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Abstract

A generalization of the classical mechanics is presented. The dynamical variables ~~(functions on the phase space)~~ are assumed to be elements of an algebra with anticommuting generators (the Grassmann algebra). The action functional and the Poisson brackets are defined. The equations of motion are deduced from the variational principle. The dynamics is described also by means of the Liouville equation for the phase-space distribution. The canonical quantization leads to the Fermi (anticommutator) commutation relations. The phase-space path integral approach to the quantum theory is also formulated. The theory is applied to describe the particle spin. In the nonrelativistic case, the elements of the phase-space are anticommuting three-vectors $\underset{\sim}{\xi}$, transformed to the Pauli matrices after the quantization: $\hat{\underset{\sim}{\xi}} = (\hbar/2)^{1/2}\underset{\sim}{\sigma}$. Classical description of the spin precession and of the spin-orbital forces is given. To introduce the relativistic spin in an invariant manner one needs a five-dimensional phase space (a four-vector plus a scalar). The Lagrangian is singular and there is a constraint, resulting from a "supersymmetry". The quantized phase-space elements are proportional to the Dirac matrices $\gamma_5\gamma_\mu$ and γ_5, while the constraint is transformed to the Dirac equation. The phase-space distribution and the interaction with an external field are also considered.

Contents

I. INTRODUCTION

1.1. Physical Background

During the past few years a not so familiar concept has emerged in high-energy physics, that of "anticommuting c-numbers". The formalism of the Grassmann algebra is well known to mathematicians and used for a long time. The analysis on the Grassmann algebra was developped and exploited in a systematic way in applying the generating functional method to the theory of second quantization [1] . This method was also used for the theory of fermion fields in a textbook by Rzewuski [2] . Seemingly, the first physical work dealing with the anticommuting numbers in connection with fer ions was that by Matthews and Salam [3] . The anticommuting c-numbers and the "Lie algebra with anticommutators" (i.e. the Z_2-graded Lie algebra) were the tools applied by Gervais and Sakita [4] to the dual theory. These authors invented the two-dimentional field-theoretical approach to the fermionic dual models (those proposed by Ramond and Neveu-Schwartz) and used the symmetry under the transformation with anticommuting parameters ("supergauge" transformations) to prove the no-ghost theorem. Naturally, anticommuting classical fields are necessary to construct the string picture of the fermionic dual models; a highly shillful approach to this problem is presented by Iwasaki and Kikkawa [5] . Interest to the concept in view was greatly increased by exciting new results obtained in 1974, elaborating the four-dimensional supersymmetry, discovered previously by Golfand and Lichtman [6] , and the formalism of superspace (see e.g. the review report by Zumino [7], where references to basic works on the subject may be found).

We present here an application of the analysis on Grassmann algebra to such a respectable problem as the Hamilton dynamics of a classical spinning particle. Both nonrelativistic and relativistic situations are considered. The first attempt to treat the classical relativistic top dates as early as 1926 and is du to Frenkel [8,9] . A review of subsequent work along this line is given by Barut [10] . However, the problem seems to be far from its exhaustive solution in terms of the conventional approach. An evidence to that is the paper by Hanson and Regge [11] , where one may find a number of further references. Our approach is essentially different as it uses the Grassmann algebra to describe the spin degrees of freedom. Apart from the concrete physical application to spin dynamics the present theory may be of some interest as an example of a generalized Hamilton dynamics with the appropriate quantization scheme.

1.2. Mathematical Background

To define the classical mechanics in an abstract way one needs three basic objects. 1) A differentiable manyfold M, called the phase space. Local coordinates x_k may be introduced in the manyfold M. In principle, no global coordinates may exist, and even they do, there may be no reasonable definition of the canonical coordinate-momentum pairs.

2) The algebra $\mathfrak{A}$(M) of complex-valued differentiable functions on M defined in terms of the usual sum and product operations .

3) A Lie algebra of the Poisson brackets in $\mathfrak{A}(M)$, given by means of a skew-symmetric tensor field $\omega_{kl}(x)$ (sum over repeated indeces is implied)

$$\{f, g\}_{P.B.} = \omega_{kl}(x)\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l} \qquad (1.1)$$

for any f(x) and $g(x)$ belonging to $\mathfrak{A}(M)$. The field $\omega_{kl}(x)$ satisfies the condition

$$\omega_{mn}\frac{\partial \omega_{kl}}{\partial x_n} + \omega_{ln}\frac{\partial \omega_{mk}}{\partial x_n} + \omega_{kn}\frac{\partial \omega_{lm}}{\partial x_n} = 0, \qquad (1.2)$$

which is equivalent to the Jacobi identity. Physical observables are real elements of $\mathfrak{A}(M)$. Dynamics is a continuous one-parameter isomorphism group on $\mathfrak{A}(M)$, determined by a Hamilton function H(x) by means of the equation

$$\frac{df}{dt} = \{H, f\}_{P.B.} \qquad (1.3)$$

for any element f(x), where the "time" t is the parameter of the group.

A way to generalize the concept of the classical mechanics is to abandon the manifold M, the "material basis" of the mechanics, retaining only the algebraic construction: ring and Lie algebra. The Grassmann variant is a simple example of such an "ideal mechanics", for which the multiplication in the algebra is not commutative. The generalization is rather straight-forward, because though the elements of the Grassmann algebra are not just functions, there are for them meaningful analogues of such concepts of the conventional analysis as the differentiation, the integration, and the Lie groups, as introduced in the work [12] (see also the review article [13]).

To quantize an ideal mechanics is to construct an associative algebra of operators in the Hilbert space, related to the classical algebra and having some general properties, discussed in paper [27] , which may be formulated in a purely algebraic way independently on the existence of a material phase space. One may see that in the Grassmann case with the "flat" Poisson brackets (ω_{kl} is a constant matrix) the operator algebra has a finite-dimensional representation.

1.3. Results and Discussion

The basic idea of the present approach is to consider elements of a Grassmann algebra as classical dynamical variables, i.e. functions of the phase space. The action functional, the Hamiltonian and the Poisson brackets, as well as some other concepts of the classical mechanics. are defined. The receipt of quantization is to substitute the Poisson bracket for the canonical variables by the anticommutator of the corresponding operators (as usual, devided by $-i\hbar$).So after the quantization the Grassmann algebra generates the Clifford algebra.

The Grassmann algebra with three generators, transformed as components of a three-vector under space rotations, gives rise to the nonrelativistic spin dynamics. Quantized canonical variables are represented by the Pauli matrices. The Grassmann algebra with five generators, an axial vector and a pseudoscalar, is necessary for description of the relativistic spin. Quantized variables are expressed in terms of the Dirac matrices. In the relativistic case the system, describing a particle, is constrained in the spin phase space, as well as in the orbital phase space. The spin constraint is just the Dirac equation. Thus after quantization the present scheme reproduces the well known Pauli-Dirac theory of the spinning electron. A brief account of our results was published previously [14].

The quantal action for anticommuting canonical variables was written first by Schwinger [15], who named them variables of the second kind. However, the classical mechanics and the theory of a relativistic spinning particle was not considered by Schwinger; this author had in mind the quantized electron fields necessary for his formulation of the quantum electrodynamics. No clear statement that the classical variables not only anticommute, but have also zero square, may be found in the work by Schwinger. He also argued that the number of the second-kind variables must be even (note, that in our construction the phase space is odd-dimensional), so that complex-conjugated coordinate-momentum pairs may be defined. No particular mechanical system was discussed by Schwinger. Perhaps our theory is useful as a simple example to Schwinger's variational formulation of the quantum field theory *). The idea to consider the generalization of the classical mechanics on a ring with arbitrary generators was suggested by Martin [16], who also presented a nonrelativistic top as example of mechanics on a ring with anticommuting generators. Unfortunately, we were not aware of this interesting work in time of our first publication [14] and did not mentioned it. The progress of the theory of supersymmetry provoked an interest to the classical mechanics in the

*) A Lagrangian formalism for spin variables, extending Schwinger's variational principle, was constructed by Volkov and Peletminsky [52].

superspace, in other words, to the theory of diffeomorphism groups on the Grassmann algebras; an evidence to that is a note by DeWitt [17].

In Section II, the nonrelativistic spinning particle is considered. The classical action principle is formulated. The phase-space dynamics based on the Liouville equation is developed. The canonical quantization is discussed in general and reconstruction of the conventional formalism is shown. The path integral in the Grassmann phase space is defined; the quantal Green's function for precession of spin in a constant field is calculated by this method. The theory of a relativistic spinning particle is presented in Section III. It is shown that in the invariant description of a free particle there are two symmetries, "gauge" and "supergauge". The quantization is done and the Dirac equation is deduced. The case of external field is also considered and the Bargmann-Michel-Telegdi equation is obtained in the classical mechanics. Some necessary results, published previously, are presented in the most appropriate form in Appendices. Results on the Grassmann algebra are compiled in Appendix A. Phase-space representation of quantal operators and the phase-space path integral for Green's function in case of the conventional theory are considered in Appendices B and C.

