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ON ELEMENTARY TOPOSES

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1. Introduction

The theory of elementary toposes has its origins in two seperate lines of mathematical development, that is, the one a geometric, the other a logical aspect, which remained distinct for nearly ten years.

The earlier of two lines begins with the rise of sheaf theory. The framework of this first line is taken by A. Grothendieck, M. Artin, J. Giraud and others in 1963-1964, as a generalization of topological space, for the purpose in algebraic geometry.

The starting point of this second line is taken to be F. W. Lawveres [6] in 1964, as the notion of higher order language. During the year 1969~1970 the two lines of mathematical development began to come together by F. W. Lawveres and M. Tierney [7] and the theory of elementary toposes was developed by P. Freyd [2], A. Kock and G. C. Wraith [10], C. J. Mikkesen and others. The next major advance was made by R. Diaconescu [1], [7, pp. 44-59], R. Pare [9], W. Mitchell [8], and P. T. Johnstone [4], [5].

The purpose of this paper is to investigate some basic properties of elementary toposes. In section 2 we prove some useful principles concerning the morphisms in elementary toposes. In section 3 we discuss the properties concerning representability for partial maps and the section 4 is devoted to give the properties for the endofunctors in an elementary topos. Finally in section 5 we state some exactness properties, in particulary we prove that an elementary topos is embedded in the elementary topos of objects over Ω.

2. Equivalence relations in an elementary topos

DEFINITION 1. A category $\boldsymbol{E}$ is called an elementary topos (or topos) if

(i) $\boldsymbol{E}$ has all finite limits.

(ii) For each object X of $\boldsymbol{E}$ we have an exponential functor $(\ )^X:\boldsymbol{E}\longrightarrow\boldsymbol{E}$ which is right adjoint to the functor $(\ )\times X:\boldsymbol{E}\longrightarrow\boldsymbol{E}$.

(iii) $\boldsymbol{E}$ has an object Ω and a morphism $t:1\longrightarrow\Omega$ such that for each monomorphism $\sigma: Y\longrightarrow X$ in $\boldsymbol{E}$, there is a unique $\phi_\sigma: X\longrightarrow\Omega$ (the classifying map of σ), making

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$$\begin{array}{ccc} Y & \longrightarrow & 1 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle t} \\ X & \underset{\phi_\sigma}{\longrightarrow} & \Omega \end{array}$$

a pullback diagram, where 1 is the terminal object.

Examples. The following categories are toposes;

(i) $\boldsymbol{S}$, the category of sets and functions.

(ii) The category of G–Sets and G–functions for a group G.

(iii) A Grothendick topos [4].

In a topos $\boldsymbol{E}$ if $g\cdot h=1_A : A\xrightarrow{h}B\xrightarrow{g}A$ then g is called a split epimorphism and h a split monomorphism.

LEMMA 1. *In a topos every monomorphism is an equalizer* [10].

PROPOSITION 1. *A topos is balanced.*

Proof. Let $f:A\to B$ be a bimorphism in a topos $\boldsymbol{E}$. Then by the lemma 1 f is an equalizer of two morphisms $B\underset{h}{\overset{g}{\rightrightarrows}}C$. Hence $g\cdot f=h\circ f$ and $g=h$ since f is an epimorphism. There is a unique morphism $f':B\longrightarrow A$ such that $f\cdot f'=1_B$ since $g\cdot 1_B=h\cdot 1_B$ and f is an equalizer of g and h. While $f'\cdot f=1_A$ since f is a monomorphism. Therefore f is an isomorphism. *q. e. d.*

A parallel pair $X\underset{f}{\overset{g}{\rightrightarrows}}Y$ in a category $\boldsymbol{C}$ is said to be reflexive if there exists $Y\xrightarrow[h]{}X$ such that $fh=gh=1_Y$.

DEFINITION 2. Let $R\underset{b}{\overset{a}{\rightrightarrows}}X$ be a parallel pair of morphisms in a topos $\boldsymbol{E}$. We say that $\langle a,b\rangle$ is an equivalence relation on X if

(i) $R\xrightarrow[\langle a,b\rangle]{}X\times X$ is a monomorphism.

(ii) The diagonal subobject $X\overset{\Delta}{\rightarrowtail}X\times X$ factors through $\langle a,b\rangle$.

(iii) There exists a morphism $\tau:R\longrightarrow R$ such that $b\cdot\tau=a$ and $a\cdot\tau=b$.

(iv) If the diagram

$$\begin{array}{ccc} T & \overset{q}{\longrightarrow} & R \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle a} \\ R & \underset{b}{\longrightarrow} & X \end{array}$$

is pullback, then $\langle ap, bq\rangle : T \to X\times X$ factors through $\langle a, b\rangle$ [4].

If the diagram in a topos $\boldsymbol{E}$

$$\begin{array}{ccc} K & \xrightarrow{k_1} & A \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is pullback then a parallel pair $K \underset{k_2}{\overset{k_1}{\rightrightarrows}} A$ of morphisms in $\boldsymbol{E}$ is called the kernel pair of f. And an equivalence relation which is a kernel pair of some morphism is said to be effective.

LEMMA 2. *The kernel pair* $K \underset{k_1}{\overset{k_0}{\rightrightarrows}} A$ *of a morphism* $f: A\to B$ *in a topos* $\boldsymbol{E}$ *is always an equivalence relation.*

Proof. (i) For $x_0 \cdot x_1 : X \to K$, if $k_0\cdot x_0 = k_0\cdot x_1$, then there exists a unique $h: X\to K$ such that $k_1\cdot h = k_0\cdot x_1$. Therefore $h=x_1=x_2$, and so k_0 is a monomorphism. Similarly k_1 is a monomorphism. Hence $K \xrightarrow[\langle k_0, k_1\rangle]{} A\times A$ is a monomorphism.

