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Penalized GNSS Ambiguity Resolution

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Abstract. Global Navigation Satellite System (GNSS) carrier phase ambiguity resolution is the process of resolving the carrier phase ambiguities as integers. It is the key to fast and high precision GNSS positioning and it applies to a great variety of GNSS models which are currently in use in navigation, surveying, geodesy and geophysics. A new principle of carrier phase ambiguity resolution is introduced. The idea is to give the user the possibility to assign penalties to the possible outcomes of the ambiguity resolution process: a high penalty for an incorrect integer outcome, a low penalty for a correct integer outcome and a medium penalty for the real valued float solution. As a result of the penalty assignment, each ambiguity resolution process has its own overall penalty. Using this penalty as the objective function which needs to be minimized, it is shown which ambiguity mapping has the smallest possible penalty. The theory presented is formulated using the class of integer aperture estimators as a framework. This class of estimators was introduced elsewhere as a larger class than the class of integer estimators. Integer aperture estimators, being of a hybrid nature, can have integer outcomes as well as non-integer outcomes. The minimal penalty ambiguity estimator is an example of an integer aperture estimator. The computational steps involved for determining the outcome of the minimal penalty estimator are given. The additional complexity in comparison with current practice is minor, since the optimal integer estimator still plays a major role in the solution of the minimal penalty ambiguity estimator.

Key words: Global Navigation Satellite System ambiguity resolution – Integer estimation – Integer aperture estimation – Penalized ambiguity resolution

1 Introduction

As our point of departure we take the following system of linear observation equations:

$$E\{y\} = Aa + Bb, \ a \in Z^n, \ b \in R^q \tag{1}$$

with $E\{\cdot\}$ the mathematical expectation operator, y the m-vector of observables, a the n-vector of unknown integer parameters and b the q-vector of unknown real valued parameters. All the linear(ized) Global Navigation Satellite System (GNSS) models can in principle be cast in the above frame of observation equations. The data vector y will then usually consist of the 'observed minus computed' single-, dual- or multi-frequency double-difference (DD) phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector a are then the DD carrier phase ambiguities, expressed in units of cycles rather than range, while the entries of the vector b will consist of the remaining unknown parameters, such as, for instance, baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere).

The procedure for solving the above GNSS model can be divided conceptually into three steps. In the first step we simply discard the integer constraints $a \in Z^n$ and perform a standard adjustment. As a result we obtain the so called float solution $\hat{a}$ and $\hat{b}$. This solution is real valued. In the second step the float solution $\hat{a}$ is further adjusted so as to take the integerness of the ambiguities into account in some pre-defined way. This gives

$$\hat{a}_S = S(\hat{a}) \tag{2}$$

in which S is an n-dimensional mapping that takes the integerness of the ambiguities into account. This estimator is then used in the final step to adjust the float estimator $\hat{b}$. As a result we obtain the so called fixed estimator of b as

$$\hat{b}_S = \hat{b} - Q_{\hat{b}\hat{a}} Q_{\hat{a}}^{-1} (\hat{a} - \hat{a}_S) \tag{3}$$

in which $Q_{\hat{a}}$ denotes the variance–covariance (VC) matrix of $\hat{a}$ and $Q_{\hat{b}\hat{a}}$ denotes the covariance matrix of $\hat{b}$ and $\hat{a}$.

The above three-step procedure is still ambiguous in the sense that it leaves room for choosing the n-dimensional map S. Different choices for S will lead to different ambiguity estimators and thus also to different baseline estimators $\hat{b}_S$. We can therefore now think of constructing a family of maps S with certain desirable properties. Three such classes of ambiguity estimators are the class of integer estimators, the class of integer equivariant estimators and the class of integer aperture estimators. These classes were introduced by the author in, respectively, Teunissen (1999, 2002, 2003a). These three classes of estimators are subsets of one another. The first class is the most restrictive class. This is due to the fact that the outcomes of any estimator within this class are required to be integer. The most relaxed class is the class of integer equivariant estimators. These estimators are real valued and they only obey the integer remove–restore principle. The class of integer aperture estimators is a subset of the integer equivariant estimators but it encompasses the class of integer estimators. The integer aperture estimators are of a hybrid nature in the sense that their outcomes are either integer or non-integer.

The optimal integer estimator and the optimal integer equivariant estimator are given in, respectively, Teunissen (1999, 2002). In the present contribution we will introduce the optimal ambiguity estimator for the class of integer aperture estimators. The idea is to give the user the possibility to assign penalties to the outcomes of an integer aperture estimator; for example, a high penalty for an incorrect integer outcome and a low penalty for a correct outcome. Each integer aperture estimator will then have its own average penalty. The optimal integer aperture estimator is the one which has the smallest possible average penalty.

This contribution is organized as follows. In order to introduce the underlying principle of integer aperture estimation we first consider the definition of integer estimators in Sect. 2. The pull-in regions of integer estimators need to obey three conditions: they need to be translational invariant and fill the complete ambiguity space R^n without gaps and overlaps. Integer aperture estimators, however, only need to satisfy two of these three conditions. For the integer aperture estimators we skip the condition that their pull-in regions need to fill the ambiguity space completely. Gaps are therefore allowed. The exact definition of integer aperture estimators is given in Sect. 2. Since there is a whole class of such estimators many different examples can be given. We will give three different examples of integer aperture estimators and show that two of them are already in use as so called ‘discernibility tests’. This also illustrates how the use of ‘discernibility tests’ fits into the theory of integer aperture estimation.

In Sect. 3 we introduce the possibility of assigning penalties to the outcomes of an integer aperture estimator. The optimal integer aperture estimator is then defined as the estimator which returns the smallest possible average penalty. It turns out that the average penalty can be minimized in three different ways. The first two are constrained minimization problems, whereas the third one is an unconstrained minimization problem. The solutions of all three problems, together with the computational steps involved, are given in Sect. 3. These solutions also clearly show the relationships that exist between optimal integer aperture estimation and optimal integer estimation. Although the proofs are given for an arbitrary probability density function (PDF) of the float solution, we also give the explicit solutions for the Gaussian case. The contribution is concluded with a summary in Sect. 4.

