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THE WITTEN GENUS AND S^3-ACTIONS ON MANIFOLDS

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0. In this paper we study the Witten genus φ_W on $BO\langle 8\rangle$-manifolds (a manifold M is a $BO\langle 8\rangle$-manifold if M is spin and $p_1(M)/2=0$). For a $4k$-dimensional $BO\langle 8\rangle$-manifold M the Witten genus is equal to a sequence of twisted Dirac operators:

$$\varphi_W(M)=ind(D(M)\otimes\bigotimes_{n=1}^{\infty}S_{q^n}(T_{\mathbb{C}}))\cdot c=(\sum_{n=0}^{\infty}ind(D(M)\otimes R_n(T_{\mathbb{C}}))\cdot q^n)\cdot c.$$

Here $T_{\mathbb{C}}$ denotes the complexified tangent bundle, $S^i(V)$ is the i-th symmetric power of a vector bundle V, $S_t(V):=\sum_{i=0}^{\infty}S^i(V)\cdot t^i$, $\sum_{n=0}^{\infty}R_n(T_{\mathbb{C}})\cdot q^n:=\bigotimes_{n=1}^{\infty}S_{q^n}(T_{\mathbb{C}})$ and $c:=\prod_{n=1}^{\infty}(1-q^n)^{4k}$.

Our main result states that the Witten genus vanishes on a $BO\langle 8\rangle$-manifold M if and only if a non-trivial multiple of M is $BO\langle 8\rangle$-bordant to a $BO\langle 8\rangle$-manifold with non-trivial S^3-action (see Corollary 5). This result is analogous to a theorem of M.F. Atiyah and F. Hirzebruch (cf. [AtHi70]) which may be phrased in the following way: The $\hat{A}$-genus vanishes on a *Spin*-manifold M if and only if a non-trivial multiple of M is *Spin*-bordant to a *Spin*-manifold with non-trivial S^1-action.

1. The Witten genus φ_W is given by the even stable characteristic power series

$$Q(x)=\frac{x}{e^{x/2}-e^{-x/2}}\cdot\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^ne^x)(1-q^ne^{-x})}\cdot e^{-G_2(\tau)\cdot x^2}.$$

Here τ is in the upper half-plane, $q:=e^{2\pi i\tau}$, $\sigma_1(n):=\sum_{d|n}d$ and $G_2(\tau):=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma_1(n)\cdot q^n$. Thus, for a $4k$-dimensional oriented closed manifold M with formal roots $\{\pm x_i\}_{i=1}^{2k}$ the Witten genus $\varphi_W(M)$ is equal to $\prod_{i=1}^{2k}Q(x_i)[M]$, where $[M]$ denotes the evaluation on the fundamental cycle of M. Also, $\varphi_W(M)$ is equal to the q-expansion in the cusp $i\infty$ of a modular form of weight $2k$ for $Sl_2(\mathbb{Z})$ and my be embedded in the ring $\mathbb{C}[[q]]$ (cf. [Br89]; consult [Se73] or [Ko84] for an introduction to modular forms). The genus $\widetilde{\varphi}_W$ given by the power series

$$\widetilde{Q}(x):=Q(x)\cdot e^{G_2(\tau)\cdot x^2}=\frac{x}{e^{x/2}-e^{-x/2}}\cdot\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^ne^x)(1-q^ne^{-x})}$$

