{A}(R,N)$. From eq. (8.17), $(W\psi N)g = Hg \circ \rho_R$. We consider the diagram in fig. 1, which commutes merely because $\epsilon: H \to I$ is natural. We start from $\mathrm{id}_R \in \mathcal{A}(R,R)$, which maps to $Hg \circ \rho_R \in \mathcal{A}(R,HN)$, $1_R \in WR$, $\mathrm{id}_R \in \mathcal{A}(R,R)$ (by axiom (8.14)(iv)), and hence to $g \in \mathcal{A}(R,N)$.

Part (b) is then a restatement of Lemma 8.20. □

Change of categories. Now assume $\mathcal{A}'$ is a second category, equipped similarly with W', H', ψ' etc. satisfying axioms (8.14) and (8.19). We assume that $\mathcal{A}$ and $\mathcal{A}'$

Figure 1. Diagram for the comonad H.

$$
\begin{array}{ccccccc}
\mathcal{A}(R,R) & \xrightarrow{=} & WHR & \xrightarrow{W\epsilon R} & WR & & \\
\big\downarrow{\scriptstyle \mathcal{A}(R,\rho_R)} & & \big\downarrow{\scriptstyle WH\rho_R} & & \big\downarrow{\scriptstyle W\rho_R} & & \\
\mathcal{A}(R,HR) & \xrightarrow{=} & WHHR & \xrightarrow{W\epsilon HR} & WHR & \xleftarrow{=} & \mathcal{A}(R,R) \\
\big\downarrow{\scriptstyle \mathcal{A}(R,Hg)} & & \big\downarrow{\scriptstyle WHHg} & & \big\downarrow{\scriptstyle WHg} & & \big\downarrow{\scriptstyle \mathcal{A}(R,g)} \\
\mathcal{A}(R,HN) & \xrightarrow{=} & WHHN & \xrightarrow{W\epsilon HN} & WHN & \xleftarrow{=} & \mathcal{A}(R,N)
\end{array}
$$

are connected by a somewhat forgetful functor $V{:}\,\mathcal{A} \to \mathcal{A}'$ such that $W'V = W$. Then given an object M of $\mathcal{A}$, there is an obvious natural transformation $\omega_V{:}\,F_M \to F_{VM}V{:}\,\mathcal{A} \to \mathit{Set}$, defined on N in $\mathcal{A}$ as $V{:}\,\mathcal{A}(M,N) \to \mathcal{A}'(VM,VN)$.

We assume that H and H' are related by a natural transformation $\theta{:}\,VH \to H'V{:}\,\mathcal{A} \to \mathcal{A}'$. If $\rho_M{:}\,M \to HM$ is a pre-coaction on M in $\mathcal{A}$, we give VM the pre-coaction

$$
\rho_{VM}{:}\,VM \xrightarrow{V\rho_M} VHM \xrightarrow{\theta M} H'VM \qquad \text{in } \mathcal{A}'. \tag{8.23}
$$

This we convert to the commutative diagram of natural transformations

$$
\begin{array}{ccc}
F_M & \xrightarrow{\rho_M} & F_M H \\
 & & \big\downarrow{\scriptstyle \omega_V H} \\
\Big\downarrow{\scriptstyle \omega_V} & & F_{VM}VH \\
 & & \big\downarrow{\scriptstyle F_{VM}\theta} \\
F_{VM}V & \xrightarrow{\rho_{VM}V} & F_{VM}H'V
\end{array}
$$

Because WH is corepresented by R, the natural transformation $W'\theta{:}\,WH = W'VH \to W'H'V$ is determined by a certain morphism $u{:}\,R' \to VR$ in $\mathcal{A}'$ (which will be obvious in applications); explicitly, given M in $\mathcal{A}$, $W'\theta M{:}\,WHM = W'VHM \to W'H'VM$ is

$$
W'\theta M{:}\,\mathcal{A}(R,M) \xrightarrow{V} \mathcal{A}'(VR,VM) \xrightarrow{\mathcal{A}'(u,VM)} \mathcal{A}'(R',VM)\ .
$$

Lemma 8.24. *Assume that u satisfies:*

(i) *$u{:}\,R' \to VR$ is a morphism of pre-coactions* (this uses eq. (8.23))*;*

(ii) *$W'u{:}\,W'R' \to W'VR = WR$ sends $1_{R'}$ to 1_R.*

Then $\theta{:}\,VH \to H'V$ is a natural transformation of comonads, in the sense that we *have commutative diagrams*

$$\text{(i)}\quad \begin{array}{ccc} VH & \xrightarrow{V\psi} & VHH \\ \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\theta H} \\ & & H'VH \\ & & \Big\downarrow{\scriptstyle H'\theta} \\ H'V & \xrightarrow{\psi' V} & H'H'V \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} VH & & \\ \Big\downarrow{\scriptstyle\theta} & \searrow^{V\epsilon} & \\ H'V & \xrightarrow{\epsilon' V} & V \end{array}$$

Proof. We apply W' and expand all the definitions. □

9. E-cohomology of spectra

In this section, we adapt the results and techniques of sections 3 and 4 to the graded stable homotopy category ***Stab**** of spectra. Our general reference is Adams [3]. Many results become simpler and most are well known, apart from the topological embellishments.

Cohomology. Any *based* space (X, o) may be regarded as a spectrum, via the stabilization functor ***Ho***$'$ $\to$ ***Stab***. Given a spectrum E, whether X is a based space or a spectrum, we define the *reduced E-cohomology* of X as $E^*(X,o) = \{X,E\}^* =$ ***Stab***$^*(X,E)$, the graded group of morphisms in ***Stab**** from X to E that has the component $E^k(X,o) = \{X,E\}^k$ in degree k. The universal class $\iota \in E^0(E,o)$ is thus the identity map of E.

The suspension isomorphism $E^*(X,o) \cong E^*(\Sigma X,o)$ is that induced by the canonical desuspension map $\Sigma X \simeq X$ of degree 1 in ***Stab**** given by (6.1) (with signs as in eq. (6.3)). Equivalently, given $x \in E^k(X,o)$, the class $\Sigma x \in E^{k+1}(X,o)$ is the composite of the maps $\Sigma X \to \Sigma E$ and $\Sigma E \simeq E$ (with no sign).

This cohomology is the only kind available in the stable context. For compatibility with the unstable notation of section 3, we *always* write the cohomology of a spectrum X, redundantly but unambiguously, as $E^*(X,o)$.

The *skeleton* filtration of $E^*(X,o)$ can be defined exactly as unstably, in eq. (3.33). It is quite satisfactory for spectra of finite type (those with each skeleton finite), which include many of our examples, but is wildly inappropriate for nonconnective spectra such as KU. We therefore give $E^*(X,o)$ the *profinite* filtration and topology, exactly as in Defn. 4.9. If necessary, we complete it as in Defn. 4.11 to the *completed cohomology* $E^*(X,o)\hat{}$.

A map $r: E \to E$ in ***Stab**** of degree h induces the stable cohomology operation $r_*: E^k(X,o) \to E^{k+h}(X,o)$. It commutes with suspension up to the sign $(-1)^h$ as in fig. 2.

Spaces. For a *space* X, it is more useful, whether or not X is based, to work with the *absolute E-cohomology* of X defined by $E^*(X) = E^*(X^+,o)$, as suggested by eq. (3.3). The absolute theory is thereby *included* in the reduced theory. In

Figure 2. Operations and suspension.

$$\begin{array}{ccc} E^k(X,o) & \xrightarrow{\ r_*\ } & E^{k+h}(X,o) \\ \Sigma \big\downarrow \cong & (-1)^h & \Sigma \big\downarrow \cong \\ E^{k+1}(\Sigma X,o) & \xrightarrow{\ r_*\ } & E^{k+h+1}(\Sigma X,o) \end{array}$$

particular, the coefficient group of E-cohomology is $E^* = E^*(T) = E^*(T^+,o) = \pi_*^S(E,o)$. Conversely, every graded cohomology theory on spaces has this form.

Theorem 9.1. *Let $E^*(-)$ be a graded cohomology theory on Ho in the sense of section 3. Then:*

(a) *There is a spectrum E, unique up to equivalence, that represents $E^*(-)$ as above;*

(b) *Any sequence of cohomology operations $r_k: E^k(X) \to E^{k+h}(X)$, that are defined and natural for spaces X and commute with suspension up to the sign $(-1)^h$ as in fig. 2, is induced by a map of spectra $r: E \to E$ of degree h.*

Sketch proof. The representing spaces $\underline{E}_n$ provided by Thm. 3.17 and the structure maps $f_n: \Sigma\underline{E}_n \to \underline{E}_{n+1}$ from Defn. 3.19 are used to construct the spectrum E for (a). In (b), Thm. 3.6(b) provides a representing map $r_k: \underline{E}_k \to \underline{E}_{k+h}$ for each operation r_k. We take $X = \underline{E}_k$ in fig. 2 and evaluate on the universal class ι_k. By Lemma 3.21, the class $(-1)^{k+h}\,\Sigma r_k \iota_k$ corresponds to the upper route $f_{k+h} \circ \Sigma r_k$ in the square

$$\begin{array}{ccc} \Sigma\underline{E}_k & \xrightarrow{\ \Sigma r_k\ } & \Sigma\underline{E}_{k+h} \\ \big\downarrow f_k & & \big\downarrow f_{k+h} \\ \underline{E}_{k+1} & \xrightarrow{\ r_{k+1}\ } & \underline{E}_{k+h+1} \end{array} \qquad \text{in } \mathit{Ho}' \tag{9.2}$$

Meanwhile, by Defn. 3.19, $r_{k+1} \circ f_k$ corresponds to the class $(-1)^k r_{k+1} \Sigma \iota_k$. Thus the square commutes, and we may take the maps r_k as the raw material for constructing the desired map of spectra $r: E \to E$. (However, r need not be unique.) A similar construction gives the uniqueness in (a). Further details depend on the choice of implementation of *Stab**. □

Stabilization. In Thms. 3.17 and 9.1 we have two ways to represent E-cohomology, in the categories *Ho* and *Stab**. Thus for any space X, we may identify:

(i) The *cohomology class* $x \in E^k(X)$;

(ii) The *map of spectra* $x_S: X^+ \to E$, of degree k, defined by $x = x_S^* \iota$;

(iii) The *map of spaces* $x_U\colon X \to \underline{E}_k$, defined by $x = x_U^* \iota_k$.

We compare the two maps by taking $x = \iota_k$ in (ii).

Definition 9.3. For each integer k, we define the *stabilization* map of spectra $\sigma_k\colon \underline{E}_k \to E$ by $\sigma_k^* \iota = \iota_k \in E^k(\underline{E}_k, o) \subset E^k(\underline{E}_k)$. It has degree k.

It follows immediately that for any $x \in E^k(X)$, x_S is the composite

$$x_S\colon X^+ \xrightarrow{x_U^+} \underline{E}_k^+ \longrightarrow \underline{E}_k \xrightarrow{\sigma_k} E \qquad \text{in } \mathit{Stab}^*. \tag{9.4}$$

If x is based, i.e. $x \in E^k(X, o)$, we can simplify this to

$$x_S\colon X \xrightarrow{x_U} \underline{E}_k \xrightarrow{\sigma_k} E \qquad \text{in } \mathit{Stab}^*. \tag{9.5}$$

In practice, we normally omit the suffixes $_S$ and $_U$ and write x for all three. (On occasion, this can cause some difficulty with signs, as x and x_S have degree k, while x_U is a map of spaces and has no degree.)

Lemma 9.6. *The structure maps $f_k\colon \Sigma\underline{E}_k \to \underline{E}_{k+1}$ and the stabilization maps σ_k are related by the commutative square*

$$\begin{array}{ccc} \Sigma\underline{E}_k & \xrightarrow{f_k} & \underline{E}_{k+1} \\ \downarrow{\simeq} & & \downarrow{\sigma_{k+1}} \\ \underline{E}_k & \xrightarrow{\sigma_k} & E \end{array} \qquad \text{in } \mathit{Stab}^*$$

in which we use the canonical desuspension map (6.1).

Proof. The upper route in the square corresponds to the class $f_k^* \iota_{k+1} = (-1)^k\, \Sigma\iota_k \in E^*(\Sigma\underline{E}_k, o)$. If we write $g\colon \Sigma\underline{E}_k \simeq \underline{E}_k$ for the desuspension, the lower route corresponds to $(\sigma_k \circ g)^* \iota = (-1)^k g^* \sigma_k^* \iota = (-1)^k g^* \iota_k = (-1)^k\, \Sigma\iota_k$. □

These maps display E as the homotopy colimit in Stab^* of the based spaces $\underline{E}_n$. The relevant Milnor short exact sequence (cf. diag. (3.38)) is

$$0 \longrightarrow \lim_n{}^1 E^{k-1}(\underline{E}_n, o) \longrightarrow E^k(E, o) \longrightarrow \lim_n E^k(\underline{E}_n, o) \longrightarrow 0\ . \tag{9.7}$$

Moreover, the profinite topology makes the map from $E^k(E, o)$ an open map and therefore a homeomorphism whenever it is a bijection. (Take the basic open set $F^a E^*(E, o)$ defined by some finite subspectrum $E_a \subset E$. This inclusion lifts (up to homotopy) to a map of spectra (of degree $-n$) $E_a \to \underline{E}_{n,b} \subset \underline{E}_n$ for some n and some finite subcomplex $\underline{E}_{n,b}$ of $\underline{E}_n$. Then the image of $F^a E^*(E, o)$ contains $F^b E^*(\underline{E}_n)$.)

The maps σ_n also relate the stable and unstable operations in Thm. 9.1(b). Suppose the stable operation r of degree h is represented stably in Stab^* by a map of spectra $r_S\colon E \to E$ of degree h, and unstably in *Ho* by the maps $r_k\colon \underline{E}_k \to \underline{E}_{k+h}$. These maps are related by the commutative square

$$\begin{array}{ccc} \underline{E}_k & \xrightarrow{r_U} & \underline{E}_{k+h} \\ \big\downarrow{\scriptstyle\sigma_k} & & \big\downarrow{\scriptstyle\sigma_{k+h}} \\ E & \xrightarrow{r_S} & E \end{array} \qquad \text{in } \mathit{Stab}^* \tag{9.8}$$

because by the definition of σ_n, both routes represent the class $r\iota_k \in E^*(\underline{E}_k, o)$. Cohomologically,

$$\sigma_k^* r = (-1)^{kh} r \circ \sigma_k = (-1)^{kh} r_k \qquad \text{in } E^{k+h}(\underline{E}_k, o). \tag{9.9}$$

(Without the sign, $r \mapsto r_k$ is *not* in general an E^*-module homomorphism.)

Ring spectra. Now let E be a ring spectrum, i. e. a commutative monoid object in the symmetric monoidal category $(\mathit{Stab}, \wedge, T^+)$, with multiplication $\phi\colon E \wedge E \to E$ and unit $\eta\colon T^+ \to E$. (All our ring spectra are assumed commutative.)

Given $x \in E^*(X,o)$ and $y \in E^*(Y,o)$, we define their *cross product* $x \times y \in E^*(X \wedge Y, o)$ as

$$x \times y\colon X \wedge Y \xrightarrow{x \wedge y} E \wedge E \xrightarrow{\phi} E \ .$$

These products are biadditive, commutative, associative, and have $\eta \in E^*(T^+, o)$ as the unit in the sense that under the isomorphism

$$E^*(T^+ \wedge X, o) \cong E^*(X, o) \tag{9.10}$$

induced by $X \simeq T^+ \wedge X$, $\eta \times x$ corresponds to x.

The coefficient group $E^* = E^*(T^+, o) = \pi_*^S(E, o)$ becomes a commutative ring, using $\times$-products and $T^+ \simeq T^+ \wedge T^+$ for multiplication; its unit element is $1_T = \eta \in E^0(T^+, o)$. Then $E^*(X, o)$ becomes a *left* E^**-module* if we define $vx \in E^*(X,o)$ for $v \in E^*$ and $x \in E^*(X,o)$ as corresponding to $v \times x \in E^*(T^+ \wedge X, o)$ under the isomorphism (9.10); expanded, this is

$$vx\colon X \simeq T^+ \wedge X \xrightarrow{v \wedge x} E \wedge E \xrightarrow{\phi} E \ .$$

Rearranging slightly, we see that scalar multiplication by v on $E^*(-, o)$ is represented by the map

$$\xi v\colon E \simeq T^+ \wedge E \xrightarrow{v \wedge E} E \wedge E \xrightarrow{\phi} E \qquad \text{in } \mathit{Stab}^*, \tag{9.11}$$

as in eq. (3.27). The map ξv corresponds to the class $v\iota$. We apply Lemma 7.7(d).

Lemma 9.12. *The actions* (9.11) *make the ring spectrum* E *an* E^**-module object in the graded category* Stab^**, which represents the* E^**-module structure on cohomology* $E^*(-, o)$. □

Now that $\times$-products are known to be E^*-bilinear, we can write them in the more familiar and useful form

$$\times\colon E^*(X,o) \otimes E^*(Y,o) \longrightarrow E^*(X \wedge Y, o) \ . \tag{9.13}$$

Together with the definition $z\colon E^* = E^*(T^+, o)$, they make E-cohomology a symmetric monoidal functor

$$(E^*(-,o), \times, z)\colon (\mathit{Stab}^{*\mathrm{op}}, \wedge, T^+) \longrightarrow (\mathit{Mod}^*, \otimes, E^*) \ . \tag{9.14}$$

For spaces X and Y, we have $X^+ \wedge Y^+ \cong (X \times Y)^+$, and we recover the unstable $\times$-pairing (3.22) as a special case of (9.13). The reduced diagonal map $\Delta^+\colon X^+ \to (X \times X)^+ \cong X^+ \wedge X^+$ and projection $q^+\colon X^+ \to T^+$ make X^+ a commutative monoid object in $\mathit{Stab}^{\mathrm{op}}$, so that $E^*(X) = E^*(X^+, o)$ becomes a commutative monoid object in Mod, i.e. a commutative E^*-algebra. We have a multiplicative graded cohomology theory in the sense of section 3.

The stable and unstable multiplication maps are related by the commutative diagram, similar to eq. (9.5),

$$\begin{array}{ccc} \underline{E}_k \times \underline{E}_m & \xrightarrow{\phi_U} & \underline{E}_{k+m} \\ \downarrow & & \downarrow = \\ \underline{E}_k \wedge \underline{E}_m & \longrightarrow & \underline{E}_{k+m} \\ \downarrow \sigma_k \wedge \sigma_m & & \downarrow \sigma_{k+m} \\ E \wedge E & \xrightarrow{\phi_S} & E \end{array} \qquad \text{in } \mathit{Stab}^* \tag{9.15}$$

However, there is a technical difficulty in extending Thm. 9.1 to make E a ring spectrum.

Theorem 9.16. *Assume there are no weakly phantom classes in the groups $E^0(E,o)$, $E^0(E\wedge E,o)$ and $E^0(E\wedge E\wedge E,o)$. Then any natural multiplicative structure that is defined on $E^*(X)$ for all spaces X (as in section 3) is induced by a unique ring spectrum structure on E.*

Proof. Theorem 3.25 provides a compatible family of unstable multiplications $\phi_U\colon \underline{E}_k \times \underline{E}_m \to \underline{E}_{k+m}$ and the unit $\eta_U\colon T \to \underline{E}_0$. We immediately recover η_S from η_U by taking $x = 1 \in E^*(T)$ in eq. (9.4), but there is a problem with ϕ_S. We may regard $E \wedge E$ as the homotopy colimit in Stab^* of the spaces $\underline{E}_n \wedge \underline{E}_n$ and obtain the Milnor short exact sequence

$$0 \longrightarrow \lim_n{}^1 E^{-1}(\underline{E}_n \wedge \underline{E}_n, o) \longrightarrow E^0(E\wedge E, o) \longrightarrow \lim_n E^0(\underline{E}_n \wedge \underline{E}_n, o) \longrightarrow 0$$

analogous to (9.7). It shows that there exists a lifting ϕ_S that makes diag. (9.15) commute for all k and m, but it is not unique in general. Our hypotheses simplify the diagrams for $E^0(E,o)$, $E^0(E\wedge E,o)$, and the analogue for $E^0(E\wedge E\wedge E,o)$ to the limit term only, to ensure respectively that ϕ_S: (i) has η_S as a unit; (ii) is unique and commutative; and (iii) is associative. □

Homology. The companion homology theory to $E^*(-)$ is easily defined (see G. W. Whitehead [36] or Adams [3]) in the stable context. The *reduced E-homology* of a spectrum or based space X is simply

$$E_*(X,o) = \{T^+, E \wedge X\}^* = \pi^S_*(E \wedge X, o), \tag{9.17}$$

the stable homotopy of $E \wedge X$. (We observe that $\pi_*^S(-,o)$ is itself the homology theory given by taking $E = T^+$, but we do not wish to write it $T_*^+(-,o)$.) It has the component $E_k(X,o) = \{T^+, E \wedge X\}^{-k} = \pi_k^S(E \wedge X, o)$ in degree $-k$. Again, we have the suspension isomorphism $E_k(X,o) \cong E_{k+1}(\Sigma X, o)$, induced by (the inverse of) the canonical desuspension (6.1).