(ii) Let $\triangle : A \rightarrowtail A\times A$ be the diagonal subobject. Then for $k_0, k_1 : K\to A$, there is a unique $\langle k_0, k_1\rangle : K \to A\times A$ and since $K \underset{k_1}{\overset{k_0}{\rightrightarrows}} A$ is the kernel pair of f, there exists a unique k': $A\to K$ such that $k_1\cdot k' = 1_A$ and $k_0\cdot k' = 1_A$. For the projection $\pi_i : A\times A \to A$, $\pi_i(\langle k_0, k_1\rangle \cdot k') = k_1\cdot k' = 1_A (i=1, 2)$. But $\pi\cdot\triangle = 1_A$, and so, $\triangle = \langle k_0, k_1\rangle\cdot k'$.

(iii) Since the diagram

$$\begin{array}{ccc} K & \xrightarrow{k_1} & A \\ {\scriptstyle k_0}\downarrow & & \downarrow{\scriptstyle f} \\ A_0 & \xrightarrow[f]{} & B \end{array}$$

is pullback there is a unique $\tau : K\to K$ such that $k_1\cdot\tau = k_0$ and $k_0\cdot\tau = k_1$.

(iv) If the diagram

$$\begin{array}{ccc} T & \xrightarrow{q} & K \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle k_0} \\ K & \xrightarrow[k_1]{} & A \end{array}$$

is pullback, then the diagram

$$\begin{array}{ccc} T & \xrightarrow{p} & K \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle k_1} \\ K & \xrightarrow[k_0]{} & A \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is also pullback. Therefore there exists a unique $t: T \to K$ such that $k_1 \cdot q = k_0 t$ and $k_1 \cdot p = k_1 t$. Since k_1 is a monomorphism $p = t$, and so $k_1 \cdot q = k_0 \cdot p$. Since the diagram

$$\begin{array}{ccc} K & \xrightarrow{k_1} & A \\ {\scriptstyle k_0}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is pullback and $f \cdot k_0 \cdot p = f \cdot k_1 \cdot q$, there exists a unique $t': T \to K$ such that $k_0 \cdot p = k_0 \cdot t'$ and $k_1 \cdot q = k_1 \cdot t'$. Hence we have $p = q = t'$. *q. e. d.*

PROPOSITION 2. *In a topos, equivalence relations are effective.*

Proof. Let $R \underset{b}{\overset{a}{\rightrightarrows}} X$ be an equivalence relation. Let $X \times X \xrightarrow{\phi} \Omega$ be the classifying map of $R \rightarrowtail X \times X$, and $X \xrightarrow{\bar{\phi}} \Omega^X$ its' exponential transpose.

(i) If $U \underset{g}{\overset{f}{\rightrightarrows}} X$ be a pair of morphisms such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow[\bar{\phi}]{} & \Omega^X \end{array}$$

is commutative, then we have $\phi \cdot (f \times 1_X) = \phi \cdot (g \times 1_X)$. If we compose with $U \xrightarrow{\langle 1_U, g \rangle} U \times X$ and $\phi \cdot (f \times 1_X)$, $\phi \cdot (g \times 1_X)$, then we have $\phi \cdot \langle f, g \rangle = \phi \cdot \langle g, g \rangle \cdot$ But $\triangle \cdot g = \langle g, g \rangle$ and $\triangle = \langle a, b \rangle \cdot a'$ since R is reflexive, and so $\langle g, g \rangle = \langle a, b \rangle \cdot a' \cdot g$ and $\phi \cdot \langle g, g \rangle = t \cdot (R \to 1) \cdot a \cdot g$.

Let the diagram

$$\begin{array}{ccccc} P & & \xrightarrow{k} & & 1 \\ {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle t} \\ U & \xrightarrow[\langle g, g \rangle]{} & X \times X & \xrightarrow[\phi]{} & \Omega \end{array}$$

be pullback. Then there exists a unique morphism $h': U \longrightarrow P$ such that $h \cdot h' = 1_U$. Hence h' is an isomorphism since h is a monomorphism. Therefore there is a unique morphism $\mu: U \longrightarrow R$ such that

$$\langle a, b \rangle \cdot \mu = \langle f, g \rangle. \tag{2.1}$$

(ii) We must show that $\bar{\phi} \cdot a = \bar{\phi} \cdot b$. But $\bar{\phi} \cdot a = \bar{\phi} \cdot \mathrm{b}$ iff $\phi \cdot (a \times 1_X) = \phi \cdot (b \times 1_X)$. Let the diagrams

$$\begin{array}{ccccc} P_a & \longrightarrow & & & 1 \\ {\scriptstyle \langle \alpha, x \rangle} \downarrow & & & & \downarrow {\scriptstyle t} \\ R \times X & \underset{a \times 1_X}{\longrightarrow} & X \times X & \longrightarrow & \Omega \end{array} \quad \text{and} \quad \begin{array}{ccccc} P_b & \longrightarrow & & & 1 \\ {\scriptstyle \langle \beta, y \rangle} \downarrow & & & & \downarrow {\scriptstyle t} \\ R \times X & \underset{b \times 1_X}{\longrightarrow} & X \times X & \longrightarrow & \Omega \end{array} \tag{2.2}$$

be pullback.