is closely related to the Witten genus in the following sense: If M is a manifold, s.t. all Pontrjagin numbers of M involving $p_1(M)$ vanish, e.g. $p_1(M)$ is a torsion class, the two genera coincide. In this case $\widetilde{\varphi}_W(M) = \prod_{i=1}^{2k} \widetilde{Q}(x_i)[M]$ is also the q-expansion in the cusp $i\infty$ of a modular form of weight $2k$ for $Sl_2(\mathbb{Z})$. While the Witten genus has good modularity properties the genus $\widetilde{\varphi}_W$ does not in general. But $\widetilde{\varphi}_W$ is related to the index of twisted Dirac operators for *Spin*-manifolds. We will explain this next. Let $T_{\mathbb{C}}$ be the complexified tangent bundle of M and let $\sum_{n=0}^{\infty} R_n(T_{\mathbb{C}}) \cdot q^n := \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}})$ (recall, $S^i(V)$ is the i-th symmetric power of a vector bundle V and $S_t(V) := \sum_{i=0}^{\infty} S^i(V) \cdot t^i$). Then $\widetilde{\varphi}_W(M) = (\sum_{n=0}^{\infty} \hat{A}(M, R_n(T_{\mathbb{C}})) \cdot q^n) \cdot c$, where $c := \prod_{n=1}^{\infty} (1-q^n)^{4k}$. If M is spin each coefficient of the q-expansion is the index of a twisted Dirac operator:

$$\widetilde{\varphi}_W(M) = ind(D(M) \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T_{\mathbb{C}})) \cdot c = (\sum_{n=0}^{\infty} ind(D(M) \otimes R_n(T_{\mathbb{C}})) \cdot q^n) \cdot c.$$

For a *Spin*-manifold M with G-action, G a compact Lie group, the equivariant genus $\widetilde{\varphi}_W(M)(g), g \in G$, is defined by replacing the indices of the twisted Dirac operators by their equivariant indices.

We will show that the vanishing of the Witten genus is directly connected to the existence of an S^3-action up to rational bordism (see Corollary 5).

Recall, that $H^4(BSpin;\mathbb{Z}) \cong \mathbb{Z}$ and $p_1 \in H^4(BSpin;\mathbb{Z})$ is two times a generator called $\frac{1}{2}p_1$. A *Spin*-manifold M is a $\boldsymbol{BO\langle 8\rangle}$**-manifold** if and only if $\frac{1}{2}p_1(M) = 0$. A *Spin*-manifold M is a **rational** $\boldsymbol{BO\langle 8\rangle}$**-manifold** if and only if $p_1(M)$ is a torsion class.

Let $MO\langle 8\rangle_*$ be the bordism ring of $BO\langle 8\rangle$-manifolds. Let R be the ring in $\Omega_*^{SO} \otimes \mathbb{Q}$ generated by manifolds M which have the property that all Pontrjagin numbers of M which involve $p_1(M)$ vanish. With the help of the Pontrjagin-Thom construction follows that the natural map $MO\langle 8\rangle_* \otimes \mathbb{Q} \to \Omega_*^{SO} \otimes \mathbb{Q}$ is injective with image equal to R. Thus, two $BO\langle 8\rangle$-manifolds are rationally SO-bordant if and only if they are rationally $BO\langle 8\rangle$-bordant and a non-trivial multiple of any element of R, for example a rational $BO\langle 8\rangle$-manifold, is oriented bordant to a $BO\langle 8\rangle$-manifold.

Theorem 1. *Let M be an oriented closed manifold. If a non-trivial multiple of M is oriented bordant to a rational $BO\langle 8\rangle$-manifold $\hat{M}$ with non-trivial S^3-action, then the equivariant Witten genus $\varphi_W(\hat{M})(g), g \in S^3$, is constant zero and $\varphi_W(M) = 0$.*

Corollary 2. *Let M be a rational $BO\langle 8\rangle$-manifold and $\varphi_W(M) \neq 0$. Then any compact connected subgroup of the diffeomorphism group of M is a torus.*

Theorem 3. *The rational Spin-bordism ring has a basis sequence $\{M_{4k}\}_{k\geq 1}$, s.t.*

1. *M_{4k} is a Spin-manifold for all k,*
2. *M_{4k} is a $BO\langle 8\rangle$-manifold for $k \geq 2$,*
3. *M_{4k} is the total space of a Cayley plane bundle with structure group $SU(2)$ and non-trivial S^3-action along the fibres for $k \geq 4$.*