II. NONRELATIVISTIC SPIN DYNAMICS

2.1. Classical Action Principle and Equations of Motion

Suppose that the dynamical variables, describing the nonrelativistic spin dynamics, are elements of the Grassmann algebra $\mathcal{G}_3$ with three real generators ξ_k , $k = 1,2,3$

(for definitions see Appendix A). Define the phase-space trajectory $\xi(t)$ as an odd element of $\mathcal{G}_3$, depending on a time parameter t. Introduce the classical action as a functional of $\xi(t)$ and suppose it is an even real element of $\mathcal{G}_3$. Write it in a form analoguous to the Hamiltonian action (cf. Eq. (C.1)):

$$\mathcal{A}_\xi(t_i, t_f) = \int_{t_i}^{t_f} dt \left[\frac{1}{2} \tilde{\omega}^{k\ell} \xi_k \dot{\xi}_\ell - H(\xi) \right], \quad (2.1)$$

where $H(\xi)$ is an even real function of ξ, the Hamiltonian, $\dot{\xi} = d\xi/dt$, and $\tilde{\omega}$ is a symmetric imaginary matrix ($\tilde{\omega}$ is anti-Hermitian, as in the conventional mechanics). By means of a linear transformation $\tilde{\omega}$ may be reduced to the simplest form

$$\tilde{\omega}^{k\ell} = i\,\delta_{k\ell}\,. \quad (2.2)$$

Note that the first term in Eq. (2.1) is not a complete derivative, because ξ and $\dot{\xi}$ anticommute. As any element of $\mathcal{G}_3$, $H(\xi)$ is a polynomial of a degree, no more than 3. Only even terms may be present; so, omitting an inessential constant, the most general form of the Hamiltonian is

$$H(\xi) = -\frac{i}{2}\, \varepsilon_{k\ell m}\, b_k\, \xi_\ell\, \xi_m\,, \quad (2.3)$$

where b_k are real numbers.

The equations of motion are obtained under the condition that the variation of $\mathcal{A}$ be zero:

$$\dot{\xi}_k = i H \overleftarrow{\partial}_k = \varepsilon_{k\ell m}\, b_\ell\, \xi_m\,, \quad (2.4)$$

where $\overleftarrow{\partial}_k \equiv \overleftarrow{\partial}/\partial\xi_k$ is the right derivative. The solution of this equation is evident

$$\underset{\sim}{\xi}(t) = R(t)\underset{\sim}{\xi}(0) \tag{2.5}$$

where $\underline{R}$ is an orthogonal matrix describing the rotation with the angular velocity $\underset{\sim}{b}$. This solution may be interpreted as the spin precession in an external magnetic field $\underset{\sim}{B}$, where $\underset{\sim}{b} = \varkappa B$ and $\varkappa$ is the magnetic moment. An explicit time dependence of $\underset{\sim}{b}$ is also possible. The Hamiltonian (2.3), a bilinear function on the phase space, is an analogue of the oscillator. A formal equivalence between the spin precession and the Fermi oscillator was also noted in another context [18] .

In accordance with Eq. (2.4) define the Poisson brackets for any pair of dynamical variables

$$\{f(\xi), g(\xi)\}_{P.B.} \equiv i(f\overleftarrow{\partial}_k)(\overrightarrow{\partial}g), \tag{2.6}$$

$$\dot{f} = \{H, f\}_{P.B.} \tag{2.7}$$

Evidently, the Poisson brackets are antisymmetric if $\underline{f}$ and $\underline{g}$ are even elements of the Grassmann algebra and if $\underline{f}$ is an even element while $\underline{g}$ is an odd element. In case $\underline{f}$ and $\underline{g}$ are odd elements the Poisson brackets are symmetric. The graded version of Jacobi's identity is also valid. So the algebra defined by Eq. (2.6) is a Z_2 -graded Lie algebra (see e.g. the definition in [12] or [13]). For the canonical variables the Poisson brackets are

$$\{\xi_k, \xi_\ell\}_{P.B.} = i\delta_{k\ell} \,. \tag{2.8}$$

The rotation group in the Grassmann phase space is generated by the spin angular momentum

$$S_k = -\frac{i}{2}\,\varepsilon_{klm}\xi_l\xi_m \equiv -\frac{i}{2}(\underset{\sim}{\xi}\times\underset{\sim}{\xi})_k, \tag{2.9}$$

$$\{S_k,\xi_l\}_{P.B.} = -\varepsilon_{klm}\xi_m. \tag{2.10}$$

The classical mechanics of a nonrelativistic particle with spin is constructed in the phase "superspace" consisting of the six-dimensional obrital subspace (q_k, p_k) and the three-dimensional spin Grassmann subspace. The most general action describing a particle in a local external field is

$$\mathcal{A}(t_i,t_f)=\int_{t_i}^{t_f} dt\Big[\underset{\sim}{p}\dot{\underset{\sim}{q}} + \frac{i}{2}\underset{\sim}{\xi}\dot{\underset{\sim}{\xi}} - p^2/2m - V_0(\underset{\sim}{q}) - \underset{\sim}{L}\underset{\sim}{S}\,V_1(\underset{\sim}{q}) - \underset{\sim}{S}\underset{\sim}{B}(\underset{\sim}{q})\Big], \tag{2.11}$$

where $L_k = \varepsilon_{klm}q_l p_m \equiv (\underset{\sim}{q}\times\underset{\sim}{p})_k$ is the orbital angular momentum, $V_0(\underset{\sim}{q})$ and $V_1(\underset{\sim}{q})$ are potential functions, $\underset{\sim}{B}(\underset{\sim}{q})$ is a vector field. The term with V_1 in Eq. (2.11) is the spin-orbital interaction. The equation of motion derived from the variational principle are

$$\begin{aligned}
\dot{q} &= p/m + (S\times q)\,V_1,\\
\dot{p} &= -\nabla V_0 - (LS)\nabla V_1 + (S\times p)\,V_1 - \nabla(SB),\\
\dot{\xi} &= (L\times\xi)\,V_1 + (B\times\xi)
\end{aligned} \tag{2.12}$$

It is remarkable that in presence of the spin-orbital interaction the orbital subspace is not invariant, so $\underset{\sim}{q}$ and $\underset{\sim}{p}$ are not just real numbers. The dynamics algebra is a ring with 6 commuting and 3 anticommuting generators. The equations are simplified in the case of spherical symmetry; it is considered in some detail in Section 2.5.

2.2. The Phase-Space Distribution and Observables

Relation between an abstract mechanics and observable quantities is established by means of a distribution function in the phase space. As in the conventional mechanics, the dynamical principle for the Grassmann variant may be formulated as a Cauchy problem for the distribution $\rho(\xi, t)$. The Loiuville equation is

$$\frac{\partial \rho}{\partial t} + \{H, \rho\}_{P.B.} = 0, \qquad (2.13)$$

the equations of motion (2.4) are just its characteristics equations. For any dynamical variable $f(\xi)$, its averaged value is observed, which is a number

$$\langle f \rangle = \int f(\xi)\rho(\xi,t)\, d^3\xi, \qquad d^3\xi = i\, d\xi_3\, d\xi_2\, d\xi_1. \qquad (2.14)$$

The integral is defined in Appendix A. It is appropriate to assume that the distribution is an odd real element of $\mathcal{G}_3$:

$$\rho(\underset{\sim}{\xi}) = -\frac{i}{6}(\underset{\sim}{\xi}\underset{\sim}{\xi}\underset{\sim}{\xi}) + \underset{\sim}{c}\underset{\sim}{\xi}, \qquad (2.15)$$

$$(\underset{\sim}{\xi}\underset{\sim}{\xi}\underset{\sim}{\xi}) \equiv \varepsilon_{klm}\,\xi_k \xi_l \xi_m .$$

The distribution is normalized and $\underset{\sim}{c}$ is the average spin momentum

$$\langle 1 \rangle = 1, \quad \langle \underset{\sim}{S} \rangle = \underset{\sim}{C}, \quad \langle \underset{\sim}{\xi} \rangle = 0. \tag{2.16}$$

In case of motion (2.5) the vector $\underset{\sim}{c}$ depends on the time $\underline{t}$, and the dependence is given by the same rotation matrix $\underline{R}$. So the average spin vector is subject to the precession.

To be an honest distribution the function $\rho(\underset{\sim}{\xi})$ must be non-negative in some sense. The usual way to generalize the concept of positivity is to demand that the integral of ρff^* be non-negative for any function $\underline{f}$. One may see that this is true only in the trivial case $\underset{\sim}{c} = 0$. This is the reason, why the Grassmann variant of the classical mechanics can not be applied to the real world. It acquires a physical meaning only after the quantization.

2.3. The Canonical Quantization

In accordance with the general rule of quantization [19], we replace the Poisson brackets for the canonical variables in the Grassmann case by the anticommutator of the corresponding operators, divided by $-i\hbar$:

$$[\hat{\xi}_k, \hat{\xi}_\ell]_+ = \hbar \delta_{k\ell} \tag{2.17}$$

Renormalizing the operators suitably, we get the Clifford algebra with 3 generators:

$$\hat{\xi}_k = (\hbar/2)^{1/2} \hat{\sigma}_k, \quad [\hat{\sigma}_k, \hat{\sigma}_\ell]_+ = 2\delta_{k\ell}. \tag{2.18}$$

The only irreducible representation of this algebra is two-dimensional, it is equivalent to that realized by the Pauli matrices. Consequently,

$$\hat{S}_k = -\frac{i}{2}\varepsilon_{klm}\hat{\xi}_k\hat{\xi}_l = \frac{1}{2}\hbar\hat{\sigma}_k\,,\ [\hat{S}_k,\hat{S}_l]_- = i\hbar\varepsilon_{klm}\hat{S}_m\,. \tag{2.19}$$

Note, that in the conventional theory of the angular momentum the starting point is the commutator, while the simple form of the anticommutator arises only in the spinor representation describing the spin 1/2. The present approach is inverse: the anticommutator (2.17) is postulated and therefore only the spinor representation is produced.

The operator corresponding to the phase-space distribution (2.15) is proportional to the usual density matrix

$$\hat{\rho} = 2(\hbar/2)^{3/2}\left(\tfrac{1}{2} + \underset{\sim}{c}\underset{\sim}{\sigma}/\hbar\right), \tag{2.20}$$

while the integral over the phase space is replaced by tracing of the representing matrix. Note that the matrix $\hat{\rho}$ is positive **semidefinite** if $|\underset{\sim}{c}| \leqslant \hbar/2$. So, the purely quantal nature of the spin manifests itself once more.