2 Integer aperture estimation

2.1 Integer estimation

The class of integer aperture (IA) estimators is larger than the class of integer (I) estimators. In order to understand the underlying principle of IA estimation, we first consider the definition of integer estimators.

Definition 1 (integer estimators). The mapping $\check{a} = S(\hat{a})$, $S : R^n \mapsto Z^n$, is said to be an *integer estimator* if its pull-in regions

$$S_z = \{x \in R^n \mid z = S(x)\}, \quad z \in Z^n \tag{4}$$

satisfy

(1) $\bigcup_{z \in Z^n} S_z = R^n$
(2) $\mathrm{Int}(S_{z_1}) \bigcap \mathrm{Int}(S_{z_2}) = \emptyset, \quad \forall z_1, z_2 \in Z^n, z_1 \neq z_2$
(3) $S_z = z + S_0, \quad \forall z \in Z^n$

This class of estimators was introduced in Teunissen (1999) with the following motivation. Each one of the above three conditions describes a property which it seems reasonable is possessed by an arbitrary I estimator. The first condition states that the pull-in regions should not leave any gaps and the second that they should not overlap. The absence of gaps is needed in order to be able to map any float solution $\hat{a} \in R^n$ to Z^n, while the absence of overlaps is needed to guarantee that the float solution is mapped to just one integer vector. Note that we allow the pull-in regions to have common boundaries. This is permitted if we assume to have zero probability that $\hat{a}$ lies on one of the boundaries. This will be the case when the PDF of $\hat{a}$ is continuous. The third and last condition of the definition follows from the requirement that $S(x + z) = S(x) + z, \forall x \in R^n, z \in Z^n$. This condition is also a reasonable one to ask for. It states that when the float solution $\hat{a}$ is perturbed by $z \in Z^n$, the corresponding integer solution is perturbed by the same amount. This property allows us to apply the *integer remove–restore* technique: $S(\hat{a} - z) + z = S(\hat{a})$. It therefore allows us to work with the fractional parts of the entries of $\hat{a}$, instead of with its complete entries.

Using the pull-in regions, we can give an explicit expression for the corresponding I estimator $\check{a}$. It reads

$$\check{a} = \sum_{z \in Z^n} z s_z(\hat{a}) \text{ with } s_z(\hat{a}) = \begin{cases} 1 & \text{if } \hat{a} \in S_z \\ 0 & \text{if } \hat{a} \notin S_z \end{cases} \tag{5}$$

Note that the $s_z(\hat{a})$ can be interpreted as weights, since $\sum_{z \in Z^n} s_z(\hat{a}) = 1$. The I estimator $\check{a}$ is therefore equal to a weighted sum of integer vectors with binary weights. Also note that the above given definition of I estimators allows us to devise our own I estimator. Once we have come up with disjoint subsets which are translated copies of one another and which cover R^n completely, we have defined our own I estimator. Well-known examples of I estimators are the estimators based on integer rounding, integer bootstrapping and integer least squares (LS). In two dimensions their pull-in regions are given respectively as squares, parallelograms and hexagons.

2.2 Optimal integer estimation

For the evaluation of the I estimator we need the distribution of $\check{a}$. This distribution is of the discrete type and it will be denoted as $P(\check{a} = z)$. It is a probability mass function (PMF), having zero masses at nongrid points and nonzero masses at some or all grid points. In order to obtain this PMF we need the PDF of the float solution $\hat{a}$. This PDF will be denoted as $f_{\hat{a}}(x \mid a)$, in which we explicitly show the dependence on the unknown but integer vector a. In the Gaussian case the PDF will be given as

$$f_{\hat{a}}(x \mid a) = \frac{1}{\sqrt{\det(Q_{\hat{a}})(2\pi)^{\frac{1}{2n}}}} \exp\left\{-\frac{1}{2} \parallel x - a \parallel^2_{Q_{\hat{a}}}\right\} \tag{6}$$

with $\parallel \cdot \parallel^2_{Q_{\hat{a}}} = (\cdot)^T Q_{\hat{a}}^{-1}(\cdot)$. The PMF $P(\check{a} = z)$ follows from integrating $f_{\hat{a}}(x \mid a)$ over the pull-in regions S_z:

$$P(\check{a} = z) = \int_{S_z} f_{\hat{a}}(x \mid a)\, \mathrm{d}x\ ,\ z \in Z^n \tag{7}$$

This distribution is of course dependent on the pull-in regions S_z and thus on the chosen I estimator. Since various I estimators exist which are admissible, some may be better than others. Having the problem of GNSS ambiguity resolution in mind, we are particularly interested in the estimator which maximizes the probability of correct I estimation. This probability equals $P(\check{a} = a)$, but it will differ for different estimators. The answer to the question of which estimator maximizes the probability of correct integer estimation was given in Teunissen (1999).

Theorem 1 (optimal integer estimation). *Let $f_{\hat{a}}(x \mid a)$ be the PDF of the float solution $\hat{a}$ and let*

$$\check{a}_{ML} = \arg\max_{z \in Z^n} f_{\hat{a}}(\hat{a} \mid z) \tag{8}$$

be an integer estimator. Then

$$P(\check{a}_{ML} = a) \geq P(\check{a} = a) \tag{9}$$

for any arbitrary integer estimator $\check{a}$.

The above theorem holds true for an arbitrary PDF of the float solution $\hat{a}$. In most GNSS applications however, we assume the data to be normally distributed. The estimator $\hat{a}$ will then be normally distributed too, with mean $a \in Z^n$ and VC matrix $Q_{\hat{a}}$, $\hat{a} \sim \mathrm{N}(a, Q_{\hat{a}})$. In this case the optimal estimator becomes identical to the integer LS estimator

$$\check{a}_{LS} = \arg\min_{z \in Z^n} \parallel \hat{a} - z \parallel^2_{Q_{\hat{a}}} \tag{10}$$

The above theorem therefore gives a probabilistic justification for using the integer LS estimator when the PDF is Gaussian. For GNSS ambiguity resolution we are thus better off using the integer LS estimator than any other admissible I estimator. A well known and very efficient method for GNSS integer LS ambiguity resolution is the LAMBDA method. Examples of its application can be found in, for example de Jonge and Tiberius (1996b), de Jonge et al. (1996), Boon and Ambrosius (1997), Boon et al. (1997) and Cox and Brading (1999). For more information on the LAMBDA method, we refer to, for example Teunissen (1993, 1995) and de Jonge and Tiberius (1996a), or to the textbooks of Strang and Borre (1997), Teunissen and Kleusberg (1998), Misra and Enge (2001), Hofmann-Wellenhof et al. (2002) and Seeber (2003).

