A Framework for Assessing Inconsistency in Disjunctive Temporal Problems

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Abstract

Inconsistency measures serve to quantify the level of contradiction present within a knowledge base. They can be used for both consistency restoration and information extraction. In this article, we specifically explore inconsistency measures applicable to Disjunctive Temporal Problems (DTPs). We present a framework that extends traditional propositional logic approaches to DTPs, incorporating both new postulates and adaptations of existing ones. We identify and elaborate on various properties that establish relationships among these postulates. Furthermore, we introduce multiple inconsistency measures, adopting both a conventional approach that particularly leverages Minimal Inconsistent Subsets and a DTP-specific strategy based on constraint relaxation. Finally, we show the applicability of the inconsistency measures in DTPs through two real-world applications.

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1 Introduction

Numerous formalisms have been proposed in the literature for representing and reasoning about temporal information under constraints. The temporal constraints considered by these formalisms differ in two primary aspects. First, the types of temporal entities represented by their variables, which can include temporal points, intervals, or even durations. Second, their nature can be qualitative [1], quantitative, or a combination of both [11, 5]. Simple Temporal Problems (STPs) [7] belong to the temporal formalisms allowing to handle quantitative constraints. They represent temporal entities as points on a timeline and allow constraining distance between each pair of temporal entities using numeric values specified by an interval. To increase the expressiveness of the considered temporal constraints, STPs have been extended numerous times [12, 20]. In particular, Disjunctive Temporal Problems (DTPs) [18] extend STPs by employing disjunctions of STP constraints, thus providing a temporal framework highly useful in a wide range of applications.

In the literature, an inconsistency measure is defined as a function that assigns a non-negative value to each knowledge base. It quantifies the degree of conflict or contradiction present within the database, offering an interesting tool for evaluating and managing inconsistencies (e.g. see [10, 19, 13, 9]). In the realm of application, these measures are used in various analytical reasoning approaches. For instance, in the data mining task of clustering, inconsistency measures are utilized to enhance the quality of clusters by actively reducing contradictions [14]. Furthermore, these measures are used as a stepping stone for defining paraconsistent consequence relations, which allow for logical deduction in the presence of inconsistent information [15].

The literature presents a wide range of proposals for defining inconsistency measures, aimed at identifying and addressing various forms of conflict, highlighting the richness of this research field (e.g., see [10, 8, 3, 2, 4]). Numerous studies on inconsistency measures have adopted a

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postulate-based approach by reasoning about various dimensions related to the measurement and management of inconsistency. This approach standardizes assessment criteria and facilitates a comprehensive understanding of the underlying causes of inconsistency.

Despite extensive research, there remains a significant gap in the exploration of inconsistency measurement within the realm of temporal reasoning. To the best of our knowledge, only a few studies have specifically focused on adapting inconsistency measures to this context, concentrating primarily on qualitative spatio-temporal reasoning and temporal logic (see [6, 17, 16]). In this paper, we introduce the first framework designed specifically for measuring inconsistency in DTPs.

Our first contribution consists in introducing a range of rationality postulates for defining inconsistency measures in DTPs. Some of these postulates are adaptations from those established in the propositional context, while others are uniquely tailored to DTPs. For example, one DTP-specific postulate asserts that applying an identical shift to all intervals within a DTP does not alter the amount of inconsistency. We also examine the relationships among these postulates. Our analysis particularly highlights that certain postulates are incompatible, and combining specific ones can yield an inconsistency measure that can only distinguish between consistent and inconsistent DTPs.

Our second contribution is the development of various inconsistency measures using different approaches. Specifically, we employ a traditional approach that involves Minimal Inconsistent Subsets, and we introduce a strategy specifically tailored for DTPs based on constraint relaxation. This relaxation is achieved by widening the temporal intervals.

Our third contribution details two applications of inconsistency measures within DTPs. The core principle of our approach involves using these measures to facilitate the selection of the most suitable solutions. Within a DTP framework, constraints may correspond to either the specific needs of an individual agent or the integrity constraints of a computational service. By applying inconsistency measures, we are able to identify the optimal service or achieve consensus among agents.

2 Preliminaries

2.1 Temporal problems

In the sequel we will denote by $\mathcal{I}^{\mathbb{Z}}$ the set of closed (possibly half-unbounded or unbounded) intervals over $\mathbb{Z}$ with endpoints in $\mathbb{Z} \cup \{-\infty, +\infty\}$. Given $I \in \mathcal{I}^{\mathbb{Z}}$, I^{-1} will denote the interval of $\mathcal{I}^{\mathbb{Z}}$ containing the opposite values of I. We consider Disjunctive Temporal Problems (DTP) [18] on $\mathcal{I}^{\mathbb{Z}}$.

▶ **Definition 1** (Disjunctive Temporal Problem (DTP)).

- *A temporal constraint c is a disjunction $x_1 - y_1 \in I_1 \vee \ldots \vee x_k - y_k \in I_k$ where $k \geq 1$, $x_1, \ldots, x_k, y_1, \ldots, y_k$ are temporal variables with domain $\mathbb{Z}$ and $I_1, \ldots, I_k$ are intervals belonging to $\mathcal{I}^{\mathbb{Z}}$.*
- *A DTP D is a pair (V, C) where $V = \{x_1, \ldots, x_n\}$ is a finite set of temporal variables ranging over $\mathbb{Z}$ and $C = \{c_1, \ldots, c_m\}$ is a set of temporal constraints involving V where $n \geq 1$ and $m \geq 1$.*
- *A solution σ of a DTP $D = (V, C)$ is an assignment of integer numbers to the variables in V such that all constraints in C are satisfied. More formally, a solution σ of D is a function $\sigma : V \to \mathbb{Z}$ such that for each $c \in C$ there exists at least one disjunct $x - y \in I$ belonging to c such that the value $\sigma(x) - \sigma(y)$ belongs to the interval I.*
- *A DTP admitting at least one solution will be said consistent. In the contrary case it will said inconsistent.*

Let $\mathcal{DTP}$ represent the set of all DTPs. In the sequel, a temporal constraint (resp. a disjunct of a temporal constraint) will also be called a temporal clause (resp. a temporal literal).

Given a DTP $D = (V, C)$ and a constraint $c \in C$, we sometimes use $D \setminus \{c\}$ to denote the DTP $(V, C \setminus \{c\})$.

A Simple Temporal Problem (STP) [7] is a specific type of DTP characterized by each constraint containing exactly one disjunct. Furthermore, a Temporal Constraint Satisfaction Problem (TCSP) [7] represents a DTP in which all disjuncts within a constraint apply to the same pair of variables.