The Heisenberg equations of motion are obtained from Eqs. (2.4) by means of the direct substitution. To get the Schroedinger picture one has to introduce the spinor ψ function, "factorizing" the density matrix $\hat{\rho}(t) = \psi(t)\times\psi^*(t)$. Its time evolution is mastered by the Pauli equation. So the usual theory of the nonrelativistic spin 1/2 is reconstructed.

Now, it is appropriate to consider the case of the Grassmann algebra with any number of generators, $\mathcal{G}_n$. Evidently, the construction of the classical mechanics, described for $\mathcal{G}_3$, may be directly expanded to $\mathcal{G}_n$. The quantization is defined by Eq. (2.17), and $\hat{\xi}_k = (\hbar/2)^{1/2}\hat{\sigma}_k$, $\underline{k}$ =

$= 1, \ldots, \underline{n}$, where $\hat{\sigma}_k$ are generators of the Clifford algebra C_n. It is known that C_n has only one irreducible Hermitian matrix representation. Its dimensionality is $\underline{d} = 2^m$ for $\underline{n} = 2m$ or $\underline{n} = 2\underline{m} + 1$, $\underline{m}$ is integer. (Note that the matrix representation of $\mathcal{G}_n$ is 2^n - dimensional). The case of an even $\underline{n}$ is in closer analogy with the conventional theory, as one may introduce pairs of conjugated canonical (complex) variables (q_r, p_r), $r = 1, \ldots, \underline{m}$, defining $q_1 = (\xi_1 + i\xi_2)/\sqrt{2}$, $p_1 = (i\xi_1 + \xi_2)/\sqrt{2} = iq_1^*$, $q_2 = (\xi_3 + i\xi_4)/\sqrt{2}$, etc. The anticommutators are

$$\left[\hat{q}_r, \hat{p}_s\right]_+ = i\hbar\,\delta_{rs}\ ,\ \left[q_r, q_s\right]_+ = \left[p_r, p_s\right]_+ = 0. \quad (2.21)$$

Just this case was considered by Schwinger [15]. It is the case of an odd $\underline{n}$ that is of interest for our purpose. A remarkable feature of C_n for odd $\underline{n}$ is that in the matrix representation the generators are not independent, $\hat{\sigma}_1 \hat{\sigma}_2 \ldots \hat{\sigma}_n = \pm i$. Indeed, the product commutes with any $\hat{\sigma}_k$ and its square is -1, while its sign is reffered to the choice between two classes of equivalent representations (right or left coordinate frames). Therefore, to make unambiguous the classical counterpart of a quantal operator, one has to declare whether it is an even element, or an odd element of $\mathcal{G}_n$.

Represent the quantal operators by their symbols, in analogy with the usual quantum mechanics (see Appendix B). For any operator $\hat{\underline{g}}$ a polynomial representation may be written

$$\hat{g} = \sum_{\nu=0}^{n} \sum_{\{k\}} g_\nu^{k_1 \ldots k_\nu}\, \hat{\xi}_{k_1} \ldots \hat{\xi}_{k_\nu}\ , \quad (2.22)$$

where $g_\nu^{\{k\}}$ are "c-number" totally antisymmetric tensors. For odd n, this form is unique if only terms of a fixed parity are present. Thus any $\hat{g}$ has two equivalent decompositions, even and odd. Define an analogue of the Weyl symbol for the operator

$$\hat{g} \to g(\xi) = \sum_{\nu,\{k\}} g_\nu^{k_1 \dots k_\nu} \xi_{k_1} \cdots \xi_{k_\nu} . \tag{2.23}$$

Each operator has two symbols, even and odd. It may be seen that they are interrelated by a Fourier transformation. As in the conventional theory, the Fourier transformation may be used also to formulate the Weyl quantization. Relation between an operator and its symbol is given by the integral form

$$\hat{g} \to g(\xi) = \int \exp[i(\xi\rho)]\, \tilde{g}(\rho)\, d^n\rho , \tag{2.24}$$

$$\hat{g} = \int \hat{\Omega}(\rho)\, \tilde{g}(\rho)\, d^n\rho , \quad \hat{\Omega}(\rho) = \exp[i(\hat{\xi}\rho)] \tag{2.25}$$

Here $\underset{\sim}{\rho} \equiv (\rho_1, \dots, \rho_n)$ are generators of the Grassmann algebra $\tilde{\mathcal{G}}_n$, anticommuting with ξ_k and $\hat{\xi}_k$. Properties of the operator $\hat{\Omega}(\rho)$ are similar to those of the operator $\hat{\Omega}(z)$, considered in Appendix B:

$$\hat{\Omega}(\underset{\sim}{\rho}_1)\hat{\Omega}(\underset{\sim}{\rho}_2) = \exp\left[\tfrac{1}{2}\hbar(\underset{\sim}{\rho}_1\underset{\sim}{\rho}_2)\right]\hat{\Omega}(\rho_1+\rho_2) \tag{2.26}$$

$$\mathrm{Tr}\,\hat{\Omega}(\underset{\sim}{\rho}) = d\left[1 + i(\hbar/2)^{n/2}\rho_1 \dots \rho_n\right] . \tag{2.27}$$

(We discuss now the case of an odd $\underline{n}$, $\underline{d} = 2^{(n-1)/2}$ is the dimensionality). Using these formulae, one easily gets

$$Tr\left[\hat{\Omega}(-\rho)\hat{g}\right] = d\left[i(\hbar/2)^{n/2}\tilde{g}(\rho) + g\left(\tfrac{i}{2}\hbar\rho\right)\right]. \tag{2.28}$$

This result enables one to find the symbol of the operator $\hat{g}$, if its parity is chosen (note, that $g(\xi)$ and $\tilde{g}(\rho)$ have opposite parities). The multiplication law for the symbols is obtainable from Eq. (2.26)

$$\hat{g}_1\hat{g}_2 \rightarrow g(\xi) = \int W(\xi_1,\xi_2,\xi)\, g_1(\xi_1)\, g_2(\xi_2)\, d^n\xi_1\, d^n\xi_2$$

$$W(\xi_1,\xi_2,\xi) = \left(\frac{\hbar}{2}\right)^n \exp\left\{\frac{2}{\hbar}\left[(\xi_1\xi_2) + (\xi_2\xi) + (\xi\xi_1)\right]\right\}. \tag{2.29}$$

Here ξ_1, ξ_2, ξ are regarded as three independent sets of generators of the Grassmann algebra $\mathcal{G}_{3n}$.

We conclude this Section with the following resumé. In the Grassmann mechanics the quantal operators have two representations: by finite dimensional matrices and by elements of the Grassmann algebra. This is quite similar to the usual quantum mechanics, where the operators may be represented either by functional kernels (say, in the coordinate space) or by their symbols, i.e. functions on the phase space.

2.4. Path Integral for Green's Function

The subject of this Section is to obtain an expression for the operator

$$\hat{G}(t) = \exp\left(-\frac{i}{\hbar}t\hat{H}\right) \tag{2.30}$$

in the Grassmann case, The method is to calculate its symbol $\mathcal{G}(\xi)$ in form of the phase-space path integral. This is a direct generalization of the approach applied to the quantum mechanics in a work by one of the authors [20] (see also the Appendix C). It is quite natural to use this approach, because in the Grassmann phase space one can not use the coordinate - momentum language, and it is impossible to define an analogue the Feynman path intergal in the coordinate (or momentum) space.

Represent the operator $\hat{\mathcal{G}}(t)$ as an infinite product of infinitesimal time translations

$$\hat{\mathcal{G}}(t) = \lim_{N\to\infty} \left[\hat{\mathcal{G}}(t/N) \right]^N . \tag{2.31}$$

Rewriting this form in terms of symbols and using the multiplication law (2.29) one gets $\mathcal{G}(\xi,t) = \lim_{N\to\infty} \mathcal{G}^{(N)}(\xi,t)$,

$$\mathcal{G}^{(N)}(\xi,t) = (\hbar/2)^{nN} \int \prod_{\nu=1}^{N} d^n\xi_\nu \, d^n\eta_\nu \\ \exp\Big\{ \frac{1}{\hbar} \sum_{\nu=1}^{N} \big[2(\xi_\nu\eta_\nu) + 2(\eta_\nu\xi_{\nu+1}) + 2(\xi_{\nu+1}\xi_\nu) \\ - i\,H(\eta_\nu)\,t/N \big] \Big\} \tag{2.32}$$

with the boundary condition $\xi_{N+1} = \xi$. Note that one should regard $\xi, \xi_1, \dots, \xi_N, \eta_1, \dots, \eta_N$ as independent sets of (anticommuting) generators of the Grassmann algebra. This formula is to be compared with Eq. (C.4). One may integrate over $\xi_1, \dots, \xi_N$, get an analogue of Eq. (C.8) and write the formal expression

$$\mathcal{G}(\xi,t) = \int \mathcal{D}[\eta(\tau)] \exp\Big\{ \frac{i}{\hbar} \mathcal{A}_{cl}[\eta(\tau)] + \\ + \frac{1}{\hbar} \big[(\xi\eta_0) + (\eta_0\eta_t) + (\eta_t\xi) \big] \Big\}, \tag{2.33}$$

where $\eta_0 = \eta(0)$, $\eta_t = \eta(t)$,

$$\mathcal{A}_{cl}[\eta(\tau)] = \int_0^t \left[\frac{i}{2}(\eta\dot{\eta}) - H(\eta)\right] d\tau, \qquad (2.34)$$

$$\mathcal{D}[\eta(\tau)] = \lim_{N\to\infty} (\hbar/2)^{nN/2} \prod_{\nu=1}^{N} d^n\eta_\nu . \qquad (2.35)$$

However, only Eq. (2.32) reveals the true meaning of the functional integral, and it is useful for further analysis.

