A HoTT Quantum Equational Theory

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MURI Project Review
University of Maryland

March 8, 2019

With Steve Zdancewic at the University of Pennsylvania.

Quantum data, classical control

...via embedded languages

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- Quipper [Green et al., 2013]
  - Embedded in Haskell, a functional lazy language.
  - Uses Haskell types, functions, data structures, type classes, template haskell... to construct quantum circuits.
  - Access to Haskell REPL and debugging tools.

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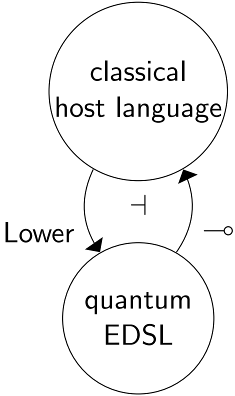
- $\mathcal{Q}$WIRE [Paykin et al., 2017, Rand et al., 2017]
  - A formal theory of embedded quantum circuits.
  - Implemented as an embedded language in Coq, a theorem prover with dependent types.
  - Uses Coq theorem proving capabilities to prove correctness of quantum circuits.

Quantum/non-quantum calculus



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- Linear types, pairs ($\otimes$), etc

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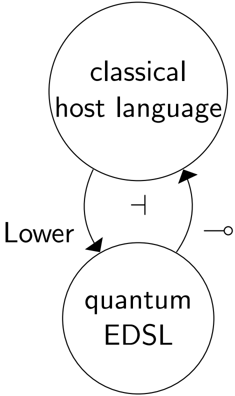

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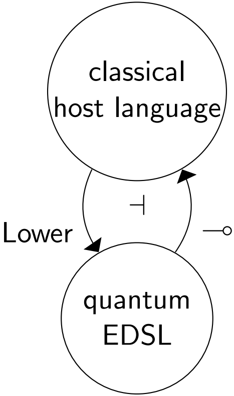

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$$\frac{a : \alpha}{\text{put } a : \text{QExp} \ \cdot \ (\text{Lower } \alpha)}$$

$$\frac{e : \text{QExp } \Delta \ (\text{Lower } \alpha) \qquad f : \alpha \to \text{QExp } \Delta' \ \tau}{e >!\ f : \text{QExp } (\Delta, \Delta') \ \tau}$$

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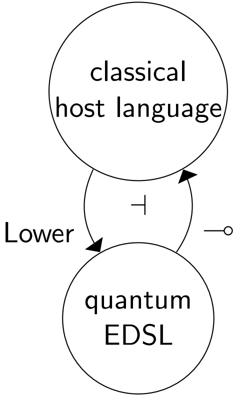

- Derived quantum operations:

$$\begin{aligned} \mathsf{Qubit} &= \mathsf{Lower}(\mathsf{Bool}) \\ |b\rangle &= \mathsf{put}\ b \\ \mathsf{let}\ b := \mathsf{meas}\ e\ \mathsf{in}\ e' &= e >!\ \lambda b.e' \end{aligned}$$

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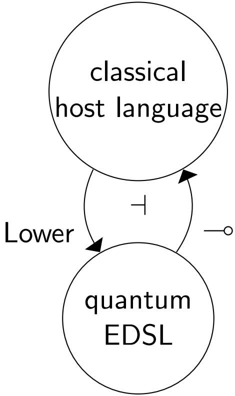

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- Unitaries (not derived):

$$\frac{U : \mathsf{UMatrix}(\sigma, \tau) \qquad e : \mathsf{QExp}\ \Delta\ \sigma}{U \# e : \mathsf{QExp}\ \Delta\ \tau}$$

Reasoning about quantum data

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- Denotational semantics
  - Spaces are exponential in size of program
- Program logics
  - Best suited to imperative quantum languages
- Equational theory
  - Syntactic rules that characterize when programs are equivalent.
  - May or may not be directed; difficult to normalize.
  - Validated with respect to denotational semantics.

Goal

Equational theory for *embedded* quantum circuit language.

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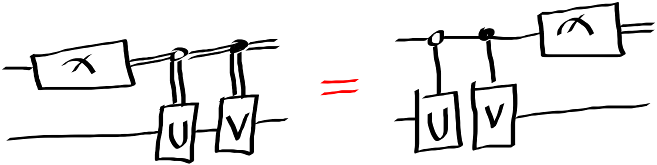
- Interaction between quantum data and host language control
- NOT equational theory for classes of unitaries

Prior work – Staton [2015]

- Equational theory for algebra with unitaries and classical control.



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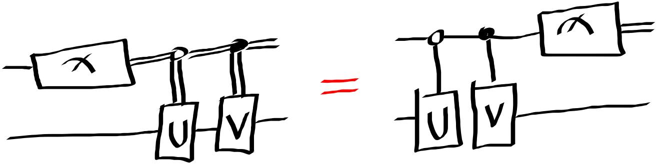

- Complete with respect to C^*-algebras.

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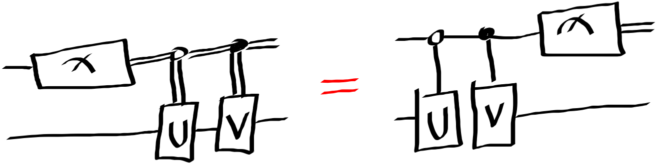
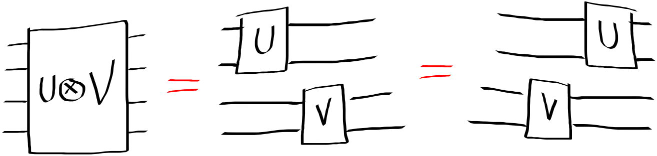


- Complete with respect to C^*-algebras.
- Procedural axioms based on diagrams
  - symmetric monoidal structure



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Equational theory for *embedded* quantum circuit language.

