Counting Objects

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Abstract

We survey results on axiomatic completeness and complexity of formal languages for reasoning about finite quantities. These languages are interpreted on domains with and without structure. Expressions dealt with include 'at least n Xs are Ys,' in the case of unstructured domains, and 'at least n 'related' Xs are Ys,' in the structured case. The results contained in this paper are brought together from generalized quantifier theory, modal logic, and knowledge representation, while a few new results are also included. A unifying approach is offered to the wide variety of formalisms available in the literature.

Keywords: Modal logic, knowledge representation, generalized quantifier theory, axiomatic completeness, complexity.

1 Introduction

This is a paper on the borderline between generalized quantifier theory, modal logic, and knowledge representation. It surveys results from these areas on axiomatizations and complexity of certain 'counting expressions'; a few new results are included as well. These counting expressions include sentences like

$$\text{some woman is young,} \tag{1.1}$$

and, more generally, like

$$\text{at least } n \text{ women are young.} \tag{1.2}$$

Traditionally, the analysis of such expressions has mainly been confined to the theory of generalized quantifiers. But over the past few years researchers in modal logic and knowledge representation have also taken an interest in them. This interest derives from a concern with expressions of a more general pattern; whereas the domain of discourse used to interpret (1.1) and (1.2) may be any unstructured set, the more general pattern involves domains equipped with a certain structure, say of a binary relation, reflected in this pattern:

$$\text{some relatives that are female, are young,} \tag{1.3}$$

and

$$\text{at least } n \text{ relatives that are female, are young.} \tag{1.4}$$

Clearly, sentences (1.1) and (1.2) arise as special cases of (1.3) and (1.4), respectively, as the cases where the structure of the domain is such that all objects are related.

This paper detects correspondences concerning axiomatic and complexity aspects of the above counting expressions between generalized quantifier theory, modal logic and knowledge representation. By doing so, and by bringing together relevant results from these areas, we hope the paper contributes to each of them.

Results on axiomatic aspects of counting expressions like (1.1)–(1.4) are found mainly in modal logic, but usually without an explicit connection with generalized quantifier theory; some results on axiomatics are presented in Section 3 below. In Section 4 the complexity of the axiom systems discussed in Section 3 is addressed; here, a look at recent papers on the complexity of concept languages in knowledge representation proves useful. Proofs of results in the paper will be sketchy, referring the reader to the literature for full details.

We start out by stating the relevant basics of the modal logic, knowledge representation and generalized quantifier theory approaches to the above counting expressions.

2 Preliminaries

Let us start with some facts on basic modal logic.[1] The first thing we need is a general definition of modal formulas.

Let Φ be a set of proposition letters, and let Op be a set of (unary) modal operators. The set $Form(\Phi, Op)$ of well-formed formulas over Φ and Op (typically denoted by ϕ) is built up using proposition letters ($p \in \Phi$), and modal operators ($O \in Op$) according to the following rule

$$\phi ::= p \mid \bot \mid \top \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid O\phi.$$

The language of basic (poly-) modal logic is given by a set $\Phi = \{p_0, p_1, \ldots\}$, and $Op = \{\langle R\rangle : R \in \mathcal{R}\}$, for some set $\mathcal{R}$; $[R]$ is short for $\neg\langle R\rangle\neg$; we use Op and $\{[R] : R \in \mathcal{R}\}$ interchangeably. Its *semantics* is based on structures $\mathfrak{M} = (W, \{R\}_{R\in\mathcal{R}}, V)$, where $W \neq \emptyset$, $R \subseteq W^2$ (for $R \in \mathcal{R}$), and V is a *valuation* assigning subsets of W to the proposition letters in Φ. The truth conditions are $\mathfrak{M}, x \models p$ iff $x \in V(p)$, $\mathfrak{M}, x \models \neg\phi$ iff $\mathfrak{M}, x \not\models \phi$, $\mathfrak{M}, x \models \phi \wedge \psi$ iff $\mathfrak{M}, x \models \phi$ and $\mathfrak{M}, x \models \psi$, and

$$\mathfrak{M}, x \models \langle R\rangle\phi \text{ iff for some } y \text{ in } \mathfrak{M} \text{ we have } xRy \text{ and } \mathfrak{M}, y \models \phi.$$

We say that a formula is *valid* on a model ($\mathfrak{M} \models \phi$) if $\mathfrak{M}, x \models \phi$ for all x in $\mathfrak{M}$. Parallel to the above truth definition one can define an embedding of the modal language into a suitable first-order language. Given this embedding the basic modal language can be viewed as a fragment of first-order logic (cf. [4] for details).

Given the analogy between the existential quantifier and the modal operator $\langle R\rangle$, the *Tarskian numerical* quantifiers[2] inspired Fine [11] to consider so-called *graded modal operators* $\langle R\rangle_n$ whose duals $\neg\langle R\rangle_n\neg$ are written as $[R]_n$ (for $R \in \mathcal{R}$, as before, and $n \in \mathbb{N}$). The semantic structures are the same as in the basic modal case, and the truth condition of the graded modal operators reads

$$\mathfrak{M}, x \models \langle R\rangle_n\phi \text{ iff } |\{y : xRy \text{ and } \mathfrak{M}, y \models \phi\}| > n,$$

where $|X|$ denotes the cardinality of the set X. Clearly, for all $\mathfrak{M}$ it holds that $\mathfrak{M}, x \models \langle R\rangle\phi \leftrightarrow \langle R\rangle_0\phi$, and

$$\mathfrak{M}, x \models [R]_n\phi \text{ iff } |\{y : xRy \text{ and } \mathfrak{M}, y \not\models \phi\}| \leq n.$$

Just like the basic modal language the graded modal language can be considered as a fragment of first-order logic.[3]

[1] [12] is an excellent and short introduction to (poly-) modal logic.

[2] Cf. [25].

[3] Some references on graded modal logic include [6, 23] on uses of the language, and [10, 15] on its theoretical aspects.

In recent years a connection has been discovered between modal logic and *terminological* or *concept languages*. Concept languages provide a means for expressing knowledge about hierarchies of sets of objects with common properties (cf. Donini *et al.* [9] for a brief overview). They have been investigated mainly in the field of knowledge representation, but recently also in database theory [3], and in logic programming with object-oriented features [1].

Just like the earlier modal languages, concept languages may be conceived as fragments of first-order logic. Expressions in concept languages are built up using *concepts* and *roles*, interpreted as subsets of and binary relations on a given universe. Compound expressions are built up using a number of constructs. To give examples we suppose that female and young are primitive concepts, and child and relative are primitive roles. Using intersection and complement, the set of 'youngsters that are not female' can be described by young $\sqcap$ ¬female. Most concept languages provide *restricted quantification*, that is quantification over roles. 'Women whose children are all young' are described by female $\sqcap$ (ALL child young). Clearly, viewing things from a modal logic perspective, primitive concepts can be conceived as atomic propositional formulas, while quantifications (ALL R C) become formulas of the form $[R]\gamma$.

Another construct found in most concept languages is *number restriction*. Number restrictions on roles denote sets of objects having at least, or at most a certain number of fillers for a role. For instance, female $\sqcap$ ($\geq$ 3 relative young) describes the set of all women having at least three young relatives. The connection between number restrictions ($\geq n\ R\ C$) and the earlier graded modalities should be obvious: the former may be written as formulas of the form $\langle R\rangle_{n-1}\gamma$.

DEFINITION 2.1
We define the basic $\mathcal{AL}$-language. The language $\mathcal{AL}$ has *concepts* (denoted by $C, D, \ldots$) built up from primitive concepts (denoted by A) and primitive *roles* (denoted by R) according to the rule

$$C ::= \top \mid \bot \mid A \mid \neg A \mid C_1 \sqcap C_2 \mid (\text{ALL}\ R\ C) \mid (\text{SOME}\ R\ \top);$$

in $\mathcal{AL}$ roles are always primitive.

Models for the $\mathcal{AL}$-languages have the form $(\mathcal{D}, \cdot^{\mathcal{I}})$, consisting of a set $\mathcal{D}$, the domain, and an interpretation function $\cdot^{\mathcal{I}}$ that maps concepts to subsets of $\mathcal{D}$, and roles to binary relations on $\mathcal{D}$ such that $\top^{\mathcal{I}} = \mathcal{D}$ and $\bot^{\mathcal{I}} = \emptyset$, as usual, while

$$\begin{array}{rclrcl} (C_1 \sqcap C_2)^{\mathcal{I}} & = & C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}, & (\text{ALL}\ R\ C)^{\mathcal{I}} & = & \{ d \in \mathcal{D} : \forall y\,(dR^{\mathcal{I}}y \to y \in C^{\mathcal{I}}) \}, \\ (\neg A)^{\mathcal{I}} & = & \mathcal{D} \setminus A^{\mathcal{I}}, & (\text{SOME}\ R\ \top)^{\mathcal{I}} & = & \{ d \in \mathcal{D} : \exists y\,(dR^{\mathcal{I}}y) \}. \end{array}$$

DEFINITION 2.2
Languages more general than $\mathcal{AL}$ are obtained by adding to $\mathcal{AL}$ one of the following constructs:

$\mathcal{U}$ *union* of concepts, written $C \sqcup D$, with $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$;
$\mathcal{E}$ *full existential quantification*, written as (SOME R C), defined by

$$(\text{SOME}\ R\ C)^{\mathcal{I}} = \{ d \in \mathcal{D}^{\mathcal{I}} : \exists y\,(dR^{\mathcal{I}}y \wedge y \in C^{\mathcal{I}}) \};$$

$\mathcal{C}$ *complement* of non-primitive concepts, written as $\neg C$, with $(\neg C)^{\mathcal{I}} = \mathcal{D} \setminus C^{\mathcal{I}}$;
$\mathcal{N}$ *number restrictions*, written as $(\geq n\ R)$ and $(\leq n\ R)$, where n ranges over the non-negative integers, with, for each $\bowtie \in \{ \geq, \leq \}$,

$$(\bowtie n\ R)^{\mathcal{I}} = \{ d \in \mathcal{D} : |\{ y : dR^{\mathcal{I}}y \}| \bowtie n \};$$

$\mathcal{M}$ *meet* or *intersection of roles*, written as $Q \sqcap R$, with $(Q \sqcap R)^{\mathcal{I}} = Q^{\mathcal{I}} \cap R^{\mathcal{I}}$.

