Full Characterisation of Extended CTL*

Massimo Benerecetti
Università di Napoli Federico II, Italy

Laura Bozzelli
Università di Napoli Federico II, Italy

Fabio Mogavero
Università di Napoli Federico II, Italy

Adriano Peron
Università di Trieste, Italy

Abstract

The precise identification of the expressive power of logic languages used in formal methods for specifying and verifying run-time properties of critical systems is a fundamental task and *characterisation theorems* play a crucial role as model-theoretic tools in this regard. While a clear picture of the expressive power of linear-time temporal logics in terms of word automata and predicate logics has long been established, a complete mapping of the corresponding relationships for branching-time temporal logics has proven to be a more elusive task over the past four decades with few scattered results. Only recently, an *automata-theoretic characterisation* of both CTL* and its full-ω-regular extension ECTL* has been provided in terms of *Symmetric Hesitant Tree Automata* (HTA), with and without a suitable *counter-freeness restriction* on their linear behaviours. These two temporal logics also correspond to the *bisimulation-invariant* semantic fragments of *Monadic Path Logic* (MPL) and *Monadic Chain Logic* (MCL), respectively. Additionally, it has been proven that the counting extensions of CTL* and ECTL*, namely CCTL* and CECTL*, enjoy equivalent graded versions of the HTAs for the corresponding non-counting logics. However, while Moller and Rabinovich have proved CCTL* to be equivalent to full MPL, thus filling the gap for the standard branching-time logic, no similar result has been given for CECTL*. This work completes the picture, by proving the expressive equivalence of CECTL* and full MCL, by means of a *composition theorem* for the latter logic. This also indirectly establishes the equivalence between HTAs and their first-order extensions HFTAs, as originally introduced by Walukiewicz.

2012 ACM Subject Classification Theory of computation → Automata over infinite objects; Theory of computation → Modal and temporal logics; Theory of computation → Tree languages

Keywords and phrases Branching-Time Temporal Logics, Monadic Chain Logic, Tree Automata

Digital Object Identifier 10.4230/LIPIcs.TIME.2024.18

Funding Indam GNCS 2024 project "Certificazione, Monitoraggio, ed Interpretabilità in Sistemi di Intelligenza Artificiale". M. Benerecetti, F. Mogavero, and A. Peron are members of the Gruppo Nazionale Calcolo Scientifico-Istituto Nazionale di Alta Matematica (GNCS-INdAM).

1 Introduction

In the domain of *formal verification* of complex systems, temporal logics [33] play the crucial role of *specification languages* [34] for the correct behaviour of system components over time. These languages are generally divided into two main categories: *linear-time logics* and *branching-time logics*. Logics in the first category focus on properties that span the entirety of each possible behaviour in isolation, while those in the second one are designed to address the interactions among those behaviours. Prominent examples of linear-time logics are *Linear-Time Temporal Logic* (LTL) [40, 41] and its full ω-regular extension ELTL [52]. On the other hand, typical representatives of branching-time logics fall within the families

31st International Symposium on Temporal Representation and Reasoning (TIME 2024).
Editors: Pietro Sala, Michael Sioutis, and Fusheng Wang; Article No. 18; pp. 18:1–18:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

of *Dynamic Logics* [17] and *Computation Tree Logics* [12, 11, 13, 14, 50]. Notable examples of such logics include CTL, CTL*, and ECTL*, together with the corresponding counting versions CCTL, CCTL*, and CECTL*.

The semantics of these temporal logics is often defined using suitable variants of *predicate logic*, usually *First-Order Logic* (FO) or *Second-Order Logic* (SO), interpreted either over *linearly-ordered structures*, such as infinite words, or over *partially-ordered structures*, such as infinite trees. At the same time, the extensive literature on automata-theoretic techniques [48] has been instrumental in providing effective technical tools for solving related decision problems. Predicate logics and automata theory also offer a comprehensive and coherent framework to evaluate and compare the expressive power and the computational properties of temporal languages, as evidenced by numerous *characterisation theorems*.

The foundational result in this context is Kamp's theorem [28], which establishes the equivalence between LTL and FO over infinite words. This result links FO-*definability* with *recognisability* by *counter-free finite-state automata* [30, 44, 45, 39], via the notions of *star-free language*, *aperiodic language*, and *aperiodic syntactic monoid*. Altogether these results fully characterise the expressive power of LTL in terms of predicate logics and automata. A similar correspondence also exists between ELTL, *Monadic Second-Order Logic* (MSO), and regular automata on infinite words [8, 9, 35, 10].

The landscape for branching-time temporal logics is much more complex due to the non-linear structure of the models and additional factors like *bisimulation invariance* [49] and *counting quantifiers* [16], and until recently it was far from being as complete as the linear-time counterpart. The more complete results were, indeed, the full correspondences among *(1)* the μ-Calculus [29], the *bisimulation-invariant* fragment of MSO interpreted over trees, and *Symmetric Alternating Parity Tree Automata* [27] and *(2)* the *alternation-free* fragment of μ-Calculus (AFμ-Calculus), the bisimulation-invariant fragments of WMSO over bounded-branching trees, and *Symmetric Alternating Weak Tree Automata* [1, 26]. These equivalences extend to the general case when counting quantifiers are incorporated into the modal logics [26, 25]. For four decades, the scenarios for CTL* and ECTL* remained significantly more fragmented. In the eighties, it was proved that, on binary trees, CTL* is equivalent to *Monadic Path Logic* (MPL) [24] and ECTL* to *Monadic Chain Logic* (MCL) [47]. The single result concerning CTL* was later extended to arbitrary-branching trees, at the turn of the century, addressing both bisimulation-invariance [37] and counting quantifiers [38]. Only very recently have corresponding classes of automata been proposed for these logics. Specifically, in [3], it was shown that, on arbitrary-branching trees, CTL* and ECTL* are equivalent to two versions of *Symmetric Hesitant Tree Automata*, namely HTA_{cf} and HTA, with and without a suitable *counter-freeness restriction* on their linear behaviours. Additionally, it was proved that HTA are equivalent to the bisimulation-invariant fragment of MCL. Thus, we finally have complete correspondences among *(a)* CTL*, the *bisimulation-invariant* fragment of MPL, and HTA_{cf}, and *(b)* ECTL*, the *bisimulation-invariant* fragment of MCL, and HTA. The first result was further extended to show, on arbitrary-branching trees, the equivalence of *(c)* CCTL*, MPL, and a graded version of HTA_{cf}, called $HGTA_{cf}$. However, the same result has not been obtained for CECTL*. This logic has been proven equivalent to a graded version of HTA, called HGTA, while MCL has only been shown equivalent to a potentially more-general first-order extension of HTA, called HFTA, inspired to a class of automata proposed by Walukiewicz [51].

The objective of this work is to complete the picture regarding the standard branching-time temporal logics by showing the equivalence of CECTL* with MCL. The key idea behind this result is to establish a composition theorem for MCL. *Composition Theorems* serve as model-theoretic tools that simplify reasoning about complex structures by breaking down a statement about the whole into several statements about its individual components. A first example of this approach is the renowned Feferman-Vaught Theorem [15], which reduces the first-order theory of any product of structures to the first-order theory of its factors. An initial application to linear orders was proposed by Läuchli [31], as an alternative to the automata-theoretic technique on words by Büchi [7, 8, 9], and subsequently advanced in a series of works by Shelah and Gurevich [43, 19, 21, 22, 20]. Thomas then applied the approach to binary-branching tree structures [46, 47], culminating in the composition theorem for MPL in collaboration with Hafer [24]. This result was later extended to arbitrary-branching trees by Moller and Rabinovich [37, 38]. In the present work, we merge and generalise the techniques considered in [47, 38] to obtain the corresponding result for MCL, by relying on Ehrenfeucht-Fraïssé games tailored to this logic. Specifically, we show that verifying an MCL formula with quantifier rank m and a unique free chain variable over a tree boils down to verifying an MSO sentence over a word that is the encoding of a suitable vector of m chains induced by the interpretation of that variable. This allows us to translate, via structural induction, every MCL formula with a single first-order variable into an equivalent CECTL* state formula. Given that the translation from CECTL* to MCL is relatively straightforward, we obtain the stated result, settling one of the problems left open in [3]. It is important to note that the automata-theoretic technique developed in [3] cannot be directly applied here. In principle, given the equivalences of MCL with HFTA and CECTL* with HGTA, one might be tempted to show the equivalence of the two logics by proving the equivalence of the two automaton classes. However, the natural compositional transformation of the first-order formulae encoded in the transition function of an HFTA to the corresponding graded modal formulae does not satisfy the hesitant constraint required by a HGTA. The approach proposed in this work circumvents that difficulty and allows us to prove, though indirectly, that HFTA and HGTA are two equivalent types of automata.

2 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers. For $i, j \in \mathbb{N}$ with $i \leq j$, $[i, j]$ denotes the set of natural numbers k such that $i \leq k \leq j$. For a finite or infinite word ρ over some alphabet, $|\rho|$ is the length of ρ ($|\rho| = \omega$ if ρ is infinite) and for all $0 \leq i < |\rho|$, $\rho(i)$ is the $(i+1)$-th letter of ρ.

Kripke Trees. A tree T is a non-empty subset of $\mathbb{N}^*$ which is prefix closed (i.e., for each $w \cdot n \in \mathrm{T}$ with $n \in \mathbb{N}$, $w \in \mathrm{T}$). Elements of T are called nodes and the empty word ε is the root of T. For $w \in \mathrm{T}$, a *child* of w in T is a node in T of the form $w \cdot n$ for some $n \in \mathbb{N}$, and a *descendant* of w in T is a node of T of the form $w \cdot w'$ for some $w' \in \mathbb{N}^*$. For $w \in \mathrm{T}$, the *subtree of* T *rooted at node* w is the tree consisting of the nodes of the form w' such that $w \cdot w' \in \mathrm{T}$. A *subtree of* T is a tree T' such that for some $w \in \mathrm{T}$, T' is a subset of the subtree of T rooted at w. A *path* of T is a subtree π of T which is totally ordered by the child-relation (i.e., each node of π has at most one child in π). In the following, a path π of T is also seen as a word over T in accordance to the total ordering in π induced by the child relation. A *chain* of T is a subset of a path of T, while a *branch* of T is a path of T starting at the root. A tree is *non-blocking* if each node has some child. A non-blocking tree T is infinite, and maximal paths in T are infinite as well.

For an alphabet Σ, a Σ-labelled tree is a pair $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$ consisting of a tree and a labelling $\mathit{Lab} : \mathrm{T} \mapsto \Sigma$ assigning to each node in T a symbol in Σ. For a subtree T' of T, we denote by $\mathcal{S}_{\mathrm{T}'}$ the Σ-labelled subtree $(\mathrm{T}', \mathit{Lab}\lceil_{\mathrm{T}'})$ of $\mathcal{S}$. In this paper, we consider formalisms whose specifications are interpreted over labeled trees. For the easy of presentation, we focus on labeled trees which are non-blocking. All the results of this paper can be easily adapted to the general case, where the non-blocking assumption is relaxed. For a finite set AP of atomic propositions, a *Kripke tree* over AP is a non-blocking 2^{AP}-labelled tree.

Automata over Infinite and Finite Words. We first recall the class of parity nondeterministic automata on infinite words (parity NWA for short) which are tuples $\mathcal{A} = \langle \Sigma, \mathrm{Q}, \delta, q_I, \Omega \rangle$, where Σ is a finite input alphabet, Q is a finite set of states, $\delta : \mathrm{Q} \times \Sigma \mapsto 2^{\mathrm{Q}}$ is the transition function, $q_I \in \mathrm{Q}$ is an initial state, and $\Omega : \mathrm{Q} \mapsto \mathbb{N}$ is a parity acceptance condition over Q assigning to each state a natural number (color). The NWA $\mathcal{A}$ is *deterministic* if for all states q and input symbols a, $\delta(q, a)$ is a singleton $\{q'\}$ (in this case, we write $\delta(q, a) = q'$). We use the acronym DWA for the subclass of deterministic NWA.