Corollary 4. *Let M be an oriented manifold, s.t. all Pontrjagin numbers of M which involve $p_1(M)$ vanish and assume $\varphi_W(M) = 0$. Then a non-trivial multiple of M is oriented bordant to a $BO\langle 8\rangle$-manifold with non-trivial S^3-action.*

A theorem of M.F. Atiyah and F. Hirzebruch (cf. [AtHi70]) may be phrased in the following way: The $\hat{A}$-genus vanishes on a *Spin*-manifold M if and only if a non-trivial multiple of M is *Spin*-bordant to a *Spin*-manifold with non-trivial S^1-action. If we restrict to $BO\langle 8\rangle$-manifolds Theorem 1 and Corollary 4 give a corresponding result for the Witten genus and S^3-actions.

Corollary 5. *Let M be a $BO\langle 8\rangle$-manifold. Then $\varphi_W(M) = 0$ if and only if a non-trivial multiple of M is $BO\langle 8\rangle$-bordant to a $BO\langle 8\rangle$-manifold with non-trivial S^3-action.*

We do not know whether Theorem 1 is true if one replaces S^3 by S^1. If this is the case a non-trivial multiple of a $BO\langle 8\rangle$-manifold with S^1-action is bordant to a $BO\langle 8\rangle$-manifold with S^3-action. If not Corollary 2 is sharp, i.e. for any given torus T there exists a $BO\langle 8\rangle$-manifold M with $\varphi_W(M) \neq 0$ such that T is contained in the diffeomorphism group of M.

2. *Proof of Theorem 1.* In this proof cohomology is always taken with rational coefficients. Assume a non-trivial multiple of M, $n \cdot M$, is bordant to a rational $BO\langle 8\rangle$-manifold $\hat{M}$ with non-trivial S^3-action. Then $n \cdot \varphi_W(M) = \varphi_W(\hat{M}) = \widetilde{\varphi}_W(\hat{M})$.

Now choose any $S^1 \hookrightarrow S^3$. Then the induced S^1-action on $\hat{M}$ is non-trivial. If the S^1-action has no fixed points the equivariant genus $\varphi_W(\hat{M})(\lambda), \lambda \in S^1$, is constant zero by the Lefschetz fixed point formula (cf. [AtSi68]).

So assume $\hat{M}^{S^1} \neq \emptyset$. We want to show that $p_1(\hat{M})_{S^1} \in im(\pi^*)$. Consider the following commutative diagram of cohomology groups induced by $S^1 \hookrightarrow S^3$:

$$\begin{array}{ccc} H^4(BS^3) & \xrightarrow{\pi^*} & H^4(\hat{M} \times_{S^3} ES^3) \\ \cong\downarrow & & \downarrow \\ H^4(BS^1) & \xrightarrow{\pi^*} & H^4(\hat{M} \times_{S^1} ES^1). \end{array}$$

Let $\{E_r^{*,*}\}$ be the Leray-Serre spectral sequence for the fibre bundle $\hat{M} \times_{S^3} ES^3 \to BS^3$. Since BS^3 is 3-connected the sequence $0 \to E_\infty^{4,0} \xrightarrow{\pi^*} H^4(\hat{M} \times_{S^3} ES^3) \xrightarrow{i^*} E_\infty^{0,4} \to 0$ is exact. From $\hat{M}^{S^1} \neq \emptyset$ follows that the horizontal arrows in the above diagram are injections. Thus, $H^4(BS^3) = E_2^{4,0} = E_\infty^{4,0}$. Since $E_\infty^{0,4} \hookrightarrow E_2^{0,4} = H^4(M)$ the sequence

$$0 \to H^4(BS^3) \xrightarrow{\pi^*} H^4(\hat{M} \times_{S^3} ES^3) \xrightarrow{i^*} H^4(M)$$

is exact.