Following [9], suffixing the name of any of the above constructs to '$\mathcal{AL}$' denotes the addition of the construct to the basic $\mathcal{AL}$-language; for instance, $\mathcal{ALUN}$ denotes the extension of $\mathcal{AL}$ that allows for union and number restrictions.

We denote the obvious translation of $\mathcal{AL}$-expressions into modal ones by δ; for instance, $\delta(\text{ALL } R\ C) = [R]\delta(C)$. Moreover, there is a standard transformation Δ that takes models $(\mathcal{D}, \cdot^{\mathcal{I}})$ for concept languages into models $\mathfrak{M} = (W, \{R\}_{R \in \mathcal{R}}, V)$ for modal languages, such that given $(\mathcal{D}, \cdot^{\mathcal{I}})$, we have $x \in C^{\mathcal{I}}$ iff $\Delta(\mathcal{D}, \cdot^{\mathcal{I}}), x \models \delta(C)$. For a modal logic $\mathbf{L}$, and concept language $\mathcal{ALX}$, we write $\mathbf{L} \cong \mathcal{ALX}$ if the translation δ from $\mathcal{ALX}$ to $\mathbf{L}$ is a bijection. By means of such bijections results for modal logics translate effortlessly into results for concept languages, and conversely.[4]

According to general wisdom a generalized quantifier is a function assigning to every *unstructured set* $\mathfrak{M}$ a binary relation $Q_{\mathfrak{M}}$ between subsets of $\mathfrak{M}$.[5] At least as far as axiomatic aspects are concerned, so far relatively little attention has been paid to quantification over *structured* domains, that is, to binary relations between subsets of structured domains. Nevertheless, we feel that these are at least as important as the unstructured ones, especially since most examples of quantification one encounters in applications and in everyday life seem to presuppose some kind of structure of the underlying domain.[6]

In this paper we assume that our domains $\mathfrak{M}$ are structured by (one or more) binary relations.[7] Then, the mainstream conception of generalized quantifiers arises when very special choices are made concerning $\mathfrak{M}$, namely when it is assumed that all relations coincide with $\mathfrak{M} \times \mathfrak{M}$. It should be noted that we only look at the 'explicit' structure present in sentences like (1.3) and (1.4); the 'implicit' structure often assumed to underly, for example, conditionals is not considered here. Even for the case of explicit structure we only look at a basic fragment in which quantification is restricted by arbitrary binary relations. This excludes sentences like 'Every woman writes a Christmas card', in which one has both unrestricted and restricted quantification. (But, as pointed out before, unrestricted quantification could be analysed as quantification 'restricted' by the universal relation.)

The only generalized quantifiers that we will consider in this paper are the ones exemplified in the earlier sentences (1.1)–(1.4): *cardinality quantifiers*, both over structured and over unstructured domains. We will deal only with *finite* cardinalities.

The analogies between such cardinality quantifiers and modal logic should be obvious now. Sentences (1.3) and (1.4) can be simulated in the graded modal language by

(1.3′) $\langle R \rangle_0 (f \wedge y)$, and

(1.4′) $\langle R \rangle_{n-1} (f \wedge y)$,

respectively, while (1.1) and (1.2) can be simulated by the same modal formulas provided it is assumed that R is the *universal relation*, that is the Cartesian square of the domain. Van der Hoek and De Rijke [18] take the latter assumption for granted and explore the connections between generalized quantifiers and modal logic on the basis of this assumption.

[4]Connections between other concept languages and propositional dynamic logic have been explored by Schild [22]; yet another concept language is linked to modal logic in [21]. Complexity results on satisfiability problems in the above $\mathcal{AL}$-languages are contained in [9].

[5]Cf. [25] for a thorough introduction to and overview of generalized quantifier theory.

[6]Cf. [7] for further discussions on quantification over structured domains.

[7]Although various plausible conditions may be imposed on generalized quantifiers over such domains, we will not go into them here, but refer the reader to [7].

3 Axioms

In this section we present axiom systems for the notions of validity introduced in Section 2. The languages we consider can be roughly classified along two dimensions. First, we consider languages over *structured domains*, in which sentences (1.3) and (1.4) can be expressed. Then we consider languages over *unstructured domains*, that are equipped to deal with sentences like (1.1) and (1.2). For each class, we present axiom systems for two basic languages. The first of them allows only for the quantifiers *some* and *all* (sufficient to represent (1.1) and (1.3)) and the second basic language has tools to reason about sets of arbitrary finite cardinality (needed to represent (1.2) and (1.4)). We will also present axiom systems for sublanguages and extensions of those four basic systems. The overall picture is that of Figure 1.

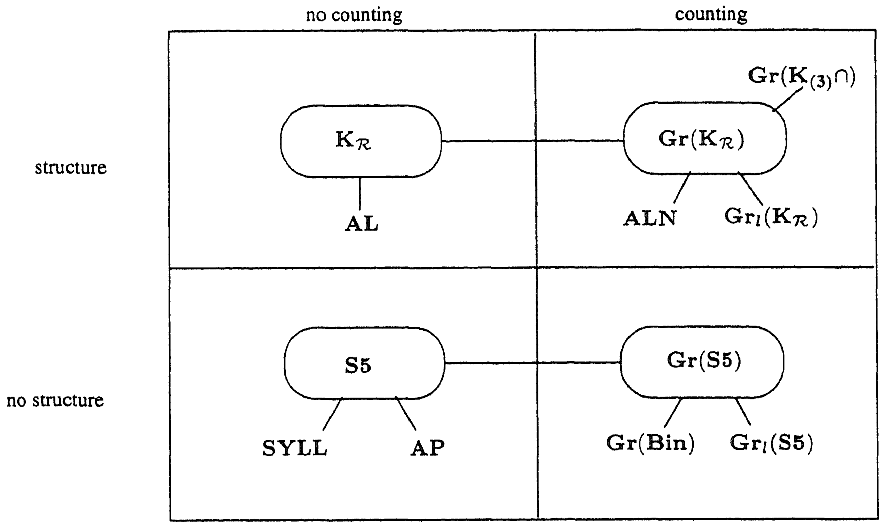

FIG. 1. A plethora of calculi

We use modal languages to relate the three traditions brought together in this paper: every system presented below will also be presented as (a fragment of) a modal system. For the remainder of this section, we suppose to have a fixed set of operators $Op = \{[R] : R \in \mathcal{R}\}$, with typical element $[R]$. We say that an axiom system $\mathbf{L}$ is *normal for* $Op' \subseteq Op$ if it has the following axioms and derivation rules:

Tautology: $\mathbf{L} \vdash \phi$, for all propositional tautologies ϕ;
Distribution: $\mathbf{L} \vdash [R](\phi \rightarrow \psi) \rightarrow ([R]\phi \rightarrow [R]\psi)$, for all $[R] \in Op'$;
Modus Ponens: $\mathbf{L} \vdash \phi \rightarrow \psi, \mathbf{L} \vdash \phi \Rightarrow \mathbf{L} \vdash \psi$;
Substitution: $\mathbf{L} \vdash \alpha \leftrightarrow \beta \Rightarrow \mathbf{L} \vdash \phi \leftrightarrow [^{\alpha}/_{\beta}]\phi$, where $[^{\alpha}/_{\beta}]\phi$ is a formula obtained by substituting any number of occurrences of β by α in ϕ;
Necessitation: $\mathbf{L} \vdash \phi \Rightarrow \mathbf{L} \vdash [R]\phi$, for all $[R] \in Op'$.

Structured domains

The first system we present is known as the modal logic $\mathbf{K}_{\mathcal{R}}$: its language deals with structured domains $(W, \{R\}_{R \in \mathcal{R}}, V)$, and does not allow for number restrictions.

DEFINITION 3.1
The system $\mathbf{K}_{\mathcal{R}}$ over the language $Form(\Phi, Op)$ is the minimal logic that is normal for Op.

THEOREM 3.2
$\mathbf{K}_{\mathcal{R}} \vdash \phi$ iff ϕ is valid on all structures $(W, \{R\}_{R\in\mathcal{R}}, V)$.

PROOF. The proof consists of two steps. First, a model satisfying a given $\mathbf{K}_{\mathcal{R}}$-consistent formula ϕ is built by means of a *Henkin construction*, which yields a *canonical model* $\mathfrak{M}^c = (W^c, \{R^c\}_{R\in\mathcal{R}}, V^c)$ in which $W^c = \{\Gamma : \Gamma$ is a maximal consistent (m.c.) set $\}$; $V^c(p) = \{\Gamma : p \in \Gamma\}$ and $R^c\Gamma\Sigma$ iff for all ϕ, $([R]\phi \in \Gamma \Rightarrow \phi \in \Sigma)$. The second step is to prove a *Truth Lemma*: for all ϕ, and Γ, $(W^c, \{R^c\}_{R\in\mathcal{R}}, V^c), \Gamma \models \phi$ iff $\phi \in \Gamma$. Since every consistent ϕ is contained in some m.c. set Γ, this proves the satisfiability of such a ϕ. ∎

REMARK 3.3
It is easily seen that $\delta : \mathcal{ALUEC} \cong \mathbf{K}_{\mathcal{R}}$. This implies that we have found a sound and complete axiomatization $(\delta^{-1}(\mathbf{K}_{\mathcal{R}}))$ for validity of $\mathcal{ALUEC}$-formulas. Moreover, since the construct $\mathcal{C}$ can be expressed in $\mathcal{AL}$ using the constructs $\mathcal{U}$ and $\mathcal{E}$ and vice versa, we also have $\mathbf{K}_{\mathcal{R}} \cong \mathcal{ALUE}$, $\mathbf{K}_{\mathcal{R}} \cong \mathcal{ALC}$.