Given a word ρ over Σ, a *path* of $\mathcal{A}$ over ρ is a word π over Q of length $|\rho| + 1$ ($|\rho| + 1$ is ω if ρ is infinite) such that $\pi(i + 1) \in \delta(\pi(i), \rho(i))$ for all $0 \leq i < |\rho|$. A *run* over ρ is a path over ρ starting at the initial state. The NWA $\mathcal{A}$ is *counter-free* if for all $n > 0$, states $q \in \mathrm{Q}$ and finite words ρ over Σ, the following holds: if there is a path from q to q over ρ^n, then there is also a path from q to q over ρ.

A run π of $\mathcal{A}$ over an infinite word ρ is *accepting* if the highest color of the states appearing infinitely often along π is even. The ω-language $\mathsf{L}(\mathcal{A})$ accepted by $\mathcal{A}$ is the set of infinite words ρ over Σ such that there is an accepting run π of $\mathcal{A}$ over ρ.

A parity acceptance condition $\Omega : \mathrm{Q} \mapsto \mathbb{N}$ is a *Büchi* condition if $\Omega(\mathrm{Q}) \subseteq \{1, 2\}$. A *Büchi* NWA is a parity NWA whose acceptance condition is Büchi.

We also consider NWA over finite words ($\mathrm{NWA_f}$ for short) which are defined as parity NWA but the parity condition Ω is replaced with a set $\mathrm{F} \subseteq \mathrm{Q}$ of accepting states. A run π over a finite word is *accepting* if its last state is accepting.

Monadic Chain Logic. We recall now Monadic Chain Logic (MCL for short) [47] interpreted over arbitrary Kripke trees. MCL is the well-known fragment of MSO where second-order quantification is restricted to chains of the given Kripke tree. For technical convenience, we consider a one-sorted variant of MCL where first-order variables are encoded as second-order variables which are singletons. It is straightforward to show that this variant is equivalent to standard MCL.

Formally, given a finite set AP of atomic propositions and a finite set Vr_2 of second-order variables (or *chain* variables), the syntax of the considered variant of MCL is the set of formulae built according to the following grammar:

$$\varphi \coloneqq \mathtt{sing}(X) \mid X \subseteq p \mid X \subseteq Y \mid X \leq Y \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists^{\mathsf{C}} X.\, \varphi$$

where $p \in \mathrm{AP}$ and $X, Y \in \mathrm{Vr}_2$. Intuitively, $\mathtt{sing}(X)$ asserts that X is a singleton, $X \subseteq p$ means that p holds at each node of X, and $X \leq Y$ means that each node of Y is a descendant of each node of X. As usual, a *free variable* of a formula φ is a variable occurring in φ that is not bound by a quantifier. A *sentence* is a formula with no free variables. The language of MCL consists of its sentences.

Semantics of MCL. Formulae of MCL are interpreted over Kripke trees on AP. Given a Kripke tree $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$ over AP , a *second-order valuation for* $\mathcal{S}$ is a mapping $\mathsf{V}_2 : \mathrm{Vr}_2 \mapsto 2^{\mathrm{T}}$ assigning to each second-order variable a chain of T. For an MCL formula φ, the satisfaction relation $(\mathcal{S}, \mathsf{V}_2) \models \varphi$, meaning that $\mathcal{S}$ satisfies the formula φ under the valuation V_2, is defined as follows (the treatment of Boolean connectives is standard):

$(\mathcal{S}, \mathsf{V}_2) \models \mathtt{sing}(X) \Leftrightarrow \mathsf{V}_2(X)$ is a singleton;
$(\mathcal{S}, \mathsf{V}_2) \models X \subseteq p \quad \Leftrightarrow p \in \mathit{Lab}(w)$ for each $w \in \mathsf{V}_2(X)$;
$(\mathcal{S}, \mathsf{V}_2) \models X \subseteq Y \quad \Leftrightarrow \mathsf{V}_2(X) \subseteq \mathsf{V}_2(Y)$;
$(\mathcal{S}, \mathsf{V}_2) \models X \leq Y \quad \Leftrightarrow$ for all $w \in \mathsf{V}_2(X)$ and $w' \in \mathsf{V}_2(Y)$, w' is a descendant of w in T;
$(\mathcal{S}, \mathsf{V}_2) \models \exists^{\mathsf{C}} X. \varphi \quad \Leftrightarrow (\mathcal{S}, \mathsf{V}_2[X \mapsto \mathrm{C}]) \models \varphi$ for some chain C of T.

where $\mathsf{V}_2[X \mapsto \mathrm{C}]$ denotes the second-order valuation for $\mathcal{S}$ defined as: $\mathsf{V}_2[X \mapsto \mathrm{C}](X) = \mathrm{C}$ and $\mathsf{V}_2[X \mapsto \mathrm{C}](Y) = \mathsf{V}_2(Y)$ if $Y \neq X$. Note that the satisfaction relation $(\mathcal{S}, \mathsf{V}_2) \models \varphi$, for fixed $\mathcal{S}$ and φ, depends only on the values assigned by V_2 to the variables occurring free in φ. In particular, if φ is a sentence, we say that $\mathcal{S}$ *satisfies* φ, written $\mathcal{S} \models \varphi$, if $(\mathcal{S}, \mathsf{V}_2) \models \varphi$ for some valuation V_2. In this case, we also say that $\mathcal{S}$ is a model of φ.

3 Branching-Time Temporal Logics

In this section, we recall syntax and semantics of *Counting*-CTL* (CCTL* for short) [38], which extends standard CTL* [14] with counting operators, as well as the counting extension CECTL* [3] of ECTL* [50], a branching-time temporal logic more expressive than CCTL*. For technical convenience, we shall consider an equivalent syntactic variant of ECTL*, which employs $\mathrm{NWA_f}$ over finite words, instead of right-linear grammars, as the building blocks of formulae.[1] We also consider a fragment of CECTL*, that we call *counter-free* CECTL*, where all the $\mathrm{NWA_f}$ over finite words are required to be counter-free. We prove that counter-free CECTL* and CCTL* have the same expressive power.

3.1 The Logic CCTL*

The syntax of CCTL* is given by specifying inductively the set of *state formulae* φ and the set of *path formulae* ψ over a given finite set AP of atomic propositions:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathtt{E}\psi \mid \mathtt{D}^n\varphi$$
$$\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \mathtt{X}\,\psi \mid \psi\,\mathtt{U}\,\psi$$

where $p \in \mathrm{AP}$, $\mathtt{X}$ and $\mathtt{U}$ are the standard "next" and "until" temporal modalities, $\mathtt{E}$ is the existential path quantifier, and $\mathtt{D}^n$, with $n \in \mathbb{N} \setminus \{0\}$, is the counting operator. The language of CCTL* consists of the state formulae of CCTL*. Standard CTL* is the fragment of CCTL* where counting operators $\mathtt{D}^n$ with $n > 1$ are not allowed.

Given a Kripke tree $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$ (over AP), a node w of T, an infinite path π of T, and $0 \leq i < |\pi|$, the satisfaction relations $(\mathcal{S}, w) \models \varphi$ for a state formula φ (meaning that φ holds at node w of $\mathcal{S}$), and $(\mathcal{S}, \pi, i) \models \psi$ for a path formula ψ (meaning that ψ holds at position i of the path π in $\mathcal{S}$) are defined as follows (Boolean connectives are treated as usual):

$(\mathcal{S}, w) \models p \quad \Leftrightarrow p \in \mathit{Lab}(w)$;
$(\mathcal{S}, w) \models \mathtt{E}\psi \quad \Leftrightarrow (\mathcal{S}, \pi, 0) \models \psi$ for some infinite path π of T starting at node w;
$(\mathcal{S}, w) \models \mathtt{D}^n\varphi \quad \Leftrightarrow$ there are at least n distinct children w' of w in T s.t. $(\mathcal{S}, w') \models \varphi$;
$(\mathcal{S}, \pi, i) \models \varphi \quad \Leftrightarrow (\mathcal{S}, \pi(i)) \models \varphi$;
$(\mathcal{S}, \pi, i) \models \mathtt{X}\,\psi \quad \Leftrightarrow (\mathcal{S}, \pi, i+1) \models \psi$;
$(\mathcal{S}, \pi, i) \models \psi_1\,\mathtt{U}\,\psi_2 \quad \Leftrightarrow$ for some $j \geq i$: $(\mathcal{S}, \pi, j) \models \psi_2$ and $(\mathcal{S}, \pi, k) \models \psi_1$ for all $i \leq k < j$.

Note that $\mathtt{D}^1\varphi$ corresponds to $\mathtt{EX}\,\varphi$. A Kripke tree $\mathcal{S}$ satisfies (or is a model of) a state formula φ, written $\mathcal{S} \models \varphi$, if $\mathcal{S}, \varepsilon \models \varphi$.

[1] In [3], the considered syntactic variant of CECTL* is called Counting Computation Dynamic logic (CCDL) since it essentially corresponds to a branching-time extension of *Linear Dynamic Logic* [18].

3.2 The Logic CECTL*

Like CCTL*, the syntax of CECTL* is composed of *state formulae* φ and *path formulae* ψ over a given finite set AP of atomic propositions, defined as follows:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathtt{E}\psi \mid \mathtt{D}^n\varphi$$
$$\psi ::= \varphi \mid \neg\psi \mid \psi \wedge \psi \mid \langle\mathcal{A}\rangle\psi$$

where $p \in \mathrm{AP}$ and $\langle\mathcal{A}\rangle$ is the *existential sequencing* modality applied to a *testing* $\mathrm{NWA_f}$ $\mathcal{A}$. We define a *testing* $\mathrm{NWA_f}$ $\mathcal{A} = \left\langle 2^{\mathrm{AP}}, \mathrm{Q}, \delta, q_I, \mathrm{F}, \tau \right\rangle$ as consisting of an $\mathrm{NWA_f}$ $\left\langle 2^{\mathrm{AP}}, \mathrm{Q}, \delta, q_I, \mathrm{F} \right\rangle$ over finite words over 2^{AP} and a test function τ mapping states in Q to CECTL* path formulae. Intuitively, along an infinite path π of a Kripke tree, the testing automaton accepts the labeling of a (possibly empty) infix $\pi(i) \ldots \pi(j-1)$ of π if the embedded $\mathrm{NWA_f}$ has an accepting run $q_i \ldots q_j$ over the labeling of such an infix so that, for each position $k \in [i, j]$, the formula holds at position k along π. A test function τ is *trivial* if it maps each state to $\top$. We also use the shorthand $[\mathcal{A}]\psi \triangleq \neg\langle\mathcal{A}\rangle\neg\psi$ (*universal sequencing* modality). The language of CECTL* consists of the state formulae of CECTL*. A CECTL* formula φ is *counter-free* if (i) the testing automata $\mathcal{A}$ occurring in φ are counter-free and (ii) *either* $\mathcal{A}$ is deterministic *or* the test function of $\mathcal{A}$ is trivial.

Given a Kripke tree $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$, an infinite path π of T, and $0 \leq i < |\pi|$, the semantics of modality $\langle\mathcal{A}\rangle$ is defined as follows, where $\mathcal{A} = \left\langle 2^{\mathrm{AP}}, \mathrm{Q}, \delta, q_I, \mathrm{F}, \tau \right\rangle$:

$$(\mathcal{S}, \pi, i) \models \langle\mathcal{A}\rangle\psi \quad \Leftrightarrow \text{for some } j \geq i,\ (i, j) \in \mathrm{R}_{\mathcal{A}}(\mathcal{S}, \pi) \text{ and } (\mathcal{S}, \pi, j) \models \psi$$

where $\mathrm{R}_{\mathcal{A}}(\mathcal{S}, \pi)$ is the set of pairs (i, j) with $j \geq i$ s.t. there is an accepting run $q_i \ldots q_j$ of the $\mathrm{NWA_f}$ embedded in $\mathcal{A}$ over $\mathit{Lab}(\pi(i)) \ldots \mathit{Lab}(\pi(j-1))$ and, for all $k \in [i, j]$, it holds that $(\mathcal{S}, \pi, k) \models \tau(q_k)$. The notion of a model of a CECTL* formula is defined as for CCTL*.

