FIELDS ADMITTING NONTRIVIAL STRONG ORDERED EULER CHARACTERISTICS ARE QUASIFINITE

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1. Introduction and Background

The note contains the details of an assertion made in [1] to the effect that fields admitting a nontrivial strong ordered Euler characteristic are quasifinite. In this section we recall the relevant definitions and in the next section we complete the proof.

Recall that a field K is quasifinite if K is perfect and its absolute Galois group is isomorphic to the profinite completion of $\mathbb{Z}$. In particular, a finite field is quasifinite. A strong ordered Euler characteristic on the field K is a function $\chi : \mathrm{Def}(K) \to R$ from the set of definable (in the language of rings) subsets of (any Cartesian power) of K to a partially ordered ring R having image amongst the nonnegative elements of R and satisfying $\chi(X) = \chi(Y)$ for X and Y definably isomorphic, $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$, $\chi(X \cup Y) = \chi(X) + \chi(Y)$ for $X \cap Y = \varnothing$, and $\chi(E) = c \cdot \chi(B)$ if $f : E \to B$ is a definable function and $c = \chi(f^{-1}\{b\})$ for every $b \in B$. The Euler characteristic is nontrivial if $0 < 1$ in R and the image of χ is not just $\{0\}$.

The main theorem of this note is the following:

Theorem 1. *Any field admitting a nontrivial strong ordered Euler characteristic is quasifinite.*

2. Proofs

As the conclusion of Theorem 1 holds for finite fields, we may restrict attention to infinite fields. Throughout the rest of this note K denotes an infinite field given together with a nontrivial strong ordered Euler characteristic $\chi : \mathrm{Def}(K) \to R$.

Lemma 1. *K is perfect.*

Proof. If K has characteristic zero, then there is nothing to prove. So we may assume that the characteristic of K is $p > 0$. The map $x \mapsto x^p$ on K is a definable bijection so $\chi([K]) = \chi([K^p])$. The inclusion $K^p \hookrightarrow K$ shows that $\chi([K^p]) \le \chi([K])$ with equality only if $K = K^p$. Thus, $K = K^p$. That is, K is perfect as claimed. □

We now aim to show by a counting argument that for each positive integer n there is a unique extension of K of degree n. We need a simple combinatorial lemma.

Lemma 2. *For a multiïndex $\alpha \in {\mathbb{Z}_+}^\omega$ define $w(\alpha) := \sum_{n=0}^\infty n\alpha_n$. Then for any natural number N we have $\sum_{\{\alpha : w(\alpha) = N\}} \prod_{n=1}^\infty \frac{1}{n^{\alpha_n}(\alpha_n!)} = 1$.*

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Proof.

$$\begin{aligned}
\sum_{N=0}^{\infty}\Big(\sum_{\{\alpha:w(\alpha)=N\}}\prod_{n=1}^{\infty}\frac{1}{n^{\alpha_n}(\alpha_n!)}\Big)X^N &= \prod_{n=1}^{\infty}\Big(\sum_{m=0}^{\infty}\frac{1}{n^m(m!)}X^{nm}\Big)\\
&= \prod_{n=1}^{\infty}\exp\Big(\frac{X^n}{n}\Big)\\
&= \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}X^n\Big)\\
&= \exp\Big(\log\Big(\frac{1}{1-X}\Big)\Big)\\
&= \frac{1}{1-X}\\
&= \sum_{N=0}^{\infty}X^N
\end{aligned}$$

Equating the coëfficients of X^N we obtain the statement of the lemma. □

Lemma 3. *Let $R' := R\otimes\mathbb{Q}$. There is a unique structure of a partially ordered ring on R' for which $\nu : R\to R'$ is morphism of partially ordered ring. Moreover, $R'\neq 0$.*

Proof. The positive elements in R' are exactly those of the form $x\otimes r$ with $x>0$ in R and $r>0$ in $\mathbb{Q}$. The rest of the proof is routine. □

We let $\tilde{\chi} := \nu\circ\chi : \mathrm{Def}(K)\to R'$.

We define $I_n := \{(a_0,\ldots,a_{n-1})\in K^n : X^n+\sum_{i=0}^{n-1}a_iX \text{ is irreducible over } K\}$.

Lemma 4. *For any positive integer n we have $\tilde{\chi}([I_n]) = \frac{1}{n}\tilde{\chi}([K])^n + O(\tilde{\chi}([K])^{n-1})$.*

Proof. We prove the lemma by induction on n with the case of $n=1$ being trivial as $I_1 = K$.