Given a set of temporal variables V, the set of all possible assignments of integer numbers to the variables in V will be denoted by $[\![V]\!]$. Moreover, the subset of $[\![V]\!]$ corresponding to the set of solutions of a DTP $D = (V, C)$ will be denoted by $\mathsf{sols}(D)$.

Note that here we consider DTPs with constraints on closed integer intervals and interpret their variables with integer values. Even if it may seem restrictive, most of the concepts and results introduced later can be extended to more general DTPs.

Consider a temporal constraint $c = x_1 - y_1 \in I_1 \vee \ldots \vee x_k - y_k \in I_k$. The notation $\mathsf{vars}(c)$ refers to the set of temporal variables involved in c, *i.e.*, the set of variables $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$. Furthermore, $\mathsf{Lit}(c)$ denotes the disjuncts (temporal literals) of the constraint of c. Moreover, given another temporal constraint c', c subsumes c' if and only if (i) $\mathsf{vars}(c) \subseteq \mathsf{vars}(c')$, and (ii) for all $(x - y \in I) \in \mathsf{Lit}(c)$, there exists $(x - y \in I')$ such that $I \subseteq I'$. A sub-DTP of a DTP $D = (V, C)$ is a DTP (V, C') such that $C' \subseteq C$. Given two DTPs $D = (V, C)$ and $D' = (V', C')$, the union of D and D', denoted by $D \cup D'$, is the DTP defined by $(V \cup V', C \cup C')$.

▶ **Example 2.** As illustration, consider the set of temporal variables $V = \{x_1, x_2, x_3, x_4, x_5\}$ and the following set of temporal constraints C:

$c_1 = x_1 - x_2 \in [4, 7] \vee x_2 - x_3 \in [-2, 2] \vee x_2 - x_4 \in [0, 8]$,
$c_2 = x_3 - x_4 \in [-20, 20]$,
$c_3 = x_1 - x_2 \in [-15, -10] \vee x_2 - x_3 \in [8, 12]$,
$c_4 = x_1 - x_3 \in [-11, -8] \vee x_2 - x_4 \in [-6, -3]$,
$c_5 = x_1 - x_3 \in [10, 12] \vee x_1 - x_4 \in [6, 7]$,
$c_6 = x_1 - x_2 \in [-11, -6] \vee x_1 - x_3 \in [6, 9]$,
$c_7 = x_2 - x_5 \in [5, 10] \vee x_3 - x_5 \in [0, 5]$,
$c_8 = x_1 - x_3 \in [0, 3] \vee x_1 - x_4 \in [12, 14]$.

Let the DTPs $D = (V, C)$ and $D' = (V, C \setminus \{c_8\})$. The DTP D' is consistent, whereas the DTP D is inconsistent. A solution σ of D' is given by the following assignment: $\sigma(x_1) = 10$, $\sigma(x_2) = \sigma(x_3) = 21$, $\sigma(x_4) = 4$ and $\sigma(x_5) = 16$. This assignment satisfies, for example, the temporal literal $x_1 - x_3 \in [-11, -8]$ of the constraint c_4 since $\sigma(x_1) - \sigma(x_3) = 10 - 21 = -11$.

2.2 Relaxations

An *c-rewriting rule* for a temporal clause c is a function μ mapping from $\mathsf{Lit}(c)$ (the set of temporal literals in c) to $\mathcal{I}^{\mathbb{Z}}$. This function μ is applied to a temporal constraint c such that $\mu(c)$ denotes the temporal clause resulting from replacing each literal $l = x - y \in I$ within c with $x - y \in \mu(l)$.

A *local c-transformation* for a DTP instance $D = (V, C)$, where $C = \{c_1, \ldots, c_n\}$, is a function λ that assigns an c-rewriting rule μ_i to each constraint c_i in C. This transformation is applied to D such that $\lambda(D)$ results in a new DTP instance (V, C'), where $C' = \bigcup_{i=1}^{n} \{\mu_i(c_i)\}$. In the sequel, by abuse of notation, we will sometimes denote $(\lambda(c_i))(c_i)$ by $\lambda(c_i)$ for notational convenience.

A *local c-relaxation* for a DTP instance $D = (V, C)$ is defined as a local c-transformation λ, where for each temporal constraint $c = x_1 - y_1 \in I_1 \vee \ldots \vee x_k - y_k \in I_k$ in C, the transformation ensures that $I_i \subseteq (\lambda(c))(x_i - y_i \in I_i)$ for every $i \in \{1, \ldots, k\}$. This transformation effectively relaxes the constraints, making them less restrictive.

Given two intervals $I = [l, u]$ and $I' = [l', u']$ with $I \subseteq I'$, we use $\delta(I, I')$ to denote the value $(u' - l') - (u - l)$. We extend δ to the cases where I or I' are half-unbounded or unbounded intervals in the following way: $\delta(I, I') = +\infty$ for the cases where I is left-bounded (resp. right-bounded) and I' is left-unbounded (resp. left-unbounded). Intuitively, this means that the value of infinity dominates in this scenario. In the case where $I =]-\infty, u]$ and $I' =]-\infty, u']$ (resp. $I = [l, +\infty[$ and $I' = [l', +\infty[$), $\delta(I, I')$ is defined by $u' - u$ (resp. $l - l'$). For the last case which corresponds to $I = I' =]-\infty, +\infty[$, $\delta(I, I')$ is defined by 0.

Let λ be a local c-relaxation of $D = (V, C)$, we use $\omega(\lambda)$ to denote the following value:

$$\sum_{c \in C} \quad \sum_{l=(x-y \in I) \in \mathsf{Lit}(c)} \delta(I, (\lambda(c))(l))$$

Furthermore, we use $\theta(\lambda)$ to denote the following value:

$$\max_{c \in C, l=(x-y \in I) \in \mathsf{Lit}(c)} \delta(I, (\lambda(c))(l))$$

▶ **Example 3.** Consider again the DTP $D = (V, C)$ defined in Example 2. Let λ be a local c-transformation for D that assigns the c-rewriting rule μ_i to the constraint c_i in C, with the rules μ_i defined as follows:
$\mu_1(x_1 - x_2 \in [4, 7]) = [4, 10]$, $\mu_1(x_2 - x_3 \in [-2, 2]) = [-4, 3]$, $\mu_1(x_2 - x_4 \in [0, 8]) = [0, 8]$;
$\mu_2(x_3 - x_4 \in [-20, 20]) = [-30, 40]$;
$\mu_5(x_1 - x_3 \in [10, 12]) = [0, 20]$, $\mu_5(x_1 - x_4 \in [6, 7]) = [2, 15]$;
and $\mu_i(x - y \in I) = I$ for each $c_i \in \{c_3, c_4, c_6, c_7, c_8\}$ and each temporal $x - y \in I$ belonging to $\mathsf{Lit}(c_i)$.
We have $\lambda(c_1) = x_1 - x_2 \in [4, 10] \vee x_2 - x_3 \in [-4, 3] \vee x_2 - x_4 \in [0, 8]$, $\lambda(c_2) = x_3 - x_4 \in [-30, 40]$, $\lambda(c_5) = x_1 - x_3 \in [0, 20] \vee x_1 - x_4 \in [2, 15]$ and $\lambda(c_i) = c_i$ for all $c_i \{c_3, c_4, c_6, c_7, c_8\}$.

