Adelic approach to the zeta function of arithmetic schemes in dimension two

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Abstract. A new theory to understand fundamental properties of the zeta functions of arithmetic schemes is proposed and developed in the case of a regular model of an elliptic curve over a global field. This is a two-dimensional commutative K_2 and $K_1 \times K_1$ extension of the classical adelic analysis of Iwasawa and Tate. Using structures from the explicit two-dimensional class field theory and working with a new $\mathbb{R}((X))$-valued translation invariant measure, integration theory and harmonic analysis on various complete objects associated to arithmetic surfaces we define and study zeta integrals which are closely related to the zeta function of the regular model. The two-dimensional adelic analysis and geometry reduces the study of poles of the zeta function to the study of poles of a boundary term which is an integral of a certain arithmetic function over the boundary of an adelic space. The structure of the boundary and function determines the analytic properties of the boundary term and location of the poles of the zeta function, which results in applications of the theory to several key directions of arithmetic of elliptic curves over global fields.

0. Introduction

0.1. To study properties of the Riemann zeta function

$$\zeta_{\mathbb{Z}}(s) = \sum_{n\geqslant 1} \frac{1}{n^s}.$$

one can work with the completed zeta function

$$\widehat{\zeta}_{\mathbb{Z}}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta_{\mathbb{Z}}(s)$$

which has an integral representation

$$\int_0^\infty (\theta(x^2)-1)x^s\frac{dx}{x}, \qquad \theta(x) = \sum_{n\in\mathbb{Z}} \exp(-\pi n^2 x).$$

The integral can be rewritten as

$$\int_1^\infty \big(\theta(x^2)-1\big)\, x^s\frac{dx}{x} + \int_1^\infty \big(\theta(x^2)-1\big)\, x^{1-s}\frac{dx}{x} + \omega(s)$$

where

$$\omega(s) = \int_0^1 \big((\theta(x^2)-1)x - (\theta(x^{-2})-1)\big)x^{s-1}\frac{dx}{x}.$$

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an updated version which takes into account some simplifications from and adheres to the notation of [F5], for 40 open research problems see [F9]

The first two integrals are absolutely convergent and their sum is an entire function on the complex plane symmetric with respect to $s \to 1 - s$.

The Gauss–Cauchy–Poisson summation formula, i.e the functional equation for the theta function $\theta(x^2)x = \theta(x^{-2})$ implies

$$\omega(s) = \int_0^1 (1-x)x^{s-1}\frac{dx}{x} = -\left(\frac{1}{s} + \frac{1}{1-s}\right)$$

is a rational function symmetric with respect to $s \to 1 - s$, hence the functional equation and meromorphic continuation of the completed Riemann zeta function.

This method, the only one among seven methods of the proof of the functional equation of $\zeta_{\mathbb{Z}}(s)$ listed in Ch. II of [Ti], can be rewritten in the adelic language, and so can be applied to the study of zeta function of an arbitrary global field. The completed zeta function can be viewed as the adelic zeta integral

$$\widehat{\zeta}_{\mathbb{Z}}(s) = \int_{\mathbb{A}_{\mathbb{Q}}^{\times}} f(x)|x|^s d\mu_{\mathbb{A}_{\mathbb{Q}}^{\times}}(x)$$

with respect to an appropriately normalised Haar measure on the group of ideles $\mathbb{A}_{\mathbb{Q}}^{\times}$, where $f(x)$ is the tensor product of the characteristic functions of the integer p-adic numbers and of $\exp(-\pi x^2)$ at the archimedean prime. The functional equation of the theta function (and of the zeta integral) corresponds to the analytic duality furnished by Fourier transform on the adelic spaces and its subspaces.

0.2. The study of adelic integrals associated to automorphic representations of algebraic groups over adeles is an important part of activity in the Langlands programme. In the commutative case the classical adelic method by Tate and Iwasawa [T1], [I2], [W1,W2] gives an easy proof of meromorphic continuation and functional equation of a twisted by character zeta function of a global field. Even though the method uses objects originating from the one-dimensional class field theory, it does not use the class field theory. For a quasi-character χ of the class group of ideles and a function f in an appropriate space one derives

$$\zeta(f,\chi) = \xi(f,\chi) + \xi(\widehat{f},\widehat{\chi}) + \omega(f,\chi), \qquad \Re(s(\chi)) > 1$$

where the first two terms are absolutely convergent integrals on the plane. The integral $\zeta(f,\chi)$ can be written as a triple integral corresponding to the flitration of $\mathbb{A}^{\times}$: ideles of norm one and global elements; correspondingly it involves an integral over the multiplicative group of the global field. Similarly to 0.1 using harmonic analysis on the adelic space one obtains that the corresponding internal integral for $\omega(f,\chi)$ is an integral over 0 of an appropriate function. One can view 0 as the boundary of the multiplicative group of the global field with respect to the weakest topology on adeles in which every character is still continuous. The boundary term ω is a simple rational function either of s or of q^{-s}. The meromorphic continuation, location of poles of the zeta integral and their residues, and the functional equation follow from easy to establish analytic properties of the boundary term. Thus, to study properties of the zeta functions, one can use zeta integrals on appropriate adelic spaces and then using appropriate duality property reduce the properties to adelic geometry and analysis.

In the general case of algebraic groups the analogue of the boundary term is an integral over the weak boundary of an algebraic group over a global field, it has finitely many poles in s or in q^{-s}. In particular, a cuspidal function has the property that the corresponding boundary term vanishes. Similarly to the commutative case one gets the functional equation and analytic properties of the zeta integral [GJ], [So], [D].

0.3. Instead of going from the one-dimensional commutative theory, GL_1 or K_1, to the one-dimensional noncommutative theory, i.e. algebraic groups over one-dimensional global fields, as in the Langlands direction, this work develops a higher dimensional commutative theory using higher algebraic K-groups. The latter play the central role in higher class field theory (those of its versions which include a compatibility between the global and local theories, which in turn correspond to the Euler factorization of the zeta function and zeta integral). Whereas the L-factors of the zeta functions of arithmetic schemes are normally treated as 1-dimensional noncommutative objects, the zeta functions can be treated as commutative n-dimensional objects where n is the Kronecker dimension of the function field of the scheme.

A new translation invariant measure and integration on higher-dimensional local and adelic objects associated to arithmetic schemes are employed to define and study properties of higher dimensional zeta integrals which translate fundamental properties of the zeta function into geometric and analytic properties of adelic subquotients. The zeta integral is an object which takes into account both additive analytic structures and class field theoretical adelic structures, and both analytic and geometric adelic structures and an interplay between their multiplicative groups.

In dimension higher than one, one can talk about two different worlds, one of L-functions and one of the zeta functions of arithmetic schemes. The latter factorize into products and quotients of their L-factors, up to an auxiliary easier factor. The L-functions are traditionally studied using concepts of the Langlands correspondence, and hence noncommutative representation theoretical methods. Langlands wrote that "progress in functoriality has been largely analytic, exploiting the more abstract consequences of abelian class-field theory but developing very few arithmetic arguments" [La]. More recently, the fundamental lemma was proved using geometric representation theoretical arguments and algebraic geometry, but the first nontrivial case of the Langlands correspondence for $GL(2)$ over rationals is still not completed. The study of automorphic properties is related but separated in terms of its methods from the study of special values, and little is known about the generalized Riemann hypothesis, especially in relation to the special values. Successful partial methods of the study of the arithmetic automorphic and special values of L-functions are typically indirect, local and/or noncommutative and work over small number fields (such as totally real) only. In contrast, all the main conjectures on automorphic properties, location of zeros and special values are stated over an arbitrary number field.

It is the main new concept of the current work and related works [F3,F4,F5], [FRS] to study the zeta functions of arithmetic schemes via lifting them to appropriate (commutative) zeta integrals on higher adelic spaces and reducing then the fundamental properties of the zeta functions to those of certain boundary terms which are integrals over subquotients of adelic spaces. The adelic objects of analytic and geometric type come naturally from explicit higher class field theory. An interplay between the multiplicative groups of the adelic objects via a map related to the symbol map in algebraic K-theory allows to study analytic properties of the zeta functions in terms of geometric adelic spaces. One of the key advantages of adelic methods is that they work over an arbitrary global field. Another key advantage is that all the three aspects, functional equation and meromorphic continuation, generalized Riemann hypothesis and the special values are closely intertwined in the zeta integral theory.

This text presents the main features of the two-dimensional commutative theory of zeta integrals. For a very short review see [F7]. The text consists of ten sections based on the theory of [F3–F5] and related papers.

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0.4. The central case discussed in this paper is the case of a regular model of elliptic curve over a global field. From the point of view of underlying Galois structures, this work is based on the maximal abelian extension of a two-dimensional global field, the field of rational function on the model, whereas the prevailing methods to deal with arithmetic of elliptic curves have been to use the (noncommutative in general) extension of a global field generated by points of finite order of elliptic curve.

Let E be an elliptic curve over a global field k and let $\mathcal{E}$ be a regular model of it. Let $\zeta(s)$ be the square of the Hasse–Weil zeta function $\zeta_E(s)$ or of the Hasse zeta function $\zeta_{\mathcal{E}}(s)$, each of them multiplied by c^{1-s} where c is the conductor associated to ζ_E or $\zeta_{\mathcal{E}}$. So $\zeta(s)$ is a generalized Dirichlet series

$$\zeta(s) = \sum_{m \in c\,\mathbb{N}} \frac{c_m}{m^s}.$$

A simple way to describe a two-dimensional analogue of the procedure applied in 0.1 to the Riemann zeta function is the following. Let, for simplicity, k = $\mathbb{Q}$. According to the theory of this work, up to an exponential factor and the product of finitely many zeta functions of affine lines over finite fields corresponding to the structure of bad reduction fibres of $\mathcal{E}$ the zeta integral can have the form of

$$\widehat{\zeta}(s) = \widehat{\zeta}_{\mathbb{Z}}(s/2)^2 \zeta(s)$$

(more generally, the first factor can be the product of squares of several completed zeta functions of global fields at $s/2$). It has an integral representation

$$\int_0^\infty u(x)\, x^s \frac{dx}{x}, \qquad u(x) = 4 \sum_{j,l \in \mathbb{N}, m \in c\,\mathbb{N}} c_m K_0(2\pi j l m^2 x^2)$$

which involves the K_0-Bessel function. The integral can be written as the sum

$$\int_1^\infty u(x)\, x^{s-2} \frac{dx}{x} + \int_1^\infty u(x)\, x^{-s} \frac{dx}{x} + \omega(s)$$

where

$$\omega(s) = \int_0^1 h(x)\, x^{s-2} \frac{dx}{x}, \quad h(x) = x^2 u(x) - u(x^{-1}).$$

The first two integrals are absolutely uniformly convergent and are entire functions on the complex plane. See [FRS] for a very simple presentation of related analytic features.

Using adelic duality on arithmetic surfaces and a two-dimensional Fourier transform one studies the boundary term $\omega(s)$. In particular, a two-dimensional theta formula plays a fundamental role, similar to the one-dimensional theta formula role in the calculation for the Riemann zeta function, see Theorem in section 6.

The two-dimensional method includes applications in several directions. Four of them are listed in this paragraph, for more see the file referred to on the first page.

- functional equation and meromorphic continuation of the zeta functions and mean-periodicity (in appropriate functional spaces). A conjectural mean-periodicity implies the meromorphic continuation of the zeta function, without any need to establish automorphic properties of its factors. Moreover, the zeta functions of arithmetic schemes which have the functional equation and meromorphic continuation correspond to mean-periodic functions.
- location of poles of the zeta function and the generalized Riemann hypothesis for elliptic curves over global fields. Analytically, the poles of the zeta integral in dimension two, given its integral representation, are easier to understand than the zeros of the zeta function in dimension one. An expected permanence of the sign of the fourth log derivative of the boundary function is essentially

the reason for the GRH to be true provided one knows the only real pole inside the critical strip is at the central point.

- special value at the central point, understanding of the Birch and Swinnerton-Dyer conjecture using the boundary integral which encodes and relates the arithmetic-geometric and analytic data.
- a two-dimensional version of the double quotient of algebraic group adeles, a theory of automorphic functions in dimension two and its applications in various directions.

See the introduction of [F5] for more introductory information.

0.5. Now we go briefly through the content of the sections.

Section 1 introduces main features of higher dimensional local fields and then sketches the theory of the translation invariant $\mathbb{R}((X))$-valued measure on them. The measure takes into account the arithmetic structure of two-dimensional local field. In this text we discuss some of the main features of the theory, for more details and different approaches see [F3], [F4], [Mo1].

Section 2 introduces basic adelic spaces in dimension two and their properties. For an arithmetic scheme S corresponding to a model of a smooth projective curve over a global field k we introduce an adelic space $\mathbb{A}$, this space in dimension two satisfies two adelic conditions, one of which is taken with respect to the rank two integral structure. Using the two-dimensional local theory of section 1 and in parallel to the one-dimensional theory of Tate, in section 3 we define an $\mathbb{R}((X))$-valued measure, $\mathbb{C}((X))$-valued integration and transform of certain functions on the adelic spaces. Those functions are essentially tensor products over a set of irreducible curves on the arithmetic surface of pullbacks with respect to residue maps of functions on residue adelic objects. This adelic space is good for analytic theory, and its multiplicative version is related to 0-cycles on the surface and its zeta function.

There is another adelic object which is more of geometrical nature and whose multiplicative version is more related to 1-cycles on the surface, see 2.2. In two-dimensional adelic analysis and geometry "relations between the analytic theory and the arithmetical" one are more straightforward in comparison to what is mentioned at the end of 0.2. The multiplicative groups of the two adelic objects are blended via adelic K_2, which plays the role of the ideles in the two-dimensional commutative class field theory, see commutative diagrammes of 4.3 and 6.1.

An explicit two-dimensional class field theory is sketched in section 4, its initial knowledge is useful but not necessary for understanding the theory of this paper. Abelian extensions of S are described by open subgroups of finite index in a certain K_2^t-delic group J_S/P_S associated to S, and for the purposes of the unramified theory one can work with even a simpler quotient object J/P.

The main object of the study, a two-dimensional zeta integral

$$\zeta(f,\chi) = \int_{\mathfrak{T}} f\chi_{\mathfrak{t}}\, d\mu$$

for a function f in a two-dimensional extension of the space of Bruhat–Schwartz functions, and a quasi-character χ of J_S/P_S is introduced in section 5. We discuss its properties in the central case of arithmetic scheme $\mathcal{E}$ corresponding to a regular model of an elliptic curve over a global field.

The first calculation of $\zeta(f, |\ |_2^s)$ in the case of arithmetic scheme $\mathcal{E}$ compares it with the zeta function $\zeta_{\mathcal{E}}(s)$ and proves its existence on the half plane $\Re(s) > 2$. For curves of higher genus g one should do a renormalization of the zeta integral using the $(g-1)$-st power of the zeta function of the projective space over the base.

The second calculation of the zeta integral for a centrally normalized function f and a two-dimensional theta formula are described in section 6. There we show that $\zeta(f, |\ |_2^s)$ is the sum of three terms

$$\xi(|\ |_2^s) + \xi(|\ |_2^{2-s}) + \omega(|\ |_2^s)$$

on the half-plane $\Re(s) > 2$ with an entire function $\xi(|\ |_2^s)$. Thus, similar to the theory of Tate and Iwasawa, the study of the functional equation and meromorphic continuation of the zeta integral for the unramified character in dimension two and of its poles are reduced to the study of the corresponding properties of the boundary term $\omega(|\ |_2^s)$. The boundary term up to a non-zero constant factor equals the integral

$$\int_{N^-} h(n)\, n^{s-2}\, d\mu_{N^-}(n)$$

where $h(n)$ is a real valued function which comes from adelic integration on the scheme. In characteristic zero the boundary term is the (one-sided) Laplace transform of $h(e^{-t})e^{2t}$. Using a two-dimensional theta formula we get a representation of the boundary term which involves integration over the (weak) boundary ∂T_0 of a certain adelic space T_0. This adelic space is of local-global nature, which should be studied in relation to the adelic object $\mathfrak{T}$ and to a discrete object $K^\times$ of invertible rational functions of $\mathcal{E}$.

0.6. Sections 7–9 present results, methods, ideas and hypotheses in three central directions of the further development of applications of the theory: meromorphic continuation and functional equation, location of poles, and the behaviour at $s = 1$. In the first and second direction we state concrete hypothesis in sections 7–8 which together with the theory of this work are aimed to imply meromorphic continuation and functional equation and Riemann Hypothesis for the poles of the zeta integral. In the third direction of the rank part of the conjecture of Birch and Swinnerton–Dyer in section 9 we propose a concrete new method to deduce it using the theory of this work.

In section 7 we easily get the first functional equation for the function h: $h(n^{-1})\, n = -h(n)\, n^{-1}$ is straightforward. The issue is what are additional properties of h which would imply the meromorphic continuation and functional equation of ω. Note that the zeta integral corresponds to the square of the zeta function, which itself mixes the automorphic structures of its factors and hence h cannot be expected to possess any clean automorphic properties. As a replacement of the second functional equation for classical modular forms, a new *hypothesis on mean-periodicity in an appropriate functional space of a function related to* h is proposed in section 7. The general theory shows that the mean-periodicity of H implies meromorphic continuation and functional equation of the zeta integral, the square of the zeta function of $\mathcal{E}$, of the zeta function of E and of the L-function of E. In positive characteristic the function H is indeed mean-periodic in the space of functions on integers. In characteristic zero a recent work of M. Suzuki, G. Ricotta and the author [FRS] shows that if the zeta-function of E has meromorphic continuation of expected shape and satisfies the functional equation then the corresponding function H is indeed mean-periodic in several functional spaces, which include the space of infinitely differentiable functions on $\mathbb{R}$ of exponential growth. The hypothesis can be viewed as a weak version of the Taniyama conjecture since the conjectured mean-periodicity of a certain function H, which in characteristic zero equals $h(e^{-t})$, is weaker than the automorphic property of the L-function of elliptic curve. In particular, one can say that this new hypothesis addresses one of the issues raised by Langlands about getting meromorphic continuation of L_E without proving their full automorphic properties.

The function H and its first two derivatives are monotone functions near infinity, and it is natural to study the monotone behaviour of its third derivative. In section 8 we show that the *permanence of the sign of the fourth derivative of H near infinity* and the real part of the Riemann hypothesis for the zeta integral (i.e. the only real pole inside the critical strip is at $s = 1$, this condition is easy to check computationally for any given $\mathcal{E}$) imply the Riemann hypothesis for the zeta integral. It is expected that the difficulty of proving the permanence of the sign of the fourth derivative of H near infinity is essentially smaller than the difficulties associated with the work on the classical Riemann hypothesis. One of the reasons is that in dimension two the weak boundary ∂T_0 of T_0 is very large unlike the dimension one case, which results in smoothening of the behaviour of related functions, like H and its derivatives. For analytic study of several aspects of the positivity of the fourth derivative of H near infinity see the recent paper [Su1]. In particular, assuming that the L-function of E has holomorphic continuation and functional equation and satisfies the Riemann hypothesis and assuming that the multiplicity of the real zero on the critical line is strictly greater than the multiplicity of nonreal zeros, M. Suzuki has proved the positivity of the fourth derivative of H near infinity, modulo a technical condition.

The analytic behaviour of the zeta integral at $s = 1$ is completely described by the behaviour at $s = 1$ of the boundary term, which itself involves an integral over the weak boundary of the space T_0. The boundary decomposes into the product of a finite number (related to the rank of E) of spaces associated to curves and some other simple spaces. The boundary term at $s = 1$ serves as a bridge between the analytic rank and arithmetic rank of $\mathcal{E}$. Thus we get a new, adelic, complex valued approach to study the conjectured equality of the analytic and arithmetic ranks of E. This method is sketched in section 9.