Since R is an equivalence relation there exists $\tau: R \longrightarrow R$ such that

$$a \cdot \tau = b \text{ and } b \cdot \tau = a. \tag{2.3}$$

From (2.2) and (2.3) we have

$$\phi \cdot a \times 1_X \cdot \langle \alpha, x \rangle = t \cdot (P_a \longrightarrow 1) \tag{2.4}$$

and $(b \times 1_X) \cdot (\tau \times 1_X) = a \times 1_X$. From (2.4) we have also

$$\phi \cdot (b \times 1_X) \cdot (\tau \times 1_X) \cdot \langle \alpha, x \rangle = t \cdot (P_a \longrightarrow 1)$$

and so, there is a unique morphism $\gamma: P_a \longrightarrow P_b$ such that $\langle \beta, y \rangle \cdot \gamma = \langle \tau\alpha, x \rangle$.

Similarly there is a unique morphism $\gamma': P_b \longrightarrow P_a$ such that $\langle \alpha, x \rangle \cdot \gamma' = \langle \tau\beta, y \rangle$. Since $\langle \beta, y \rangle$ is a monomorphism $\gamma \cdot \gamma' = 1_{P_b}$ and $\gamma' \cdot \gamma = 1_{P_a}$.

Hence P_a and P_b are isomorphic. In (2.2) since we can put $P_a = P_b$ and $\langle \alpha, x \rangle = \langle \beta, y \rangle$, by the condition (iii) of the definition 1 we have $\phi \cdot (a \times 1_X) = \phi \cdot (b \times 1_X)$. *q. e. d.*

DEFINITION 3. The exponential transpose of the classifying map $\delta: X \times X \rightarrow \Omega$ of the diagonal map $\Delta: X \rightarrowtail X \times X$ is called the singleton map on X and denote by $\{\ \}_X: X \longrightarrow \Omega^X$ or simply $\{\ \}$.

COROLLARLY 1. *A singleton map is a monomorphism.*

Proof. The diagonal map $\Delta: X \rightarrowtail X \times X$ and $1_x: X \longrightarrow X$ are equivalence relations by the proposition 1. $\langle 1_X, 1_X \rangle: X \longrightarrow X \times X$ is the kernel pair of $\{\ \}: X \longrightarrow \Omega^X$. If $\{\ \} \cdot x = \{\ \} \cdot y$, then there exists a unique $h: Y \longrightarrow X$ such that $h = x$ and $h = y$, and so, $x = y$. *q. e. d.*

3. Partial maps

A partial map $X \underset{d}{\overset{f}{\longrightarrow}} Y$ in a category $\boldsymbol{E}$ is a diagram of the form

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\big\downarrow & & \\ X & & \end{array}$$

We say that partial maps with codomain Y are representable if there exists a mono $Y \overset{\eta_Y}{\rightarrowtail} \tilde{Y}$ such that, for any $X \underset{d}{\overset{f}{\longrightarrow}} Y$, there exists a unique $X \overset{\tilde{f}}{\rightarrowtail} \tilde{Y}$ making

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle \eta_Y} \\ X & \underset{\tilde{f}}{\longrightarrow} & \tilde{Y} \end{array}$$

a pullback diagram.

LEMMA 3. *Let* $\phi : \Omega^Y \times Y \longrightarrow \Omega$ *classify the singleton map* $\{\ \} : Y \rightarrowtail \Omega^Y$, $\bar{\phi} : \Omega^Y \longrightarrow \Omega^Y$ *its exponential transpose and let* $\tilde{Y} \overset{e}{\rightarrowtail} \Omega^Y$ *the equalizer of* $\bar{\phi}$ *and* 1_{Ω^Y} *in a topos* $\mathbf{E}$. *Then* $\bar{\phi} \cdot \{\ \} = \{\ \}$ *and so* $\{\ \}$ *factors through e.*

Proof. Since $\{\ \} \times 1_Y \cdot \Delta = \langle \{\ \}, 1_Y \rangle$, from the hypothesis we have the pullback diagram

$$\begin{array}{ccccc} Y & \xrightarrow{1_Y} & Y & \longrightarrow & 1 \\ {\scriptstyle \Delta}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle t} \\ Y \times Y & \longrightarrow & \Omega^Y \times Y & \longrightarrow & \Omega. \end{array}$$

But since $\delta \cdot \Delta = t \cdot (Y \longrightarrow 1)$ we have $\phi \cdot \{\ \} \times 1_Y = \delta : Y \times Y \to \Omega$ and their exponential transposes are equal; that is,

$$\bar{\phi} \cdot \{\ \} = \{\ \}. \tag{3.1}$$

Since $\tilde{Y} \overset{e}{\longrightarrow} \Omega^Y$ is the equalizer of $\bar{\phi}$ and 1_{Ω^Y}, from (3.1) there exists a unique morphism $\eta_Y : Y \longrightarrow \tilde{Y}$ such that $\{\ \} = e \cdot \eta_Y$. (3.2). *q.e.d.*

LEMMA 4. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\big\downarrow & & \\ X & & \end{array}$$

be a partial map in a topos $\boldsymbol{E}$ *and* $\phi: X\times X \to \Omega$ *be a classifying map of the graph* $X' \xrightarrow{\langle d,f\rangle} X\times Y$ *of* f. *Then the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \{\ \}} \\ X & \xrightarrow[\bar{\phi}]{} & \Omega^Y \end{array} \tag{3.3}$$

is pullback.

Proof. (i) Since the exponential transposes of $\bar{\phi}\cdot d$ and $\{\ \}\cdot f$ are $\phi\cdot d\times 1_X$ and $\delta\cdot f\times 1_{X'}$ respectively. We must show that

$\phi\cdot(d\times 1_X)=\delta\cdot(f\times 1_{X'})$ instead of $\bar{\phi}\cdot d=\{\ \}\cdot f$.