Clearly, the local c-transformation λ is a local c-relaxation of D. It results in the less constraining DTP $\lambda(D) = (V, \{\lambda(c_1), \lambda(c_2), \lambda(c_5)\} \cup \{c_3, c_4, c_6, c_7, c_8\})$. We can notice that $\lambda(D)$ is consistent, whereas D is an inconsistent DTP. A solution σ of $\lambda(D)$ is given by the following assignment: $\sigma(x_1) = 20$, $\sigma(x_2) = \sigma(x_3) = 31$, $\sigma(x_4) = 8$ and $\sigma(x_5) = 26$. Moreover, we have $\omega(\lambda) = 3 + 3 + 30 + 18 + 12 = 66$ and $\theta(\lambda) = max\{3, 3, 30, 18, 12\} = 30$.

In the following, we show that to achieve a consistent DTP from an inconsistent DTP, we can restrict ourselves to local c-relaxations that extend the interval of at most one temporal literal per temporal clause by modifying only one of its bounds. We also show that such a restriction preserves optimal relaxations with respect to minimizing the values generated by the functions $\omega(.)$ et $\theta(.)$.

▶ **Proposition 4.** *Let $D = (V, C)$ be a DTP and a local c-relaxation λ of D. If $\lambda(D)$ is a consistent DTP then there exists a local c-relaxation λ' of D such that:*
(1) *$\lambda'(D)$ is a consistent DTP,*
(2) *for each $c \in C$, we have $|\{l = (x - y \in I) \in \mathsf{Lit}(c) : I \neq (\lambda'(c))(l)\}| \leq 1$,*
(3) *for each $c \in C$ and $l = (x - y \in I) \in \mathsf{Lit}(c)$, we have $I = (\lambda'(c))(l)$ or $(\lambda'(c))(l) \setminus I$ is a bounded interval of $\mathcal{I}^{\mathbb{Z}}$,*
(4) *we have $\theta(\lambda) \geq \theta(\lambda')$ and $\omega(\lambda) \geq \omega(\lambda')$.*

Proof. Suppose that $\lambda(D)$ is a consistent DTP. Let σ a solution of $\lambda(D)$. From σ we will define a local c-relaxation λ' with the desired properties. As σ is a solution of $\lambda(D)$, we know that for each $c \in C$ there exists at least one temporal literal $l_c = (x - y \in I) \in \mathsf{Lit}(c)$ such that $\sigma(x) - \sigma(y) \in (\lambda(c))(l_c)$. Select such a temporal literal $l_c = (x - y \in I)$ and define $\lambda'(c)$ by $(\lambda'(c))(l_c) = I$ if $(\lambda(c))(l_c) = I$ or $\sigma(x) - \sigma(y) \in I$, by the smallest interval of $\mathcal{I}^{\mathbb{Z}}$ containing the values of I and $\sigma(x) - \sigma(y)$ in the contrary case. Moreover, we define $(\lambda'(c))(u - v \in I')$ by I' for all $(u - v \in I') \in \mathsf{Lit}(c) \setminus \{l_c\}$. Remark that for each $c \in C$, we have $\sigma(x) - \sigma(y) \in (\lambda'(c))(l_c)$, by consequence σ is a solution of $\lambda'(D)$. Hence Property (1) is satisfied. Moreover, by construction of λ', we can assert that the properties (2) and (3) are also satisfied. Always by construction, we can observe that $I \subseteq (\lambda'(c))(l) \subseteq (\lambda(c))(l)$ for all $c \in C$ and $l = (x - y \in I) \in \mathsf{Lit}(c)$. It follows that $\delta(I, (\lambda'(c))(l)) \leq \delta(I, (\lambda(c))(l))$ for all $c \in C$ and $l = (x - y \in I) \in \mathsf{Lit}(c)$. As a result, we have $\theta(\lambda') \leq \theta(\lambda)$ and $\omega(\lambda') \leq \omega(\lambda)$. ◀

In the sequel, given a DTP D, the set of the local c-relaxations λ of D such that $\lambda(D)$ is consistent will be denoted by $\mathsf{LCR}(D)$.

▶ **Example 5.** Consider the DTP $D = (V, C)$ defined in Example 2 and its local c-relaxation λ given in Example 3 with the solution σ of $\lambda(D)$. By following the approach described in the proof of Proposition 4, we can construct a new local c-relaxation λ' from λ and σ that satisfies the properties specified in Proposition 4. The resulting local c-relaxation λ' assigns the c-rewriting rule μ'_i to the constraint c_i in C in the following way:
$\mu'_2(x_3 - x_4 \in [-20, 20]) = [-20, 23]$;
$\mu'_5(x_1 - x_3 \in [10, 12]) = [10, 12]$, $\mu'_5(x_1 - x_4 \in [6, 7]) = [6, 12]$;
and $\mu'_i(x - y \in I) = I$ for each $c_i \in \{c_1, c_3, c_4, c_6, c_7, c_8\}$ and each temporal $x - y \in I$ belonging to $\mathsf{Lit}(c_i)$.

We have $\lambda'(c_2) = x_3 - x_4 \in [-20, 23]$, $\lambda'(c_5) = x_1 - x_3 \in [10, 12] \vee x_1 - x_4 \in [6, 12]$ and $\lambda'(c_i) = c_i$ for all $c_i \in \{c_1, c_3, c_4, c_6, c_7, c_8\}$. It is clear that λ' is a local c-relaxation of D ensuring that $\lambda'(D)$ is a consistent DTP with σ serving as a solution. Hence, we have $\lambda' \in \mathsf{LCR}(D)$. Moreover, we have $\omega(\lambda') = 3 + 5 = 8$ and $\theta(\lambda') = max\{3, 5\} = 5$, whereas $\omega(\lambda) = 66$ and $\theta(\lambda) = 30$.

3 Rationality Postulates for Inconsistency Measurement

In this section, we describe various rationality postulates that can be used for defining inconsistency measures in the context of DTPs. Many of these rationality postulates are adaptations of those introduced in the propositional case. Additionally, we highlight several interesting relationships between the considered postulates.