- Specialized to an embedded programming language
  - not algebra or diagrams (e.g. ZX calculus [Backens, 2015])
- Fewer "procedural" axioms, focus on interesting axioms.
- Completeness of axioms by comparing with Staton's theory.

Homotopy type theory (HoTT): a type theory of equality

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- Path induction:

$$\frac{H : \forall(a, b : A).\ a = b \to \mathrm{Type} \qquad \forall(a : A).\ H(1_a)}{\texttt{path\_ind}_H : \forall(a, b : A).\ \forall(p : a = b).\ H(p)}$$

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- Equivalence class of an element $a : A$ with respect to a relation R: $[a]_R = [b]_R$ if $(a, b) \in R$.

Higher Inductive Type (HIT)

Definition

The quotient of a type A by a relation $R : A \to A \to \text{Prop}$ is a type A/R with data constructor:

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Note

If $p, q : R(a, b)$ and $p \neq q$, then $[p] \neq [q]$.

So what?

- HITs use paths to represent *equivalence relations* or *groupoids*.

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- Path induction still holds of HITs:
  - Prove theorems about groupoids by showing property holds of $1_a : a = a$.
- Unitary transformations form a groupoid.

Idea: Represent Unitaries as paths

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- Quantum types: QType = FinType/UMatrix.
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- Unitaries are paths:

$$\frac{U : \text{UMatrix}(\alpha, \beta)}{[U] : [\alpha] = [\beta]}$$

  - $[H] : \mathit{Qubit} = \mathit{Qubit}$

HoTT QNQ calculus

$$\sigma \in \mathsf{QType} = \mathsf{FinType}/\mathsf{UMatrix}$$

$$\mathsf{Lower}\ \alpha \equiv [\alpha]_{\mathsf{UMatrix}}$$

$$\begin{aligned} e &:= x \mid \mathsf{let}\ x := e\ \mathsf{in}\ e' \\ &\mid (e_1, e_2) \mid \mathsf{let}\ (x_1, x_2) := e\ \mathsf{in}\ e' \\ &\mid \mathsf{put}\ a \mid e >!\ f \mid \cdots \end{aligned}$$

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- Derive $|b\rangle$ and meas e using Lower
- Derive unitaries...

Unitaries in HoTT QNQ

Theorem

Let U be a unitary transformation $U : \sigma = \tau$.
(σ, τ : QType $\equiv$ FinType/UMatrix)

If $\Delta \vdash e : \sigma$ then there exists another expression $\Delta \vdash U \# e : \tau$.
(apply the unitary U to e)

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$$1_\sigma \# e \equiv e$$

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Note

$[H] \# e \neq e$ *because* $[H] \neq 1_{Qubit}$

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Theorem

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$$V \# (U \# e) = (V \circ U) \# e.$$

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Proof.

By path induction on V. If $V \equiv 1_\tau$ then

$$LHS = 1_\tau \# (U \# e) = U \# e$$
$$RHS = (1_t \circ U) \# e = U \# e$$

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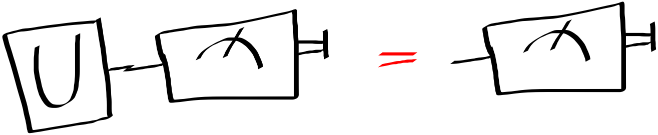

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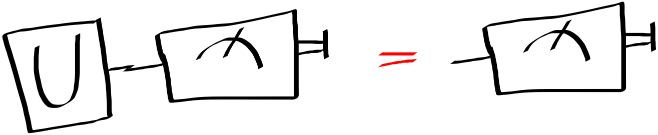

Theorem
$X \# |0\rangle = |1\rangle \qquad meas(X \# e) = \neg meas(e)$

...but not everything

Theorem

$SWAP \# (e_1, e_2) = (e_2, e_1)$

Proof.

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Similar results for behavior of other “structural” unitaries:

$$\begin{aligned} \text{ASSOC} &: \sigma_1 \otimes (\sigma_2 \otimes \sigma_3) = (\sigma_1 \otimes \sigma_2) \otimes \sigma_3 \\ \text{LUNIT} &: () \otimes \sigma = \sigma \\ &\vdots \end{aligned}$$

Partial initialization axiom

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The *partial initialization* a state $X \otimes Y$ is a pair of expressions.

$$\begin{aligned} \mathsf{init}_X\ e &\equiv e \\ \mathsf{init}_{\mathsf{Qubit}}\ (b : \mathsf{Bool}) &\equiv |b\rangle \\ \mathsf{init}_{\sigma\otimes\tau}\ (a, b) &\equiv (\mathsf{init}_\sigma\ a, \mathsf{init}_\tau\ b) \end{aligned}$$

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Axiom

Let f be a structural equivalence. Then

$$\widehat{f} \mathbin{\#} \mathit{init}(b) \approx \mathit{init}(f(b))$$

Partial measurement axiom

Partial measurement or *partial observation*:

$$
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\mathsf{match}_X\ e\ \mathsf{with}\ f &= \mathsf{let}\ x := e\ \mathsf{in}\ f\ x \\
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Axiom

Let f be a structural equivalence. Then:

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- Two axioms:
  - structural unitaries + initialization
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Results

- Two axioms:
  - structural unitaries + initialization
  - structural unitaries + measurement
- Quantum programming language embedded in HoTT
  - (Finite) classical data, tuples, and sums
- Complete with respect to Staton's equational theory
- Sound with respect to density matrices

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- Pros: theorems for free with path induction
- Cons:
  - theorems not actually free
  - no normalization
  - steep learning curve

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Questions?

Supported by FA9550-16-1-0082
Semantics and Structures for Higher-level Quantum Programming Languages

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