For a space X, we have the *absolute E-homology*

$$E_*(X) = E_*(X^+, o) = \{T^+, E \wedge X^+\}^*,$$

as suggested by eq. (3.3) for cohomology, and it satisfies axioms dual to (3.1). The coefficient group is

$$E_*(T) = E_*(T^+, o) = \{T^+, E \wedge T^+\}^* \cong \{T^+, E\}^* = E^* = \pi_*^S(E, o),$$

the same as E-cohomology. (But we note that $E_k(T) \cong E^{-k}$.)

When E is a ring spectrum, it too is a symmetric monoidal functor

$$(E_*(-,o), \times, z) \colon (\mathit{Stab}^*, \wedge, T^+) \longrightarrow (\mathit{Mod}^*, \otimes, E^*), \tag{9.18}$$

with an obvious $\times$-product pairing

$$\times \colon E_*(X,o) \otimes E_*(Y,o) \longrightarrow E_*(X \wedge Y, o), \tag{9.19}$$

if we use the above identification $z \colon E^* \cong E_*(T^+, o)$. We can ask whether eq. (9.19) is an isomorphism. The following two results provide all the homology isomorphisms we need.

Theorem 9.20. *Assume that $E_*(X,o)$ or $E_*(Y,o)$ is a free or flat E^*-module. Then the pairing* (9.19) *induces the Künneth isomorphism $E_*(X \wedge Y, o) \cong E_*(X,o) \otimes E_*(Y,o)$ in homology.*

Proof. The proof of Thm. 4.2 works just as well for spectra. □

Lemma 9.21. *For $E = H(\mathbb{F}_p)$, $K(n)$, MU, BP, or KU, $E_*(E,o)$ is a free E^*-module.*

Remark. For $E = KU$, this is a substantial result of Adams-Clarke [4, Thm. 2.1].

Proof. For $E = H(\mathbb{F}_p)$ or $K(n)$, all E^*-modules are free. For $E = MU$ or BP, the result is well known [3]. For KU, we defer the proof until we have a good description of $KU_*(KU,o)$, in section 14. □

The homology version of the Milnor short exact sequence (9.7) is simply

$$E_*(E,o) = \operatorname*{colim}_n E_*(\underline{E}_n, o), \tag{9.22}$$

analogous to eq. (4.4). More generally, from the definition (9.17),

$$E_*(X,o) = \operatorname*{colim}_a E_*(X_a, o) \tag{9.23}$$

for any X, where X_a runs over all finite subspectra of X.

Strong duality. The Kronecker pairing $\langle -,- \rangle \colon E^*(X,o) \otimes E_*(X,o) \to E^*$ is easily constructed for spectra E and X, directly from the definitions. As in section 4, it makes sense to ask whether the right adjunct form

$$d \colon E^*(X,o) \longrightarrow DE_*(X,o) \tag{9.24}$$

is an isomorphism, or better, a homeomorphism. Again, one theorem is all we need. It includes the unstable result Thm. 4.14.

Theorem 9.25. *Assume that $E_*(X,o)$ is a free E^*-module. Then X* has strong duality, *i. e. d in* (9.24) *is a homeomorphism between the profinite topology on $E^*(X,o)$ and the dual-finite topology on $DE_*(X,o)$. In particular, $E^*(X,o)$ is complete Hausdorff.*

E-modules. To establish Thm. 9.25, we must take E-modules seriously. An *E-module* is a spectrum G equipped with an action map $\lambda_G\colon E \wedge G \to G$ in Stab that satisfies the usual two axioms (8.3), using the functor $T = E \wedge -$. Everything is formally identical to the R-module case, with the monoid object R in the symmetric monoidal category $(\mathit{Ab}, \otimes, \mathbb{Z})$ replaced by E in $(\mathit{Stab}^*, \wedge, T^+)$. We form the category E-Mod of E-modules, and the graded version E-Mod^*.

Theorem 9.26. *The forgetful functor $V\colon E\text{-}\mathit{Mod}^* \to \mathit{Stab}^*$ has the free functor $E \wedge -\colon \mathit{Stab}^* \to E\text{-}\mathit{Mod}^*$ as a left adjoint, and for any spectrum X and E-module G, we have a natural homeomorphism*

$$G^*(X) = \mathit{Stab}^*(X, VG) \cong E\text{-}\mathit{Mod}^*(E \wedge X, G)\ . \tag{9.27}$$

Proof. Theorem 8.5 provides the isomorphism. We make it trivially a homeomorphism by topologizing $E\text{-}\mathit{Mod}^*(E \wedge X, G)$, *not* as a subspace of $G^*(E \wedge X)$, but by filtering it by the submodules

$$F^a E\text{-}\mathit{Mod}^*(E \wedge X, G) = \operatorname{Ker}[E\text{-}\mathit{Mod}^*(E \wedge X, G) \longrightarrow E\text{-}\mathit{Mod}^*(E \wedge X_a, G)],$$

where X_a runs through the finite subspectra of X. □

Corollary 9.28. *Let $g\colon E \wedge X \to E \wedge Y$ be an E-module morphism* (not necessarily of the form $E \wedge f$). *Then for any E-module G, $g^*\colon E\text{-}\mathit{Mod}^*(E \wedge Y, G) \to E\text{-}\mathit{Mod}^*(E \wedge X, G)$ is continuous.*

Proof. The right adjunct of g is a map $f\colon X \to E \wedge Y$ of spectra. Given a finite $X_a \subset X$, we choose a finite $Y_b \subset Y$ such that $f|X_a$ factors through $E \wedge Y_b$; then by taking left adjuncts, g restricts to a morphism of E-modules $E \wedge X_a \to E \wedge Y_b$. It follows that $g^*(F^b) \subset F^a$, in the notation of the Theorem. □

The desired theorem follows directly, as in Adams [3, Lemma II.11.1].

Proof of Thm. 9.25. We choose a basis of $E_*(X,o)$ consisting of maps $S^{n_\alpha} \to E \wedge X$ of degree zero, and use them as the components of a map $f\colon W = \bigvee_\alpha S^{n_\alpha} \to E \wedge X$. By Thm. 9.26, the left adjunct of f is a morphism of E-modules $g\colon E \wedge W \to E \wedge X$. By construction, g induces an isomorphism $g_*\colon E_*(W,o) \cong E_*(X,o)$ on homotopy groups, and is therefore an isomorphism in Stab. It follows formally that g is also an isomorphism in E-Mod. We factor d to obtain the commutative diagram

$$\begin{array}{ccccc}
d\colon E^*(X,o) & \xrightarrow{\cong} & E\text{-}\mathit{Mod}^*(E \wedge X, E) & \xrightarrow{\pi_*^S(-,o)} & DE_*(X,o) \\
 & & \Big\downarrow{\scriptstyle \operatorname{Mor}(g,E)} & & \Big\downarrow{\scriptstyle Dg_*} \\
d\colon E^*(W,o) & \xrightarrow{\cong} & E\text{-}\mathit{Mod}^*(E \wedge W, E) & \xrightarrow{\pi_*^S(-,o)} & DE_*(W,o)
\end{array}$$

Theorem 9.26 provides the two marked homeomorphisms. By Cor. 9.28, $\mathrm{Mor}(g,E)$ is a homeomorphism. It is clear from Lemma 4.10 that W has strong duality. We have a diagram of homeomorphisms. □

Künneth homeomorphisms. As the Künneth pairing (9.13) is continuous, we can complete it to

$$\times: E^*(X,o)\,\widehat{\otimes}\,E^*(Y,o) \longrightarrow E^*(X\wedge Y,o)\hat{}, \tag{9.29}$$

and the symmetric monoidal functor (9.14) to another one,

$$(E^*(-,o)\hat{}, \times, z): (\mathit{Stab}^{*\mathrm{op}}, \wedge, T^+) \longrightarrow (\mathit{FMod}^*, \widehat{\otimes}, E^*), \tag{9.30}$$

for completed cohomology. As in Thm. 4.19, we combine Thm. 9.25 with Thm. 9.20 to deduce Künneth homeomorphisms.

Theorem 9.31. *Assume that $E_*(X,o)$ and $E_*(Y,o)$ are free E^*-modules. Then the pairing* (9.29) *induces the cohomology Künneth homeomorphism*

$$E^*(X\wedge Y,o) \cong E^*(X,o)\,\widehat{\otimes}\,E^*(Y,o)\ . \qquad \square$$

10. What is a stable module?

In this section, we give various interpretations of what it means to have a module over the stable operations on E-cohomology, with a view to future generalization in [9] to unstable operations. We are primarily interested in the absolute cohomology $E^*(X) = E^*(X^+,o)$ of a space X, and state most results for this case only. Nevertheless, we sometimes need the more general reduced cohomology $E^*(X,o)$ of a spectrum X.

An operation $r: E^*(-,o) \to E^*(-,o)$ is *stable* if it is natural on Stab^*. It is automatically additive, Stab^* being an additive category, but need not be an E^*-module homomorphism.

Recall from section 3 (or section 9) that the profinite filtration makes $E^*(X)$ (or $E^*(X,o)$) a filtered E^*-module. When Hausdorff, it is an object of FMod^*. We remind that all tensor products are taken over the coefficient ring $E^* = E^*(T) = E^*(T^+,o)$ unless otherwise indicated, where T denotes the one-point space and T^+ the sphere spectrum.

First Answer. Since E-cohomology $E^*(-,o)$ is represented in Stab^* by the spectrum E, Yoneda's Lemma identifies the ring $\mathcal{A}$ of all stable operations with the endomorphism ring $\mathrm{End}(E) = \{E,E\}^* = E^*(E,o)$ of E. Its unit element is ι, the universal class of E. It acts on $E^*(X) = E^*(X^+,o)$ by composition,

$$\lambda_X: \mathcal{A}\otimes E^*(X) = E^*(E,o)\otimes E^*(X) \longrightarrow E^*(X)\ . \tag{10.1}$$

In particular, for each $v \in E^h$ we have the *scalar multiplication* operation $x \mapsto vx$ on $E^*(X)$, which by Lemma 9.12 is represented by the map of spectra $\xi v: E \to E$ of degree h in eq. (9.11) or the element $v\iota \in E^h(E,o)$. This defines an embedding of rings (usually not central)

$$\xi: E^* \longrightarrow E^*(E,o) = \mathcal{A}, \tag{10.2}$$

which we used already in eq. (10.1) to make $\mathcal{A}$ an E^*-bimodule under composition and λ_X a homomorphism of E^*-modules.

Notation. Standard notation for tensor products is ambiguous here, and will soon become hopelessly inadequate for coping with the future plethora of bimodules and multimodules. When it is necessary to convey detailed information about the many E^*-actions involved, we rewrite λ_X as

$$\lambda_X : E1^*(E2, o) \otimes_2 E2^*(X) \longrightarrow E1^*(X), \tag{10.3}$$

which we call the E^*-*action scheme* of λ_X. Here, Ei denotes a copy of E tagged for identification, and $\otimes_i$ indicates a tensor product that is to be formed using the two E^*-actions labeled by i. If desired, we can add information about the degrees by writing

$$\lambda_X : E1^i(E2, o) \otimes_2 E2^j(X) \longrightarrow E1^{i+j}(X).$$

For example, the composition

$$\mathcal{A} \otimes \mathcal{A} = E^*(E, o) \otimes E^*(E, o) \xrightarrow{\circ} E^*(E, o) = \mathcal{A} \tag{10.4}$$

has action scheme $E1^*(E2, o) \otimes_2 E2^*(E3, o) \to E1^*(E3, o)$. We promise to use this over-elaborate notation sparingly.

The important special case $X = T$ of the action (10.1) gives

$$\lambda_T : \mathcal{A} \cong \mathcal{A} \otimes E^* \longrightarrow E^*, \tag{10.5}$$

which encodes the action of $\mathcal{A}$ on the coefficient ring $E^* = E^*(T)$.

The action (10.1) satisfies the usual two laws:

$$(sr)x = s(rx); \qquad \iota x = x; \tag{10.6}$$

for any operations s and r and any $x \in E^*(X)$. This suggests that a stable module structure on a given E^*-module M should consist of an action $\lambda_M : \mathcal{A} \otimes M \longrightarrow M$ that satisfies these laws and is a homomorphism of left E^*-modules. Because the tensor product is taken over E^*, this implies that λ_M extends the given module action of E^* on M.

Unfortunately, this description is inadequate even for finite X. In the classical case $E = H(\mathbb{F}_p)$, $\mathcal{A}$ is the Steenrod algebra over $\mathbb{F}_p$, which is generated by the Steenrod operations subject to explicitly given Adem relations. In general, $\mathcal{A}$ is uncountable, which suggests that we should make use of the profinite topology on it. We described a filtration for tensor products in eq. (4.15). However, the tensor product in the action (10.1) is formed using the *right* E^*-action on $\mathcal{A}$, for which we have not defined a filtration; worse, the usual E^*-module structure on the tensor product is not the one that makes λ_X an E^*-module homomorphism. We have to find something else.

Second Answer. In [1, 3], Adams suggested that for suitable ring spectra E, one could avoid the various limit problems and infinite products that are inherent in cohomology by replacing the action (10.1) by the dual coaction on homology. Stably, the only difference between homology operations and cohomology operations is the possibility of weakly phantom cohomology operations; in practice, these usually do not exist. Unstably, however, the difference is vast. Our ignorance of unstable homology operations in general forces us to learn to live with cohomology. We therefore dualize only partially. We defer the details until section 11.

If $E_*(E,o)$ is a free E^*-module, we can convert the action λ_X in (10.1) into a coaction (after completion)

$$\rho_X\colon E^*(X) \longrightarrow E^*(X) \mathbin{\widehat{\otimes}} E_*(E,o) \tag{10.7}$$

(whose action scheme is $E2^*(X) \to E1^*(X) \mathbin{\widehat{\otimes}}_1 E1_*(E2,o)$). There is much structure on $E_*(E,o)$, as explicated in [1, 3]. Dual to the composition (10.4) with unit (10.2) in $E^*(E,o)$, there is a coassociative comultiplication with counit

$$\psi = \psi_S\colon E_*(E,o) \longrightarrow E_*(E,o) \otimes E_*(E,o); \qquad \epsilon = \epsilon_S\colon E_*(E,o) \longrightarrow E^*;$$

on $E_*(E,o)$. The action axioms (10.6) on λ_X translate into the diagrams

$$\text{(i)}\quad \begin{array}{ccc} E^*(X) & \xrightarrow{\rho_X} & E^*(X) \mathbin{\widehat{\otimes}} E_*(E,o) \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle 1\otimes\psi_S} \\ E^*(X) \mathbin{\widehat{\otimes}} E_*(E,o) & \xrightarrow{\rho_X\otimes 1} & E^*(X) \mathbin{\widehat{\otimes}} E_*(E,o) \mathbin{\widehat{\otimes}} E_*(E,o) \end{array}$$

$$\text{(ii)}\quad \begin{array}{ccc} E^*(X) & \xrightarrow{\rho_X} & E^*(X) \mathbin{\widehat{\otimes}} E_*(E,o) \\ \downarrow & & \downarrow{\scriptstyle 1\otimes\epsilon_S} \\ E^*(X)\hat{} & \xrightarrow{=} & E^*(X) \mathbin{\widehat{\otimes}} E^* \end{array} \tag{10.8}$$

These are in effect the usual axioms for a comodule coaction over $E_*(E,o)$ on $E^*(X)\hat{}$, the only novelty being the two distinct E^*-actions on $E_*(E,o)$.

Historically, the original example was developed by Milnor [22] in the case $E = H(\mathbb{F}_p)$, to give a description of the Steenrod operations that is both elegant and more informative; we summarize it in section 14. Even in this case, the completed tensor product is needed in the coaction (10.7) when X is infinite-dimensional. For finite spaces or spectra X, one can use Spanier-Whitehead duality to switch between homology and cohomology. This leads to Adams's coaction on homology [1, Lecture 3], except that he used a *left* coaction in an attempt to make the E^*-actions easier to track. It turns out that in cohomology, the right coaction, even with its notational difficulties, is both more customary and more convenient.

Third Answer. We rewrite our Second Answer in a more categorical form in order to allow generalization. We still leave the details to section 11.

As the target of ρ_X is complete, we lose nothing if we complete the cohomology $E^*(-)$ to $E^*(-)\hat{}$ everywhere. We define the functor $S'\colon \mathit{FMod}^* \to \mathit{FMod}^*$ by $S'M = M \mathbin{\widehat{\otimes}} E_*(E,o)$. Then we can use ψ_S and ϵ_S to define natural transformations

$$\psi'_S M = M \otimes \psi_S\colon S'M \to S'S'M, \qquad \epsilon'_S M = M \otimes \epsilon_S\colon S'M \to M.$$

The coalgebra properties of ψ_S and ϵ_S will supply the necessary axioms (8.6) to make S' a *comonad* in FMod^*.

We rewrite the coaction (10.7) as a morphism $\rho'_X: E^*(X)\hat{} \to S'(E^*(X)\hat{})$ in the category FMod^*. This converts the axioms (10.8) into diags. (8.7), which then state that $E^*(X)\hat{}$ is precisely an *S'-coalgebra* in FMod^*.

We have condensed our answer down to the single word *S'-coalgebra.*

Fourth Answer. We are not done rewriting yet. The problem with our Third Answer is that it still depends heavily on the tensor product, an essentially bilinear construction that is simply unavailable for operations that are not additive (not that this has stopped us from trying).

We therefore go back to our First Answer and convert λ_X to adjoint form, as suggested by section 8. We treat $x \in E^*(X)$ as a map of spectra $x: X^+ \to E$, and note that the E^*-module homomorphism $x^*: \mathcal{A} = E^*(E, o) \to E^*(X)$ is continuous. (There is the usual sign, $x^*r = (-1)^{\deg(x)\deg(r)} r \circ x = (-1)^{\deg(x)\deg(r)} rx$, from eq. (6.3).)

For convenience, we assume that $\mathcal{A}$ is Hausdorff and work in FMod^*. Given any complete Hausdorff filtered E^*-module M (i. e. object of FMod), we define

$$SM = \mathsf{FMod}^*(\mathcal{A}, M) = \mathsf{FMod}^*(E^*(E, o), M) \ . \tag{10.9}$$

Then for any space X, we define the coaction

$$\rho_X: E^*(X) \longrightarrow S(E^*(X)\hat{}) = \mathsf{FMod}^*(\mathcal{A}, E^*(X)\hat{}) \tag{10.10}$$

on $x \in E^*(X)$ by $\rho_X x = x^*: \mathcal{A} = E^*(E, o) \to E^*(X)\hat{}$, completing as necessary. In the important special case $X = T$, we find

$$\rho_T: E^* = E^*(T) \longrightarrow \mathsf{FMod}^*(\mathcal{A}, E^*(T)) = SE^*(T) = SE^*. \tag{10.11}$$

Similarly, we have $\rho_X: E^*(X, o) \to S(E^*(X, o)\hat{})$ for spectra and based spaces X.

Theorem 10.12. *Assume that the E^*-module $\mathcal{A} = E^*(E, o)$ is Hausdorff* (as is true for $E = H(\mathbb{F}_p)$, MU, BP, KU, or $K(n)$ by Lemma 9.21 and Thm. 9.25)*. Then we can make the functor S defined in eq.* (10.9) *a comonad in the category FMod of complete Hausdorff filtered E^*-modules.*

Now that we have a suitable comonad, the definition of stable module is clear. This is the answer that will generalize satisfactorily.

Definition 10.13. A *stable (E-cohomology) module* is an S-coalgebra in FMod^*, i. e. a complete Hausdorff filtered E^*-module M that is equipped with a morphism

$$\rho_M: M \longrightarrow SM \qquad \text{in } \mathsf{FMod}^* \tag{10.14}$$

that is E^*-linear and continuous and satisfies the coaction axioms (8.7). We then *define* the action of $r \in \mathcal{A}^h = E^h(E, o)$ on $x \in M^k$ by $rx = (-1)^{kh}(\rho_M x)r \in M$.

A closed submodule $L \subset M$ is called *(stably) invariant* if ρ_M restricts to $\rho_L: L \to SL$. Then the quotient M/L also inherits a stable module structure.

The group SM may be thought of as the set of all candidates for the action of $\mathcal{A}$ on a typical element of M. Then ρ_M selects for each $x \in M$ an appropriate action on x. The axioms (8.7) translate into the usual action axioms (10.6). If we evaluate the first only partially, we obtain the commutative square

$$\begin{array}{ccc} M & \xrightarrow{r} & M \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_M} \\ SM & \xrightarrow{\omega_r M} & SM \end{array} \tag{10.15}$$

where the natural transformation ω_r is defined on $f \in SM$ as

$$(\omega_r M)f = (-1)^{\deg(r)\deg(f)} f \circ r^* \colon \mathcal{A} \longrightarrow M,$$

using $r^* \colon \mathcal{A} = E^*(E,o) \to E^*(E,o) = \mathcal{A}$. It may be viewed as the analogue of diag. (8.8).