$\phi\cdot(d\times 1_X)\cdot\langle 1_{X'}, f\rangle=\phi\cdot\langle d, f\rangle=t(x'\to 1)=t(X\to 1)=ty\cdot f$, where $y: Y\to 1$

$\delta\cdot(f\times 1_X)\cdot(1_{X'}, f)=\delta\cdot\langle f, f\rangle=(\delta\cdot\triangle)\cdot f=ty\cdot f$.

Hence $\phi\cdot d\ \times 1_X=\delta\cdot f\times 1_{X'}$ and $\bar{\phi}\cdot d=\{\ \}\cdot f$.

(ii) Let $U\xrightarrow{\langle a,b\rangle} X\times Y$ such that $\bar{\phi}\cdot a=\{\ \}\cdot b$.

By transposition, we obtain

$$\phi\cdot a\times 1_Y=\delta\cdot b\times 1_Y,$$
$$\phi\cdot a\times 1_Y\langle 1, b\rangle=\delta\cdot b\times 1_Y\langle 1, b\rangle,$$
$$\phi\cdot\langle a, b\rangle=\delta\cdot\langle b, b\rangle=\delta\cdot\langle\triangle, b\rangle=\delta\cdot\triangle\cdot b.$$

But since the diagram

$$\begin{array}{ccc} Y & \longrightarrow & 1 \\ {\scriptstyle \triangle}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ Y\times Y & \xrightarrow[\delta]{} & \Omega \end{array}$$

is pullback, the diagram

$$\begin{array}{ccccc} U & \longrightarrow & Y & \longrightarrow & 1 \\ {\scriptstyle b}\big\downarrow & & & & \big\downarrow{\scriptstyle t} \\ Y & \xrightarrow[\triangle]{} & Y\times Y & \xrightarrow[\delta]{} & \Omega \end{array}$$

is pullback. Therefore we have $\phi\cdot\langle a, b\rangle=\delta\cdot\langle b, b\rangle=U\longrightarrow 1\xrightarrow{t}\Omega$.

From the hypothesis there exists a unique morphism $U\xrightarrow{u} X'$ such that $\langle d, f\rangle\cdot u=\langle a, b\rangle$. *q. e. d.*

THEOREM *In a topos, all partial maps are representable.*

Proof. (i) By the lemmas 3 and 4, the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f} & Y & \longrightarrow & 1 \\ {\scriptstyle\langle d,f\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \{\ \}\times 1_Y} & & \big\downarrow{\scriptstyle t} \\ X\times Y & \xrightarrow[\bar{\Phi}\times 1_Y]{} & \Omega^Y\times Y & \xrightarrow[\phi]{} & \Omega \end{array} \tag{3.4}$$

is pullback.

From (3.2) and (3.4) we have

$$\phi=\phi\cdot\bar{\Phi}\times 1_Y. \tag{3.5}$$

Hence $$\bar{\Phi}\cdot\bar{\Phi}=\bar{\Phi}. \tag{3.6}$$

Since $\tilde{Y}\xrightarrow{e}\Omega^Y$ is an equalizer of ϕ and 1_{Ω^Y}, there exists a unique $X\xrightarrow{\tilde{f}}\tilde{Y}$ such that $\bar{\Phi}=e\cdot\tilde{f}$ (3.7)

From (3.1) and (3.7) we have $\bar{\Phi}\cdot d=\{\ \}\cdot f$ and $(e\cdot\tilde{f})\cdot d=(e\cdot\eta_Y)\cdot f$, and so, we have $\tilde{f}\cdot d=\eta_Y\cdot f$, since e is a monomorphism.

(ii) We shall show the uniqueness of $\tilde{f}$.

Suppose that the diagrams

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[\tilde{f}_1]{} & \tilde{Y} \end{array} \quad \text{and} \quad \begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[\tilde{f}_2]{} & \tilde{Y} \end{array}$$

are pullback. Put $\bar{\Phi}_i=e\cdot\tilde{f}_i$ $(i=1,2)$.

From (3.4) and (3.5)

$$tx'=\phi_1\cdot\langle d,f\rangle=\phi_1\cdot(d\times 1_Y)\cdot\langle 1_{X'},f\rangle$$
$$=(\phi_1\cdot(d\times 1_Y))\cdot\langle 1_{X'},f\rangle=(\overline{\bar{\Phi}_1\cdot d})\cdot\langle 1_{X'},f\rangle$$

where $x':X'\to 1$.

But, by the hypothesis,

$$\tilde{f}_1\cdot d=\eta_Y\cdot f=\tilde{f}_2\cdot d$$
$$e(\tilde{f}_1\cdot d)=e(\eta_Y\cdot f)=e\cdot(\tilde{f}_2\cdot d)$$
$$\bar{\Phi}_1\cdot d=\bar{\Phi}_2\cdot d=\{\ \}\cdot f \tag{3.8}$$
$$(\overline{\bar{\Phi}_2\cdot d})\cdot\langle 1_{X'},f\rangle=\phi_2\cdot\langle d,f\rangle=tx'$$
$$\therefore\ \phi_1=\phi_2 \text{ and so } \bar{\Phi}_1=\bar{\Phi}_2.$$

From (3.7) $e\tilde{f}_1=e\tilde{f}_2$, and $\tilde{f}_1=\tilde{f}_2$ since e is a monomorphism. *q.e.d.*

4. Endofunctors in a topos

We can consider a partial map $X \xrightarrow[d]{f} Y$ in $\boldsymbol{E}$ as a monomorphism $\langle d, f\rangle : X' \rightarrowtail X \times Y$ since $d : x' \rightarrowtail x$ is mono. Hence the equivalence classes of partial maps $X \xrightarrow[d]{f} Y$ means the equivalence classes of the subobjects $\langle d, f\rangle : X' \rightarrowtail X \times Y$ and denote it by $Pt(X, Y)$.