Before presenting our rationality postulates, we first outline the concepts used to express them.

▶ **Definition 6** (Minimal Inconsistent Sub-DTP (MIS)). *Let $D = (V, C)$ be a DTP. A Minimal Inconsistent Sub-DTP (MIS) of D is an inconsistent sub-DTP D' of D such that each sub-DTP D'' of D', with $D' \neq D''$, is consistent.*

A constraint c is said to be *free* in a DTP D if there exists no MIS $D' = (V, C)$ of D such that $c \in C$.

Consider now the dual concept of MIS.

▶ **Definition 7** (Maximal Consistent Sub-DTP (MCS)). *Let $D = (V, C)$ be a DTP. A Maximal Consistent Sub-DTP (MCS) of D is an consistent sub-DTP D' of D such that each sub-DTP D'' of D, with $D' \subsetneq D''$, is inconsistent.*

Clearly, a constraint is free if it belongs to all MCSes.

A constraint c is said to be *safe* in a DTP D if there is a variable $x \in \mathsf{vars}(c)$ such that $v \notin \mathsf{vars}(c')$ for every $c' \in C \setminus \{c\}$.

Let us note that our definition of safe constraints diverges from the concept of a safe formula as defined in [10]. Indeed, a safe formula is defined as one that shares no propositional variables with the remaining elements of the knowledge base.

We adopt the following notational conventions:

- $\mathsf{MIS}(D)$: the set of MISes of D,
- $\mathsf{MCS}(D)$: the set of MCSes of D,
- $\mathsf{Free}(D)$: the set of free constraints of D,
- $\mathsf{Safe}(D)$: the set of safe constraints of D.

▶ **Example 8.** Revisiting the DTP $D = (V, C)$ defined in Example 2, we observe the following:

- $\mathsf{MCS}(D) = \{(V, C \setminus \{c_5\}), (V, C \setminus \{c_8\})\}$,
- $\mathsf{MIS}(D) = \{(V, \{c_1, c_4, c_5, c_8\}), (V, \{c_1, c_5, c_6, c_8\}), (V, \{c_3, c_4, c_5, c_8\}), (V, \{c_4, c_5, c_6, c_8\})\}$,
- $\mathsf{Free}(D) = \{c_2, c_7\}$,
- $\mathsf{Safe}(D) = \{c_7\}$.

▶ **Proposition 9.** *Let D be a DTP. If c is a safe constraint in D, then c is free in D.*

Proof. Let $D = (V, C)$ and consider a constraint $c \in C$ which is identified as a safe but not free constraint in C. Given that c is not free, there exists a MIS $D' = (V, C')$ of D such that $c \in C'$. Since D' is an MIS, we can define $D'' = (V, C' \setminus \{c\})$, which admits a solution σ. Given the safety of c in D, it follows that there exists a variable $x \in \mathsf{vars}(c)$ that does not appear in any constraints of D''. Assuming without loss of generality that $x - y \in I$ is a temporal literal in c, we can identify a specific value v such that $v - \sigma(y) \in I$. We then define a new assignment σ' for D' as follows: $\sigma'(x) = v$, and for all $y \in V \setminus \{x\}$, $\sigma'(y) = \sigma(y)$. Since σ' satisfies c and σ satisfies D'', we conclude that σ' is a solution for D'. This leads to a contradiction, as D' being a MIS. ◀

Given a DTP D and an integer $k \in \mathbb{Z}$, we use $D \oplus k$ to denote the DTP obtained from D by replacing each interval $[l, u]$ with $[l + k, u + k]$.

▶ **Proposition 10.** *Let D be a DTP and $k \in Z$. Then D is consistent iff $D \oplus k$ is consistent.*

Proof. This is mainly a consequence of the fact that for every solution σ, the assignment σ' is a solution of $D \oplus k$, where $\sigma'(x) = \sigma(x) + k$ for every variable x. ◀

An inconsistency measure $\mathcal{I}$ is a function that maps a DTP to a non-negative real value. By denoting $\mathbb{R}^{\infty}_{\geq 0}$ the set of non-negative real value, an inconsistency measure is a function $\mathcal{I} : \mathcal{DTP} \rightarrow \mathbb{R}^{\infty}_{\geq 0}$ that satisfies the following property:

- $\mathcal{I}(D) = 0$ iff D is a consistent DTP (Consistency - Cons).

The property Cons stipulates that an inconsistency measure must distinguish between consistent and inconsistent DTPs.

In this work, many rationality postulates for defining inconsistency measures are analogous to those introduced in the propositional case (e.g., see [10, 19]). The considered postulates are as follows: for all DTPs $D = (V, C)$ and $D = (V', C')$,

- $\mathcal{I}(D \cup D') \geq \mathcal{I}(D)$ (Monotonicity - Mono).
- If $c \in C$ is a safe temporal constraint of D then $\mathcal{I}(D) = \mathcal{I}((V, C \setminus \{c\}))$ (Safe Constraint Independence - SCI).
- If $c \in C$ is a free temporal constraint of D then $\mathcal{I}(D) = \mathcal{I}((V, C \setminus \{c\}))$ (Free Constraint Independence - FCI).
- If $c \in C$ is not free in D then $\mathcal{I}(D) > \mathcal{I}((V, C \setminus \{c\}))$ (Problematic Constraint Dependence - PCD).

- If D' is consistent and $V \cap V' = \emptyset$, then $\mathcal{I}(D \cup D') = \mathcal{I}(D)$ (Sub-DTP Independence - SDI).
- If c subsumes c' then $\mathcal{I}((V \cup \mathsf{vars}(c), C \cup \{c\})) \geq \mathcal{I}((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$ (Subsumption - Sub).
- If c subsumes c' and $c \notin C$, then $\mathcal{I}((V \cup \mathsf{vars}(c), C \cup \{c\})) \geq \mathcal{I}((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$ (Weak Subsumption - WSub).
- If $V \cap V' = \emptyset$, then $\mathcal{I}(D \cup D') = \mathcal{I}(D) + \mathcal{I}(D')$ (Variable Independence-Additivity - VIA).
- If $C \cap C' = \emptyset$, then $\mathcal{I}(D \cup D') \geq \mathcal{I}(D) + \mathcal{I}(D')$ (Super-Additivity - SA).
- For any $k \in \mathbb{Z}$, $\mathcal{I}(D) = \mathcal{I}(D \oplus k)$ (Shift Independence - SI).