3.3 Expressiveness equivalence of CCTL* and counter-free CECTL*

We first show that CCTL* can be embedded into counter-free CECTL*. Let $\mathcal{A}$ be the testing counter-free $\mathrm{NWA_f}$ having trivial tests and accepting all and only the words of length 1. Moreover, for a counter-free CECTL* path formula ψ_1, let $\mathcal{A}_{\psi_1}$ be the testing counter-free $\mathrm{DWA_f}$ $\mathcal{A} = \left\langle 2^{\mathrm{AP}}, \{q_1\}, \delta, q_1, \{q_1\}, \tau \right\rangle$ defined as follows: $\delta(q_1, a) = q_1$ for each input symbol a, and $\tau(q_1) = \psi_1$. Then, the next and until formulae $\mathtt{X}\,\psi_1$ and $\psi_1\,\mathtt{U}\,\psi_2$ can be expressed as: $\mathtt{X}\,\psi_1 \equiv \langle\mathcal{A}\rangle\psi_1$ and $\psi_1\,\mathtt{U}\,\psi_2 \equiv \psi_2 \vee \langle\mathcal{A}_{\psi_1}\rangle\langle\mathcal{A}\rangle\psi_2$. Hence, we obtain the following result.

▶ **Proposition 3.1.** *Given a* CCTL* *formula, one can build an equivalent counter-free* CECTL* *formula.*

For the converse translation from counter-free CECTL* to CCTL*, by the known equivalence between *Monadic Path Logic* (MPL) and CCTL* [38], it suffices to show that each counter-free CECTL* formula can be translated into an equivalent MPL sentence. We first recall the logic MPL [23], the well-known fragment of MSO where second-order quantification is restricted to paths of the given Kripke tree.

Monadic Path Logic (MPL) [23]. Given a finite set AP of atomic propositions, a finite set Vr_1 of first-order variables, and a finite set Vr_2 of second-order variables, the syntax of standard MPL is the set of formulae built according to the following grammar:

$$\varphi \coloneqq p(x) \mid x \leq y \mid x \in X \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x.\, \varphi \mid \exists^{\mathrm{P}} X.\, \varphi$$

where $p \in \text{AP}$, $x, y \in \text{Vr}_1$, $X \in \text{Vr}_2$, and $\exists^{\mathsf{P}} X$ is the path quantifier which ranges over paths of the given Kripke tree. We also exploit the standard logical connectives $\vee$ and $\rightarrow$ as abbreviations, the universal first-order quantifier $\forall x$, defined as $\forall x.\varphi \triangleq \neg\exists x.\neg\varphi$, and the universal path quantifier $\forall^{\mathsf{P}} X$, defined as $\forall^{\mathsf{P}} X.\,\varphi \triangleq \neg\exists^{\mathsf{P}} X.\,\neg\varphi$. We also make use of the shorthands (i) $x = y$ for $x \leq y \wedge y \leq x$, (ii) $x < y$ for $x \leq y \wedge \neg(y \leq x)$; (iii) $\exists x \in X.\,\varphi$ for $\exists x.\,(x \in X \wedge \varphi)$, and (iv) $\forall x \in X.\,\varphi$ for $\forall x.\,(x \in X \rightarrow \varphi)$. Moreover, the child relation is definable in MPL by the binary predicate $\texttt{child}(x, y) \triangleq x < y \wedge \neg\exists z.\,(x < z \wedge z < y)$ which exploits only first-order quantification.

Given a Kripke tree $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$, a *first-order valuation for* $\mathcal{S}$ is a mapping $\mathsf{V}_1 : \text{Vr}_1 \mapsto \mathrm{T}$ assigning to each first-order variable a node of T. A *path valuation for* $\mathcal{S}$ is a second-order valuation $\mathsf{V}_2 : \text{Vr}_2 \mapsto 2^{\mathrm{T}}$ assigning to each second-order variable a path of T. Given an MPL formula φ, a first-order valuation V_1 for $\mathcal{S}$, and a path valuation V_2 for $\mathcal{S}$, the satisfaction relation $(\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2) \models \varphi$ is defined as follows (the treatment of Boolean connectives is standard):

$(\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2) \models p(x) \quad \Leftrightarrow p \in \mathit{Lab}(\mathsf{V}_1(x))$;
$(\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2) \models x \leq y \quad \Leftrightarrow \mathsf{V}_1(y)$ is a descendant of $\mathsf{V}_1(x)$ in T;
$(\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2) \models x \in X \quad \Leftrightarrow \mathsf{V}_1(x) \in \mathsf{V}_2(X)$;
$(\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2) \models \exists x.\,\varphi \quad \Leftrightarrow (\mathcal{S}, \mathsf{V}_1[x \mapsto w], \mathsf{V}_2) \models \varphi$ for some $w \in \mathrm{T}$;
$(\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2) \models \exists^{\mathsf{P}} X.\,\varphi \Leftrightarrow (\mathcal{S}, \mathsf{V}_1, \mathsf{V}_2[X \mapsto \pi]) \models \varphi$ for some path π of T.

where $\mathsf{V}_1[x \mapsto w]$ denotes the first-order valuation for $\mathcal{T}$ defined as: $\mathsf{V}_1[x \mapsto w](x) = w$ and $\mathsf{V}_1[x \mapsto w](y) = \mathsf{V}_1(y)$ if $y \neq x$.

From counter-free CECTL* to MPL. The translation presented in the following is based on known results about counter-free DWA [32, 6].

▶ **Proposition 3.2.** *Given a counter-free* CECTL* *formula, one can build an equivalent* MPL *sentence.*

Proof. Let ψ be a counter-free CECTL* path formula. We construct an MPL formula $\widehat{\psi}(x, X)$ with exactly one *free* first-order variable x and at most one *free* second-order variable X such that for each Kripke tree $\mathcal{S}$, infinite path π of $\mathcal{S}$, and position $i \geq 0$, it holds that

$(\mathcal{S}, \pi, i) \models \psi$ if and only if $(\mathcal{S}, x \rightarrow \pi(i), X \rightarrow \pi) \models \widehat{\psi}(x, X)$

Moreover, $\widehat{\psi}(x, X)$ does not depend on X if ψ is a state formula. Thus, given a state CECTL* formula φ, the MPL sentence equivalent to φ is given by $\exists x.\,(\texttt{root}(x) \wedge \widehat{\varphi}(x))$ with $\texttt{root}(x) \triangleq \neg\exists y.\,y < x$.

The MPL formula $\widehat{\psi}(x, X)$ is defined by structural induction on ψ as follows, where we exploit the predicate $\texttt{Inf}(Y)$ expressing that the path Y is infinite ($\texttt{Inf}(Y)$ can be easily specified in MPL by using only first-order quantification).

- $\psi = p$ with $p \in \text{AP}$: $\widehat{\psi}(x, X) \triangleq p(x)$.
- $\psi = \neg\psi_1$: $\widehat{\psi}(x, X) \triangleq \neg\widehat{\psi_1}(x, X)$.
- $\psi = \psi_1 \wedge \psi_2$: $\widehat{\psi}(x, X) \triangleq \widehat{\psi_1}(x, X) \wedge \widehat{\psi_2}(x, X)$.
- $\psi = \texttt{E}\psi_1$: $\widehat{\psi}(x, X) \triangleq \exists^{\mathsf{P}} Y.\,(\texttt{Inf}(Y) \wedge x \in Y \wedge \widehat{\psi_1}(x, Y) \wedge \forall y \in Y.\,x \leq y)$.
- $\psi = \texttt{D}^n\psi_1$: $\widehat{\psi}(x, X) \triangleq \exists x_1 \ldots \exists x_n.\,(\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i=1}^{n} (\texttt{child}(x, x_i) \wedge \widehat{\psi_1}(x_i, X)))$.
- $\psi = \langle\mathcal{A}\rangle\psi_1$, where $\mathcal{A}$ is a counter-free testing NWA_{f} such that either $\mathcal{A}$ is deterministic or the test function of $\mathcal{A}$ is trivial. Since a counter-free NWA_{f} on finite words can be converted into an equivalent counter-free DWA_{f} [36], we can assume that the NWA_{f} with

tests $\mathcal{A}$ is deterministic. Let $\mathcal{A} = \langle 2^{\mathrm{AP}}, \mathrm{Q}, \delta, q_I, \mathrm{F}, \tau \rangle$. By [32, 6], for each state $q \in \mathrm{Q}$, we can construct an FO formula $\xi_q(x, y)$ with two free variables x and y such that for each infinite word ρ over 2^{AP} and positions $i, j \geq 0$, it holds that $(\rho, x \to i, y \to j) \models \xi_q(x, y)$ iff $i \leq j$ and the unique run of $\mathcal{A}$ over $\rho[i, j-1]$ leads to state q. Let $\widehat{\xi_q}(x, y, X)$ be the MPL formula obtained by "relativizing" the FO formula $\xi_q(x, y)$ with respect to path X, i.e., by replacing each subformula $\exists x.\, \theta$ of $\xi_q(x, y)$ with $\exists x.\, (x \in X \wedge \theta)$. Then, $\widehat{\psi}(x, X)$ is defined as follows:

$$\bigvee_{q \in \mathrm{F}} \exists y \in X. \Big(x \leq y \wedge \widehat{\xi_q}(x, y, X) \wedge \widehat{\tau(q)}(y, X) \wedge$$
$$\forall z \in X. \big[x \leq z < y \to \bigvee_{q \in \mathrm{Q}} (\widehat{\xi_q}(x, z, X) \wedge \widehat{\tau(q)}(z, X)) \big] \wedge \widehat{\psi_1}(y, X) \Big) \quad \blacktriangleleft$$

It remains an open question whether counter-free NWA$_{\mathrm{f}}$ with non-trivial tests can be captured in MPL. Thus, by Propositions 3.1–3.2 and the known equivalence of MPL and CCTL* [38], we obtain the following result.

▶ **Theorem 3.3.** CCTL* *and counter-free* CECTL* *are equivalent formalisms, i.e., they specify the same class of tree languages.*

4 Expressiveness equivalence of MCL and CECTL*

It is known [3] that each CECTL* state formula has an equivalent MCL sentence. In this section, we show that the two logics CCDL and MCL are in fact expressively equivalent. We provide a proof of this result which relies on an adaptation of the compositional argument given in [38] for showing that each MPL sentence has an equivalent CCTL* state formula.

4.1 Model-theoretic fundamentals

We first introduce some notations and definitions. The *quantifier rank* $\mathsf{qr}(\varphi)$ of an MCL formula φ is the maximum number of nested quantifiers occurring in it. In the following, a Kripke tree (over AP) is called *structure* (over AP).

Fix a finite set AP of atomic propositions. Given $h \in \mathbb{N}$, an *h-structure* $\mathcal{S}_h$ is a tuple of the form $\mathcal{S}_h = (\mathcal{S}, \mathrm{C}_1, \ldots, \mathrm{C}_h)$ such that $\mathcal{S}$ is a structure and $\mathrm{C}_1, \ldots, \mathrm{C}_h$ are chains of $\mathcal{S}$. An *h-word structure* is defined similarly but we require that the structure $\mathcal{S}$ is an infinite word over 2^{AP} (recall that an infinite word over 2^{AP} corresponds to a structure where each node has exactly one child). Note that a structure can be seen as a 0-structure.

An h-MCL formula is an MCL formula having at most h free variables (recall that in the one-sorted formalization of MCL, all the variables range over chains). Note that a 0-MCL formula is a sentence. An h-structure $\mathcal{S}_h = (\mathcal{S}, \mathrm{C}_1, \ldots, \mathrm{C}_h)$ satisfies an h-MCL formula $\varphi(X_1, \ldots, X_h)$ if $\mathcal{S} \models \varphi(\mathrm{C}_1, \ldots, \mathrm{C}_h)$ (which means that $(\mathcal{S}, \mathsf{V}_2) \models \varphi(X_1, \ldots, X_h)$ for any valuation V_2 such that $\mathsf{V}_2(X_i) = \mathrm{C}_i$ for each $i \in [1, h]$). Two h-MCL formulas are *equivalent* if they are satisfied by the same h-structures. Two h-MCL formulas are *word-equivalent* if they are satisfied by the same h-word structures.