Now that we have related the basic modal logic to one of the concept languages of Section 2, we want to match the basic concept language $\mathcal{AL}$ with a modal counterpart. Given proposition letters $p \in \Phi$ and operators $\langle R\rangle \in Op$ we define the language $Form_{\mathbf{AL}}$ by $\phi ::= p \mid \neg p \mid \bot \mid \top \mid \phi_1 \wedge \phi_2 \mid [R]\phi \mid \langle R\rangle\top$.

DEFINITION 3.4
As the language $Form_{\mathbf{AL}}$ is rather poor, we cannot express important properties (like the mutual exclusiveness of p and $\neg p$) *within* the language itself. Instead we reason about consequences $\Lambda \vdash \psi$ directly. We stipulate

$$\mathfrak{M}, x \models \Lambda \vdash \psi \text{ iff } (\mathfrak{M}, x \models \Lambda \Rightarrow \mathfrak{M}, x \models \psi).$$

Notice that, although we cannot express the inter-definability of $\langle R\rangle\top$ and $[R]\bot$ inside our language, semantically we can still treat them as being dual.

We define the logic **AL**. First, we suppose that '$\vdash$' has the following so-called *structural properties*:

Monotonicity	$\Lambda \vdash \psi \Rightarrow \Lambda \cup \{\phi\} \vdash \psi$,
Cut	$\Lambda \vdash \psi$ and $\Lambda \cup \{\psi\} \vdash \chi \Rightarrow \Lambda \vdash \chi$,

and the inferences rules

*Distribution**	$\Lambda \vdash \psi \Rightarrow \{[R]\lambda : \lambda \in \Lambda\} \vdash [R]\psi$,
Complement	$\Gamma, p \vdash \phi$ and $\Gamma, \neg p \vdash \phi \Rightarrow \Gamma \vdash \phi$,
	$\Gamma, [R]\bot \vdash \phi$ and $\Gamma, \langle R\rangle\top \vdash \phi \Rightarrow \Gamma \vdash \phi$.

Omitting braces where this does not lead to confusion, **AL** has the following 'axioms':

*A*1	$\phi \vdash \phi$,
*A*2	$p, \neg p \vdash \bot$,
*A*3	$\phi \vdash \top$,
*A*4	$\bot \vdash \phi$,
*A*5	$\phi, \psi \vdash \phi \wedge \psi$,
*A*6	$\phi \wedge \psi \vdash \phi$ and $\phi \wedge \psi \vdash \psi$,
*A*7	$[R]\bot, \langle R\rangle\top \vdash \bot$.

For infinite sets Γ, '$\Gamma \vdash \phi$' will mean '$\Gamma_0 \vdash \phi$ for some finite $\Gamma_0 \subseteq \Gamma$.' A set of formulas Λ is called *consistent* if for *no* $\Lambda' \subseteq \Lambda$ we have $\Lambda' \vdash \bot$. To prove the completeness of **AL** we cannot use the familiar *maximal* consistent sets as our language does not have full negation. Instead we use *large consistent* (l.c.) sets Γ whose defining clauses read

- $\bot \notin \Gamma$;
- $\top \in \Gamma$;
- $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash \phi \Rightarrow \phi \in \Gamma$;
- for every p: $p \in \Gamma$ or $\neg p \in \Gamma$;
- for every R: $[R]\bot \in \Gamma$ or $\langle R\rangle\top \in \Gamma$.

LEMMA 3.5
Let Λ be a set of formulas. Assume $\Lambda \nvdash \psi$ in **AL**. Then there is an l.c. set Γ such that $\Lambda \subseteq \Gamma$ and $\Gamma \nvdash \psi$.

PROOF. As this is not completely standard, we will give some details. Let $\chi_0, \chi_1, \ldots$ enumerate all **AL**-formulas in such a way that every formula occurs infinitely often. Define $\Gamma = \bigcup_n \Gamma_n$, where $\Gamma_0 = \Lambda$, and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\chi_n\}, & \text{if } \Gamma_n \cup \{\chi_n\} \text{ is consistent and } \Gamma_n \cup \{\chi_n\} \nvdash \psi, \\ \Gamma_n, & \text{otherwise.} \end{cases}$$

Then
1. $\Gamma \nvdash \psi$: this follows from the claim that for no n, $\Gamma_n \vdash \psi$, as may be established by induction; together with $A4$ this gives $\Gamma \nvdash \bot$.
2. $\Gamma' \subseteq \Gamma, \Gamma' \vdash \phi \Rightarrow \phi \in \Gamma$: all formulas involved in deriving ϕ from Γ' are contained in some Γ_n; let $k \geq n$ be such that $\phi \equiv \chi_k$. Then $\Gamma_{k+1} = \Gamma_k \cup \{\chi_k\}$. Otherwise *either* $\Gamma_k \cup \{\chi_k\}$ is inconsistent — but then, by the Cut rule, Γ_k would already be inconsistent as $\Gamma_n \subseteq \Gamma_k$ and $\Gamma_n \vdash \phi$, *or* $\Gamma_k, \chi_k \vdash \psi$, but then, by Cut again, $\Gamma_k \vdash \psi$ — which is impossible because of **1.**.
3. For every p and every R we have $p \in \Gamma$ or $\neg p \in \Gamma$, and $[R]\bot \in \Gamma$ or $\langle R\rangle\top \in \Gamma$; we only establish the first claim. Assume $p \equiv \chi_n$, $\neg p \equiv \chi_k$ for some $k > n$, and that $p, \neg p \notin \Gamma_{k+1}$. Then

$$\left\{ \begin{array}{l} \text{(i) } \Gamma_n, p \vdash \bot \\ \text{or} \\ \text{(ii) } \Gamma_n, p \vdash \psi, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \text{(iii) } \Gamma_k, \neg p \vdash \bot \\ \text{or} \\ \text{(iv) } \Gamma_k, \neg p \vdash \psi. \end{array} \right.$$

Let's check each case: if case (i) and (iii) hold, we have that (i) implies that $\Gamma_k, p \vdash \bot$ by Monotonicity; together with (iii) and the Complement rule this gives $\Gamma_k \vdash \bot$ contradicting **1**. Case (ii) and (iii): (ii) and Monotonicity give $\Gamma_k, p \vdash \psi$. On the other hand, (iii), $A4$ and Cut give $\Gamma_k, \neg p \vdash \psi$, which then yields $\Gamma_k \vdash \psi$ — contradicting **1**. Case (i) and (iv): this is the mirror image of case (ii) and (iii). Case (ii) and (iv): by Monotonicity (ii) yields $\Gamma_k, p \vdash \psi$, and with the Complement rule and (iv) this gives $\Gamma_k \vdash \psi$ — contradicting **1**. ∎

THEOREM 3.6
Let Λ be a set of **AL**-formulas. Then $\Lambda \vdash \psi$ in **AL** iff $\Lambda \vdash \psi$ is valid on all **AL**-models.

PROOF. This is a two-step proof. First we build a canonical model consisting of l.c. sets. Whereas in the standard modal case one may need to have an arbitrary number of R-successors for each l.c. set Γ, and R, here we need at most one. If needed this successor can be found as follows. Suppose $\langle R\rangle\top \in \Gamma$, and consider the set $\Delta' = \{\phi : [R]\phi \in \Gamma\}$. Then Δ' is consistent.

Otherwise, we can find $\phi_1, \ldots, \phi_n \in \Delta'$ for which $\phi_1, \ldots, \phi_n \vdash \bot$. Applying *Distribution** we get $[R]\phi_1, \ldots, [R]\phi_n \vdash [R]\bot$. Using Lemma 3.5 we conclude that $[R]\bot \in \Gamma$. But then $\Gamma \vdash \bot$ by axiom $A7$ and Monotonicity. It follows that there is an l.c. set $\Delta \supseteq \Delta'$; for this one we put $R^c\Gamma\Delta$. As usual in the second part of the proof one proves a Truth Lemma. ∎

REMARK 3.7
Observe that the translated system $\delta^{-1}(\mathbf{AL})$ is a sound and complete axiomatization for validity in the language $\mathcal{AL}$.

The next system we present is the graded variant of $\mathbf{K}_{\mathcal{R}}$: its language adds the operators $\langle R\rangle_n$, $[R]_n$ to that of $\mathbf{K}_{\mathcal{R}}$.

DEFINITION 3.8
The system $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ is a logic over the language $Form(\Phi, \{\langle R\rangle_n : R \in \mathcal{R}, n \in \mathbb{N}\})$. We define $[R]_n = \neg\langle R\rangle\neg$; $\langle R\rangle!_0\phi = \neg\langle R\rangle_0\phi$ and $\langle R\rangle!_n\phi = \langle R\rangle_{n-1}\phi \wedge \neg\langle R\rangle_n\phi$ if $n \geq 1$. $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ is normal over $\{[R]_0 : R \in \mathcal{R}\}$, and on top of that it has the following axioms ($n, m \in \mathbb{N}$):

$A8$ $\quad \langle R\rangle_{n+1}\phi \rightarrow \langle R\rangle_n\phi$,
$A9$ $\quad [R]_0(\phi \rightarrow \psi) \rightarrow (\langle R\rangle_n\phi \rightarrow \langle R\rangle_n\psi)$,
$A10$ $\quad [R]_0\neg(\phi \wedge \psi) \rightarrow ((\langle R\rangle!_n\phi \wedge \langle R\rangle!_m\psi) \rightarrow \langle R\rangle!_{n+m}(\phi \vee \psi))$.

Semantically speaking, axiom $A9$ guarantees that a formula ψ is true in at least as many points as any stronger formula ϕ. In fact, $A9$ is even stronger than the *Distribution* axiom for $[R]_0$.[8] $A10$ expresses a notion of *additivity*: the number of points satisfying one of two mutually exclusive formulas is simply the sum of the number of points satisfying each of those formulas separately.