2.3 Aperture pull-in regions

The outcome of an I estimator is always integer. It may happen, however, that we are not willing to accept the integer outcome. In that case we would prefer to work with the real valued float solution, than with the integer solution even though it is known that the parameter to be estimated is integer. The rationale of this choice is that the usage of an incorrect integer outcome is more harmful than the usage of the non-integer float solution. The decision whether or not to make use of the integer outcome can be made in different ways. One approach is to base the decision on the probability of correct integer estimation, also referred to as the success rate. The decision is then made in favour of the float solution if this probability falls below a certain user-defined threshold. This approach can be referred to as being model driven, since the probability of correct integer estimation depends on the strength of the underlying mathematical model but not on the actual outcome of the estimator. With this approach, the decision whether or not to make use of the I estimator can thus be made before the actual measurements are collected and processed. Next to this model-driven approach, we can also make use of a more data driven approach. In many GNSS ambiguity resolution procedures we also have such a data-driven approach in place. They are referred to as the 'discernibility tests'. They come to reject the integer outcome when it appears difficult, using the float solution, to discern between the 'best' and the 'second best' integer solution. In the case of a rejection the decision is made in favour of the float solution. As with the model driven approach, the rationale of the 'discernibility tests' is that we want to avoid the situation of having to work with an incorrect integer solution. Since the aim of both approaches is essentially the same, we may wonder whether or not it is possible to formulate an overall framework in which both approaches find their natural place. This indeed turns out to be possible. The required framework is given by the class of integer aperture (IA) estimators as introduced in Teunissen (2003a). The IA estimators are defined by dropping one of the three conditions of Definition 1, namely the condition that the pull-in regions should cover R^n completely. The pull-in regions of the IA estimators are therefore allowed to have gaps, thus making it possible that their outcomes could be equal to the float solution as well.

In order to introduce the class of IA estimators from first principles, let $\Omega \subset R^n$ be the region of R^n for which $\hat{a}$ is mapped to an integer if $\hat{a} \in \Omega$. It seems reasonable to ask of the region Ω that it has the property that if $\hat{a} \in \Omega$ then also $\hat{a} + z \in \Omega$, for all $z \in Z^n$. If this property would not hold, then float solutions could be mapped to integers whereas their fractional parts could not. We thus require Ω to be translational invariant with respect to an arbitrary integer vector: $\Omega + z = \Omega$, for all $z \in Z^n$. Knowing Ω is however not sufficient for defining our estimator. Ω only determines whether or not the float solution is mapped to an integer; it does not tell us yet to which integer the float solution is mapped. We therefore define

$$\Omega_z = \Omega \cap S_z \ , \ \forall z \in Z^n \tag{11}$$

where S_z is a pull-in region satisfying the conditions of Definition 1. Then

$$\begin{aligned}
&(1) \quad \bigcup_z \Omega_z = \bigcup_z (\Omega \cap S_z) = \Omega \cap (\bigcup_z S_z) = \Omega \cap R^n = \Omega \\
&(2) \quad \Omega_{z_1} \cap \Omega_{z_2} = (\Omega \cap \Omega_{z_1}) \cap (\Omega \cap \Omega_{z_2}) \\
&\qquad = \Omega \cap (S_{z_1} \cap S_{z_2}) = \emptyset, \ \forall z_1, z_2 \in Z^n, z_1 \neq z_2 \\
&(3) \quad \Omega_0 + z = (\Omega \cap S_0) + z = (\Omega + z) \cap (S_0 + z) \\
&\qquad = \Omega \cap S_z = \Omega_z, \ \ \forall z \in Z^n
\end{aligned}$$

This shows that the subsets $\Omega_z \subset S_z$ satisfy the same conditions as those of Definition 1, be it that R^n has now been replaced by $\Omega \subset R^n$. Hence, the mapping of the IA estimator can now be defined as follows. The IA estimator maps the float solution $\hat{a}$ to the integer vector z when $\hat{a} \in \Omega_z$ and it maps the float solution to itself when $\hat{a} \notin \Omega$. The class of IA estimators can therefore be defined as follows.

Definition 2 (integer aperture estimators) *Integer aperture estimators* are defined as

$$\hat{a}_{IA} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a}) \tag{12}$$

with $\omega_z(x)$ the indicator function of $\Omega_z = \Omega \cap S_z$ and $\Omega \subset R^n$ translational invariant.

Note that the class of IA estimators is larger than the class of I-estimators. That is, every I estimator is also an IA estimator, but not vice versa. That every I estimator is an IA estimator can also be seen by showing that every I estimator can be written as Eq. (12). Since the indicator functions $s_z(x)$ of the pull-in regions S_z sum up to unity, $\sum_{z \in Z^n} s_z(x) = 1$, the I estimator of Eq. (5) may indeed be written as

$$\check{a} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) s_z(\hat{a}) \tag{13}$$

Comparing this expression with that of Eq. (12) shows that the difference between the two estimators lies in their binary weights, $s_z(x)$ versus $\omega_z(x)$. Since the $s_z(x)$ sum up to unity for all $x \in R^n$, the outcome of an I estimator will always be integer. This is not true for an IA estimator, since the binary weights $\omega_z(x)$ do not sum up to unity for all $x \in R^n$. The IA estimator is therefore a hybrid estimator having as outcome either the real valued float solution $\hat{a}$ or an integer solution. The IA estimator returns the float solution if $\hat{a} \notin \Omega$ and it will be equal to z when $\hat{a} \in \Omega_z$. Note that, since Ω is the collection of all $\Omega_z = \Omega_0 + z$, the IA estimator is completely determined once Ω_0 is known. Thus $\Omega_0 \subset S_0$ plays the same role for the IA estimators as S_0 does for the I-estimators. By changing the size and shape of Ω_0 we change the outcome of the IA estimator. The subset Ω_0 can therefore be seen as an adjustable pull-in region with two limiting cases: the limiting case in which Ω_0 is empty and the limiting case when Ω_0 equals S_0. In the first case the IA estimator becomes identical to the float solution $\hat{a}$, and in the second case the IA estimator becomes identical to an I estimator. The subset Ω_0 therefore determines the *aperture* of the pull-in region.