Since η_Y is a monomorphism, given $X \xrightarrow{f} \tilde{Y}$, we can produce a unique partial map $X \xrightarrow[d]{f} Y$ and by the theorem, given a partial map $X \xrightarrow[d]{f} Y$, there exists a unique morphism $\bar{f} : X \to \tilde{Y}$ such that

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[\bar{f}]{} & \tilde{Y} \end{array}$$

is pullback. Hence we have the one-to-one correspondence $\lambda : Pt(X, Y) \longrightarrow \mathrm{Hom}_{\mathrm{E}}(X, \tilde{Y})$.

Let $W : \boldsymbol{E} \to \boldsymbol{E}$ be $W(X) = \tilde{X}$, for every $X \in Ob(\boldsymbol{E})$ and $W(\mu) = \tilde{\mu} : \tilde{X} \to \tilde{Y}$ for $\mu : X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \tilde{X} & \xrightarrow[\tilde{\mu}]{} & \tilde{Y} \end{array}$$

is pullback. Then for $X \xrightarrow{\mu} Y \xrightarrow{\nu} Z$ by the theorem there exists a unique morphism $\widetilde{\nu \cdot \mu}$ such that

$$\begin{array}{ccccc} X & \xrightarrow{\mu} & Y & \xrightarrow{\nu} & Z \\ {\scriptstyle \eta_x}\downarrow & & & & \downarrow{\scriptstyle \eta_z} \\ \tilde{X} & & \xrightarrow[\widetilde{\nu \cdot \mu}]{} & & Z \end{array}$$

is pullback. And the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\mu} & Y & \xrightarrow{\nu} & Z \\ {\scriptstyle \eta_X}\downarrow & & {\scriptstyle \eta_Y}\downarrow & & {\scriptstyle \eta_Z}\downarrow \\ \tilde{X} & \xrightarrow[\tilde{\mu}]{} & \tilde{Y} & \xrightarrow[\tilde{\nu}]{} & \tilde{Z} \end{array}$$

is pullback since $Z \rightarrow \tilde{Z}$ is a monomorphism. Hence $W(\nu \cdot \mu) = \tilde{\nu} \cdot \tilde{\mu}$ and W is an endofunctor in $\boldsymbol{E}$.

Let I_{E} be the identity functor of $\boldsymbol{E}$. For $Y \xrightarrow{g} Y'$ there exists a unique $\tilde{g}: \tilde{Y} \rightarrow \tilde{Y}'$ such that

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y' \\ {\scriptstyle \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_{Y'}} \\ Y & \xrightarrow[\tilde{g}]{} & Y' \end{array}$$

is pullback. Hence $\eta: I_{\mathrm{E}} \rightarrow W$ is a natural transformation.

Define the composition $\Theta: Pt(X, Y) \times Pt(Y, Z) \longrightarrow Pt(X, Z)$ by

$$\Theta(\langle d, f\rangle, \langle d', g\rangle) = \langle d, g \cdot f\rangle.$$

We can define the category $Pt(\boldsymbol{E})$ with objects the same as the objects of $\boldsymbol{E}$ and with hom-sets $Pt(X, Y)$.

PROPOSITION 3. *Define* $I: \boldsymbol{E} \longrightarrow Pt(\boldsymbol{E})$ *by*

$$I(X) = X, \quad I(X \xrightarrow{f} Y) = X \xrightarrow[\langle \eta_X, f\rangle]{} \tilde{X} \times Y$$

and $J: Pt(\boldsymbol{E}) \longrightarrow \boldsymbol{E}$ *by*

$$J(X) = \tilde{X}, \text{ and } J(\langle d, f\rangle : X' \rightarrowtail X \times Y) = \bar{f}$$

such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & Y \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow[\bar{f}]{} & \tilde{Y} \end{array}$$

is pullback. Then I is a left adjoint of J.

Proof. Define for $(X, Y) \in Ob(\boldsymbol{E}^{op} \times Pt(\boldsymbol{E}))$, a map

$$\begin{array}{c} \psi_{X,Y}: \mathrm{Hom}_{Pt(\mathrm{E})}(I(x), Y) \longrightarrow \mathrm{Hom}_{\mathrm{E}}(X, \tilde{Y}) \\ \parallel \\ Pt(X, Y) \end{array}$$

by $\psi_{X,Y}(X' \overset{<d,f>}{\rightarrowtail} X\times Y)=\bar{f}$.

Then for every $\bar{h}: X \longrightarrow \tilde{Y} \in \mathrm{Hom}_E(X, \tilde{Y})$, there exists the unique partial map $X \underset{d}{\overset{h}{\longrightarrow}} Y$ and if $\psi_{X,Y}(\langle d, f\rangle)=\psi_{X,Y}(\langle d', f\rangle)$, then $\bar{f}=\bar{f}'$. Hence by the theorem $d\cdot f=d'\cdot f'$, so that, $\psi_{X,Y}{}'$ is an onto one-one map. For $X \underset{u}{\longrightarrow} U$ of $\boldsymbol{E}$ and $V' \underset{<v,g>}{\rightarrowtail} V\times Y \in Pt(V, Y)$ we have

$$\psi_{X,Y}\cdot \mathrm{Hom}_{Pt(\mathbf{E})}(u, \langle v, g\rangle)=\mathrm{Hom}_{\mathbf{E}}(u, \tilde{g})\cdot \psi_{U,V},$$