THEOREM 3.9 (De Caro [8])
For all formulas $\phi \in Form(\Phi, \{\langle R\rangle_n : R \in \mathcal{R}, n \in \mathbb{N}\}$, we have $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}}) \vdash \phi$ iff ϕ is valid on all models $\mathfrak{M}$.

PROOF. We sketch De Caro's 2-step proof. To construct a canonical model

$$\mathfrak{M}^c = (W^c, \{R^c\}_{R^c \in \mathcal{R}^c}, V^c),$$

let $\Theta = \{\Gamma : \Gamma \text{ is a maximal consistent set}\}$. For each $R \in \mathcal{R}$ we define a function $Succ_R$: $\Theta \times \Theta \rightarrow \omega \cup \{\omega\}$ by

$$Succ_R(\Gamma, \Delta) = \begin{cases} \omega, & \text{if } \exists\alpha \in \Delta\forall n\,(\langle R\rangle_n\alpha \in \Gamma) \\ min\{n \in \mathbb{N} : \langle R\rangle!_n\alpha \in \Gamma,\ \alpha \in \Delta\}, & \text{otherwise.} \end{cases}$$

Then, the *satisfiability set* with respect to R for Γ, $SF_R(\Gamma)$ is defined as

$$SF_R(\Gamma) = \{(\Delta, i) : \Delta \in \Theta,\ i \leq Succ_R(\Gamma, \Delta)\}.$$

This satisfiability set $SF_R(\Gamma)$ contains 'sufficiently many' copies of maximal consistent sets, as is guaranteed by

$$\langle R\rangle_n\alpha \in \Gamma \text{ iff } |\{(\Delta, i) \in SF_R(\Gamma) : \alpha \in \Delta\}| > n. \tag{3.1}$$

The canonical model then consists of copies of m.c. sets: $W^c = \{(\Delta, i) : \text{for some } \Gamma \in \Theta,$ and some R, $Succ_R(\Gamma, \Delta) = i\}$. We stipulate $R^c(\Gamma, j)(\Delta, i)$ iff $(\Delta, i) \in SF_R(\Gamma)$, and the valuation V^c is standard: $V^c(p) = \{(\Delta, i) : p \in \Delta\}$.

The final step, the proof of the Truth Lemma, is a straightforward induction on ϕ, where the only interesting case of $\langle R\rangle_n$-formulas is taken care of by (3.1). ∎

[8] Observe that the Distribution axiom is not valid for $[R]_n$ with $n \geq 1$.

REMARK 3.10
Of the concept languages considered by Donini *et al.* [9], the language $\mathcal{ALUECN}$ is the one that resembles $\mathbf{Gr}(\mathbf{K}_\mathcal{R})$ most. However, in that concept language, one can only reason about numbers of R-related things, not about numbers of R-related things that satisfy some property C. Simulating $\mathcal{ALUECN}$ by modal means would require a restriction of the arguments of $\langle R\rangle_n$ to $\top$ (for $n \geq 1$). From a modal logic point of view, this restriction is not a very natural one, but in some cases it yields an improvement in complexity, as can be deduced from [9]. Nevertheless, many researchers in the field don't consider this restriction.

Let us quickly move on to a graded modal system that *does* have a counterpart in the $\mathcal{AL}$-hierarchy, a system called **ALN**. **ALN**-formulas are generated by the following rule

$$\phi ::= \top \mid \bot \mid p \mid \neg p \mid \phi_1 \wedge \phi_2 \mid [R]\phi \mid \langle R\rangle_n\top \mid [R]_n\bot.$$

So, the **ALN**-language extends the **AL**-language by allowing a restricted form of counting.

DEFINITION 3.11
The logic **ALN** has the rules of **AL** plus the following Complement rule

Complement $\quad \Gamma, [R]_n\bot \vdash \phi$ and $\Gamma, \langle R\rangle_n\top \vdash \phi \Rightarrow \Gamma \vdash \phi,$

instead of the second Complement rule of **AL**; its axioms are those of **AL** and on top of that ($n \in \mathbb{N}$):

$A11 \quad \langle R\rangle_{n+1}\top \vdash \langle R\rangle_n\top$ and $[R]_n\top \vdash [R]_{n+1}\top,$
$A12 \quad [R]_n\bot, \langle R\rangle_n\top \vdash \bot.$

To prove axiomatic completeness for **ALN** we need to slightly modify the notion of large consistent sets as used in Lemma 3.5: we say that a set of formulas Γ is a large consistent set for **ALN** if it is a large consistent set in the earlier sense, and in addition

- for all n and R: $[R]_n\bot \in \Gamma$ or $\langle R\rangle_n\top \in \Gamma$.

THEOREM 3.12
Let Λ be a set of **ALN**-formulas. Then $\Lambda \vdash \psi$ in **ALN** iff $\Lambda \vdash \psi$ is valid on all **ALN**-models.

PROOF. This is another two-step proof. We construct a canonical model as in the case of **AL** (Theorem 3.6). But now we may have to add multiple copies of R-successors. This can be done as follows. Let $n_0 = min(\omega, max\{\, n+1 : \langle R\rangle_n\top \in \Gamma\,\})$. Then we simply add n_0 copies of an l.c. set Δ_Γ extending $\{\, \phi : [R]\phi \in \Gamma\,\}$ to our model.

As usual, the second step consists of a Truth Lemma. The only interesting cases in its inductive proof are formulas of the form $\langle R\rangle_n\top$ and $[R]_n\bot$; here we only consider the first kind. Now, $\langle R\rangle_n\top \in \Gamma$ implies that in our canonical model we have added at least $n+1$ R-successors of Γ; this implies that $\mathfrak{M}, \Gamma \models \langle R\rangle_n\top$. Conversely: if $\langle R\rangle_n\top \notin \Gamma$ then, by construction $[R]_n\bot \in \Gamma$. Let m_0 be the minimal index m such that $[R]_m\bot \in \Gamma$. Then $m_0 \leq n$, and $n_0 \leq \max(0, m_0)$ by $A11$, where n_0 is defined as before. Hence, at most m_0 R-successors Δ_Γ of Γ were added to the canonical model. But then $\mathfrak{M}, \Gamma \models [R]_{m_0}\bot$ and $\mathfrak{M}, \Gamma \models [R]_n\bot$, hence $\mathfrak{M}, \Gamma \not\models \langle R\rangle_n\top$. ∎

Observe that in **ALN**, we can only express that there *are at least n* R-related things, *at most n* R-related things and that *all* R-related things share some property ϕ. So the only possible conflicts the logic has to deal with, are combinations of $\langle R\rangle_n\top$ with $[R]_k\top$ ($n \geq k$), and not with more complicated combinations like $\langle R\rangle_n\phi$ with $[R]_k\phi$. In Section 4 the benefits of this simplification with respect to $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ in terms of complexity are stated.

We now give an example of a logic $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}})$ in a language 'in between' $\mathbf{K}_{\mathcal{R}}$ and $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$. This system is akin to the system $QUANT_k$ studied in [18], but the latter is designed for reasoning about domains without structure. Its introduction is motivated by the idea that having infinitely many operators $\langle R\rangle_n$ for one relation R is sometimes too much for one's purposes. Especially for applications in areas where formal laws are applied, decisions are often made when some *fixed* number of requirements is fulfilled, like passing an exam when *at least* l tests have been successfully done, for some fixed l (cf. [16] for further motivation).

For the language of $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}})$ we suppose that for each R, we have a threshold l_R. In $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}})$ we are able to distinguish between the cases that *no, some, at least l* and *all* R-related things have some property.

DEFINITION 3.13
Let $l_R \in \mathbb{N}$ $(R \in \mathcal{R})$. $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}})$ is a logic for $Form(\Phi, \{\langle R\rangle_0, \langle R\rangle_{l_R} : R \in \mathcal{R}\})$. It is normal in $\{[R]_0 : R \in \mathcal{R}\}$, and, on top of that, it has the following axioms (with $i \in \{0, l_R : R \in \mathcal{R}\}$):

$A13$ $\quad [R]_0(\phi \to \psi) \to (\langle R\rangle_i\phi \to \langle R\rangle_i\psi)$,
$A14$ $\quad \bigwedge_{0\leq j\neq k\leq l_R}[R]_0\neg(\psi_j \wedge \psi_k) \to (\bigwedge_{0\leq j\leq l_R}\langle R\rangle_0(\psi_j \wedge \psi) \to \langle R\rangle_{l_R}\psi)$.

Axiom $A13$ is just the restriction to the proper indices of axiom $A9$ for $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$. Axiom $A14$ is an appropriate version of $A10$: whereas in $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$, to conclude $\langle R\rangle_{l_R}\psi$, it suffices to find mutually exclusive formulas α, β, and numbers n, k such that

$$[R]_0\neg(\alpha \wedge \beta) \wedge \langle R\rangle_n(\alpha \wedge \psi) \wedge \langle R\rangle_k(\beta \wedge \psi)$$

holds together with $n + k \geq l_R - 1$, in $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}})$ we need to find $l_R + 1$ of such mutually exclusive formulas, according to A14.

THEOREM 3.14
For all $\phi \in Form(\Phi, \{\langle R\rangle_0, \langle R\rangle_{l_R} : R \in \mathcal{R}\})$ we have that $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}}) \vdash \phi$ iff ϕ is valid in all models $\mathfrak{M} = (W, \{R\}_{R\in\mathcal{R}}, V)$.

PROOF. Again, a two-step proof can be given. The construction of a canonical structure can be taken from [18, Lemma 3.6]: it consists of at most $min(\omega, max\{l_R : R \in \mathcal{R}\})$ copies of m.c. sets. The relations are defined by putting, for m.c. sets Γ, Δ, and $i, j \in \{0, \ldots, l_R\}$: $R^c(\Gamma, i)(\Delta, j)$ iff either $[(j = 0)$ and $(\delta \in \Delta \Rightarrow \langle R\rangle_0\delta \in \Gamma)]$ or $[(0 \leq j \leq l)$ and $(\delta \in \Delta \Rightarrow \langle R\rangle_{l_R}\delta \in \Gamma)]$. In this way, copies of Γ are R^c-related either to no, one, or l_R copies of Δ.