Apply the path integral approach to a simple example: the spin precession in a constant magnetic field. The Hamiltonian is given by Eq. (2.3) and may be rewritten also as $H(\xi) = bS$, where S is the spin momentum (2.9). Using the resemblance to the harmonic oscillator we proceed along the same lines as in obtaining Eq. (C.16). In the present case

$$G(\xi, t) = \cos(bt/2)\exp\left[-\frac{2i}{\hbar}(Sn)\,tg(bt/2)\right]$$
$$= \cos(bt/2) - \frac{2i}{\hbar}(Sn)\sin(bt/2), \qquad (2.36)$$

where $b = |b|$, $n = b/b$; we have used the fact that $(Sn)^2 = 0$. To get this result, we calculated the Gaussian integrals over the Grassmann phase space by means of Eq.(A.9). Note that the cosine is now in the nominator, contrary to Eq.(C.16). Using the symbol and Eq. (2.19) it is quite easy to write the operator

$$\hat{G}(t) = \cos(bt/2) - i(\sigma n)\sin(bt/2) \qquad (2.37)$$

and to reproduce the result that one obtains calculating

$\exp(-\frac{i}{2}\,\underset{\sim}{b}\underset{\sim}{\sigma}t)$ in a usual manner.

A number of authors used the concept of the path integral in space of anticommuting functions: Khalatnikov [21], Matthews and Salam [3], Candlin [22], Martin [23] and others. A consistent mathematical formulation and the definition of the integral on the Grassmann algebra were given in paper [24]. On the other hand, some authors described the spin dynamics by means of phase spaces with commuting elements: Schulman [25], Bezak [26] (both considered the path integrals), Berezin [27], Tarski [28], Hanson and Regge [11]. The present approach seems to be the most adequate, and spin of the electron finds "a simple and ready representation" in the method of path integrals, absent before, as was stated in the book by Feynman and Hibbs [29] (p.355).

2.5. Motion in a central potential with Spin-Orbital Forces

For a simple example, consider the motion of a spinning particle influenced by forces presented by the Hamiltonian (2.11), assuming that $\underset{\sim}{B} = 0$ and that the potential functions depend on $R = |\underset{\sim}{q}|$ only. The intergals of motion are the total angular momentum $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$, L^2 and $\Lambda = \underset{\sim}{L}\underset{\sim}{S}$ (note that $S^2 \equiv 0$ and $\Lambda^2 \equiv 0$). From the equations (2.12) we get for the radial motion:

$$\dot{R} = P/m, \quad \dot{P} = -V_0'(R) + L^2/mR^3 - \Lambda V_1'(R) \tag{2.38}$$

The problem is now reduced to that of motion with the effective potential $U(R) = V_0 + L^2/2mR^2 + \Lambda V_1$ containing a nilpotent perturbation in the last term. Evidently, the solution is to be represented in form

$$R(t) = r(t) + \Lambda a(t), \quad P(t) = p(t) + \Lambda b(t), \tag{2.39}$$

where r, p, a, and b are number functions. Substituting into (2.38) one can see that r(t) and p(t) are just the solution of the problem with no account of the spin-orbital potential, while a and b satisfy the linear equations

$$\dot{a} = b/m, \quad \dot{b} = -g(t)a - f(t) \tag{2.40}$$

where

$$g(t) = V_0''(r) + 3L^2/mr^4, \quad f(t) = V_1'(r)$$

If the orbit is stable against small perturbation in the classical sense, $g(t) > 0$ and (2.40) is the equation for an oscillator with the frequency $[g(t)]^{1/2}$ and the driving force f(t), which are constants in case of a circular orbit. Thus the solution is identical to the usual perturbation theory in Λ, however it is exact, because higher powers of Λ vanish identically. To get the "observable" trajectory one has to average over the spin variables, i.e. to integrate R(t) with the distribution (2.15). The result is that in the final expression one should dubstitute Λ by a constant $\langle\Lambda\rangle = (cL)$, determined by the initial conditions.

As for the angular coordinate and spin, their motion is mastered by the equations

$$\begin{aligned} \dot{\xi} &= V_1(L \times \xi) \equiv V_1(\mathcal{L} \times \xi) \\ \dot{S} &= V_1(L \times S) \equiv V_1(\mathcal{L} \times S) \\ \dot{L} &= V_1(S \times L) \equiv V_1(\mathcal{L} \times L) \end{aligned} \tag{2.41}$$

where $V_1 = V_1[r(t)]$ is a function of t. It is possible to substitute R by r in the argument of V_1 in Eq.(2.41), because $\Lambda S \equiv 0$, $(S \times \xi) \equiv 0$, and $\Lambda (L \times \xi) \equiv 0$. The vectors ξ, S, and L precess around the same fixed axis $\mathcal{J}$ with the same angular velocity, which is constant in case of a circular orbit.

III. RELATIVISTIC SPIN AND THE DIRAC EQUATION

3.1. Classical Action and the Symmetrics

Construct the action for a relativistic spinning particle, invariant under the full Poincare group and having the nonrelativistic limit, considered in Section 2.1. Assume that the spin variables are components of a four-vector ξ_μ. However, an introduction of a new phase-space coordinate ξ_0 is not so inoffensive in view of two reasons. First, in the nonrelativistic limit a "second spin" $\xi_0 \xi_k$ would arize and the representation of the rotation group would be reducible. Second, and more serious, is that one is not able to quantize such a system because of the Minkowski indefinite metrics. To get a consistent scheme (and to reconstruct the Dirac theory) we assume that the action has an additional symmetry, so that ξ_0 be in fact excluded from the equations of motion, even though the equations are Lorentz-invariant.

Start from the action

$$\mathcal{A}_{free} = \int_{\tau_i}^{\tau_f} \left\{ -mz + \frac{i}{2} \left[(\xi \dot{\xi}) + (u\xi)(u\dot{\xi}) \right] \right\} d\tau, \tag{3.1}$$

$$z = \left[-(\dot{q})^2 \right]^{1/2}, \quad u_\mu = \dot{q}_\mu / z.$$

Here τ is a monotonic parameter labelling the points on the particle world line, $q_\mu(\tau)$ are coordinates of the point, $\dot{q}_\mu = dq_\mu/d\tau$, $\dot{\xi}_\mu = d\xi_\mu/d\tau$, the light velocity c = = 1. Note that ξ_μ are the phase-space elements, and (3.1) may be considered as Routh's form of the action. Our metric convention is (-, +, +, +), so $u^2 = -1$. The action is invariant under the reparametrization of the trajectory: $\tau \to \tilde{\tau} = \varphi(\tau)$, where $\varphi(\tau)$ is a monotonous function. The fundamental bilinear form for the ξ variables is degenerate, and there is no equation of motion for the longitudinal component of the vector ξ_μ. To formulate the dynamics, an additional constraint is necessary, and to make this constraint explicitly invariant introduce a new Grassmann variable ξ_5. The constraint is

$$(u\xi) + \xi_5 = 0. \tag{3.2}$$

To get a manifestly covariant canonical formalism for the system having a singular Lagrangian one may apply Dirac's method (the general approach is given in a book by Dirac [30], modern applications to the relativistic particle dynamics may be found in papers by Hanson and Regge [11] and by Casalbuoni et al. [31]). Add the constraint (3.2) with the Lagrange multiplier λ (anticommuting) to the original Lagrangian

$$\mathcal{A}_{free} = \int_{\tau_i}^{\tau_f} \left\{ -m z + \frac{i}{2}\left[(\xi\dot{\xi}) + \xi_5\dot{\xi}_5 - \left(u\xi + \xi_5 \right)\lambda \right] \right\} d\tau \tag{3.3}$$

The canonical momentum is

$$p^\mu = \partial\mathcal{L}/\partial\dot{q}_\mu = m u^\mu - \frac{i}{2}\left(\xi^\mu + (u\xi)u^\mu \right)\lambda/z, \tag{3.4}$$

and the phase-space constraints are

$$p^2 + m^2 = 0, \quad (p\xi) + m\xi_5 = 0 . \tag{3.5}$$

With account of the constraints the Hamilton action is

$$\mathcal{A}_{free} = \int_{\tau_i}^{\tau_f} \Big\{ (p\dot{q}) - \ell(p^2+m^2) + \frac{i}{2}\Big[(\xi\dot{\xi}) + \xi_5\dot{\xi}_5 - (p\xi + m\xi_5)\lambda/m\Big]\Big\} d\tau , \tag{3.6}$$

where l is another Lagrange multiplier (commuting). The equations of motion derived from the action principle are

$$\begin{gathered} \dot{q}_\mu = 2\ell p_\mu + i\xi_\mu \lambda/2m , \quad \dot{p}_\mu = 0, \\ \dot{\xi}_\mu = p_\mu \lambda/2m , \quad \dot{\xi}_5 = \lambda/2. \end{gathered} \tag{3.7}$$

The first equation is consistent with (3.4) if

$$\ell = (z - \tfrac{i}{2}\xi_5\lambda/m)/2m , \tag{3.8}$$

and the equation takes another form

$$\dot{q}_\mu = p_\mu z/m + \frac{i}{2}\left(\xi_\mu + p_\mu (p\xi)/m^2\right)\lambda/m . \tag{3.9}$$

An appearance of the second term in Eq. (3.9) might be anticipated; it is the classical analogue of Schroedinger's <u>Zitterbewegung</u> (a discussion of this concept was presented by Dirac [19], § 69, the algebraic aspects are considered by Jordan and Mukunda [32]). Note that the time evolution mixes coordinate and spin degrees of freedom, just as in the nonrelativistic case with a spin-orbital potential (cf. Eqs. (2.12)); the whole phase space of the relativistic spinning particle is a "superspace".