Since $i^*(p_1(\hat{M})_{S^3}) = 0$ the class $p_1(\hat{M})_{S^3}$ lives in the image of π^*. By naturality $p_1(\hat{M})_{S^1} \in im(\pi^*)$. In [Li92] K. Liu showed that the equivariant Witten genus is constant zero for S^1-manifolds with $p_1(\hat{M})_{S^1} \in im(\pi^*)$. Thus $\widetilde{\varphi}(\hat{M})(\lambda), \lambda \in S^1$, is constant zero. Since every element of S^3 is an element of a maximal torus $S^1 \hookrightarrow S^3$ the equivariant Witten genus

$\widetilde{\varphi}(\hat{M})(g), g \in S^3$, is constant zero and $n \cdot \varphi_W(M) = \varphi_W(\hat{M}) = \widetilde{\varphi}_W(\hat{M})(1) = 0$. This proves the theorem.

Proof of Theorem 3. Let $M_4 = V^{(4)}$ be the quartic in $P^3(\mathbb{C})$. Let M_8 and M_{12} be almost parallelizable manifolds of dimension 8 and 12, respectively, with non-vanishing top Pontrjagin class. Let M_{16} be the Cayley plane. We define $M_{\geq 20}$ as follows:

Let T be the standard maximal torus of $Spin(9) \hookrightarrow F_4$. Consider a co-character $f : S^1 \to T$ which maps a generator u of the integral lattice of S^1 to $2x_1$. By Lemma 6 the structure group of $\eta := (Bf)^*(\xi)$ allows a reduction to S^1. The bundle has a non-trivial S^3-action along the fibres. Since $p_1(T^\triangle \eta) = 8u^2 \in H^4(BS^1;\mathbb{Z})$ and $p_1(T^\triangle \eta) \equiv w_2(T^\triangle \eta)^2 \bmod 2$ the bundle $T^\triangle \eta$ is spin.

Let $k \geq 5$ and $V^{\hat{d}}$ a complete intersection of complex dimension $2k-8$ and of degree $\hat{d} = (d_1, d_2, d_3, d_4)$. Let $j^*_{k-4} : V^{\hat{d}} \to BS^1$ classify $a_{k-4} \cdot h$, where h is the pull-back of the generator of $H^*(CP^\infty;\mathbb{Z})$ with respect to the inclusion $V^{\hat{d}} \hookrightarrow CP^\infty$. For each $k \geq 5$ we choose $\hat{d}$ and a_{k-4}, s.t. $\sum d_i^2 = 8a_{k-4}^2 + 2k - 3$, where $d_i > 0$ for all i and $a_{k-4} \neq 0$. It is a simple consequence of the Lemma of Lagrange (about the sum of four squares) that we can always find such d_i and a_{k-4}.

Now define M_{4k} for $k \geq 5$ as the pull-back bundle of η under $j^*_{k-4} : V^{\hat{d}} \to BS^1$. As a pull-back bundle M_{4k} has a non-trivial S^3-action along the fibres for $k \geq 5$. We claim that M_{4k} is $BO\langle 8\rangle$ and has non-vanishing Milnor number:

Since $c_1(V^{\hat{d}}) = (2k-3-\sum d_i)h \equiv (2k-3-\sum d_i^2)h \equiv -8a_{k-4}^2 \cdot h \equiv 0 \bmod 2$ the manifold $V^{\hat{d}}$ is spin. The tangent bundle of M_{4k} splits as the direct sum of the bundle along the fibres, $T^\triangle M_{4k} \cong j^*_{k-4}(T^\triangle \eta)$ and the pull-back of the tangent bundle of $V^{\hat{d}}$ under the projection $\pi : M_{4k} \to V^{\hat{d}}$, i.e. $TM_{4k} \cong j^*_{k-4}(T^\triangle \eta) \oplus \pi^*(TV^{\hat{d}})$. Since $T^\triangle \eta$ and $TV^{\hat{d}}$ are spin the same holds for M_{4k}. Next $p_1(M_{4k}) = \pi^*(p_1(V^{\hat{d}})) + j^*_{k-4}(p_1(T^\triangle \eta)) = \pi^*(p_1(V^{\hat{d}}) + 8a_{k-4}^2 \cdot h^2)$. Since by construction $p_1(V^{\hat{d}}) = ((2k-8+5) - \sum d_i^2) \cdot h^2 = -8a_{k-4}^2 \cdot h^2$, we get $p_1(M_{4k}) = 0$. Since $H^4(M_{4k};\mathbb{Z})$ is torsion free this proves $p_1/2(M_{4k}) = 0$ and M_{4k} is $BO\langle 8\rangle$.