Then, to prove the Truth Lemma, assume that $\mathfrak{M}, (\Gamma, i) \models \langle R\rangle_{l_R}\psi$. To prove that $\langle R\rangle_{l_R}\psi \in \Gamma$ we distinguish two cases: either there is some Δ containing ψ and of which l_R copies are R-related to (Γ, i) and then, by definition of R^c, $\langle R\rangle_{l_R}\psi \in \Gamma$. The second case is that there is no such Δ with $\psi \in \Delta$ and $R^c(\Gamma, i)(\Delta, l)$. In this case one can prove a *Separation Lemma* which guarantees that there are pairwise different sets $\Delta_0, \Delta_1, \ldots, \Delta_{l_R}$ such that $\psi \in \Delta_j$ and $R^c(\Gamma, i)(\Delta_j, 0)$ $(j \leq l_R)$. These different m.c. sets contain mutually exclusive formulas ψ_j for which $\langle R\rangle_0(\psi_j \wedge \psi) \in \Gamma$, so that we can use axiom $A14$ to obtain $\langle R\rangle_{l_R}\psi$. ∎

In $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$, one may build arbitrary complex expressions, using arbitrary (nestings of) relational operators, each with its own grade, as in '3 relatives, all living in my home town, received at least 4 Christmas cards'. This provides us with a very powerful tool, be it that the relations are considered to be primitive: 'relatives living in my home town' cannot be expressed in terms of 'relative' and 'living in my home town'. The construct $\mathcal{M}$ in $\mathcal{AL}$-languages does allow for intersection of roles. We will now sketch some (im-) possibilities to deal with intersections in modal languages. To start with the possibilities, let us consider models $\mathfrak{M} = (W, \{R, S, T\}, V)$, in which $T = R \cap S$. Our first claim is that validity on those models can be axiomatized.

DEFINITION 3.15
The logic $\mathbf{Gr}(\mathbf{K}_{(3)}\cap)$ is normal over $\{ [R]_0, [S]_0, [T]_0 \}$. On top of axioms $A8$, $A9$ and $A10$ of $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ for the operators $[R]_0$, $[S]_0$ and $[T]_0$ it has the axioms (with $n \in \mathbb{N}$):

$A15$ $\langle T\rangle_n\phi \rightarrow \langle R\rangle_n\phi$,
$A16$ $\langle T\rangle_n\phi \rightarrow \langle S\rangle_n\phi$.

THEOREM 3.16
$\mathbf{Gr}(\mathbf{K}_{(3)}\cap) \vdash \phi$ iff ϕ is valid on all models $\mathfrak{M} = (W, \{R, S, T\}, V)$, in which $T = R \cap S$.

PROOF. The proof that a consistent formula is satisfiable consists of three steps. We omit the second step, the proof of a Truth Lemma. We first build a model along the lines of the proof of Theorem 3.9. It is easily verified that axioms $A15$ and $A16$ guarantee that, for all Γ and Γ', we have $Succ_T(\Gamma, \Gamma') \leq Succ_R(\Gamma, \Gamma')$ and $Succ_T(\Gamma, \Gamma') \leq Succ_S(\Gamma, \Gamma')$, respectively. It follows that $T^c \subset R^c$ and $T^c \subset S^c$, and hence $T^c \subset (R^c \cap S^c)$. However, the converse is not true in $\mathfrak{M}^c$.[9]

The last step of the proof is a construction that transforms $\mathfrak{M}$ into a model $\mathfrak{M}' = (W', \{ R', S', T' \}, V')$, in which $T' = R' \cap S'$. The crucial idea behind this transformation is the following. We want to get rid of all pairs $(u, v) \in W \times W$ that are in R and S, but not in T. This can be done by replacing this pair by two pairs $(u, v_1), (u, v_2)$ of which the first is added to R and the second to S. Then u is still both S and R-accessible to a 'v-like' point, but not $R \cap S$-accessible anymore. Formally, we define $W' = \{ (x_1, x_2) : x \in W \}$, and, letting i and j range over $\{1, 2\}$, $R' = \{ (x_i, y_1), (x_i, z_2) : (x, y) \in R \text{ and } (x, z) \in T \}$; $S' = \{ (x_i, y_2), (x_i, z_1) : (x, y) \in S \text{ and } (x, z) \in T \}$ and $T' = \{ (x_i, y_j) : (x, y) \in T \}$. We leave it to the reader to verify that $\mathfrak{M}', (w, i) \models \phi$ and that moreover $T' = R' \cap S'$. ∎

The construction above is an adaptation of a construction given in [16], for a modal logic without numeric operators, but with other requirements on the relations R, S and T. Also, in dynamic logic, a lot of attention has been paid to axiomatizing intersection of relations [20].

In Theorem 3.16 we only axiomatized the very restricted case of 'graded quantification' over two relations plus their intersection. The more general case is far more difficult, and cannot be handled by a trivial generalization of the construction of Theorem 3.16. To sketch some problems that arise, consider the case in which we have relations R_1, R_2, R_3, and all their intersections R_{12}, R_{13}, R_{23} and R_{123}. Suppose that we have a pair (u, v) that is only in the relations $R_1, R_2, R_3, R_{12}, R_{13}$. Since $(u, v) \notin R_{123}$, we would have to add a pair (u, v') for R_1, R_2, R_{12} and a pair (u, v'') for R_1, R_3, R_{13}. However, in that case we alter the *number* of 'v-points' that are R_1-accessible from u! Thus what seems to be needed are additional constraints that prevent models from containing such unwanted pairs.

Many powerful techniques are available in modal logic to deal with modally undefinable properties like intersection, usually based on additions like extra derivation rules, or special propositional symbols. Attempts to axiomatize validity in numerical modal languages with arbitrary many relations and intersections probably have to involve these techniques.

[9] Since the set $\{ [T]_0\bot, \langle R\rangle_1 p, \langle R\rangle_n\phi \leftrightarrow \langle S\rangle_n\phi : \phi \in Form(\Phi, \{\langle R\rangle, \langle S\rangle, \langle T\rangle\}) \}$ is consistent, it will have the same R^c and S^c- successors, but no T^c-successor.

Unstructured domains

Now we move on to languages for domains without structure, or, as pointed out above, domains with a very special structure. This fact that we have more constraints on the binary relation is mirrored by the addition of special axioms to the formal systems for those languages; these axioms force the relation to have the desired properties.

Saying that all things are related to each other requires only one binary relation: hence, from now on, we will just write $\Box$ for $[R]$ and $\Diamond$ for $\langle R \rangle$.

DEFINITION 3.17
The logic $\mathbf{S5}$ over $Form(\Phi, \{\Diamond\})$ is normal over $\{\Box\}$ and has the following axioms on top of that:

$A17$ $\quad \Box\phi \to \phi,$
$A18$ $\quad \Diamond\phi \to \Box\Diamond\phi.$

Remember that our modal formulas are evaluated at some point x in a model $\mathfrak{M}$. But then $A17$ just expresses that if all things in the model have some property ϕ, x must also have this property. According to $A18$, if there is a ϕ-thing, then where ever one evaluates formulas, there must be some ϕ-point. Below we show that $\mathbf{S5}$ axiomatizes validity on all structures of the form (W, R, V), in which R is the universal relation $W \times W$. This implies that the $\mathbf{S5}$-box and -diamond are the modal counterparts of the universal and existential quantifiers, respectively. Hence, $\mathbf{S5}$ is a suitable formalism for reasoning with expressions like (1.1) (cf. [13, 18] for further uses of $\mathbf{S5}$ along these quantificational lines).

THEOREM 3.18
$\mathbf{S5} \vdash \phi$ iff ϕ is valid on all structures $\mathfrak{M} = (W, \{R\}, V)$ in which R is the universal relation.

PROOF. The proof consists of three steps. As before, in the first two steps the Henkin construction is used to build a model in which ϕ is satisfied, and a Truth Lemma is proved. However, the canonical relation R^c of this model is only an *equivalence* relation. This property is forced by the additional axioms $A17$ and $A18$. As an example we show that R^c is symmetric. Suppose $R^c\Gamma\Delta$. If not $R^c\Delta\Gamma$, there is some ψ such that $\Box\psi \in \Delta$, but $\psi \notin \Gamma$. Since Γ is an m.c. set, we have $\neg\psi \in \Gamma$. But then $\Diamond\neg\phi \in \Gamma$ (by $A17$). Axiom $A18$ then guarantees that $\Box\Diamond\neg\psi \in \Gamma$, and by definition of R^c, $\Diamond\neg\psi \in \Delta$. This contradicts our assumption that $\Box\psi \in \Delta$.

The third and last step in the proof is to turn this canonical model into a universal one. We define a model $\mathfrak{M}'$, with $W' = \{ y \in W : Rxy \}$, where x is some state verifying ϕ, and let R' and V' be the restrictions of R and V to this domain W'. By standard modal arguments $\mathfrak{M}, x$ verifies the same modal formula as $\mathfrak{M}', x$; moreover, the relation R' is universal on the latter model! ∎

From the point of view of more traditional quantifier formalisms dealing with unstructured domains, $\mathbf{S5}$ has at least two non-standard features. Firstly, $\mathbf{S5}$ allows for arbitrary deep nestings of modal operators (or quantifiers) — in the traditional quantifier formalisms this is usually not allowed for. But, by a folklore result, over $\mathbf{S5}$ each formula is equivalent to one without nestings of operators. A second feature of $\mathbf{S5}$ that is not usually present in traditional quantificational formalisms, is that proposition letters (or variables over subsets of the domain) need not occur inside the scope of a modal operator (or quantifier). The latter feature suggests that restrictions on admissible $\mathbf{S5}$-formulas are a natural thing to consider here. Many (generalized) quantifier formalisms designed for reasoning with *all* and *some* may actually be viewed as *fragments* of $\mathbf{S5}$ in this sense.