The constraints (3.5)result from the invariance of the action under two kinds of transformations. The first one was mentioned; it is the "gauge" group, $\tau \to \tilde{\tau} = \varphi(\tau)$. An infinitesomal transformation of the second kind is

$$\xi_\mu \to \tilde{\xi}_\mu = \xi_\mu + u_\mu \eta\,, \quad \xi_5 \to \tilde{\xi}_5 = \xi_5 + \eta\,,$$
$$q_\mu \to \tilde{q}_\mu = q_\mu + i\,\xi_\mu \eta/m\,, \tag{3.10}$$

where $\eta(\tau)$ is an anticommuting "parameter" depending on τ ambiguously. In analogy with the transformations introduced in a dual model by Gervais and Sakita [4] we call (3.10) the "supergauge" group. The variation of the action (3.3), induced by (3.10) is

$$\delta\mathcal{A} = i\int_{\tau_i}^{\tau_f} d\tau\, d(\xi_5\,\eta)/d\tau\,; \tag{3.11}$$

it vanishes if $\eta(\tau_i) = \eta(\tau_f) = 0$. It is remarkable that both transformations, gauge and supergauge, change the scale factor $\underline{z}$, contrary to the Poincaré group.

To fix a solution of the equations of motion (3.7), (3.9) one has to choose the indefinite factors λ and $\underline{z}$ (or $\underline{l}$) in some way, i.e. to fix the supergauge and the gauge. As for $\underline{z}$, two variants are used in case of no spin: i) $\underline{z} = 1$, τ is the self-time, and ii) $\underline{z} = m/p_0$, $\tau = q_0$ is the "laboratory" time. An appropriate choice of λ is not so evident. There are only 2 Poincaré-invariant anticommuting elements in the phase space: ξ_5 and $(p\xi\xi\xi) \equiv \equiv \varepsilon^{\alpha\beta\gamma\delta} p_\alpha \xi_\beta \xi_\gamma \xi_\delta$. If one takes $\lambda \sim \xi_5$, the equations of motion have no apparent $\underline{T}$ invariance. More appropriate is the choice

$$\lambda = ikz(p\xi\xi\xi), \tag{3.12}$$

where $\underline{k}$ is a real constant of dimension (action)$^{-2}$. Note that if $\underline{z}$ = const, λ is also conserved. This choice is rather convenient, because the second term in (3.9) vanishes identically and the motion of a free particle is quite simple:

$$q_\mu(\tau) = q_\mu(0) + \tau p_\mu z/m, \quad \xi_\mu(\tau) = \xi_\mu(0) + \tau p_\mu \lambda/2m. \tag{3.13}$$

A defect of the choice (3.12) is that it breaks the symmetry under space reflections. Just as in the nonrelativistic case, before the quantization one cannot decide whether ξ_μ is an axial vector (and ξ_5 is a pseudoscalar), or a vector (and ξ_5 is a scalar). However, parity of λ given by (3.12) is opposite to the parity of ξ_5 in both the variants. To identify τ with the laboratory time q_0 one may put

$$1 = (2p_0)^{-1}, \qquad \lambda = fm\xi_0 \tag{3.14}$$

where $\underline{f}$ is a scalar function of dimension (action)$^{-1}$.

3.2. Quantization and the Dirac Equation

The action (3.6) results in the canonical Poisson brackets

$$\{p_\mu, q_\nu\}_{P.B.} = g_{\mu\nu}, \; \{\xi_\mu, \xi_\nu\}_{P.B.} = ig_{\mu\nu}, \; \{\xi_5, \xi_5\}_{P.B.} = i, \tag{3.15}$$

where $g_{\mu\nu}$ = diag(-1, +1, +1, +1), other brackets vanish. The commutation relations for the quantal operators are

$$[\hat{p}_\mu, \hat{q}_\nu]_- = -i\hbar g_{\mu\nu}, \ [\hat{\xi}_\mu, \hat{\xi}_\nu]_+ = \hbar g_{\mu\nu}, \ [\hat{\xi}_5, \hat{\xi}_5]_+ = \hbar, \tag{3.16}$$

while the constraints (3.5) are converted into conditions on the physical states

$$(\hat{p}^2 + m^2)\psi = 0, \tag{3.17a}$$

$$(\hat{p}\hat{\xi} + m\hat{\xi}_5)\psi = 0. \tag{3.17b}$$

The operators $\hat{\xi}_\mu$, $\hat{\xi}_5$ are generators of the Clifford algebra C_5, its representation is four-dimensional and is given by the Dirac-Pauli matrices:

$$\hat{\xi}_\mu = (\hbar/2)^{1/2}\gamma_5\gamma_\mu, \quad \hat{\xi}_5 = (\hbar/2)^{1/2}\gamma_5, \tag{3.18}$$

where, as usual,

$$[\gamma_\mu, \gamma_\nu]_+ = -2g_{\mu\nu}, \ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3, \ \gamma_5^2 = 1, \tag{3.19}$$

γ_0 and γ_5 are Hermitean, $\gamma_{1,2,3}$ are anti-Hermitean. Multiplying (3.17b) by $(\hbar/2)^{-1/2}\gamma_5$ we get the Dirac equation $(p\gamma + m)\psi = 0$. Conditions (3.17a) and (3.17b) are consistent, as may be (and should be) checked directly. Note that without the condition (3.17b) the quantization would be inconsistent, because in view of (3.16) $\hat{\xi}_0^2 = -\hbar$ and an indefinite metrics arises.

Generators of the Lorentz group $J_{\mu\nu}$ are constructed along the conventional lines

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \tag{3.20}$$

In the classical theory

$$L_{\mu\nu} = q_\mu p_\nu - q_\nu p_\mu , \quad S_{\mu\nu} = -i\xi_\mu \xi_\nu \tag{3.21}$$

$$\begin{aligned} \{L_{\mu\nu}, q_\lambda\}_{P.B.} &= g_{\nu\lambda} q_\mu - g_{\mu\lambda} q_\nu \\ \{S_{\mu\nu}, \xi_\lambda\}_{P.B.} &= g_{\nu\lambda}\xi_\mu - g_{\mu\lambda}\xi_\nu . \end{aligned} \tag{3.22}$$

To get the quantal operators an (anti) symmetrisation is necessary

$$\hat{L}_{\mu\nu} = \frac{1}{2}(\hat{q}_\mu \hat{p}_\nu + \hat{p}_\nu \hat{q}_\mu - \hat{q}_\nu \hat{p}_\mu - \hat{p}_\mu \hat{q}_\nu) \tag{3.23}$$

$$\hat{S}_{\mu\nu} = -\frac{i}{2}(\hat{\xi}_\mu \hat{\xi}_\nu - \hat{\xi}_\nu \hat{\xi}_\mu) \equiv \frac{i}{4}\hbar(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) . \tag{3.24}$$

To construct a relativistic phase-space distribution, like (2.15), note that the components $(p\xi)$ and ξ_5 are not observable and were introduced in order to make the formalism invariant. So we assume that $\rho(\xi) = \delta[(p\xi)/m]\cdot \delta(\xi_5)\,\tilde{\rho}(\xi)$, where $\tilde{\rho}$ depends on the transversal components of ξ_μ only and is an odd element. As it is evident from definition of the integral on the Grassmann algebra (Eq.(A.6)), $\delta(\xi_k) = \xi_k$. With all this in view, write the phase-space distribution in a form ready for the quantisation

$$\begin{aligned} \rho(\xi) &= \frac{1}{2}\left[(p\xi)/m + \xi_5\right]\tilde{\rho}(\xi)\left[(p\xi)/m - \xi_5\right] , \\ \tilde{\rho}(\xi) &= (v\xi) - i\,b(p\xi\xi\xi) , \end{aligned} \tag{3.25}$$

where v_μ is a real four-vector, $(v\,p) = 0$, and $b = 1/6m$, as given by the normalization

$$\int \rho(\xi)\, d^5\xi = 1, \quad d^5\xi = -i\, d\xi_5\, d\xi_3\, d\xi_2\, d\xi_1\, d\xi_0 . \tag{3.26}$$

The function $\tilde{\rho}(\xi)$ is written in a form invariant under the supergauge transformations (3.10). The vector v_μ is a classical analogue of the Pauli-Lubanski vector; it determines the averaged value of the spin momentum, defined by (3.21)

$$\langle S_{\mu\nu} \rangle \equiv \int S_{\mu\nu}\, \rho(\xi)\, d^5\xi = \varepsilon_{\mu\nu\lambda\rho}\, v^\lambda p^\rho . \tag{3.27}$$

For a free particle v_μ is constant, as follows from the Liouville equation for $\rho(\xi)$.

To get the quantal density matrix, substitute the operators (3.18) into (3.25)

$$\hat{\rho} = 2\left(\hbar/2\right)^{5/2} \frac{(p\gamma)-m}{2m}\left[1+\gamma_5 (a\gamma)\right]\frac{(p\gamma)-m}{2m} , \tag{3.28}$$

where $a_\mu = 2v_\mu/\hbar$. This is just the form introduced by Michel and Wightman [33].

3.3. Particle in an External Field

To describe the interaction of a charged particle with an electromagnetic field $A^{\mu}(q)$ write the action as a sum $A = A_{free} + A_{int}$, where A_{free} is given by Eq. (3.1), and

$$\mathcal{A}_{int} = \int_{\tau_i}^{\tau_f} \left[e A^{\mu} \dot{q}_{\mu} - i \varkappa z F^{\mu\nu} \xi_{\mu} \xi_{\nu} \right] d\tau. \tag{3.29}$$

Here e is the charge, $\varkappa$ is the (total) magnetic moment $F^{\mu\nu} = \partial A^{\nu}/\partial q_{\mu} - \partial A^{\mu}/\partial q_{\nu}$. The interaction of spin with the field was written as ($F^{\mu\nu} S_{\mu\nu}$) by Frenkel [8] . This form was also analysed by Barut [10] and Hanson and Regge [11] . (Among many other papers on motion of spin in a field mention the works by Suttorp and de Groot [34] and Ellis [35]). Dealing with the Grassmann variables, one escapes some difficulties present in the previous approaches.