Theorem 10.16. *Assume that the E^*-module $\mathcal{A} = E^*(E,o)$ is Hausdorff* (as is true for $E = H(\mathbb{F}_p)$, MU, BP, KU, or $K(n)$ by Lemma 9.21 and Thm. 9.25). *Then:*

(a) *We can factor ρ_X (defined in eq.* (10.10)*) through $E^*(X)\hat{}$ as $\rho_X \colon E^*(X)\hat{} \to S(E^*(X)\hat{})$, to make $E^*(X)\hat{}$ a stable module for any space X (and similarly $E^*(X,o)\hat{}$ for spectra);*

(b) *ρ is universal: given an object N of FMod^*, any transformation*

$$\theta X \colon E^*(X,o) \longrightarrow \mathsf{FMod}^*(N, E^*(X,o)\hat{})$$

(or $\widehat{\theta} X \colon E^*(X,o)\hat{} \to \mathsf{FMod}^*(N, E^*(X,o)\hat{})$) of any degree, that is defined for all spectra X and natural on Stab^*, is induced from ρ_X by a unique morphism $f \colon N \to \mathcal{A}$ in FMod^* as the composite

$$\begin{aligned} \theta X \colon E^*(X,o) &\xrightarrow{\rho_X} SE^*(X,o)\hat{} = \mathsf{FMod}^*(\mathcal{A}, E^*(X,o)\hat{}) \\ &\xrightarrow{\mathrm{Hom}(f,1)} \mathsf{FMod}^*(N, E^*(X,o)\hat{}) \ . \end{aligned} \tag{10.17}$$

Proofs of Thms. 10.12 and 10.16. The discussion in section 8 is intended to suggest that these two proofs are interlaced. The main proof is in seven steps. Lemma 9.12 provides the E^*-module object E in Stab^*. We find it useful to write $\mathrm{id}_{\mathcal{A}}$ for the identity map $\mathcal{A} \to \mathcal{A}$, considered as an element of $S\mathcal{A}$.

Step 1. We introduce an E^*-module structure (different from the obvious one) on the graded group SM defined by eq. (10.9); by hypothesis, $\mathcal{A}$ is an object of FMod^* and S is defined. By Lemma 7.6(a), the additive functor

$$\mathsf{FMod}^*(E^*(-,o)\hat{}, M) \colon \mathsf{Stab}^* \xrightarrow{E^*(-,o)\hat{}} \mathsf{FMod}^{*\mathrm{op}} \xrightarrow{\mathrm{Mor}(-,M)} \mathsf{Ab}^*$$

takes the E^*-module object E to an E^*-module object in Ab^*, i.e. makes SM an E^*-module. (By Lemma 7.1(a), the additive structure on SM must be the obvious one.) As M varies, Lemma 7.7(b) shows that SM is functorial, and we have a functor $S \colon \mathsf{FMod}^* \to \mathsf{Mod}^*$. We enrich it later, in Step 3, to take values in FMod^*.

Step 2. We show that ρ_X is an E^*-module homomorphism. Given a spectrum (or space) X, the cohomology functor $E^*(-,o)\hat{}\colon \mathsf{Stab}^{*\mathrm{op}} \to \mathsf{FMod}^*$ induces the natural transformation of additive functors

$$\mathsf{Stab}^*(X,-) \longrightarrow \mathsf{FMod}^*(E^*(-,o)\hat{}, E^*(X,o)\hat{})\colon \mathsf{Stab}^* \longrightarrow \mathsf{Ab}^* .$$

We apply this to the E^*-module object E in Stab^*; then Lemma 7.6(c) shows that ρ_X is a homomorphism of E^*-modules.

Step 3. In order to make S and ρ_X take values in FMod^*, we must filter SM. If M is filtered by the submodules F^aM, we filter SM in the obvious way by the $F^a(SM) = S(F^aM)$, which are E^*-submodules because S is a functor. We trivially have the exact sequence

$$0 \longrightarrow SF^aM \longrightarrow SM \longrightarrow S(M/F^aM),$$

which we use to rewrite the filtration in the more useful form

$$F^aSM = \mathrm{Ker}[SM \longrightarrow S(M/F^aM)]. \tag{10.18}$$

(In fact, there is a short exact sequence in all our examples. However, we do not exploit this fact because (a) it requires a stronger hypothesis on E, but more importantly, (b) it does not generalize correctly.)

It is not difficult to see directly that SM is complete Hausdorff. Because M is complete Hausdorff, we have the limit $M = \lim_a M/F^aM$, which is automatically preserved by S. This yields by eq. (10.18) the inclusion

$$\lim_a \frac{SM}{F^aSM} \cong \lim_a \mathrm{Im}\left[SM \longrightarrow S\frac{M}{F^aM}\right] \subset \lim_a S\frac{M}{F^aM} = SM$$

in Ab^*. But this inclusion is visibly epic and therefore an isomorphism, which makes SM complete Hausdorff.

We have now defined S as a functor taking values in FMod^* as required. Our choice of the profinite topology on $E^*(X,o)$ and the naturality of ρ make it clear that ρ_X is continuous and factors as asserted in Thm. 10.16(a).

Step 4. We convert the object $E^*(X)\hat{}$ of FMod^* to the corepresented functor $F_X = \mathsf{FMod}^*(E^*(X)\hat{},-)\colon \mathsf{FMod}^* \to \mathsf{Ab}^*$ (and similarly $E^*(X,o)\hat{}$ for spectra X). As suggested by eq. (8.16), we also convert the coaction ρ_X to the natural transformation $\rho_X\colon F_X \to F_XS\colon \mathsf{FMod}^* \to \mathsf{Ab}^*$. Given M, the homomorphism

$$\rho_XM\colon F_XM = \mathsf{FMod}^*(E^*(X)\hat{}, M) \longrightarrow \mathsf{FMod}^*(E^*(X)\hat{}, SM) = F_XSM \tag{10.19}$$

is defined by $(\rho_XM)f = Sf \circ \rho_X\colon E^*(X)\hat{} \to S(E^*(X)\hat{}) \to SM$.

Step 5. We define the natural transformation $\psi\colon S \to SS$ by taking $X = E$ in eq. (10.19), so that

$$\psi M\colon SM = \mathsf{FMod}^*(\mathcal{A}, M) \longrightarrow \mathsf{FMod}^*(\mathcal{A}, SM) = SSM \tag{10.20}$$

is given on the element $f\colon \mathcal{A} \to M$ of SM as the composite

$$(\psi M)f\colon \mathcal{A} = E^*(E,o) \xrightarrow{\rho_E} SE^*(E,o) = S\mathcal{A} \xrightarrow{Sf} SM .$$

(In terms of elements, this is $r \mapsto [s \mapsto f(r^*s) = (-1)^{\deg(r)\deg(s)} f(sr)]$.) When we substitute the E^*-module object E for X in diag. (10.19), Lemma 7.6(c) shows that ψM takes values in Mod^*. Naturality in M shows that ψ is filtered and takes values in FMod^*, as required.

Step 6. The other required natural transformation,

$$\epsilon M \colon SM = \mathsf{FMod}^*(\mathcal{A}, M) \longrightarrow M, \tag{10.21}$$

is defined simply as evaluation on the universal class $\iota \in \mathcal{A}$, i. e. $(\epsilon M) f = f\iota$. Once again, naturality in M shows that ϵM is filtered, but we have to verify that ϵM is an E^*-module homomorphism. (All proofs involving ϵ are necessarily somewhat computational, because the definition is.) Additivity is clear. Take any $v \in E^h$. By Lemma 9.12, the structure map $\xi v \colon E \to E$ induces $(\xi v)^* \iota = v\iota$ in $E^*(E, o)$. Given an element $f \colon \mathcal{A} = E^*(E, o) \to M$ of SM, we defined $vf = \pm f \circ (\xi v)^*$ in Step 1; then

$$\epsilon(vf) = \pm\, \epsilon(f \circ (\xi v)^*) = \pm\, f(\xi v)^* \iota = \pm\, f(v\iota) = vf\iota = v\epsilon f,$$

using the given E^*-linearity of f.

Step 7. We show that S is a comonad and that $E^*(X)\hat{}$ is an S-coalgebra. Naturality of ρ with respect to the map of spectra $x \colon X^+ \to E$ for any $x \in E^*(X)$ shows that ρ_X is a coaction on $E^*(X)\hat{}$ in the sense of Defn. 8.15, using $R = \mathcal{A} = E^*(E, o)$, $\rho_R = \rho_E$, and $1_R = \iota$. By Lemma 8.20, ρ_X makes $E^*(X)\hat{}$ (or $E^*(X, o)\hat{}$) an S-coalgebra; we constructed ψ and ϵ to satisfy the conditions (8.19). Finally, S is a comonad by Lemma 8.22(a).

Yoneda's Lemma gives Thm. 10.16(b) for θ. Because $E^*(-, o)$ is represented by E, θ is classified by the element $f = (\theta E)\iota \in \mathsf{FMod}^*(N, \mathcal{A})$ and so given by eq. (10.17). If we are given $\widehat{\theta}$ instead, we compose with $E^*(X, o) \to E^*(X, o)\hat{}$ to obtain θ. Conversely, any θ factors through $\widehat{\theta}$ by naturality. □

11. Stable comodules

Although the Fourth Answer of section 10, in terms of stable modules, is the cleanest and most general, the Second Answer, in terms of stable comodules, is usually available and more practical in the cases of interest. (One could argue that this feature is what makes these cases interesting.) At least for $E = MU$ or *BP*, such comodules are called *cobordism comodules.* This is the context for Landweber theory, as developed in [17, 18] and discussed in section 15 for *BP*.

Rather than develop the Second and Third Answers from scratch, we deduce them from the Fourth Answer by comparing the algebraic structures on $E^*(E, o)$ and $E_*(E, o)$. This section is entirely algebraic in the sense that the only spectrum we study in any depth is E. In Thm. 11.35 we show that the structure maps η_R, ψ_S, and ϵ_S on $E_*(E, o)$ agree with those of Adams.

We assume later in this section that $E_*(E, o)$ is a free E^*-module, which is true for our five examples by Lemma 9.21. The duality $d \colon E^*(E, o) \cong DE_*(E, o)$ in Thm. 9.25 allows us to identify the following, with only slight abuse of notation:

$$\begin{array}{ll} \text{(i) The } \textit{cohomology operation } r \text{ on } E^*(-) \text{ (or } E^*(-,o)\text{);} \\ \text{(ii) The } \textit{class } r\iota \in E^*(E,o)\text{, which we also write simply as } r\text{;} \\ \text{(iii) The } \textit{map of spectra } r\colon E \to E\text{, a morphism in } \mathsf{Stab}^*\text{;} \\ \text{(iv) The } E^*\textit{-linear functional } \langle r,-\rangle\colon E_*(E,o) \to E^*. \end{array} \tag{11.1}$$

The *degree* of r is the same in any of these contexts (once we remember that $E_i(E,o)$ has degree $-i$).

The bimodule algebra $E_*(E,o)$. As $E_*(E,o)$ is better understood and smaller than $E^*(E,o)$, (iv) is the preferred choice in (11.1). There is much structure on $E_*(E,o)$. First, like all E-homology, it is a left E^*-module.

When we apply the additive functor $E_*(-,o)$ to the E^*-module object E in Lemma 9.12, we obtain by Lemma 7.6(a) the E^*-module object $E_*(E,o)$ in *Mod*, equipped with the E^*-module homomorphism $(\xi v)_*$ of degree h for each $v \in E^h$. To extract a bimodule as commonly understood, we define the right action by

$$c \cdot v = (-1)^{hm}(\xi v)_* c \qquad \text{for } v \in E^h,\ c \in E_m(E,o),$$

to ensure that $v'(c\cdot v) = (v'c)\cdot v$, with no signs. Nevertheless, we find it technically convenient to keep all functions and operations on the left and work with $(\xi v)_*$.

The ring spectrum structure (ϕ,η) on E induces the *multiplication*

$$\phi = \phi_S\colon E_*(E,o) \otimes E_*(E,o) \xrightarrow{\times} E_*(E \wedge E, o) \xrightarrow{\phi_*} E_*(E,o)$$

and *left unit*

$$\eta = \eta_S\colon E^* \cong E_*(T^+,o) \xrightarrow{\eta_*} E_*(E,o)$$

for $E_*(E,o)$. In particular, we have the unit element $1 = \eta 1 \in E_0(E,o)$.

The equation $vc = v(1c) = (v1)c = (\eta v)c$ describes the left E^*-action in terms of ϕ and η, and implies that η is a ring homomorphism. We shall see presently that the right action is similarly determined by its effect on 1.

Definition 11.2. We define the *right unit* function $\eta_R\colon E^* \to E_*(E,o)$ on $v \in E^* = E^*(T^+,o)$ by $\eta_R v = v_* 1$, using the homology homomorphism $v_*\colon E^* \cong E_*(T^+,o) \to E_*(E,o)$ induced by the map $v\colon T^+ \to E$ in Stab^*.

We summarize all this structure. We recall that in general, the left and right units and E^*-actions on $E_*(E,o)$ are quite different.

Proposition 11.3. *In $E_*(E,o)$, for any ring spectrum E:*

(a) *$E_*(E,o)$ is an E^*-bimodule;*

(b) *The unit element $1 = \eta 1 = \eta_R 1$ is well defined;*

(c) *The multiplication ϕ makes $E_*(E,o)$ a commutative E^*-algebra with respect to the left or right E^*-action;*

(d) *$\eta\colon E^* \to E_*(E,o)$ and $\eta_R\colon E^* \to E_*(E,o)$ are ring homomorphisms;*

(e) *The left action of $v \in E^*$ is left multiplication by $v1$;*

(f) *The right action of $v \in E^*$ is right multiplication by $\eta_R v$.*

Proof. For (c), we apply the E-homology symmetric monoidal functor (9.18) to the commutative monoid object E in *Stab*, to obtain the commutative monoid object $E_*(E,o)$ in *Mod*, i.e. commutative E^*-algebra, with respect to the left E^*-action.

We trivially have (b), because the map $\eta: T^+ \to E$ is $1_T \in E^0(T)$. For (f), we apply E-homology to eq. (9.11), which expresses ξv in terms of the multiplication. This implies that η_R is a ring homomorphism. □

Remark. There is a well-known conjugation $\chi: E_*(E,o) \to E_*(E,o)$ which interchanges the left and right E^*-actions. We avoid it because it does not generalize to the unstable situation.

The functor S'. Duality and Lemma 6.16(b) provide the natural isomorphism

$$S'M = M \mathbin{\widehat{\otimes}} E_*(E,o) \cong \mathit{FMod}^*(E^*(E,o), M) = SM \tag{11.4}$$

for any complete Hausdorff filtered E^*-module M, with action scheme

$$(S'M)2 = M1 \mathbin{\widehat{\otimes}_1} E1_*(E2,o) \cong \mathit{FMod}_1^*(E1^*(E2,o), M1) = (SM)2\ .$$

The functors S and S' are those of section 10. Moreover, this is an isomorphism of filtered E^*-modules in *FMod* if we filter $S'M$ as in eq. (4.15), which is the same as filtering it by the submodules $S'F^aM$. (We remind that $E_*(E,o)$, like all homology, invariably carries the discrete topology.) Explicitly, with the help of Prop. 11.3, the isomorphism of E^*-actions is expressed by

$$\langle r \circ (v\iota), c\rangle = \langle r, (\eta_R v)c\rangle \qquad \text{for } r \in E^*(E,o),\ v \in E^*,\ c \in E_*(E,o). \tag{11.5}$$

In view of the proliferation of E^*-actions, one must be careful in applying duality; the correct way to establish all properties of S' is to deduce them from the corresponding properties of S in section 10 by applying the isomorphism (11.4). (Once our equivalences are well established, we shall normally omit the $'$ everywhere.)

The coalgebra structure on $E_*(E,o)$. The comonad structure (ψ_S, ϵ_S) on S in Thm. 10.12 corresponds under eq. (11.4) to a comonad structure on S' consisting of natural transformations $\psi'M: S'M \to S'S'M$ and $\epsilon'M: S'M \to M$. By naturality and the case $M = E^*$, $\psi'M$ must take the form $M \otimes \psi$ for a certain well-defined comultiplication

$$\psi = \psi_S: E_*(E,o) \longrightarrow E_*(E,o) \otimes E_*(E,o) \tag{11.6}$$

(with action scheme $E1_*(E3,o) \to E1_*(E2,o) \otimes_2 E2_*(E3,o)$). It is not cocommutative (in any ordinary sense). Similarly, $\epsilon'M$ must have the form

$$M \otimes \epsilon_S: S'M = M \mathbin{\widehat{\otimes}} E_*(E,o) \longrightarrow M \otimes E^* \cong M$$

for some well-defined counit

$$\epsilon = \epsilon_S: E_*(E,o) \longrightarrow E^*. \tag{11.7}$$

(Here and elsewhere, the isomorphism $M \otimes E^* \cong M$ always involves the usual sign, $x \otimes v \mapsto (-1)^{\deg(x)\deg(v)} vx$.) Both ψ_S and ϵ_S are morphisms of E^*-bimodules.

Lemma 11.8. *Assume that $E_*(E,o)$ is a free E^*-module. Then the homomorphisms ψ_S and ϵ_S in diags.* (11.6) *and* (11.7) *make $E_*(E,o)$ a coalgebra over E^*.*

Proof. By taking $M = E^*$, the comonad axioms (8.6) for S' translate into the coassociativity of ψ_S,

$$\begin{array}{ccc} E_*(E,o) & \xrightarrow{\psi_S} & E_*(E,o)\otimes E_*(E,o) \\ \downarrow{\scriptstyle \psi_S} & & \downarrow{\scriptstyle 1\otimes\psi_S} \\ E_*(E,o)\otimes E_*(E,o) & \xrightarrow{\psi_S\otimes 1} & E_*(E,o)\otimes E_*(E,o)\otimes E_*(E,o) \end{array} \tag{11.9}$$

and the two counit axioms on ϵ_S:

$$\begin{array}{ccc} E_*(E,o) & \xrightarrow{\psi_S} & E_*(E,o)\otimes E_*(E,o) \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \epsilon_S\otimes 1} \\ & & E^*\otimes E_*(E,o) \end{array} \qquad \begin{array}{ccc} E_*(E,o) & \xrightarrow{\psi_S} & E_*(E,o)\otimes E_*(E,o) \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle 1\otimes\epsilon_S} \\ & & E_*(E,o)\otimes E^* \end{array}$$

(i) (ii)

(11.10)

These commutative diagrams express precisely what we mean by saying that $E_*(E,o)$ is a coalgebra. □

Comodules. We are now ready to convert Defn. 10.13 of a stable module and Thm. 10.16, by means of the isomorphism (11.4). The coaction $\rho_M\colon M \to SM$ in (10.14) on a stable module M corresponds to a coaction $\rho_M\colon M \to S'M = M \mathbin{\widehat{\otimes}} E_*(E,o)$.

Definition 11.11. A *stable (E-cohomology) comodule* structure on a complete Hausdorff filtered E^*-module M (i.e. object of FMod) consists of a coaction $\rho_M\colon M \to M \mathbin{\widehat{\otimes}} E_*(E,o)$ that is a continuous morphism of filtered E^*-modules (i.e. morphism in FMod, with action scheme $M2 \to M1 \mathbin{\widehat{\otimes}}_1 E1_*(E2,o)$) and satisfies the axioms

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M\mathbin{\widehat{\otimes}}E_*(E,o) \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle M\otimes\psi_S} \\ M\mathbin{\widehat{\otimes}}E_*(E,o) & \xrightarrow{\rho_M\otimes 1} & M\mathbin{\widehat{\otimes}}E_*(E,o)\mathbin{\widehat{\otimes}}E_*(E,o) \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M\mathbin{\widehat{\otimes}}E_*(E,o) \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle M\otimes\epsilon_S} \\ & & M\otimes E^* \end{array} \tag{11.12}$$

(i) (ii)

Theorem 11.13. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then given a complete Hausdorff* *filtered E^*-module M (i. e. object of FMod), a stable module structure on M in the sense of Defn. 10.13 is precisely equivalent under eq.* (11.4) *to a stable comodule structure on M in the sense of Defn. 11.11.*

Proof. The axioms (11.12) are just the axioms (8.7) interpreted for S'. □

Theorem 11.14. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then:*

(a) *For any space (or spectrum) X, there is a natural coaction*

$$\rho_X : E^*(X) \longrightarrow E^*(X) \widehat{\otimes} E_*(E,o) \tag{11.15}$$

(or $\rho_X : E^*(X,o) \to E^*(X,o) \widehat{\otimes} E_*(E,o)$) in FMod that makes $E^*(X)\hat{}$ (or $E^*(X,o)\hat{}$) a stable comodule, which corresponds by Thm. 11.13 to the coaction ρ_X of Thm. 10.16;

(b) *ρ is universal: given a discrete E^*-module N, any transformation $\theta X : E^*(X,o) \to E^*(X,o) \widehat{\otimes} N$ (or $\widehat{\theta} X : E^*(X,o)\hat{} \to E^*(X,o)\hat{} \widehat{\otimes} N$), that is defined and natural for all spectra X, is induced from ρ_X by a unique morphism $f : E_*(E,o) \to N$ of E^*-modules, as*

$$\theta X : E^*(X,o) \xrightarrow{\rho_X} E^*(X,o) \widehat{\otimes} E_*(E,o) \xrightarrow{1 \otimes f} E^*(X,o) \widehat{\otimes} N\ . \tag{11.16}$$

Proof. For (a), we combine Thm. 11.13 with Thm. 10.16(a).