In this paper we consider two such formalisms. To introduce the first one, let us agree that a *syllogism* is an inference scheme with three quantified sentences, two premisses and one conclusion, using three variables, all three occurring in the premisses and one (the 'middle term') occurring in the premisses but not in the conclusion:

$$\frac{Q_1XY \qquad Q_2YZ}{Q_3XZ},$$

where the Q_i range over the quantifiers *all*, *some*, *no* and *not all*.

Let the *propositional syllogistic language* be propositional logic with atomic sentences of the form QXY, where $Q \in \{\,\textit{all}, \textit{some}\,\}$. In this language syllogisms may be written as implications. A *model* for this language is given by a domain W and an assignment V of subsets of W to the variables, and the truth conditions are the obvious ones.

Clearly, the propositional syllogistic language 'is' a fragment of $\mathbf{S5}$ that contains formulas generated by the following rule (for proposition letters $p, q, \ldots$):

$$\chi ::= \Diamond(p \wedge q) \mid \Box(p \rightarrow q) \mid \neg\chi \mid \chi_1 \wedge \chi_2.$$

The following axiomatization of validity in the above language may be found in [24]. (We have included the modal transcription of the axioms in the right-most column.)

$A19$	$\textit{all}\,XX$	$\Box(p \rightarrow p)$,
$A20$	$\textit{all}\,XY \wedge \textit{all}\,YZ \rightarrow \textit{all}\,XZ$	$\Box(p \rightarrow q) \wedge \Box(q \rightarrow r) \rightarrow \Box(p \rightarrow r)$,
$A21$	$\textit{some}\,YX \wedge \textit{all}\,YZ \rightarrow \textit{some}\,XZ$	$\Diamond(q \wedge p) \wedge \Box(q \rightarrow r) \rightarrow \Diamond(p \wedge r)$,
$A22$	$\textit{some}\,XY \rightarrow \textit{some}\,YX$	$\Diamond(p \wedge q) \rightarrow \Diamond(q \wedge p)$,
$A23$	$\neg\textit{some}\,XX \rightarrow \textit{all}\,XY$	$\neg \Diamond\, (p \wedge p) \rightarrow \Box(p \rightarrow q)$.

Let $\mathbf{SYLL}$ be the theory with $A19$–$A23$ as axioms, plus some standard axioms for propositional logic, and Modus Ponens as its sole rule of inference. The following may be proved by a Henkin-type argument.

THEOREM 3.19
$\mathbf{SYLL}$ completely axiomatizes validity in the propositional syllogistic language.

The next subsystem of $\mathbf{S5}$ we consider here is in some ways even more constrained than $\mathbf{SYLL}$. Atzeni and Parker [2] thoroughly study a system, that we call $\mathbf{AP}$, for *binary set containment inference* in knowledge representation, that is, a system for determining the consequences of positive constraints $X \subseteq Y$ and negative constraints $X \cap -Y \neq \emptyset$ on sets. The language of *binary set containment* consists of formulas of the form '$X\ isa\ Y$', and '$not(X\ isa\ Y)$', where X, Y are either $\top$, $\bot$, primitive variables, or obtained from primitive variables by application of non, where

$$non(X) = \begin{cases} \bot & \text{if } X = \top, \\ \top & \text{if } X = \bot, \\ X' & \text{if } X \text{ is of the form } non(X'), \\ non(X) & \text{otherwise.} \end{cases}$$

This language corresponds to the fragment of $\mathbf{S5}$ that is generated by the rule

$$\phi ::= \top \mid \bot \mid \Box(\sim l_1 \rightarrow \sim l_2) \mid \neg\Box\,(\sim l_1 \rightarrow \sim l_2),$$

where l_1, l_2 are literals of the form p or $\sim p$ (for p a proposition letter), and where $\sim \phi$ is defined analogously to $non(X)$.[10] '$X\ isa\ Y$' is true in a model (W, V) if the set $V(X)$ denoted by X is a subset of the set $V(Y)$ denoted by Y.

As with **AL** axiomatizations for binary set containment cannot have the usual form of a set of implicational axioms plus some rules. Therefore, the proof system we are about to present, has rules only.

DEFINITION 3.20
Let $X, Y, \ldots$ be as before. Then $X\ int\ Y$ is short for $not(X\ isa\ non(Y))$. Let $\mathbf{AP}$ be the theory with the following inference rules:

$A24$	$X\ int\ Y \vdash X\ int\ \top$	$\neg\Box(p \to \sim q) \vdash \neg\Box(p \to \sim\top)$,
$A25$	$X\ int\ Y \vdash X\ int\ X$	$\neg\Box(p \to \sim q) \vdash \neg\Box(p \to \sim p)$,
$A26$	$X\ int\ Y \vdash Y\ int\ X$	$\neg\Box(p \to \sim q) \vdash \neg\Box(q \to \sim p)$,
$A27$	$X\ int\ Y, Y\ isa\ Z \vdash X\ int\ Z$	$\neg\Box(p \to \sim q), \Box(q \to r) \vdash \neg\Box(p \to \sim r)$,
$A28$	$X\ int\ non(X) \vdash Y\ isa\ Z$	$\neg\Box(p \to p) \vdash \Box(q \to r)$,
$A29$	$X\ int\ non(X) \vdash Y\ int\ Z$	$\neg\Box(p \to p) \vdash \neg\Box(q \to \sim r)$,
$A30$	$\vdash X\ isa\ \top$	$\vdash \Box(p \to \top)$,
$A31$	$\vdash X\ isa\ X$	$\vdash \Box(p \to p)$,
$A32$	$X\ isa\ Y, Y\ isa\ Z \vdash X\ isa\ Z$	$\Box(p \to q), \Box(q \to r) \vdash \Box(p \to r)$,
$A33$	$X\ isa\ Y \vdash non(Y)\ isa\ non(X)$	$\Box(p \to q) \vdash \Box(\sim q \to \sim p)$,
$A34$	$X\ isa\ non(X) \vdash X\ isa\ Y$	$\Box(p \to \sim p) \vdash \Box(p \to q)$.

THEOREM 3.21 (Atzeni and Parker [2])
The system $\mathbf{AP}$ completely axiomatizes validity in the language of binary set containment.

PROOF. As the language of binary set containment is very restricted the usual Henkin-method does not seem to be applicable. Instead, Atzeni and Parker [2] use a quite long alternative argument. It is also possible to use semantic tableaux, but this yields an even longer proof. ∎

Let us now add the graded modalities to $\mathbf{S5}$, and obtain the last 'main' language of this paper, one that is suitable for reasoning about finite quantities of unstructured sets: $\mathbf{Gr(S5)}$.

DEFINITION 3.22
The modal logic $\mathbf{Gr(S5)}$ over the language $Form(\Phi, \{\Diamond_n : n \in \mathbb{N}\})$ is normal in $\{\Box_0\}$ and has moreover the following axioms ($n \in \mathbb{N}$):

$A35 \quad \Box_0(\phi \to \psi) \to (\Diamond_n\phi \to \Diamond_n\psi)$,
$A36 \quad \Box_0\neg(\phi \wedge \psi) \to (\Diamond!_n\phi \wedge \Diamond!_m\psi \to \Diamond!_{n+m}(\phi \vee \psi))$,
$A37 \quad \Diamond_{n+1}\phi \to \Diamond_n\phi$,
$A38 \quad \Box_0\phi \to \phi$,
$A39 \quad \Diamond_n\phi \to \Box_0\Diamond_n\phi$.

The axioms $A38$ and $A39$ added to $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ force the relation R^c in the canonical model to be an equivalence relation.

THEOREM 3.23 ([11, 8])
For all formulas ϕ in the language of $\mathbf{Gr(S5)}$, we have $\mathbf{Gr(S5)} \vdash \phi$ iff ϕ is valid on all models (W, V).

[10] Hence, unlike the earlier syllogistic language the language of binary set containment does allow for some structure in the 'quantified variables' p and q.

PROOF. As with $\mathbf{S5}$ the proof consists of three steps. First, we construct a canonical model using the method of Theorem 3.9, and prove a Truth Lemma. Finally, by using axioms $A38$ and $A39$ it may be shown that the relation R^c in the canonical model is reflexive, and satisfies

$$Succ_R(\Gamma_1, \Gamma_2) \neq 0 \Rightarrow Succ_R(\Gamma_1, \Delta) = Succ_R(\Gamma_2, \Delta),$$

for all m.c. sets $\Gamma_1, \Gamma_2, \Delta$. From this it follows that R^c-related sets Γ_1 and Γ_2 will be R^c-related to the same number of copies of any Δ. Hence, R^c must be an equivalence relation. But then, by taking $W^c = \{\Delta : Succ_R(\Gamma, \Delta) \neq 0\}$ for some Γ containing the formula ϕ we want to satisfy, we obtain a universal model that verifies ϕ at Γ. ∎

As with $\mathbf{S5}$ the move to study fragments of $\mathbf{Gr}(\mathbf{S5})$ can be well-argued for (cf. the remarks following Theorem 3.18). Van der Hoek and De Rijke [18] study a system called $\mathbf{Gr}(\mathbf{Bin})$ which is in fact formulated in a fragment of the $\mathbf{Gr}(\mathbf{S5})$-language. To define its syntax, assume that we have *primitives* $(X, Y, \ldots)$ built up from unary predicate letters $P_0, P_1, \ldots$ using complement $(\cdot)^c$ and intersection $\cap$. The *atomic formulas* have the form $more_n\, XY$ ('$|X \cap Y| > n$'), with $n \in \mathbb{N}$ and X, Y primitives. We interpret such formulas on models $\mathfrak{M} = (W, V)$ only in a *global* manner:

$$\mathfrak{M} \models more_n\, XY \text{ iff } |V(X) \cap V(Y)| > n.$$

A useful abbreviation is $all\ but_n\, XY := \neg more_n\, XY^c$. Further, $precisely_n\, XY$ is defined in the obvious way. Notice that this language corresponds to the fragment of $\mathbf{Gr}(\mathbf{S5})$ that is given by the rule $\chi ::= \Diamond_n(\phi \wedge \psi) \mid \neg\chi \mid \chi_1 \wedge \chi_2$, for ϕ, ψ purely propositional formulas. We obtain an axiomatization $\mathbf{Gr}(\mathbf{Bin})$ for this language simply by removing from the $\mathbf{Gr}(\mathbf{S5})$ axioms $A35$–$A39$ all axioms that alter the number of nestings of operators (that is, $A38$ and $A39$). This plus an axiom governing the way in which the operators $more_n$ combine with conjunction, is sufficient.