where g is a morphism such that the diagram

$$\begin{array}{ccc} V' & \overset{g}{\longrightarrow} & Y \\ {\scriptstyle v}\downarrow & & \\ V & & \Big\downarrow {\scriptstyle \eta_Y} \\ {\scriptstyle \eta_V}\downarrow & & \\ \tilde{V} & \underset{\tilde{\tilde{g}}}{\longrightarrow} & \tilde{Y} \end{array}$$

is pullback, since for every $\langle b, t\rangle \in Pt(U, V)$,

$$\psi_{X,Y}\cdot \mathrm{Hom}_{Pt(E)}(u, \langle v, g\rangle)(\langle b, t\rangle)=\psi_{X,Y}(\langle d, f\rangle)\bar{f}=g\cdot \bar{t}\cdot u$$

where $\langle d, f\rangle$ is a partial map determined by the pullback diagram by pulling η_Y along $\tilde{g}\cdot t\ u$. and similarly

$$\mathrm{Hom}_{\mathbf{E}}(u, \tilde{g})\cdot \psi_{U,v}(\langle b, t\rangle)=\mathrm{Hom}_E(u, \tilde{g})(\bar{t})=\tilde{g}\cdot \bar{t}\cdot u.$$

Therefore, ψ is a natural equivalence. *q.e.d.*

5. Exactness properties

Let $P: \boldsymbol{E}^{op} \longrightarrow \boldsymbol{E}$, $P(X)=\Omega^X$ for each object X of $\boldsymbol{E}$ and $P(f)$ be the exponential transpose of the morphism

$$\Omega^Y\times X \overset{1_{\Omega^Y}\times f}{\longrightarrow} \Omega^Y\times Y \overset{ev_Y}{\longrightarrow} \Omega$$

for each morphism $f: X \to Y$ in $\boldsymbol{E}$, where ev_Y is the evalution map to Y. Then P is the contravariant functor.

We shall denote the exponential transpose of the classifying map $\varepsilon(f)$ of the monomorphism $\epsilon_X \longrightarrow \Omega^X\times X^X \overset{1\times f}{\longrightarrow} \Omega^X\times Y$ by $\exists f: \Omega^X \longrightarrow \Omega^Y$ where $f: X \rightarrowtail Y$ is mono and ϵ_X is the subobject classified by $\Omega^X\times X \overset{ev_X}{\longrightarrow} \Omega$. $\epsilon_X \overset{e_X}{\rightarrowtail} \Omega^X\times X$ is called the membership relation for X [9]. And $\varepsilon(f)$ is called an exponential adjoint of $\exists f$.

PROPOISITION 4. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g\downarrow & & \downarrow h \\ Z & \xrightarrow[k]{} & T \end{array}$$

be a pullback diagram with h mono. Then we have

$$P(k)\cdot\exists h=\exists g\cdot P(f)\ [4,9].$$

COROLLARY 2. *Let* $f: A \rightarrowtail C$ *be a monomorphism and* $g: B \longrightarrow C$ *morphism in* $\boldsymbol{E}$. *If* $(A\times B,\ \pi_A, \pi_B)$ *is a product of* (A, B) *and if* (E, e) *is the equalizer of* $f.\pi_A,\ g.\pi_B$, *then we have*

$$P(g)\cdot\exists f=(\exists\pi_B\cdot e)\cdot P(\pi_A\cdot e).$$

Proof. Since the diagram

$$\begin{array}{ccc} E & \xrightarrow{\pi_A\cdot e} & A \\ \pi_B\cdot e\downarrow & A\times B & \downarrow f \\ B & \xrightarrow[g]{} & C \end{array}$$

is pullback, by the proposition 4 we complete this proof.

COROLLARY 3. *If* $X\xrightarrow{f}Y$ *is a mono then* $P(f)\cdot\exists f=1_{\Omega^X}$.

Proof. The diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ 1_X\downarrow & & \downarrow f \\ X & \xrightarrow[f]{} & Y \end{array}$$

is pullback.

Since the morphism $P(1_X)$ is the exponential transpose of $ev_X(1_{\Omega^X}\times 1_X)=ev_X$. Similarly $\exists(1_X)$ is the exponential transpse of $\varepsilon(1_X)=ev_X$. Hence $\exists(1_X)=1_{\Omega^X}$. By the proposition 4,

$$P(f)\cdot\exists(f)=\exists(1_X)\cdot P(1_X)=1_{\Omega^X}.\quad q.e.d.$$

PROPOSITION 5. *For any monomorphism f in* $\boldsymbol{E}$, $P(f)$ *is mono if the evaluation map* ev_A *for all A of* $Ob(\boldsymbol{E})$ *is monomorphism.*

Proof. Let $f:A\to B$, then the exponential adjoint of $P(f)$ is $ev_A\cdot P(f)\times 1_A=ev_B\cdot 1_{\Omega^c}\times f$.

By the hypothesis, $ev_B\cdot 1_{\Omega^c}\times f$ is mono. Hence $P(f)\times 1_A$ is mono, so that $P(f)$ is mono. *q.e.d.*

PROPOSITION 6. *Let* $P(X)\underset{P(g)}{\overset{P(f)}{\rightrightarrows}}P(Y)\xrightarrow{P(h)}P(Z)$ *be a coequalizer diagram in* $\boldsymbol{E}$. *Then* $X\underset{g}{\overset{f}{\rightrightarrows}}Y\xrightarrow{h}Z$ *is a coequalizer* $\boldsymbol{E}^{op}$ *if the morphism* $Z\longrightarrow Y$ *in* $\boldsymbol{E}$ *is the subobject classified by* $\delta_X\cdot\langle f,g\rangle$, *where* δ_X *is the classifying map of* $\triangle:X\longrightarrow X\times X$.