It is a remarkable phenomenon of the two-dimensional theory that the study of three aspects: meromorphic continuation and functional equation, location of poles and local behaviour at $s = 1$ becomes much more related to each other in the two-dimensional theory than in the traditional approaches, where they are quite separated.

On the basis of the two-dimensional adelic analysis and geometry in section 10 we suggest new concrete objects functions on which should be related to the space of automorphic $\mathbb{G}_m \times \mathbb{G}_m$- and $G \times G$-functions on the surface $\mathcal{E}$. These objects mix in a nontrivial way two integral structures on the arithmetic surface $\mathcal{E}$: one structure which corresponds to divisors and is more of geometric nature, and another structure which corresponds to 0-cycles on the surface and which is of more arithmetic nature.

1. Local theory

1.1. Two dimensional local fields. A local field (archimedean, $\mathbb{R}$, $\mathbb{C}$) or nonarchimedean, with finite residue field, can be geometrically viewed as associated to a closed point x (including in one or another way archimedean valuations) of a one-dimensional arithmetic scheme B, i.e. of the spectrum of the ring of integers of algebraic number field or a smooth projective curve over a finite field.

Let S be an integral normal scheme of dimension two. We assume that we are in a relative situation: there is a proper flat morphism $S \longrightarrow B$ with fibre dimension one, where B is a one-dimensional arithmetic scheme as above, and the generic fibre of S is a nonsingular projective geometrically irreducible curve over the field of functions k *on B.* Unless stated otherwise, we assume in addition that S is a regular scheme.

Denote by K the field of rational functions of the arithmetic scheme S. We will use the word "fibre" for closed fibres, and we will denote the fibre of S over $b \in B_0$ by S_b. Unless stated otherwise, *we will assume that the components of every fibre intersect transversally* if necessary performing blowing ups.

A *two-dimensional local field* is associated to a point x on an irreducible curve y on S. For example, consider the completion $\mathcal{O}_x$ of the local ring of S at x and localize and complete this ring with respect to the prime ideal corresponding to a local branch of y at x, the field of fractions of the latter ring is a two-dimensional local field. Alternatively, consider the completion $\mathcal{O}_y$ of the local ring of S at a curve y, its residue field is $k(y)$; given its completion associated to a point on the curve one can form a two-dimensional completed version of $\mathcal{O}_y$ whose fraction field will be a two-dimensional local field, isomorphic to the previous one if x is a nonsingular point of y.

There are four types of two-dimensional local fields:

(1) power series field in two variables over a finite field $\mathbb{F}_q((t_1))((t_2))$, these fields are associated to points on vertical and horizontal curves on S when K is of positive characteristic;

(2) power series field in one variable $E((t))$ over a local nonarchimedean number field E, these fields are associated to points on horizontal curves on S when K is of characteristic zero;

(3) power series field in one variable $E((t))$ over a local archimedean field E, these fields are associated to "archimedean points" on horizontal curves on S and to points on "vertical curves" on S over archimedean primes of B when K is of characteristic zero;

(4) mixed characteristic fields – finite extensions of the field $\mathbb{Q}_p\{\{t\}\}$ which is the fraction field of the completion of the local ring $\mathbb{Z}_p[[t]]$ with respect to its prime ideal generated by p, these fields are associated to points on vertical curves when K is of characteristic zero.

For a detailed presentation of main properties of two-dimensional fields see [IHLF] and references therein.

Power series fields over one-dimensional local fields can be viewed as arithmetic loop fields. Recall that the space of complex valued continuous functions on the unit circle contains the space of complex functions continuous on the unit circle which have meromorphic continuation to the unit ball and holomorphic outside its centre, the latter space is a subspace (via Taylor series at the origin) of a formal loop space $\mathbb{C}((t))$.

Let F be a two-dimensional local field. Denote by $\mathcal{O} = \mathcal{O}_F$ the ring of integers with respect to the discrete valuation of rank one of F. Call fields of type (1), (2), (4) nonarchimedean two-dimensional local fields. For a nonarchimedean two-dimensional local field denote by $O = O_F$ the ring of integers with respect to any discrete valuation of rank two of F. For example, for fields of the second type $\mathcal{O} = E[[t_2]]$ and $O = O_E + t_2 E[[t_2]]$, where O_E is the ring of integers of E. Even though a surjective discrete valuation v from $F^\times$ to the lexicographically ordered group $\mathbb{Z} \oplus \mathbb{Z}$ (with $(1,0) > (0,1)$) is not unique, the ring of integers O_v does not depend on the choice of v. A choice of a surjective discrete valuation v corresponds to a choice of two local parameters t_2, t_1: $v(t_2) = (1,0)$, $v(t_1) = (0,1)$. For example, if $F = \mathbb{Q}_p\{\{t\}\}$ then one can take $t_2 = p$, $t_1 = t$.

For a nonarchimedean two-dimensional local field we have the following 2d picture of O-modules inside F:

$$\begin{array}{llllll}
\dots & & & \dots & & \dots \\
\cup_j\, t_2 t_1^j O = t_2\mathcal{O} & \dots\supset & t_2 t_1^{-1} O \;\supset & t_2 O \;\supset & t_2 t_1 O \;\supset\dots & t_2^2\mathcal{O} = \cap_j t_2 t_1^j O \\
\cup_j\, t_1^j O = \mathcal{O} & \dots\supset & t_1^{-1} O \;\supset & O \;\supset & t_1 O \;\supset\dots & t_2\mathcal{O} = \cap_j t_1^j O \\
\cup_j\, t_2^{-1} t_1^j O = t_2^{-1}\mathcal{O} & \dots\supset & t_2^{-1} t_1^{-1} O \supset & t_2^{-1} O \supset & t_2^{-1} t_1 O \supset\dots & \mathcal{O} = \cap_j t_2^{-1} t_1^j O \\
\dots & & & \dots & & \dots
\end{array}$$

Denote by E the residue field of $\mathcal{O}$ and by $p\colon \mathcal{O} \to E$ the residue map. So, for nonarchimedean fields, O is the preimage of O_E with respect to p. Denote by $\mathbb{F}_q$ the residue field of O, which is the same as the residue field of O_E, its cardinality is q.

Every two-dimensional local field F can be endowed with a two-dimensional translation invariant topology which takes into account the topology of the one-dimensional local residue field E. For example in the equal characteristic case take a sequence of open subgroups U_i in the residue field E and declare all subgroups of type $F \cap \big(\sum_{i \leqslant i_0} U_i t_2^i + t_2^{i_0+1} E[[t_2]]\big)$ open in F. Every element of F can be written as a convergent with respect to this topology series $\sum \theta_{i,j} t_1^j t_2^i$ where $\theta_{i,j}$ are from a set of representatives of the finite residue field. Normally we take multiplicative representatives. The set of coefficients $\theta_{i,j}$ satisfies a certain condition of being zero outside an admissible set of indices i, j.

So we can view F not just as an infinite dimensional topological space (in the equal characteristic case) but as a more refined topological structure which in particular incorporates certain arithmetic information in it. Define a topology on the multiplicative group $F^\times$ as induced from the topology of F by $F^\times \to (F, F)$, $\alpha \mapsto (\alpha, \alpha^{-1})$. In dimension two the group $F^\times$ is a topological group with respect to this topology. In the nonarchimedean case $F^\times$ is the product of discrete cyclic groups generated by t_1 and t_2 and the group of units $O^\times$ with the induced topology from F.

Let ψ be a nontrivial continuous complex character of F. We can choose it such that it has conductor O, i.e. O is the largest O-submodule of F on which this character is trivial. One can show that every continuous character of F can be uniquely written as a multiplicative shift of ψ, i.e. as $\alpha \to \psi(\beta\alpha)$ for some $\beta \in F$. In this sense the field F is self dual.

1.2. Measure and integration on two-dimensional local fields. Suppose we had a translation invariant real valued measure μ on F in which principal O-modules are measurable sets. If, say, $\mu(O) = 1$ then since the index of $t_1 O$ in O is q, $\mu(t_1^m O) = q^{-m}$ tends to zero when m tends to infinity. Then, due to the monotone property of the measure we would have $\mu(t_2^i t_1^j O) = 0$ for all $i > 0$, which is not good.

A way out is to work with $\mathbb{R}((X))$-valued translation invariant measures on two-dimensional local fields, viewing $\mathbb{R}((X))$ endowed with a two-dimensional topology. Define a function μ on the ring $\mathcal{A}$ of sets generated by closed balls $a + t_2^i t_1^j O$ with respect to the rank two integral structure

$$\mu(a + t_2^i t_1^j O) := q^{-j} X^i.$$

Then the function μ is well defined, translation invariant and finitely additive. Moreover, it is countably additive in the following refined sense. Call a series $\sum \alpha_n$, $\alpha_n = \sum a_{i,n} X^i \in \mathbb{C}((X))$, absolutely convergent in the two-dimensional local field $\mathbb{C}((X))$ if there is i_0 such that $a_{i,n} = 0$ for all $i < i_0$ and all n and if for every i the series $\sum_n a_{i,n}$ absolutely converges in $\mathbb{C}$. Then for countably many disjoint sets A_n in $\mathcal{A}$ such that $\cup A_n \in \mathcal{A}$ and $\sum \mu(A_n)$ absolutely converges in $\mathbb{C}(X))$ we have $\mu(\cup A_n) = \sum \mu(A_n)$. See [F3,F4] for more details.

We define the space of integrable functions in several steps. First, consider the space R_F of all functions $f\colon F \longrightarrow \mathbb{C}((X))$ which can be written as a sum of a function which is zero outside finitely many points and of $\sum c_n\, char_{A_n}$ with countably many disjoint measurable sets A_n, $c_n \in \mathbb{C}((X))$, such that the series $\sum c_n \mu(A_n)$ absolutely converges in $\mathbb{C}((X))$. Define $\int f\, d\mu = \sum c_n \mu(A_n)$, this definition is consistent. For example, $\int char_O d\mu = 1$ and $\int char_{\mathcal{O}} d\mu = 0$, but note that $\mathcal{O} \notin \mathcal{A}$.

In order to have an analogue of the Fourier transform, we have to extend this class of integrable functions to include functions of type $\alpha \mapsto \psi(\beta\alpha)$, where ψ is a continuous character of F with conductor O. This larger space is generated by functions $f\colon F \longrightarrow \mathbb{C}((X))$ which are zero outside a subgroup A in $\mathcal{A}$, such that the function $g(x) = \sum_i f(a_i + x)$ for some $a_i \in F$, $1 \leqslant i \leqslant m$, belongs to the previously defined space of functions. Then define $\int f\, d\mu = \frac{1}{m} \int g\, d\mu$ and check the consistency, see [F3,F4].

In particular, if a function $f_1\colon E \longrightarrow \mathbb{C}$ is (absolutely) integrable over E with respect to the normalized Haar measure μ_E, then the function $f_1 \circ p$ extended by zero outside $\mathcal{O}$ is integrable and $\int_{\mathcal{O}} f_1 \circ p\, d\mu = \int_E f_1\, d\mu_E$. Denote by Q_F the subspace of integrable functions consisting of functions f with support in $\mathcal{O}$ and such that $f|_{\mathcal{O}} = g \circ p|_{\mathcal{O}}$ for a Bruhat–Schwartz complex valued function g on E.

Now for an integrable function f define its transform

$$\mathcal{F}(f)(\beta) = \int_F f(\alpha)\, \psi(\alpha\beta)\, d\mu(\alpha).$$

Given $f \in Q_F$, the function $\mathcal{F}(f)$ belongs to Q_F, and reducing to the one-dimensional case one easily gets a double transform formula $\mathcal{F}^2(f)(\alpha) = f(-\alpha)$.

For two-dimensional local fields of type (3) $F = E((t))$, where E is an archimedean local field, the ring of measurable sets is generated by $B = a + t^i D + t^{i+1} K[[t]]$ where D is an open ball in E. The measure is a translation invariant additive measure μ on this ring such that $\mu(B) = \mu_E(D) X^i$ where μ_E is the ordinary Lebesgue measure on E if E is real, and is twice the ordinary Lebesque measure on E if E is complex. Define a character $\psi\colon E((t)) \longrightarrow \mathbb{C}^\times$ as $\sum a_i T^i \mapsto \exp(-2\pi i \operatorname{Tr}_{E/\mathbb{R}}(a_0))$. The transform of an integrable function f is defined by the same formula as above.

More generally, given an integral domain A with principal ideal $P = tA$ and projection $p\colon A \longrightarrow A/P = B$ and an $\mathbb{R}$-valued translation invariant measure on B, similar to the previous theory one defines a measure and integration on A and its field of fractions. For example, the analogue of the ring $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t^i p^{-1}(S)$, where S is from a class of measurable subsets of B, its measure is by definition $X^i \mu_B(S)$; the space of integrable functions is generated by functions $\alpha \to g \circ p(t^{-i}\alpha)$ extended by zero outside $t^i A$, where g is an integrable function on B. Such an $\mathbb{R}((X))$-valued measure and integration on A is natural to call a lift of the measure and integration from B to A. See [Mo1] for a systematic development of this point of view.

Let $\alpha \in F^\times$. Then for any $A \in \mathcal{A}$ the set αA is measurable and $\mu(\alpha A) = |\alpha|\mu(A)$ with $|\alpha|$ independent of A. Thus we get a two-dimensional module $|\ |\colon F^\times \to \mathbb{R}((X))^\times$, $|0| = 0$; it is a generalization of the usual module on locally compact fields.

1.3. Remarks.

1. There are several other related approaches to translation invariant measures on higher dimensional local fields and algebraic groups over them (where many new interesting phenomena show up).

For a nonstandard approach to measure and integration on higher local fields see Remark 1 sect. 4 and Remark 3 sect. 13 of [F3]. A related theory of Hrushovski–Kazhdan unifies the translation invariant measure with the so called motivic measure, and it uses model theory, therefore it best works when the residue field E is of characteristic zero. For this and representation theoretical applications see [HK1], [HK2].

Another direction of applications to representation theory is due to Kim–Lee, see [KL1], [KL2].

The third lifting approach was stated in sect. 13 of [F3] and systematically developed by M. Morrow in [Mo1], its great advantage is that is leads to measure and integration theory satisfying Fubini property on finite dimensional vector spaces over higher local fields, see [Mo2-Mo3].

For the purposes of working with translation invariant measures on algebraic groups over two-dimensional local fields the class of measurable sets of a finite dimensional vector space over it should be a much larger set than the ring of sets generated by products of measurable sets in F. However, for the purposes of the study of the (commutative) zeta integral in dimension two it is sufficient to work with the latter ring. Thus, extend in the natural way the measure on F to a measure on $F \times F$. Define spaces of functions $R_{F\times F}$ and $Q_{F\times F}$ similarly to the previous definitions.

2. The Feynman measure is a translation invariant measure on the loop space of continuous functions on the unit circle. This measure still lacks a solid mathematical foundation which would justify all its applications in physics. The translation invariant $\mathbb{R}((X))$-valued measure μ on the arithmetic loop fields, i.e. two-dimensional local fields, has a number of striking similarities to the desired properties of the Feynman measure, for their list see sect. 18 of [F4].

3. Using projective and inductive limits constructions one can easily define the Fourier transform on spaces of functions and distributions over a two-dimensional local field algebraically, without defining and using a translation invariant measure and integration, see [Kz], [GK]. A crucial underlying feature of the approach of this work is to make the integration theory on arithmetic surfaces

as explicit and simple as possible, often being motivated by arithmetical considerations which in particular come from the explicit higher class field theory.

2. Adelic spaces in dimension two

2.1. General notation. Let $S \to B$ be as in 1.1. Irreducible closed subschemes of dimension one of S endowed with the structure of an integral scheme will be called curves y, and we call closed points of S points x. We will also use notation y for fibres. Denote by $S^{\shortmid}$ the set of vertical curves of $S \longrightarrow B$.

If y is a curve on S, denote by K_y the field of fractions of the completion $\mathcal{O}_y$ of the local ring of S at y. The field K_y is a complete discrete valuation field with residue field $k(y)$. For a closed point x denote by K_x the field of fractions of the completion $\mathcal{O}_x$ of the local ring of S at x. For $x \in y$ and a local branch z of y at x let $K_{x,z}$ be the z-adic completion of K_x, so $K_{x,z}$ a two-dimensional local field. Denote by $O_{x,z}$ the ring of integers of $K_{x,z}$ with respect to the two-dimensional structure and by $\mathcal{O}_{x,z}$ be the ring of integers in $K_{x,z}$ with respect to the discrete valuation of rank 1. Denote by $E_{x,z}$ the residue field of $\mathcal{O}_{x,z}$ and by $k_z(x)$ the finite residue field of $O_{x,z}$.

For a map $\mathcal{K}$ from the set of all $x \in z$, z a local branch of a curve or fibre y on S passing through a point x, to abelian groups we denote by $\mathcal{K}_{x,y}$ the direct sum of $\mathcal{K}_{x,z}$ where z runs through the local branches of y at x. We write $\prod_{x\in y} \mathcal{K}_{x,z}$ for the direct sum (= direct product) of $\mathcal{K}_{x,z}$ where for each x one branch z of y at x is taken (in this case the data on each of the branches will be the same).

Let y be a horizontal curve in characteristic zero and let t_y be a local parameter of the discrete valuation field K_y. Take an embedding $\sigma\colon \mathrm{k} \longrightarrow \mathbb{C}$ and extend it to an embedding $\omega\colon k(y) \longrightarrow \mathbb{C}$. Denote by $E_{\omega,y}$ the archimedean completion of $k(y)$ with respect to ω and by $K_{\omega,y}$ the corresponding two-dimensional local field $E_{\omega,y}((t_y))$. Denote by $\mathcal{O}_{\omega,y}$ its ring of integers. To simplify notation we often view ω as a "point x on y" and include the field $K_{\omega,y}$ in the list of fields $K_{x,z}$ associated to y. We get natural embeddings $K_x \longrightarrow K_{x,y}, K_y \longrightarrow K_{x,y}$.

2.2. Adelic spaces in dimension two. We will define several adelic spaces in dimension two, as we will be working with several of them. One of them, $\mathbf{A}_S$ in the case of positive characteristic was introduced 30 years ago by Parshin.

For a nonsingular curve y on S and integer r define an adelic space

$$\mathbb{A}_y^r = \left\{ \sum_{i\geqslant r} a_i t_y^i : \text{where } a_i \text{ are lifts of } \overline{a_i} \in \mathbb{A}_{k(y)} \text{ to } a_i \in \prod_{x\in y} \mathcal{O}_{x,y} \right\}.$$

Here $\mathbb{A}_{k(y)}$ is the one-dimensional adelic space of the one-dimensional global field $k(y)$. In characteristic zero the lifts have to be defined in a suitably nice way, see sect. 25 of [F5] for details.

Put

$$\mathbf{A}_y = \cup_{r\in\mathbb{Z}} \mathbb{A}_y^r = \left\{ \sum_{i\geqslant i_0 \in\mathbb{Z}} a_i t_y^i : \text{where } a_i \text{ are lifts of } \overline{a_i} \in \mathbb{A}_{k(y)} \text{ to } a_i \in \prod_{x\in y} \mathcal{O}_{x,y} \right\}.$$

Denote by $p_y\colon \mathbf{A}_y \to \mathbb{A}_{k(y)}$ the projection map $\sum a_i t_y^i \to \overline{a_0}$.

If y is a singular curve define $\mathbf{A}_y$ similarly working with all two-dimensional local fields $K_{x,z}$ associated to y, i.e. associated to the normalization of the curve. So for every branch z of y at a singular point x we get such a field. We will identify two such fields associated to transversally intersected branches at a singular point, mapping a local parameter of the residue field of the first field to a local parameter of the residue field of the second field. Define p_y similarly to the above.

If y is a fibre, define the adelic space $\mathbf{A}_y$ consisting of $\{(a_{x,y})_{x\in y}\}$ with $a_{x,z} \in K_{x,z}$, so that for every component of y the corresponding local elements belong to the adelic space on the curve.