2.4 Three examples of IA estimators

Various examples can be given of IA estimators. In fact, we can devise our own IA estimator by specifying the aperture pull-in region. Since the output of an IA estimator is given as $\hat{a}_{IA} = z$ if $\hat{a} \in \Omega_z \subset S_z$ and as $\hat{a}_{IA} = \hat{a}$ if $\hat{a} \notin \Omega \subset R^n$, there are essentially two steps involved when computing the integer aperture estimate: (1) the computation of an integer estimate and (2) the verification whether or not the ambiguity residual resides in Ω_0.

In the first step the integer estimate $\check{a}$ is computed from the float solution as $\check{a} = z \Leftrightarrow \hat{a} \in S_z$. These computations depend very much on the type of pull-in region. They are straightforward in the case of integer rounding and integer bootstrapping, whereas in the case of integer LS we would need an efficient integer search procedure such as the mechanized one in the LAMBDA method. Once the integer estimate $\check{a}$ has been computed, the second step amounts to the verification of whether or not the ambiguity residual $\check{\epsilon} = \hat{a} - \check{a}$ resides in Ω_0. This is equivalent to the verification of whether or not $\hat{a} \in \Omega_{\check{a}}$. The output of the IA estimator then equals the integer estimate when the residual resides in Ω_0 and it equals the real valued float solution otherwise. These two computational steps can be recognized in any IA estimator. We will now give three examples of IA estimators.

2.4.1 The ratio testimator

In the practice of GNSS carrier phase ambiguity resolution various tests are in use for discriminating between the 'best' and the so called 'second-best' solution. These tests are usually referred to as discernibility tests. One such test is the popular ratio test. The ratio test is defined as follows. Let $\hat{a}$ be the float solution, $\check{a} = \arg\min_{z \in Z^n} \| \hat{a} - z \|^2_{Q_{\hat{a}}}$ the integer solution and $\breve{a}' = \arg\min_{z\in Z^n\setminus\breve{a}} \| \hat{a} - z \|^2_{Q_{\hat{a}}}$ the 'second-best' solution. Then $\breve{a}$ is accepted as the fixed solution if

$$\frac{\| \hat{a} - \breve{a} \|^2_{Q_{\hat{a}}}}{\| \hat{a} - \breve{a}' \|^2_{Q_{\hat{a}}}} \leq \rho \tag{14}$$

This test has been used in, for example, Euler and Schaffrin (1990), Wei and Schwarz (1995), Han and Rizos, (1996). Thus, with the ratio test, $\breve{a}$ is accepted as the fixed solution if the float solution $\hat{a}$ is sufficiently closer to $\breve{a}$ than to the 'second-best' solution $\breve{a}'$. The non-negative scalar ρ is a user-defined tolerance level.

In Teunissen (2003c) it was shown that the procedure underlying the above test is actually that of an IA estimator. The rejection region of the above test is integer translational invariant and thus an example of $R^n \setminus \Omega$. For this region the outcome will be $\hat{a}$. The outcome will be the integer $z \in Z^n$, however, when the test is passed and $\hat{a}$ lies in the integer LS pull-in region of z. The aperture of the pull-in region of the ratio test is governed by the choice of the single parameter ρ. We have a zero aperture in the case that $\rho = 0$ and a maximum aperture in the case that $\rho = 1$. In the first case the procedure of the ratio test will always output the float solution, while in the second case it will always output the integer LS solution $\breve{a}$. Changing the value of the aperture parameter ρ will thus change the performance of the ratio test.

2.4.2 The difference testimator

Although perhaps less popular, tests other than the ratio test have been proposed in the GNSS literature as well. One such test is the difference test. This test was introduced in Tiberius and de Jonge (1995). This test also makes use of the integer LS solution and the 'second best' solution. It is defined as follows. The integer LS solution $\breve{a}$ is accepted as the fixed solution with the difference test if

$$\| \hat{a} - \breve{a}' \|^2_{Q_{\hat{a}}} - \| \hat{a} - \breve{a} \|^2_{Q_{\hat{a}}} \geq \delta \tag{15}$$

where the non-negative scalar δ is a user-defined tolerance level. As with the ratio test, the difference test accepts $\breve{a}$ as the fixed solution if the float solution is sufficiently more closer to $\breve{a}$ than to the 'second best' solution $\breve{a}'$. 'Closeness' is, however, measured differently. The procedure underlying the difference test can also be shown to be that of an IA estimator (see Teunissen 2003b).

2.4.3 The ellipsoidal IA estimator

The procedures currently in place for GNSS ambiguity resolution all make use of comparing, in some predefined sense, the 'best' solution with the 'second best' solution. But when we think of the concept of the aperture region, there is in principle no need to compute or to make use of the 'second-best' solution. That is, we can do without the 'second-best' solution, as long as we are able to measure and evaluate the closeness of the float solution to an integer. The ellipsoidal IA estimator is one such IA estimator. The aperture pull-in regions of the ellipsoidal integer aperture (EIA) estimator are defined as

$$E_z = E_0 + z, \quad E_0 = S_0 \cap C_{\epsilon,0}, \quad \forall z \in Z^n \tag{16}$$

with S_0 being the integer LS pull-in region and $C_{\epsilon,0} = \{x \in R^n \mid \| x \|^2_{Q_{\hat{a}}} \leq \epsilon^2\}$ an origin-centred ellipsoidal region the size of which is controlled by the aperture parameter ϵ.