The canonical momentum is

$$p^{\mu} = \partial \mathcal{L}/\partial \dot{q}_{\mu} = P^{\mu} + e A^{\mu},$$

$$P_{\mu} = \left[m + i\varkappa (F\xi\xi) \right] u_{\mu} - \frac{i}{2} \left[\xi_{\mu} + (u\xi) u_{\mu} \right] \lambda / z, \tag{3.30}$$

and the equations of motion are

$$\dot{p}^{\mu} = e F^{\mu\nu} \dot{q}_{\nu} + \varkappa z (\partial F^{\sigma\nu}/\partial q_{\mu}) S_{\sigma\nu},$$

$$\dot{\xi}_{\mu} = 2\varkappa z F_{\mu\nu} \xi^{\nu} + u_{\mu} \lambda/2, \quad \dot{\xi}_5 = \lambda/2. \tag{3.31}$$

Besides the Zitterbewegung , present also in case of no field, there are two effects on the space-time trajectory due to the spin variables: a renormalization of the mass and forces proportional to the derivatives of the field.

To obtain the Bargmann-Michel-Telegdi equation [36], describing the spin precession in a homogeneous field, write the phase-space distribution (3.25) as follows

$$\rho(\xi)=\left[(v\xi)-\frac{i}{6}(u\xi\xi\xi)\right](u\xi)\xi_5 , \tag{3.32}$$

where u_μ is a solution of the equation $m\dot{u}^\mu = eF^{\mu\nu}\dot{q}_\nu$ and apply to it the Liouville equation in ξ variables

$$-\dot{\rho}=\{H,\rho\}_{P.B.}=ig_{\mu\nu}\left(\frac{H\overleftarrow{\partial}}{\partial\xi_\mu}\right)\left(\frac{\overrightarrow{\partial}\rho}{\partial\xi_\nu}\right)+i\left(\frac{H\overleftarrow{\partial}}{\partial\xi_5}\right)\left(\frac{\overrightarrow{\partial}\rho}{\partial\xi_5}\right),$$
$$H=i\varkappa z(F\xi\xi)+\frac{i}{2}\left[(u\xi)+\xi_5\right]\lambda . \tag{3.33}$$

Under the condition $(u\xi)+\xi_5=0$, this equation is equivalent to the following one

$$\dot{v}^\mu = 2\varkappa F^{\mu\nu}v_\nu + 2(\varkappa - e/2m)u^\mu(Fvu), \tag{3.34}$$

and we get the familiar result [36].

The interaction Lagrangian (3.29) is Lorentz-invariant and gauge-invariant, but it breaks the supergauge symmetry. The variation of the first term under (3.10) is

$$\delta(A\dot{q})=d(A\delta q)/d\tau-(F\dot{q}\,\delta q),\quad \delta q_\mu = i\xi_\mu\eta/m. \tag{3.35}$$

Note that the breaking is in a sense "minimal", i.e. proportional to higher powers of ξ, if there is no anomalous magnetic moment and $\varkappa = e/2m$. It is possible also to reduce the supergauge breaking, writing the first term in (3.29) in a "superspace" form: $eA^\mu(Q)\dot{q}_\mu$, where $Q_\mu = q_\mu - i\xi_\mu\xi_5/m$. Now $\delta Q_\mu = iu_\mu\xi_5\eta$, and the variation (3.35) vanishes.

IV. CONCLUDING REMARKS

We have presented the Grassmann variant of the Hamilton mechanics and applied the general theory to the simplest system, a relativistic spinning particle. Mention some other physical objects that may be considered along the similar lines.

Higher spins. The quantization in our scheme leads to the spin 1/2 only. To get a higher spin s one may consider the Grassmann algebra generated by $2s$ vectors. After the quantization, a multispinor wave function arizes, in the relativistic case the formalism by Bargmann and Wigner [37] is reconstructed (its relation to other formalisms is considered, for instance, in [38]).

Internal symmetry. If generators of the Grassmann algebra are components of a vector in an internal "isospace", the quantization results in a multiplet of particles. The internal symmetry groups SO(n) are directly obtained by this method; the simplest example is the isotopic group SO(3)$\sim$SU(2). Another possibility is to consider the Grassmann variables with a pair of indeces, one spatial and another related to the internal symmetry.

Field theory. In fact, the classical field theory dealing with the anticommuting fields was formulated by Schwinger [39,15] in developing the quantum dynamical principle for electrodynamics. However, it is not necessary to investigate the classical theory in this case, because the quantization is quite simple. A more sophisticated example is the theory of relativistic spinning string [5,40] . Non-linear field Lagrangians and the classical solutions are now intensively investigated (see, e.g. the review by Rajaraman [41]). In this connection, an extension of the scope of classical fields may be of interest.

With all this in view, we suppose that the Grassmann algebra and "anticommuting C-numbers" are not "an unnecessary addition to mathematical physics", as it was stated by Klauder [42] .

Appendix A. The Grassmann Algebra: Basic Definitions and Results

Generators. Let $\xi_1, \ldots, \xi_n$ be n generators of a Grassmann algebra $\mathcal{G}_n$, i.e. for any $j,k=1, \ldots, n$

$$\xi_j \xi_k + \xi_k \xi_j = 0, \qquad (A.1)$$

in particular, $\xi_k^2 = 0$. Any element $g \in \mathcal{G}$ may be represented as a finite sum of homogeneous monomials

$$g(\xi) = \sum_{\nu=0}^{n} \sum_{\{k\}} g_\nu^{k_1 \ldots k_\nu} \xi_{k_1} \ldots \xi_{k_\nu}, \qquad (A.2)$$

where $g_\nu^{\{k\}}$ are numbers (real or complex) and it is assumed that they are antisymmetric in indeces $\{k\}$. The set of elements, for which only terms with even ν are present in the sum (the even elements) is a subalgebra $\mathcal{G}_n^{(+)}$. The set of odd elements, defined in an analoguous way, $\mathcal{G}_n^{(-)}$, is not a subalgebra. Even elements commute with all elements of $\mathcal{G}_n$; odd elements commute with even elements and anticommute with odd elements.

Involution (an analogue of the complex conjugated). Define a one-to-one mapping of the algebra onto itself, $g \leftrightarrow g^*$, satisfying the following conditions

$$(g^*)^* = g, \qquad (A.3a)$$

$$(g_1 g_2)^* = g_2^* g_1^*, \qquad (A.3b)$$

$$(\alpha g)^* = \alpha^* g^*, \qquad (A.3c)$$

where α is a complex number. An element g is real if $g^* = g$. The algebra is real if all its elements are real, in particular, $\xi_k^* = \xi_k$.

Derivatives. The following linear operators are introduced in $\mathcal{G}_n$:

$$(\vec{\partial}/\partial\xi_\ell)\xi_{k_1}\cdots\xi_{k_\nu} = \delta_{k_1\ell}\,\xi_{k_2}\cdots\xi_{k_\nu} - \cdots \quad \text{(A.4a)}$$

$$- \delta_{k_2\ell}\,\xi_{k_1}\xi_{k_3}\cdots\xi_{k_\nu} + (-)^\nu \delta_{k_\nu\ell}\,\xi_{k_1}\cdots\xi_{k_{\nu-1}},$$

$$\xi_{k_1}\cdots\xi_{k_\nu}(\overleftarrow{\partial}/\partial\xi_\ell) = \delta_{k_\nu\ell}\,\xi_{k_1}\cdots\xi_{k_{\nu-1}} - \quad \text{(A.4b)}$$

$$\cdots + (-)^\nu \delta_{k_1\ell}\,\xi_{k_2}\cdots\xi_{k_\nu}.$$

Their action on any $g \in \mathcal{G}_n$ is determined by means of Eq. (A.2). The operators $(\vec{\partial}/\partial\xi_\ell)$ and $(\overleftarrow{\partial}/\partial\xi_\ell)$ are called the left derivative and the right derivative. In simple words, to find the left derivative $(\vec{\partial}/\partial\xi_\ell)$ of a monomial one has to permute ξ_ℓ to the first place and then to drop it; to find the right derivative one has to permute ξ_ℓ to the last place and then to drop it. If ξ_ℓ is absent, the derivative of the monomial vanishes. It is easily seen that

$$\frac{\vec{\partial}}{\partial\xi_j}\left(\frac{\vec{\partial}}{\partial\xi_k}g\right) = -\frac{\vec{\partial}}{\partial\xi_k}\left(\frac{\vec{\partial}}{\partial\xi_j}g\right), \quad \text{(A.5a)}$$

$$\frac{\vec{\partial}}{\partial\xi_j}\left(g\frac{\overleftarrow{\partial}}{\partial\xi_k}\right) = \left(\frac{\vec{\partial}}{\partial\xi_j}g\right)\frac{\overleftarrow{\partial}}{\partial\xi_k}. \quad \text{(A.5b)}$$

Integral (an analogue of the definite integral over the whole region of a variable). It is sufficient to definite the single integrals

$$\int 1\,d\xi_k = 0, \quad \int \xi_k\,d\xi_k = 1. \tag{A.6}$$

The multiple integral is defined by means of iteration of the single integrals. Evidently,

$$\int \xi_{k_1}\cdots\xi_{k_\nu}\,d\xi_\nu\cdots d\xi_1 = \varepsilon_{k_1\cdots k_\nu}\,,$$
$$\int g(\xi)\,d\xi_n\cdots d\xi_1 = \varepsilon_{k_1\cdots k_n}\,g_n^{k_1\cdots k_n}, \tag{A.7}$$

where $\varepsilon_{\{k\}}$ is the Levi-Civita tensor. The integration by parts is possible

$$\int f\left(\frac{\vec{\partial} g}{\partial \xi_\ell}\right) d\xi_\ell = \int \left(\frac{f\overleftarrow{\partial}}{\partial \xi_\ell}\right) g\,d\xi_\ell. \tag{A.8}$$

The "Gauss intergal" is important for applications. It may be shown that

$$\int \exp\left(\sum a_{jk}\xi_j\xi_k\right) d\xi_n\cdots d\xi_1 = \left(\det\|2a_{jk}\|\right)^{1/2}, \quad a_{jk} = -a_{kj}. \tag{A.9}$$

Note that the square root of the determinant of a skew-symmetric matrix (Pfaffian) is a polynomial of its elements.