The postulate Mon asserts that adding a new constraint cannot decrease the existing level of contradiction within the DTP. SCI and FCI maintain that safe constraints and free constraints, respectively, do not influence the level of conflict. PCD says that introducing non-free constraints must increase the level of contradiction. SDI states that an independent, consistent sub-DTP does not affect the overall amount of contradiction. Sub postulates that stricter constraints results in more conflicts. WSub is a weaker variant of Sub; the condition $c \notin C$ allows us to indicate that c is replaced with c'. VIA asserts that the total contradiction in the union of two DTPs, which do not share any variables, equals the sum of their individual contradictions. SA says that the total amount of contradiction in two disjoint DTPs cannot be less than the sum of their individual contradictions. Finally, SI posits that applying a shift to a DTP does not alter the level of contradiction.

It is important to note that in a consistent DTP, introducing a constraint that is less restrictive than an existing one does not modify the solution set. This observation also explains why, in Sub as opposed to WSub, we do not require the condition $c \notin C$: adding a weaker constraint does not affect the amount of contradiction.

The properties previously described are not entirely independent and display various interrelationships. For example, it is evident that Sub implies WSub. The proposition below outlines additional relationships among these properties.

▶ **Proposition 11.** *The following properties hold:*
1. *FCI implies SCI.*
2. *SA implies Mono.*
3. *Cons and VIA implies SDI.*

Proof.

Property 1. It is a direct consequent of Proposition 9: every safe constraint is a free constraint.

Property 2. Let $\mathcal{I}$ be an inconsistency measure that satisfies SA. Let $D = (V, C)$ and $D' = (V', C')$ be two DTPs. We define a new DTP D'' as $D'' = (V', C' \setminus C)$. It follows that $D \cup D' = D \cup D''$. Furthermore, using property SA and noting that $D \cap D'' = \emptyset$, we deduce $\mathcal{I}(D \cup D') = \mathcal{I}(D \cup D'') \geq I(D) + I(D'')$. Consequently, we obtain $\mathcal{I}(D \cup D') \geq I(D)$.

Property 3. Let $\mathcal{I}$ be an inconsistency measure that satisfies both Cons and VIA. Let D and D' be two DTPs such that D' is consistent and $\mathsf{vars}(D) \cap \mathsf{vars}(D') = \emptyset$. Applying VIA under the condition that $\mathsf{vars}(D) \cap \mathsf{vars}(D') = \emptyset$, we deduce that $\mathcal{I}(D \cup D') = \mathcal{I}(D) + \mathcal{I}(D')$. Moreover, by invoking Cons, $\mathcal{I}(D') = 0$ holds. Therefore, this leads to $\mathcal{I}(D \cup D') = \mathcal{I}(D)$. ◀

As demonstrated by the following proposition, some of our properties are incompatible.

▶ **Proposition 12.** *There is no inconsistency measure that satisfies both PCD and Sub.*

Proof. Assume, for the sake of contradiction, that there exists an measure $\mathcal{I}$ that satisfies both PCD and Sub. Consider the DTP $D = (\{x, y\}, C)$ where $C = \{x - y \in [0, 1], x - y \in [2, 3]\}$. Clearly, the constraint $x - y \in [2, 3]$ subsumes $x - y \in [2, 4]$. Employing Sub, it follows that $\mathcal{I}(D') \geq \mathcal{I}(D'')$, with $D' = (\{x, y\}, C \cup \{x - y \in [2, 3]\})$ and $D'' = (\{x, y\}, C \cup \{x - y \in [2, 4]\})$. Since $C \cup \{x - y \in [2, 3]\} = C$, we have $\mathcal{I}(D') = \mathcal{I}(D)$. Furthermore, $x - y \in [2, 4]$ is clearly problematic in D''; hence, by applying PCD,we deduce that $\mathcal{I}(D) < \mathcal{I}(D'')$. This leads to a contradiction. ◀

Over-constraining inconsistency measures can lead to uninteresting results, as demonstrated by the following proposition.

▶ **Proposition 13.** *An inconsistency measure $\mathcal{I}$ satisfies Cons, Sub and SA iff $\mathcal{I}$ is defined as follows:*

$$\mathcal{I}(D) = \begin{cases} \infty & \text{if } D \text{ is inconsistent} \\ 0 & \text{otherwise} \end{cases}$$

Proof.

The If Part. First, we establish that $\mathcal{I}(D) = \infty$ if and only if D is inconsistent, which implies that $\mathcal{I}$ satisfies Cons.

The satisfaction of Sub follows from the observation: if c subsumes c' and $D = (V, C \cup \{c\})$ is a consistent DTP, then $\mathcal{I}(D) = \mathcal{I}((V \cup \mathsf{vars}(c'), C \cup \{c'\})) = 0$; if D is inconsistent, then $\mathcal{I}(D) = \infty \geq \mathcal{I}((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$.

For SA, we consider: (i) if $D \cup D'$ is consistent, then both D and D' are consistent, leading to $\mathcal{I}(D \cup D') = 0$ and $\mathcal{I}(D) + \mathcal{I}(D') = 0$; (ii) if $D \cup D'$ is inconsistent, then $\mathcal{I}(D \cup D') = \infty \geq \mathcal{I}(D) + \mathcal{I}(D')$.

The Only-If Part. Let $D = (V, C)$ be a DTP. If D is consistent, then by Cons, $\mathcal{I}(D) = 0$. Now consider that D is inconsistent. Define a mapping f that associates each constraint c in C with any two distinct constraints not in C by adding two incompatible literals: $c' = c \vee (x - y \in [l, l])$ and $c'' = c \vee (x - y \in [l + 1, l + 1])$, where x and y are arbitrary variables, and l is an integer chosen such that c' and c'' are not in C. Using Sub, it follows that $\mathcal{I}(D) = \mathcal{I}((V, C \cup \bigcup_{c \in C} f(c)))$. Additionally, by SA, $\mathcal{I}((V, C \cup \bigcup_{c \in C} f(c))) \geq \mathcal{I}(D) + \mathcal{I}(D')$ where $D' = (V, \bigcup_{c \in C} f(c))$. Given Cons and the inconsistency of D, $\mathcal{I}(D') > 0$. Therefore, if $\mathcal{I}(D) \neq \infty$, this leads to $\mathcal{I}(D) > \mathcal{I}(D)$, a contradiction. Consequently, we deduce $\mathcal{I}(D) = \infty$. ◀

The following proposition demonstrates the need for caution when allowing infinity as an inconsistency value.