Thus the EIA estimator equals $\hat{a}_{\text{EIA}} = z$ if $\hat{a} \in E_z$ and $\hat{a}_{\text{EIA}} = \hat{a}$ otherwise. From the definition it follows that $E_z = \{x \in S_z \mid \| x - z \|^2_{Q_{\hat{a}}} \leq \epsilon^2\}$. This shows that the procedure for computing the EIA estimator is rather straightforward. Using the float solution $\hat{a}$, its VC matrix $Q_{\hat{a}}$ and the aperture parameter ϵ as input, we only need to compute the integer LS solution $\breve{a}$ and verify whether or not the inequality

$$\| \hat{a} - \breve{a} \|^2_{Q_{\hat{a}}} \leq \epsilon^2 \tag{17}$$

is satisfied. If the inequality is satisfied then $\hat{a}_{\text{EIA}} = \breve{a}$, otherwise $\hat{a}_{\text{EIA}} = \hat{a}$. A comparison with the ratio test [Eq. (14)] and with the difference test [Eq. (15)] shows that [Eq. (17)] is indeed the simplest of the three inequalities. Instead of working with a distance ratio or a distance difference, the EIA estimator simply evaluates the distance to the closest integer directly. There is therefore no need to make use of a 'second-best' solution.

The simple choice of the ellipsoidal criterion of Eq. (17) is motivated by the fact that the squared norm of a normally distributed random vector is known to have a Chi-square distribution. That is, if $\hat{a}$ is distributed as $\hat{a} \sim \mathrm{N}(a, Q_{\hat{a}})$ then $P(\hat{a} \in C_{\epsilon,z}) = P(\chi^2(n, \mu_z) \leq \epsilon^2)$, in which $\chi^2(n, \mu_z)$ denotes a random variable having as PDF the noncentral Chi-square distribution with n degrees of freedom and noncentrality parameter $\mu_z = (z - a)^T Q_{\hat{a}}^{-1} (z - a)$. This implies that we can give an exact solution to the success rate of the EIA estimator, provided the ellipsoidal regions $C_{\epsilon,z}$ do not overlap (see Teunissen, 2003c).

3 Penalized IA estimation

3.1 Minimizing the average penalty

So far we have discussed the class of IA estimators and have given three examples of IA estimators of which the aperture pull-in regions were chosen a priori. In fact, using the general definition of IA estimators it is not too difficult to define our own IA estimator. In order to do so we only have to define an aperture pull-in region Ω_0 which satisfies the conditions of Definition 2. Since many different IA estimators exist, which one do we choose? In order to answer this question we first need to define a criterion by means of which we can compare different IA estimators, followed by solving the problem of which of the IA estimators meets this criterion best. In order to tackle this problem we introduce the principle of *penalized IA estimation*.

To understand this principle it is helpful to first classify the possible outcomes of an IA estimator. An IA estimator can produce one of the following three outcomes: $a \in Z^n$ (correct integer), $z \in Z^n \setminus \{a\}$ (incorrect integer) or $\hat{a} \in R^n \setminus Z^n$ (no integer). A correct integer outcome may be considered a success, an incorrect integer outcome a failure, and an outcome where no correction at all is given to the float solution as indeterminate or undecided. The idea of penalized IA estimation is now to assign penalties to each of the above three possible outcomes. The three different penalties assigned are: a success penalty, p_S, a failure penalty, p_F, and a penalty for undecided, p_U. The penalty assignment therefore reads

$$\begin{aligned} &p_S \text{ if } \hat{a} \in \Omega_a \\ &p_F \text{ if } \hat{a} \in \Omega \setminus \Omega_a \\ &p_U \text{ if } \hat{a} \in R^n \setminus \Omega \end{aligned} \tag{18}$$

Through this assignment we have now constructed a discrete random variable, the penalty p, having the three possible outcomes, $p = \{p_S, p_F, p_U\}$. These outcomes have the following probabilities of occurrence:

$$\begin{aligned} P_S &= \int_{\Omega_a} f_{\hat{a}}(x \mid a) \mathrm{d}x \quad \text{(success)} \\ P_F &= \sum_{z \neq a} \int_{\Omega_z} f_{\hat{a}}(x \mid a) \mathrm{d}x \quad \text{(failure)} \\ P_U &= 1 - P_S - P_F \quad \text{(undecided)} \end{aligned} \tag{19}$$

where $f_{\hat{a}}(x \mid a)$ denotes the PDF of the float solution $\hat{a}$. In the case of GNSS this PDF is usually assumed to be Gaussian, $\hat{a} \sim N(a, Q_{\hat{a}})$. Note that the success probability P_S equals the integral of the PDF over the aperture pull-in region centred at $a \in Z^n$, the true but unknown integer ambiguity vector. The failure probability P_F equals the integral of the PDF over all pull-in regions except the one centred at a, and the undecided probability P_U equals the integral of the PDF over $R^n \setminus \Omega$.

We may now consider the average of the discrete random variable p, the average penalty $E\{p\}$. It is a weighted sum of the individual penalties, with the weights being equal to the three probabilities P_S, P_F and P_U

$$E\{p\} = p_S P_S + p_F P_F + p_U P_U \tag{20}$$

The average penalty depends on the chosen individual penalties and—through the probabilities—on the chosen aperture pull-in region $\Omega_0 \subset S_0$. Changes in any of these will change the average penalty. The penalties are chosen by the user. Their size will depend on the application at hand, e.g. a severe penalty p_F will most likely be chosen if a wrong integer outcome of the IA estimator is considered to have unacceptable consequences. The penalties satisfy $p_S < p_U < p_F$. That is, a failure will be given the highest penalty and a success the lowest penalty.

Since the average penalty depends on Ω_0, any changes in Ω_0 will also change the average penalty. Thus different IA estimators will have a different performance as far as their average penalty is concerned. The idea now is to select the IA estimator which minimizes the average penalty.