Define a large adelic space $\mathbf{A}_S$ associated to the *integral structure of rank one* on S. As a subspace of $\prod K_{x,z}$ it is the restricted product of $\mathbf{A}_y$, where y runs through all fibres and horizontal curves on S, with respect to $\mathbb{A}_y^0$ in the following sense: $(a_{x,y})_{x\in y}$ with $a_{x,z} \in K_{x,z}$ belongs to $\mathbf{A}_S$ if and only if

(a) for almost all y the element $a_{x,y}$ belongs to $\mathcal{O}_{x,y}$ for all $x \in y$ and

(b) there is an integer r such that $(a_{x,y})_{x\in y}$ belongs to $\mathbb{A}_y^r$ for every y.

For a set S' of fibres and horizontal curves define the adelic space $\mathbf{A}_{S'}$ similarly to the above replacing everywhere y by $y \in S'$.

Define $\mathbf{OA}_{S'} = \mathbf{A}_{S'} \cap \prod \mathcal{O}_{x,y}$. In particular, $\mathbf{OA}_y = \mathbb{A}_y^0$. The space $\mathbf{A}_S$ is the restricted product of $\mathbf{A}_y$ with respect to $\mathbf{OA}_y$.

Here in characteristic zero we ignore certain data which come from "archimedean fibres" over archimedean places of k, as this is not relevant for the study of the zeta integral.

The adelic space $\mathbf{A}_S$ in positive characteristic was introduced in [P1–P3], [Be]; in [P2] and [P3] this space is denoted by A_{012}.

Using diagonal embeddings $\prod K_y \to \prod K_{x,y}$, $\prod K_x \to \prod K_{x,y}$, inside the space $\mathbf{A}_S$ we have two smaller local-global spaces $\mathbf{B}_S$ and $\mathbf{C}_S$ as the intersection of $\mathbf{A}_S \subset \prod K_{x,y}$ with the image of the first and second product:

$$\begin{array}{ccccc} & & \mathbf{A}_S & & \\ & \diagup & & \diagdown & \\ \mathbf{B}_S & & & & \mathbf{C}_S \\ & \diagdown & & \diagup & \\ & & K & & \end{array}$$

For a subset S' of the set of curves on S define the adelic space $\mathbf{B}_{S'}$ as the diagonal image of the intersection of $\prod_{y\in S'} K_y$ with $\mathbf{A}_{S'}$.

The adelic spaces denoted above by the bold font are one of two types of adelic spaces on arithmetic surfaces. They are predominantly associated to *the integral structures of rank 1* on the surface, i.e. to divisors on the surface, and are quite useful for algebraic geometric studies. In dimension two there are adelic spaces $\mathbb{A}, \mathbb{B}$ of the second type, they take into account a more refined information associated to the integral structures of rank 2 on the surface. We will have the following picture of adelic spaces

$$\begin{array}{ccc} \mathbf{A}_S & - & \mathbb{A}_S \\ | & & | \\ \mathbf{B}_S & - & \mathbb{B}_S \\ | & & \\ K & & \end{array}$$

in which $K = k(S)$.

The spaces $\mathbb{A}, \mathbb{B}$ will be very useful in the study of the zeta function of the surface. As sets they are subsets of the bold font adelic spaces, but their adelic structure is not the induced one. One of their important features is that one can integrate over them, unlike the adelic spaces of the first type $\mathbf{A}$, $\mathbf{B}$.

Now we introduce an adelic space $\mathbb{A}$ as follows. For a curve y let $\mathbb{A}_y = \mathbb{A}_y^0 = \mathbf{OA}_y$ and for a fibre y let $\mathbb{A}_y$ be the product of the $\mathbb{A}$-spaces associated to its components. Denote by $p_y\colon \mathbb{A}_y \longrightarrow k(y)$ the restriction of the homomorphism p_y defined above. Put $O_{\omega,y} = \mathcal{O}_{\omega,y}$. At singular x of a fibre y let $\mathrm{O}_{x,y}$ be the preimage with respect to p_y of the completion of the localization of the curve y at its singular point x. Define $O\mathbb{A}_y$ as the intersection of $\mathbb{A}_y$ with the product of $O_{x,y}$ at nonsingular $x \in y$ and of $\mathrm{O}_{x,y}$ at singular $x \in y$.

Introduce a two-dimensional adelic object $\mathbb{A}_S$, a subset of $\mathbf{A}_S$, as the restricted product of $\mathbb{A}_y$, $y \in S$, with respect to *the integral structure* $O\mathbb{A}_y$ *of rank* 2: an element $(a_{x,y})$, $a_{x,y} \in K_{x,y}$ belongs to $\mathbb{A}_S$ if

(a) for almost all x, y such that $y \ni x$ the element $a_{x,y}$ belongs to $O_{x,y}$;

(b) for every y the element $(a_{x,y})_{x\in y}$ belongs to $\mathbb{A}_y$.

These conditions imply that if $(a_{x,y})_{x\in y\in S} \in \mathbb{A}_S$ then for almost all y the element $\alpha_y = (a_{x,y})_{x\in y}$ can be written as $\sum_{i\geqslant 0} a_i t_y^i$ where a_i are lifts of $\overline{a_i} \in \mathbb{A}_{k(y)}$ to $a_i \in \prod_{x\in y} \mathcal{O}_{x,y}$ and $a_0 \in \prod_{x\in y} O_{x,y}$.

For a subset S' of curves define

$$\mathbb{A}_{S'} = \prod_{y\in S'}{}' \mathbb{A}_y := \mathbb{A}_S \cap \prod_{y\in S'} \mathbb{A}_y .$$

The space $\mathbb{A}_{S'}$ is the restricted product of $\mathbb{A}_y$ with respect to $O\mathbb{A}_y$, $y \in S'$.

If S' is infinite then the adelic structure of $\mathbb{A}_{S'}$ is not induced from $\mathbf{A}_{S'}$, $\mathbb{A}_{S'} \neq \mathbf{A}_{S'} \cap \prod_{y\in S'} \mathbb{A}_y$.

From now on fix a subset S' of the set of all fibres and finitely many horizontal nonsingular curves on S. Put

$$\mathbb{A} = \mathbb{A}_{S'}, \qquad O\mathbb{A} = \mathbb{A} \cap \prod_{y\in S'} O\mathbb{A}_y .$$

For a horizontal curve or a fibre y put $\mathbb{B}_y = \mathcal{O}_y$ and define $\mathbb{B} = \mathbb{B}_{S'}$ as the intersection of $\prod \mathbb{B}_y$ in $\prod \mathbb{A}_y$ with $\mathbb{A}_{S'}$.

2.3. Additive duality. For each horizontal curve and fibre y one can choose a complex character $\psi_y = \otimes_{x\in y}\psi_{x,y}$ of $\mathbb{A}_y$ trivial on $\mathbb{B}_y$, this character is just an appropriate lift of a character on $\mathbb{A}_{k(y)}$ trivial on $k(y)$. For a singular fibre one can choose an appropriate character which corresponds to the canonical (in this case = dualizing) sheaf on its reduced part, see sect. 27 of [F5]. Of course, this is related to dualities of $\mathbf{A}_S$, see Remark 3 in sect. 28 of [F5] and [Mo4], [Mo5] for nicely written texts.

The conductor $A_{x,z}$ of the local character $\psi_{x,z}$ equals $O_{x,z}$ for almost all $x \in y$. However, since there are infinitely many vertical curves on S, in general the local conductor $A_{x,z} = {t_1}_{x,z}^{d_{x,z}} O_{x,z}$ differs from $O_{x,z}$ for infinitely many (x, z). We may choose ψ_y so that $d_{x,z} = 0$ at singular $x \in y$ and such that the orthogonal complement $\mathrm{O}_{x,y}^{\perp}$ of $\mathrm{O}_{x,y}$ is $t^{-1}\mathrm{O}_{x,y}$ where t serves at t_1 parameter both in $\mathcal{O}_{x,z}$ and $\mathcal{O}_{x,z'}$.

Endow $\mathbb{A}_S, \mathbb{A}$ with the following translation invariant topology: it has $\left(\prod W_{x,z}\right) \cap \mathbb{A}$ as a fundamental system of neighbourhoods of zero, where $W_{x,z}$ are open neighbourhoods of zero in $K_{x,z}$ with respect to its topology and $W_{x,z} = O_{x,z}$ for almost all $x \in y$. Then $\mathbb{A}$ is a reflexive space: the dual to the dual of it (in the topological sense) is canonically isomorphic to it. Unlike the one-dimensional case, since S' contains infinitely many curves, the space $\mathbb{A}$ is not self dual with respect to $\otimes\psi_y$, but it is easy to describe the dual space to $\mathbb{A}$, see sect. 29 of [F5].

We also have a stronger topology on $\mathbb{A}_S$: it has $\left(\prod W_{x,z}\right) \cap \mathbb{A}$ as a fundamental system of neighbourhoods of zero, where $W_{x,z}$ are open neighbourhoods of zero in $K_{x,z}$ with respect to its topology and $W_{x,z} = \sum_{i=0}^{r_y} t_y^i U + t_y^{r_y+1} \mathcal{O}_{x,z}$, U are the lift of $O\mathbb{A}_{k(y)} = p(O\mathbb{A}_y)$ for almost all $x \in y$, with $r_y \geqslant 1$ not depending on x. This topology is the restricted product topology of $\mathbb{A}_y$ with respect to $O\mathbb{A}_y$ where these groups are endowed with the topology in which open neighbourhoods of zero are $\sum_{i=0}^{r_y} t_y^i U_i + t_y^{r_y+1}\mathbb{A}_y$, U_i are lifts of open subgroups in $\mathbb{A}_{k(y)}$, $r_y \geqslant 1$.

Denote by the same notation $|\ |_{x,z}$ the module on $K_{x,z} \times K_{x,z}$ which is the product of the modules of the components. Denote by $|\ |\colon \mathbb{A}_S^\times \to \mathbb{R}$ (note that the image is in $\mathbb{R}^\times \subset \mathbb{R}((X))^\times$) and $|\ |\colon (\mathbb{A}_S \times \mathbb{A}_S)^\times \to \mathbb{R}$ the product of the local modules.

Define a topology on the multiplicative group $\mathbb{A}_S^\times$ as (the sequential saturation of) the induced topology from the stronger topology of $\mathbb{A}_S$ via $\mathbb{A}_S^\times \to \mathbb{A}_S \times \mathbb{A}_S$, $\alpha \mapsto (\alpha, \alpha^{-1})$.

3. Measure and integration on adelic spaces

From now on we will assume that the singular points of of every fibre of $S \longrightarrow B$ are split ordinary double. We will use the notation y for a horizontal curve in the finite horizontal part S^- of S' fixed in the previous section or for a fibre.

Certain parts of the following theory are parallel to the classical theory in [T1] and [W2].

To work with the zeta integral we will need measure and integration on $\mathbb{A} \times \mathbb{A}$, $(\mathbb{A} \times \mathbb{A})^\times$, and on $\mathbb{B} \times \mathbb{B}$ and $(\mathbb{B} \times \mathbb{B})^\times$. The central object of two-dimensional adelic analysis is an unramified zeta integral. The zeta integral will be an integral with respect to a measure on $(\mathbb{A} \times \mathbb{A})^\times$.

Define normalized additive and multiplicative measures: let the measure $\mu_{x,z}$ be normalized by the condition $\mu_{x,z}(O_{x,z}) = q_{x,z}^{d_{x,z}/2}$, $d_{x,z}$ is defined in 2.3, and let $\mu_{K_{\omega,y}}$ be as defined in section 1. Let $\mu_{(K_{x,z}\times K_{x,z})^\times} = (1-q_{x,z}^{-1})^{-2}\mu_{K_{x,z}\times K_{x,z}}/|\ |_{x,z}$ for nonarchimedean x, z, and $\mu_{(K_{\omega,y}\times K_{\omega,y})^\times} = \mu_{K_{\omega,y}\times K_{\omega,y}}/|\ |_{\omega,y}$.

If y is a nonsingular curve then the space $\mathbb{A}_y^\times$ coincides with the preimage of its image with respect to the projection map p_y. Functions which we will integrate in the study of the zeta integral will all be constant on groups associated to $\mathbb{A}_y^1$. Hence for the purposes of this work it is sufficient to work with an $\mathbb{R}$-valued measure on $(\mathbb{A}_y \times \mathbb{A}_y)^\times$ which is the pullback with respect to (p_y, p_y) of a normalized one-dimensional adelic measure on $(\mathbb{A}_{k(y)} \times \mathbb{A}_{k(y)})^\times$ (of course the integral of the function against it equals the integral against the measure which is the tensor product of $\mu_{(K_{x,y}\times K_{x,y})^\times}$, $x \in y$) and with the measure on $(\mathbb{A} \times \mathbb{A})^\times$ which is their tensor product.

3.1. Space $Q_{\mathbb{A}\times\mathbb{A}}$ of functions on $\mathbb{A} \times \mathbb{A}$ and adelic transform $\mathcal{F}$. Using the local transforms $\mathcal{F}_{x,z}$ associated to the fixed in the previous section local characters $\psi_{x,z}$ and normalized measures $\mu_{x,z}$ defined above one gets an adelic transform $\mathcal{F}$.

Define the space $Q_{\mathbb{A}\times\mathbb{A}}$ of functions on $\mathbb{A} \times \mathbb{A}$ generated by functions $\otimes_{x\in y\in S'}(f_{x,y}^{(1)}, f_{x,y}^{(2)})$ where $f_{x,z}^{(m)}$ are in the local space $Q_{K_{x,z}}$, nonarchimedean-archimedean components $f_{\omega,y}^{(m)}$ are of the form $h_{\omega,y}^{(m)} \circ (p_{\omega,y}, p_{\omega,y})$ where $p_{\omega,y}$ is the projection to the first residue field and $h_{\omega,y}^{(m)}$ is in its Schwartz space, and for almost all x, y we have $f_{x,z}^{(m)} = char_{t_{1\,x,z}^{c_{x,z,m}} O_{x,z}}$, such that for all y for almost all $x \in y$ the integer $c_{x,z,m}$ equals zero, and $\prod_{x\in y} q_{x,y}^{d_{x,y}/2 - c_{x,y,1} - c_{x,y,2}} = 1$ for almost all y.

For a function f as above define its adelic transform $\mathcal{F}(f)$ as the product of its local transforms and then extend to the space $Q_{\mathbb{A}\times\mathbb{A}}$.

For example,

$$\text{if } f = \otimes_{x\in y} char_{(t_{1x,y}^{c_{x,y}} O_{x,y},\, t_{1x,y}^{c'_{x,y}} O_{x,y})}$$

$$\text{then } \mathcal{F}(f) = \otimes_{x\in y}\, q_{x,y}^{d_{x,y}-c_{x,y}-c'_{x,y}}\; char_{(t_{1x,y}^{d_{x,y}-c_{x,y}} O_{x,y},\, t_{1x,y}^{d_{x,y}-c'_{x,y}} O_{x,y})}.$$

3.2. Space $R_{(\mathbb{A}\times\mathbb{A})^\times}$ and measure and integration on $(\mathbb{A}\times\mathbb{A})^\times$. From the definition of $\mathbb{A}$ we deduce that the multiplicative group $\mathbb{A}^\times$ is the restricted product of $\mathbb{A}_y^\times$ with respect to $(O\mathbb{A}_y)^\times$, $y\in S'$. Similarly to the definition of $\mathbb{A}_y = \mathbb{A}_y^0$ define an adelic space

$$\mathbb{A}_y\times\mathbb{A}_y := \{(\alpha_{x,y}^{(1)},\alpha_{x,y}^{(2)})_{x\in y} : \alpha_{x,y}^{(m)}\in K_{x,y}, (\alpha_{x,y}^{(m)})\in\mathbb{A}_y, m=1,2\}.$$

Define $(\mathbb{A}\times\mathbb{A})^\times$ as the restricted product of $(\mathbb{A}_y\times\mathbb{A}_y)^\times$ with respect to $(\mathbb{A}_y\times\mathbb{A}_y\cap O\mathbb{A}_y\times O\mathbb{A}_y)^\times$.

Define $\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ as the tensor product of the normalized local measures $\mu_{(K_{x,y}\times K_{x,y})^\times}$, $x\in y$. The definition of $(\mathbb{A}_y\times\mathbb{A}_y)^\times$ implies that $\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ is a real valued measure.

Define $\mu_{(\mathbb{A}\times\mathbb{A})^\times}$ as the tensor product of $\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$, $y\in S'$. Define a space of functions $R_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ as the linear space generated by $g_y = \otimes_{x\in y}(f_{x,y}^{(1)}, f_{x,y}^{(2)})$ with $g_y = h_y\circ(p_y,p_y)$ for an integrable function h_y on $(\mathbb{A}_{k(y)}\times\mathbb{A}_{k(y)})^\times$, and such that $f_{x,z}^{(m)}$ is continuous on $K_{x,z}^\times$, $f_{x,z}^{(m)} char_{K_{x,z}^\times}\in R_{K_{x,z}}$ for all $x\in y$ and $f_{x,z}^{(m)}|_{O_{x,z}^\times} = 1$ for almost all $x\in y$, $m=1,2$. For $f_y = \otimes_{x\in y} f_{x,y}\in R_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ define $\int f_y\, d\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times} = \prod_{x\in y}\int f_{x,y}\, d\mu_{(K_{x,y}\times K_{x,y})^\times}$ and extend by linearity to the space $R_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$.

Define a space of functions $R_{(\mathbb{A}\times\mathbb{A})^\times}$ as the space generated by $\otimes f_y$ with $f_y = (f_y^{(1)}, f_y^{(2)})\in R_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ such that $\otimes f_y$ induces a continuous map $(\mathbb{A}\times\mathbb{A})^\times\longrightarrow\mathbb{C}$ and $\prod\int f_y\, d\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ absolutely converges in the compactified complex plane $\mathbb{C}\cup\{\infty\}$.

For $f=\otimes f_y\in R_{(\mathbb{A}\times\mathbb{A})^\times}$ with $f_y\in R_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$ define

$$\int f\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times} = \prod\int f_y\, d\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}$$

and extend by linearity to $R_{(\mathbb{A}\times\mathbb{A})^\times}$.

3.3. Example. Let $f=\otimes_{x\in y} f_{x,y}$ where for all nonarchimedean x,z

$$f_{x,z} = |\;|_{x,z}^s\, char_{(t_{1x,z}^{c_{x,z,1}} O_{x,z},\, t_{1x,z}^{c_{x,z,2}} O_{x,z})},$$

and for all $y\in S'$ $c_{x,z,m}=0$ for almost all $x\in y$, $m=1,2$, for almost all $y\in S'$ $\prod_{x\in y} q_{x,y}^{c_{x,y,m}} = 1$, $m=1,2$; and $c_{x,z,m}=c_{x,z',m}$ for two local branches z,z' of a fibre at x. Define the components of f over archimedean places as

$$f_{\omega,y}(\alpha,\beta) = |\;|_{\omega,y}^s\,\exp\bigl(-e_\omega\,\pi\,(|p_y(\alpha)|^2+|p_y(\beta)|^2)\bigr),$$

for $(\alpha,\beta)\in\mathcal{O}_{\omega,y}\times\mathcal{O}_{\omega,y}$ where $|\;|$ is the usual absolute value, p_y is the projection map, $e_\omega=1$ if ω is a real embedding and $e_\omega=2$ if ω is a complex embedding. Then

$$\int f_y\, d\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times} = \prod_{y\in S'}\prod_{x\in y,\mathrm{na}} q_{x,y}^{d_{x,y}-(c_{x,y,1}+c_{x,y,2})s}\left(\frac{1}{1-q_{x,y}^{-s}}\right)^2\prod_{\omega\in y}\Gamma_{\omega,y}(s),$$

where for $y\in S^-$ the factor $\Gamma_{\omega,y}(s)=\pi^{-s}\Gamma(s/2)^2$ if ω is a real embedding and $\Gamma_{\omega,y}(s) = (2\pi)^{2-2s}\Gamma(s)^2$ if ω is a complex embedding. So we get

$$\int f\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times} = \prod_{x\in y} q_{x,y}^{d_{x,y}-(c_{x,y,1}+c_{x,y,2})s}\left(\frac{1}{1-q_{x,y}^{-s}}\right)^2\prod_{\omega\in y\in S^-}\Gamma_{\omega,y}(s).$$

The product of the Euler factors equals the square of the Hasse zeta function of S times additional factors at singular points on fibres; we take care of the latter factors later in 4.3 and 5.1 where we introduce a subgroup T of $(\mathbb{A} \times \mathbb{A})^\times$ over which the zeta integrals will be taken.