Equivalence relations between h-structures. Let $m \in \mathbb{N}$. Given two h-structures, $\mathcal{S}_h = (\mathcal{S}, \mathrm{C}_1, \ldots, \mathrm{C}_h)$ and $\mathcal{S}'_h = (\mathcal{S}', \mathrm{C}'_1, \ldots, \mathrm{C}'_h)$, we say that $\mathcal{S}_h$ and $\mathcal{S}'_h$ are *m-rank equivalent*, written $\mathcal{S}_h \equiv_m \mathcal{S}'_h$, if no h-MCL formula $\varphi(X_1, \ldots, X_k)$ of quantifier rank at most m can distinguish them, i.e., $\mathcal{S} \models \varphi(\mathrm{C}_1, \ldots, \mathrm{C}_k)$ iff $\mathcal{S}' \models \varphi(\mathrm{C}'_1, \ldots, \mathrm{C}'_k)$. If $\mathcal{S}_h \equiv_m \mathcal{S}'_h$ and $\mathcal{S}_h$ and $\mathcal{S}'_h$ are h-word structures, we write $\mathcal{S}_h \equiv^{\omega}_m \mathcal{S}'_h$. The equivalence relation $\equiv_m$ over the class of

h-structures has finite index and each equivalence class can be characterized by an h-MCL formula of quantifier rank at most m, called *m-type for h-*MCL *formulas*. In particular, the following result follows from standard arguments.

▶ **Proposition 4.1.** *Let $h \in \mathbb{N}$. Then, the following properties hold for each $m \geq 0$:*

1. *the equivalence $\equiv_m$ over the set of h-structures defines finitely-many equivalence classes;*
2. *for each equivalence class Λ_m of $\equiv_m$ over h-structures, there is an h-*MCL *formula β (called m-*type for *h-*MCL formulas*) with $\mathsf{qr}(\beta) \leq m$ which characterizes it: that is, $\mathcal{S}_h \models \beta$ iff $\mathcal{S}_h \in \Lambda_m$, for all h-structures $\mathcal{S}_h$;*
3. *each h-*MCL *formula φ with $\mathsf{qr}(\varphi) \leq m$ is equivalent to a disjunction of m-types;*
4. *the variants of Properties 1–3 for the class of h-word structures.*

Proof. We focus on Properties 1–3. We observe that by variable renaming, we can assume that h-MCL formulas φ with $\mathsf{qr}(\varphi) \leq m$ only use variables from a *finite* set. Hence, by a straightforward induction on $\mathsf{qr}(\varphi)$, the following holds.

▷ Claim. There is a *finite* set Υ of h-MCL formulas with quantifier rank at most m such that each h-MCL formula φ with $\mathsf{qr}(\varphi) \leq m$ is equivalent to some formula in Υ.

Let $\Upsilon = \{\psi_1, \ldots, \psi_N\}$ be the finite set of h-MCL formulas with quantifier rank at most m satisfying the previous claim. We consider the h-MCL formulas of the form

$$\overline{\psi}_1 \wedge \ldots \wedge \overline{\psi}_N$$

where $\overline{\psi}_i$ is either ψ_i or $\neg\psi_i$ for all $i \in [1, k]$. Let us denote by $\beta_1, \ldots, \beta_\ell$ these formulas (note that $\ell = 2^N$). By construction, for each h-structure $\mathcal{S}_h$, there is exaclty one $i \in [1, \ell]$ such that $\mathcal{S}_h \models \beta_i$. Moreover, each formula $\psi_i \in \Upsilon$ can be expressed as the disjunction of all and only the formulas in $\{\beta_1, \ldots, \beta_\ell\}$ whose associated conjunct $\overline{\psi}_i$ is ψ_i. Thus, by the previous claim, Properties 1–3 easily follow. ◀

Local isomorphism on h-structures. The equivalence relation $\equiv_0$ over h-structures can be characterized as follows. Given two h-structures, $\mathcal{S}_h = (\mathcal{S}, \mathrm{C}_1, \ldots, \mathrm{C}_k)$ and $\mathcal{S}'_h = (\mathcal{S}', \mathrm{C}'_1, \ldots, \mathrm{C}'_h)$, we say that $\mathcal{S}_h$ and $\mathcal{S}_h$ are *locally-isomorphic* (for MCL) if the following conditions hold, where $\mathcal{S} = \langle \mathrm{T}, \mathit{Lab} \rangle$ and $\mathcal{S}' = \langle \mathrm{T}', \mathit{Lab}' \rangle$:

- for all $i \in [1, h]$, C_i is a singleton iff C'_i is a singleton;
- for all $i \in [1, h]$ and $p \in \mathrm{AP}$, $\mathrm{C}_i \subseteq \mathrm{T}_p$ iff $\mathrm{C}'_i \subseteq \mathrm{T}'_p$, where $\mathrm{T}_p = \{w \in \mathrm{T} \mid p \in \mathit{Lab}(w)\}$ and $\mathrm{T}'_p = \{w \in \mathrm{T}' \mid p \in \mathit{Lab}'(w)\}$.
- for all $i, j \in [1, h]$, $\mathrm{C}_i \subseteq \mathrm{C}_j$ iff $\mathrm{C}'_i \subseteq \mathrm{C}'_j$;
- for all $i, j \in [1, h]$, $\mathrm{C}_i \leq \mathrm{C}_j$ iff $\mathrm{C}'_i \leq \mathrm{C}'_j$.

Note that two structures are always locally-isomorphic and two h-structures are 0-rank equivalent iff they are locally-isomorphic.

Ehrenfeucht-Fraissé Games for MCL. The rank-equivalence relation $\equiv_m$ over the class of h-structures has an elegant characterization in terms of *Ehrenfeucht-Fraissé games* (EF-games) over h-structures. The EF-game $\mathcal{G}_m(\mathcal{S}_h, \mathcal{S}'_h)$ over two h-structures $\mathcal{S}_h = (\mathcal{S}, \mathrm{C}_1, \ldots, \mathrm{C}_h)$ and $\mathcal{S}'_h = (\mathcal{S}', \mathrm{C}'_1, \ldots, \mathrm{C}'_h)$ is played by two players called the *spoiler* and the *duplicator*. Each play consists of m-rounds. At i-th round, with $i \in [1, m]$, the spoiler chooses a chain in one of the two structures $\mathcal{S}$ and $\mathcal{S}'$, after which the duplicator responds by choosing a chain in the other structure which she believes *matches* the chain chosen by the spoiler. After m-rounds, there will be m chains $\overline{\mathrm{C}}_1, \ldots, \overline{\mathrm{C}}_m$ selected in the structure $\mathcal{S}$, and corresponding m chains

$\overline{\mathrm{C}}'_1, \ldots, \overline{\mathrm{C}}'_i$ selected in the structure $\mathcal{S}'$. The duplicator wins if the two $(h+m)$-structures $(\mathcal{S}, \mathrm{C}_1, \ldots, \mathrm{C}_h, \overline{\mathrm{C}}_1, \ldots, \overline{\mathrm{C}}_m)$ and $(\mathcal{S}', \mathrm{C}'_1, \ldots, \mathrm{C}'_h, \overline{\mathrm{C}}'_1, \ldots, \overline{\mathrm{C}}'_m)$ are locally-isomorphic (note that this entails that the original h-structures $\mathcal{S}_h$ and $\mathcal{S}'_h$ need to be locally-isomorphic). Otherwise, the spoiler wins. We say that the duplicator has a *winning strategy* in the game $\mathcal{G}_m(\mathcal{S}_h, \mathcal{S}'_h)$ if it is possible for him to win each play whatever choices are made by the opponent. The h-structures $\mathcal{S}_h$ and $\mathcal{S}'_h$ are *m-game equivalent*, written $\mathcal{S}_h \sim_m \mathcal{S}'_h$ if the duplicator has a winning strategy in the game $\mathcal{G}_m(\mathcal{S}_h, \mathcal{S}'_h)$. If $\mathcal{S}_h \sim_m \mathcal{S}'_h$ and $\mathcal{S}_h$ and $\mathcal{S}'_h$ are h-word structures, we write $\mathcal{S}_h \sim^{\omega}_m \mathcal{S}'_h$. By classical arguments, one can show that the m-game equivalence relation corresponds to the rank equivalence $\equiv_m$.

▶ **Proposition 4.2.** *For all $h, m \in \mathbb{N}$ and h-structures $\mathcal{S}_h$ and $\mathcal{S}'_h$, $\mathcal{S}_h \equiv_m \mathcal{S}'_h$ iff $\mathcal{S}_h \sim_m \mathcal{S}'_h$.*

Proof. First, we observe that for each $m \geq 0$, $\sim_m$ is the unique *equivalence* relation satisfying the following properties for all h-structures $\mathcal{S}_h = (\mathcal{S}, \ldots)$ and $\mathcal{S}'_h = (\mathcal{S}', \ldots)$:

1. if $m = 0$, then $\mathcal{S}_h \sim_0 \mathcal{S}'_h$ iff $\mathcal{S}_h$ and $\mathcal{S}_h$ are locally isomorphic;
2. if $m > 0$, then:
   (forth) for each chain C of $\mathcal{S}$, there is a chain C′ of $\mathcal{S}'$ such that $(\mathcal{S}_h, \mathrm{C}) \sim_{m-1} (\mathcal{S}'_h, \mathrm{C}')$, where $(\mathcal{S}_h, \mathrm{C})$ and $(\mathcal{S}'_h, \mathrm{C}')$ denote the $(h+1)$-structures defined in the obvious way;
   (back) for each chain C′ of $\mathcal{S}'$, there is a chain C of $\mathcal{S}$ such that $(\mathcal{S}_h, \mathrm{C}) \sim_{m-1} (\mathcal{S}'_h, \mathrm{C}')$.

Thus, it suffices to show that the equivalence relation $\equiv_m$ satisfies the previous conditions with $\sim_m$ replaced with $\equiv_m$. If $m = 0$, the result trivially follows. Now, assume that $m > 0$. We focus on the forth condition. Let C be a chain of $\mathcal{S}$. According to Proposition 4.1, let β be the unique $(m-1)$-type for $(h+1)$-structures such that $(\mathcal{S}_h, \mathrm{C}) \models \beta$. Hence, $\mathcal{S}_h \models \exists^{\mathsf{c}} X_{h+1}. \beta$. By Proposition 4.1, β is an $(h+1)$-MCL formula with $\mathsf{qr}(\beta) \leq m-1$. Since $\mathcal{S}_h \equiv_m \mathcal{S}'_h$, it follows that $\mathcal{S}'_h \models \exists^{\mathsf{c}} X_{h+1}. \beta$. Hence, there exists a chain $\mathrm{C}' \in \mathcal{S}'$ such that $(\mathcal{S}'_h, \mathrm{C}') \models \beta$. Thus, being β the $(m-1)$-type for $(h+1)$-structures, we obtain that $(\mathcal{S}_h, \mathrm{C}) \equiv_{m-1} (\mathcal{S}'_h, \mathrm{C}')$, and we are done. ◀

4.2 A Composition Theorem for MCL

We provide now a characterization of the game-equivalence relation $\sim_m$ over 1-structures on AP, for a given $m \geq 1$, in terms of the game-equivalence relation $\sim^{\omega}_{2m}$ over word-structures defined over a suitable set of atomic propositions.