However, (b) is *not* a translation of Thm. 10.16(b), although the proof is similar. Because $E^*(-,o)$ is represented in Stab^* by E, θ is determined by the value $(\theta E)\iota \in E^*(E,o) \widehat{\otimes} N$, which corresponds to the desired homomorphism $f : E_*(E,o) \to N$ under the isomorphism $E^*(E,o) \widehat{\otimes} N \cong \mathit{Mod}^*(E_*(E,o), N)$ of Lemma 6.16(a). □

Remark. The universal property (b) shows that diags. (11.12), with $M = E^*(X,o)$, may be viewed as *defining* ψ_S and ϵ_S in terms of ρ. Three applications of the uniqueness in (b) then show that ψ_S is coassociative and has ϵ_S as a counit.

Remark. From a purely theoretical point of view, one should write the coaction (11.15) as $\rho_X : E^*(X)\hat{} \to E^*(X)\hat{} \widehat{\otimes} E_*(E,o)$, using *three* completions, in order to stay inside the category *FMod* of filtered modules at all times. This seems excessive. The way we are writing ρ_X, using just the $\widehat{\otimes}$ (and that only when necessary) and leaving the other completions implicit, conveys exactly the same algebraic and topological information after completion. But we warn that in using diag. (11.12)(ii), $M \widehat{\otimes} E^* \cong M$ is valid if and only if M is complete Hausdorff. In particular, $E^*(X)$ can only be a stable comodule if it is already Hausdorff.

Linear functionals. Theorem 11.13 establishes the equivalence between stable modules and stable comodules. For applications, we need to make this correspondence explicit. Given a stable comodule M, we recover the action of $r \in E^*(E,o)$ on the stable module M from ρ_M as

$$r : M \xrightarrow{\rho_M} M \widehat{\otimes} E_*(E,o) \xrightarrow{M \otimes \langle r, - \rangle} M \otimes E^* \cong M, \tag{11.17}$$

by means of the isomorphism (11.4), whose details are supplied by Lemma 6.16(b).

To make everything explicit, we choose $x \in M^k$ and write

$$\rho_M x = \sum_\alpha (-1)^{\deg(x_\alpha)\deg(c_\alpha)} x_\alpha \otimes c_\alpha \qquad \text{in } M \mathbin{\widehat{\otimes}} E_*(E,o), \tag{11.18}$$

where the sum may be infinite. (We introduce signs here to keep other formulae cleaner. It is noteworthy that in the explicit formulae of section 14, these signs are invariably +1.) Then from Cor. 6.17, the corresponding element $x^* \in \mathit{FMod}^k(E^*(E,o),M)$ is given by

$$x^* r = (-1)^{k \deg(r)} \sum_\alpha \langle r, c_\alpha \rangle \, x_\alpha \ ,$$

and conversely. We rewrite this more conveniently as

$$rx = \sum_\alpha \langle r, c_\alpha \rangle \, x_\alpha \qquad \text{in } M, \text{ for all } r \tag{11.19}$$

(with no signs at all!), where we emphasize that the c_α and x_α depend only on x, not on r. The sums here may be infinite, but will converge because eq. (11.18) does.

The statement that ρ_M is an E^*-homomorphism may be expressed as

$$r(vx) = \sum_\alpha \langle r, (\eta_R v) c_\alpha \rangle \, x_\alpha \qquad \text{in } M, \text{ for all } r, \tag{11.20}$$

for any $v \in E^*$, with the help of eq. (11.5).

It is important for our purposes *not* to require the c_α to form a basis of $E_*(E,o)$, or even be linearly independent; but if they do form a basis, the x_α are uniquely determined by eq. (11.19) as $x_\alpha = c_\alpha^* x$, where c_α^* denotes the operation dual to c_α.

As ϵ_S is dual to $\xi\colon E^* \to E^*(E,o)$, we see that

$$\langle \iota, - \rangle = \epsilon_S \colon E_*(E,o) \longrightarrow E^*, \tag{11.21}$$

which is obvious by comparing diag. (11.12)(ii) with eq. (11.17). In other words, the functional ϵ_S corresponds to the identity operation ι in the list (11.1). In practice, ϵ_S is always easy to write down; it is ψ_S that causes difficulties. Of course, ψ_S is dual to composition (10.4), as we make explicit later in eq. (11.34).

The cohomology of a point. Our first test space is the one-point space T. We have enough to determine the stable structure of $E^*(T)$ completely.

Proposition 11.22. *Let r be a stable operation on E-cohomology and $v \in E^*$. Assume that $E_*(E,o)$ is a free E^*-module. Then in the stable comodule $E^*(T) = E^*$:*

(a) *The action of the operation r is given by*

$$rv = \langle r, \eta_R v \rangle \qquad \text{in } E^*(T) = E^*; \tag{11.23}$$

(b) *The coaction $\rho_T \colon E^* \to E^* \otimes E_*(E,o) \cong E_*(E,o)$ coincides with the right unit $\eta_R \colon E^* \to E_*(E,o)$;*

(c) *If we write $E^* = \pi_*^S(E,o)$ and regard $r\colon E \to E$ as a map of spectra, the induced homomorphism $r_*\colon E^* \to E^*$ on stable homotopy groups is given by $r_* v = \langle r, \eta_R v \rangle$.*

Proof. If we regard v as a map $v: T^+ \to E$ and use Defn. 11.2, we find

$$rv = \pm\, v^* r = \pm\, \langle v^* r, 1\rangle = \langle r, v_* 1\rangle = \langle r, \eta_R v\rangle,$$

which is (a). We compare eq. (11.19) with eq. (11.18) to rewrite this as $\rho_T v = 1 \otimes \eta_R v$, which gives (b). Parts (a) and (c) are equivalent, because both rv and $r_* v$ are the same morphism $r \circ v: T^+ \to E \to E$ in Stab^*. □

The cohomology of spheres. Our second test space is the sphere S^k. By definition, stable operations commute up to sign with the suspension isomorphism, as in fig. 2 in section 9. In view of the multiplicativity of E and eq. (3.24), this reduces to

$$\rho_S u_k = u_k \otimes 1 \qquad \text{in } E^*(S^k, o) \otimes E_*(E, o) \tag{11.24}$$

for all integers k (positive or negative), where S^k denotes the k-sphere *spectrum* and $u_k \in E^k(S^k, o) \cong E^0$ the standard generator. Equivalently, from eqs. (11.18) and (11.19), the action of any operation r is given by

$$r u_k = \langle r, 1\rangle\, u_k \qquad \text{in } E^*(S^k, o). \tag{11.25}$$

Both formulae then hold in $E^*(S^k)$ for the *space* S^k, which exists for $k \geqslant 0$. Also, eq. (11.20) gives $r(vu_k)$.

Homology homomorphisms. In some applications, it is useful to regard the element $x \in E^k(X, o)$ as a map of spectra $x: X \to E$ and compute the homomorphism induced on E-homology.

Proposition 11.26. *Assume that $E_*(E,o)$ is a free E^*-module. Given $x \in E^*(X, o)$, suppose that rx is given by eq.* (11.19)*. Then the E-homology homomorphism $x_*: E_*(X, o) \to E_*(E, o)$ induced by the map of spectra $x: X \to E$ is given on $z \in E_m(X, o)$ by*

$$x_* z = \sum_\alpha (-1)^{\deg(c_\alpha)(\deg(x_\alpha)+m)} \langle x_\alpha, z\rangle\, c_\alpha\ . \tag{11.27}$$

Proof. For a general operation r, we have

$$\begin{aligned} \langle r, x_* z\rangle &= \pm \langle x^* r, z\rangle = \langle rx, z\rangle \\ &= \sum_\alpha \langle r, c_\alpha\rangle\, v_\alpha = \Big\langle r, \sum_\alpha (-1)^{\deg(c_\alpha)\deg(v_\alpha)} v_\alpha c_\alpha \Big\rangle, \end{aligned}$$

where $v_\alpha = \langle x_\alpha, z\rangle$. Since this holds for all r, eq. (11.27) follows by duality. □

Conversely, we can recover $\rho_X x$ from x_* when X is well behaved.

Proposition 11.28. *Assume that $E_*(X)$ is a free E^*-module. Take $x \in E^*(X)$. Then under the isomorphism $E^*(X) \widehat{\otimes} E_*(E, o) \cong \mathit{Mod}^*(E_*(X), E_*(E, o))$ of Lemma 6.16(a), the element $\rho_X x$ corresponds to the homomorphism $x_*: E_*(X) \to E_*(E, o)$ of E^*-modules.*

Proof. We apply Lemma 6.16(a) to eq. (11.18), using the strong duality $E^*(X) \cong DE_*(X)$ of Thm. 9.25, and compare with eq. (11.27). □

Similarly, it is important to know the E-homology homomorphism $r_*\colon E_*(E,o) \to E_*(E,o)$ induced by an operation r, regarded as a map $r\colon E \to E$ of spectra. This provides a convenient faithful representation of the operations on $E_*(E,o)$, as it is clear that $\iota_* = \mathrm{id}$ and $(sr)_* = s_* \circ r_*$. From diag. (10.15) and the isomorphism (11.4) we deduce the commutative square

$$\begin{array}{ccc} M & \xrightarrow{r} & M \\ \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_M} \\ M \mathbin{\widehat{\otimes}} E_*(E,o) & \xrightarrow{M \otimes r_*} & M \mathbin{\widehat{\otimes}} E_*(E,o) \end{array} \tag{11.29}$$

We need to know how to pass between $\langle r, - \rangle$ and r_*. From the identity $\langle r, c \rangle = \langle r^* \iota, c \rangle = \langle \iota, r_* c \rangle$ and eq. (11.21), we easily recover the functional $\langle r, - \rangle$ from r_* as

$$\langle r, - \rangle\colon E_*(E,o) \xrightarrow{r_*} E_*(E,o) \xrightarrow{\epsilon_S} E^*. \tag{11.30}$$

The following result gives the reverse direction.

Lemma 11.31. *Let $r \in E^*(E,o)$ be an operation and assume that $E_*(E,o)$ is a free E^*-module. Then:*

(a) *The diagram*

$$\begin{array}{ccc} E_*(E,o) & \xrightarrow{r_*} & E_*(E,o) \\ \downarrow{\scriptstyle \psi_S} & & \downarrow{\scriptstyle \psi_S} \\ E_*(E,o) \otimes E_*(E,o) & \xrightarrow{1 \otimes r_*} & E_*(E,o) \otimes E_*(E,o) \end{array} \tag{11.32}$$

commutes; in other words, r_* is a morphism of left $E_*(E,o)$-comodules;

(b) *$r_*\colon E_*(E,o) \to E_*(E,o)$ is the unique homomorphism of left E^*-modules that satisfies eq.* (11.30) *and is a morphism of left $E_*(E,o)$-comodules as in (a);*

(c) *The homomorphism r_* is given in terms of the functional $\langle r, - \rangle$ as*

$$\begin{aligned} r_*\colon E_*(E,o) &\xrightarrow{\psi_S} E_*(E,o) \otimes E_*(E,o) \\ &\xrightarrow{1 \otimes \langle r, - \rangle} E_*(E,o) \otimes E^* \cong E_*(E,o)\ . \end{aligned} \tag{11.33}$$

Proof. After applying $M \mathbin{\widehat{\otimes}} -$, diag. (11.32) corresponds under eq. (11.4) to the square

$$\begin{array}{ccc} SM & \xrightarrow{\theta_r M} & SM \\ \Big\downarrow{\scriptstyle \psi_S M} & & \Big\downarrow{\scriptstyle \psi_S M} \\ SSM & \xrightarrow{\theta_r SM} & SSM \end{array}$$

where $\theta_r\colon S \to S$ is defined on $f \in SM = \mathit{FMod}^*(\mathcal{A}, M)$ by $(\theta_r M)f = (-1)^{\deg(r)\deg(f)} f \circ r^*$. (Note that θ_r only takes values in Ab^*, because it fails to preserve the preferred E^*-module structure on SM.) Since S is corepresented by $\mathcal{A}$, commutativity of this diagram reduces to the equality

$$\rho_E \circ r^* = Sr^* \circ \rho_E\colon \mathcal{A} = E^*(E,o) \longrightarrow SE^*(E,o) = S\mathcal{A}$$

in $SS\mathcal{A}$, which expresses the naturality of ρ. (Explicitly, $(sr)^* = \pm\, r^* \circ s^*$ for all $s \in \mathcal{A}$, which is the associativity $t(sr) = (ts)r$ for fixed r.)

If we compose diag. (11.32) with

$$1 \otimes \epsilon_S\colon E_*(E,o) \otimes E_*(E,o) \longrightarrow E_*(E,o) \otimes E^* \cong E_*(E,o)$$

and use eq. (11.30) and diag. (11.10)(ii), we obtain (c). This also establishes the uniqueness of r_* in (b). □

To summarize, eqs. (11.30) and (11.33) express r_* and $\langle r, -\rangle$ in terms of each other, with the help of ψ_S and ϵ_S. Conversely, these equations may be viewed as characterizing ψ_S and ϵ_S in terms of the r_* and $\langle r, -\rangle$ for all r.

We can at last make explicit how ψ_S is dual to the composition (10.4). It is immediate from eq. (11.30) that $\langle sr, -\rangle = \langle s, -\rangle \circ r_*$. We substitute eq. (11.33) to obtain

$$\begin{aligned} \langle sr, -\rangle\colon E_*(E,o) &\xrightarrow{\psi_S} E_*(E,o) \otimes E_*(E,o) \\ &\xrightarrow{1\otimes\langle r,-\rangle} E_*(E,o) \otimes E^* \cong E_*(E,o) \xrightarrow{\langle s,-\rangle} E^* \end{aligned} \tag{11.34}$$

(Note that we cannot simply write $\langle s, -\rangle \otimes \langle r, -\rangle$ here, which is undefined unless $\langle s, -\rangle$ happens to be right E^*-linear.)

Remark. From a more sophisticated point of view, several of our formulae may be explained by noting that ψ_S makes $E_*(E,o)$ a stable comodule, provided we use the right E^*-module action. The comodule axioms are (11.9) and (11.10)(ii). Then by comparing eq. (11.33) with eq. (11.17), we see that the action of r on $E_*(E,o)$ is just r_*, and diag. (11.32) becomes a special case of diag. (11.29), which in turn comes from diag. (8.8).

Compatibility. It is clear that eqs. (11.30) and (11.33) determine ϵ_S and ψ_S uniquely, and that eq. (11.23) determines η_R. We now show that they agree with the homomorphisms introduced by Adams [3, III.12].

Theorem 11.35. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Suppose that $E_*(E,o)$ is equipped with ψ_S and ϵ_S that satisfy eqs.* (11.30) *and* (11.33)*, and η_R as in Defn. 11.2. Then:*

(a) *η_R must be*

$$E^* = \pi_*^S(E,o) \cong \pi_*^S(T^+ \wedge E, o) \xrightarrow{\pi_*^S(\eta\wedge E,o)} \pi_*^S(E \wedge E, o) = E_*(E,o);$$

(b) *ϵ_S must be*

$$\epsilon_S \colon E_*(E,o) = \pi_*^S(E \wedge E, o) \xrightarrow{\pi_*^S(\phi,o)} \pi_*^S(E,o) = E^*;$$

(c) *ψ_S must be*

$$\psi_S \colon E_*(E,o) \cong E_*(T^+ \wedge E, o) \xrightarrow{E_*(\eta\wedge E)} E_*(E\wedge E,o) \cong E_*(E,o) \otimes E_*(E,o),$$

where we use the twisted *Künneth isomorphism with action scheme*

$$E1_*(E2 \wedge E3, o) \cong E1_*(E2,o) \otimes_2 E2_*(E3,o).$$

Proof. The definition $\eta_R v = v_* 1$ expands to

$$T^+ \xrightarrow{\eta} E \simeq E \wedge T^+ \xrightarrow{E\wedge v} E \wedge E \ .$$

We prove (a) by rearranging this as

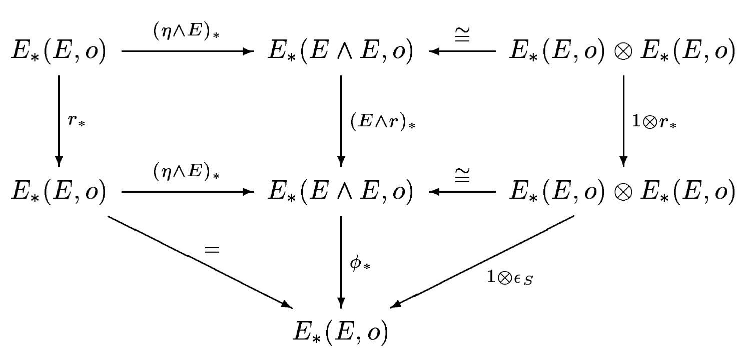
$$T^+ \xrightarrow{v} E \simeq T^+ \wedge E \xrightarrow{\eta\wedge E} E \wedge E \ .$$

Given $r \in E^*(E,o)$ and $c \in E_*(E,o)$, we may construct $\langle r, c\rangle$ as the composite

$$\langle r,c\rangle \colon T^+ \xrightarrow{c} E \wedge E \xrightarrow{r\wedge E} E \wedge E \xrightarrow{\phi} E \ .$$

If we take $r = \iota$ and compare with (11.30), we obtain (b).

The commutative diagram



shows, with the help of eq. (11.30), that r_* is the composite of $1 \otimes \langle r, -\rangle$ with Adams's ψ, which appears as the top row. (Here, both Künneth isomorphisms are twisted.) Since this holds for all r, comparison with eq. (11.33) gives (c). □

12. What is a stable algebra?

In section 10, we gave four answers for the structure of a module over the ring $\mathcal{A} = E^*(E,o)$ of stable operations in E-cohomology, to encode the algebraic structure present on the E^*-module $E^*(X)$ or $E^*(X,o)$ for a space or spectrum X. When X is a space, $E^*(X)$ is an E^*-algebra. In this section, we enrich each answer and theorem to include this multiplicative structure.

The organizing principle of this section is to make everything symmetric monoidal. We have three symmetric monoidal categories in view: $(\mathit{Stab}^*, \wedge, T^+)$, $(\mathit{Mod}^*, \otimes, E^*)$, and the filtered version $(\mathit{FMod}^*, \widehat{\otimes}, E^*)$. We also have three symmetric monoidal functors: E-cohomology (9.14), completed E-cohomology (9.30), and E-homology (9.18).

In this section, *we generally assume that $E_*(E,o)$ is a free E^*-module;* then Thm. 9.31 provides Künneth homeomorphisms $E^*(E\wedge E,o) \cong \mathcal{A}\widehat{\otimes}\mathcal{A}$ and $E^*(E\wedge E\wedge E,o) \cong \mathcal{A}\widehat{\otimes}\mathcal{A}\widehat{\otimes}\mathcal{A}$.

First Answer. For a spectrum X, we have the action (10.1)

$$\lambda_X : \mathcal{A}\otimes E^*(X,o) \longrightarrow E^*(X,o).$$

Given an operation r, we would like to have an external Cartan formula

$$r(x\times y) = \sum_\alpha \pm r'_\alpha x \times r''_\alpha y \qquad \text{in } E^*(X\wedge Y,o) \tag{12.1}$$

for suitable choices of operations r'_α and r''_α (and signs). For a space X, this leads to the corresponding internal Cartan formula,

$$r(xy) = \sum_\alpha \pm (r'_\alpha x)(r''_\alpha y) \qquad \text{in } E^*(X). \tag{12.2}$$

For the universal example $X = Y = E$, with $x = y = \iota$, eq. (12.1) reduces to

$$\phi^* r = \sum_\alpha r'_\alpha \times r''_\alpha \qquad \text{in } E^*(E\wedge E,o).$$

This requires $\phi^* r$ to lie in the image of the cross product (9.13)

$$\times : E^*(E,o)\otimes E^*(E,o) \longrightarrow E^*(E\wedge E,o),$$

which rarely happens. However, the pairing becomes an isomorphism if we use the *completed* tensor product and so allow infinite sums. This is another reason to topologize $E^*(X)$.