For the product $\prod_{x\in y\in S'} q_{x,y}^{d_{x,y}-(c_{x,y,1}+c_{x,y,2})s}$ to converge we need to impose the following condition: $\prod_{x\in y} q_{x,y}^{d_{x,y}-(c_{x,y,1}+c_{x,y,2})s} = 1$ for almost every fibre y. For a nonsingular fibre $\prod_{x\in y} q_{x,y}^{d_{x,y}}$ equals 1 only if S is a regular model $\mathcal{E}$ of elliptic curve over a global field, see 5.3 below. Hence the study of zeta integrals will be the simplest in the case of such arithmetic surfaces. If $S = \mathcal{E}$ then the function f belongs to the space $R_{(\mathbb{A}\times\mathbb{A})^\times}$ for $\Re(s) > 2$. In the general case one has to renormalize fibre integrals to ensure the convergence of their infinite product, see 5.6.

3.4. Measure and integration on $\mathbb{B} \times \mathbb{B}$. Using 1.2 define an $\mathbb{R}((X))$-valued translation invariant measure $\mu_{\mathbb{B}_y\times\mathbb{B}_y}$ on $\mathbb{B}_y \times \mathbb{B}_y$ which lifts the discrete counting measure on $k(y) \times k(y)$. Components of a measurable set with respect to this measure for almost all $y \in S'$ are sets $(p_y, p_y)^{-1}(pt)$. Define a measure $\mu_{\mathbb{B}\times\mathbb{B}} = \otimes\mu_{\mathbb{B}_y\times\mathbb{B}_y}$.

For a subset S_o of S' of fibres and horizontal curves and a function $f = \otimes f_y \in Q_{\mathbb{A}\times\mathbb{A}}$, $f_y = \otimes_{x\in y}(f^{(1)}_{x,y}, f^{(2)}_{x,y})$, $f^{(m)}_{x,y} \in Q_{K_{x,y}}$, $f_y = g_y \circ (p_y, p_y)$, where $g_y = (g^{(1)}_y, g^{(2)}_y)$, $g^{(m)}_y$ are integrable functions on $\mathbb{A}_{k(y)}$, define $\int_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}} f(\beta)\, d\mu_{\mathbb{B}\times\mathbb{B}}(\beta)$ as equal to $\prod_{y\in S_o} \int_{k(y)\times k(y)} g_y\, d\mu_{k(y)\times k(y)}$ and extend to the space generated by such functions. The right hand side can diverge if S_o is infinite.

Since the measure on $k(y)$ is discrete counting, it induces the measure on $k(y)^\times$. Define the measure on $(\mathbb{B}_y \times \mathbb{B}_y)^\times$ as induced from the measure on $\mathbb{B}_y \times \mathbb{B}_y$. So this measure is just the pullback with respect to (p_y, p_y) of the discrete measure on $(k(y) \times k(y))^\times$. Define the measure on $(\mathbb{B} \times \mathbb{B})^\times$ as the induced from the measure on $\mathbb{B} \times \mathbb{B}$. For a subset $B = \prod (p_y, p_y)^{-1}(B_y)$ of $(\mathbb{B} \times \mathbb{B})^\times$ and $f = \otimes f_y$ as above, define $\int_B f\, d\mu_{(\mathbb{B}\times\mathbb{B})^\times} = \prod_y \int_{B_y} g_y\ d\mu_{k(y)\times k(y)}$.

3.5. Summation formula. For a function $f \in Q_{\mathbb{A}\times\mathbb{A}}$ and a finite subset S_o of S', $\alpha \in (\mathbb{A}_{S_o} \times \mathbb{A}_{S_o})^\times$ we get a summation formula, which follows from the one-dimensional formula and the duality associated with the canonical sheaf on a fibre (see sect. 32 in [F5])

$$\int_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}} f(\alpha\beta)\, d\mu_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}(\beta) = \frac{1}{|\alpha|} \int_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}} \mathcal{F}(f)(\alpha^{-1}\beta)\, d\mu_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}(\beta).$$

If we use f which is the tensor product of $char_{(O_{x,y},O_{x,y})}$ at nonsingular $x \in y$ and $char_{(\mathcal{O}_{x,y},\mathcal{O}_{x,y})}$ at singular $x \in y$ then the summation formula corresponds to the Riemann–Roch theorem for y.

For more details on the measure and integration see [F5].

4. Explicit two-dimensional class field theory

The higher class field theory is not used in the study of the zeta integral, one just uses some of its objects or related objects. However, some basic knowledge of the higher class field theory can help a broader understanding of the whole theory. By various reasons the two-dimensional class field theory is still not well known to number theorists. Some of objects of higher class field theory in positive characteristic are due to Parshin, this approach is very explicit and uses topological Milnor K-groups and the higher Artin–Schreier–Witt pairing, as a higher dimensional generalization of the classical explicit approach to the one-dimensional class field theory in positive characteristic by Kawada and Satake, see sections of [IHLF] for a review. A comprehensive treatment of higher class field theory is due to Kato and Saito, see [K2] for a review. We rather need an *explicit higher* *class field theory*, and we now briefly sketch the main local and global theorems in the explicit two-dimensional class field theory.

4.1. Local theory. Let F be a nonarchimedean two-dimensional local field with one-dimensional residue field E. The one-dimensional reciprocity map is an injective homomorphism $E^\times \longrightarrow \mathrm{Gal}\,(E^{\mathrm{ab}}/E)$ with dense image, such that for a finite abelian extension R/E it induces an isomorphism $E^\times/N_{R/E}R^\times \longrightarrow \mathrm{Gal}\,(R/E)$. To implement a natural idea of lifting it to the level of F such that the right hand side is replaced by $\mathrm{Gal}\,(F^{\mathrm{ab}}/F)$ one more or less quickly understands that $E^\times = K_1(E)$ should be replaced by something close to the Milnor K_2-group of F. Recall that $K_2(F)$ is the quotient of $F^\times \otimes F^\times$ by its subgroup generated by $x \otimes 1-x$. The image of $x\otimes y$ in $K_2(F)$ is denoted by $\{x,y\}$, and the map $F^\times \times F^\times \longrightarrow K_2(F)$, $(x,y)\mapsto \{x,y\}$ is called the symbol map. The group operation in $K_2(F)$ is normally called addition and one uses the additive notation.

There is a natural boundary map $K_2(F) \longrightarrow K_1(E)$ which serves as one side of the commutative diagramme connecting the reciprocity maps in dimension two and one. The two-dimensional reciprocity map $K_2(F) \longrightarrow \mathrm{Gal}\,(F^{\mathrm{ab}}/F)$ is not injective but its image is dense. It can be proved that the kernel of the reciprocity map for two-dimensional fields whose first residue field is of positive characteristic equals $\cap_{l\geqslant 1} lK_2(F)$, and this subgroup equals the intersection of all open neighbourhoods of zero in the strongest topology on $K_2(F)$ in which the addition in $K_2(F)$ is continuous and the symbol map is continuous in each argument with respect to the topology on $F^\times$ defined in the first section, see [F1] for more details. Thus, even before developing the class field theory it makes sense to introduce a *topological* K_2^t*-group* $K_2^t(F)$ *as* $K_2(F)/\Lambda_2(F)$, where $\Lambda_2(F) = \cap_{l\geqslant 1} lK_2(F)$+ the intersection of all neighbourhoods of zero, with the induced topology. It is much easier to work with the topological K-groups, since using topological generators one can operate with infinite topologically convergent products and sums. Using explicit pairings: the higher dimensional tame symbol, the Artin–Schreier-Witt and Vostokov pairings (see sect. 6 of [IHLF]) one can get various information on the structure of $K_2^t(F)$, which are stronger than the results on its natural factor filtration obtained in purely algebraic way.

In the *explicit class field theory* (see sect. 10 of [IHLF]) a major role is played by a surjective homomorphism

$$\mathfrak{t}\colon T = (\mathcal{O}\times\mathcal{O})^\times \longrightarrow K_2^t(F).$$

In the nonarchimedean case we have $\mathcal{O}^\times = t_1^{\mathbb{Z}} O^\times$. Define $\mathfrak{t}$ as $(t_1^i u, t_1^j v) \mapsto (i+j)\{t_1,t_2\} + \{t_1,u\} + \{v,t_2\}$, $u,v\in O^\times$. Denote by $UK_2^t(F)$ the image of $(O\times O)^\times$. We have a commutative diagramme

$$\begin{array}{ccccc} & & \mathcal{O}^\times \otimes F^\times/\mathcal{O}^\times & & \\ & & \downarrow & \searrow & \\ \mathcal{O}^\times\times\mathcal{O}^\times = T & \longrightarrow & \mathcal{O}^\times\times\mathcal{O}^\times/O^\times & \longrightarrow & K_2^t(F)/UK_2^t(F). \end{array}$$

The vertical map sends $\alpha\otimes {t_2}^m$ to $(\alpha^m,1)$; the surjective diagonal map is induced by the symbol map; the first horizontal map is the projection on the second component, and the composition of the first and second horizontal maps is induced by $\mathfrak{t}$. The second horizontal map does not depend on the choice of local parameters. An analogue of the unramified quasi-character in the classical theory is a homomorphism $|\ |_2^s\colon K_2^t(F) \longrightarrow \mathbb{C}^\times$, $s\in\mathbb{C}$, which sends $\{t_1,t_2\}$ to q^{-s}, q is the cardinality of the finite residue field of F, and which sends $UK_2^t(F)$ to 1.

In the case of fields of type (3), $F = E((t))$ where E is an archimedean local field, define $\mathfrak{t}\colon T = \mathcal{O}^\times\times\mathcal{O}^\times \longrightarrow E^\times\times E^\times \longrightarrow K_2^t(F)$, where the first map is (p,p), p is the residue map and the second map is $(\alpha,\beta)\mapsto\{\alpha\beta,t\}$. Denote by $K_2^t(F)_0$ is the cyclic group generated by $\{-1,-1\}$, it is nontrivial if and only if E is real. We have a commutative diagramme

$$\begin{array}{ccccc} & & \mathcal{O}^\times \otimes F^\times/\mathcal{O}^\times & & \\ & & \downarrow & \searrow & \\ \mathcal{O}^\times \times \mathcal{O}^\times = T & \longrightarrow & \mathcal{O}^\times \times \mathcal{O}^\times & \longrightarrow & K_2^t(F)/K_2^t(F)_0 \end{array}$$

where the vertical map sends $\alpha \otimes t^m$ to $(\alpha^m, 1)$, and the surjective diagonal map is induced by the symbol map. Define a homomorphism $|\ |_2 \colon K_2^t(F) \longrightarrow \mathbb{C}^\times$, it sends the image of $\mathfrak{t}$, $\{\alpha, t\}$, $\alpha \in E^\times$, to $|\alpha|_E$, and it sends its orthogonal complement $K_2^t(F)_0$ to 1. Here $|\alpha|_E$ is the module associated to the measure μ_E defined in section 1.

In each case the composite $|\ |_2 \circ \mathfrak{t}$ is the module map $|\ |$ on T. The group $N = |T|$ equals the multiplicative group of positive real numbers or the cyclic group generated by $q > 1$.

If $\chi \colon K_2^t(F) \longrightarrow \mathbb{C}^\times$ is a continuous quasi-character, then similar to dimension one it is easy to show that it is the product of $|\ |_2^s$ and a character χ_0 of finite order which is trivial on $\{t_1, t_2\}$.

One can show that the topology of $K_2^t(F)$ on the level of subgroups coincides with the induced via $\mathfrak{t}$ topology of T (which is induced from $F^\times \times F^\times$).

The symbol $\{t_1, t_2\}$ modulo units plays in the two-dimensional class field theory the role of a (K_2^t-) prime element: its image in the Galois group restricted to the purely unramified extension F' of F corresponding to the maximal algebraic extension of the finite residue field is the Frobenius automorphism. For a nonarchimedean two-dimensional local field F the inverse morphism Ψ_F to the *reciprocity homomorphism* $\Phi_F \colon K_2^t(F) \longrightarrow \mathrm{Gal}\,(F^{\mathrm{ab}}/F)$ has a very explicit description, Ψ_F sends an element σ of a finite Galois extension L of F to the norm of a prime element in an appropriate finite extension of F, the fixed field of a good lifting of σ to LF'/F, see sect. 10 of [IHLF].

When F is of type (3) its topological K_2-group is not too useful for a description of abelian extension of F, which are anyway very easy to describe.

4.2. Global theory. Now we turn to the global two-dimensional class field theory for an arithmetic surface S as in section two (even though the main theorems of this theory can be stated more generally for integral schemes projective over $\mathbb{Z}$). For main general results see [KS1], [KS2]. We actually need an adelic (K-delic) form of the main theory, which is not explicitly available from those sources. We will define two objects J_S and P_S which are analogues of the idele group and the multiplicative group of global elements.

Using the local maps $\mathfrak{t}$ define

$$\mathfrak{t} \colon (\mathbb{A}_S \times \mathbb{A}_S)^\times \cap \prod T_{x,y} \longrightarrow \prod K_2^t(K_{x,y}).$$

Denote by

$$J_S = {\prod}' K_2^t(K_{x,y})$$

the image of this homomorphism. This J_S is a simplified and a reduced version of J_S introduced in [F5]: the difference is that in the case where K has a structure of ordered field we ignore a part of the Galois group of K which corresponds to the symbol $\{-1, -1\} \in K_2^t(\mathbb{R}((t)))$ locally associated to the ramified extension $\mathbb{R}((t^{1/2}))/\mathbb{R}((t))$. Anyway, we will soon concentrate on the unramified part only. Below the word "almost" means that we ignore this part of the Galois group of K.

Define the topology of J_S as induced via $\mathfrak{t}$ from the topology of $\left(\mathbb{A}_S \times \mathbb{A}_S\right)^\times$. Define

$$P_S = \Delta {\prod_y}' K_2(K_y) + \Delta {\prod_x}' K_2(K_x)$$

where the restricted product signs mean the intersection of J_S with the image with respect to Δ induced by diagonal field embeddings. Endow J_S/P_S with the induced topology.

Then we have the *global reciprocity map*

$$\Phi_S = \prod \Phi_{x,y}\colon J_S \longrightarrow \mathrm{Gal}\,(K^{\mathrm{ab}}/K)$$

where $\Phi_{x,z}$ are local reciprocity maps. The map Φ_S vanishes on P_S. Continuous characters of finite order of the Galois group of the function field K of S are (almost) in one-to-one correspondence with continuous characters of finite order of the K-delic class group $C_S = J_S/P_S$ via the reciprocity map. For (almost) every finite Galois extension L/K the map Φ_S induces an isomorphism $J_S/(P_S + N_{S\otimes_K L/S} J_{S\otimes_K L}) \longrightarrow \mathrm{Gal}\,(L \bigcap K^{\mathrm{ab}}/K)$. This follows from the results of Kato and Saito.

Denote by $|\ |_2\colon J_S \to \mathbb{R}^\times$ the product of the local modules $|\ |_{2,x,y}$. Every continuous quasi-character χ on J_S/P_S is the product of $|\ |_2^s$, $s \in \mathbb{C}$ and a character χ_0 of finite order trivial on $\{t_1, t_2\}$.

4.3. Objects for the unramified theory. As we will be mainly interested in the unramified situation we now describe a simplified version of objects J_S, P_S which provides background objects for the definition of the unramified zeta integral in dimension two.

Recall that for regular points x one has the following exact sequence

$$K_2(\mathcal{O}_x) \longrightarrow K_2(K_x) \longrightarrow \oplus_{w\ni x} K_1(E_{x,w}) \longrightarrow K_0(k(x)) \longrightarrow 0,$$

where w runs through all prime ideals of height 1 in the ring $\mathcal{O}_x$, see e.g. [Bl1]. Note that the image of $\oplus K_1(E_{x,z})$ coincides with the image of its vertical part. This gives an exact sequence $K_2(K_x) \longrightarrow \oplus_{z\ni x} K_1(E_{x,z}) \longrightarrow K_0(k(x)) \longrightarrow 0$ where z runs through all local branches at x of curves y on S. Thus, in J_S/P_S we can move modulo $K_2(K_x)$ the contribution of local (K_2^t-) prime elements in $K_2^t(K_{x,z})$ for branches z of all horizontal curves y passing through x to the vertical branches. Then we are left with $UK_2^t(K_{x,z})$ on branches of all horizontal curves on S. Also, if x is a singular point of a fibre then factorizing the product of $K_2^t(K_{x,z})$ for the local branches of the fibre at x modulo the image of $K_2(K_x)$ we see that that we can work modulo units with just one copy of $K_2^t(K_{x,z})$ for every x in every fibre. In the rest of the paper we will assume that every singular point of a fibre is a rational (split) ordinary double point.

For a subset S_o of curves on S define

$$T_{S_o} = \big(\mathbb{A}_{S_o} \times \mathbb{A}_{S_o}\big)^\times.$$

Denote $N_{S_o} = |T_{S_o}|$.

For S' as in section 2, denote

$$T = T_{S'}, \quad J = J_S \cap \prod_{y\in S'} \prod K_2^t(K_{x,y}), \quad P = J \cap P_S.$$

The K_2-delic object J equals $J_{S,}$ defined in sect. 35 of [F5].

Denote by J_S^1 the kernel of $|\ |_2$ and $J^1 = J \cap J_S^1$. Then $P_S < J_S^1$, $P < J^1$. Define a subgroup $UJ_S = \mathfrak{t}\big(T \cap \prod (O_{x,y} \times O_{x,y})^\times\big) \cap J_S^1$ of J_S^1 (recall that $O_{\omega,y} = \mathcal{O}_{\omega,y}$ by definition). Put $UJ = UJ_S \cap J$. Denote by VJ_S the sum of the image $\mathfrak{t}\big(T_S \cap \prod (O_{x,y} \times O_{x,y})^\times\big)$ and $\oplus_{y,\omega} K_2^t(K_{\omega,y})_0$ and put $VJ = VJ_S \cap J$.

Define a subgroup $V\mathbb{A}_S^\times$ of $\mathbb{A}_S^\times$ which at nonarchimedean data equals $\mathbb{A}_S^\times \cap \prod O_{x,y}^\times$ and whose mixed archimedean-nonarchimedean data components are 1. Put $V\mathbb{A}^\times = \mathbb{A}^\times \cap V\mathbb{A}_S^\times$. Using the adelic space $\mathbf{A}_S$ in the previous section define $\mathbf{VA}_S^\times = \mathbf{A}_S^\times \bigcap \prod \mathcal{O}_{x,y}^\times$ and let $\mathbf{A}_{S'}$, $\mathbf{VA}_{S'}^\times$ be the S'-part of $\mathbf{A}_S$ and the previously defined object. Define $\mathbb{A}_S^\times \times \mathbf{A}_S^\times$ as the subspace of the adelic space

$\big(\mathbf{A}_S \times \mathbf{A}_S\big)^\times$ consisting of $(\alpha_{x,y}, \beta_{x,y})$ with $(\alpha_{x,y}), (\beta_{x,y}) \in \mathbf{A}_S^\times$; similarly define $\mathbb{A}_S^\times \times \mathbf{A}_S^\times/\mathbf{VA}_S^\times$. Note that $\mathbf{A}_S^\times/\mathbf{VA}_S^\times$ is isomorphic to $\oplus_{x,z}\mathbb{Z}$ and hence $\mathbb{A}_S^\times \otimes \mathbf{A}_S^\times/\mathbf{VA}_S^\times$ consists of $(\alpha_{x,y} \otimes \gamma_{x,y})$, $(\gamma_{x,y}) \in \mathbf{A}_S^\times/\mathbf{VA}_S^\times$.