3.2 Minimizing the penalty for a given integer estimator

The average penalty can be minimized in three different ways. In order to understand this properly we have to take a closer look at the role played by Ω and S_0 in defining an IA estimator. We have seen that any IA estimator is uniquely characterized by its aperture pull-in region $\Omega_0 = \Omega \cap S_0$. The translational invariant region $\Omega = \cup_z \Omega_z$ is the region for which the float solution is mapped to an integer. This region is independent of S_z. Thus if S_z and S'_z are two different sets of pull-in regions, then $\Omega_z = \Omega \cap S_z \neq \Omega'_z = \Omega \cap S'_z$, but $\Omega = \cup_z \Omega_z = \cup_z \Omega'_z$. The fact that Ω and S_0 can be chosen independently when defining an IA estimator implies that we can minimize the average penalty in three different ways, namely for a given S_0 as function of Ω, or for a given Ω as function of S_0, or as function of both Ω and S_0. The first two are constrained minimization problems, whereas the third one is an unconstrained minimization problem. We will first consider the minimization of the average penalty as function of Ω for a given S_0. The solution to this problem is given in the following theorem.

Theorem 2a (penalized IA estimation with given I-estimator) *Let $f_{\hat{a}}(x \mid a)$ be the PDF of the float solution $\hat{a}$, let $\check{a} = \sum_{z \in Z^n} z s_z(\hat{a})$ be the chosen I estimator which is uniquely characterized by its pull-in region S_0, and let $\Omega = \Omega + z \subset R^n$ be the translational invariant aperture region for which the float solution is mapped to an integer. The IA estimator having*

$$\min_{\Omega} E\{p\} \ \text{ subject to } S_0 \tag{21}$$

as average penalty is then given as

$$\hat{a}_{IA} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a})$$

with

$$\begin{aligned} \Omega_0 = \{x \in S_0 \mid & \sum_{z \in Z^n} f_{\hat{a}}(x + z \mid a) \\ & \leq \frac{p_F - p_S}{p_F - p_U} f_{\hat{a}}(x + a \mid a)\} \end{aligned} \tag{22}$$

and where $\omega_z(x)$ denotes the indicator function of $\Omega_z = \Omega_0 + z$.

Proof. See Appendix.

This theorem is applicable to the situation where we have already decided which I estimator to use. This could be integer rounding, integer bootstrapping, integer LS or variations thereof. Given the integer estimator chosen, the theorem shows how we need to choose the aperture pull-in region in order to have the smallest possible average penalty. The algorithmic steps for computing the corresponding IA estimator are then as follows. Consider that we have decided to use the bootstrapped estimator $\check{a}_B$ as I estimator. The first step consists then of computing the outcome of this estimator, say $\check{a}_B = z$. Then in the second step we compute the corresponding bootstrapped residual $\check{\epsilon}_B = \hat{a} - z$ and verifies whether or not $\check{\epsilon}_B \in \Omega_0$. The outcome of the IA estimator equals then $\hat{a}_{IA} = z$ if the residual resides in Ω_0 and it equals the float solution $\hat{a}_{IA} = \hat{a}$ otherwise.

Note that the theorem shows that the assigned penalties come together in the single ratio

$$\lambda = \frac{p_F - p_S}{p_F - p_U} \tag{23}$$

This ratio governs the aperture of the penalized pull-in region Ω_0. The larger this value, the larger the aperture. Note that the absolute level of the penalties is of no consequence. It is the relative values of the penalties that count. If the success penalty is set to zero, $p_S = 0$, the ratio becomes driven by the penalty ratio p_U/p_F. The smaller this ratio is, the more serious a failure is considered and the smaller the aperture of Ω_0 is taken to be.

3.3 *Minimizing the penalty for a given aperture region*

We will now consider the second constrained minimization of the average penalty. This applies to the situation where we have already decided for which region the IA estimator will have to output the float solution. What remains to be determined is the integer outcome which minimizes the average penalty of the IA estimator.

Theorem 2b (penalized IA estimation with given aperture region). *Let $f_{\hat{a}}(x \mid a)$ be the PDF of the float solution $\hat{a}$, let Ω be the chosen translational invariant aperture region for which the float solution is mapped to an integer, and let S_0 be the pull-in region of the I estimator. The IA estimator having*

$$\min_{S_0} E\{p\} \quad \text{subject to } \Omega \tag{24}$$

as average penalty is then given as

$$\hat{a}_{IA} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a})$$

with

$$S_0 = \{x \in R^n \mid 0 = \arg\max_{z \in Z^n} f_{\hat{a}}(x \mid z)\} \tag{25}$$

and where $\omega_z(x)$ denotes the indicator function of $\Omega_z = \Omega \cap S_z$.

Proof. See Appendix

This result shows that the I estimator which minimizes the average penalty for a given Ω does not depend on the assigned penalties or on the choice made for Ω. Hence, whatever choice is made for the size and shape of the translational invariant region Ω, we will always have to use the same I estimator in order to have the smallest possible average penalty for the IA estimator. Note that this estimator is identical to the one which maximizes the success rate within the class of I estimators, [see Eq. (8)]. In fact, the above result can be seen as a generalization of Theorem 1. If the translational invariant aperture region is chosen without gaps, then $\Omega = R^n$ and thus $\Omega_z = S_z$. But the absence of gaps implies that $P_U = 0$ and thus that $P_F = 1 - P_S$, from which it follows that the average penalty becomes $E\{p\} = p_F + (p_S - p_F)P_S$. Minimizing the average penalty in this case then corresponds to a maximization of the success rate.

3.4 *IA estimation with minimal penalty*

So far we considered the minimization of the average penalty given the I estimator or given the aperture region. Both minimization problems are of the constrained type. We will now consider the unconstrained minimization of the average penalty. The minimum so obtained will be smaller than, or at most equal to, the two constrained minima. The solution of the unconstrained minimization problem is given in the following theorem.