DEFINITION 3.24
The logic $\mathbf{Gr}(\mathbf{Bin})$ has the rules Modus Ponens, Substitution and a *restricted* version of Necessitation: if the primitive X (considered as a propositional formula) is a propositional tautology, then $all\ but_0 \top X$ is a theorem of $\mathbf{Gr}(\mathbf{Bin})$. Besides the axioms of propositional logic (on the level of formulas now) its axioms are the following (we have included their $\mathbf{Gr}(\mathbf{S5})$-equivalents):

$A40$ $\quad all\ but_0\, XY \to (more_n \top X \to more_n \top Y)$
$$\Box(p \to q) \to (\Diamond_n(\top \wedge p) \to \Diamond(\top \wedge q)),$$
$A41$ $\quad all\ but_0\, XY^c \to (precisely_n \top X \wedge precisely_m \top Y \to precisely_{m+n} \top(X \cup Y)),$
$$\Box(p \to \neg q) \to (\Diamond!_n(\top \wedge p) \wedge \Diamond!_n(\top \wedge q) \to \Diamond!_{m+n}(\top \wedge (p \vee q))),$$
$A42$ $\quad more_{n+1}\, XY \to more_n\, XY, \qquad \Diamond_{n+1}(p \wedge q) \to \Diamond_n(p \wedge q),$
$A43$ $\quad more_n\, XY \leftrightarrow more_n \top(X \cap Y), \qquad \Diamond_n(p \wedge q) \leftrightarrow \Diamond_n(\top \wedge (p \wedge q)).$

THEOREM 3.25 ([18])
Let ϕ be a formula in the language of $\mathbf{Gr}(\mathbf{Bin})$. Then, $\mathbf{Gr}(\mathbf{Bin}) \vdash \phi$ iff ϕ is valid on all models.

PROOF. The proof consists of two steps. In the first one a canonical model is constructed. Since this construction diverges from the ones used so far, we will give some details.

Let ϕ be a consistent formula in disjunctive normal form. At least one of its disjuncts ψ must be consistent. To build a canonical model for this ψ, we restrict ourselves to the unary predicate letters occurring in ψ. We may assume that ψ is a conjunction of formulas of the form $(\neg)(more_n\, XY)$, containing only the predicate letters $P_0, \ldots, P_k$. Let N be the greatest number for which $more_{N-1}\, XY$ is a subformula of ψ. We use $\mathcal{P}_s$ $(s \leq 2^k)$

for formulas $(\neg)P_0 \wedge \ldots \wedge (\neg)P_k$, and $\mathcal{U}_t$ $(t \leq 2^{2^k})$ for disjunctions of such $\mathcal{P}_s$'s. Define $Form_\psi = \{(\neg)more\,\mathcal{U}_i\mathcal{U}_j : n \leq N, i, j \leq 2^{2^k}\}$. Let Ψ be a subset of $Form_\psi$ that is m.c. in $Form_\psi$ and contains all conjuncts of ψ. This set Ψ contains all the instructions we need to build our model $\mathfrak{M}^c$. First we associate with each $\mathcal{P}_s$ a set Π_s such that Π_s contains exactly the positive atoms of $\mathcal{P}_s$. Then we put all pairs (Π_s, n) in W^c for which $more_n \top \mathcal{P}_s \in \Psi$, and stipulate that $(\Pi_s, n) \in V^c(p_i)$ iff $p_i \in \Pi_s$, $(0 \leq n \leq N, s \leq 2^k)$.

The subsequent proof of the Truth Lemma is not quite trivial; we refer the reader to [18, Lemma 3.6] for details. ∎

4 Complexity

Before establishing complexity results for the axiom systems discussed in Section 3, we review some facts needed to understand these results.

The complexity classes involved are among P, NP, PSPACE and EXPTIME: P is the class of problems decidable in deterministic polynomial time, NP are the problems decidable in non-deterministic polynomial time, PSPACE are the problems decidable in deterministic polynomial space, and EXPTIME are the problems decidable in deterministic polynomial time. The relation between these classes is: P $\subseteq$ NP $\subseteq$ PSPACE $\subseteq$ EXPTIME; the only *known* strict inclusion is P $\subset$ EXPTIME. A *reduction* is a polynomial time computable many-one function. A problem L is X-*hard* (for X $\in$ { NP, PSPACE, EXPTIME }) if for every problem in X there is a reduction to L. A problem is X-*complete*, for X as before, if it is both *in* X and X-hard. (Notice that it makes no sense to talk about P-completeness: given our notion of reduction, being in P is the least property we can measure.)

One more detail before we start: we assume that the operators $\langle R\rangle_n$, $[R]_n$ have their indices n coded in *unary* (that is, the integer n is assumed to be represented by a string of length n). This yields the following definition of the length $\#\phi$ of a formula ϕ:

$$\begin{array}{rclcrcl} \#p & = & 1 & \quad & \#(\phi \wedge \psi) & = & 1 + \#\phi + \#\psi \\ \#(\neg\phi) & = & 1 + \#\phi & \quad & \#(\langle R\rangle_n\phi) & = & 1 + n + \#\phi. \end{array}$$

Structured domains

THEOREM 4.1
Satisfiability in $\mathbf{K}_\mathcal{R}$ is PSPACE-complete.

PROOF. Ladner [19] proves the PSPACE-completeness of satisfiability in $\mathbf{K}_\mathcal{R}$, in case $|\mathcal{R}| = 1$. But, as Halpern and Moses [14] note, Ladner's result extends to arbitrary $|\mathcal{R}| \geq 1$. ∎

The strong syntactical restrictions in the $\mathbf{AL}$-language may have forced us to use a non-standard approach in proving the completeness of $\mathbf{AL}$ in Section 3—complexity-wise it has the following pleasant consequence:

THEOREM 4.2
Satisfiability in $\mathbf{AL}$ is in P.

PROOF. Immediate from [9, Theorem 6.2]. ∎

THEOREM 4.3
Satisfiability in $\mathbf{Gr}(\mathbf{K}_\mathcal{R})$ is PSPACE-complete.

PROOF. As $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ extends $\mathbf{K}_{\mathcal{R}}$ PSPACE-hardness is obvious by Theorem 4.1. We now sketch an algorithm for checking satisfiability in $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$; the algorithm is a modification of an algorithm used by Donini *et al.* [9] to determine the complexity of $\mathcal{ALCNR}$; for full details the reader is referred to the latter paper.

By rewriting every formula in negation normal form (NNF), we may assume that every formula has negation signs only in front of proposition letters. (This rewriting can be done in linear time.)

Our algorithm tries to build a model by operating on finite sets of constraints '$x \models \phi$' and 'xRy'. The algorithm starts with a constraint set (c.s.) $S = \{ x \models \phi \}$, where ϕ is a formula in NNF that we want to satisfy. Subsequent steps add constraints to S until a 'clash' is generated, or a model satisfying ϕ can be obtained from the resulting set.

Let $n_{R,\phi,S}(x)$ be the number of variables y such that $xRy, y \models \phi \in S$; $[z/y]S$ is the c.s. obtained from S by replacing each occurrence of y in S by z; this replacement is said to be *safe* if for every variable x, formula ϕ and relation symbol R with $x \models \langle R\rangle_n\phi, xRy, xRz \in S$ we have $n_{R,\phi,[z/y]S}(x) > n$. A *clash* is a c.s. extending one of $\{ x \models \bot \}$, $\{ x \models p, x \models \neg p \}$, $\{ x \models [R]_0\bot, xRy \}$, or $\{ x \models \langle R\rangle_m\phi, x \models [R]_n{\sim}\phi \}$, where ${\sim}\phi$ is $\neg\phi$ in NNF and $m \geq n$.

The algorithm is based on the following completion rules:

1. $S \rightarrow_\wedge \{ x \models \phi, x \models \psi \} \cup S$,
   if $(x \models \phi \wedge \psi) \in S$ and $x \models \phi, x \models \psi \notin S$;
2. $S \rightarrow_\vee \{ x \models \chi \} \cup S$,
   if $x \models \phi \vee \psi \in S$, neither $x \models \phi \in S$ nor $x \models \psi \in S$, and $\chi = \phi$ or $\chi = \psi$;
3. $S \rightarrow_> \{ xRy, y \models \phi \} \cup S$,
   if $x \models \langle R\rangle_n\phi \in S$, $n_{R,\phi,S}(x) \leq n$ and y is a fresh variable;
4. $\rightarrow_{\leq 0} \{ y \models \phi \} \cup S$,
   if $x \models [R]_0\phi, xRy \in S$ and $y \models \phi \notin S$;
5. $S \rightarrow_\leq [z/y]S$,
   if $x \models [R]_n\phi, xRy, xRz \in S, n_{R,\phi,S}(x) > n > 0$ and replacing y by z is safe in S.

A c.s. is *complete* if no completion rules can be applied to it. A clash-free complete c.s. S derived from $\{ x \models \phi \}$ represents a model of ϕ, whose elements are the variables in S, whose valuation is defined by $y \in V(p)$ iff $(y \models p) \in S$, and whose relations are defined by xRy iff $xRy \in S$. The decidability and finite model property of $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ now follow immediately from

CLAIM 1. *Let ϕ be in NNF. Then ϕ is satisfiable in $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$ iff $\{ x \models \phi \}$ can be transformed into a clash-free complete c.s. using the above rules 1.–5.*

To get the PSPACE upper bound for $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$-satisfiability requires replacing rule 3 by 3′ below.