The Fourier transformation. Let $\mathcal{G}_n$ and $\mathfrak{H}_n$ be the Grassmann algebras with generators ξ_k and ρ^k, $k = 1, \ldots, n$, respectively. Consider a linear mapping $g = \mathcal{F}(h)$, $g \in \mathcal{G}_n$, $h \in \mathfrak{H}_n$, defined in terms of the decomposition (A.2):

$$h(\rho)=\sum_{\mu=0}^{n}\sum_{\{j\}} h^{(\mu)}_{j_1\dots j_\mu}\,\rho^{j_1}\dots\rho^{j_\mu},$$

$$g^{k_1\dots k_\nu}_{(\nu)}=\frac{\epsilon_\nu}{\nu!}\sum_{\{j\}}\varepsilon^{k_1\dots k_\nu j_1\dots j_\mu}\,h^{(\mu)}_{j_1\dots j_\mu}, \qquad \text{(A.10)}$$

where $\mu+\nu=n$, $\epsilon_\nu=1$ at ν even and $\epsilon_\nu=i$ at ν odd. The inverse transformation is

$$h^{(\mu)}_{j_1\dots j_\mu}=\frac{\epsilon_\mu}{\epsilon_n\mu!}\sum_{\{k\}}\varepsilon_{j_1\dots j_\mu k_1\dots k_\nu}\,g^{k_1\dots k_\nu}_{(\nu)}. \qquad \text{(A.11)}$$

This mapping is remarkable because $\vec{\partial} g/\partial\xi_j=\mathcal{F}(i\rho^j h)$.
It may be also presented by means of the integral

$$g(\xi)=\int\exp\Big(i\sum_k\xi_k\rho^k\Big)\,h(\rho)\,d\rho^n\dots d\rho^1,$$

$$h(\rho)=\epsilon_n^{-1}\int\exp\Big(-i\sum_k\xi_k\rho^k\Big)\,g(\xi)\,d\xi_n\dots d\xi_1, \qquad \text{(A.12)}$$

and does generalize the concept of the Fourier transformation.

Details, proofs and further information may be found in the book [1].

Appendix B. Operators and their Symbols

Operators of the quantum mechanics are elements of the Heisenberg algebra with the generators $\hat{q}_j$, $\hat{p}_j$ ($j=1,\dots,f$; f is the number of degrees of freedom), obeying the canonical commutation relations

$$\left[\hat{p}_j,\hat{q}_k\right]=-i\hbar\,\delta_{jk}. \qquad \text{(B.1)}$$

It is well known that the operators may be represented by means of functions on the phase space, with an appropriate multiplication law. Namely, let x be a vector in the phase space, $x = (x_1, \dots, x_n)$, $n = 2f$; and let the representation is $\hat{g}_1 \to g_1(x)$, $\hat{g}_2 \to g_2(x)$. Then

$$\hat{g}_1 \hat{g}_2 = \hat{g} \to g(x) = \int W(x_1, x_2, x)\, g_1(x_1)\, g_2(x_2)\, d^n x_1\, d^n x_2 \,. \tag{B.2}$$

The kernel $W(x_1, x_2, x)$ determines the representation. The operator algebra is associative, so

$$\int W(x_1, x_2, x)\, W(x, x_3, x_4)\, d^n x = \int W(x_1, x, x_4)\, W(x_2, x_3, x)\, d^n x. \tag{B.3}$$

It is natural to adopt the correspondence principle: in the classical limit $g(x)$ coincides with the classical dynamical variable, corresponding to $\hat{g}$: $\lim_{\hbar \to 0} g(x) = g_{cl}(x)$. Then in the classical limit the multiplication law is trivial, $g_{cl}(x) = g_{1cl}(x)\, g_{2cl}(x)$, and

$$\lim_{\hbar \to 0} W(x_1, x_2, x) = \delta^{(n)}(x_1 - x)\, \delta^{(n)}(x_2 - x). \tag{B.4}$$

One may also require that $\hat{1} \to 1$, so

$$\int W(x_1, x_2, x)\, d^n x_1 = \int W(x_2, x_1, x)\, d^n x_1 = \delta^{(n)}(x_2 - x); \tag{B.5}$$

Of course, the representation is not unique. Concentrate now on the Weyl representation. For the sake of symmetry, we shall not divide the components of x between coordinates and momenta, and rewrite the canonical commutation relation (B.1):

$$[\hat{x}_k, \hat{x}_l]_- = i\hbar\,\omega_{kl}, \tag{B.6}$$

where ω_{kl} is a constant antisymmetric matrix (inverse to that of the fundamental symplectic form). Define the symmetric product $(\hat{x}_{k_1} \dots \hat{x}_{k_\nu})$ by means of the generating function

$$(\underline{\tau}\hat{x})^\nu = \sum_{\{k\}} \tau^{k_1} \dots \tau^{k_\nu} (\hat{x}_{k_1} \dots \hat{x}_{k_\nu}), \tag{B.7}$$

where τ^k is a vector from a "dual" space, the monomials $(\hat{x}_{k_1} \dots \hat{x}_{k_\nu})$ form a complete basis of the operator algebra. Any operator $\hat{g}$ may be represented as a formal series

$$\hat{g} = \sum_{\nu=0}^{\infty} \sum_{\{k\}} g_{(\nu)}^{k_1 \dots k_\nu} (\hat{x}_{k_1} \dots \hat{x}_{k_\nu}), \tag{B.8}$$

where $g_{(\nu)}^{\{k\}}$ are the "$\underline{c}$ - number" totally symmetric tensors. The Weyl representation is defined by means of this decomposition:

$$\hat{g} \to g(x) = \sum_{\nu=0}^{\infty} \sum_{\{k\}} g_{(\nu)}^{k_1 \dots k_\nu} x_{k_1} \dots x_{k_\nu}. \tag{B.9}$$

Evidently, the correspondence is one-to-one. The Weyl representation may be described in an equivalent form, making a direct use of the definition (B.7), and it was just the original prescription [43]. Consider the Fourier transform

$$g(x) = \int \exp[i(xr)]\, \tilde{g}(r)\, d^n r. \tag{B.10}$$

The corresponding operator $\hat{g}$ is constructed by means of the exponential operator $\hat{\Omega}$:

$$\hat{g} = \int \hat{\Omega}(r)\, \tilde{g}(r)\, d^n r ,$$
$$\hat{\Omega}(r) = \exp[i(\hat{x} r)] \equiv \sum_{\nu=0}^{\infty} \frac{i^\nu}{\nu!} (r\hat{x})^\nu \qquad \text{(B.11)}$$

Now we are in position to find the kernel of the multiplication law Eq. (B.2). Note that in view of the commutator Eq. (B.6), the operators $\hat{\Omega}(r)$ form a projective group

$$\hat{\Omega}(r_1)\hat{\Omega}(r_2) = \exp\Big(-\frac{i}{2}\hbar \sum_{k,\ell} \omega_{k\ell}\, r_1^k r_2^\ell\Big)\hat{\Omega}(r_1 + r_2). \qquad \text{(B.12)}$$

The following equalities are also useful:

$$\hat{\Omega}^{-1}(r)\, \hat{x}_k\, \hat{\Omega}(r) = \hat{x}_k - \hbar\, \omega_{k\ell}\, r^\ell , \qquad \text{(B.13a)}$$

$$\partial \hat{\Omega}/\partial r^k = i\big(\hat{x}_k + \tfrac{1}{2}\hbar\, \omega_{k\ell}\, r^\ell\big)\hat{\Omega}(r) \qquad \text{(B.13b)}$$

$$\mathrm{Tr}\, \hat{\Omega}(r) = (2\pi\hbar)^{-n/2} (2\pi)^n\, \delta^{(n)}(r). \qquad \text{(B.13c)}$$

Substituting the representation (B.11) and using the Fourier transform, inverse to (B.10), one gets

$$W(x_1, x_2, x_3) = \omega^{-1}(\pi\hbar)^{-n} \exp\Big\{\frac{2i}{\hbar}\big[(x_1 \cdot x_2) + (x_2 \cdot x_3) + (x_3 \cdot x_1)\big]\Big\}, \qquad \text{(B.14)}$$

where $\omega = \det \|\omega_{k\ell}\|$,

$$(x \cdot y) = -(y \cdot x) = \tilde{\omega}^{k\ell} x_k y_\ell , \qquad \text{(B.15)}$$
$$\tilde{\omega}^{k\ell}\omega_{m\ell} = \delta^k_m ,$$

i.e. $\tilde{\omega}$ is the matrix, inverse to ω. It is remarkable that in case of one degree of freedom ($\underline{n} = 2$) the bilinear form in the exponential of Eq. (B.14) has a simple geometrical meaning: it is proportional to the area of the triangle with vertices (x_1, x_2, x_3) on the phase plane.

A somewhat more familiar way to represent the operators is to use their kernels, say, in the coordinate basis:

$$\hat{g}_1 \hat{g}_2 = \hat{g} \rightarrow \langle q'' | g | q' \rangle = \int \langle q'' | g_1 | q \rangle \langle q | g_2 | q' \rangle d^f q. \tag{B.16}$$

The multiplication law is much simpler, than Eq. (B.2), however, the correspondence to the classical mechanics is not so transparent. To get a relation between the symbol and the kernel one needs to calculate the kernel for the operator $\hat{\Omega}$ (r). Return to usual coordinates and momenta and note that in view of Eq. (B.12)

$$\hat{\Omega}(u,v) \equiv \exp(iu\hat{q} + iv\hat{p}) = e^{\frac{i}{2}u\hat{q}} e^{iv\hat{p}} e^{\frac{i}{2}u\hat{q}} \tag{B.17}$$

It follows from (B.9) and (B.11) that

$$\langle q'' | g | q' \rangle = (2\pi\hbar)^{-f} \int g\left(\tfrac{1}{2}q' + \tfrac{1}{2}q'', p\right) \exp\left[-\frac{i}{\hbar} p (q' - q'')\right] d^f p.. \tag{B.18}$$

In conclusion, mention two nice properties of the Weyl symbols, that are generalized also to the Grassmann case. First, the Hermitean conjugation of the operators induces the complex conjugation of symbols $\hat{g} \rightarrow g(x)$, $\hat{g}^+ \rightarrow g^*(x)$. Second,

$$\mathrm{Tr}\,\hat{g} = (2\pi\hbar)^{-n/2} \int g(x)\, d^n x. \tag{B.19}$$

Representation of quantal operators by means of functions in the phase space was developed by Weyl [43] and Wigner [44], and further investigated by Moyal [45]. Generalization to infinite number of degrees of freedom and to the Fermi case, as well as some proofs and details may be found in the works by one of the authors [46,20]. A more recent paper on this subject is that by Schmutz [47].