Theorem 2c (minimal penalty IA estimation) *Let $f_{\hat{a}}(x \mid a)$ be the PDF of the float solution $\hat{a}$ and let $\Omega_0 = \Omega \cap S_0$ be the aperture pull-in region of the IA estimator. The IA estimator having*

$$\min_{\Omega_0 = \Omega \cap S_0} E\{p\} \tag{26}$$

as average penalty is then given as

$$\hat{a}_{IA} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a})$$

with

$$\Omega_0 = \{x \in S_0 \mid \sum_{z \in Z^n} f_{\hat{a}}(x + z \mid a) \leq \frac{p_F - p_S}{p_F - p_U} f_{\hat{a}}(x + a \mid a)\} \tag{27}$$

and

$$S_0 = \{x \in R^n \mid 0 = \arg\max_{z \in Z^n} f_{\hat{a}}(x \mid z)\}$$

and where $\omega_z(x)$ denotes the indicator function of $\Omega_z = \Omega \cap S_z$.

Proof. See Appendix

Note how the above result relates to the solutions of the two previous minimization problems. The two constrained minimizers combined make up for the unconstrained minimizer of Eq. (27). This can be explained by the fact that the solution for the pull-in region of the constrained minimization problem of Eq. (25) is independent of the constraint put on the aperture region Ω.

The above result applies to an arbitrary PDF of $\hat{a}$. In most cases, however, the PDF of the float solution is assumed to be Gaussian. In the case that the float solution is normally distributed as $\hat{a} \sim N(a, Q_{\hat{a}})$, the minimal penalty aperture pull-in region becomes

$$\Omega_0 = \{x \in S_0 \mid \sum_{z \in Z^n \backslash \{0\}} \exp\{-\frac{1}{2} \parallel x - z \parallel^2_{Q_{\hat{a}}}\} \leq \frac{p_{\mathrm{U}} - p_{\mathrm{S}}}{p_{\mathrm{F}} - p_{\mathrm{U}}} \exp\{-\frac{1}{2} \parallel x \parallel^2_{Q_{\hat{a}}}\}\} \quad (28)$$

with S_0 being the integer LS pull-in region. The computational steps involved in computing the optimal IA estimator are now as follows. First compute the integer LS solution

$\check{a}_{\mathrm{LS}} = \arg\min_{z \in Z^n} \parallel \hat{a} - z \parallel^2_{Q_{\hat{a}}}$. Then form the ambiguity residual $\check{\epsilon}_{\mathrm{LS}} = \hat{a} - \check{a}_{\mathrm{LS}}$ and check whether $\check{\epsilon}_{\mathrm{LS}} \in \Omega_0$. If this is the case then the outcome of the optimal estimator is $\check{a}_{\mathrm{LS}}$, otherwise the outcome is $\hat{a}$. For the purpose of computational efficiency it is advisable to compute $\check{a}_{\mathrm{LS}}$ with the LAMBDA method and use the LAMBDA-transformed ambiguities also for the evaluation of $\check{\epsilon}_{\mathrm{LS}} \in \Omega_0$.

Note that the contribution of the exponentials in the sum of Eq. (28) gets smaller the more peaked the PDF of the float solution is. The aperture pull-in region Ω_0 will therefore get larger the more peaked the PDF is. This is also what we would expect. With Eq. (28) we are now also in a position to make an interesting link with one of the IA estimators presented in the previous section, namely the difference testimator. If we approximate Ω_0 by retaining only the largest term in the sum of the inequality of Eq. (28) we obtain the inequality of the difference test. This shows that the difference testimator is a close to optimal IA estimator in the case that the PDF is peaked.

4 Summary

In this contribution a new principle of GNSS carrier phase ambiguity resolution was introduced. The idea is to give the user the possibility to assign penalties to the outcomes of the ambiguity resolution process. The optimal ambiguity estimator is then the one which achieves the smallest possible penalty. The theory was presented in the framework of IA estimation.

When comparing I estimators with IA estimators and, in particular, with the minimal penalty IA estimator, the following conclusions can be drawn. Since the outcome of an I estimator is always integer, we usually use the probability of correct integer estimation of the I estimator as criterion for deciding whether or not to make use of the integer outcome of the I estimator. This probability, also referred to as the ambiguity success rate, describes the expected performance of the I estimator. It depends on the choice of I estimator and on the strength of the underlying mathematical model. The success rate can be computed a priori, i.e. before the actual measurements are taken. When the success rate turns out to be too low, we ignore the integer outcome and rely on the float solution. Although this is a sound procedure, it is somewhat conservative in the sense that the decision is based on the expected outcome of the I estimator and not on its actual outcome.

With IA estimators the user has gained more flexibility. This flexibility stems from the fact that the class of IA estimators encompasses the class of I estimators. Thus every I estimator is an IA estimator, but not vice versa. Every IA estimator can be represented as

$$\hat{a}_{\mathrm{IA}} = \hat{a} + \sum_{z \in Z^n} (z - \hat{a}) \omega_z(\hat{a}) \quad (29)$$

with $\omega_z(x)$ the indicator function of $\Omega_z = \Omega \cap S_z$ and $\Omega \subset R^n$ being invariant for an arbitrary integer translation. An IA estimator reduces to an I estimator if Ω is chosen equal to R^n. Every IA estimator is uniquely characterized by its aperture pull-in region Ω_0. The flexibility of integer aperture estimation shows itself in the possibility given to the user to set the size and shape of the aperture pull-in region. Once the aperture pull-in regions have been defined, the decision of which outcome to use is dictated by the actual data. That is, the outcome is $z \in Z^n$ if $\hat{a} \in \Omega_z$, and it is $\hat{a} \in R^n$ otherwise. We can therefore conclude, when comparing IA estimators with the practical use of I estimators, that the decision between an integer outcome or a noninteger outcome is data driven in the case of IA estimators, but model driven in the case of I estimators.