Then the following diagrammes are the adelic version of the diagrammes in 4.1

$$\begin{array}{ccccc} & & \mathbb{A}_S^\times \otimes \mathbf{A}_S^\times/\mathbf{VA}_S^\times & & \\ & & \downarrow & \searrow & \\ T_S & \longrightarrow & \mathbb{A}_S^\times \times \mathbb{A}_S^\times/V\mathbb{A}_S^\times & \longrightarrow & J_S/VJ_S \end{array}$$

and

$$\begin{array}{ccccc} & & \mathbb{A}^\times \otimes \mathbf{A}_{S'}^\times/\mathbf{VA}_{S'}^\times & & \\ & & \downarrow & \searrow & \\ T & \longrightarrow & \mathbb{A}^\times \times \mathbb{A}^\times/V\mathbb{A}^\times & \longrightarrow & J/VJ. \end{array}$$

These commutative diagrammes glue together in a special and important way the adelic structures of the first and second type, corresponding to the integral structures of rank 1 and rank 2 on the surface.

For the unramified (with respect to the structure of rank two) theory elements of $UK_2^t(K_{x,z})$ do not matter, and hence for this purpose we can replace J_S/P_S by J/P. The induced homomorphisms from T_S to $J_S/(P_S{+}UJ_S)$ and from T to $J/(P{+}UJ)$ are surjective. We also deduce that $J^1/(P{+}UJ)$ is naturally isomorphic to $J_S^1/(P_S{+}UJ_S)$. The latter is isomorphic to the (zero degree part in positive characteristic) $CH_0(S)^0$ of the Chow group of 0-cycles on S, which from the point of view of class field theory plays the role of the (zero degree) class group in dimension one. In dimension one the finiteness of the class group, as well as the Dirichlet theorem follow from the calculation of the zeta integral, see [I2]. Similarly in dimension two the finiteness of several groups including $CH_0(S)^0$ can be deduced from a calculation of the two-dimensional zeta integral.

5. Zeta integrals

We will define zeta integrals in the local case and then in the adelic case.

5.1. The generic formula. The general formula for the zeta integral has a shape similar to the dimension one zeta integral:

$$\zeta(g,\chi) = \int_{\mathfrak{T}} g\chi_{\mathfrak{t}}\, d\mu$$

where g is a function in the spaces R or Q defined in 1.3 and 2.2, χ is a quasi-character on the group which describes abelian extensions ($K_2^t(F)$ or J_S/P_S) and $\chi_{\mathfrak{t}}$ is its pullback to a quasi-character on the group T, local or adelic; tildes and $\mathfrak{T}$ is a certain rescaled version of T and the local integrals at singular points of fibres are modified appropriately. In the unramified theory without essential loss one can work with the zeta integral

$$\zeta(g, |\ |^s) = \int_{\mathfrak{T}} g\,|\ |^{s/2}\, d\mu.$$

We will have *different rescaling on vertical and horizontal curves*. For a curve y denote by $T_{1,y}$ the kernel of the module map on T_y. Choose a set of multiplicative representatives $M_y < T_y$ of

$N_y = |T_y|$. So if $k(y)$ is of positive characteristic then N_y is a cyclic group generated by $q_y > 1$, if $k(y)$ is of characteristic zero then N_y is the multiplicative group of positive real numbers.

For a fibre $\star$ put $|| \, ||_\star = | \, |_\star$ and denote $\mathfrak{T}_\star = T_\star$. We also have to modify the local integrals at singular points of fibres appropriately, for details see sect. 37 of [F5].

For a horizontal curve $\star$ put $|| \, ||_\star = | \, |_\star^{1/2}$ and choose a maximal subgroup $\mathfrak{T}_\star$ of $T_\star$ such that $||\mathfrak{T}_\star|| = |T_\star|$. In other words, $\mathfrak{T}_\star = T_\star$ for horizontal curves in characteristic zero and $\mathfrak{T}_\star = T_{1,\star} \times M_\star^2$ (of course, this depends on the choice of $M_\star$) for horizontal curves in positive characteristic. Put

$$\mathfrak{T} = T \cap \prod_{\star \in S'} \mathfrak{T}_\star = \prod_{\star \in S'}{}' \mathfrak{T}_\star, \quad || \, || = \prod_{\star \in S'} || \, ||_\star.$$

For a continuous homomorphism $\chi\colon J \longrightarrow \mathbb{C}^\times$ write $\chi = \chi_0 | \, |_2^s$ as the product of the unramified quasi-character $| \, |_2^s$ and χ_0. Define

$$\chi_{\mathfrak{t}} := (\chi_0 \circ \mathfrak{t}) || \, ||^s : \mathfrak{T} \longrightarrow \mathbb{C}^\times,$$

χ uniquely determines $| \, |_2^s$ and χ_0, and hence $|| \, ||^s$ and $\chi_{\mathfrak{t}}$ as functions on $\mathfrak{T}$ are uniquely determined by χ.

On the vertical part of T we have $\chi_{\mathfrak{t}} = \chi \circ \mathfrak{t}$.

Now, for a function g in $R_{F\times F}$ or in $R_{(\mathbb{A}\times\mathbb{A})^\times}$, spaces defined in 1.3 and 3.2, and a quasi-character $\chi\colon K_2^t(F) \longrightarrow \mathbb{C}^\times$ or $\chi\colon J_S \longrightarrow \mathbb{C}^\times$, $\chi|_{P_S} = 1$, define a *generic local zeta integral* as

$$\zeta(g,\chi) = \zeta(g,\chi,\mu) = \int_T g\chi_{\mathfrak{t}} \, d\mu_{(F\times F)^\times}$$

and an *adelic zeta integral* as

$$\zeta(g,\chi) = \zeta_{S,S'}(g,\chi,\mu) = \int_{\mathfrak{T}} g\chi_{\mathfrak{t}} \, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}.$$

The latter for a singular fibre is not $\int g\chi_{\mathfrak{t}} \, char_T \, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}$, but differs at the $(x,\star)$-data for singular points $x \in \star$. See sect. 37 of [F5] for details.

Of course, the adelic zeta integral depends on the choice of the set S' of curves, which includes all vertical curves and finitely many horizontal curves, and which was fixed in section 2.

For $g \in Q_{F\times F}$ and $g \in R_{(\mathbb{A}\times\mathbb{A})^\times}$ the zeta integrals take complex values if converges.

Note that for a function g which is the product of its local components the adelic zeta integral is not the product of the generic local zeta integrals: it differs at singular points of fibres and on horizontal curves. For a subset S_o of S' define similarly $\zeta_{S,S_o}(g,\chi)$.

5.2. Examples. All these formulas are very easy to use for concrete calculations of zeta integrals. For example, for the local nonarchimedean zeta integral

$$\zeta(g,\chi) = (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l} \int_{O^\times\times O^\times} g(t_1^j u_1, t_1^l u_2)\, \chi_0\big(\mathfrak{t}(u_1,u_2)\big) \, d\mu_{F\times F}(u_1,u_2),$$

where $\chi = \chi_0 | \, |_2^s$, χ_0 is of finite order and trivial on $\{t_1, t_2\}$. If, moreover, for fixed j,l the value $g(t_1^j u_1, t_1^l u_2)$ is constant $= g_0(j,l)$, then

$$\zeta(g,\chi) = (1-q^{-1})^{-2} \sum_{j,l\in\mathbb{Z}} (q^{-s})^{j+l} g_0(j,l) \int_{O^\times\times O^\times} \chi_0\big(\mathfrak{t}(u_1,u_2)\big) \, d\mu_{F\times F}(u_1,u_2).$$

Keeping in mind the normalization of the self dual measure $\mu_{x,z}$ we also easily get

$$\zeta(char_{(t_1^{c_{x,z}}O_{x,z},t_1^{c'_{x,z}}O_{x,z})}, |\ |_{2x,z}^{s}, \mu_{x,z}) = q_{x,z}^{d_{x,z}-(c_{x,z}+c'_{x,z})s} \left(\frac{1}{1-q_{x,z}^{-s}}\right)^2.$$

One can show, see [F3], that if $g \in Q_{F\times F}$, then the local zeta integral is a rational function of q^{-s}, and for two such functions f, g one has a local functional equation

$$\zeta(f,\chi)\,\zeta(\widehat{g},\widehat{\chi}) = \zeta(\widehat{f},\widehat{\chi})\,\zeta(g,\chi)$$

where $\widehat{\chi} := |\ |_2^s\,\chi^{-1}$ and where for g its transform $\widehat{g}$ is $\mathcal{F}(g)$ composed with a certain rescaling map.

Instead of integrating over K_2^t-objects we integrate over $K_1 \times K_1$-objects using the morphism $\mathfrak{t}$. The kernel of $\mathfrak{t}$ consists of units which can be ignored as far as the unramified theory is concerned. If one wants to develop a full theory, certain modifications of the previous constructions are required.

There are several alternative approaches to a ramification theory in dimension two, none of which is sufficiently general and satisfactory. A development of a comprehensive theory of ramified $\mathbb{C}((X))$-valued zeta integral is likely to be closely related to such a general higher ramification theory. In relation to the ramification issues see also Theorem 5.5, which contains a new interpretation of the (tame part of the) conductor of elliptic curve.

5.3. Zeta integrals on curves and fibres. If $g = \otimes_{y\in S'} g_y$, where y runs through all horizontal curves in S' and all fibres, then

$$\zeta_{S,S'}(g,\chi) = \prod_{y\in S'} \zeta_y(g_y,\chi), \qquad \zeta_y(g_y,\chi) = \int_{\mathfrak{T}_y} g_y \chi \mathfrak{t}\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}.$$

Note that this integral diverges unless S is a model of elliptic curve over a global field, see below in this subsection, thus in the general case of arithmetic surfaces one will have to renormalize it as explained in the end of this section.

Write the quasi-character χ as $\otimes\chi_{x,y}$. If, furthermore, $g_y = \otimes_{x\in y} g_{x,y}$, then we have the following formulas for fibre and curve integrals.

If y is a nonsingular fibre then $\zeta_y(g_y,\chi) = \prod_{x\in y} \zeta_{x,z}(g_{x,z},\chi_{x,z})$ is the product of the generic local zeta integrals. For a fibre y we get

$$\zeta_y(g_y,\chi) = \prod_{x\in y} \zeta_{x,z}(g_{x,z},\chi_{x,z}),$$

where if x lies on several components/branches then only one local zeta integral $\zeta_{x,z}(g_{x,z},\chi_{x,z})$ participates in the product.

If y is horizontal in characteristic zero then $\zeta_y(g_y, |\ |^s)$ is equal to $\prod_{x\in y} \zeta_{x,z}(g_{x,z}, |\ |^{s/2})$.

If y is horizontal in positive characteristic, introduce an auxiliary zeta integral $\zeta_y^a(g_y, |\ |_2^s) = \int_{T_y} g_y(\alpha)\,|\alpha|^{s/2}\, d\mu_{(\mathbb{A}_y\times\mathbb{A}_y)^\times}(\alpha)$. The latter is the product of $\int_{T_{x,z}} g_y(\alpha)\,|\alpha|^{s/2}\, d\mu_{K_{x,z}^\times\times K_{x,z}^\times}(\alpha)$, to calculate which one can use the formulas for horizontal y in characteristic zero. If $\zeta_y^a(g_y, |\ |_2^s) = \sum_{n\in N_y} c_n n^{-s/2}$, $c_n = \int_{T_{1,y}} g_y(m_n\gamma)\, d\mu_{\mathbb{A}_y\times\mathbb{A}_y)^\times}(\gamma)$, $m_n \in T_y$, $|m_n|_y = n$, then $\zeta_y(g_y, |\ |_2^s) = \sum_{n\in N_y} c_{2n} n^{-s}$.

Denote by $q_{x,z}$ the cardinality of the finite residue field $k_z(x)$ of a nonarchimedean two-dimensional field $K_{x,z}$. For a horizontal curve or a fibre y define

$$\mathfrak{c}_y = \prod_{x\in y} q_{x,z}^{e_{x,z}},$$

where each point x participates only once, and $e_{x,z} = d_{x,z}$ at nonsingular $x \in y$ as fixed in 2.3, and $e_{x,z} = -1$ at singular $x \in y$. From the classical one-dimensional theory it is very easy to see that for a nonsingular fibre $\mathfrak{c}_y$ equals $q_y^{2(1-g_y)}$, where $q_y > 1$ defined in 5.1 is the cardinality of the algebraic closure of $\mathbb{F}_p$ in $k(y)$ and g_y is the genus the curve y. For a singular fibre $\mathfrak{c}_y$ equals $\prod q_{y'}^{2(1-g_{y'})} \prod_x q_x^{-1}$, where y' runs through irreducible components of y, x runs through all singular points of the fibre y, q_x is the cardinality of the residue field at x.

The example in 3.3 discusses an integral over $(\mathbb{A} \times \mathbb{A})^\times$ which is closely related to the adelic zeta integral. In order that the integral converges we need to have $\prod_{x\in y} q_{x,z}^{d_{x,z}} = 1$ for almost every fibre y. Hence g_y should be 1 for almost every fibre y. This explains why we introduce from now on the following restriction.

Now we specialize to the case where $S = \mathcal{E}$ is a regular model of elliptic curve E over a global field k. *We will assume the set S^- of horizontal curves in S' contains the image of the zero section of $\mathcal{E} \to B$. We continue to assume that singular points of its fibres are split ordinary double points.* See the end of this section for a sketch of the general case.

5.4. A centrally normalized function f. Now we define a centrally normalized function $f \in Q_{\mathbb{A}\times\mathbb{A}}$ for which the calculation of the adelic zeta integral is straightforward. We will work with its zeta integral in the sections to follow. Put

$$f = \otimes_{y\in S'} f_y, \qquad f_y = \otimes_{x\in y} f_{x,y}$$

and define the local factors as follows.

For nonarchimedean (x,z) on vertical curves y in a nonsingular fibre and horizontal curves in characteristic zero put $f_{x,z} = char_{(O_{x,z},O_{x,z})}$. Then on a vertical curve y in a nonsingular fibre $f_y = f_y$ and $\mathcal{F}(f_y)(\alpha) = f_y(\nu_y^{-1}\alpha)$ with some $\nu_y \in T_{1,y}$.

Let y be a singular fibre. At singular $x \in y$ put $f_{x,y} = q_x^{-1} char_{(\mathrm{O}_{x,y},\mathrm{O}_{x,y}^{\perp})}$. The transform $\mathcal{F}(f_{x,y})$ is $q_x^{-1} char_{(\mathrm{O}_{x,y}^{\perp},\mathrm{O}_{x,y})}$. The pull-back of $f_{x,y}$ with respect to $\mathfrak{v}$ in 4.3 is $q_x^{-1} char_{(O_{x,z},t_{1,x,z}^{-1}O_{x,z})}$. At appropriate finite number of nonsingular $x \in y$ modify $char_{(O_{x,z},O_{x,z})}$:

$$f_{x,z}(\alpha) = char_{(O_{x,z},O_{x,z})}(\varepsilon_{x,z}\alpha)$$

with $(\varepsilon_{x,z}) \in T_y$, $\varepsilon_{x,z} = (t_{1,x,z}^{-c_{x,z}}, t_{1,x,z}^{-c'_{x,z}})$, such that for $f_y = \otimes_{x\in y} f_{x,y}$ we have

$$\mathcal{F}(f_y)(\alpha) = f_y(\nu_y^{-1}\alpha)$$

where $\nu_y \in T_{1,y}$. Just choose $c_{x,z}, c'_{x,z}$ such that $\prod_{x\in y} q_{x,z}^{d_{x,z}} = \prod_{x\in y} q_{x,z}^{c_{x,z}+c'_{x,z}}$.

On a horizontal curve y in positive characteristic define $f_{x,z}(\alpha) = char_{(O_{x,z},O_{x,z})}(\varepsilon_{x,z}\alpha)$ with $(\varepsilon_{x,z}) \in T_y$ such that for $f_y = \otimes_{x\in y} f_{x,y}$ we have $\mathcal{F}(f_y)(\alpha) = f_y(\rho_y^{-1}\alpha)$ with $\rho_y \in T_{1,y}$.

Using the notation of 3.3, put over archimedean places $f_{\omega,y}^{pr}(\alpha,\beta) = \exp\big(-e_\omega\, \pi\, \big(|p_y(\alpha)|^2 + |p_y(\beta)|^2\big)\big)$, for $(\alpha,\beta) \in \mathcal{O}_{\omega,y} \times \mathcal{O}_{\omega,y}$. For a fixed archimedean σ choose $\eta_{\omega,y} \in \mathbb{R}_{>0}$ equal each other, such that $\prod_\omega \eta_{\omega,y}^{2e_\omega} = \eta_{\omega,y}^{2n} = \mathfrak{c}_y$ where $n = |\mathrm{k} : \mathbb{Q}|$ and $\mathfrak{c}_y$ is defined in 5.3.

For a horizontal y in characteristic zero define f_y as having components $char_{(O_{x,z},O_{x,z})}$ at nonarchimedean data and $f_{\omega,y}(\alpha) = f_{\omega,y}^{pr}((\eta_{\omega,y},\, \eta_{\omega,y})\,\alpha)$ at ω, y. Then on horizontal curves in characteristic zero we have $\mathcal{F}(f_y)(\alpha) = f_y(\rho_y^{-1}\alpha)$ with $\rho_y \in T_{1,y}$. Put $\rho_y = \nu_y$ and define

$$\rho = \otimes_{y\in S'} \rho_y \in T.$$

It is easy to check that f and f belong to the space $Q_{\mathbb{A}\times\mathbb{A}}$ defined in 3.1.

5.5. The first calculation of the zeta integral. Now, using the previous formulas it is easy to obtain the following description of $\zeta(f, |\ |_2^s)$.

THEOREM. *Let $\mathcal{E}$ be a regular model of elliptic curve E over a global field, as in 5.3. Assume in addition that E has good or multiplicative reduction in residue characteristic 2 and 3.*

For every fibre y we have

$$\zeta_y(f, |\ |_2^s) = \mathfrak{c}_y^{1-s} \prod_{x \in y} \left(\frac{1}{1 - q_{x,z}^{-s}} \right)^2.$$

For a nonsingular fibre $\mathfrak{c}_y = 1$. For a singular fibre $y = \mathcal{E}_b$ we get $\mathfrak{c}_y = |k(b)|^{f_b + m_b - 1}$ where m_b is the number of irreducible geometric components of the fibre, and f_b is its conductor.

For every horizontal curve y the zeta integral $\zeta_y(f, |\ |_2^s)$ is a meromorphic function which satisfies the functional equation $\zeta_y(f, |\ |_2^s) = \zeta_y(f, |\ |_2^{2-s})$ and which is holomorphic outside its poles of multiplicity two at $s = 0, 2$ in characteristic zero and at $q_y^s = 1, q_y^2$ in positive characteristic. For a horizontal curve y in characteristic zero the zeta integral $\zeta_y(f, |\ |_2^s)$ is the square of a one-dimensional integral at $s/2$ on $k(y)$.

Recall that the (unramified) Hasse zeta function of a scheme S of finite type is

$$\zeta_S(s) = \prod_{x \in S_0} \left(1 - |k(x)|^{-s}\right)^{-1},$$

where x runs through the set of closed points on S. It is equal to the product $\prod_{b \in B_0} \zeta_{S_b}(s)$, $S_b = S \times_B b$.