For each n-tuple $a=(a_0,\ldots,a_{n-1})\in K^n$, let $\alpha(a):\mathbb{Z}_+\to\omega$ be defined by $\alpha(a)_m :=$ the number of irreducible factors of $X^n+\sum_{i=0}^{n-1}a_iX^i$ of degree m. Let $\beta(a):\mathbb{Z}_+^2\to\omega$ be defined by $\beta(a)(m,r) :=$ the number of irreducible factors of $X^n+\sum_{i=0}^{n-1}a_iX^i$ of degree m appearing with multiplicity exactly r.

For a given function $f:\mathbb{Z}_+\to\omega$ with $w(f)=n$, let $P_f := \{a\in K^n : \alpha(a)=f\}$. Likewise, for a given $g:\mathbb{Z}_+^2\to\omega$ with $\tilde{w}(g) := \sum_{m=1,r=1}^{\infty} r\cdot m\cdot g(r,m) = n$, let $Q_g := \{a\in K^n : \beta(a)=g\}$. We define $u(g) := \sum_{m=1,r=1}^{\infty} m\cdot g(r,m)$.

Given g with $\tilde{w}(g)=n$, let $\psi_g : \prod_{m,r} I_m^{g(m,r)}\to K^n$ be the coëfficient map associated to the composition of multiplication of polynomials with exponentiation of polynomials to the power r. Note that the image of ψ_g is Q_g. Moreover, ψ_g is $\prod_{m,r} g(m,r)!$-to-one over it image. Therefore, $(\prod_{m,r} g(m,r)!)\chi([Q_g]) = \prod_{m,r}\chi([I_m])^{g(m,r)} = \prod_{m,r}\frac{1}{m^{g(m,r)}}\chi([K])^{u(g)} + O(\chi([K])^{u(g)-1}$.

We have $K^n = I_n \dot{\cup} \coprod_{\{g:\tilde{w}(g)=n,g(n,1)=0\}} Q_g$. Thus,

$$\begin{aligned}
\chi([I_n]) &= \chi([K])^n - \sum_{\{g:\tilde{w}(g)=n,g(n,1)=0\}} (\prod_{m,r} \frac{1}{m^{g(m,r)}(g(m,r)!)})\chi([K])^{u(g)} + O(\chi([K])^{n-1}) \\
&= (1 - \sum_{\{f:w(f)=n,f(n)=0\}} \frac{1}{m^{f(m)}(f(m)!)})\chi([K])^n + O(\chi([K])^{n-1}) \\
&= \frac{1}{n}\chi([K])^n + O(\chi([K])^{n-1})
\end{aligned}$$

as claimed. □

Lemma 5. *Let L/K be an extension of degree n. Let $S := \{a \in K^n : X^n + \sum_{i=0}^{n-1} a_i X^i$ is the monic minimal polynomial of some $a \in L\}$. Then $\tilde{\chi}([S]) \geq \frac{1}{n}\tilde{\chi}([K])^n + O(\chi([K])^{n-1})$.*

Proof. Let $B := \{b \in L : K(c) \neq L\}$. As the extension L/K is finite and separable, $B = \bigcup K \leq M < LM$ where the union runs over the finitely many proper subfields of L containing K. Each of these is a finite dimensional vector space over K of dimension strictly less than n. Thus, $\tilde{\chi}([L \setminus B]) = \chi([K])^n + O(\tilde{\chi}([K])^{n-1})$.

For each $1 \leq s \leq n$ let $E_s := \{a \in L \setminus B : a$ has exactly s conjugates in L over $K\}$. Let $f : (L \setminus B) \to K^n$ be defined by $f(a) = (b_0, \ldots, b_{n-1})$ where $X^n + \sum_{i=0}^{n-1} b_i X^i$ is the monic minimal polynomial of a over K. Note that when restricted to E_s, the function f is s-to-one. Then $S = \coprod_{s=1}^n f(E_s)$. Thus, $\tilde{\chi}([S]) = \sum_{s=1}^n \frac{1}{s}\tilde{\chi}([E_s]) \geq \sum_{s=1}^n \frac{1}{n}\chi([E_s]) = \frac{1}{n}\chi([L \setminus B]) = \frac{1}{n}\tilde{\chi}([K])^n + O(\tilde{\chi}([K])^{n-1})$ as claimed. □

Proof of main theorem: Combining the last two lemmata we see that there is a unique (Galois!) field extension of each degree. ⊣

References

[1] J. Krajíček and T. Scanlon, Combinatorics with definable sets: Grothendieck rings and Euler characteristics, *Bulletin of Symbolic Logic*, (to appear).

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