In order for IA estimators to make sense, a guiding principle is needed to set the size and shape of the aperture pull-in region. In this contribution we have given the user the possibility to assign penalties to the outcomes of an IA estimator. Using this framework we have shown which aperture pull-in region minimizes the average penalty. This aperture pull-in region can be computed once the user has assigned the penalties for his or her application. The actual outcome of the optimal IA estimator can then be computed in two steps. The first step consists of computing the outcome of an I estimator and the second step consists of verifying whether or not the residual resides in the origin-centred aperture pull-in region Ω_0. It is rewarding to see that, in the Gaussian case, the first step consists of computing the well-known integer LS solution $\check{a}_{\mathrm{LS}}$. The second step then consists of verifying whether or not the ambiguity residual $\check{\epsilon}_{\mathrm{LS}} = \hat{a} - \check{a}_{\mathrm{LS}}$ resides in Ω_0. The first step thus consists of computing

$$\check{a}_{\mathrm{LS}} = \arg\min_{z \in Z^n} \parallel \hat{a} - z \parallel^2_{Q_{\hat{a}}} \quad (30)$$

while the second step consists of verifying whether

$$\sum_{z \in Z^n \backslash \{0\}} \exp\left\{-\frac{1}{2} \parallel \check{\epsilon}_{\mathrm{LS}} - z \parallel^2_{Q_{\hat{a}}}\right\} \leq \lambda \exp\left\{-\frac{1}{2} \parallel \check{\epsilon}_{\mathrm{LS}} \parallel^2_{Q_{\hat{a}}}\right\} \quad (31)$$

in which the scalar λ is determined by the assigned penalties. Both Eqs. (30) and (31) can be computed efficiently with the LAMBDA method. The outcome of the minimal penalty IA estimator is then equal to $\check{a}_{\mathrm{LS}}$ if the above inequality is satisfied and it equals $\hat{a}$ otherwise. When comparing this procedure with the practice of GNSS ambiguity resolution, we can thus conclude that for an optimal result we need to replace the existing 'discernibility tests' with the above given inequality. Thus although the procedures underlying the usage of the existing 'discernibility tests' are those of an IA estimator, they are not optimal in the sense of achieving the smallest possible penalty.

5 Appendix

Proof of Theorem 2

Theorem 2 consists of three parts. It gives the solution to the following three minimization problems:

$$1.\ \min_{\Omega_0=\Omega\cap S_0} E\{p\} \text{ subject to } S_0 \tag{A1a}$$

$$2.\ \min_{\Omega_0=\Omega\cap S_0} E\{p\} \text{ subject to } \Omega \tag{A1b}$$

$$3.\ \min_{\Omega_0=\Omega\cap S_0} E\{p\} \tag{A1c}$$

with the average penalty

$$E\{p\} = p_S P_S + p_F P_F + p_U P_U \tag{A2}$$

and the probabilities of occurrence

$$\begin{aligned} P_S &= \int_{\Omega_a} f_{\hat{a}}(x \mid a)\mathrm{d}x \\ P_F &= \sum_{z\neq a} \int_{\Omega_z} f_{\hat{a}}(x \mid a)\mathrm{d}x \\ P_U &= 1 - P_S - P_F \end{aligned} \tag{A3}$$

We will first solve Eq. (A1a). Note that the average penalty can be written as

$$E\{p\} = p_U + \int_{\Omega_0\subset S_0} F(x)\mathrm{d}x,$$

with

$$F(x) = (p_S - p_F)f_{\hat{a}}(x+a \mid a) + (p_F - p_U)\sum_{z\in Z^n} f_{\hat{a}}(x+z \mid a).$$

The smallest possible value of $\int_{\Omega_0\subset S_0} F(x)\mathrm{d}x$ is then obtained if the region of integration Ω_0 is chosen to cover that part of S_0 for which the function values of $F(x)$ are all negative. Thus $\Omega_0 = \{x \in S_0 \mid F(x) \leq 0\}$ or

$$\Omega_0 = \left\{ x\in S_0 \mid \sum_{z\in Z^n} f_{\hat{a}}(x+z \mid a) \leq \frac{p_F - p_S}{p_F - p_U} f_{\hat{a}}(x+a \mid a) \right\} \tag{A4}$$

which proves Theorem 2a.

We will now solve Eq. (A1b). Note that the average penalty can be written as

$$E\{p\} = p_U + (p_S - p_U)\int_\Omega f(x \mid a)\mathrm{d}x + (p_F - p_S)\int_{\Omega\cap S_0} \sum_{z\neq a} f(x+z \mid a)\mathrm{d}x.$$

Hence, when minimizing the average penalty as function of S_0, we may restrict attention to the third term in this sum. We will now show that this term is minimized for

$$S_0 = \{x \in R^n \mid 0 = \arg\max_{z\in Z^n} f_{\hat{a}}(x \mid z)\} \tag{A5}$$

From Theorem 1 it follows that

$$\int_{S_0} f_{\hat{a}}(x+a \mid a)\mathrm{d}x \geq \int_{S_0'} f_{\hat{a}}(x+a \mid a)\mathrm{d}x$$

for any arbitrary pull-in region S_0' satisfying the conditions of Definition 1. We therefore also have

$$\int_{\Omega\cap S_0} f_{\hat{a}}(x+a \mid a)\mathrm{d}x \geq \int_{\Omega\cap S_0'} f_{\hat{a}}(x+a \mid a)\mathrm{d}x.$$

Using this result in the identity

$$\sum_{z\in Z^n}\int_{\Omega\cap S_z} f_{\hat{a}}(x+a \mid a)\mathrm{d}x = \sum_{z\in Z^n}\int_{\Omega\cap S_z'} f_{\hat{a}}(x+a \mid a)\mathrm{d}x$$

gives the inequality

$$\sum_{z\neq 0}\int_{\Omega\cap S_z} f_{\hat{a}}(x+a \mid a)\mathrm{d}x \leq \sum_{z\neq 0}\int_{\Omega\cap S_z'} f_{\hat{a}}(x+a \mid a)\mathrm{d}x$$

and therefore

$$\int_{\Omega\cap S_0}\sum_{z\neq a} f(x+z \mid a)dx \leq \int_{\Omega\cap S_0'}\sum_{z\neq a} f(x+z \mid a)\mathrm{d}x.$$

This proves Theorem 2b.

The solution of Eq. (A1c) is now easy to obtain. Note that Eq.(A5) is independent of Ω. Hence, S_0 of Eq. (A5) is the solution of $\min_{\Omega\cap S_0} E\{p\}$ for any translational invariant region Ω. The solution of the unconstrained minimization problem of Eq. (A1c) is therefore given by Eqs. (A4) and (A5). This proves Theorem 2c.

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