If S is an arithmetic surface it is easy to see that $\zeta_S(s)$ absolutely and normally converges on $\Re(s) > 2$. If $\mathcal{E}$ is a model of elliptic curve over a global field then it is well know that $\zeta_{\mathcal{E}}(s)$ extends to a meromorphic function on $\Re(s) > 3/2$ with the only simple pole(s) at $s = 2$ in characteristic zero and $q^s = q^2$ in positive characteristic.

The Hasse-Weil zeta function $\zeta_E(s)$ factorizes as $\prod_{b \in B_0} \zeta_{E_b}(s)$ and $\zeta_{E_b}(s)$ is the Hasse zeta function of the model corresponding to a minimal Weierstrass equation of E at b. If E has a global minimal Weierstrass equation then $\zeta_E(s) = \zeta_{\mathcal{E}_0}(s)$ where $\mathcal{E}_0$ is it the arithmetic scheme corresponding to such an equation.

The previous theorem implies a comparison of the zeta integral and the square of the Hasse function of $\mathcal{E}$ which, in particular, shows the convergence of the zeta integral on the half plane $\Re(s) > 2$.

COROLLARY. *Let $\mathcal{E}$ be as in the theorem. On $\Re(s) > 2$ we get*

$$\zeta_{\mathcal{E},S'}(f, |\ |_2^s) = c_{\mathcal{E},S'}(|\ |_2^s)\, \zeta_{\mathcal{E}}(s)^2, \qquad c_{\mathcal{E},S'}(|\ |_2^s) = c_{\mathcal{E},S^{\shortmid}}(|\ |_2^s)\, c_{\mathcal{E},S^-}(|\ |_2^s).$$

The first factor $c_{\mathcal{E},S^{\shortmid}}(|\ |_2^s)$ is $\mathfrak{c}_{\mathcal{E}}^{1-s}$, the product of $\mathfrak{c}_y^{1-s}$ over all fibres. The second factor is the product of zeta integrals for horizontal curves and hence has a meromorphic continuation to the complex plane and satisfies the functional equation $c_{\mathcal{E},S^-}(|\ |_2^s) = c_{\mathcal{E},S^-}(|\ |_2^{2-s})$ and is holomorphic outside its poles at $s = 0, 2$ in characteristic zero and at $q^s = 1, q^2$ in positive characteristic. The zeta integral $\zeta_{\mathcal{E},S'}(f, |\ |_2^s)$ absolutely and normally converges on $\Re(s) > 2$.

5.6. A very short sketch of the general case. In the general case of S whose generic fibre is a curve of genus $g > 1$ the adelic zeta integral diverges, due to the appearance of the same factor $\mathfrak{c}_y^{1-s}$ for infinitely many vertical curves.

Let $\mathfrak{f}$ be the tensor product of $char_{(O_{x,z},O_{x,z})}$ for $\mathbb{P}^1(B)$. In the general case of S introduce a (renormalized) zeta integral

$$\zeta_{S,S'}(f, |\ |_{2,S}^s) = \prod_{b\in B_0} \zeta_{\mathbb{P}^1(B),\mathbb{P}^1(B)_b}\big(\mathfrak{f}, |\ |_{2,\mathbb{P}^1(B)}^s\big)^{g-1} \zeta_{S,S_b}\big(f, |\ |_{2,S}^s\big) \cdot \prod_{y\in S'\setminus S'} \zeta_{S,y}\big(f, |\ |_{2,S}^s\big).$$

For a nonsingular fibre y the y-factor $\mathfrak{c}_y^{1-s}$ of $\zeta_{S,y}(f, |\ |_{2,S}^s)$ *is cancelled out* by the y-factor of $\zeta_{\mathbb{P}^1(B)}\big(\mathfrak{f}, |\ |_2^s\big)^{g-1}$ which is equal to its inverse, and the product over the fibres in the definition of the renormalized zeta integral converges for $\Re(s) > 2$. Similar to the previous calculation one deduces that for $\Re(s) > 2$ the zeta integral $\zeta_{S,S'}\big(f, |\ |_{2,S}^s\big)$ equals the product of $\zeta_{\mathbb{P}^1(B)}(s)^{2g-2}\,\zeta_S(s)^2$ and of a factor $c_{S,S'}(|\ |_2^s)$ which is the product of exponential and Euler factors for vertical curves in singular fibres and of the factors for horizontal curves in S'.

6. Second calculation of the zeta integral and the boundary term

We continue to assume that $S = \mathcal{E}$ as fixed in 5.3. As in the one-dimensional theory the second calculation of the zeta integral uses the decomposition of $\mathfrak{T}$ into the product of the group $\mathfrak{T}_1$ of elements of module 1 and of the module value group $N = |T|$, and decomposition of $\mathfrak{T}_1$ into the product of T_0 and of $\mathfrak{T}_1/T_0$ where $T_0 = (\mathbb{B} \times \mathbb{B})^\times$, and then when integrating over T_0 one uses the summation formula of 3.5.

6.1. More adelic objects and integrals. Denote by $\mathfrak{T}_1$ the kernel of $||\ ||\circ\colon \mathfrak{T} \longrightarrow \mathbb{R}$ and by T_1 the kernel of $|\ |\circ\colon T \longrightarrow \mathbb{R}$. Similarly define local $T_{1\,x,z}$. Put $N = |T| = ||\mathfrak{T}||$. Via the diagonal map T_0 is a subgroup of T_1. Denote $T_0 = (\mathbb{B} \times \mathbb{B})^\times$.

Denote by UT the intersection of T_1 with the product of the nonarchimedean part of $T \cap \prod T_{1\,x,z}$ and of the archimedean part of T for (ω, y)'s. The group UT is open in T_1.

Since $\mathfrak{t}(\alpha,\beta) \equiv \{\alpha\beta, t_2\} \mod UK_2^t(K_{x,z})$, we have $\mathfrak{t}(T_0 + UT) \subset P + UJ$ and a surjective homomorphism $T_1/(T_0 + UT) \longrightarrow J^1/(P + UJ)$ induced by $\mathfrak{t}$. Thus we have the compatibility of the $K_1 \times K_1$-objects T and T_0 and the K_2-objects J and P, up to units.

From the adelic commutative diagramme in 4.3 we get a commutative diagramme

$$\begin{array}{ccccc} & & \mathbb{B}^\times \otimes \mathbf{B}_{S'}^\times/(\mathbf{B}_{S'}^\times \cap \mathbf{VA}_{S'}^\times) & & \\ & & \downarrow & \searrow & \\ T_0 & \longrightarrow & \mathbb{B}^\times \times \mathbb{B}^\times/(\mathbb{B}^\times \cap V\mathbb{A}^\times) & \longrightarrow & P/(P \cap VJ), \end{array}$$

where the diagonal map is the symbol map, and the maps are the restriction of the appropriate maps in that diagramme.

Let $\mathfrak{M}$ be the unique subgroup of M such that $||\mathfrak{M}|| = |M|$. Then $\mathfrak{T} = \mathfrak{T}_1\mathfrak{M}$. Introduce similar objects for $y \in S'$.

Let y_0 be the image of the zero section, $y_0 \in S^-$; then $N_{y_0} = N$. Choose multiplicative representatives $\mathfrak{m}_n \in \mathfrak{T}_{y_0}$ of N such that $||\mathfrak{m}_n||_{y_0} = n \in N_{y_0} = N$. The groups N, M are locally compact groups. Let μ_N be the appropriate measure on the group N, as in the one-dimensional theory, corresponding to the counting measure in positive characteristic case and corresponding to the induced from $\mathbb{R}^\times$, dn/n in characteristic zero case.

As in dimension one, but with a little more effort, see sect. 43 in [F5], one can define integrals $\int_{\mathfrak{T}_1}$, $\int_{\mathfrak{M}}$, $\int_M$ such that $\int_{\mathfrak{T}} g(\alpha)\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}(\alpha) = \int_{\mathfrak{M}} \big(\int_{\mathfrak{T}_1} g(m\alpha)\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}(\alpha)\big) d\mu_{\mathfrak{M}}(m) = \int_M \big(\int_{\mathfrak{T}_1} g(\mathfrak{m}\alpha)\, d\mu_{(\mathbb{A}\times\mathbb{A})^\times}(\alpha)\big) d\mu_M(m)$, $||\mathfrak{m}|| = |m|$. Let $\mu_{\mathfrak{M}} = ||\ ||^*\mu_N$ and $\mu_M = |\ |^*\mu_N$.

6.2. Two dimensional theta formula. It is easy to see that for a vertical curve y and for the function $g = \otimes char_{(O_{x,y},O_{x,y})}$ the integral $\int_{\mathbb{B}_y\times\mathbb{B}_y} g\, d\mu_{\mathbb{B}_y\times\mathbb{B}_y}$ is $\geqslant q_y^2$, q_y is the cardinality of the maximal finite subfield of $k(y)$. Therefore the product of such fibre integrals for all vertical y diverges. We will have to introduce on T_0 a rescaled measure so that the integral is 1 for almost all vertical curves.

Put $d_y = 1$ for horizontal curves. For a vertical curve y put $d_y = (q_y - 1)^{-2}$. Let S_o be a finite subset of S' which contains all horizontal curves in S' and all vertical curves in singular fibres. Denote $d_{S_o} = \prod_{y\in S_o} d_y$. Suppose that $g = \otimes g_y \in Q_{\mathbb{A}\times\mathbb{A}}$ and there is a finite set of curves outside which $d_y \int g\, d\mu_{(\mathbb{B}_y\times\mathbb{B}_y)^\times} = 1$. Define

$$\int_{T_{0,S_o}} g\, d\mu_{T_0} := d_{S_o} \prod_{y\in S_o} \int_{(\mathbb{B}_y\times\mathbb{B}_y)^\times} g\, d\mu_{\mathbb{B}_y\times\mathbb{B}_y}, \qquad \int_{T_0} g\, d\mu_{T_0} = \lim_{S_o} \int_{T_{0,S_o}} g\, d\mu_{T_0}.$$

Denote $\partial\,(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times := \mathbb{B}_{S_o}\times\mathbb{B}_{S_o} \setminus (\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times$, this is the boundary with respect to the weakest topology of $\mathbb{A}\times\mathbb{A}$ in which every character is still continuous.

Define

$$\int_{\partial T_0} g\, d\mu_{\partial T_0} := \lim_{S_o} d_{S_o} \int_{\partial\,(\mathbb{B}_{S_o}\times\mathbb{B}_{S_o})^\times} g\, d\mu_{\mathbb{B}\times\mathbb{B}}$$

and for two such function g_1, g_2 define $\int_{\partial T_0} (g_1 - g_2)\, d\mu_{\partial T_0} := \int_{\partial T_0} g_1\, d\mu_{\partial T_0} - \int_{\partial T_0} g_2\, d\mu_{\partial T_0}$.

Let f be as fixed in 5.4. If y is a nonsingular fibre then both integrals $\int_{T_{0,y}} f(\beta)\, d\mu_{T_0}(\beta)$, $\int_{T_{0,y}} f(\rho^{-1}\beta)\, d\mu_{T_0}(\beta)$ are equal to 1, where ρ is defined in 5.4. The functions $\beta \mapsto f(\alpha\beta)$ and $\beta \mapsto f(\rho^{-1}\alpha^{-1}\beta)$ satisfy the conditions in the previous paragraph and for them (denote it by g) we get

$$\int_{T_0} g\, d\mu_{T_0} + \int_{\partial T_0} g\, d\mu_{\partial T_0} = \lim_{S_o} d_{S_o} \int_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}} g\, d\mu_{\mathbb{B}\times\mathbb{B}}.$$

Note that the measures μ_{T_0}, $\mu_{\partial T_0}$, and $\lim_{S_o} d_{S_o}\mu_{\mathbb{B}_{S_o}\times\mathbb{B}_{S_o}}$ are not the lifts of the discrete counting measure on $\prod_{y\in S'} k(y)\times k(y)$.

Using the summation formula of 3.5 for infinitely many finite subsets of horizontal and vertical curves in S' we obtain the following *two-dimensional theta formula* for $\alpha \in T$

$$\int_{T_0} f(\alpha\beta)\, d\mu_{T_0}(\beta) - |\alpha|^{-1} \int_{T_0} f(\rho^{-1}\alpha^{-1}\beta) d\mu_{T_0}(\beta)$$
$$= \int_{\partial T_0} \big(|\alpha|^{-1} f(\rho^{-1}\alpha^{-1}\beta) - f(\alpha\beta)\big)\, d\mu_{\partial T_0}(\beta).$$

6.3. The second calculation of the zeta integral. As in dimension one, but with more effort, see sect. 43 of [F5], one can define $\mu_{\mathfrak{T}_1/T_0}$, such that for appropriate functions g

$$\int_{\mathfrak{T}_1} g\, d\mu = \int_{\mathfrak{T}_1/T_0} \int_{T_0} g(\beta\gamma)\, d\mu_{T_0}(\beta)\, d\mu_{\mathfrak{T}_1/T_0}(\gamma).$$

Introduce "halves" of M:

$$M^{\pm} = \{m \in M : \pm(|m| - 1) \geqslant 0\}$$

with the measure induced from M on $M \setminus M \bigcap T_1$ and half of the measure μ_M on $M \bigcap T_1$ for each of M^+ and M^-. So the space M is the disjoint union of the spaces M^- and M^+ which are mapped to each other by the involution $m \mapsto m^{-1}$. Similarly define measure spaces $N^\pm$, $\mathfrak{N}^\pm$, $\mathfrak{M}^\pm$. In particular, in characteristic zero N^- is $(0,1]$ with the measure dx/x and in positive characteristic N^- is $\{q^k, k \leqslant 0\}$ with $\mu_{N^-}(\{q^k\}) = 1$ for $k < 0$ and $\mu_{N^-}(\{1\}) = 1/2$.

For $m \in M$ denote

$$\zeta_m(f, |\ |_2^s) := \int_{\mathfrak{T}_1} f(\mathfrak{m}\alpha)\, |m|^s \, d\mu(\alpha)$$

where, as agreed earlier, $\mathfrak{m} \in \mathfrak{M}$, $||\mathfrak{m}|| = |m|$. Then for $\Re(s) > 2$

$$\zeta(f, |\ |_2^s) = \int_M \zeta_m(f, |\ |_2^s)\, d\mu_M(m),$$

Following the classical path we observe that the absolute convergence of $\zeta(f, |\ |_2^s)$ for $\Re(s) > 2$ implies the absolute convergence of the integral $\int_{M^+} \zeta_m(f, |\ |_2^s)\, d\mu_{M^+}(m)$ in the same area, and therefore this integral as well as the integral $\int_{M^+} \zeta_m(f, |\ |_2^{2-s})\, d\mu_{M^+}(m)$ absolutely converge for all complex s.

Section 45 in [F5] contains the proof of the two-dimensional version of the main result of one-dimensional adelic analysis.

THEOREM. *On the half plane* $\Re(s) > 2$

$$\zeta(f, |\ |_2^s) = \xi(|\ |_2^s) + \xi(|\ |_2^{2-s}) + \omega(|\ |_2^s)$$

where

$$\xi(|\ |_2^s) = \int_{M^+} \zeta_m(f, |\ |_2^s)\, d\mu_{M^+}(m)$$

is an entire function on the complex plane. The boundary term $\omega(|\ |_2^s)$ *for* $\Re(s) > 2$ *is given by*

$$\begin{aligned}
\omega(|\ |_2^s) &= \int_{M^-} \omega_m(|\ |_2^s)\, d\mu_{M^-}(m),\\
\omega_m(|\ |_2^s) &= |m|^s \int_{\mathfrak{T}_1} \big(f(\mathfrak{m}\alpha) - |m|^{-2} f(\mathfrak{m}^{-1}\alpha^{-1})\big)\, d\mu(\alpha)\\
&= |m|^s \int_{\mathfrak{T}_1/T_0} \int_{T_0} \big(f(\mathfrak{m}\gamma\beta) - |m|^{-2} f(\mathfrak{m}^{-1}\gamma^{-1}\beta)\big)\, d\mu_{T_0}(\beta)\, d\mu_{\mathfrak{T}_1/T_0}(\gamma).
\end{aligned}$$

Using the theta formula established earlier we can rewrite it as

$$\begin{aligned}
\omega_m(|\ |_2^s) = {} & |m|^{s-2} \int_{\mathfrak{T}_1} \big(|\alpha|^{-1} - 1\big)\, f(\mathfrak{m}^{-1}\alpha^{-1})\, d\mu(\alpha)\\
&+ |m|^s \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}\gamma|^{-1} f(\mathfrak{m}^{-1}\rho^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}\gamma\beta)\big)\, d\mu_{\partial T_0}(\beta)\, d\mu_{\mathfrak{T}_1/T_0}(\gamma)
\end{aligned}$$

and the integral $\int_{M^-}$ *of first term on the right hand side extends to an entire function on the complex plane.*

Thus, the meromorphic continuation and functional equation of $\zeta(f, |\ |_2^s)$ *and the study of its poles are reduced to establishing those properties for* $\omega(|\ |_2^s)$.

The structure of the boundary term is more complicated and much richer than that in dimension one. Its study is a new challenge.

7. Hypothesis on mean-periodicity and meromorphic continuation

In this section we state a hypothesis about mean-periodicity of a function h. When this hypothesis is proved it will imply meromorphic continuation and functional equation for the zeta function of $\mathcal{E}$. In the case when $\mathrm{k} = \mathbb{Q}$ mean-periodicity of h is implied by modularity of the L-function; and the hypothesis on mean-periodicity can be viewed as a weaker version of the Taniyama conjecture.

7.1. The function h. For every $n \in N$ make in appropriate places change of variable $\gamma \to \gamma^{-1}$ and $\gamma \to \rho\gamma$ and define an important object of study, a function $h(n)$

$$\begin{aligned} h(n) &:= \int_{\mathfrak{T}_1} \big(n^2 f(\mathfrak{m}_n\gamma) - f(\mathfrak{m}_n^{-1}\gamma)\big)\, d\mu(\gamma) \\ &= \int_{\mathfrak{T}_1/T_0} \left(\int_{T_0} \big(n^2 f(\mathfrak{m}_n\gamma\beta) - f(\mathfrak{m}_n^{-1}\rho^{-1}\gamma^{-1}\beta)\big)\, d\mu_{T_0}(\beta) \right) d\mu_{\mathfrak{T}_1/T_0}(\gamma) \\ &= h_1(n) + h_2(n), \\ h_1(n) &= \int_{\mathfrak{T}_1} \big(|\alpha|^{-1} - 1\big)\, f(\mathfrak{m}_n^{-1}\alpha^{-1})\, d\mu(\alpha), \\ h_2(n) &= n^2 \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}_n\gamma|^{-1} f(\mathfrak{m}_n^{-1}\rho^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}_n\gamma\beta)\big)\, d\mu_{\partial T_0}(\beta)\, d\mu_{\mathfrak{T}_1/T_0}(\gamma). \end{aligned}$$

The function $h(n)$ does not depend on the choice of m_n corresponding to n, described in 6.1, and as mentioned above, the integral $\int_{N^-} h_1(n)\, n^{s-2}\, d\mu_{N^-}(n)$ extends to an entire function on the complex plane, see sect. 46 of [F5].

The boundary term $\omega(|\ |_2^s)$ has the following integral presentation for $\Re(s) > 2$

$$\omega(|\ |_2^s) = \int_{N^-} h(n)\, n^{s-2}\, d\mu_{N^-}(n).$$

Hence for $\Re(s) > 2$ $c_0^{-1}\omega(|\ |_2^s)$ equals the *Laplace–Stieltjes transform* $\int_0^\infty e^{-st}\, dj(t)$, where $j(t) = \int_0^t e^{2u}\, h(e^{-u})\, du$ in characteristic zero and $\frac{h(1)}{2}\,\lambda(t) + \sum_{k\geqslant 1} h(q^{-k})\, q^{2k}\, \lambda(t - k\log q)$ in positive characteristic, where $q > 1$ is the generator of N in positive characteristic, $\lambda(t)$ is the characteristic function of positive real numbers.