Proof. $h\cdot g=h\cdot f$ and h is an epimorphism in $\boldsymbol{E}^{op}$ because $P(h)\cdot P(g)=P(hg)=P(h)\cdot P(f)=P(hf)$ and P is faithful. Since $h:Z\longrightarrow Y$ is a monomorphism in $\boldsymbol{E}$, by the hypothesis it is the subobject classified by $\delta_X\cdot\langle f,g\rangle$. Hence h is the equalizer of of f and g in $\boldsymbol{E}$, so that, h in $\boldsymbol{E}^{op}$ is the coequalizer of f,g in $\boldsymbol{E}^{op}$. *q.e.d.*

DEFINITION 4. For any category $\boldsymbol{C}$ and object A of $\boldsymbol{C}$, the category whose objects are morphisms with codomain A, and in which a morphism $p\xrightarrow{f}q$ from $X\xrightarrow{p}A$ to $Y\xrightarrow{q}A$ is given by a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & A & \end{array}$$

is called the category of objects over A, denote by $\boldsymbol{C}/A$, [3].

For $f:A\longrightarrow B$ in $\boldsymbol{E}$ and the commutative diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{\approx} & Z_1 \\ & {\scriptstyle \zeta_0}\searrow \quad \swarrow{\scriptstyle \zeta_1} & \\ & B & \end{array}$$

in $\boldsymbol{E}/B$, if $f^*(\zeta_0)=P_0\longrightarrow A$ such that the diagram

$$\begin{array}{ccc} P_0 & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z_0 & \xrightarrow[\zeta_0]{} & B \end{array}$$

is pullback and $f^*(z)=P_0 \longrightarrow P_1$ such that $f^*(\zeta_0)=f^*(\zeta_1)\cdot f^*(z)$, then f^* is a functor from $\boldsymbol{E}/B$ to $\boldsymbol{E}/A$. The f^* is called the pullback functor. In any topos $\boldsymbol{E}$ for any $f:X\rightarrow Y$, it's pullback functor f^* has a left adjoint $S_f:\boldsymbol{E}/X\rightarrow\boldsymbol{E}/Y$, [10].

If we define $P_f:\boldsymbol{E}/A\rightarrow\boldsymbol{E}/B$, for a morphism $f:A\rightarrow B$ in $\boldsymbol{E}$ by

(1) $P_f(X\xrightarrow{\xi}A)=P_f(\xi)\longrightarrow B$ for every $\xi\in\boldsymbol{E}/A$, such that the diagram

$$\begin{array}{ccc} P_f(\xi) & \longrightarrow & \tilde{X}^A \\ \downarrow & & \downarrow (\tilde{\xi})^A \\ B & \xrightarrow[\hat{\varphi}]{} & \tilde{A}^A \end{array}$$

is pullback, where $\hat{\varphi}:B\longrightarrow\tilde{A}^A$ is the exponentional transpose of $\varphi:B\times A\rightarrow\tilde{A}$ and $\tilde{\xi}$ is so that the diagrams

$$\begin{array}{ccc} A & \longrightarrow & A \\ \langle f,1_A\rangle \downarrow & & \downarrow \eta_A \\ B\times A & \xrightarrow[\varphi]{} & \tilde{A} \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \longrightarrow & A \\ \eta_X \downarrow & & \downarrow \eta_A \\ \tilde{X} & \xrightarrow[\tilde{\xi}]{} & \tilde{A} \end{array}$$

are pullback respectively, and

(2) for $\xi\xrightarrow{x}\xi'$ in $\boldsymbol{E}/A$, $P_f(x)$; $P_f(\xi)\longrightarrow P_f(\xi')$ so that $(P_f(\xi)\longrightarrow B)=(P_f(\xi')\longrightarrow B).P_f(x)$. Then P_f is a functor [10]. The functor P_f is a right adjoint of f^* for any $f:A\longrightarrow B$ in $\boldsymbol{E}$ [4], [10].

PROPOSITION 7. *Let $t:1\longrightarrow\Omega$ be a subobject classifier in an elememtary topos $\boldsymbol{E}$, then $\boldsymbol{E}$ is embedded in $\boldsymbol{E}/\Omega$.*

Proof. For $\xi:X\longrightarrow 1$ and $\xi':X'\longrightarrow 1$ in $\boldsymbol{E}/1$, if $S_t(\xi)=S_t(\xi')$ then $t\cdot\xi=t\cdot\xi'$. Hence $\xi=\xi'$ since t is a monomorphism. For $x_1, x_2, \in\mathrm{Hom}_{E/1}(\xi,\xi')$ if $S_t(x_1)=S_t(x_2)$, then $x_1=x_2$.

Hence S_t is the embedding. But $\boldsymbol{E}/1$ is equivalent to $\boldsymbol{E}$

COROLLARY 4. *If the functor t^*; $\boldsymbol{E}/\Omega\rightarrow\boldsymbol{E}/1$ for the subobject classifier $t:1\rightarrow\Omega$ in $\boldsymbol{E}$ reflects epimorphism then $\boldsymbol{E}$ is equivalent to $\boldsymbol{E}/\Omega$.*

Proof. For $\zeta:Z\rightarrow\Omega$ and $\zeta':Z'\rightarrow\Omega$, if $t^*(\zeta)=t^*(\zeta')$ pullback diagrams t along ζ and ζ' are equal. Hence $\zeta=\zeta'$. Since t^* has a left adjoint S_t, it is faithful by the hypothesis.