Fix $m \geq 1$. Referring to Proposition 4.1, let $\beta_1, \ldots, \beta_\ell$ be the m-types of the equivalence relation $\equiv_m$ over the class of structures. Recall that for each structure $\mathcal{S}$, there is exactly one m-type β_i for some $i \in [i, \ell]$ such that $\mathcal{S} \models \beta_i$ (we say that β_i is the m-type of $\mathcal{S}$). Given a structure $\mathcal{S}$, a node w of $\mathcal{S}$, a child w_{C} of w in $\mathcal{S}$, and $i \in [1, \ell]$, let $\mathrm{N}_{\mathcal{S}}(w, w_{\mathrm{C}}, i)$ be the (possibly infinite) cardinality of the set of children w' of w in $\mathcal{S}$ such that $w' \neq w_{\mathrm{C}}$ and the substructure (i.e., the labeled subtree) of $\mathcal{S}$ rooted at w' has m-type β_i. We denote by $f_{\mathcal{S}}(w, w_{\mathrm{C}})$ the mapping in $[1, \ell] \to [0, m]$, where for each $i \in [1, \ell]$, $f_{\mathcal{S}}(w, w_{\mathrm{C}})(i)$ is defined as:

$$f_{\mathcal{S}}(w, w_{\mathrm{C}})(i) \triangleq \begin{cases} \mathrm{N}_{\mathcal{S}}(w, w_{\mathrm{C}}, i) & \text{if } \mathrm{N}_{\mathcal{S}}(w, w_{\mathrm{C}}, i) < m \\ m & \text{otherwise.} \end{cases}$$

Thus, $f_{\mathcal{S}}(w, w_{\mathrm{C}})(i)$ approximates $\mathrm{N}_{\mathcal{S}}(w, w_{\mathrm{C}}, i)$ with the greatest number in $[0, m]$ which is smaller or equal to $\mathrm{N}_{\mathcal{S}}(w, w_{\mathrm{C}}, i)$. We consider the finite set AP_m of propositions defined as:

$$\mathrm{AP}_m \triangleq 2^{\mathrm{AP} \cup \{\mathsf{c}\}} \times \big([1, \ell] \mapsto [0, m]\big) \times [1, \ldots, \ell].$$

Let $(\mathcal{S}, \mathrm{C})$ be a 1-structure with labeling *Lab*. For each infinite branch π of $\mathcal{S}$ such that π contains the chain C (note that if C is infinite, then π is uniquely determined), we denote by $\omega(\mathcal{S}, \pi, \mathrm{C})$ the infinite word over 2^{AP_m} defined as follows for all positions $j \geq 0$:

$$\omega(\mathcal{S}, \pi, \mathrm{C})(j) = \{(Lab(\pi(j)) \cup \flat, f_{\mathcal{S}}(\pi(j), \pi(j+1)), k_j)\}$$

where (i) $\flat = \{\mathsf{c}\}$ if $\pi(j) \in \mathrm{C}$, and $\flat = \emptyset$ otherwise, and (ii) k_j is such that β_{k_j} is the m-type of the substructure of $\mathcal{S}$ rooted at $\pi(j+1)$. Thus the label of the jth position of $\omega(\mathcal{S}, \pi, \mathrm{C})(j)$ is a singleton and corresponds to the label of the jth node of the infinite path π of $\mathcal{S}$ extended with additional information concerning the m-type of the subtree rooted at node $\pi(j+1)$, the m-types of the subtrees rooted at the children of $\pi(j)$ which are not in π, and the indication whether node $\pi(j)$ belong to the chain C or not. Note that the infinite word $\omega(\mathcal{S}, \pi, \mathrm{C})$ can be seen as the word-structure (π, Lab'), where $Lab'(\pi(j)) = \omega(\mathcal{S}, \pi, \mathrm{C})(j)$ for each $j \geq 0$.

The importance of the word-structure $\omega(\mathcal{S}, \pi, \mathrm{C})$ is that it captures the whole of the 1-structure $(\mathcal{S}, \mathrm{C})$ with respect to the distinguishing power of 1-MCL formulas with quantifier rank at most m. In particular, we establish the following crucial result.

▶ **Lemma 4.3** (From MCL-games on 1-structures to MCL-games on word-structures)**.** *Let $m \geq 1$, $(\mathcal{S}, \mathrm{C})$ and $(\mathcal{S}', \mathrm{C}')$ be two 1-structures, and π and π' be two infinite branches of $\mathcal{S}$ and $\mathcal{S}'$, respectively, such that $\mathrm{C} \subseteq \pi$ and $\mathrm{C}' \subseteq \pi'$. Then:*

$$\omega(\mathcal{S}, \pi, \mathrm{C}) \sim^{\omega}_{2m} \omega(\mathcal{S}', \pi', \mathrm{C}') \Rightarrow (\mathcal{S}, \mathrm{C}) \sim_m (\mathcal{S}', \mathrm{C}').$$

Proof. We need some additional definitions and preliminary observations. Let $\mathcal{S}_1 = (\mathcal{S}, \mathrm{C})$ be a 1-structure, and π be an infinite branch of $\mathcal{S}$ such that $\mathrm{C} \subseteq \pi$. We observe that each chain of $\mathcal{S}$ which is not contained in the branch π can be partitioned into two chains C_1 and C_2 of $\mathcal{S}$ such that C_1 is a subset of the path π and there exists a child w' of some node w of π such that w' is not a π-node and C_2 is a non-empty chain of the subtree of $\mathcal{S}$ rooted at node w'. This justifies the following definition. A *π-term of $\mathcal{S}_1$* is either a chain of π, or a tuple of the form $(\mathrm{C}_1, w, \mathrm{T}, \mathrm{C}_2)$ such that the following holds:

- C_1 is a finite chain of π and w is a node of π which is a descendant of all nodes in C_1;
- there is a child w' of w in $\mathcal{S}$ such that $w' \notin \pi$, T is the labeled subtree of $\mathcal{S}$ rooted at node w', and C_2 is a non-empty chain of T.

For a compound π-term $t = (\mathrm{C}_1, w, \mathrm{T}, \mathrm{C}_2)$ of $\mathcal{S}_1$, we write $\mathrm{T}(t)$ for T, $\mathrm{C}_1(t)$ for C_1, $\mathrm{C}_2(t)$ for C_2, and $\mathrm{C}_w(t)$ for w. For a simple π-term t consisting of a chain of π, $\mathrm{C}_1(t)$ is for t, and $\mathrm{T}(t)$, $\mathrm{C}_2(t)$, and $\mathrm{C}_w(t)$ denote the empty set. For each $h \geq 0$, a *(π, h)-term* of $\mathcal{S}_1$ is a tuple of the form $(t_1, \ldots, t_h)$ where $t_1, \ldots, t_h$ are π-terms of $\mathcal{S}_1$. We make the following observation:

Disjointness property. For π-terms $(\mathrm{C}_1, w, \mathrm{T}, \mathrm{C}_2)$ and $(\mathrm{C}'_1, w', \mathrm{T}', \mathrm{C}'_2)$ of $\mathcal{S}_1$, either $T = T'$ or $T \cap T' = \emptyset$. Moreover, if $T \cap T' = \emptyset$, then $\mathrm{C}_2 \cap \mathrm{C}'_2 = \emptyset$ (in particular, C_2 and C'_2 are not related by the descendant relation).

Fix two 1-structures $\mathcal{S}_1 = (\mathcal{S}, \mathrm{C})$ and $\mathcal{S}'_1 = (\mathcal{S}', \mathrm{C}')$, an infinite branch π of $\mathcal{S}_1$ with $\mathrm{C} \subseteq \pi$, and an infinite branch π' of $\mathcal{S}'_1$ with $\mathrm{C}' \subseteq \pi'$. Let $\mathcal{S}_\omega = \omega(\mathcal{S}, \pi, \mathrm{C})$ and $\mathcal{S}'_\omega = \omega(\mathcal{S}', \pi', \mathrm{C}')$. Given $m \geq 1$, $h \in [0, m]$, a (π, h)-term $\mathsf{tr} = (t_1, \ldots, t_h)$ of $\mathcal{S}_1$, and a (π', h)–term $\mathsf{tr}' = (t'_1, \ldots, t'_h)$ of $\mathcal{S}'_1$, we say that tr and tr' are *m-consistent* if the following holds:

1. for all $i \in [1, h]$, $\mathrm{T}(t_i) \neq \emptyset$ iff $\mathrm{T}(t'_i) \neq \emptyset$. Moreover, if $\mathrm{T}(t_i) \neq \emptyset$, then:
   - let $t_{i_1}, \ldots, t_{i_p}$ and $t'_{i'_1}, \ldots, t'_{i'_{p'}}$ be the ordered sequences of compound terms in tr and tr', respectively, having tree-component $\mathrm{T}(t_i)$ and $\mathrm{T}(t'_i)$, respectively. Then, $p = p'$, and $i_j = i'_j$ for all $j \in [1, p]$. Moreover, $(\mathcal{S}_{\mathrm{T}(t_i)}, \mathrm{C}_2(t_{i_1}), \ldots, \mathrm{C}_2(t_{i_p})) \sim_{m-p} (\mathcal{S}'_{\mathrm{T}(t'_i)}, \mathrm{C}_2(t'_{i_1}), \ldots, \mathrm{C}_2(t'_{i_p}))$.
2. $(\mathcal{S}_\omega, \mathrm{C}_1(t_1), \mathrm{C}_w(t_1), \ldots, \mathrm{C}_1(t_h), \mathrm{C}_w(t_h)) \sim^{\omega}_{2m-2h} (\mathcal{S}'_\omega, \mathrm{C}_1(t'_1), \mathrm{C}_w(t'_1), \ldots, \mathrm{C}_1(t'_h), \mathrm{C}_w(t'_h))$.

By the disjointness property and since $m \geq 1$ (recall that the special proposition c of AP_m marks the nodes of the chains C and C′ of $\omega(\mathcal{S}, \pi, \mathrm{C})$ and $\omega(\mathcal{S}', \pi', \mathrm{C}')$, respectively), the following result easily follows.

▷ Claim 4.4. Let $\mathsf{tr} = (t_1, \ldots, t_h)$ be a (π, h)-term of $\mathcal{S}_1$ and $\mathsf{tr}' = (t'_1, \ldots, t'_h)$ be a (π', h)-term of $\mathcal{S}'_1$. If tr and tr' are m-consistent, then the $(h+1)$-structures $(\mathcal{S}, \mathrm{C}, \mathrm{C}_1(t_1) \cup \mathrm{C}_2(t_1), \ldots, \mathrm{C}_1(t_h) \cup \mathrm{C}_2(t_h))$ and $(\mathcal{S}', \mathrm{C}', \mathrm{C}_1(t'_1) \cup \mathrm{C}_2(t'_1), \ldots, \mathrm{C}_1(t'_h) \cup \mathrm{C}_2(t'_h))$ are locally isomorphic.

Assume that $\omega(\mathcal{S}, \pi, \mathrm{C}) \sim^{\omega}_{2m} \omega(\mathcal{S}', \pi', \mathrm{C}')$. We need to prove that $\mathcal{S}_1 \sim_m \mathcal{S}'_1$. By Claim 4.4 (for the case where $h = 0$), $\mathcal{S}_1$ and $\mathcal{S}'_1$ are locally isomorphic. Now, given $0 \leq h < m$, assume that after h-rounds in the EF-game $\mathcal{G}_m(\mathcal{S}_1, \mathcal{S}'_1)$, there are h π-terms $t_1, \ldots, t_h$ selected in the structure $\mathcal{S}$ and corresponding h π'-terms $t'_1, \ldots, t'_h$ selected in the structure $\mathcal{S}'$ such that $\mathsf{tr} = (t_1, \ldots, t_h)$ and $\mathsf{tr}' = (t'_1, \ldots, t'_h)$ are m-consistent with respect to $\mathcal{S}_1$ and $\mathcal{S}'_1$. Moreover, assume that at the $(h+1)$-round the spoiler chooses a π-term t_{h+1} in $\mathcal{S}$ (the case where the choice is made on the structure $\mathcal{S}'$ is similar). We show that the duplicator can respond by choosing a π'-term t'_{h+1} in $\mathcal{S}'$ such that the tuples $(t_1, \ldots, t_h, t_{h+1})$ and $(t'_1, \ldots, t'_h, t'_{h+1})$ are still m-consistent with respect to $\mathcal{S}_1$ and $\mathcal{S}'_1$. Hence, by Claim 4.4, the result follows. We focus on the case where t_{h+1} is a compound term π-term of the form $(\mathrm{C}_1, w, \mathrm{T}, \mathrm{C}_2)$ (the case where t_{h+1} is a chain of π is simpler). We distinguish two cases:

- *there is some term t_i in* tr *such that* $\mathrm{T}(t_i) = \mathrm{T}$: let $t_{i_1}, \ldots, t_{i_N}$ be the ordered sequence of compound terms in tr having T as tree-component. Note that $\mathrm{C}_w(t_{i_j}) = w$ for all $j \in [1, N]$. Since tr and tr' are m-consistent with respect to $\mathcal{S}_1$ and $\mathcal{S}'_1$, it holds that $\mathrm{T}(t'_i) = \mathrm{T}'_i \neq \emptyset$, $t'_{i_1}, \ldots, t'_{i_N}$ is the ordered sequence of compound terms in tr' having T' as tree-component, and there is a node w' of π (the parent of the T'-root) such that $\mathrm{C}_w(t'_{i_j}) = w'$ for all $j \in [1, N]$. Moreover, $(\mathcal{S}_{\mathrm{T}}, \mathrm{C}_2(t_{i_1}), \ldots, \mathrm{C}_2(t_{i_N})) \sim_{m-N} (\mathcal{S}'_{\mathrm{T}'}, \mathrm{C}_2(t'_{i_1}), \ldots, \mathrm{C}_2(t'_{i_1}))$ and $(\mathcal{S}_\omega, \mathrm{C}_1(t_1), \mathrm{C}_w(t_1), \ldots, \mathrm{C}_1(t_h), \mathrm{C}_w(t_h)) \sim^{\omega}_{2m-2h} (\mathcal{S}'_\omega, \mathrm{C}_1(t'_1), \mathrm{C}_w(t'_1), \ldots, \mathrm{C}_1(t'_h), \mathrm{C}_w(t'_h))$. Thus, since $N < m$, in the EF-game over the substructures $\mathcal{S}_{\mathrm{T}}$ and $\mathcal{S}'_{\mathrm{T}'}$, the duplicator can pick a chain C'_2 of T' such that $(\mathcal{S}_{\mathrm{T}}, \mathrm{C}_2(t_{i_1}), \ldots, \mathrm{C}_2(t_{i_N}), \mathrm{C}_2) \sim_{m-(N+1)} (\mathcal{S}'_{\mathrm{T}'}, \mathrm{C}_2(t'_{i_1}), \ldots, \mathrm{C}_2(t'_{i_N}), \mathrm{C}'_2)$. Moreover, since $h < m$, in the EF-game over the word structures $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$, the duplicator can pick a chain C'_1 of π' such that $(\mathcal{S}_\omega, \mathrm{C}_1(t_1), \mathrm{C}_w(t_1), \ldots, \mathrm{C}_1(t_h), \mathrm{C}_w(t_h), \mathrm{C}_1, \{w\}) \sim^{\omega}_{2m-2(h+1)} (\mathcal{S}'_\omega, \mathrm{C}_1(t'_1), \mathrm{C}_w(t'_1), \ldots, \mathrm{C}_1(t'_h), \mathrm{C}_w(t'_h), \mathrm{C}'_1, \{w'\})$. Note that w' must be a descendant of all nodes in C'_1. Hence, by setting t'_{h+1} to $(\mathrm{C}'_1, w', \mathrm{T}', \mathrm{C}'_2)$, the result follows.
- *there is no term in* tr *having tree-component* T: let $t_{i_1}, \ldots, t_{i_N}$ be the (possibly empty) ordered sequence of compound terms in tr whose second component is w. Since $h < m$ and tr and tr' are m-consistent, in the EF-game over the word structures $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$, the duplicator can pick a chain C'_1 of π' and a node $w' \in \pi'$ which is a descendant of all nodes in C'_1 such that $(\mathcal{S}_\omega, \mathrm{C}_1(t_1), \mathrm{C}_w(t_1), \ldots, \mathrm{C}_1(t_h), \mathrm{C}_w(t_h), \mathrm{C}_1, \{w\}) \sim^{\omega}_{2m-2(h+1)} (\mathcal{S}'_\omega, \mathrm{C}_1(t'_1), \mathrm{C}_w(t'_1), \ldots, \mathrm{C}_1(t'_h), \mathrm{C}_w(t'_h), \mathrm{C}'_1, \{w'\})$. Hence, $t'_{i_1}, \ldots, t'_{i_N}$ is the (possibly empty) ordered sequence of compound terms in tr' whose second component is w'. Since the label of w in $\mathcal{S}_\omega$ and the label of w' in $\mathcal{S}'_\omega$ coincide and $N < m$, by construction of the word-structures $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$, there must be a child w'' of w' in $\mathcal{S}'$ such that $w'' \notin \pi'$ and for the subtree T' of $\mathcal{S}'$ rooted at w'', it holds that $\mathcal{S}_{\mathrm{T}} \equiv_m \mathcal{S}'_{\mathrm{T}'}$ and $\mathrm{T}' \neq \mathrm{T}(t'_{i_j})$ for all $j \in [1, N]$. Being $\mathrm{C}_2 \subseteq T$ and $m \geq 1$, in the EF-game over the substructures $\mathcal{S}_{\mathrm{T}}$ and $\mathcal{S}'_{\mathrm{T}'}$, the duplicator can pick a chain C'_2 of T' such that $(\mathcal{S}_{\mathrm{T}}, \mathrm{C}_2) \sim_{m-1} (\mathcal{S}_{\mathrm{T}}, \mathrm{C}_2)$. We set $t'_{h+1} = (\mathrm{C}'_1, w', \mathrm{T}', \mathrm{C}'_2)$, and the result follows. ◀

We can now state a composition theorem for 1-MCL formulas over 1-structures, which allows to express such formulas in terms of MCL sentences over word-structures on 2^{AP_m} (or, equivalently, MSO sentences over infinite words on 2^{AP_m}).

▶ **Theorem 4.5** (Composition Theorem for MCL). *For all $m \geq 1$ and* 1-MCL *formulas $\varphi(X)$ over* AP *with $\mathsf{qr}(\varphi) \leq m$, there is an* MCL *sentence ψ over AP_m such that for each* 1*-structure $(\mathcal{S}, \mathrm{C})$ and infinite branch π of $\mathcal{S}$ with $\mathrm{C} \subseteq \pi$, we have $(\mathcal{S}, \mathrm{C}) \models \varphi(X) \Leftrightarrow \omega(\mathcal{S}, \pi, \mathrm{C}) \models \psi$.*

Proof. According to Proposition 4.1, we consider the following formulas:

- the m-types $\alpha_1(X), \ldots, \alpha_\ell(X)$ for 1-structures. By Proposition 4.1, there is $\mathrm{I} \subseteq \{1, \ldots, \ell\}$ such that $\varphi(X)$ is equivalent to $\bigvee_{p \in \mathrm{I}} \alpha_p(X)$.
- The $2m$-types $\gamma_1, \ldots, \gamma_h$ for the MCL-sentences over word-structures on AP_m.

We denote by Γ the finite set of $2m$-types γ_i with $i \in [1, h]$ such that there exist a 1-structure $(\mathcal{S}, \mathrm{C})$ and an infinite branch π of $\mathcal{S}$ with $\mathrm{C} \subseteq \pi$ so that $(\mathcal{S}, \mathrm{C}) \models \varphi(X)$ and $\omega(\mathcal{S}, \pi, \mathrm{C}) \models \gamma_i$. The desired MCL sentence ψ is then given by $\bigvee_{\gamma_i \in \Gamma} \gamma_i$. We prove the following, hence the result directly follows.

▷ Claim. For each 1-structure $(\mathcal{S}, \mathrm{C})$ and infinite branch π of $\mathcal{S}$ so that $\mathrm{C} \subseteq \pi$, it holds that $(\mathcal{S}, \mathrm{C}) \models \varphi(X)$ if and only if $\omega(\mathcal{S}, \pi, \mathrm{C}) \models \gamma_i$, for some $\gamma_i \in \Gamma$.

To prove the claim, let $(\mathcal{S}, \mathrm{C})$ and π be as in the claim. By Proposition 4.1, there is a unique $i \in [1, h]$ such that $\omega(\mathcal{S}, \pi, \mathrm{C}) \models \gamma_i$.

If $(\mathcal{S}, \mathrm{C}) \models \varphi(X)$, then by construction, $\gamma_i \in \Gamma$. Assume now that $\gamma_i \in \Gamma$. It remains to show that $(\mathcal{S}, \mathrm{C}) \models \varphi(X)$. We assume the contrary and derive a contradiction. Hence, there exists $p' \in [1, \ell] \setminus \mathrm{I}$ such that $(\mathcal{S}, \mathrm{C}) \models \alpha_{p'}(X)$. Since $\gamma_i \in \Gamma$, there exist $p \in \mathrm{I}$, a 1-structure $(\mathcal{S}', \mathrm{C}')$, and an infinite branch π' of $\mathcal{S}'$ with $\mathrm{C}' \subseteq \pi'$ so that $(\mathcal{S}', \mathrm{C}') \models \alpha_p(X)$ and $\omega(\mathcal{S}', \pi', \mathrm{C}') \models \gamma_i$. Since $\omega(\mathcal{S}, \pi, \mathrm{C}) \models \gamma_i$, it follows that $\omega(\mathcal{S}, \pi, \mathrm{C}) \equiv^{\omega}_{2m} \omega(\mathcal{S}', \pi', \mathrm{C}')$. Thus, by Proposition 4.2 and Lemma 4.3, we obtain that $(\mathcal{S}, \mathrm{C}) \equiv_m (\mathcal{S}', \mathrm{C}')$, which is a contradiction since $(\mathcal{S}, \mathrm{C})$ and $(\mathcal{S}', \mathrm{C}')$ have distinct m-types. ◀

4.3 From MCL to CECTL*

By exploiting Theorem 4.5, we show that the logics MCL and CECTL* have the same expressiveness.

▶ **Theorem 4.6.** MCL *and* CECTL* *are equally expressive.*

Proof. By [3], each CECTL* state formula has an equivalent MCL sentence. Thus, it suffices to show that for each MCL sentence, there is an equivalent CECTL* state formula. Let φ be an MCL sentence. The result is shown by an induction argument on the quantifier rank $\mathsf{qr}(\varphi)$.

Base case. Let φ be an MCL sentence such that $\mathsf{qr}(\varphi) = 1$. Since the existential chain quantifier $\exists^{\mathsf{C}}$ distributes over disjunction, φ is equivalent to a Boolean combination of sentences of the form $\exists^{\mathsf{C}} X. \psi$, where ψ is a conjunction of *atoms in* X, i.e., atoms of the form $X \subseteq p$ or $X \leq X$ or $\mathtt{sing}(X)$, or negations of atoms in X.

Fix an MCL sentence of the form $\exists^{\mathsf{C}} X. \psi$, where ψ is a conjunction of atoms in X or negations of atoms X. We show that there is an equivalent CECTL* formula. Hence, the result follows. We assume that $\exists^{\mathsf{C}} X. \psi$ is satisfiable (otherwise, $\exists^{\mathsf{C}} X. \psi$ is equivalent to $\neg\top$, and the result trivially follows). We distinguish two cases:

- ψ holds when X is bound to the empty chain. In this case, $\exists^{\mathsf{C}} X. \psi$ is equivalent to $\top$, and the result follows.
- ψ does not hold when X is bound to the empty chain: in this case, the atomic formula $X \leq X$ corresponds to $\mathtt{sing}(X)$ and there exist distinct atomic propositions $p_1, \ldots, p_n, q_1, \ldots, q_m$ such that ψ can be rewritten as

$$\xi \wedge \bigwedge_{i=1}^{i=n} X \subseteq p_i \wedge \bigwedge_{j=1}^{j=m} \neg(X \subseteq q_j)$$

where either $\xi = \top$, or $\xi = \mathtt{sing}(X)$, or $\xi = \neg\mathtt{sing}(X)$. We focus on the case where $\xi = \neg\mathtt{sing}(X)$ (the other cases being similar). Note that for an atom $X \subseteq p_i$ and two chains C and C′ of a structure (Kripke tree) $\mathcal{S} = \langle \mathrm{T}, \mathit{Lab}\rangle$ such that $\mathrm{C} \subseteq \mathrm{C}'$, $(\mathcal{S}, \mathsf{V}_2[X \mapsto \mathrm{C}']) \models X \subseteq p_i$ entails that $(\mathcal{S}, \mathsf{V}_2[X \mapsto \mathrm{C}]) \models X \subseteq p_i$ (intuitively, the satisfaction relation is *downward-closed* for atoms $X \subseteq p_i$). Moreover, $(\mathcal{S}, \mathsf{V}_2[X \mapsto \mathrm{C}]) \models \neg(X \subseteq q_j)$ iff there is a node $w \in \mathrm{C}$ such that $q_j \notin \mathit{Lab}(w)$. It follows that a structure (Kripke tree) $\mathcal{S} = \langle \mathrm{T}, \mathit{Lab}\rangle$ is a model of $\exists^{\mathsf{C}} X.\, \psi$ iff there exist $\ell \in [2, m+2]$ and a finite chain C of $\mathcal{S}$ having cardinality ℓ such that (i) $p_i \in \mathit{Lab}(w)$ for all $i \in [1, n]$ and $w \in \mathrm{C}$, and (ii) for all $j \in [1, m]$, there exists $w \in \mathrm{C}$ so that $q_j \notin \mathit{Lab}(w)$. These requirements can be easily captured by a CTL* formula (and thus by a CECTL* formula as well). Hence, the result follows.