The function $c_0 h(n)$ is equal to $l(n)n^2 - l(n^{-1})$ where the function $l(n)$ is the inverse Mellin transform of $\zeta(f, |\ |_2^s)$:

$$\int_N l(n)\, n^s\, d\mu_N(n) = \zeta(f, |\ |_2^s).$$

For more explicit formulas for h see the next section.

Immediately from the definition we get the following functional equation

$$h(n^{-1}) = -n^{-2}\, h(n).$$

Hence $e^t h(e^{-t})$ in characteristic zero and $q^t h(q^{-t})$ in positive characteristic are odd functions of t, and $\omega(|\ |_2^{1+s})$ *is the Laplace–Stieltjes transform of an odd function.*

7.2. Mean-periodic functions. Now we discuss the analytic shape of the function h. We are interested to see what should be an additional condition on an odd infinitely differentiable function g so that its Laplace–Stieltjes transform extends to a meromorphic function on the plane satisfying $G(s) = G(-s)$. There is no necessary condition, apparently, but there is a reasonably sufficient one which we describe below. Recall the definition of a *mean-periodic function* in a classical space of functions on $\mathbb{R}$. Start with the space $\mathcal{C}(\mathbb{R})$ of continuous functions on the line endowed with the topology of uniform convergence on compact subsets. A complex valued function g is called mean-periodic in $\mathcal{C}(\mathbb{R})$ if one of the following equivalent conditions is satisfied: a) there exists a closed proper linear subspace of $\mathcal{C}(\mathbb{R})$ which contains all translates of g; b) g is a solution of a homogeneous convolution equation $g * \mu = 0$ where μ is a non-zero element in the dual space of $\mathcal{C}(\mathbb{R})$, i.e. μ is a non-zero complex valued regular Radon measure carried by a compact subset on the line. These two conditions imply and are implied by the following condition c) g is the limit in $\mathcal{C}(\mathbb{R})$ of linear combinations of polynomial exponentials $p_i(x)e^{z_i x}$ each of which is annihilated by μ.

In the case of $\mathcal{C}(\mathbb{R})$ the class of mean-periodic functions is an extension of the class of periodic functions; it is related to but does not contain the class of so called almost periodic functions.

Similarly one gives the definition of a mean-periodic function in other spaces of functions on $\mathbb{R}$ and on $\mathbb{Z}$. An element g of a given functional space X is called mean-periodic if it is a solution of a homogeneous convolution equation $g * \tau = 0$ where τ is a non-zero element of the dual space X^*.

For the theory of mean-periodic functions in $\mathcal{C}(\mathbb{R})$ see [Kh], for a short review see [Me, p.169–181]; for mean-periodic functions in $\mathcal{E}(\mathbb{R})$ see [BG], [BT], [BS]; for more general case of functional spaces see [N1], [N2].

Define the causal function g^+ associated to g

$$g^+(t) = g(t) \text{ for } t > 0, \quad g^+(0) = g(0)/2, \quad g^+(t) = 0 \text{ for } t < 0.$$

If g is mean-periodic in X and if g is of finite exponential growth then $g^+ \circ \tau \in X^*$ and for sufficiently large $\Re(s)$ its Laplace–Stieltjes transform equals to the function

$$G(s) = \frac{\displaystyle\int_{-\infty}^{\infty} g^+ \circ \tau(t)\, e^{-st}\, dt}{\displaystyle\int_{-\infty}^{\infty} \tau(t)\, e^{-st}\, dt}.$$

This does not depend on the choice of $\tau \neq 0$. Both the numerator and denominator extend to entire functions on the plane, and hence $G(s)$ has meromorphic extensions to the plane. It is called the *Laplace–Stieltjes–Carleman transform* of g.

In the case of an odd mean-periodic function g the function $G(s)$ is a symmetric: $G(-s) = G(s)$.

7.3. A hypothesis on mean-periodicity.

Now we state

HYPOTHESIS. *The zeta integral $\zeta(f, |\ |_2^{1+s})$ is the sum of an entire symmetric function and the Laplace–Stieltjes–Carleman transform of an odd mean-periodic function in an appropriate functional space.*

More precisely, the spaces $X = \mathcal{C}^{\infty}_{\exp}(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$ of exponential growth and $X = \mathfrak{F}(\mathbb{Z})$ of functions on $\mathbb{Z}$ should do the job in characteristic zero and positive characteristic.

These mean-periodic function and entire function are then uniquely determined.

If the hypothesis were proved then by the above discussion the zeta integral and hence the square of the zeta function of $\mathcal{E}$ would have meromorphic extension to the plane and satisfy appropriate functional equations.

Keeping in mind the material of the previous section, to verify the hypothesis one only needs to show that the boundary term $\omega(|\ |_2^{1+s})$ is the sum of a symmetric entire function and the Laplace–Stieltjes–Carleman transform of an odd mean-periodic function.

Recall that one can define the Hasse–Weil zeta function

$$\zeta_E(s) = \frac{\zeta_{\mathbb{P}^1(B)}(s)}{L_E(s)} = \frac{\zeta_{\mathrm{k}}(s)\,\zeta_{\mathrm{k}}(s-1)}{L_E(s)}$$

of E which depends only on the generic fibre of $\mathcal{E}$, unlike the zeta function $\zeta_{\mathcal{E}}(s)$, see [Se2–3]. See 5.5 for its description as the Hasse zeta function of a model of E. Denote by m_b the number of irreducible components without multiplicities in the geometric fibre $\mathcal{E}_b$ over a closed point b of B; so $m_b = 1$ for almost all b. Then

$$\zeta_{\mathcal{E}}(s) = n_{\mathcal{E}}(s)\,\zeta_E(s), \quad n_{\mathcal{E}}(s) = \prod_{b\in B_0, 1\leqslant i\leqslant n_b} (1-|k(b)|^{n_{i,b}(1-s)})^{-1},$$

where if $m_b \neq 1$ then $n_{i,b}$ are certain positive integers, $1 \leqslant i \leqslant n_b$ (n_b is the number of irreducible components in the singular fibre $\mathcal{E}_b$ with the component intersecting the zero section excluded), such that $\sum_{1\leqslant i\leqslant n_b} n_{i,b} = m_b - 1$. This easily follows from [Li, Thms 3.7, 4.35 in Ch. 9 and sect. 10.2.1 in Ch. 10], see also [Se2–Se3], [T2], and for a cohomological interpretation [Bl2, p.300]. In particular, $n_{\mathcal{E}}(s)$ and $n_{\mathcal{E}}(s)^{-1}$ are holomorphic for $\Re(s) > 1$.

Thus in view of the previous discussion we get (see also sect. 48 in [F5])

THEOREM. *Let $S = \mathcal{E}$ be as above. Suppose that the function*

$$H(t) = \begin{cases} h(e^{-t}), t \in \mathbb{R}, & \textit{in characteristic zero,} \\ h(q^{-t}), t \in \mathbb{Z}, & \textit{in positive characteristic,} \end{cases}$$

is a mean-periodic function in an appropriate functional space on $\mathbb{R}$ and $\mathbb{Z}$.

Then the zeta integral $\zeta(f, |\ |_2^s)$ of $\mathcal{E}$ and the zeta function of $\mathcal{E}$ and of E extend meromorphically to the plane and satisfy the functional equations:

$$\zeta_{\mathcal{E}}(f, |\ |_2^s) = \zeta_{\mathcal{E}}(f, |\ |_2^{2-s}),$$
$$\mathfrak{c}_{\mathcal{E}}(s)\,\zeta_{\mathcal{E}}(s)^2 = \mathfrak{c}_{\mathcal{E}}(2-s)\,\zeta_{\mathcal{E}}(2-s)^2,$$
$$m_E(s)\,\zeta_E(s)^2 = \zeta_E(2-s)^2,$$

where $\mathfrak{c}_{\mathcal{E}}(s) = c_{\mathcal{E},S'}(|\ |_2^s)$ is the vertical part of $c_{\mathcal{E},S'}(|\ |_2^s)$ defined in Corollary in 5.5*,*

$$m_E(s) = \frac{c_{\mathcal{E}}(s)}{c_{\mathcal{E}}(2-s)}\,\frac{n_{\mathcal{E}}(s)^2}{n_{\mathcal{E}}(2-s)^2} = N(\mathrm{cond}_E)^{2-2s}$$

where $N(\mathrm{cond}_E) = \prod |k(b)|^{f_b}$ is the norm of the conductor of E, see 5.5.

To establish the mean-periodicity of H further study of the integral over ∂T_0 will be useful.

7.4. Remarks. 1. Pursuing an analogy with L-functions, the Laplace transform of h in some sense corresponds to the Mellin transform for a modular L-function, and the functional equation of h and mean-periodicity of H is in some sense a weak analogue of the automorphic property of the L-function.

2. In positive characteristic the known rationality in q^s and functional equation of the zeta function imply the mean-periodicity of H in the space $\mathfrak{F}(\mathbb{Z})$.

3. The functional equations of $\zeta_E(s)$ and $\zeta_{\mathcal{E}}(s)$ do not involve Γ-functions. In characteristic zero the completed L-function $\Lambda_E(s)$ is the product of $L_E(s)$ and a certain factor $\Gamma_E(s)$, see e.g. [Se3]. However, the ratio $\Gamma_E(s)/\Gamma_{\mathbb{P}^1(B)}(s)$, where $\Gamma_{\mathbb{P}^1(B)}(s) = \Gamma_k(s)\Gamma_k(s-1)$, is a simple rational function; for example $(s-1)/(4\pi)$ if $\mathrm{k} = \mathbb{Q}$. Its square is invariant with respect to $s \to 2-s$. Thus, the factor $\Gamma_E(s)$ in the functional equation of the denominator $L_E(s)$ of the zeta function is essentially due

to the factor $\Gamma_{\mathbb{P}^1(B)}(s)$ in the functional equation of the numerator of the zeta function. Using the previous Theorem and the formula for $c_{\mathcal{E},S'}(s)$ in 5.5, one can easily see that the mean-periodicity of $H(t)$ implies the functional equation of L_E^2 with its conjectured exponential factor exactly as described in [Se2–Se3]. For more detail see 5.4 of [FRS].

4. An analogue of this theorem in the case of an arbitrary regular model S of a curve of genus $g > 1$ would be

$$\zeta_{S,S'}(f, |\,|_2^s) = \zeta_{S,S'}(f, |\,|_2^{2-s}),$$
$$c_S(s)\left(\zeta_S(s)\,\zeta_{\mathbb{P}^1(B)}(s)^{g-1}\right)^2 = c_S(2-s)\left(\zeta_S(2-s)\,\zeta_{\mathbb{P}^1(B)}(2-s)^{g-1}\right)^2,$$

the factor $c_S(s)$ is the product of exponential factors associated to curves in singular fibres, see sect. 57 of [F5] for more details.

5. In characteristic zero work [FRS] shows that if the zeta-function of E has meromorphic continuation of expected shape and satisfies the functional equation then the corresponding functions h and H are indeed mean-periodic as elements of appropriate functional spaces. In particular the function H is mean-periodic in the space $\mathcal{C}^\infty_{\exp}(\mathbb{R})$. More generally, [FRS] demonstrates new links between Hasse zeta functions which have meromorphic continuation of expected shape and satisfy the functional equation and mean-periodic functions in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$. Let $g(x)$ is the inverse Mellin transform of the product of an appropriate positive power of the completed Riemann zeta function and the completed Hasse zeta function of an arithmetic surface which is rescaled to have the expected functional equation with respect to $s \to 1-s$ with sign ϵ. Then $g(e^{-t}) - \epsilon e^t g(e^t)$ is a mean-periodic function in $\mathcal{C}^\infty_{\exp}(\mathbb{R})$ if and only if the completed zeta function extends to a meromorphic function of expected shape and satisfied the functional equation.

For modular curves the convolutor for H can be obtained using the Connes–Soulé approach ([C], [So], [D]) to zeros of $GL(2)$ cuspidal automorphic representations, which in this sense is dual to the two-dimensional commutative theory, see [Su2].

8. Monotone behaviour and poles of the zeta integral

In this section we will assume without loss of generality that S' contains one horizontal curve, the image of the zero section.

We study the asymptotic behaviour of $h(n)$ near 0. The weak boundary ∂T_0 of the space T_0 is very large in dimension two, which is likely to result in more decent (monotone) behaviour of certain integrals of functions on it, and functions associated to h, and therefore to give more information on the location of poles of the zeta integral which essentially correspond to zeros of the L-function.

For $n \in N$ define

$$\operatorname{Log}(n) = \begin{cases} \log n & \text{in characteristic zero,} \\ \log_q(n) & \text{in positive characteristic.} \end{cases}$$

In positive characteristic define the derivative of a function $g\colon \mathbb{Z} \longrightarrow \mathbb{R}$ as $g'(k) = g(k) - g(k-1)$.

The singular behaviour of $\int_{N^-} h(n)\, n^{s-2}\, d\mu_{N^-}(n)$ at $s = 2$ (resp. $q^s = q^2$ in positive characteristic) corresponds to the singular behaviour of the zeta integral and hence its first pole from the right is at $s = 2$ of order 4. Let its principal part at $s = 2$ be $\sum_{1\leqslant i\leqslant 4} a_i(s-2)^{-i}$ with nonzero a_4. Let $\mathfrak{w}(t) = \sum_{0\leqslant i\leqslant 3} c_i t^i$ be a polynomial of degree 3 such that $\int_{N^-} \mathfrak{w}(-\operatorname{Log} n)\, n^{s-2}\, d\mu_{N^-}(n)$ equals this principal part. Put $c = c_3$. It is then easy to deduce that for the function $H(t)$ defined in the previous section ($t \in \mathbb{Z}$ in positive characteristic) we have

$$H(t) - \mathfrak{w}(t) \to 0, \qquad t \to \infty,$$
$$H^{(i)}(t) - \mathfrak{w}^{(i)}(t) \to 0, \qquad t \to \infty, i \geqslant 1.$$

In particular, in positive characteristic the third derivative $h(n) - 3h(nq) + 3h(nq^2) - h(nq^3)$ tends to $6c$ when $n \to 0$ and in characteristic zero the third derivative $h(e^{-t})'''$ tends to $6c$ when $t \to \infty$. We also deduce that $H(t)$ and its first derivatives are monotone for all sufficiently large t.

8.1. The monotone behaviour of the third derivative of $H(t)$. We can ask a natural question about the monotone behaviour of the third derivative of $H(t)$ near infinity.

HYPOTHESIS $(*) = (*)_{\mathcal{E}}$. *The fourth derivative of $H(t)$ keeps its sign for all sufficiently large t.*

In characteristic zero this hypothesis can be translated into the following. Let k be of a number field. Let Θ be the the theta function associated to k, so if k = $\mathbb{Q}$ then $\Theta(x)$ is the classical $\vartheta(x) = \sum_{k\in\mathbb{Z}} \exp\big(-\pi k^2 x\big)$.

For positive real a, b denote

$$w_{a,b}(x) = \big(\Theta(x^{-2}a^2) - 1\big)\big(\Theta(x^{-2}b^2) - 1\big) - x^2\big(\Theta(x^2a^2) - 1\big)\big(\Theta(x^2b^2) - 1\big).$$

For $\nu > 0$ denote

$$V_\nu(x) = \int_0^\infty w_{a,\nu a^{-1}}(x)\,\frac{da}{a}, \qquad Z_\nu(e^{-t}) = V_\nu(e^{-t})'''' = \left(x\frac{d}{dx}\right)^4 V_\nu(x)\Big|_{x=e^{-t}}.$$

Denote by $c(\nu)$ the coefficients of the generalized Dirichlet series

$$\zeta_{\mathcal{E},S'}(f, |\ |_2^s) = \zeta_{\mathcal{E}}(s)^2\mathfrak{c}^{1-s} = \sum_{\nu\in\mathfrak{c}_{\mathcal{E}}\mathbb{N}} c(\nu^2)\nu^{-s}$$

where $\mathfrak{c}_{\mathcal{E}}$ is defined in section 5, for all fibres. One can show that

$$h(x) = -\mathfrak{c}\sum_{\nu\in\mathfrak{c}_{\mathcal{E}}\mathbb{N}} c(\nu^2)\,V_{\nu^2}(x),$$

where $\mathfrak{c}$ is the square of the one-dimensional normalized measure of the norm one idele class group of the field $k(y_0)$, see sect. 51–52 of [F5]. This implies that $h(e^{-t})'''' = -\mathfrak{c}\,Z(e^{-t})$, where

$$Z(x) = Z_{\mathcal{E}}(x) = \sum_{\nu\in\mathfrak{c}_{\mathcal{E}}\mathbb{N}} c(\nu^2)\,Z_{\nu^2}(x).$$

Hence the hypothesis $(*)$ is equivalent to permanence of the sign of $Z(x)$ in some open interval $(0, x_c)$, with x_c depending on the sequence $c(\nu)$.

See [Su1] for various analytic aspects of condition $(*)$.

8.2. A presentation of h as a Bessel series. Condition $(*)$ involves a modification of the Dirichlet series associated to $\mathcal{E}$, $\nu^{-s} \to Z_\nu(x)$, using relatively nicely behaved functions coming from a horizontal curve on $\mathcal{E}$. Let, for simplicity, k = $\mathbb{Q}$. Let K_0 be the Bessel function

$$K_0(x) = \frac{1}{2}\int_0^\infty e^{-x(t+\frac{1}{t})/2}\,\frac{dt}{t}.$$

Then

$$\int_0^\infty \big(\vartheta(xa^2) - 1\big)\big(\vartheta(x\nu^2a^{-2}) - 1\big)\,\frac{da}{a} = 4\sum_{l_1,l_2\geqslant 1} K_0(2\pi l_1 l_2 \nu x) = 4\sum_{l\geqslant 1}\sigma_0(l)K_0(2\pi l\nu x),$$

where σ_0 is the number of positive divisors. We get

$$V_\nu(x) = 4\sum_{l\geqslant 1}\sigma_0(l)\big(K_0(2\pi l\nu x^{-2}) - x^2K_0(2\pi l\nu x^2)\big).$$

and using the formula for $h(x)$ above one gets the explicit presentation of the function $h(x)$ in 0.4 involving the K_0-Bessel functions.

Now we have easy equalities

$$\big(x\frac{d}{dx}\big)^4\big(x^2K_0(ax^2)\big) = a^{-1}\mathcal{K}_1(ax^2),$$
$$\mathcal{K}_1(x) = \big(16x + 288x^3 + 16x^5\big)K_0(x) - \big(64x^2 + 128x^4\big)K_1(x)$$

with K_1-Bessel function involved. Define

$$\widetilde{Z}_\nu(x) = -\frac{2}{\pi\nu}\sum_{l\geqslant 1}\frac{\sigma_0(l)}{l}\,\mathcal{K}_1(2\pi l\nu x^2), \quad \widetilde{Z}_\nu(x) = \frac{1}{\nu}\widetilde{Z}_1\big(x\sqrt{\nu}\big),$$
$$\widetilde{Z}(x) = \sum_{\nu\in\mathfrak{c}\mathbb{N}} c(\nu^2)\,\widetilde{Z}_{\nu^2}(x) = \sum_{\nu\in\mathfrak{c}\mathbb{N}}\frac{c(\nu^2)}{\nu^2}\,\widetilde{Z}_1\big(x\,\nu\big).$$

It is easy to show that the behaviour of $Z_\nu(x)$ and $Z(x)$ when $x\to 0$ is determined by the behaviour of $\widetilde{Z}_\nu(x)$, $\widetilde{Z}(x)$.