Therefore t^* is embedding and by the proposition 6, $\boldsymbol{E}$ is equivalent to $\boldsymbol{E}/\Omega$. *q.e.d.*

Since for $k:B\to 1$ in $\boldsymbol{E}$, S_k is a left adjoint of $k^*:\boldsymbol{E}/1\to\boldsymbol{E}/B$, the functor $a\cdot S_k:\boldsymbol{E}/B\to\boldsymbol{E}/1\cong\boldsymbol{E}$, preserves epimorphism [3], and reflects an epimorphism since it is faithful.

Hence if $\zeta:Z\to B$ is an epimorphism then ζ in $\boldsymbol{E}/B$ such that

$$\begin{array}{ccc} Z & \xrightarrow{\zeta} & B \\ & \searrow \quad \swarrow 1_B & \\ & B & \end{array}$$

commute, is epimorphism. Therefore we have the following proposition.

PROPOSITION 8. *If $\zeta:Z\to B$ is an epimorphism and the diagram*

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \xi\downarrow & & \downarrow\zeta \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback, then ξ is an epimorphism.

Proof. Since f^* is a left adjoint of P_f, $f^*(\xi)$ is an epimorphism. But $f^*(\zeta)=\xi$. *q.e.d.*

Since the functors $\boldsymbol{E}\underset{(\ )\times A}{\overset{A\times(\ )}{\rightrightarrows}}\boldsymbol{E}$ are left adjoints of the functor $(\ )^A:\boldsymbol{E}\to\boldsymbol{E}$, they reflect an epimorphism. Hence if $P_i:X_i\to Y_i$ are epimorphisms for $i=0$, 1, then $X_0\times X_1\xrightarrow{1_{X_0}\times p_1}X_0\times Y_1$ and $X_0\times Y_1\xrightarrow{p_0\times 1_{Y_1}}Y_0\times Y_1$ are an epimorphism. Therefore $p_0\times p_1:X_0\times X_1\to Y_0\times Y_1$ is an epimorphism.

LEMMA 5. *Any morphism $f:X\to Y$ in $\boldsymbol{E}$ can be factored as an epimorphism followed by a monomorphism. (Such a factorization is unique up to isomorphism.)* [11].

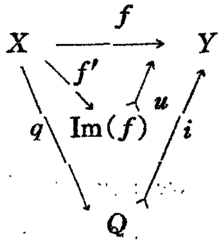
If every morphism $X\xrightarrow{f}Y$ in a category has an image and the morphism $f':X\to\mathrm{Im}(f)$ is always an epimorphism, we say that the category has epimorphic images.

If $f:X\longrightarrow Y$ and $X'\xrightarrow{x}X$ is a monomorphism, we shall denote the image of the composite $X'\xrightarrow{x}X\xrightarrow{f}Y$ by $f(X')$.

PROPOSITION 9. *An elementary topos $\boldsymbol{E}$ has epimorphic images and if $X\xrightarrow{f}Y$, $Y\xrightarrow{g}Z$, and $X'\rightarrowtail^{x} X$ is a monomorphism, then*

$$g\cdot(f(X'))=g\cdot f(X').$$

Proof. By the lemma 5 the morphism f can be factored as $X \xrightarrow{q} Q \xrightarrow{i} Y$ with q an epimorphism and i a monomorphism. By the definition of $\mathrm{Im}(f)$ there is a morphism $h: \mathrm{Im}(f) \to Q$ such that the diagram



is commutative.

But since q is an epimorphism and u is a monomorphism, h is a monomorphism and epimorphism, so that, by the proposition 1 h, is an isomorphism. Since $g \cdot f \cdot x: X' \longrightarrow f(X') \longrightarrow g(f(X')) \rightarrowtail Z$ and $g \cdot f(X') = \mathrm{Im}(g \cdot f \cdot x)$ by the lemma 5 we have $g \cdot f(X') = g \cdot (f(X'))$. *q. e. d.*

Using lemma 5, we shall define a "power-set functor" $Q: \boldsymbol{E} \to \boldsymbol{E}$ as follows; $Q(X) = \Omega^X$ for each $X \in Ob(\boldsymbol{E})$ and we shall take $Q(f): \Omega^X \to \Omega^Y$ for any $f: X \to Y$ in $\boldsymbol{E}$, may be described as the morphism whose exponential adjoint $\overline{Q(f)}: \Omega^X \times Y \to \Omega$ is the classifying map of the image $(1_{\Omega^X} \times f)(\epsilon_X) \rightarrowtail \Omega^X \times Y$, that is, the image of composite $\epsilon_X \longrightarrow \Omega^X \times X \xrightarrow[1_{\Omega^X} \times f]{} \Omega^X \times Y$, where for each pair $(f: X \to Y, g: Y \to Z)$ of morphism in $\boldsymbol{E}$ the classifying map $\Omega^X \times Z \to \Omega$ of the image $1_{\Omega^X} \times g(1_{\Omega^X} \times f(\epsilon_X)) \longrightarrow \Omega^X \times Z$ is

$$\Omega^X \times Z \xrightarrow[Q(f) \times 1_Z]{} \Omega^Y \times Z \xrightarrow[Q(g) \times 1_Z]{} \Omega^z \times Z \xrightarrow[ev_X]{} \Omega.$$

Then by Proposition 9, $Q(g \cdot f) = Q(g) \cdot Q(f)$ and $Q(1_X) = ev_X$ so that, $Q(1_X) = 1_{Q(X)}$.

Therefore, Q is a covariant power-set functor of $\boldsymbol{E}$.

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