Induction step. Let $m \geq 1$ and assume that for each MCL sentence with quantifier rank at most m, there is an equivalent CECTL* state formula. Fix an MCL sentence of the form $\exists^{\mathsf{C}} X.\, \varphi(X)$ with $\mathsf{qr}(\varphi) \leq m$. We show that $\exists^{\mathsf{C}} X.\, \varphi(X)$ has an equivalent CECTL* state formula. Hence, the result follows. For the fixed $m \geq 1$, let $\beta_1, \ldots, \beta_\ell$ be the m-types for MCL sentences over structures on AP. Since $\beta_1, \ldots, \beta_\ell$ have quantifier rank at most m, by the induction hypothesis, there exist CECTL* state formulas $\hat{\beta}_1, \ldots, \hat{\beta}_\ell$ such that β_i and $\hat{\beta}_i$ are equivalent for each $i \in [1, \ell]$. Recall that $\mathrm{AP}_m = 2^{\mathrm{AP} \cup \{\mathsf{c}\}} \times ([1, \ell] \mapsto [0, m]) \times [1, \ell]$. Let AP'_m obtained from AP_m by removing the special proposition c from the first component $2^{\mathrm{AP} \cup \{\mathsf{c}\}}$ of AP_m. For a structure $\mathcal{S}$ (over AP) and an infinite branch π of $\mathcal{S}$, we write $\omega(\mathcal{S}, \pi)$ to mean the word-structure $\omega(\mathcal{S}, \pi, \emptyset)$. Note that $\omega(\mathcal{S}, \pi)$ corresponds to an infinite word over AP'_m. Recall that for each MSO sentence ϕ over infinite words, one can construct a Büchi NWA accepting the models of ϕ. Moreover, Büchi NWA are closed under projection and a Büchi NWA can be converted into an equivalent parity DWA [42]. Thus, since MCL over word structures corresponds to MSO over infinite words, by applying Theorem 4.5 to the 1-MCL formula $\varphi(X)$, there exists a parity NWA $\mathcal{D}_\varphi$ over AP'_m such that the following holds.

▷ Claim 4.7. For each structure $\mathcal{S}$, there exists a chain C of $\mathcal{S}$ such that $(\mathcal{S}, \mathrm{C}) \models \varphi(X)$ *if and only if* there exists an infinite branch π of $\mathcal{S}$ so that $\omega(\mathcal{S}, \pi) \in \mathsf{L}(\mathcal{D}_\varphi)$.

Let $\Upsilon \triangleq ([1, \ell] \mapsto [0, m]) \times [1, \ell]$. Now, for each $(f, k) \in \Upsilon$, we define a CECTL* path formula $\theta_{(f,k)}$ expressing, for a given structure $\mathcal{S}$, infinite branch π of $\mathcal{S}$, and node $w \in \pi$, that:

- for each $i \in [1, \ell]$, let N be the (possibly infinite) number of distinct children w' of w such that the substructure of $\mathcal{S}$ rooted at w' has m-type β_i. Then:
  - case $i \neq k$: $N = f(i)$ if $f(i) < m$, and $N \geq f(i)$ otherwise;
  - case $i = k$: $N = f(i) + 1$ if $f(i) < m$, and $N \geq f(i) + 1$ otherwise.
- The substructure of $\mathcal{S}$ rooted at the child w' of w along π has m-type β_k.

$$\theta_{(f,k)} \triangleq (\mathtt{X}\, \hat{\beta}_k) \wedge \bigwedge_{i \in [1,\ell]} \theta^i_{(f,k)}, \text{ where}$$

$$\theta^i_{(f,k)} \triangleq \begin{cases} \mathtt{D}^{f(i)} \hat{\beta}_i, & \text{if } f(i) = m \wedge i \neq k; \\ \mathtt{D}^{f(i)} \hat{\beta}_i \wedge \neg \mathtt{D}^{f(i)+1} \hat{\beta}_i, & \text{if } f(i) < m \wedge i \neq k; \\ \mathtt{D}^{f(i)+1} \hat{\beta}_i, & \text{if } f(i) = m \wedge i = k; \\ \mathtt{D}^{f(i)+1} \hat{\beta}_i \wedge \neg \mathtt{D}^{f(i)+2} \hat{\beta}_i, & \text{if } f(i) < m \wedge i = k. \end{cases}$$

Since $\beta_1, \ldots, \beta_\ell$ are the m-types for MCL sentences, by construction, the following holds.

▷ Claim 4.8. Given a structure $\mathcal{S}$, an infinite branch π of $\mathcal{S}$, and $i \geq 0$, there is exactly one element $(f,k) \in \Upsilon$ such that $(\mathcal{S},\pi,i) \models \theta_{(f,k)}$.

Let $\mathcal{D}_\varphi = \langle 2^{\mathrm{AP}} \times \Upsilon, \mathrm{Q}_D, \delta_D, q_{D,I}, \Omega_D \rangle$ be the parity DWA of Claim 4.7. For each $(f,k) \in \Upsilon$, we consider the parity NWA $\mathcal{N}_{(f,k)} = \langle 2^{\mathrm{AP}}, \mathrm{Q}_N, \delta_N, (q_{D,I},f,k), \Omega_N \rangle$ over 2^{AP} with initial state $(q_{D,I},f,k)$ which simulates $\mathcal{D}_\varphi$ by keeping track in the current state of the guessed second component of the next input symbol. Formally $\mathrm{Q}_N = \mathrm{Q}_D \times \Upsilon$, $\delta_N((q',f',k'),a) = \bigvee_{(f'',k'')\in\Upsilon} (\delta_D(q',(a,f',k')),f'',k'')$ and $\Omega_N(q',(f',k')) = \Omega_D(q')$ for all $q' \in \mathrm{Q}_D$, $a \in 2^{\mathrm{AP}}$, and $(f',k') \in \Upsilon$. Note that for $(f,k) \neq (f',k')$, the parity NWA $\mathcal{N}_{(f,k)}$ and $\mathcal{N}_{(f',k')}$ differ only for the initial state. Moreover, let τ be the testing function assigning to each state $(q',f',k') \in \mathrm{Q}_N$ the CECTL* path formula $\theta_{(f',k')}$. By construction and Claim 4.7, we obtain the following characterization of the structures satisfying the MCL sentence $\exists^{\mathsf{C}} X.\, \varphi(X)$.

▷ Claim 4.9. For each structure $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$, $\mathcal{S} \models \exists^{\mathsf{C}} X.\, \varphi(X)$ *iff* for some infinite branch π of $\mathcal{S}$ and some $(f,k) \in \Upsilon$, there is an accepting run ν of $\mathcal{N}_{(f,k)}$ over $\mathit{Lab}(\pi(0))\mathit{Lab}(\pi(1))\ldots$ such that $(\mathcal{S},\pi,i) \models \tau(\nu(i))$ for all $i \geq 0$.

Given a finite path π_f of a structure $\mathcal{S} = (\mathrm{T}, \mathit{Lab})$, a *good run* of $\mathcal{N}_{(f,k)}$ over π_f is a finite path ν_f of $\mathcal{N}_{(f,k)}$ over the Lab-labeling of π_f such that $(\mathcal{S},\pi_f(i)) \models \tau(\nu(i))$ for all $0 \leq i < |\pi_f|$.

We now show that the characterization of the set of models of $\exists^{\mathsf{C}} X.\, \varphi(X)$ in Claim 4.9 can be captured by a CECTL* formula. For all states $(q,f,k) \in \mathrm{Q}_N$ and set $\mathrm{P} \subseteq \mathrm{Q}_N$, we denote by ${}_{(q,f,k)}\mathcal{N}_{\mathrm{P}}$ the testing NWA$_\mathrm{f}$ with test function τ and whose embedded NWA$_\mathrm{f}$ is obtained from the automata $\mathcal{N}_{(f,k)}$ by setting a fresh copy of (q,f,k) as initial state, and P as set of accepting states. This fresh copy behaves as (q,f,k) and has the same test as (q,f,k), and ensures that the automaton cannot accept the empty word. Finally, let $\mathrm{Q}_{N,even}$ be the set of states in Q_N having even color, and for each $(q,f,k) \in \mathrm{Q}_N$, let $\mathrm{Q}_N > (q,f,k)$ be the set of states in Q_N having color greatest than the color of (q,f,k). We consider the CECTL* state formula $\mathtt{E}\psi$ where the CECTL* path formula ψ is defined as follows:

$$\psi \triangleq \bigvee_{(f,k)\in\Upsilon} \bigvee_{(q',f',k')\in \mathrm{Q}_{N,even}} (\psi_1(f,k,q',f',k') \wedge \psi_2(f,k,q',f',k'))$$
$$\psi_1(f,k,q',f',k') \triangleq \langle {}_{(q_{D,I},f,k)}\mathcal{N}_{\{(q',f',k')\}} \rangle [{}_{(q',f',k')}\mathcal{N}_{\mathrm{Q}_N > (q',f',k')}] \neg\top$$
$$\psi_2(f,k,q',f',k') \triangleq [{}_{(q_{D,I},f,k)}\mathcal{N}_{\{(q',f',k')\}}] \langle {}_{(q',f',k')}\mathcal{N}_{\{(q',f',k')\}} \rangle \top$$

Thus, an infinite branch π of a structure $\mathcal{S}$ satisfies the path formula ψ iff there exist $(f,k) \in \Upsilon$ and $(q',f',k') \in \mathrm{Q}_{N,even}$ such that the following conditions hold:

- there is a good run of $\mathcal{N}_{(f,k)}$ over some non-empty prefix $\pi(0)\ldots\pi(i)$ of π from state $(q_{D,I},f,k)$ to the state with even color (q',f',k'). Moreover, no good run of $\mathcal{N}_{(f,k)}$ over some non-empty infix of π from position i which starts and ends at state (q',f',k') visits a state with color greatest than color of (q',f',k').
- for each good run of $\mathcal{N}_{(f,k)}$ over some non-empty prefix $\pi(0)\ldots\pi(i)$ of π from state $(q_{D,I},f,k)$ to state (q',f',k'), there is a good run f $\mathcal{N}_{(f,k)}$ over some non-empty infix of π from position i which starts and ends at state (q',f',k').

The first condition is expressed by the conjunct $\psi_1(f,k,q',f',k')$ of ψ, while the second condition is expressed by the conjunct $\psi_2(f,k,q',f',k')$. By Claim 4.9, correctness of the construction directly follows from the following claim whose proof relies on the mutual-exclusivity condition expressed in Claim 4.8.

▷ Claim 4.10. For each structure $\mathcal{S} = (\mathrm{T}, Lab)$ and infinite branch π of $\mathcal{S}$, $(\mathcal{S}, \pi, 0) \models \psi$ iff for some $(f, k) \in \Upsilon$, there exists an accepting run ν of $\mathcal{N}_{(f,k)}$ over $Lab(\pi(0))Lab(\pi(1))\ldots$ such that $(\mathcal{S}, \pi, i) \models \tau(\nu(i))$ for all $i \geq 0$.