Appendix C. Representation of Green's Function by Means of the Phase-Space Path Integral

Consider a classical mechanical system with a Hamiltonian H($\underset{\sim}{x}$), where $\underset{\sim}{x}$ is a vector from the phase space. Write the classical action in the symmetric form

$$\mathcal{A}_{cl}\left[x(\tau)\right] = \int_0^t \left[\tfrac{1}{2}(x\cdot\dot{x}) - H(x)\right] d\tau, \qquad \text{(C.1)}$$

where $(x\cdot\dot{x}) \equiv \underline{p}\dot{\underline{q}} - \underline{q}\dot{\underline{p}}$, the notation is used in Appendix B (Eq. (B.15)). This action appears in the phase space path integral representation of the propagator (Green's operator).

Let $\hat{H}$ be the Hamiltonian operator of the quantised system; H(x) is the Weyl symbol of $\hat{H}$, and $\hat{G}(t)$ = = exp (-it$\hat{H}$/$\hbar$) is the proparator. Calculate the Weyl symbol of $\hat{\underline{G}}(\underline{t})$, i.e. a function $\underline{G}(x\ ;\ \underline{t})$ on the phase space. To this end start from an infinitesimal time interval $\delta\underline{t}$. Evidently

$$\begin{gathered}\hat{G}(\delta t) \approx 1 - i\delta t\,\hat{H}/\hbar \rightarrow 1 - i\delta t\,H(x)/\hbar, \\ G(x;\delta t) = \exp\left(-i\delta t\,H(x)/\hbar\right) + O(\delta t^2).\end{gathered} \qquad \text{(C.2)}$$

For a finite t the operator $\hat{G}(t)$ may be calculated by means of the limiting process, representing the step-by--step evolution of the system

$$\hat{G}(t) = \lim_{N \to \infty} \left[\hat{G}(t/N) \right]^{N} . \tag{C.3}$$

To get the symbol $G(x; t)$ we apply the multiplication law (B.2) with the kernel (B.14) to the symbols $G(x, t/N)$ given by Eq. (C.2). The result is

$$G(x;t) = \lim_{N \to \infty} G^{(N)}(x;t) \tag{C.4}$$

$$G^{(N)}(x;t) = \left[\omega(\pi\hbar)^{n}\right]^{-N} \int \prod_{\nu=1}^{N} d^{n}x_{\nu}\, d^{n}y_{\nu}$$

$$\exp\left\{\frac{i}{\hbar}\sum_{\nu=1}^{N}\left[2(x_{\nu}\cdot y_{\nu}) + 2(y_{\nu}\cdot x_{\nu+1}) + 2(x_{\nu+1}\cdot x_{\nu}) - H(y_{\nu})t/N\right]\right\},$$

where $x_{N+1} = x$. One may imagine that the system is propagating in the phase space, being influenced by its Hamiltonian at points y_{ν} and being observed at points x_{ν}.

Formally, Eq. (C.4) is a representation of the propagator by means of a "double" continual integral

$$G(x;t) = \iint \mathcal{D}[x(\tau)]\, \mathcal{D}[y(\tau)] \exp\left\{\frac{i}{\hbar}\int_{0}^{t}\left[2(y\cdot\dot{x}) - 2(x\cdot\dot{x}) - H(y)\right]d\tau\right\}, \tag{C.5}$$

$$\mathcal{D}[f(\tau)] \equiv \lim_{N\to\infty}\left[\omega(\pi\hbar)^{n}\right]^{-N/2}\prod_{\nu=1}^{N} d^{n}f_{\nu}\,, \quad f_{\nu} \equiv f(\nu t/N),$$

where the boundary condition $x(t) = x$ is implied. However, to evaluate this expression one should consider the original form (C.4). The integral in x_{ν} is Gaussian and may be calculated exactly. Substitute $x_\nu = u_\nu + z_\nu$, where z_ν are the new integration variables and the bilinear form in the exponent has an extremum at $x_\nu = u_\nu$. The equations to determine u_ν are

$$u_{\nu+1} - u_{\nu-1} = y_\nu - y_{\nu-1}, \quad \nu = 2, \dots, N-1,$$
$$u_2 = y_1, \quad u_{N-1} = x + y_{N-1} - y_N. \tag{C.6}$$

Assume that N is an even number; then one gets

$$\int \prod_{\nu=1}^{N} d^n z_\nu \exp\left[\frac{2i}{\hbar}\sum_{\nu=1}^{N-1}(z_{\nu+1}\cdot z_\nu)\right] = \left[\omega(\pi\hbar)^n\right]^{N/2} \tag{C.7}$$

$$G^{(N)}(x;t) = \left[\omega(\pi\hbar)^n\right]^{-N/2} \int \prod_{\nu=1}^{N} d^n y_\nu$$
$$\exp\left\{\frac{i}{\hbar}\left[2\sum_{\varkappa=1}^{N/2}\left(u_{2\varkappa}\cdot(y_{2\varkappa}-y_{2\varkappa-1})\right) + \right.\right.$$
$$\left.\left. + 2\left(x\cdot(u_N - y_N)\right) - \sum_{\nu=1}^{N} H(y_\nu)\, t/N\right]\right\}. \tag{C.8}$$

In the continual limit Eqs.(C.6) for $u_{2\varkappa}$ are written as

$$\dot{u} = 1/2\, \dot{y}, \quad u(0) = y(0), \tag{C.9}$$

so that $u(\tau) = 1/2(y(\tau) + y(0))$, and

$$G(x;t) = \int \mathcal{D}[y(\tau)] \exp\left\{\frac{i}{\hbar}\mathcal{A}_{cl}[y(\tau)] + \right.$$
$$\left. + \frac{i}{\hbar}\left[(x\cdot y_0) + (y_0\cdot y_t) + (y_t\cdot x)\right]\right\} \tag{C.10}$$

where $\mathcal{A}_{cl}$ is the classical action defined by Eq. (C.1). This form of the phase-space path integral is quite symmetric, one does not need to distinguish between the coordinate and the momentum, nor to prescribe that the trajectories are piecewise linear in q and piecewise constant in p , as in the conventional approach (see the work by Garrod [50]). However, the exact meaning of the functional $\int (y\cdot\dot{y})d\tau$ is clear only before the limit $N\to\infty$ and is given by Eq. (C.8).

In some applications the representation (C.4) is more useful than (C.8) or (C.10). For instance, to get the Feynman original path integral in the coordinate space one may introduce the variables $(p,q) = y$, $(p', q') = x$, write $H(y) = p^2/2m + V(q)$ and integrate (C.4) first over p, then over q' and p'. Consider now the isotrope harmonic oscillator

$$H(y) = ky^2 \tag{C.11}$$

Integrating (C.4) over y_ν we obtain

$$G^{(N)}(x;t) = (i\pi\hbar\delta)^{-nN/2}\int\prod_{\nu=1}^{N} d^n x_\nu \exp\left\{\frac{2i}{\hbar}\sum_{\nu=1}^{N}\left[(x_{\nu+1}\cdot z_\nu) + z_\nu^2/2\delta\right]\right\}, \tag{C.12}$$

where $z_\nu = x_\nu - x_{\nu+1}$, $\delta = kt/N$. [redacted] After the integration over $x_1, \dots, x_M$, $1 < M \leqslant N$ the integral takes the form

$$C_M\int\prod_{\nu=M+1}^{N} d^n x_\nu \exp\left\{\frac{2i}{\hbar}\left[(x_{\nu+1}\cdot z_\nu) + z_\nu^2/2\delta\right] - \frac{i}{\hbar}A_M x_{M+1}^2\right\}, \tag{C.13}$$

while for the constants A_M and C_M the following recursive relations hold

$$A_M = A_{M-1} + \delta \frac{1+A^2_{M-1}}{1-\delta A_{M-1}}, \quad A_0 = 0,$$
$$C_M = \left(\frac{i\pi\hbar\delta}{1-\delta A_{M-1}}\right)^{n/2} C_{M-1}. \tag{C.14}$$

In the continual limit, $\delta \to 0$, $A_M = A(Mt/N)$, $C_M = (i\pi\hbar\delta)^{+nM/2} F(Mt/N)$, and the functions $A(\tau)$ and $F(\tau)$ are obtained from the differential equations

$$\frac{dA}{d\tau} = k(1+A^2), \quad \frac{d \ln F}{d\tau} = \frac{1}{2} n k A, \tag{C.15}$$

$A(0) = 0$, $F(0) = 1$.

Thus Green's function for the oscillator is

$$G(x;t) = (\cos kt)^{-n/2} \exp\left(-\frac{i}{\hbar} x^2 \,\mathrm{tg}\, kt\right). \tag{C.16}$$

Applying Eq. (B.18), relating the symbol of an operator to its matrix element, one can see that this result is in accordance with that given by Feynman [29].

The phase-space path integrals were introduced by Feynman [48] and discussed in a number of works [49-51]. The present exposition follows the work [20].

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