▶ **Proposition 14.** *If $\mathcal{I}$ is an inconsistency measure that satisfies Cons, and there exists a DTP D such that $\mathcal{I}(D) = \infty$, then $\mathcal{I}$ does not satisfy PCD.*

Proof. Let $D = (V, C)$ be a DTP such that $\mathcal{I}(D) = \infty$. Consider c to be a non-free constraint within D. We define c' as a constraint not present in C but logically equivalent to c. This equivalence can be achieved by utilizing redundancy in temporal literals; for example, $c \vee l \equiv c \vee l \vee l$. Given that c is non-free in D, we obtain that c' is non-free in $D = (V, C \cup \{c'\})$. Applying PCD, we deduce that $\mathcal{I}(D) < \mathcal{I}(D')$. However, this leads to a contradiction since $\mathcal{I}(D) = \infty$. ◀

4 Inconsistency Measures

In this section, we present several inconsistency measures, each based on a different approach. Some measures are adaptations of those previously established in the propositional case, while others are developed by leveraging the concept of local c-relaxation.

The considered inconsistency measures are defined as follows:

- $\mathcal{I}_{\mathsf{mcs}}(D) = min\{|C'| : C' \subseteq C, (V, C \setminus C') \in \mathsf{MCS}(D)\}$
- $\mathcal{I}_{\mathsf{mis}}(D) = |\mathsf{MIS}(D)|$
- $\mathcal{I}_{\mathsf{p}}(D) = |C \setminus \mathsf{Free}(D)|$
- $\mathcal{I}_{\omega}(D) = min\{\omega(\lambda) : \lambda \in \mathsf{LCR}(D)\}$
- $\mathcal{I}_{\theta}(D) = min\{\theta(\lambda) : \lambda \in \mathsf{LCR}(D)\}$

The measure $\mathcal{I}_{\mathsf{mcs}}(D)$ quantifies the minimum number of constraints that must be removed to restore consistency. $\mathcal{I}_{\mathsf{mis}}(D)$ counts the total number of minimal inconsistent subsets within the DTP. $\mathcal{I}_{\mathsf{p}}(D)$ calculates the number of constraints in C that do not participate in any minimal inconsistent subset. $\mathcal{I}_{\omega}(D)$ measures the minimum weight of a local c-relaxation required to achieve consistency. $\mathcal{I}_{\theta}(D)$ determines the minimum width of a local c-relaxation necessary for restoring consistency.

Table 1 Properties of inconsistency measures. ✓ means "satisfies" and ✗ means "does not satisfy".

Measure	Cons	Mono	SCI	FCI	PCD	SDI	Sub	WSub	VIA	SA	SI
$\mathcal{I}_{\mathsf{mcs}}$	✓	✓	✓	✓	✗	✓	✗	✗	✓	✓	✓
$\mathcal{I}_{\mathsf{mis}}$	✓	✓	✓	✓	✓	✓	✗	✗	✓	✓	✓
$\mathcal{I}_{\mathsf{p}}$	✓	✓	✓	✓	✓	✓	✗	✗	✓	✓	✓
$\mathcal{I}_{\omega}$	✓	✓	✓	✗	✗	✓	✗	✓	✓	✓	✓
$\mathcal{I}_{\theta}$	✓	✓	✓	✗	✗	✓	✓	✓	✗	✗	✓

In Table 1, we present the properties satisfied by each considered measure.

The initial observation is that all our measures uphold the properties Cons, Mono, SCI, SDI, and SI. Cons is fulfilled for $\mathcal{I}_{\mathsf{mcs}}$ because D is the unique MCS of D if and only if D is consistent; For $\mathcal{I}_{\mathsf{mis}}$ and $\mathcal{I}_{\mathsf{p}}$, it is satisfied as a consistent DTP has no Minimal MIS; in the cases of $\mathcal{I}_{\omega}$ and $\mathcal{I}_{\theta}$, no local c-relaxation is required to achieve consistency in consistent DTPs. Mono is observed in $\mathcal{I}_{\mathsf{mcs}}$ since adding a constraint cannot reduce the number of constraints needed to be ignored for consistency; it holds for $\mathcal{I}_{\mathsf{mis}}$ and $\mathcal{I}_{\mathsf{p}}$ as adding a constraint does not eliminate any existing MIS; for $\mathcal{I}_{\omega}$ and $\mathcal{I}_{\theta}$, any sub-DTP of a consistent DTP remains consistent, which means a local c-relaxation that leads to consistency after adding a constraint will also lead to consistency when applied to the DTP prior to the addition. SCI is met as the safe constraints are not involved in any conflicts, notably, they do not require relaxation to achieve consistency. SI is applicable for $\mathcal{I}_{\mathsf{mcs}}$, $\mathcal{I}_{\mathsf{mis}}$, and $\mathcal{I}_{\mathsf{p}}$ since applying a shift does not alter the MCSes and MISes of a DTP; for $\mathcal{I}_{\omega}$ and $\mathcal{I}_{\theta}$, it is mainly because the value $\delta(l, l')$ remains unchanged when the same shift is applied to the literals l and l'.

The second observation is that the property Sub is satisfied exclusively by the measure $\mathcal{I}_{\theta}$. A main reason why this property is not met by the other measures stems from its implication that adding an subsumed constraint should not alter the inconsistency value. However, in the case of the first four measures, adding such a constraint can impact the situation by introducing new MISes and necessitating the relaxation of the newly added subsumed constraint. In contrast, $\mathcal{I}_{\theta}$ handles the addition of a subsumed constraint without issue, as a local c-relaxation leading to consistency does not require widening intervals after incorporating a subsumed constraint.

It should be noted that the inconsistency measure $\mathcal{I}_\theta$ satisfies the weaker variant WSub. This is because any local c-relaxation in a DTP that achieves consistency will also maintain consistency when any constraint is replaced by one of its subsumed constraints.

A key distinction between the measures $\mathcal{I}_{\mathsf{mcs}}$, $\mathcal{I}_{\mathsf{mis}}$, and $\mathcal{I}_{\mathsf{p}}$ on one hand, and $\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ on the other, is that the latter two incorporate internal information from the constraints. This specifically accounts for why both $\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ do not satisfy FCI.

▶ **Proposition 15.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ do not satisfy FCI.*

Proof. Let be the DTP $D = (V = \{x_1, x_2, x_3\}, C = \{c_1, c_2, c_3, c_4\})$ defined by:

- $c_1 = x_1 - x_2 \in [5, 5] \vee x_1 - x_2 \in [20, 20]$,
- $c_2 = x_1 - x_2 \in [10, 10] \vee x_1 - x_2 \in [21, 21]$,
- $c_3 = x_2 - x_3 \in [0, 0]$,
- $c_4 = x_1 - x_3 \in [5, 10]$.