The left-right implication in Claim 4.10 easily follows from construction. For the right-left implication, assume that for some $(f, k) \in \Upsilon$, there is an accepting run $\nu = (q_0, f_0, k_0)(q_1, f_1, k_1)\ldots$ of $\mathcal{N}_{(f,k)}$ over $\rho = Lab(\pi(0))Lab(\pi(1))\ldots$ with $(q_0, f_0, k_0) = (q_{D,I}, f, k)$ such that $(\mathcal{S}, \pi, i) \models \theta_{(f_i,k_i)}$ for all $i \geq 0$. Since ν is accepting, there exists a state $(q', f', k') \in \mathrm{Q}_{N,even}$ having an even color n such that n is the maximum color associated to the states which occur infinitely many times along ν. We show that $(\mathcal{S}, \pi, 0) \models \psi_1(f, k, q', f', k') \wedge \psi_2(f, k, q', f', k')$. Hence, the result follows. We focus on the conjunct $\psi_2(f, k, q', f', k')$ (the proof for the conjunct $\psi_1(f, k, q', f', k')$ is similar). By construction of $\psi_2(f, k, q', f', k')$, it suffices to show that for all $j \geq 0$ and accepting runs ν_f of ${}_{(q_{D,I},f,k)}\mathcal{N}_{\{(q',f',k')\}}$ over $\rho[0, j]$ whose states satisfy the associated tests, then ν_f is a prefix of ν. Let $\nu_f = (q'_0, f'_0, k'_0)\ldots(q'_{j+1}, f'_{j+1}, k'_{j+1})$ be such a finite run over $\rho[0, j]$ such that $(q'_0, f'_0, k'_0) = (q_{D,I}, f, k)$ and for all $i \in [0, j + 1]$, $(\mathcal{S}, \pi, i) \models \theta_{(f'_i,k'_i)}$. Since $(\mathcal{S}, \pi, i) \models \theta_{(f_i,k_i)}$ for all $i \geq 0$, by Claim 4.8, it follows that $(f'_i, k'_i) = (f_i, k_i)$ for all $i \in [0, j + 1]$. Thus, since $\mathcal{D}_\varphi$ is deterministic, we deduce that $q'_i = q_i$ for all $i \in [0, j + 1]$, and the result follows. This concludes the proof of Claim 4.10.

At this point, the equivalence between $\exists^{\mathsf{C}} X. \varphi(X)$ and $\mathsf{E}\psi$ directly follows from Claims 4.9 and 4.10. This concludes the proof of Theorem 4.6. ◀

5 Conclusion

In this work, we adopted a compositional approach to prove the expressive equivalence of *Monadic Chain Logic* (MCL) and the counting extension CECTL* of ECTL*. Recent work [3] has established that the graded version (HGTA) of *Hesitant Tree Automata* (HTA) and their first-order extension (HFTA) represent the automata counterparts of the logics CECTL* and MCL, respectively. As a corollary of our main results, we obtain the following chain of equivalence:

▶ **Corollary 5.1.** *The logics* CECTL* *and* MCL *and the classes of automata* HGTA *and* HFTA *are all equivalent formalisms.*

It would be interesting to explore the applicability of a compositional approach to *Monadic Tree Logic* (MTL) [2], a fragment of MSO where second-order quantifiers range over trees. The goal here is to gain insights into the expressiveness of various extensions of standard temporal logics for strategic reasoning, such as Substructure Temporal Logic (STL), a temporal logic that allows implicit predication over substructures/subtrees [4, 5].

References

1 A. Arnold and D. Niwiński. Fixed Point Characterization of Weak Monadic Logic Definable Sets of Trees. In *Tree Automata and Languages*, pages 159–188. North-Holland, 1992.

2 M. Benerecetti, L. Bozzelli, F. Mogavero, and A. Peron. Quantifying over Trees in Monadic Second-Order Logic. In *LICS'23*, pages 1–13. IEEECS, 2023.

3 M. Benerecetti, L. Bozzelli, F. Mogavero, and A. Peron. Automata-Theoretic Characterisations of Branching-Time Temporal Logics. In *ICALP'24*, LIPIcs 297, pages 128:1–20. Leibniz-Zentrum fuer Informatik, 2024.

4 M. Benerecetti, F. Mogavero, and A. Murano. Substructure Temporal Logic. In *LICS'13*, pages 368–377. IEEECS, 2013.

5 M. Benerecetti, F. Mogavero, and A. Murano. Reasoning About Substructures and Games. *TOCL*, 16(3):25:1–46, 2015.

6 U. Boker, K. Lehtinen, and S. Sickert. On the Translation of Automata to Linear Temporal Logic. In *FOSSACS'22*, LNCS 13242, pages 140–160. Springer, 2022. $\mathsf{path}(doi: 10.1007/978-3-030-99253-8_8)$.

7 J.R. Büchi. Weak Second-Order Arithmetic and Finite Automata. *MLQ*, 6(1-6):66–92, 1960.

8 J.R. Büchi. On a Decision Method in Restricted Second-Order Arithmetic. In *ICLMPS'62*, pages 1–11. Stanford University Press, 1962.

9 J.R. Büchi. On a Decision Method in Restricted Second Order Arithmetic. In *Studies in Logic and the Foundations of Mathematics*, volume 44, pages 1–11. Elsevier, 1966.

10 Y. Choueka. Theories of Automata on ω-Tapes: A Simplified Approach. *JCSS*, 8(2):117–141, 1974.

11 E.M. Clarke, E.A. Emerson, and A.P. Sistla. Automatic Verification of Finite-State Concurrent Systems Using Temporal Logic Specifications: A Practical Approach. In *POPL'83*, pages 117–126. ACM, 1983.

12 E.A. Emerson and E.M. Clarke. Design and Synthesis of Synchronization Skeletons Using Branching-Time Temporal Logic. In *LP'81*, LNCS 131, pages 52–71. Springer, 1982.

13 E.A. Emerson and J.Y. Halpern. "Sometimes" and "Not Never" Revisited: On Branching Versus Linear Time. In *POPL'83*, pages 127–140. ACM, 1983.

14 E.A. Emerson and J.Y. Halpern. "Sometimes" and "Not Never" Revisited: On Branching Versus Linear Time. *JACM*, 33(1):151–178, 1986.

15 S. Feferman and R. Vaught. The First-Order Properties of Products of Algebraic Systems. *FM*, 47(1):57–103, 1959.

16 K. Fine. In So Many Possible Worlds. *NDJFL*, 13:516–520, 1972. $\mathsf{path}(doi: 10.1305/NDJFL/1093890715)$.

17 M.J. Fischer and R.E. Ladner. Propositional Dynamic Logic of Regular Programs. *JCSS*, 18(2):194–211, 1979. $\mathsf{path}(doi: 10.1016/0022-0000(79)90046-1)$.

18 G. De Giacomo and M.Y. Vardi. Linear Temporal Logic and Linear Dynamic Logic on Finite Traces. In *IJCAI'13*, pages 854–860. IJCAI' & AAAI Press, 2013.

19 Y. Gurevich. Modest Theory of Short Chains. I. *JSL*, 44(4):481–490, 1979. $\mathsf{path}(doi: 10.2307/2273287)$.

20 Y. Gurevich. Monadic Second-Order Theories. In *Model-Theoretical Logics*, pages 479–506. Springer, 1985.

21 Y. Gurevich and S. Shelah. Modest Theory of Short Chains. II. *JSL*, 44(4):491–502, 1979. $\mathsf{path}(doi: 10.2307/2273288)$.

22 Y. Gurevich and S. Shelah. Rabin's Uniformization Problem. *JSL*, 48(4):1105–1119, 1979.

23 Y. Gurevich and S. Shelah. The Decision Problem for Branching Time Logic. *JSL*, 50(3):668–681, 1985. $\mathsf{path}(doi: 10.2307/2274321)$.

24 T. Hafer and W. Thomas. Computation Tree Logic CTL* and Path Quantifiers in the Monadic Theory of the Binary Tree. In *ICALP'87*, LNCS 267, pages 269–279. Springer, 1987. $\mathsf{path}(doi: 10.1007/3-540-18088-5_22)$.

25 D. Janin. *A Contribution to Formal Methods: Games, Logic and Automata*. Habilitation thesis, Université Bordeaux I, Bordeaux, France, 2005.

26 D. Janin and G. Lenzi. On the Relationship Between Monadic and Weak Monadic Second Order Logic on Arbitrary Trees, with Applications to the mu-Calculus. *FI*, 61(3-4):247–265, 2004. URL: `http://content.iospress.com/articles/fundamenta-informaticae/fi61-3-4-04`.

27 D. Janin and I. Walukiewicz. On the Expressive Completeness of the Propositional mu-Calculus with Respect to Monadic Second Order Logic. In *CONCUR'96*, LNCS 1119, pages 263–277. Springer, 1996. $\mathsf{path}(doi: 10.1007/3-540-61604-7_60)$.

28 H.W. Kamp. *Tense Logic and the Theory of Linear Order*. PhD thesis, University of California, Los Angeles, CA, USA, 1968.

29 D. Kozen. Results on the Propositional muCalculus. *TCS*, 27(3):333–354, 1983. path(doi : 10.1016/0304 − 3975(82)90125 − 6).
30 R.E. Ladner. Application of Model Theoretic Games to Discrete Linear Orders and Finite Automata. *IC*, 33(4):281–303, 1977. path(doi : 10.1016/S0019 − 9958(77)90443 − 0).
31 H. Läuchli. A Decision Procedure for the Weak Second-Order Theory of Linear Order. In *LC'66*, volume 50, pages 189–197. North-Holland, 1968.
32 O. Maler and A. Pnueli. On the Cascaded Decomposition of Automata, its Complexity, and its Application to Logic. Unpublished, 1995.
33 Z. Manna and A. Pnueli. *The Temporal Logic of Reactive and Concurrent Systems - Specification.* Springer, 1992.
34 Z. Manna and A. Pnueli. *Temporal Verification of Reactive Systems - Safety.* Springer, 1995.
35 R. McNaughton. Testing and Generating Infinite Sequences by a Finite Automaton. *IC*, 9(5):521–530, 1966. path(doi : 10.1016/S0019 − 9958(66)80013 − X).
36 R. McNaughton and S. Papert. *Counter-Free Automata.* MIT Press, 1971.
37 F. Moller and A.M. Rabinovich. On the Expressive Power of CTL*. In *LICS'99*, pages 360–368. IEEECS, 1999.
38 F. Moller and A.M. Rabinovich. Counting on CTL*: On the Expressive Power of Monadic Path Logic. *IC*, 184(1):147–159, 2003. path(doi : 10.1016/S0890 − 5401(03)00104 − 4).
39 D. Perrin and J. Pin. First-Order Logic and Star-Free Sets. *JCSS*, 32(3):393–406, 1986. path(doi : 10.1016/0022 − 0000(86)90037 − 1).
40 A. Pnueli. The Temporal Logic of Programs. In *FOCS'77*, pages 46–57. IEEECS, 1977.
41 A. Pnueli. The Temporal Semantics of Concurrent Programs. *TCS*, 13:45–60, 1981. path(doi : 10.1016/0304 − 3975(81)90110 − 9).
42 S. Safra. On the Complexity of ω-Automata. In *FOCS'88*, pages 319–327. IEEECS, 1988.
43 S. Shelah. The Monadic Theory of Order. *AM*, 102(3):379–419, 1975.
44 W. Thomas. Star-Free Regular Sets of ω-Sequences. *IC*, 42(2):148–156, 1979.
45 W. Thomas. A Combinatorial Approach to the Theory of ω-Automata. *IC*, 48(3):261–283, 1981.
46 W. Thomas. Logical Aspects in the Study of Tree Languages. In *CAAP'84*, pages 31–50. CUP, 1984.
47 W. Thomas. On Chain Logic, Path Logic, and First-Order Logic over Infinite Trees. In *LICS'87*, pages 245–256. IEEECS, 1987.
48 W. Thomas. Automata on Infinite Objects. In *Handbook of Theoretical Computer Science (vol. B)*, pages 133–191. MIT Press, 1990.
49 J. van Benthem. *Modal Correspondence Theory.* PhD thesis, University of Amsterdam, Amsterdam, Netherlands, 1977.
50 M.Y. Vardi and P. Wolper. Yet Another Process Logic. In *LP'83*, LNCS 164, pages 501–512. Springer, 1984.
51 I. Walukiewicz. Monadic Second Order Logic on Tree-Like Structures. *TCS*, 275(1-2):311–346, 2002. path(doi : 10.1016/S0304 − 3975(01)00185 − 2).
52 P. Wolper. Temporal Logic Can Be More Expressive. *IC*, 56(1-2):72–99, 1983. path(doi : 10.1016/S0019 − 9958(83)80051 − 5).