From this point of view, a stable algebra should consist of a filtered E^*-algebra M equipped with a continuous E^*-linear action $\lambda_M : \mathcal{A}\otimes M \to M$ that satisfies eq. (12.2) for all r. We must not forget the unit 1_M of the algebra M, for which eq. (11.23) requires $r1_M = \langle r,1\rangle\, 1_M$.

In the classical case $E = H(\mathbb{F}_p)$, there is a good finite Cartan formula, and this description is adequate for many applications. For MU and BP, however, this approach is not very practical and must be reworked.

Second Answer. We have the coaction $\rho_X : E^*(X) \to E^*(X) \widehat{\otimes} E_*(E,o)$ from (10.7). We shall find that the rather opaque Cartan formula (12.1) translates (for spaces) into the commutative diagram

$$\begin{array}{ccc} E^*(X) \otimes E^*(Y) & \xrightarrow{\rho_X \otimes \rho_Y} & \big(E^*(X) \widehat{\otimes} E_*(E,o)\big) \otimes \big(E^*(Y) \widehat{\otimes} E_*(E,o)\big) \\ & & \downarrow \\ \Big\downarrow \times & & E^*(X) \mathbin{\widehat{\otimes}} E^*(Y) \mathbin{\widehat{\otimes}} \big(E_*(E,o) \otimes E_*(E,o)\big) \\ & & \Big\downarrow \times \otimes \phi \\ E^*(X \times Y) & \xrightarrow{\rho_{X \times Y}} & E^*(X \times Y) \mathbin{\widehat{\otimes}} E_*(E,o) \end{array} \qquad (12.3)$$

By taking $Y = X$, we deduce that ρ_X is a homomorphism of E^*-algebras. This includes the units, which come from $1 \in E^0(T)$, since ρ_T is given by Prop. 11.22(b).

Explicitly, given $x \in E^*(X)$ and $y \in E^*(Y)$, assume that rx and ry are given as in eq. (11.19) by

$$rx = \sum_\alpha \langle r, c_\alpha \rangle\, x_\alpha; \qquad ry = \sum_\beta \langle r, d_\beta \rangle\, y_\beta; \qquad \text{(for all } r\text{)}$$

for suitable elements $x_\alpha \in E^*(X)$, $y_\beta \in E^*(Y)$, and $c_\alpha, d_\beta \in E_*(E,o)$. Evaluation of diag. (12.3) on $x \otimes y$ using eq. (11.18) yields the external Cartan formula

$$r(x \times y) = \sum_\alpha \sum_\beta (-1)^{\deg(d_\beta)\deg(x_\alpha)} \langle r, c_\alpha d_\beta \rangle\, x_\alpha \times y_\beta \quad \text{in } E^*(X \times Y)\hat{} \qquad (12.4)$$

for all r. This works too for $x \in E^*(X,o)$, $y \in E^*(Y,o)$, and $x \times y \in E^*(X \wedge Y, o)$, where X and Y are based spaces or spectra. For a space X, we can take $Y = X$ and deduce the internal Cartan formula

$$r(xy) = \sum_\alpha \sum_\beta (-1)^{\deg(d_\beta)\deg(x_\alpha)} \langle r, c_\alpha d_\beta \rangle\, x_\alpha y_\beta \quad \text{in } E^*(X)\hat{}, \text{ for all } r. \qquad (12.5)$$

All this makes it clear what the definition of a stable comodule algebra should be. The following lemma makes it reasonable. As in section 10, we defer most proofs until we have our preferred definitions, at the end of the section.

Lemma 12.6. *Assume that $E_*(E,o)$ is a free E^*-module. Then the comultiplication $\psi = \psi_S : E_*(E,o) \to E_*(E,o) \otimes E_*(E,o)$ and counit $\epsilon = \epsilon_S : E_*(E,o) \to E^*$ are homomorphisms of E^*-algebras.*

As an immediate corollary of $\psi 1 = 1 \otimes 1$, we have

$$\psi_S(vw) = v \otimes w \qquad \text{in } E_*(E,o) \otimes E_*(E,o)$$

for any $v \in E^*$ and $w \in \eta_R E^*$. If we combine this with eq. (11.33), we obtain

$$r_*(vw) = v\, \eta_R \langle r, w \rangle \qquad \text{in } E_*(E,o)$$

for any stable operation r. What makes these formulae useful is that the elements vw always span $E_*(E,o)\otimes\mathbb{Q}$ as a $\mathbb{Q}$-module. Thus in the important case when E^* has no torsion, these innocuous equations are powerful enough to determine ψ_S and r_* completely.

Definition 12.7. We call a stable comodule M in the sense of Defn. 11.11 a *stable (E-cohomology) comodule algebra* if M is an object of FAlg and its coaction ρ_M is a morphism in FAlg.

In detail, M is a complete Hausdorff filtered E^*-algebra equipped with a coaction $\rho_M\colon M\to M\mathbin{\widehat{\otimes}}E_*(E,o)$ that is a continuous homomorphism of E^*-algebras and satisfies the coaction axioms (11.12), which are now diagrams in FAlg.

Theorem 12.8. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E=H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then:*

(a) *For any space X, the coaction ρ_X in* (11.15) *makes $E^*(X)\hat{}$ a stable comodule algebra;*

(b) *ρ is universal: given a discrete commutative E^*-algebra B, any multiplicative transformation $\theta X\colon E^*(X,o)\to E^*(X,o)\mathbin{\widehat{\otimes}}B$ (or $\widehat{\theta}X\colon E^*(X,o)\hat{}\to E^*(X,o)\mathbin{\widehat{\otimes}}B$) that is defined for all spectra X and natural on Stab^* is induced from ρ_X by a unique E^*-algebra homomorphism $f\colon E_*(E,o)\to B$ as*

$$\theta X\colon E^*(X,o)\xrightarrow{\ \rho_X\ }E^*(X,o)\mathbin{\widehat{\otimes}}E_*(E,o)\xrightarrow{\ 1\otimes f\ }E^*(X,o)\mathbin{\widehat{\otimes}}B\ .$$

Third Answer. We restate our Second Answer in terms of the functor $S'M=M\mathbin{\widehat{\otimes}}E_*(E,o)$ introduced in section 11. What we have really done is construct the symmetric monoidal functor

$$(S',\zeta_{S'},z_{S'})\colon(\mathsf{FMod},\widehat{\otimes},E^*)\longrightarrow(\mathsf{FMod},\widehat{\otimes},E^*)\tag{12.9}$$

where $\zeta_{S'}\colon S'M\mathbin{\widehat{\otimes}}S'N\to S'(M\mathbin{\widehat{\otimes}}N)$ is given by

$$\begin{aligned}M\mathbin{\widehat{\otimes}}E_*(E,o)\mathbin{\widehat{\otimes}}N\mathbin{\widehat{\otimes}}E_*(E,o)&\cong M\mathbin{\widehat{\otimes}}N\mathbin{\widehat{\otimes}}(E_*(E,o)\otimes E_*(E,o))\\&\xrightarrow{\ M\otimes N\otimes\phi\ }M\mathbin{\widehat{\otimes}}N\mathbin{\widehat{\otimes}}E_*(E,o)\end{aligned}$$

and $z_{S'}\colon E^*\to S'E^*$ is just $\eta_R\colon E^*\to E_*(E,o)$. We saw $\zeta_{S'}$ in diag. (12.3).

We can now reinterpret Lemma 12.6 as saying that the natural transformations $\psi'\colon S'\to S'S'$ and $\epsilon'\colon S'\to I$ are monoidal, thus making S' a comonad in FAlg. Then diag. (12.3) simply states that ρ is monoidal. Since $E^*=E^*(T)$ by definition, the other needed axiom reduces to $\rho_T=z_{S'}$, which we have by Prop. 11.22(b).

Fourth Answer. We enrich the object $SM=\mathsf{FMod}^*(\mathcal{A},M)$ in section 10 to include the multiplicative structure.

Theorem 12.10. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E=H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then we can make S a symmetric monoidal comonad in FMod and hence a comonad in FAlg.*

The definition of stable algebra is now clear.

Definition 12.11. A *stable (E-cohomology) algebra* is an S-coalgebra in FAlg, i. e. a complete Hausdorff filtered E^*-algebra M equipped with a continuous homomorphism $\rho_M: M \to SM$ of E^*-algebras that satisfies the coaction axioms (8.7).

If a closed ideal $L \subset M$ is invariant (see Defn. 10.13), then M/L inherits a stable algebra structure.

Theorem 12.12. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then given a complete Hausdorff filtered E^*-algebra M (i. e. object of FAlg), a stable comodule algebra structure on M in the sense of Defn. 12.7 is equivalent to a stable algebra structure on M in the sense of Defn. 12.11.*

Theorem 12.13. *Assume that $E_*(E,o)$ is a free E^*-module* (which is true for $E = H(\mathbb{F}_p)$, BP, MU, KU, or $K(n)$ by Lemma 9.21). *Then:*

(a) *The natural transformation $\rho: E^*(-)\hat{} \to S(E^*(-)\hat{})$ defined on spaces by diag.* (10.10) *(or $\rho: E^*(-,o)\hat{} \to S(E^*(-,o)\hat{})$ for spectra) is monoidal and makes $E^*(X)\hat{}$ a stable algebra for any space X;*

(b) *ρ is universal: given a cocommutative comonoid object C in FMod, any multiplicative transformation*

$$\theta X: E^*(X,o) \to \mathsf{FMod}^*(C, E^*(X,o))$$

(or $\widehat{\theta}X: E^*(X,o)\hat{} \to \mathsf{FMod}^*(C, E^*(X,o)\hat{})$) that is defined for all spectra X and natural on Stab is induced from ρ_X by a unique morphism $f: C \to \mathcal{A}$ of comonoids in FMod as

$$\theta X: E^*(X,o) \xrightarrow{\rho_X} S(E^*(X)\hat{}) = \mathsf{FMod}^*(\mathcal{A}, E^*(X)\hat{}) \xrightarrow{\mathrm{Hom}(f,1)} \mathsf{FMod}^*(C, E^*(X)\hat{}) \ .$$

Proof of Thms. 12.10 and 12.13. In proving Thm. 10.12, we made $\mathcal{A} = E^*(E,o)$ an E^*-module object. We add the necessary monoidal structure to $S = \mathsf{FMod}^*(\mathcal{A},-)$ in five steps.

Step 1. We construct the symmetric monoidal functor

$$(S, \zeta_S, z_S): (\mathsf{FMod}^*, \widehat{\otimes}, E^*) \longrightarrow (\mathsf{Mod}^*, \otimes, E^*) \ . \tag{12.14}$$

We start from the ring spectrum E, with multiplication $\phi: E \wedge E \to E$, unit $\eta: T^+ \to E$, and v-action $\xi v: E \to E$, and note that it is automatically an E^*-algebra object in the symmetric monoidal category $(\mathsf{Stab}^*, \wedge, T^+)$ in the sense of Defn. 7.12. We apply the E-cohomology functor (9.14) to make $\mathcal{A}$ an E^*-algebra object in $\mathsf{FMod}^{*\mathrm{op}}$, with the comultiplication

$$\psi_{\mathcal{A}}: \mathcal{A} = E^*(E,o) \xrightarrow{\phi^*} E^*(E \wedge E, o) \cong \mathcal{A} \widehat{\otimes} \mathcal{A}$$

and counit $\epsilon_{\mathcal{A}} = \eta^*\colon \mathcal{A} = E^*(E,o) \to E^*(T^+,o) = E^*$. Then Lemma 7.14 produces the desired functor (12.14), with $z_S\colon E^* \to SE^*$ given on $v \in E^*$ by eq. (7.15) as

$$z_S v = \eta^* \circ (\xi v)^* = v^*\colon \mathcal{A} = E^*(E,o) \longrightarrow E^*(T^+,o) = E^*. \tag{12.15}$$

This identifies z_S with η_R. Then S takes monoid objects in *FMod** (i. e. objects of *FAlg*) to monoid objects in *Mod** (i. e. E^*-algebras).

Step 2. To prove that $\rho\colon E^*(-,o) \to S(E^*(-,o)\hat{\ })$ is monoidal, we need to check commutativity of the diagram in *Mod*

$$\begin{array}{ccc} E^*(X,o) \otimes E^*(Y,o) & \xrightarrow{\rho_X \otimes \rho_Y} & S(E^*(X,o)\hat{\ }) \otimes S(E^*(Y,o)\hat{\ }) \\ & & \downarrow \zeta_S \\ \downarrow \times & & S(E^*(X,o)\hat{\ } \mathbin{\widehat{\otimes}} E^*(Y,o)\hat{\ }) \\ & & \downarrow S\times \\ E^*(X \wedge Y,o) & \xrightarrow{\rho_{X\wedge Y}} & S(E^*(X \wedge Y,o)\hat{\ }) \end{array} \tag{12.16}$$

By naturality, it is enough to take $X = Y = E$ and evaluate on the universal element $\iota \otimes \iota$. By construction, $\rho_E \iota = \mathrm{id}_{\mathcal{A}} \in S\mathcal{A}$. By the definition (7.11) of ζ_S, the upper route gives $\psi_{\mathcal{A}} \in S(\mathcal{A} \otimes \mathcal{A})$, which by definition corresponds under $S\times$ to $\phi^* \in SE^*(E \wedge E,o)$ as required.

Because $E^* = E^*(T^+,o)$, the other needed diagram reduces to $z_S = \rho_T$, which we have by eq. (12.15).

Step 3. For later use, we combine diag. (12.16) (still in the case $X = Y = E$) with the commutative square

$$\begin{array}{ccc} E^*(E,o) & \xrightarrow{\rho_E} & SE^*(E,o) \\ \downarrow \phi^* & & \downarrow S\phi^* \\ E^*(E \wedge E,o) & \xrightarrow{\rho_{E\wedge E}} & SE^*(E \wedge E,o) \end{array}$$

and the definition of $\psi_{\mathcal{A}}$ to obtain the following commutative diagram, which involves only $\mathcal{A}$,

$$\begin{array}{ccccc} \mathcal{A} & & \xrightarrow{\rho_E} & & S\mathcal{A} \\ \downarrow \psi_{\mathcal{A}} & & & & \downarrow S\psi_{\mathcal{A}} \\ \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} & \xrightarrow{\rho_E \otimes \rho_E} & S\mathcal{A} \mathbin{\widehat{\otimes}} S\mathcal{A} & \xrightarrow{\zeta_S} & S(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}) \end{array} \tag{12.17}$$

Step 4. The monoidality of ψ is a formal consequence of that of ρ. The two commutative diagrams to check are

$$\begin{array}{ccc}
SM \otimes SN & \xrightarrow{\psi M \otimes \psi N} & SSM \otimes SSN \\
 & & \downarrow{\scriptstyle \zeta_S(SM,SN)} \\
\downarrow{\scriptstyle \zeta_S(M,N)} & & S(SM \mathbin{\widehat{\otimes}} SN) \\
 & & \downarrow{\scriptstyle S\zeta_S(M,N)} \\
S(M \mathbin{\widehat{\otimes}} N) & \xrightarrow{\psi(M \mathbin{\widehat{\otimes}} N)} & SS(M \mathbin{\widehat{\otimes}} N)
\end{array}
\qquad
\begin{array}{ccc}
E^* & \xrightarrow{z_S} & SE^* \\
\downarrow{\scriptstyle z_S} & & \downarrow{\scriptstyle Sz_S} \\
SE^* & \xrightarrow{\psi E^*} & SSE^*
\end{array}
\qquad (12.18)$$

(i) (ii)

where we again leave some tensor products uncompleted.

As (i) is natural in M and N, we may work with the universal example $M = N = \mathcal{A}$ and evaluate on $\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}}$. The upper route gives the element

$$\mathcal{A} \xrightarrow{\psi_{\mathcal{A}}} \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A} \xrightarrow{\rho_E \otimes \rho_E} S\mathcal{A} \mathbin{\widehat{\otimes}} S\mathcal{A} \xrightarrow{\zeta_S} S(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})$$

of $SS(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})$. The lower route gives

$$\mathcal{A} \xrightarrow{\rho_E} S\mathcal{A} \xrightarrow{S\psi_{\mathcal{A}}} S(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}),$$

which we just saw in diag. (12.17) is the same.

Since $z_S = \rho_T$, (ii) reduces to axiom (8.7)(i) for the S-coalgebra $E^* = E^*(T)$.

Step 5. We next check that ϵ is monoidal; this too is formal. As ever, there are two diagrams:

$$\text{(i)}\quad
\begin{array}{ccc}
SM \otimes SN & & \\
\downarrow{\scriptstyle \zeta_S(M,N)} & \searrow{\scriptstyle \epsilon \otimes \epsilon} & \\
S(M \mathbin{\widehat{\otimes}} N) & \xrightarrow{\epsilon} & M \mathbin{\widehat{\otimes}} N
\end{array}
\qquad
\text{(ii)}\quad
\begin{array}{ccc}
E^* & & \\
\downarrow{\scriptstyle z_S} & \searrow{\scriptstyle =} & \\
SE^* & \xrightarrow{\epsilon} & E^*
\end{array}
\qquad (12.19)$$

Again we take $M = N = \mathcal{A}$ in (i) and evaluate on $\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}}$. The lower route gives $\psi_{\mathcal{A}}\iota = \iota \otimes \iota$, by the definition of $\psi_{\mathcal{A}}$. This agrees with $\epsilon \otimes \epsilon$, since $\epsilon\, \mathrm{id}_{\mathcal{A}} = \iota$. For (ii), it is clear from eq. (12.15) that $\epsilon z_S v = v$.

In Thm. 12.13(b), we are given a comonoid object C, equipped with morphisms $\psi_C : C \to C \mathbin{\widehat{\otimes}} C$ and $\epsilon_C : C \to E^*$ in FMod^*. Let us write (V, ζ_V, z_V) for the symmetric monoidal functor with $V = \mathit{FMod}^*(C, -)$ that results from Lemma 7.9. Theorem 10.16(b) provides the unique morphism $f : C \to \mathcal{A}$ in FMod^* that induces V from S as in eq. (10.17).

We compare diag. (12.16) and a similar diagram with V in place of S. Evaluation of the universal case $X = Y = E$ on $\iota \otimes \iota$ shows that $(f \otimes f) \circ \psi_C = \psi_{\mathcal{A}} \circ f : C \to \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$. Since θT^+ takes $1 \in E^* = E^*(T)$ to the unit element $z_V \in VE^*$, eq. (10.17) shows that $\epsilon_C = \epsilon_{\mathcal{A}} \circ f$. □

Comodule algebras. We can now fill in the missing proofs on comodule algebras. By construction, the isomorphism $S'M \cong SM$ in (11.4) transforms the symmetric monoidal structure (12.9) on S' into the symmetric monoidal structure (12.14) on S. Also, ρ' is monoidal and we have diag. (12.3).

Proof of Lemma 12.6. If we replace S by S' in the four diagrams (12.18) and (12.19) for $M = N = E^*$, we obtain exactly the diagrams we need. □

Proof of Thm. 12.12. The isomorphism $S'M \cong SM$ is now an isomorphism of algebras, and the two definitions agree. □

Proof of Thm. 12.8. For (a), we combine Thm. 12.13(a) with Thm. 12.12.

In (b), Thm. 11.14(b) provides the unique homomorphism $f\colon E_*(E,o) \to B$ of E^*-modules that induces θ from ρ as in eq. (11.16); it corresponds to the element $(\theta E)\iota$ under the isomorphism $E^*(E,o)\mathbin{\widehat{\otimes}} B \cong \mathit{Mod}^*(E_*(E,o),B)$ of Lemma 6.16(a). If we evaluate $\theta(T^+)$ on 1, we see that $f1 = 1$. The multiplicativity of θ is expressed as a diagram resembling (12.3) with B in place of $E_*(E,o)$. We evaluate it in the universal case $X = Y = E$ on $\iota\otimes\iota$ and again use Lemma 6.16(a) to convert elements of $E^*(E\wedge E,o)\mathbin{\widehat{\otimes}} B$ to module homomorphisms $E_*(E,o)\otimes E_*(E,o) \to B$, with the help of $E^*(E\wedge E,o) \cong D(E_*(E,o)\otimes E_*(E,o))$ from Thms. 9.20 and 9.25. The upper route yields $\phi_B\circ(f\otimes f)\colon E_*(E,o)\otimes E_*(E,o) \to B$. Since $\iota\times\iota = \phi \in E^*(E\wedge E,o)$, the lower route yields

$$E_*(E,o)\otimes E_*(E,o) \xrightarrow{\ \times\ } E_*(E\wedge E,o) \xrightarrow{\ \phi_*\ } E_*(E,o) \xrightarrow{\ f\ } B\ .$$

Thus f is multiplicative and so is an E^*-algebra homomorphism. □

13. Operations and complex orientation

In this section, we show how a complex orientation on E determines the elements $b_i \in E_*(E,o)$ from our point of view. *We assume that $E_*(E,o)$ is free*, so that sections 11 and 12 apply. We pay particular attention to the p-local case, and the main relations that apply there.