By Lemma 7 the coefficient α of x_1^{2k-8} in $\pi_!(s_{2k}(T^\triangle \xi))$ does not vanish. By construction $a_{k-4} \neq 0$. Hence $j^*_{k-4}(u^{2k-8})$ is a generator of $H^{4k-16}(V^{\hat{d}};\mathbb{Q})$ and the Milnor number $s_{2k}(TM)[M] = \pi_!(s_{2k}(T^\triangle M))[V^{\hat{d}}] = j^*_{k-4}(2^{2k-8}\alpha \cdot u^{2k-8})$ of M_{4k} is non-zero. For $k \leq 4$ it follows directly that the Milnor number of M_{4k} is non-zero. Thus $\{M_{4k}\}$ defines a rational basis sequence for Ω_*^{Spin}, i.e. $\Omega_*^{Spin} \otimes \mathbb{Q} \cong \mathbb{Q}[M_4, M_8, \ldots]$.

Proof of Corollary 4. The ring of modular forms $M_*(Sl_2(\mathbb{Z}))$ is a polynomial ring over $\mathbb{C}$ in $\varphi_W(M_8)$ and $\varphi_W(M_{12})$. The genus vanishes on $\{M_{4k}\}_{k\geq 4}$ by Theorem 1. Hence, the kernel of φ_W restricted to $R = \mathbb{Q}[M_8, M_{12}, \ldots]$ is the ideal I in R generated by $\{M_{4k}\}_{k\geq 4}$.

The condition on the Pontrjagin numbers of M imply that the rational oriented bordism class x of M lives in R. Since $\varphi_W(M) = 0$, the bordism class x is an element of the ideal I. Now for any element $y \in I$ exists a natural number $n \neq 0$, s.t. $n \cdot y$ can be realized as a $BO\langle 8\rangle$-manifold with non-trivial S^3-action. This proves the corollary.

Lemma 6. *Let $H \hookrightarrow G$ be an inclusion of compact Lie groups and E the total space of an universal G-bundle. Let ξ be the bundle $E/H \to E/G$ with fibre G/H. Given another closed subgroup $L \overset{f}{\hookrightarrow} G$ the map $\phi : E \times_L G/H \to (Bf)^*(\xi)$, $(e, gH) \mapsto (eL, egH)$ of bundles over E/L with fibre G/H induces a reduction $L \hookrightarrow G$ of the structure group G for the bundle $(Bf)^*(\xi)$.*

Proof. Since $\phi((el, gH)) = (eL, elgH) = \phi((e, lgH))$ the map is well-defined. Also, it is surjective and induces the identity on the base space E/L. Now assume $\phi((e, gH)) = \phi((e', g'H))$, i.e. $(eL, egH) = (e'L, e'g'H)$. This implies $e' = el$ for some $l \in L$ and $gH = lg'H$. Hence, $(e', g'H) = (el, l^{-1}gH) \sim_L (e, gH)$. This proves the injectivity of ϕ and completes the proof.