Denote

$$z_E(x) = \sum_{n\geqslant 1} c(n^2)Z_{n^2}(x), \quad \text{where} \quad \sum_{n\geqslant 1}\frac{c(n^2)}{n^s} = \zeta_E(s)^2$$
$$Z_E(x) = \sum_{n\in\mathfrak{c}_E\mathbb{N}} d(n^2)Z_{n^2}(x), \quad \text{where} \quad \sum_{n\in\mathfrak{c}_E\mathbb{N}}\frac{d(n^2)}{n^s} = \mathfrak{c}_E^{1-s}\,\zeta_E(s)^2$$

where $\mathfrak{c}_E$ is the norm of the conductor of E. Then when $x\to 0$ the function $Z_E(x)$ is essentially $\widetilde{Z}_E(x)$ which is defined similar to $Z(x)$ above, and hence when $x\to 0$ the function $Z_E(x)$ is essentially $\mathfrak{c}_E^{-1}z_E(\mathfrak{c}_E x)$.

8.3. Computational results. The values of z_E for elliptic curves over $\mathbb{Q}$ of conductor 11, 14 and 15; see [F6] for more data.

0.0001	-0.02209936	-0.00430284	-0.01224467
0.00015	-0.03239639	-0.01762461	-0.00690680
0.0002	-0.04033723	-0.02019897	-0.03442790
0.00025	-0.06067197	-0.03836163	-0.02397859
0.0003	-0.05233928	-0.03681643	-0.00666095
0.00035	-0.05038168	-0.04857686	-0.05275435
0.0004	-0.07032783	-0.02993865	-0.08676343
0.0005	-0.12873965	-0.06429276	-0.01910193
0.0006	-0.08550765	-0.07625369	-0.01575656
0.0007	-0.05162608	-0.08042476	-0.09561303
0.0008	-0.16379852	-0.07117065	-0.14347671
0.0009	-0.24833464	-0.09863808	-0.09859195

0.001	-0.23124762	-0.13265245	-0.02801169
0.002	-0.37692750	-0.25255845	-0.07654677
0.003	-0.36465433	-0.30605619	-0.17510329
0.004	-0.50204280	-0.24187774	-0.22879872
0.005	-0.37769569	-0.24376896	-0.52118535
0.006	-0.60951615	-0.56471421	-0.26714385
0.007	-1.27808818	-0.67356415	-0.37544907
0.008	-0.61599746	-0.16889861	-0.24828527
0.009	-0.47875874	-0.52690224	-0.27983453
0.01	-1.06504316	-0.90987143	-0.93162037
0.02	-2.04246175	-2.04235986	-1.62653748
0.03	-2.78921872	-1.34496918	-0.27099005
0.04	-4.31556183	-2.22605813	-3.21836209
0.05	+1.21262051	-2.20966702	-1.14605886
0.06	-2.76069819	-2.32130104	-0.44322476
0.07	-5.73792947	+1.29951014	-1.21297325
0.08	-7.83473755	-3.04100641	-4.37079971
0.09	-4.32440954	-6.03835417	-4.03242881
0.1	+4.05851635	-2.82394023	+0.94470596

8.4. The role of hypothesis $(*)$**.** In positive characteristic sect. 53 of [F5] contains a similar more explicit description of $H^{(4)}(t)$ and the proof of the following property: Suppose that there are no poles of the zeta function $\zeta_{\mathcal{E}}(s)$ inside the strip $1 < \Re(s) < 2$ and that the order of the pole at $q^s = q$ is greater than the order of any other s with $q^s \neq q$, $\Re(s) = 1$. Then $H^{(4)}(t)$ keeps its sign for all sufficient large t.

THEOREM. *In characteristic zero let $r(t)$ be the second derivative with respect to t of $h(n) + c\,\mathrm{Log}\,(n)^3$, $n = e^{-t}$.*

In positive characteristic let

$$r(t) = \frac{\mathfrak{h}(1)}{2}\,\lambda(t) + \big(\mathfrak{h}(q^{-1}) - \frac{3\mathfrak{h}(1)}{2}\big)\,\lambda(t-\log q) + \big(\mathfrak{h}(q^{-2}) - 3\mathfrak{h}(q^{-1}) + \frac{3\mathfrak{h}(1)}{2}\big)\,\lambda(t-2\log q)$$
$$+ \big(\mathfrak{h}(q^{-3}) - 3\mathfrak{h}(q^{-2}) + 3\mathfrak{h}(q^{-1}) - \frac{\mathfrak{h}(1)}{2}\big)\,\lambda(t-3\log q)$$
$$+ \sum_{k\geqslant 4}\big(\mathfrak{h}(q^{-k}) - 3\mathfrak{h}(q^{-k+1}) + 3\mathfrak{h}(q^{-k+2}) - \mathfrak{h}(q^{-k+3})\big)\,\lambda(t-k\log q),$$

where $\mathfrak{h}(n) = h(n) + c(\mathrm{Log}\,n)^3$ and the function $\lambda(t)$ is the characteristic function of positive integers.

Suppose that hypothesis $(*)$ holds. Then in every characteristic $r(t)$ is monotone for all sufficiently large t.

Denote by x_0 the abscissa of convergence of

$$R(s) = \int_0^{\infty} e^{-(s-2)t}\,dr(t).$$

Then $x_0 < 2$ and x_0 is a real pole of $R(s)$. The boundary term $\omega(|\ |_2^s)$ and zeta integral $\zeta(f, |\ |_2^s)$ extend meromorphically to the right half plane $\Re(s) > x_0$, have a real pole x_0 and do not have poles inside the strip $\Re(s) \in (x_0, 2)$.

Proof. The integral $\int_{N^-} \big(h(n) + c\, \mathrm{Log}\,(n)^3\big)\, n^{s-2}\, d\mu_{N^-}(n)$ multiplied by $(s-2)^3$ in characteristic zero equals $\int_0^\infty e^{-(s-2)t}\, dr(t) - \sum_{0\leqslant j\leqslant 2}(s-2)^{2-j}\big(h(e^{-t}) - ct^3\big)^{(j)}(0)$. This integral multiplied by $(1-q^{2-s})^3$ in positive characteristic equals $\int_0^\infty e^{-(s-2)t}\, dr(t)$ where $r(t)$ is as above. In each case $R(s)$ is a holomorphic function near $s = 2$ (resp. all s such that $q^s = q^2$ in positive characteristic). Hence its abscissa of convergence x_0 is smaller than 2.

If $r(t)$ is monotone for t close to infinity, then the classical properties of the Laplace–Stieltjes transform of monotone functions imply that $R(s)$ has a real singular point x_0 on its line of convergence $\Re(s) = x_0$ and is holomorphic in $\Re(s) > x_0$. Thus, x_0 is a real pole of $R(s)$. Using the relation between $R(s)$ and $\omega(|\ |_2^s)$, and the zeta integral, we deduce that all these functions are holomorphic inside the strip $\Re(s) \in (x_0, 2)$. See also sect. 54 in [F5].

8.5. From the relation between the zeta integral and zeta function and the relation between the zeta function and L-function in the previous section we obtain

COROLLARY. *Let the assumptions of the theorem hold. Assume that the zeta function $\zeta_{\mathcal{E}}(s)$ (or equivalently, $L_E(s)$) extends to a meromorphic function on the half-plane $\Re(s) > 1$. Suppose that $\zeta_{\mathcal{E}}(s)$ (resp. $L_E(s)$) has no real poles (resp. real zeros) in $(1,2)$. Then the the zeta integral $\zeta(f, |\ |_2^s)$ does not have complex poles with $\Re(s) \in (1,2)$.*

Assume, in addition, that the zeta function $\zeta_{\mathcal{E}}(s)$ extends to a meromorphic function on the plane and satisfies the functional equation, then the poles of $\zeta(f, |\ |_2^s)$ inside the critical strip $\Re(s) \in (0,2)$ lie on the critical line $\Re(s) = 1$.

In dimension one it is elementary to show that the zeta function does not have real zeroes in the critical strip outside the critical line. In contrast, the real zeros part of the Riemann hypothesis for the L-function of elliptic curves E over number fields is not known in general. However, from the computational point of view it is not difficult to check the real part for a given L-function. For computational results on low lying zeros (including real zeros) of L-functions of elliptic curves over rationals of conductor < 8000, see [R]. They imply the real part of the Riemann hypothesis for those curves.

Thus, if condition $(*)$ holds for any of those elliptic curves then the Riemann hypothesis holds for poles of $\zeta_{\mathrm{k}}(s/2)\,\zeta_{\mathcal{E}}(s)$ and $\zeta_{\mathrm{k}}(s/2)\,\zeta_{\mathrm{k}}(s)\,\zeta_{\mathrm{k}}(s-1)/L_E(s)$.

M. Suzuki proved the following result in [Su1] which gives an inverse result to the preceding statement. Suppose that L_E extends to an entire function satisfying the functional equation and let the Riemann hypothesis hold for the L-function. If all nonreal zeros of $L_{\mathcal{E}}(s) = L_E(s) n_{\mathcal{E}}(s)^{-1}$ on the critical line are single, and if the estimate $\sum_{0<\Im(z)\leqslant x} |L'_{\mathcal{E}}(z)|^{-2} = O(x)$ holds, where z runs through all zeros of $L_{\mathcal{E}}(s)$ on the critical line, then the function $Z_{\mathcal{E}}(x)$ is negative for all sufficiently small positive x. If all nonreal zeros of $L_E(s)$ on the critical line are single, $L_E(1) = 0$ and if the estimate $\sum_{0<\Im(z)\leqslant x} |L'_E(z)|^{-2} = O(x)$ holds, where z runs through all zeros of L_E on the critical line, then the function $Z_E(x)$ is negative for all sufficiently small positive x.

Even if the analytic rank of E is zero, the computations in 8.3 indicate that sometimes the function $Z_E(x)$ tends to keep its sign near zero.

9. Boundary integral and the pole at $s = 1$

Recall (section 2) that there are three levels of objects associated to S: the full adelic space $\mathbf{A}$, the space $\mathbf{B}$ which has a feature of both local and global objects, and finally the countable object K. In the preceding study of the zeta function of the surface (and the class field theory of the field of rational functions on the surface) the objects related to the first two levels play a dominant role. This corresponds to the work with 0-cycles on S, and with structures of codimension two or integral structures of rank two. Now we will involve the bottom level K and its relation to the middle level $\mathbf{B}$ as well. The study of functoriality issues is usually quite separated from the study of special values of zeta functions, and in the terminology of this work they proceed on different levels of adelic objects. Using the objects of all three levels and the integral representation of the boundary term this section outlines a new method to settle the equality of the analytic and arithmetic ranks of the zeta function of $\mathcal{E}$ at $s = 1$.

We assume that $\mathcal{E}$ is as fixed in 5.3. In addition, we assume that the boundary term $\omega(|\ |_2^s) = \int_{N^-} h(n)n^{s-2}\, d\mu_{N^-}(n)$ and hence $\zeta_{\mathcal{E}}(s)$ and $L_E(s)$ have analytic continuation and satisfy the functional equation.

As in 6.3 we get $\omega(|\ |_2^s) = \int_{M^-} \omega_m(|\ |_2^s)\, d\mu_{M^-}(m)$ where $\omega_m(|\ |_2^s) = \omega_m^{(1)}(|\ |_2^s) + \omega_m^{(2)}(|\ |_2^s)$ and

$$\omega_m^{(1)}(|\ |_2^s) = |m|^{s-2} \int_{\mathfrak{T}_1} f(m^{-1}\alpha^{-1})\big(|\alpha|^{-1} - 1\big)\, d\mu(\alpha),$$

$$\omega_m^{(2)}(|\ |_2^s) = |m|^{s} \int_{\mathfrak{T}_1/T_0} \int_{\partial T_0} \big(|\mathfrak{m}\gamma|^{-1} f(\mathfrak{m}^{-1}\rho^{-1}\gamma^{-1}\beta) - f(\mathfrak{m}\gamma\beta)\big)\, d\mu_{\partial T_0}(\beta)\, d\mu_{\mathfrak{T}_1/T_0}(\gamma).$$

Recall that the integral $\int_{M^-} \omega_m^{(1)}(|\ |_2^s)\, d\mu_{M^-}(m)$ extends to an entire function on the complex plane, see 7.1.

Let r be the arithmetic rank of E.

Choose horizontal curves y_i, $i \in I$, $|I| = r + 1$, which include the image of the zero section of $\mathfrak{p}\colon \mathcal{E} \longrightarrow B$, and the curves corresponding to a choice of free generators of the group $E(\mathrm{k})$, so r is the arithmetic rank of the E. Take $S^- = \{y_i : i \in I\}$.

For every singular fibre take all its components, except one which intersects the zero section, and denote them y_j, $1 \leqslant j \leqslant \sum n_b$, n_b as in 7.3, b runs through closed points in the base for which the fibre $\mathcal{E}_b$ is singular. In addition, in positive characteristic choose one nonsingular fibre y_*. Denote the whole collection by y_j, $j \in J$.

Now we sketch a new method to prove the equality of the arithmetic and analytic ranks of E which is based on the theory of this work. We would like to show that the order of the pole of the zeta integral at $s = 1$ equals $2|I| + 2|J|$. If so, then using the comparison of the zeta integral $\zeta_{\mathcal{E},S'}(f, |\ |_2^s)$ with the zeta function of $\mathcal{E}$ in section 5, and the relation between the zeta function of $\mathcal{E}$ and L-function of E in section 7, we would obtain that the analytic rank of E equals r. The desired equality would follow from (a) and (b).

(a) The Picard group of $\mathcal{E}$ is isomorphic to the cokernel of the map $K^\times \longrightarrow \mathbf{B}_{\mathcal{E}}^\times/\big(\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{V}\mathbf{A}_{\mathcal{E}}^\times\big)$. The quotient of the Picard group of $\mathcal{E}$ by the image of the Picard group of B with respect to $\mathfrak{p}^*$ is a finitely generated group, which in positive characteristic is the quotient of the Neron–Severi group of $\mathcal{E}$ modulo its subgroup generated by y_*. So we obtain that $\mathbf{B}_{\mathcal{E}}^\times$ is generated by the product of $\mathbf{B}_{\mathcal{E}}^\times \cap \mathbf{V}\mathbf{A}_{\mathcal{E}}^\times$, of a discrete group, of a compact group, corresponding to $\mathfrak{p}^*\mathrm{Pic}(B)$ in characteristic zero and to $\mathfrak{p}^*\mathrm{Pic}^0(B)$ in positive characteristic, and of the images of $\mathbf{B}_{y_i}^\times$, $i \in I$ and $\mathbf{B}_{y_j}^\times$, $j \in J$.

Recall that in positive characteristic the rank of the free part of the Neron–Severi group of $\mathcal{E}$ is equal to $r+2+\sum n_b$, see [T2] and [Shi].

(b) Using the commutative diagramme in 6.1 and the description of $\mathbf{B}_{\mathcal{E}}^{\times}$, the pole of the integral $\int_{M^-} \omega_m^{(2)}(|\ |_2^s)\, d\mu_{M^-}(m)$ at $s=1$ is expected to be equal to the pole at $s=1$ of the integral in which the internal integral $\int_{\partial T_0}$ is replaced by $\int_{\partial \mathrm{T}_0}$ where T_0 is the image of the product of T_{0,y_i}, T_{0,y_j}. Using all this, the order of the pole at $s=1$ of $\omega(|\ |_2^s)$ is expected to be equal to the order of the pole at $s=1$ of the product of the zeta integrals $\zeta_{\mathcal{E},y_i}(f,|\ |^s)$, $i\in I$, and of $\zeta_{\mathcal{E},y_j}(f,|\ |^{1+s})$, $j\in J$, i.e. to $2(|I|+|J|)$.

See [F8] for more detail.

10. Automorphic functions of the surface $\mathcal{E}$

It is an important fundamental open problem to understand what is a two-dimensional analogue of spaces $\mathbb{A}_k^{\times}/k^{\times}$, $G(O\mathbb{A}_k)\backslash G(\mathbb{A}_k)/G(k)$ where $O\mathbb{A}_k$ are integral adeles.

In dimension two unlike in dimension one we cannot go directly, in one step, from the level of $K=k(S)$ to the level of the full adelic space $\mathbf{A}_S$. For some of the first attempts, which used the levels $\mathbf{A}$ and $\mathbf{B}$ only, but not the level of K, see [Ka1], [G], see also [Sh].

We now suggest a candidate for an object functions on which can be viewed as automorphic functions of $\mathcal{E}$. This object comes from the study of the zeta integral in the previous sections.

Let the set S' of curves on S be sufficiently large to include all vertical curves and the curves y_i as in the previous section. Denote

$$\mathbb{T}_{\times} = \mathbb{A}^{\times}\times\mathbb{A}^{\times}/V(\mathbb{A}^{\times}\times\mathbb{A}^{\times}),$$

where $V(\mathbb{A}^{\times}\times\mathbb{A}^{\times})$ is defined similar to $V\mathbb{A}$ in subsection 4.3. Let $\mathbb{K}_{\times}$ be the image of $\mathbb{B}^{\times}\times K^{\times}$ with respect to the map

$$\mathbb{B}^{\times}\times K^{\times}\longrightarrow \mathbb{A}^{\times}\times\mathbf{A}^{\times}/\mathbf{VA}^{\times}\longrightarrow \mathbb{T}_{\times},$$

where the second map is the vertical map in the commutative diagramme in section 4.

In light of the work with the zeta integral it is natural to propose to study the space of continuous $\mathbb{C}((X))$-valued functions, possibly satisfying some restrictions, on $\mathbb{T}_{\times}/\mathbb{K}_{\times}$ as a candidate for the space of *unramified automorphic forms* associated to GL_1 and $\mathcal{E}$.

The quotient space $\mathbb{T}_{\times}/\mathbb{K}_{\times}$ glues together the structure of the space T/T_0 useful in the study of the zeta integral and associated to the upper $\mathbf{A}$ and middle $\mathbf{B}$ level of the rhombus diagramme in 2.2 for integral structures of rank 2 with the structure of objects associated to the middle and bottom level for integral structures of rank 1 including the quotient $\mathbf{B}^{\times}/K^{\times}(\mathbf{B}^{\times}\cap\mathbf{VA}^{\times})$ isomorphic to $\mathrm{Pic}\,(\mathcal{E})$ in positive characteristic.

More generally, let $G=GL_n$ (or even more generally an algebraic reductive group). Let $\mathbb{T}_G = G(\mathbb{A})\times G(\mathbb{A})/V(G(\mathbb{A})\times G(\mathbb{A}))$, $V(G(\mathbb{A})\times G(\mathbb{A}))$ is defined similarly to $V(\mathbb{A}\times\mathbb{A})$ but with appropriate data at archimedean points of curves. Let $\mathbb{K}_G$ be the image of the map

$$G(\mathbb{B})\times G(K)\longrightarrow G(\mathbb{A})\times G(\mathbf{A})/G(\mathbf{OA})\longrightarrow \mathbb{T}_G,$$

where the second map is the G-analogue of the vertical map in the commutative diagramme in 4.3. Its local description is this: using Cartan decomposition for a two-dimensional local field $GL_n(F)=$ disjoint union of $GL_n(\mathcal{O})M_{l_1,\dots,l_n}GL_n(\mathcal{O})$, $M_{l_1,\dots,l_n}$ is the diagonal matrix with entries $t_2^{l_1},\dots,t_2^{l_n}$, $l_1\leqslant\dots\leqslant l_n$, similarly $GL_n(\mathcal{O})=$ disjoint union of $GL_n(O)N_{j_1,\dots,j_n}GL_n(O)$, $N_{j_1,\dots,j_n}$ is the diagonal matrix with entries $t_1^{j_1}, \dots, t_1^{j_n}$, $j_1 \leqslant \dots \leqslant j_n$, and $M_{l_1,\dots,l_n}$ acts on $N_{j_1,\dots,j_n}$ sending it to $N_{j_1 l_1,\dots,j_n l_n}$.

Irreducible representations of $G \times G$ in the space of continuous $\mathbb{C}((X))$-valued functions on $\mathbb{T}_G/\mathbb{K}_G$, satisfying some restrictions, could be viewed as a candidate for unramified automorphic functions associated to G and $\mathcal{E}$.

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