3′. $S \rightarrow_{>'} \{ xRy, y \models \phi \} \cup S$,
   if $x \models \langle R\rangle_n\phi \in S, n_{R,\phi,S}(x) \leq n$, there are no u, x' $(x' \neq x)$ such that $uR'x', uR'x \in S$ for some R' and x' has successors in S, and y is a fresh variable.

The idea behind rule 3′ is the following. For a variable x, and a relation symbol R rule 3′ produces *at most one* successor y with $y \models \phi$ that has itself successors.

Let a *trace* of $\{ x \models \phi \}$ be a c.s. obtained from $\{ x \models \phi \}$ by application of 1, 2, 3′, 4, 5, to which non of the latter rules applies.

CLAIM 2.

- *The length of a derivation starting from* $\{ x \models \phi \}$ *involving (only) rules 1, 2, 3', 4, and 5 is bounded by the length* $\#\phi$ *of* ϕ.
- *Every complete c.s. extending* $\{ x \models \phi \}$ *is the union of finitely many traces.*
- *If* S *is a c.s. extending* $\{ x \models \phi \}$, *and* $\mathcal{T}$ *is a finite set of traces such that* $S = \bigcup_{T \in \mathcal{T}} T$, *then* S *contains a clash iff some* $T \in \mathcal{T}$ *contains a clash.*

Based on this claim it is straightforward to write an algorithm that generates all complete c.s.s derivable from $\{ x \models \phi \}$ while keeping only one trace in memory at a time. If pointers are used to represent the subformulas of ϕ occurring in a trace, the algorithm needs space at most cubic in the size of ϕ to store a trace and the necessary control information. ∎

REMARK 4.4
Notice that the first item of the above Claim 2 would not be true if we had not assumed that numbers are coded as unary strings. If they are coded in binary the number of successors of a given variable x could be exponential in size of the input. We conjecture that binary coding will yield an EXPTIME-completeness result for $\mathbf{Gr}(\mathbf{K}_{\mathcal{R}})$-satisfiability.

As with the earlier system **AL**, the heavy syntactic restrictions in **ALN** have the following pleasant consequence:

THEOREM 4.5 ([9])
Satisfiability in **ALN** is in P.

COROLLARY 4.6
Satisfiability in $\mathbf{Gr}_l(\mathbf{K}_{\mathcal{R}})$ is PSPACE-complete.[11]

THEOREM 4.7
Satisfiability in $\mathbf{Gr}(\mathbf{K}_{(3)\cap})$ is PSPACE-complete.

PROOF. The proof is a variation on the proof of Theorem 4.3. Instead of considering constraints xRy one has to look at sequences of constraints xRy, xSy, xTy, where $T = R \cap S$. Cf. [9, Theorem 3.2] for details. ∎

Unstructured domains

We now turn to the complexity of logics designed to reason about unstructured domains. Throwing out structure yields an improvement in complexity as compared to Theorem 4.1:

THEOREM 4.8 ([19])
Satisfiability in **S5** is NP-complete.

PROOF. Let ϕ be an **S5**-formula. We first prove that ϕ is satisfiable iff it is satisfiable in a model with at most $1 + \#\phi$ elements. Let ϕ be satisfied in an **S5**-model $\mathfrak{M} = (W, V)$. We will use the subformulas of ϕ as instructions for extracting a set of elements W' from W that will serve as the domain of the desired small model. A function Γ is defined inductively on the instances of subformulas of ϕ.

1. Choose some $w \in W$ with $\mathfrak{M}, w \models \phi$; put $\Gamma(\phi) = \{ w \}$.

[11] This result would not be affected by converting to binary coding.

Now suppose that $\Gamma(\psi)$ has already been defined; then

2. $\Gamma(\chi) = \Gamma(\psi)$ if $\psi \equiv \neg\chi$;
3. $\Gamma(\chi_1) = \Gamma(\chi_2) = \Gamma(\psi)$ if $\psi \equiv \chi_1 \wedge \chi_2$;
4. $\Gamma(\chi) = \Gamma(\psi)$ if $\psi \equiv \Diamond_n\chi$ and $\mathfrak{M}, w \not\models \psi$;
5. if $\psi \equiv \Diamond_n\chi$ and $\mathfrak{M}, w \models \psi$, then choose $\Gamma(\chi)$ in such a way that $\mathfrak{M}, \Gamma(\chi) \models \chi$.

Define W' to be the union of all $\Gamma(\psi)$, where ψ ranges over subformulas of ϕ. Put $V' = V \restriction W'$, and $\mathfrak{M}' = (W', V')$. Then $|W'| \leq 1 + \#\phi$. Also, one may establish inductively that for all subformulas ψ of ϕ, and all $v \in W \cap W'$, we have $\mathfrak{M}, v \models \psi$ iff $\mathfrak{M}', v \models \psi$.

To show that satisfiability in **S5** is in NP, first guess a model with at most $1 + \#\phi$ elements, and then check the truth of ϕ in some state in this model. This guessing and checking can be 'done' in polynomial time. NP-hardness is obvious as **S5** extends propositional logic. ■

COROLLARY 4.9
Satisfiability in **SYLL** is NP-complete.

PROOF. As the system **SYLL** is a subsystem of **S5** its satisfiability problem is *in* NP by Theorem 4.8. But it is also NP-hard because it contains ordinary propositional logic. ■

When viewed as a fragment of the **S5**-language the language of **AP** is a very restricted one with limited expressive power. But these strong restrictions do pay off in terms of complexity:

THEOREM 4.10
Satisfiability in **AP** is in P.

PROOF. Atzeni and Parker [2] show that derivability in **AP** is in P. Thus to test for (un-) satisfiability one can test (in polynomial time) whether $not(\phi)$ is derivable. ■

THEOREM 4.11
Satisfiability in $\mathbf{Gr(S5)}$ is NP-complete.

PROOF. This is a slight variation on the proof of Theorem 4.8. Let ϕ be a $\mathbf{Gr(S5)}$-formula. We prove the result by first showing that ϕ is satisfiable iff ϕ is satisfiable in a model with at most $1 + \#\phi$ elements. Let ϕ be satisfied in a $\mathbf{Gr(S5)}$-model $\mathfrak{M} = (W, V)$. We will use the subformulas of ϕ as instructions for extracting a set of elements W' from W that will serve as the domain of the desired small model. A function Γ is defined inductively almost as in the proof of Theorem 4.8. Clauses 1, 2 and 3 are the same, while 4 and 5 are replaced by

4. $\Gamma(\chi) = \Gamma(\psi)$ if $\psi \equiv \Diamond_n\chi$ and $\mathfrak{M}, w \not\models \psi$;
5. if $\psi \equiv \Diamond_n\chi$ and $\mathfrak{M}, w \models \psi$, then choose $n + 1$ distinct points $w_1, \ldots, w_{n+1}$ such that $\mathfrak{M}, w_i \models \chi$ $(1 \leq i \leq n + 1)$, and put $\Gamma(\chi) = \{ w_1, \ldots, w_{n+1} \}$.

The model $\mathfrak{M}' = (W', V')$ is defined as in Theorem 4.8. Then $|W'| \leq 1 + \#\phi$, and, for all subformulas ψ of ϕ, and all $v \in W \cap W'$, we have $\mathfrak{M}, v \models \psi$ iff $\mathfrak{M}', v \models \psi$. It follows that satisfiability in $\mathbf{Gr(S5)}$ is in NP. Since $\mathbf{Gr(S5)}$ extends **S5** this implies the NP-completeness result. ■

REMARK 4.12
As in Theorem 4.3, the result of Theorem 4.11 depends in an essential way on our encoding of numbers as unary strings. Using binary coding we have only been able to show that $\mathbf{Gr(S5)}$-satisfiability is in PSPACE instead of NP [18, Theorem 3.11]. In fact, we conjecture that assuming binary coding of numbers the problem will be PSPACE-complete.

The lower bounds needed in the next corollary are immediate from the fact that the systems $\mathbf{Gr}_l(\mathbf{S5})$ and $\mathbf{Gr}(\mathbf{Bin})$ both contain propositional logic, while the upper bounds are given by Theorem 4.11. The result for $\mathbf{Gr}_l(\mathbf{S5})$ below would not be affected by converting to binary coding of numbers.

COROLLARY 4.13
The satisfiability problems for $\mathbf{Gr}_l(\mathbf{S5})$ and $\mathbf{Gr}(\mathbf{Bin})$ are NP-complete.

5 Concluding remarks

This paper is a first step towards carrying out a suggestion formulated in [18], namely to study the hierarchy of formal systems developed to reason about finite cardinalities of subsets of a domain. It combines results on such systems from generalized quantifier theory (GQT), modal logic (ML) and knowledge representation (KR).

Of course, the general interest in and framework for the counting expressions considered here, derives from GQT. We have used ML to obtain axiomatic completeness results. In that way ML has contributed to the theoretical understanding of formalisms from KR; in addition we think that this work on axiomatic aspects of (generalized) quantification over structured domains has appeared as a natural extension of existing work in GQT. In turn, results from KR settled complexity issues for a number of languages in ML and GQT, fitting in nicely with the growing sensitivity towards complexity issues in the later two fields.

What's next? We have only given axiomatic completeness results for a small number of concept languages here; finding (modal) axioms for the remaining languages considered by Donini *et al.* [9] is still an open problem. We think that, given the axiomatizations of this paper, finding those axioms will be an easy matter in many cases. But for some concept languages the search for a complete axiomatization may involve enriching the (modal) language with 'non-standard means' such as *constants* or *special derivation rules*, as suggested in our remarks following Theorem 3.16. This, in turn, may lead to new complexity issues.

Moreover, we have only looked at first-order counting expressions here: higher-order quantifiers like 'many' and 'most' are the next obvious candidates for an analysis in the spirit of this paper. In [18] a first step in this direction is made, but a lot still remains to be done, and this will most likely involve the model theoretic work done in GQT on such irreducible binary quantifiers (as found in [5, 25]).

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