We will now show that the Milnor numbers of $M_{4k}, k \geq 4$, do not vanish. Recall, that the Milnor class $s_{2k}(\eta)$ of a bundle η with formal roots $\{\pm y_i\}_{i=1}^l$ is given by $\sum_{i=1}^l y_i^{2k}$. The Milnor number of a $4k$-dimensional manifold M is given by $s_{2k}(TM)[M]$. As a tool we use integration over the fibres. We will explain this in the general situation first: Let G be a compact connected Lie group, U a closed connected subgroup of maximal rank and T a maximal torus of U. The Weyl groups of G and U are denoted by $W(G)$ and $W(U)$. Now choose a set $\{r_i\}_{i=1,\ldots,s}$ by fixing a sign for every complementary root $\pm r_i$ of $U \hookrightarrow G$. Let $\xi : BU \overset{\pi}{\to} BG$ be the induced bundle with fibre G/U and $T^\triangle\xi$ the bundle along the fibres. The rational cohomology of BU and BG may be identified with the invariants of $H^*(BT;\mathbb{Q})$ under the action of the Weyl group of U and G, respectively. Let $\pi_! : H^*(BU;\mathbb{Q}) \to H^*(BG;\mathbb{Q})$ denote the integration over the fibres of ξ, where $T^\triangle\xi$ is oriented by $e(T^\triangle\xi) = \prod_{i=1}^{s} r_i$. Then

$$\pi_!(x) = \sum_{w \in W(G)/W(U)} w\Big(\frac{x}{\prod_{i=1}^{s} r_i}\Big)$$

(cf. [BoHi58] or [KrSt93]).

Lemma 7. *The Milnor numbers of M_{4k} do not vanish for $k \geq 4$.*

Proof. Recall from the proof of Theorem 3 that the Milnor number of $M := M_{4k}$ is equal to

$$s_{2k}(TM)[M] = \pi_!(s_{2k}(T^\triangle M))[V^{\hat{d}}] = j_{k-4}^*(2^{2k-8}\alpha \cdot u^{2k-8}),$$

where α is the coefficient of x_1^{2k-8} in $\pi_!(s_{2k}(T^\triangle\xi))$. Recall also that it suffices to show that $\alpha \neq 0$.

For the standard maximal torus T^4 of $Spin(9)$ with generators $x_1, \ldots, x_4$ the complementary roots of $Spin(9)$ in F_4 are given by $\frac{1}{2}\sum \pm x_i$ (cf. [BoHi58]). As a set of representatives of $W(F_4)/W(Spin(9))$ we choose $\{1, w, w^2\}$, where

$$w := s_{2x_4} \circ s_{\frac{1}{2}(x_1 - x_2 - x_3 + x_4)}$$

and s_α denotes reflection along the hyperplane $\langle \alpha, \quad \rangle = 0$. More explicitly, w is given by

$$x_1 + x_2 \mapsto x_1 + x_2,\ x_1 - x_2 \mapsto x_3 + x_4,\ x_3 + x_4 \mapsto x_3 - x_4 \text{ and } x_3 - x_4 \mapsto x_1 - x_2.$$

If we put

$$y := \frac{x_1 + x_2}{2},\ y_1 := \frac{x_1 - x_2}{2},\ y_2 := \frac{x_3 + x_4}{2},\ y_3 := \frac{x_3 - x_4}{2}$$

and define

$$P(y, y_1, y_2, y_3) := \frac{\sum\limits_{z \in \{y, y_1\}} \left((z + y_2)^{2k} + (z - y_2)^{2k} + (z + y_3)^{2k} + (z - y_3)^{2k}\right)}{(y^2 - y_2^2)(y^2 - y_3^2)(y_1^2 - y_2^2)(y_1^2 - y_3^2)}$$

then, $\pi_!(s_{2k}(T^\triangle \xi))$ is equal to

$$Q(y, y_1, y_2, y_3) := P(y, y_1, y_2, y_3) + P(y, y_2, y_3, y_1) + P(y, y_3, y_1, y_2).$$

The coefficient α of x_1^{2k-8} in $\pi_!(s_{2k}(T^\triangle \xi))$ is equal to $Q(\frac{1}{2}, \frac{1}{2}, 0, 0)$. It follows that α is equal to $2^{8-2k}(2^2 - 2^{2k} + k(2k-1)(2k-2))$ and is negative for $k \geq 4$.

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