Complex projective space. We recall from Defn. 5.1 that a complex orientation for E yields a first Chern class $x(\theta) \in E^2(X)$ for each complex line bundle θ over any space X. As the Hopf line bundle ξ over $\mathbb{C}P^\infty$ is universal, we need only study $x = x(\xi) \in E^2(\mathbb{C}P^\infty)$. Thus $\mathbb{C}P^\infty$ is our third test space. Since $E^*(\mathbb{C}P^\infty) = E^*[[x]]$ by Lemma 5.4, the coaction ρ on $E^*(\mathbb{C}P^\infty)$ is completely determined by ρx, multiplicativity, E^*-linearity, and continuity.

Definition 13.1. Given a complex orientation for E, we define the elements $b_i \in E_{2(i-1)}(E,o)$ for all $i \geqslant 0$ by the identity

$$\rho x = b(x) = \sum_{i=0}^{\infty} x^i \otimes b_i \quad \text{in } E^*(\mathbb{C}P^\infty)\mathbin{\widehat{\otimes}} E_*(E,o) \cong E_*(E,o)[[x]], \qquad (13.2)$$

where $b(x)$ is a convenient formal abbreviation that will rapidly become essential.

Equivalently, according to eq. (11.19), the action of any operation $r \in \mathcal{A} = E^*(E,o)$ on x is given as

$$rx = \sum_{i=0}^{\infty} \langle r, b_i \rangle \, x^i \qquad \text{in } E^*(\mathbb{C}P^\infty) = E^*[[x]] \, . \tag{13.3}$$

Remark. Our indexing convention is taken from [32]. We warn that b_i is often written b_{i-1} (e. g. in [3]), as its degree suggests; the latter convention is appropriate in the current stable context, where $b_0 = 0$ (see below), but less so in the unstable context of [9], where (our) b_0 *does* become non-zero.

Since the Hopf bundle is universal, eqs. (13.2) and (13.3) carry over by naturality to the Chern class $x(\theta)$ of *any* complex line bundle θ over any space X (except that when X is infinite-dimensional and $E^*(X)$ is not Hausdorff, the infinite series force us to work in the completion $E^*(X)\hat{}$).

Proposition 13.4. *The elements $b_i \in E_{2(i-1)}(E,o)$ have the following properties:*

(a) *$b_0 = 0$ and $b_1 = 1$, so that $b(x) = x \otimes 1 + x^2 \otimes b_2 + x^3 \otimes b_3 + \dots$;*

(b) *The Chern class $x \in E^2(\mathbb{C}P^\infty, o)$, regarded as a map of spectra $x\colon \mathbb{C}P^\infty \to E$, induces $x_*\beta_i = b_i \in E_*(E,o)$, where $\beta_i \in E_{2i}(\mathbb{C}P^\infty)$ is dual to x^i* (as in Lemma 5.4(c))*;*

(c) *$\psi_S b_k$ is given by*

$$\psi_S b_k = \sum_{i=1}^{k} B(i,k) \otimes b_i \qquad \text{in } E_*(E,o) \otimes E_*(E,o),$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$, or, in condensed notation, $\psi_S b(x) = \sum_i b(x)^i \otimes b_i$;

(d) *$\epsilon_S b_i = 0$ for all $i > 1$, so that $\epsilon_S b(x) = x$.*

Proof. We prove (a) by restricting to $\mathbb{C}P^1 \cong S^2$ and comparing with eq. (11.24). Part (b) is an application of Prop. 11.26, using eq. (13.3). For (c) and (d), we take $M = E^*(\mathbb{C}P^\infty)$ in diags. (11.12) and evaluate on x. □

The formal group law. Now $\mathbb{C}P^\infty = K(\mathbb{Z},2)$ is an *H*-space, whose multiplication map $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ may be defined by $\mu^*\xi = p_1^*\xi \otimes p_2^*\xi$ for the Hopf bundle ξ. We therefore have from eq. (5.13)

$$\mu^* x = F(x \times 1, 1 \times x) = x \times 1 + 1 \times x + \sum_{i,j} a_{i,j} \, x^i \times x^j, \tag{13.5}$$

where $F(x,y)$ denotes the formal group law (5.14). When we apply ρ and write x for $x \times 1$ and y for $1 \times x$, we obtain from eq. (13.2) and naturality

$$b(F(x,y)) = F_R(b(x), b(y)) = b(x) + b(y) + \sum_{i,j} b(x)^i \, b(y)^j \, \eta_R a_{i,j} \tag{13.6}$$

in $E_*(E,o)[[x,y]]$, which is difficult to express without using the formal notations $b(x)$ and $F(x,y)$. On the right, $F_R(X,Y)$ is another convenient abbreviation. (In the language of formal groups, the series $b(x)$ is an isomorphism between the formal group laws F and F_R.)

The p-local case. The above rather formidable machinery does simplify in common situations. When the ring E^* is p-local, most of the b_i are redundant.

Lemma 13.7. *Assume that E^* is p-local. Then if k is not a power of p, the element $b_k \in E_*(E,o)$ can be expressed in terms of E^*, $\eta_R E^*$, and elements of the form b_{p^i}.*

Proof. Consider the coefficient of $x^i y^j$ in eq. (13.6), where $i+j=k$. On the left, there is a term $\binom{k}{i} b_k$ from $b_k(x+y)^k$, and all other terms involve only the lower b's. On the right, no b beyond b_i or b_j occurs. If $\binom{k}{i}$ is not divisible by p and so is a unit in E^*, we deduce an inductive reduction formula for b_k. This can be done whenever k is not a power of p, by choosing $i=p^m$ and $j=k-p^m$, where m satisfies $p^m < k < p^{m+1}$. □

We therefore reindex the b's.

Definition 13.8. When E^* is p-local, we define $b_{(i)} = b_{p^i}$ for each $i \geqslant 0$.

We still need to use the internal details of Lemma 13.7 to express each $\psi b_{(k)}$ inductively in terms of the $b_{(i)}$, $a_{i,j}$, and $\eta_R a_{i,j}$.

The main relations. In the p-local case, it is appropriate to study instead of μ the much simpler p-th power map $\zeta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ constructed from μ. In cohomology, it must induce

$$\zeta^* x = [p](x) = px + \sum_{i>0} g_i x^{i+1} \qquad \text{in } E^*(\mathbb{C}P^\infty) \cong E^*[[x]] \tag{13.9}$$

for suitable coefficients $g_i \in E^{-2i}$ (which are usually written a_i; but we need to avoid conflict with certain other elements also known as a_i that appear in section 14). The formal power series $[p](x)$ is known as the *p-series* of the formal group law. The bundle interpretation is $\zeta^*\xi = \xi^{\otimes p}$, so that

$$x(\theta^{\otimes p}) = px(\theta) + \sum_{i>0} g_i\, x(\theta)^{i+1} \qquad \text{in } E^*(Z)\hat{} \tag{13.10}$$

for any line bundle θ over any space Z. (Again, completion is not necessary for finite-dimensional Z, or if the series $[p](x)$ happens to be finite.)

When we apply ρ, we obtain

$$b([p](x)) = [p]_R(b(x)) = pb(x) + \sum_{i>0} b(x)^{i+1}\, \eta_R g_i \quad \text{in } E_*(E,o)[[x]], \tag{13.11}$$

where $[p]_R(X)$ denotes the formal power series $pX + \sum_i (\eta_R g_i) X^{i+1}$. We extract the relations we need.

Definition 13.12. For each $k>0$, we define the kth *main stable relation* in $E_*(E,o)$ as

$$(\mathcal{R}_k): \qquad L(k) = R(k) \qquad \text{in } E_*(E,o), \tag{13.13}$$

where $L(k)$ and $R(k)$ denote the coefficient of x^{p^k} in $b([p](x))$ and $[p]_R(b(x))$ respectively.

The results of section 14 will show that, despite appearances, the relations $(\mathcal{R}_k)$ contain all the information of eq. (13.6), with the understanding that the latter is used only to express (inductively) each redundant b_j in terms of the $b_{(i)}$, E^*, and $\eta_R E^*$, in accordance with Lemma 13.7.

14. Examples of ring spectra for stable operations

In section 10, we developed a comonad S that, for favorable E, expresses all the structure of stable E-cohomology operations. In section 11, we described an equivalent comonad S' in terms of structure on the algebra $E_*(E,o)$. In this section, we give the complete description of $E_*(E,o)$ for each of our five examples, namely $E = H(\mathbb{F}_p)$, MU, BP, KU, and $K(n)$. (The first splits into two, and we break out the degenerate special case $H(\mathbb{Q}) = K(0)$ merely for purposes of illustration.)

All the results here are well known, but serve as a guide for [9]. *Our purpose is to exhibit the structure of the results, not to derive them.* As Milnor discovered [22] in the case $E = H(\mathbb{F}_p)$, the most elegant and convenient formulation of stable cohomology operations is the Second Answer of sections 10 and 12, consisting of the multiplicative (i. e. monoidal) coaction (10.7)

$$\rho_X: E^*(X) \longrightarrow E^*(X) \mathbin{\widehat{\otimes}} E_*(E,o)$$

for each space X (or on $E^*(X,o)$, for a spectrum X).

The point is that the knowledge of ρ_X on a few simple test spaces and test maps is sufficient to suggest the complete structure of $E_*(E,o)$. The test spaces studied so far include the point T in Prop. 11.22, the sphere S^k in eq. (11.24), and complex projective space $\mathbb{C}P^\infty$ in eq. (13.2).

In each case, we specify (when not obvious):

(i) The coefficient ring E^*;
(ii) The E^*-algebra $E_*(E,o)$;
(iii) $\eta_R: E^* \to E_*(E,o)$, the right unit ring homomorphism;
(iv) $\psi: E_*(E,o) \to E_*(E,o) \otimes E_*(E,o)$, the comultiplication;
(v) $\epsilon: E_*(E,o) \to E^*$, the counit.

(See Prop. 11.3 for $E_*(E,o)$ and η_R. By construction and Lemma 12.6, ψ and ϵ are homomorphisms of E^*-algebras and of E^*-bimodules.) In most cases, the results allow us to express the universal property of $E_*(E,o)$ very simply.

Example: $H(\mathbb{F}_2)$. We take $E = H = H(\mathbb{F}_2)$, the Eilenberg-MacLane spectrum representing ordinary cohomology with coefficients $\mathbb{F}_2$. The main reference is Milnor [22], and many of our formulae, diagrams and results can be found there. The appropriate test space is $\mathbb{R}P^\infty = K(\mathbb{F}_2, 1)$, an H-space with multiplication $\mu: \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$, and we use a mod 2 analogue of complex orientation.

We have $H^*(\mathbb{R}P^\infty) = \mathbb{F}_2[t]$, with a polynomial generator $t \in H^1(\mathbb{R}P^\infty)$, and $\mu^* t = t\times 1 + 1\times t$ is forced. By analogy with Prop. 13.4(a), we must have

$$\rho t = t\otimes 1 + \sum_{i>1} t^i \otimes c_i \qquad \text{in } H^*(\mathbb{R}P^\infty)\,\widehat{\otimes}\, H_*(H,o) \cong H_*(H,o)[[t]]$$

for certain coefficients $c_i \in H_*(H,o)$. The analogue of eq. (13.6) is simply

$$(t+u)\otimes 1 + \sum_{i>1} (t+u)^i \otimes c_i = t\otimes 1 + \sum_{i>1} t^i \otimes c_i + u\otimes 1 + \sum_{i>1} u^i \otimes c_i$$

in $H_*(H,o)[[t,u]]$. Because the left side contains the terms $\binom{i}{j} t^{i-j} u^j \otimes c_i$, we must have $c_i = 0$ unless i is a power of 2. Imitating Defn. 13.8, we write $\xi_i = c_{2^i} \in H_{2^i-1}(H,o)$ for $i > 0$, so that now

$$\rho t = t\otimes 1 + \sum_{i=1}^{\infty} t^{2^i} \otimes \xi_i \quad \text{in } H^*(\mathbb{R}P^\infty)\,\widehat{\otimes}\, H_*(H,o) \cong H_*(H,o)[[t]]\ . \qquad (14.1)$$

Because $\underline{H}_1 = \mathbb{R}P^\infty$, this formula is valid for every $t \in H^1(X)$, for all spaces X. It is reasonable to define also $\xi_0 = c_1 = 1$. Milnor proved that this is all there is.

Theorem 14.2 (Milnor). *For the Eilenberg-MacLane ring spectrum $H = H(\mathbb{F}_2)$:*

(a) $H_*(H,o) = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$, *a polynomial algebra over $\mathbb{F}_2$ on the generators $\xi_i \in H_{2^i-1}(H,o)$ for $i > 0$;*

(b) *In the complex orientation for $H(\mathbb{F}_2)$, $b_{(i)} = \xi_i^2$ for all $i > 0$, $b_{(0)} = 1$, and $b_j = 0$ if j is not a power of 2;*

(c) *ψ is given by*

$$\psi \xi_k = \xi_k \otimes 1 + \sum_{i=1}^{k-1} \xi_{k-i}^{2^i} \otimes \xi_i + 1 \otimes \xi_k \qquad \text{in } H_*(H,o) \otimes H_*(H,o);$$

(d) $\epsilon \xi_k = 0$ *for all $k > 0$.*

Proof. Milnor proved (a) in [22, App. 1]. The complexified Hopf line bundle over $\mathbb{R}P^\infty$ has Chern class t^2. We compare ρt^2 with eq. (13.2) and read off (b). (For i not a power of 2, this is a stronger statement than Lemma 13.7 provides.) For (c) and (d), we substitute $M = H^*(\mathbb{R}P^\infty)$ into diags. (11.12) and evaluate on t. □

Corollary 14.3. *Let B be a discrete commutative graded $\mathbb{F}_2$-algebra. Assume that the operation $\theta\colon H^*(X,o) \to H^*(X,o)\,\widehat{\otimes}\, B$ is multiplicative* (i.e. monoidal) *and natural on* **Stab****. Then on $t \in H^1(\mathbb{R}P^\infty, o) = H^1(\mathbb{R}P^\infty)$, θ has the form*

$$\theta t = t\otimes 1 + \sum_{i=1}^{\infty} t^{2^i} \otimes \xi'_i \qquad \text{in } H^*(\mathbb{R}P^\infty)\,\widehat{\otimes}\, B \cong B[[t]],$$

where the elements $\xi'_i \in B^{-(2^i-1)}$ determine θ uniquely for all X and may be chosen arbitrarily.

Proof. We combine the universal property Thm. 12.8(b) of $H_*(H,o)$ with the universal property of the polynomial algebra $\mathbb{F}_2[\xi_1,\xi_2,\ldots]$. □

Example: $H(\mathbb{F}_p)$ (for p odd). We take $E = H = H(\mathbb{F}_p)$, the Eilenberg-MacLane spectrum that represents ordinary cohomology with coefficients $\mathbb{F}_p$. The main reference is still Milnor [22].

We have a complex orientation, therefore by Defn. 13.8 the elements $b_{(i)} \in H_{2(p^i-1)}(H,o)$; $b_{(i)}$ is normally written ξ_i for $i \geqslant 0$, where $\xi_0 = b_{(0)} = 1$. As in the previous example, eq. (13.6) simplifies to show that $b_j = 0$ whenever j is not a power of p, so that for the Chern class $x = x(\theta) \in H^2(X)$ of any complex line bundle θ over X, eq. (13.2) reduces to

$$\rho_X x = x\otimes 1 + \sum_{i=1}^{\infty} x^{p^i}\otimes\xi_i \qquad \text{in } H^*(X)\,\widehat{\otimes}\,H_*(H,o)\ . \tag{14.4}$$

We need one more test space, the infinite-dimensional lens space $L = K(\mathbb{F}_p,1)$, which contains S^1 and is another H-space. The cohomology $H^*(L) = \mathbb{F}_p[x]\otimes\Lambda(u)$ has an exterior generator $u \in H^1(L)$ which restricts to $u_1 \in H^1(S^1)$. As the polynomial generator $x \in H^2(L)$ is the Chern class of a certain complex line bundle, $\rho_L x$ is given by eq. (14.4). This leaves only $\rho_L u$, which must take the form

$$\rho_L u = \sum_i x^i\otimes a_i + \sum_i ux^i\otimes c_i$$

for certain well-defined coefficients $a_i, c_i \in H_*(H,o)$. By restricting to $S^1 \subset L$ and comparing with eq. (11.24), we see that $c_0 = 1$ and $a_0 = 0$.

The multiplication μ on L induces $\mu^*u = u\times 1 + 1\times u$ and $\mu^*x = x\times 1 + 1\times x$. Expansion of $\mu^*\rho_L u = \rho_{L\times L}\mu^*u = (\rho_L u)\times 1 + 1\times(\rho_L u)$ yields

$$\begin{aligned}\sum_i (x\times 1 + 1\times x)^i \otimes a_i &+ \sum_i (u\times 1 + 1\times u)(x\times 1 + 1\times x)^i \otimes c_i \\ &= \sum_i (x^i\times 1)\otimes a_i + \sum_i (ux^i\times 1)\otimes c_i + \sum_i (1\times x^i)\otimes a_i + \sum_i (1\times ux^i)\otimes c_i\ .\end{aligned}$$

For $i > 0$, there is no term with $u \times x^i$ on the right, but there is on the left, which forces $c_i = 0$ for $i > 0$. When we take coefficients of $x^i \times x^j$, we find as in Lemma 13.7 that $a_i = 0$ unless i is a power of p. Again we reindex, defining $\tau_i = a_{p^i} \in H_{2p^i-1}(H,o)$ for all $i \geqslant 0$, so that now

$$\rho_L u = u\otimes 1 + \sum_{i=0}^{\infty} x^{p^i}\otimes\tau_i \qquad \text{in } H^*(L)\,\widehat{\otimes}\,H_*(H,o)\ . \tag{14.5}$$

Again, the elements ξ_n and τ_n give everything.

Theorem 14.6 (Milnor). *For the Eilenberg-MacLane ring spectrum $H = H(\mathbb{F}_p)$ with p odd:*

(a) *As a commutative algebra over* $\mathbb{F}_p$,

$$H_*(H,o) = \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \ldots] \otimes \Lambda(\tau_0, \tau_1, \tau_2, \ldots),$$

with polynomial generators $\xi_i = b_{(i)} \in H_{2(p^i-1)}(H,o)$ *for* $i \geqslant 1$ *and exterior generators* $\tau_i \in H_{2p^i-1}(H,o)$ *for* $i \geqslant 0$*;*

(b) $\psi\colon H_*(H,o) \to H_*(H,o) \otimes H_*(H,o)$ *is given by*

$$\psi\xi_k = \xi_k \otimes 1 + \sum_{i=1}^{k-1} \xi_{k-i}^{p^i} \otimes \xi_i + 1 \otimes \xi_k$$

$$\psi\tau_k = \tau_k \otimes 1 + \sum_{i=0}^{k-1} \xi_{k-i}^{p^i} \otimes \tau_i + 1 \otimes \tau_k;$$

(c) $\epsilon\xi_k = 0$ *for all* $k > 0$ *and* $\epsilon\tau_k = 0$ *for all* $k \geqslant 0$.

Proof. Part (a) is Thm. 2 of Milnor [22]. Parts (b) and (c) comprise Thm. 3 [ibid.], but also follow by substituting ρ_L into diags. (11.12) and evaluating on x and u. (Proposition 13.4 also gives $\psi\xi_k$ and $\epsilon\xi_k$.) □

We have the analogue of Cor. 14.3.

Corollary 14.7. *Let* B *be a discrete commutative graded* $\mathbb{F}_p$*-algebra. Assume that the operation* $\theta\colon H^*(X,o) \to H^*(X,o) \widehat{\otimes} B$ *is multiplicative and natural on* ***Stab*****. Then on* $H^*(L) = \mathbb{F}_p[x] \otimes \Lambda(u)$, θ *has the form*

$$\theta x = x \otimes 1 + \sum_{i=1}^{\infty} x^{p^i} \otimes \xi'_i \ ; \qquad \theta u = u \otimes 1 + \sum_{i=0}^{\infty} x^{p^i} \otimes \tau'_i \ ;$$

where the elements $\xi'_i \in B^{-2(p^i-1)}$ *and* $\tau'_i \in B^{-(2p^i-1)}$ *determine* θ *uniquely for all* X *and may be chosen arbitrarily.* □

Example: $H(\mathbb{Q})$**.** We take $E = H = H(\mathbb{Q})$, the Eilenberg-MacLane spectrum that represents ordinary cohomology with rational coefficients $\mathbb{Q}$. There are no interesting stable operations.

Theorem 14.8. *For the Eilenberg-MacLane ring spectrum* $H = H(\mathbb{Q})$*, we have* $H_*(H,o) = H_*(H(\mathbb{Q}),o) = \mathbb{Q}$. □

Example: MU**.** Our main reference is Adams [3, II.§11]. The coefficient ring is $MU^* = \mathbb{Z}[x_1, x_2, x_3, \ldots]$, with polynomial generators x_n in degree $-2n$ that are not canonical. We have complex orientation, almost by definition, and therefore the elements $b_n \in MU_{2n-2}(MU,o)$.