D is inconsistent and admits as unique MIS $(V, \{c_1, c_2\})$. Hence, $\mathsf{Free}(D) = \{c_3, c_4\}$. We have $\mathcal{I}_\omega(D) = 5$ and $\mathcal{I}_\theta(D) = 3$. Moreover, we can easily see that have $\mathcal{I}_\omega((V, C \setminus \{c_4\}) = \mathcal{I}_\theta((V, C \setminus \{c_4\}) = 1$. Consequently, $\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ do not satisfy the postulate FCI. ◀

▶ **Theorem 16.** *The functions listed in Table 1 are inconsistency measures that satisfy the properties outlined in the same table.*

5 Applications

In this section, we explore two applications of inconsistency measures. The core concept involves using these measures to select optimal solutions. Constraints in a DTP may represent the requirements of an individual agent or the integrity constraints of a computational service (e.g., $op1 - op2 \in]-\infty, 10]$ can be used to represent the constraint that operation op_2 must start no less than 10 time units after operation op_1). When conflicts arise either between the constraints of different agents or between the constraints of an agent and a service, inconsistency measures are employed to identify the most suitable resolution.

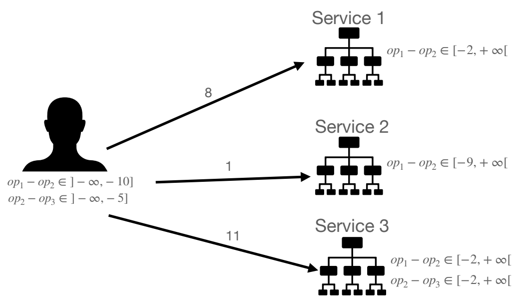

Figure 1 Scenario depicting a service selection problem.

5.1 Service Selection

In the first application, we address the scenario where an agent with specific temporal constraints needs to select computational services to perform a set of operations. These services come with their own integrity constraints, which are also temporal in nature. We formally represent this situation with the tuple $\Omega = \langle V, C, S, f\rangle$,where V is a set of temporal variables, C is a finite set of temporal constraints over V (the constraints of the considered agent), S is a set of computation services, and f is a function that assigns each service a finite set of temporal constraints (reflecting its integrity constraints).

Algorithm 1 Consensus Achievement via Constraint Modification.

```
procedure AchieveConsensus(V, A, f, 𝓘 )                    ▷ with A = {a_1, ..., a_k}
    D_ξ ← (V, ⋃_{a∈A} f(a))                                ▷ Define the initial DTP
    i ← 1
    d ← 𝓘(D_ξ)
    d_0 ← d
    while D_ξ does not admit a solution do
        D_t ← D_ξ
        for each constraint c in f(a_i) do
            D' ← D_ξ \ {c}
            if I(D') < d then
                D_t ← D'
                d ← I(D')
            end if
        end for
        D_ξ ← D_t                                          ▷ Update the DTP
        if i = k then
            if d = d_0 then        ▷ no constraint reduces the amount of contradiction
                D_ξ ← D_ξ \ {c}           ▷ c is an arbitrary constraint in D_ξ
                d ← I(D_ξ)
            end if
            d_0 ← d
            i ← 1
        else
            i ← i + 1
        end if
    end while
    return D_ξ
end procedure
```

In scenarios where the constraints of the considered agent clash with the integrity constraints of the computational services, inconsistency measures can be used to identify the most suitable service. This is achieved by computing $\mathcal{I}((V, C \cup f(s)))$ for each service $s \in S$. The inconsistency measures offer a quantitative assessment of the conflict severity between an agent's requirements and a service's constraints, facilitating an informed decision-making process.

For example, take the service selection problem illustrated in Figure 1. By employing the inconsistency measure $\mathcal{I}_\omega$, the second service is identified as the most appropriate choice. This service exhibits fewer contradictions (1) with the constraints of the agent compared to others (8 and 11).

5.2 Multi-Agent Consensus

In this section, we explore an application of inconsistency measures to DTPs, aimed at facilitating consensus within a multi-agent system. This involves using these measures to strategically guide the modification of constraints, thus driving the system towards consensus.

We define a *consensus problem* as a tuple $\xi = \langle V, A, f \rangle$, where V is a set of temporal variables, A is a set of agents, and f is a function that assigns each agent in A a set of temporal constraints. We consider that there is a *consensus* if the DTP $D_\xi = (V, \bigcup_{a \in A} f(a))$ admits a solution.

Our approach to achieving consensus involves proposing that each agent, sequentially, weaken or remove one of its constraints. The critical aspect of this strategy is to provide agents with guidance on which modifications will bring them closest to consensus. This is where the role of inconsistency measures becomes crucial. More precisely, consider $\mathcal{I}$ as the inconsistency measure in use, with $\xi = \langle V, A, f \rangle$ representing a consensus problem, and a an agent in A. An ordering $\prec$ on the constraints in $f(a)$ can be defined as follows: $c \prec c'$ if and only if $\mathcal{I}((V, (f(a) \setminus \{c\}) \cup \bigcup_{a' \in A \setminus \{a\}} f(a'))) < \mathcal{I}((V, (f(a) \setminus \{c'\}) \cup \bigcup_{a' \in A \setminus \{a\}} f(a')))$.

In Algorithm 1, we outline a variant of our approach designed to systematically achieve consensus. This algorithm iteratively removes the most problematic constraint, as determined by the inconsistency measure. This systematic elimination is designed to gradually resolve conflicts and align the system towards a solution.

6 Conclusion and perspectives

In this paper, we introduced a framework for defining inconsistency measures in Disjunctive Temporal Problems (DTPs), marking three main contributions. First, we established rationality postulates that lay foundational criteria for these measures. Second, we developed various inconsistency measures using diverse approaches. Finally, we demonstrated the applicability of these measures through two real-world applications, which underscores their potential to improve reasoning in temporal tasks.

For future work, we plan to explore additional rationality postulates to further enhance our framework. Additionally, we aim to define and investigate more inconsistency measures, expanding our current set. We also intend to assess the computational complexity of these measures and implement them.

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7 Appendix

The complete proofs of the satisfaction or the non satisfaction of each postulate for the inconsistency measures $\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ are given in this appendix.