The good description of MU^* was given by Quillen [30, Thm. 6.5], as the *universal* formal group: it is generated as a ring by the coefficients $a_{i,j} \in MU^*$ that appear in the formal group law (5.14), subject to the relations (5.15). Hence the elements $\eta_R a_{i,j}$ determine η_R.

Theorem 14.9. *For the unitary cobordism ring spectrum MU:*

(a) *As a commutative MU^*-algebra, $MU_*(MU,o) = MU^*[b_2, b_3, b_4, \ldots]$, with polynomial generators $b_i \in MU_{2(i-1)}(MU,o)$ for $i > 1$;*

(b) *$\eta_R a_{i,j} \in MU_*(MU,o)$ is determined by eq. (13.6);*

(c) *ψ is given by*

$$\psi b_k = b_k \otimes 1 + \sum_{i=2}^{k} B(i,k) \otimes b_i \qquad \text{in } MU_*(MU,o) \otimes MU_*(MU,o),$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(d) *$\epsilon b_k = 0$ for all $k \geqslant 2$.*

Proof. Part (a) is standard. In (b), the coefficient of $x^i y^j$ in eq. (13.6) provides an inductive formula for $\eta_R a_{i,j}$. Proposition 13.4 provides (c) and (d). □

As $MU_*(MU,o)$ is a polynomial algebra, Cor. 14.3 carries over to this case.

Corollary 14.10. *Let B be a discrete commutative MU^*-algebra. Assume that the operation $\theta\colon MU^*(X,o) \to MU^*(X,o) \widehat{\otimes} B$ is multiplicative and natural on Stab^*. Then on $x \in MU^2(\mathbb{C}P^\infty)$, θ has the form*

$$\theta x = x \otimes 1 + \sum_{i=2}^{\infty} x^i \otimes b'_i \qquad \text{in } MU^*(\mathbb{C}P^\infty) \widehat{\otimes} B \cong B[[x]],$$

where the elements $b'_i \in B^{-2(i-1)}$ determine θ uniquely for all X and may be chosen arbitrarily. □

In other words, there are no relations over MU^* between the b_i. The dual $MU^*(MU,o)$ is known as the Landweber-Novikov algebra. The results for ψ are no longer amenable to explicit expression as in Thms. 14.2 and 14.6.

Example: *BP*. The main reference is still Adams [3, II.§16]. The coefficient ring is now $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, a polynomial algebra on Hazewinkel's generators v_i of degree $-2(p^i - 1)$ for $i > 0$. (One could instead use Araki's generators [5] or any other system of polynomial generators, with only slight modifications.)

We still have complex orientation, but because BP^* is p-local, we need only the generators $b_{(i)}$ from Defn. 13.8, where $b_{(0)} = 1$. Moreover, it is sufficient to work with the p-series (13.9), because its coefficients g_i generate BP^* as a $\mathbb{Z}_{(p)}$-algebra (as we shall see in more detail in section 15). We write $w_i = \eta_R v_i \in BP_*(BP,o)$.

Theorem 14.11. *For the Brown-Peterson ring spectrum BP:*

(a) *As a commutative BP^*-algebra, $BP_*(BP,o) = BP^*[b_{(1)}, b_{(2)}, b_{(3)}, \ldots]$, with polynomial generators $b_{(i)} = b_{p^i} \in BP_{2(p^i-1)}(BP,o)$ for each $i > 0$;*

(b) *The nth main relation $(\mathcal{R}_n)$ in eq. (13.13) provides an inductive formula for $w_n = \eta_R v_n \in BP_*(BP,o)$;*

(c) *ψ is given by*

$$\psi b_{(k)} = b_{(k)} \otimes 1 + \sum_{i=2}^{p^k} B(i,p^k) \otimes b_i \qquad \text{in } BP_*(BP,o) \otimes BP_*(BP,o),$$

where $B(i,p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$ (and Lemma 13.7 is used to express $b(x)$ and b_i in terms of the $b_{(j)}$ and BP^*);

(d) *$\epsilon b_{(k)} = 0$ for all $k > 0$.* □

We shall find that the generators $b_{(i)}$ are better suited to [9] than Quillen's original generators t_i, or their conjugates h_i, which were used in [8]. We have the analogue of Cor. 14.10.

Corollary 14.12. *Let B be a discrete commutative BP^*-algebra. Assume the operation $\theta: BP^*(X,o) \to BP^*(X,o) \widehat{\otimes} B$ is multiplicative and natural on Stab^*. Then on $x \in BP^2(\mathbb{C}P^\infty)$, θ has the form*

$$\theta x = x \otimes 1 + \sum_{i=2}^{\infty} x^i \otimes b'_i \qquad \text{in } BP^*(\mathbb{C}P^\infty) \widehat{\otimes} B \cong B[[x]]$$

for certain elements $b'_i \in B^{-2(i-1)}$. The elements $b'_{(i)} = b'_{p^i}$ for $i \geqslant 1$ determine θ uniquely for all X and may be chosen arbitrarily. □

Example: KU. We take $E = KU = K$, the complex Bott spectrum, which we constructed in sections 3 and 9. Its coefficient ring is the ring $\mathbb{Z}[u,u^{-1}]$ of Laurent polynomials in $u \in KU^{-2}$, and one writes $v = \eta_R u$. The complex orientation (5.2) furnishes elements $b_i \in KU_*(KU,o)$, of which $b_1 = 1$. We computed its formal group law $F(x,y) = x + y + uxy$ in eq. (5.16); thus eq. (13.6) reduces to

$$b(x+y+uxy) = b(x) + b(y) + b(x)b(y)v \ . \qquad (14.13)$$

This is small enough for explicit calculation. The coefficient of xy^i yields the relation

$$(i+1)b_{i+1} + iub_i = b_i v \qquad (14.14)$$

since on the left,

$$b_j(x+y+uxy)^j \equiv b_j y^j + j b_j y^{j-1} x(1+uy) \qquad \text{mod } x^2 \ .$$

(Compare [3, Lemma II.13.5].) This includes the special case $2b_2 + u = v$ for $i = 1$. Generally, for $i > 1$ and $j > 1$, the coefficient of $x^i y^j$ yields the relation

$$b_i b_j = \sum_{k=0}^{\min(i,j)} \binom{i+j-k}{i} \binom{i}{k} u^k b_{i+j-k} v^{-1}, \qquad (14.15)$$

which serves to reduce any product of b's to a linear expression. Thus the general expression c in our generators may be assumed linear in the b's. Further, for large enough m, cv^m will have no negative powers of v; if we use eq. (14.14) to remove all the positive powers of v, c takes the form

$$c = u^q(\lambda_1 u^{-1} + \lambda_2 u^{-2} b_2 + \lambda_3 u^{-3} b_3 + \ldots + \lambda_n u^{-n} b_n) v^{-m} \qquad (14.16)$$

for some integers λ_i, n, and q. This suggests part (a) of the following.

Lemma 14.17. *In $KU_*(KU,o)$:*

(a) *Every element can be written in the form* (14.16)*;*

(b) *The element c in eq.* (14.16) *is zero if and only if $\lambda_i = 0$ for all i.*

This, with eq. (14.14), is a complete description of $KU_*(KU,o)$. We shall give a proof in [9].

Theorem 14.18. *For the complex Bott spectrum KU:*

(a) *As a commutative algebra over $KU^* = \mathbb{Z}[u,u^{-1}]$, $KU_*(KU,o)$ has the generators:*

$$v = \eta_R u \in KU_2(KU,o);$$
$$v^{-1} = \eta_R u^{-1} \in KU_{-2}(KU,o);$$
$$b_i \in KU_{2i-2}(KU,o) \text{ for } i > 1;$$

subject to the relations (14.14) *and* (14.15)*;*

(b) *As a KU^*-module, $KU_*(KU,o)$ is spanned by the monomials v^n and $b_i v^n$, for all $i > 1$ and $n \in \mathbb{Z}$, subject to the relations* (14.14) *(multiplied by any v^n);*

(c) *ψ is given by*

$$\psi b_k = b_k \otimes 1 + \sum_{i=2}^{k} B(i,k) \otimes b_i \qquad \text{in } KU_*(KU,o) \otimes KU_*(KU,o),$$

where $B(i,k)$ denotes the coefficient of x^k in $b(x)^i$;

(d) *ϵ is given by $\epsilon b_i = 0$ for all $i > 1$.*

Proof. Parts (a) and (b) follow from Lemma 14.17. Parts (c) and (d) are included in Prop. 13.4. □

Although we no longer have a polynomial algebra, we still have part of Cor. 14.10.

Corollary 14.19. *Let B be a discrete commutative KU^*-algebra. Then any operation $\theta\colon KU^*(X,o) \to KU^*(X,o) \mathbin{\widehat{\otimes}} B$ that is multiplicative and natural on* **Stab*** *is uniquely determined by its values on $KU^*(\mathbb{C}P^\infty)$.* □

The module $KU_*(KU,o)$. What makes the description (14.16) unsatisfactory is that m is not unique; we can always increase m and use eq. (14.14) to remove the extra v's to obtain another expression of the same form that looks quite different. For example, $(b_3 + ub_2)/2 = (2b_4 + 3ub_3 + u^2 b_2)v^{-1} \in KU_*(KU,o)$, in spite of the denominator 2. It is notoriously difficult to write down stable operations in $KU^*(-)$ (equivalently, linear functionals $KU_*(KU,o) \to KU^*$) other than $\Psi^1 = \mathrm{id}$ and $\Psi^{-1}[\xi] = [\,\overline{\xi}\,]$ (the complex conjugate bundle). Following Adams [3], we develop an alternate description from which the freeness of $KU_*(KU,o)$ will follow easily.

First, we note that Lemma 14.17 implies that $KU_*(KU,o)$ has no torsion, which allows us to work rationally and consider

$$KU^*[v,v^{-1}] \subset KU_*(KU,o) \subset KU^*[v,v^{-1}] \otimes \mathbb{Q}\,.$$

The key idea is that if we localize at a prime p, we have available (algebraically) the Adams operation Ψ^k for any invertible $k \in \mathbb{Z}_{(p)}$. Rationally, we have Ψ^k for all nonzero $k \in \mathbb{Q}$. It is characterized by the properties that it is additive, multiplicative, and satisfies $\Psi^k[\theta] = [\theta^{\otimes k}] = [\theta]^k$ for any line bundle θ.

To compute $\Psi^k u$, we rewrite eq. (3.32) as $uu_2 = [\xi] - 1$ and apply Ψ^k. As stability requires $\Psi^k u_2 = u_2$, and $u_2^2 = 0$, we find

$$(\Psi^k u)u_2 = [\xi]^k - 1 = (1 + uu_2)^k - 1 = kuu_2,$$

Hence $\Psi^k u = ku$. Then eq. (11.23) becomes

$$\langle \Psi^k, v \rangle = \Psi^k u = ku \ . \tag{14.20}$$

The linear functional $\langle \Psi^k, - \rangle : KU_*(KU, o) \to KU^* \otimes \mathbb{Q}$ is multiplicative because Ψ^k is, as can be seen by expanding $\Psi^k(\iota \times \iota)$ by eq. (12.4). (These are precisely the multiplicative linear functionals.)

We apply $\langle \Psi^k, - \rangle$ to eq. (14.14) to obtain, by induction starting from $b_1 = 1$,

$$\langle \Psi^k, b_n \rangle = k^{-1} \binom{k}{n} u^{n-1}. \tag{14.21}$$

Alternatively, for any $n > 1$ we can write formally

$$b_n = \frac{(v-u)(v-2u)\dots(v-(n-1)u)}{n!} \in KU^*[v, v^{-1}] \otimes \mathbb{Q} \tag{14.22}$$

and replace v by ku everywhere.

Lemma 14.23. *An element $c \in KU^*[v, v^{-1}] \otimes \mathbb{Q}$ lies in $KU_*(KU, o)$ if and only if $\langle \Psi^k, c \rangle \in KU^* \otimes \mathbb{Z}_{(p)}$ for all primes p and integers $k > 0$ such that p does not divide k.*

From this we deduce the freeness of $KU_*(KU, o)$.

Proof of Lemma 9.21 for $E = KU$. Denote by $F_{m,n}$ the free KU^*-module with basis $\{v^m, v^{m+1}, \dots, v^n\}$. It is enough to show that for any m, $KU_*(KU, o) \cap (F_{-m,m} \otimes \mathbb{Q})$ is a free KU^*-module; then any basis extends to a basis of $KU_*(KU, o) \cap (F_{-m-1,m+1} \otimes \mathbb{Q})$, and thence by induction to a basis of $KU_*(KU, o)$. We may multiply by v^m and work with $F_{0,2m}$ instead.

We therefore work in degree zero and take any element

$$c = \lambda_0 + \lambda_1 w + \lambda_2 w^2 + \dots + \lambda_{n-1} w^{n-1} \tag{14.24}$$

in $KU_0(KU, o) \cap (F_{0,n-1} \otimes \mathbb{Q})$, where each $\lambda_i \in \mathbb{Q}$ and we write $w = u^{-1}v$. We have only to find a common denominator Δ that guarantees $\Delta\lambda_i \in \mathbb{Z}$ for all i.

Given any prime p, we choose n distinct positive integers k_1, k_2, $\dots k_n$, not divisible by p; then by eq. (14.20),

$$\langle \Psi^{k_j}, c \rangle = \sum_{i=0}^{n-1} \lambda_i k_j^i \in \mathbb{Z}_{(p)} \ .$$

We solve these n linear equations for the λ_i in terms of the $\langle\Psi^{k_j}, c\rangle$, which requires division by the Vandermonde determinant

$$\Delta(p) = \det_{i,j}(k_j^{i-1}) = \prod_{1\leqslant j<i\leqslant n} (k_i - k_j) \ .$$

Then $\Delta(p)\lambda_i \in \mathbb{Z}_{(p)}$ for all i. If $p > n$, the obvious choices $k_j = j$ yield $\lambda_i \in \mathbb{Z}_{(p)}$, because then p does not divide $\Delta(p)$. We take $\Delta = \prod_{p\leqslant n} \Delta(p)$. □

Before we establish Lemma 14.23, we need a result [3, Lemma II.13.8] which explains the role of the b's.

Lemma 14.25. *Let c be an element of $KU^*[v, v^{-1}] \otimes \mathbb{Q}$. Then c is a KU^*-linear combination of the elements 1, $v = b_1 v$, $b_2 v$, $b_3 v, \dots$ if and only if $\langle\Psi^k, c\rangle \in KU^*$ for all integers $k > 0$.*

Proof. Necessity is clear from eq. (14.21). We may reduce sufficiency to the case when c has degree 0 and write c as a Laurent series in $w = u^{-1}v$. By taking k very large, it is clear that c has no negative powers of w; this allows us to write (see eq. (14.22))

$$c = \sum_{i=0}^{n} \lambda_i \binom{w}{i} = \lambda_0 + \sum_{i=1}^{n} \lambda_i b_i v$$

for some n and suitable coefficients $\lambda_i \in \mathbb{Q}$. By eq. (14.21), $\langle\Psi^k, c\rangle = \sum_{i=0}^{n} \lambda_i \binom{k}{i}$. By induction on k from 1 to $n+1$, $\langle\Psi^k, c\rangle \in \mathbb{Z}$ yields $\lambda_k \equiv (-1)^k \lambda_0 \bmod \mathbb{Z}$. But $\lambda_{n+1} = 0$. Therefore $\lambda_0 \in \mathbb{Z}$, and $\lambda_i \in \mathbb{Z}$ for all i. □

Proof of Lemma 14.23. Again, necessity is clear. For sufficiency, we assume given c in the form eq. (14.24). Let m be the maximum exponent of any prime in the denominators of the λ_i, so that $p^m\lambda_i \in \mathbb{Z}_{(p)}$ for all i and all primes p. Then $p^m\langle\Psi^k, c\rangle \in \mathbb{Z}_{(p)}$ for all integers $k > 0$ and all primes p.

If p does not divide k, we have $\langle\Psi^k, cw^m\rangle = k^m\langle\Psi^k, c\rangle \in \mathbb{Z}_{(p)}$ by hypothesis. If $k = pq$, we have instead $k^m\langle\Psi^k, c\rangle = q^m p^m\langle\Psi^k, c\rangle \in \mathbb{Z}_{(p)}$, by our choice of m. Thus for each $k > 0$, $\langle\Psi^k, cw^m\rangle \in \bigcap_p \mathbb{Z}_{(p)} = \mathbb{Z}$. Then Lemma 14.25 shows that $cw^m \in KU_*(KU, o)$. □

Example $K(n)$. The coefficient ring is now the p-local ring $K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}]$, still with $\deg(v_n) = -2(p^n - 1)$, where p is odd. We write $w_n = \eta_R v_n$, as we did for *BP*. We have a complex orientation, and therefore elements $b_{(i)}$ for $i \geqslant 0$, where $b_{(0)} = 1$. Although the formal group law remains complicated, it is well known [32, Thm. 3.11(b)] that over $\mathbb{F}_p$, the p-series (13.9) reduces to exactly

$$\zeta^* x = v_n x^{p^n} \qquad \text{in } K(n)^*[[x]], \tag{14.26}$$

so that eq. (13.11) simplifies drastically to $b(v_n x^{p^n}) = b(x)^{p^n} w_n$. The coefficient of x^{p^n} yields $w_n = v_n$, and the coefficient of $x^{p^{n+i}}$ then yields

$$b_{(i)}^{p^n} = v_n^{p^i - 1} b_{(i)} \qquad \text{in } K(n)_*(K(n), o). \tag{14.27}$$

Lemma 14.28. *Assume that k is not a power of p. Then:*

(a) $b_k \in K(n)_{2k-2}(K(n), o)$ *can be expressed in terms of v_n and the $b_{(i)}$;*

(b) $b_k = 0$ *if $k < p^n$.*

Proof. Part (a) comes from Lemma 13.7. For (b), we trivially have $a_{i,j} = 0$ whenever $i + j < p^n$; in this range, eq. (13.6) behaves exactly as in Thm. 14.6 for $H(\mathbb{F}_p)$. □

We need one more test space. The infinite lens space L is *not* appropriate, as $K(n)^*(L) = K(n)^*[x : x^{p^n} = 0]$, where x is inherited from $\mathbb{C}P^\infty$. (Because ζ is trivial on L, we must have $x^{p^n} = 0$, which makes the structure of the Atiyah-Hirzebruch spectral sequence clear.) Instead, we use the finite skeleton $Y = L^{2p^n-1}$, the orbit space of the unit sphere in $\mathbb{C}^{p^n}$ under the action of the group $\mathbb{Z}/p \subset S^1 \subset \mathbb{C}$. The spectral sequence for $K(n)^*(Y, o)$ collapses because it can support no differential, to give $K(n)^*(Y) = \Lambda(u) \otimes K(n)^*[x : x^{p^n} = 0]$, where $u \in K(n)^1(Y)$ restricts to $u_1 \in K(n)^1(S^1)$. (This fails to define u uniquely, because we can replace u by $u' = u + hv_n ux^{p^n-1}$ for any $h \in \mathbb{F}_p$.)

We know $\rho_Y x$ is given by eq. (13.2). We write

$$\rho_Y u = \sum_{i=0}^{p^n-1} x^i \otimes a_i + \sum_{i=0}^{p^n-1} ux^i \otimes c_i, \tag{14.29}$$

which defines elements $a_i, c_i \in K(n)_*(K(n), o)$. (They are independent of the choice of u.) By restriction to $S^1 \subset Y$, we see that $a_0 = 0$ and $c_0 = 1$.

Unfortunately, Y is no longer an H-space. The multiplication on L restricts (after a non-canonical deformation) to a partial multiplication on skeletons $\mu: L^{2k+1} \times L^{2m} \to L^{2(k+m)+1} = Y$, whenever $k + m = p^n - 1$. Clearly, $K(n)^*(L^{2k+1}) = \Lambda(u) \otimes K(n)^*[x : x^{k+1} = 0]$, with the coaction ρ obtained from ρ_Y by truncation; and similarly for $K(n)^*(L^{2m})$, except that $ux^m = 0$ also.

As x is inherited from L, we have $\mu^* x = x \times 1 + 1 \times x$, for lack of any other possible terms in degree 2. For u, we must have

$$\mu^* u = u \times 1 + 1 \times u + \lambda v_n ux^k \times x^m$$

for some $\lambda \in \mathbb{F}_p$. (The third term disappears if we replace u by $u + (-1)^k \lambda v_n ux^{p^n-1}$, but in any case is harmless.) We apply ρ to μ, bearing in mind that $w_n = v_n$, and carry out exactly the same algebra as for $E = H(\mathbb{F}_p)$; the coefficients of $u \times x^j$ and $x^i \times x^j$ show that $c_j = 0$ for all $j > 0$ and that $a_h = 0$ for h not a power of p. We therefore reindex, as usual.

Definition 14.30. We define $a_{(i)} = a_{p^i} \in K(n)_{2p^i-1}(K(n), o)$, for $0 \leqslant i < n$.

There is no $a_{(n)}$ because u does not lift to the next skeleton L^{2p^n+1}. In the new notation, eq. (14.29) becomes

$$\rho_Y u = u \otimes 1 + \sum_{i=0}^{n-1} x^{p^i} \otimes a_{(i)} \qquad \text{in } K(n)^*(Y) \otimes K(n)_*(K(n), o). \tag{14.31}$$

Having odd degree, the $a_{(i)}$ satisfy $a_{(i)}^2 = 0$.

Theorem 14.32 (Yagita). *For the Morava K-theory ring spectrum $K(n)$:*

(a) *The commutative $K(n)^*$-algebra $K(n)_*(K(n),o)$ has the generators:*

$$a_{(i)} \in K(n)_{2p^i-1}(K(n),o), \text{ for } 0 \leqslant i < n;$$
$$b_{(i)} \in K(n)_{2(p^i-1)}(K(n),o), \text{ for } i > 0;$$

subject to the relations (14.27)*;*

(b) *η_R is given by $\eta_R v_n = w_n = v_n \in K(n)_*(K(n),o)$;*

(c) *ψ is given by:*

$$\psi a_{(k)} = a_{(k)} \otimes 1 + \sum_{i=0}^{k-1} b_{(k-i)}^{p^i} \otimes a_{(i)} + 1 \otimes a_{(k)} \qquad \text{for } 0 \leqslant k < n;$$

$$\psi b_{(k)} = b_{(k)} \otimes 1 + \sum_{i=2}^{p^k-1} B(i,p^k) \otimes b_i + 1 \otimes b_{(k)} \qquad \text{for } k > 0;$$

where $B(i,p^k)$ denotes the coefficient of x^{p^k} in $b(x)^i$ (and we use Lemma 14.28 to express $b(x)$ and b_i in terms of the $b_{(i)}$ and v_n)*;*

(d) *$\epsilon a_{(k)} = 0$ for $0 \leqslant k < n$ and $\epsilon b_{(k)} = 0$ for $k > 0$.*

Proof. The whole theorem is essentially due to Yagita [39], who used different generators. We proved (b) above. For (c) and (d), we substitute ρ_Y in diags. (11.12) as usual and evaluate on u and x. □

Corollary 14.33. *Let B be a discrete commutative $K(n)^*$-algebra. Then any operation $\theta\colon K(n)^*(X,o) \to K(n)^*(X,o) \widehat{\otimes} B$ that is multiplicative and natural on Stab^* is uniquely determined by its values on $K(n)^*(\mathbb{C}P^\infty)$ and $K(n)^*(Y)$.* □

Remark. For low k, the formula for $\psi b_{(k)}$ simplifies by Lemma 14.28(b) to

$$\psi b_{(k)} = b_{(k)} \otimes 1 + \sum_{i=1}^{k-1} b_{(k-i)}^{p^i} \otimes b_{(i)} + 1 \otimes b_{(k)} \qquad \text{for } 0 < k \leqslant n\ .$$

15. Stable *BP*-cohomology comodules

In this section we study stable modules in the case $E = BP$ in more detail. We find it more practical to work with stable comodules, which by Thm. 11.13 are equivalent. This is the context in which Landweber showed [17, 18] that the presence of a stable comodule structure on M imposes severe constraints on its BP^*-module structure.

We recall that $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$, a polynomial ring on the Hazewinkel generators v_n of degree $-2(p^n-1)$ (see [14]). It contains the well-known ideals

$$I_n = (p, v_1, v_2, \ldots, v_{n-1}) \subset BP^* \tag{15.1}$$

for $0 \leqslant n \leqslant \infty$ (with the convention that $I_\infty = (p, v_1, v_2, \ldots)$, $I_1 = (p)$, and $I_0 = 0$.). We show in Lemma 15.8 that they are *invariant* under the action of the stable operations on $BP^*(T) = BP^*$. Indeed, Landweber [17] and Morava [27] showed that the I_n for $0 \leqslant n < \infty$ are the only finitely generated invariant prime ideals in BP^*.

Nakayama's Lemma. The fact that BP^* is a local ring with maximal ideal I_∞ is extremely useful. The advantage is that once we know certain modules are free, many questions can be answered by working over the more convenient quotient field $BP^*/I_\infty \cong \mathbb{F}_p$. We say a BP^*-module M is *of finite type* if it is bounded above and each M^k is a finitely generated $\mathbb{Z}_{(p)}$-module. (Remember that $\deg(v_i)$ is negative.)

Lemma 15.2. *Assume that $f\colon M \to N$ is a homomorphism of BP^*-modules of finite type, with N free. Then:*

(a) *f is an isomorphism if and only if $f \otimes \mathbb{F}_p\colon M \otimes \mathbb{F}_p \to N \otimes \mathbb{F}_p$ is an isomorphism;*

(b) *f is a split monomorphism of BP^*-modules if and only if $f \otimes \mathbb{F}_p$ is monic;*

(c) *If the conditions in (b) hold, both M and* Coker f *are BP^*-free;*

(d) *f is epic if and only if $f \otimes \mathbb{F}_p$ is epic (even if N is not free).*

Proof. The "only if" statements are obvious. For the "if" statements, we first consider $f/p\colon M/pM \to N/pN$. We filter M/pM and N/pN by powers of the ideal $(v_1, v_2, v_3, \ldots)$, so that for the associated graded groups, $\mathrm{Gr}(f/p)\colon \mathrm{Gr}(M/pM) \to \mathrm{Gr}(N/pN)$ is a module homomorphism over the bigraded ring $\mathrm{Gr}(BP^*/(p)) = \mathbb{F}_p[v_1, v_2, v_3, \ldots]$, with $\mathrm{Gr}(N/pN)$ free. As M and N are bounded above, these filtrations are finite in each degree. It follows that if $f \otimes \mathbb{F}_p$ is epic (or monic), so is f/p.

Then the standard Nakayama's Lemma, applied to $\mathbb{Z}_{(p)}$-modules in each degree, gives (d). If f/p is monic and N is free, we must have $\operatorname{Ker} f \subset p^n M$ for all n; as M is of finite type, f must be monic, which gives (a) and some of (b). To see that in (c), M must be free, we lift a basis of $M \otimes \mathbb{F}_p$ to M and use the liftings to define a homomorphism of BP^*-modules $g\colon L \to M$, with L free, that makes $g \otimes \mathbb{F}_p$ an isomorphism. Then $f \circ g$ is monic by what we have proved so far, and g is epic by (d); therefore g must be an isomorphism.

To finish (b) and (c), we choose an $\mathbb{F}_p$-basis of $\operatorname{Coker}(f \otimes \mathbb{F}_p)$, lift it to N, and use it to define a homomorphism $h\colon K \to N$ of BP^*-modules with K free. We use f and h to define $M \oplus K \to N$, which by (a) is an isomorphism and identifies $\operatorname{Coker} f$ with K. □

The main relations. We need to make the structure of $BP_*(BP, o)$ more explicit than in Thm. 14.11. The first few terms of the formal group law for BP in terms of the Hazewinkel generators are easily found:

$$F(x,y) \equiv x + y - v_1 x^{p-1} y \qquad \bmod (x^p, y^2). \tag{15.3}$$

Also, the p-series for BP begins with

$$[p](x) = px + (1-p^{p-1}) v_1 x^p + \dots . \tag{15.4}$$

All we need to know about $[p](x)$ beyond this is the standard fact (e.g. [32, Thm. 3.11(b)]) that

$$[p](x) \equiv px + \sum_{i>0} v_i x^{p^i} \qquad \mathrm{mod}\ I_\infty^2 \ . \tag{15.5}$$

For lack of alternative, $b_i = 0$ whenever $i-1$ is not a multiple of $p-1$, so that $b(x) = x + b_{(1)}x^p + \dots$. The first main relation is well known and readily computed from Defn. 13.12, with the help of eq. (15.4), as

$$(\mathcal{R}_1): \qquad v_1 = pb_{(1)} + w_1 \ , \tag{15.6}$$

or more easily, as the coefficient of $x^{p-1}y$ in eq. (13.6), expanded using eq. (15.3). Subsequent relations $(\mathcal{R}_k)$ are far more complicated and answers in closed form are not to be expected. To handle the right side $R(k)$, we introduce the ideal $\mathfrak{W} = (p, w_1, w_2, \dots) \subset BP_*(BP, o)$, the analogue of I_∞ for the right BP^*-action. The right side of eq. (13.11) simplifies by eq. (15.5) to

$$pb(x) + \sum_i b(x)^{p^i} w_i \qquad \mathrm{mod}\ \mathfrak{W}^2.$$

When we expand $b(x)^{p^i}$, all cross terms may be ignored, because they contain a factor $p \in \mathfrak{W}$, and we find

$$R(k) \equiv pb_{(k)} + \sum_{i=1}^{k-1} b_{(k-i)}^{p^i} w_i + w_k \qquad \mathrm{mod}\ \mathfrak{W}^2. \tag{15.7}$$

With slightly more attention to detail, we obtain a sharper, more useful result. It also implies that $\mathfrak{W} = I_\infty BP_*(BP, o)$, so that $\mathfrak{W}$ is redundant.

Lemma 15.8. *For any $n > 0$, we have $w_n \equiv v_n$ mod $I_n BP_*(BP, o)$.*

Proof. We show by induction on n that the relation $(\mathcal{R}_n)$ simplifies as stated, starting from eq. (15.6) for $n = 1$. If the result holds for all $i < n$, we have $w_i \equiv v_i \equiv 0$ mod I_n for $i < n$. Then $R(n) \equiv w_n$ from eq. (15.7), as $\mathfrak{W}^2$ contains nothing of interest in this degree. Meanwhile, the left side $L(n) \equiv v_n$ by eq. (15.5). □

Recall from Defn. 10.13 and Thm. 11.13 that an ideal $J \subset BP^*$ is invariant if it is a stable subcomodule of $BP^* = BP^*(T)$; in view of Prop. 11.22(b), the necessary and sufficient condition for this is $\eta_R J \subset JBP_*(BP, o)$. In this case, we have the quotient stable comodule BP^*/J. For example, Lemma 15.8 shows that the ideals I_n are invariant, and we have the stable comodules $BP^*/I_n \cong \mathbb{F}_p[v_n, v_{n+1}, v_{n+2}, \dots]$ (for $n > 0$) and $BP^*/I_0 \cong BP^*$.

Primitive elements. The key idea is to explore a general stable comodule M by looking for comodule morphisms $BP^* \to M$ from the (relatively) well understood stable comodule $BP^*(T) = BP^*$. A BP^*-module homomorphism $f\colon BP^* \to M$ is obviously uniquely determined by the element $x = f1 \in M$, since $fv = f(v1) = vf1 = vx$, and we can choose x arbitrarily. In BP^*, we clearly have $\rho 1 = 1 \otimes 1$, which suggests the following definition.

Definition 15.9. Given a stable comodule M, we call an element $x \in M$ *stably primitive* if $\rho_M x = x \otimes 1$.

This is the necessary and sufficient condition for the above homomorphism $f\colon BP^* \to M$ to be a stable morphism. It then induces an isomorphism of stable comodules $BP^*/\operatorname{Ker} f \cong (BP^*)x$. In particular, $\operatorname{Ker} f = \operatorname{Ann}(x)$, the annihilator ideal of x, must be an invariant ideal. We are therefore interested in finding primitives.

The primitive elements of M^k clearly form a subgroup. Moreover, there is a good supply of primitives; if M is bounded above, axiom (11.12)(ii) forces every element $x \in M$ of top degree to be primitive. (This may be viewed as an algebraic analogue of Hopf's theorem, that for a finite-dimensional space X, $\pi^k(X) \cong H^k(X;\mathbb{Z})$ in the top degree.) If x is primitive, the BP^*-linearity of ρ_M gives $\rho_M(vx) = x \otimes \eta_R v$ for any $v \in BP^*$. It follows that the comodule structure on BP^*/J is unique if the ideal J is invariant (and none exists otherwise). Landweber [17] located all the primitive elements in the stable comodule BP^*/I_n.

Theorem 15.10 (Landweber). *For $0 \leqslant n < \infty$, the only nonzero primitive elements in the stable comodule BP^*/I_n are those of the form:*

(i) λv_n^i*, where $i \geqslant 0$ and $\lambda \in \mathbb{F}_p$ (if $n > 0$); or*

(ii) λ*, where $\lambda \in \mathbb{Z}_{(p)}$ (if $n = 0$).*

It follows easily as in [17, Thm. 2.7] that the I_n are the *only* finitely generated invariant prime ideals in BP^*. This suggests that the BP^*/I_n should be the basic building blocks for a general stable comodule. This is the content of Landweber's filtration theorem (cf. [17, Lemma 3.3] and [18, Thm. 3.3′]).

Theorem 15.11. (Landweber) *Let M be a stable BP-cohomology (co)module that is finitely presented as a BP^*-module (e. g. $BP^*(X)$ for any finite complex X). Then M admits a finite filtration by invariant submodules*

$$0 = M_0 \subset M_1 \subset M_2 \ldots \subset M_m = M,$$

in which each quotient M_i/M_{i-1} is generated, as a BP^*-module, by a single element x_i such that $\operatorname{Ann}(x_i) = I_{n_i}$ *for some $n_i \geqslant 0$.*

We outline Landweber's proof [18] for reference. For nonzero M, $\operatorname{Ass}(M)$, which here may be taken as the set of all prime annihilator ideals of elements of M, is a *finite non-empty* set of invariant finitely generated prime ideals of BP^*. The recipe for constructing a filtration of M is:

(a) Let I_n be the maximal element of $\operatorname{Ass}(M)$;

(b) Construct the BP^*-submodule $N = 0{:}I_n$ of M, which is defined as $\{y \in M : I_n y = 0\}$, and prove it invariant;

(c) Take a nonzero primitive $x_1 \in N$ (e. g. any element of top degree), so that the maximality of I_n forces $\operatorname{Ann}(x_1) = I_n$;

(d) Put $M_1 = (BP^*)x_1$, so that M_1 is invariant and isomorphic to BP^*/I_n;

(e) Replace M by M/M_1 and repeat, as long as M is nonzero, making sure that the process terminates (which requires some care).

Remarks. 1. The filtration of M is *never* a composition series. The module BP^*/I_n is not irreducible, because we have the short exact sequence

$$0 \longrightarrow BP^*/I_n \xrightarrow{v_n} BP^*/I_n \longrightarrow BP^*/I_{n+1} \longrightarrow 0$$

of stable comodules. Thus we have no uniqueness statement.

2. We cannot expect to arrange $n_1 \geqslant n_2 \geqslant \ldots$, since in (e), $\mathrm{Ass}(M/M_1)$ need not be contained in $\mathrm{Ass}(M)$.

Index of symbols

This index lists most symbols in roughly alphabetical order (English, then Greek), with brief descriptions and references. Several symbols have multiple roles.

g_i coefficient in p-series, (13.9).
H generic comonad, (8.6).
H, $H(R)$ Eilenberg-MacLane spectrum, §§2, 14.
Ho, *Ho'* homotopy category of (based) spaces, §6.
$h(-)$ generic ungraded cohomology theory, §3.
h Yokota clutching function, (5.9).
I identity functor.
I_n, I_∞ ideal in BP^*, (15.1).
i_1, i_2 injection in coproduct, §2.
id identity morphism.
$K_{\mathcal{C}}$ unit object in (symmetric) monoidal category $\mathcal{C}$, §7.
$K(n)$ Morava K-theory, §2.
KU complex K-theory Bott spectrum, §2, Defn. 3.30.
L infinite lens space, §14.
M etc. generic (filtered) module or algebra.
$M\hat{}$, $\widehat{M}$ completion of filtered M, Defn. 3.37.
Mod, *Mod** (graded) category of E^*-modules, §6.
MU unitary Thom spectrum, §2.
o generic basepoint, point spectrum.
$-^{\mathrm{op}}$ categorical dual, §6.
PA the primitives in coalgebra A, (6.13).
p fixed prime number.
p_1, p_2 projection from product, §2.
$[p](x)$ p-series, (13.9).
$[p]_R(x)$ right p-series, (13.11).
QA the indecomposables of algebra A, (6.10).
$\mathbb{Q}$ the field of rational numbers.
q map to one-point space T, §2.
R generic ring.
R-Mod category of R-modules, §8.
$\mathbb{R}P^\infty$ real projective space.
r etc. generic cohomology operation.
$\langle r, -\rangle$ E^*-linear functional defined by operation r, (11.1).
S stable comonad, Thm. 10.12.
S' stable comonad, (11.4).
$-_S$ (subscript) stable context.
S^1 unit circle, as space or group.
S^n unit n-sphere.
Stab, *Stab** (graded) stable homotopy category, §6.
Set category of sets, §6.
Set$^{\mathbb{Z}}$ category of graded sets, §7.
T monad, (8.4).
T the one-point space.
T^+ 0-sphere, T with basepoint added.
$T(n)$ torus group.
t $\in H^1(\mathbb{R}P^\infty)$, generator of $H^*(\mathbb{R}P^\infty)$, (14.1).
U, $U(n)$ unitary group.
$-_U$ (subscript) unstable context.
u $\in KU^{-2}$, after Defn. 3.30.
u $\in E^1(L)$, exterior generator of $E^*(L)$, section 14.
u $\in E^1(Y)$, exterior generator of $E^*(Y)$, section 14.
u universal element of $DL \,\widehat{\otimes}\, L$, Lemma 6.16.
u_1 canonical generator of $E^*(S^1)$, Defn. 3.23.
u_n canonical generator of $E^*(S^n)$, §3.
V generic (often forgetful) functor.
v generic element of E^*.
v $= \eta_R u \in KU_2(KU, o)$, Thm. 14.18.
v_n Hazewinkel generator of BP^*, $K(n)^*$, §14.
W forgetful functor, §8.
$\mathfrak{W}$ ideal in $BP_*(BP, o)$, §15.
w $= u^{-1}v \in KU_0(KU, o)$, Lemma 14.23.
w_n $= \eta_R v_n$, §14.
X etc. generic space or spectrum.
X^+ space X with basepoint adjoined.
x generic cohomology class or module element.
x $\in E^*(\mathbb{C}P^\infty)$, Chern class of Hopf line bundle, Lemma 5.4.
$x(\theta)$ Chern class of line bundle θ, Defn. 5.1.
Y skeleton of lens space L, §14.

$\mathbb{Z}$ the ring of integers.
$\mathbb{Z}/p$ the group of integers mod p.
$\mathbb{Z}_{(p)}$ $\mathbb{Z}$ localized at p.
z_F morphism for a (symmetric) monoidal functor F, §7.
α etc. generic index.
α generic algebraic operation, §7.
β_i $\in E_{2i}(\mathbb{C}P^n)$, Lemma 5.3.
γ_i $\in E_{2i+1}(U(n))$, Lemma 5.11.
$\Delta: X \to X \times X$ diagonal map.
ϵ generic counit morphism.
$\epsilon: FV \to I$ natural transformation, §2.
ζ pth power map on $\mathbb{C}P^\infty$, (13.9).
ζ_F pairing for (symmetric) monoidal functor F, §7.
η generic monoid unit morphism.
$\eta: I \to VF$ natural transformation, §2.
η generic vector bundle.
η_R right unit, Defn. 11.2.
θ generic anything.
θ complex line bundle, §5.
θ cohomology operation (usually idempotent), §3.
ι $\in h(H)$, universal class, Thm. 3.6.
ι $\in E^0(E, o)$, universal class, §9.
ι_n $\in E^n(\underline{E}_n)$, universal class, Thm. 3.17.
$\Lambda(-)$ exterior algebra.
λ generic action.
λ numerical coefficient.
μ addition or multiplication in generic group object, §7.
ν inversion morphism in generic group object, §7.
ξ Hopf line bundle over $\mathbb{C}P^n$.
ξ generic line or vector bundle.
ξ_i stable element for $H(\mathbb{F}_2)$, (14.1).
ξ_i stable element for $H(\mathbb{F}_p)$, (14.4).
ξv action of v on E^*-module, (7.4).
$\pi_*(X)$ homotopy groups of space X.
$\pi_*^S(X, o)$ stable homotopy groups of X.
ρ generic coaction.
ρ_M coaction on module M.
ρ_X coaction on $E^*(X)$ or $E^*(X)\hat{}$.
Σ, Σ^k suspension isomorphism, (3.13), Defn. 6.6.
ΣX, $\Sigma^k X$ suspension of space X.
ΣM, $\Sigma^k M$ suspension of module M, Defn. 6.6.
$\sigma_k: \underline{E}_k \to E$ stabilization, Defn. 9.3.
τ_i stable element for $H(\mathbb{F}_p)$, (14.5).
ϕ generic monoid multiplication.
χ canonical antiautomorphism of Hopf algebra.
Ψ^k Adams operation, (14.20).
ψ generic comultiplication.
ΩX loop space on based space X.
ω zero morphism of generic group object, §7.

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