	$\mathcal{I}_\omega$	$\mathcal{I}_\theta$
Consistency (Cons)	✓(Proposition 17)	✓(Proposition 17)
Monotonicity (Mono)	✓(Proposition 18)	✓(Proposition 18)
Safe Constraint Independence (SCI)	✓(Proposition 20)	✓(Proposition 20)
Free Constraint Independence (FCI)	✗ (Proposition 15)	✗ (Proposition 15)
Problematic Constraint Dependence (PCD)	✗ (Proposition 21)	✗ (Proposition 21)
Sub-DTP Independence (SDI)	✓ (Proposition 22)	✓ (Proposition 22)
Subsumption (Sub)	✗ (Proposition 24)	✓ (Proposition 26)
Weak Subsumption (WSub)	✓ (Proposition 23)	✓ (Proposition 23)
VI-Additivity (VIA)	✓ (Proposition 29)	✗ (Proposition 27)
Super-Additivity (SA)	✓ (Proposition 28)	✗ (Proposition 27)
Shift Independence (SI)	✓ (Proposition 30)	✓ (Proposition 30)

▶ **Proposition 17.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfy Cons.*

Proof. Let $D = (V, C)$ be a DTP. Consider the particular local c-relaxation λ^D_{id} of D defined by $(\lambda^D_{id}(c))(x - y \in I) = I$ for all $c \in C$ and for all $(x - y \in I) \in \mathsf{Lit}(c)$. Clearly, we have $\lambda^D_{id}(D) = D$. Hence, $\lambda^D_{id} \in \mathsf{LCR}(D)$. Moreover, as $\omega(\lambda^D_{id}) = \theta(\lambda^D_{id}) = 0$ we can assert that $\mathcal{I}_\omega(D) = \mathcal{I}_\theta(D) = 0$.
Now, suppose that $\mathcal{I}_\omega(D) = 0$ or $\mathcal{I}_\theta(D) = 0$. We can assert that there exists a local c-relaxation $\lambda \in \mathsf{LCR}(D)$ such that $\omega(\lambda) = 0$ or $\theta(\lambda) = 0$. By definition of ω and θ it follows that for all $c \in C$ and for all $l = (x - y \in I) \in \mathsf{Lit}(c)$ we have $\delta(I, (\lambda(c))(l)) = 0$ and consequently, $I = (\lambda(c))(l)$. It results that $\lambda = \lambda^D_{id}$ and $\lambda(D) = D$. Furthermore, we know that $\lambda(D)$ is consistent since λ belongs to $\mathsf{LCR}(D)$. It results that D is also consistent. ◀

▶ **Proposition 18.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfy Mono.*

Proof. Let $D = (V, C)$ and $D' = (V', C')$ be two DTPs and $\lambda \in \mathsf{LCR}(D \cup D')$. Let λ' be the local c-transformation of D defined by $\lambda'(c) = \lambda(c)$ for all $c \in C$. Clearly, $\lambda'(c)$ is a local c-relaxation of D. Moreover, we have $\lambda'(D)$ which is consistent since $\lambda(D \cup D')$ is consistent and $\lambda'(D) \subseteq \lambda(D \cup D')$. Hence, λ' belongs to $\mathsf{LCR}(D)$. Consequently, we have $\omega(\lambda') \geq \mathcal{I}_\omega(D)$ and $\theta(\lambda') \geq \mathcal{I}_\theta(D)$. On the other hand, by construction of λ' we can notice that $\omega(\lambda) \geq \omega(\lambda')$ and $\theta(\lambda) \geq \theta(\lambda')$. Now, suppose that λ is such that $\omega(\lambda) = min\{\omega(\lambda'') : \lambda'' \in \mathsf{LCR}(D \cup D')\} = \mathcal{I}_\omega(D \cup D')$ (resp. such that $\theta(\lambda) = min\{\theta(\lambda'') : \lambda'' \in \mathsf{LCR}(D \cup D')\} = \mathcal{I}_\theta(D \cup D')$). As $\omega(\lambda) \geq \omega(\lambda')$ (resp. $\theta(\lambda) \geq \theta(\lambda')$) and $\omega(\lambda') \geq \mathcal{I}_\omega(D)$ (resp. $\theta(\lambda') \geq \mathcal{I}_\theta(D)$), we can conclude that $\mathcal{I}_\omega(D \cup D') \geq \mathcal{I}_\omega(D)$ (resp. $\mathcal{I}_\theta(D \cup D') \geq \mathcal{I}_\theta(D)$). ◀

▶ **Proposition 19.** *Let $D = (V, C)$ be a DTP and $c \in \mathsf{Safe}(D)$. D is a consistent DTP iff $(V, C \setminus \{c\})$ is a consistent DTP.*

Proof. Obviously, $(V, C \setminus \{c\})$ is a consistent DTP in the case where D is consistent. Now, suppose that $(V, C \setminus \{c\})$ is a consistent DTP and let us show that D is also consistent. Let σ a solution of $(V, C \setminus \{c\})$ and a variable $x \in \mathsf{vars}(c)$ which does not belong to $\mathsf{vars}(c')$ for all $c' \in C \setminus \{c\}$. We know that there exists in c a temporal literal of the form $x - y \in I$ (Case 1) or the form $y - x \in I$ (Case 2) with $y \in V$ and $I \in \mathcal{I}^{\mathbb{Z}}$. Let a value $a \in I$ and consider the assignment σ' of V defined by $\sigma'(u) = \sigma(u)$ for each $u \in V \setminus \{x\}$ and $\sigma'(x) = a + \sigma(y)$ if Case 1 occurs, $\sigma'(x) = \sigma(y) - a$ in the contrary case. Clearly, σ' satisfies the temporal constraints of C and is a solution of D. It results that D is consistent. ◀

▶ **Proposition 20.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfy SCI.*

Proof. Let $D = (V, C)$ be a DTP and $c \in C$ a safe temporal constraint of D. Let us prove that $\mathcal{I}_\omega(D) = \mathcal{I}_\omega((V, C \setminus \{c\}))$ and $\mathcal{I}_\theta(D) = \mathcal{I}_\theta((V, C \setminus \{c\}))$. As $D = (V, C \setminus \{c\}) \cup (V, \{c\})$, from Mono we have $\mathcal{I}_\omega(D) \geq \mathcal{I}_\omega((V, C \setminus \{c\}))$ and $\mathcal{I}_\theta(D) \geq \mathcal{I}_\theta((V, C \setminus \{c\}))$. Now, let us show that $\mathcal{I}_\omega(D) \leq \mathcal{I}_\omega((V, C \setminus \{c\}))$ and $\mathcal{I}_\theta(D) \leq \mathcal{I}_\theta((V, C \setminus \{c\}))$. Let $\lambda \in \mathsf{LCR}((V, C \setminus \{c\}))$ such that $\omega(\lambda) = min\{\omega(\lambda') : \lambda' \in \mathsf{LCR}((V, C \setminus \{c\}))\}$ (resp. such that $\theta(\lambda) = min\{\theta(\lambda') : \lambda' \in \mathsf{LCR}((V, C \setminus \{c\}))\}$). Consider the local c-relaxation λ' of D defined by $(\lambda'(c))(l) = I$ for all $l = (x - y \in I) \in \mathsf{Lit}(c)$ and $\lambda'(c') = \lambda(c')$ for all $c' \in C \setminus \{c\}$. Clearly, $\omega(\lambda') = \omega(\lambda)$ and $\theta(\lambda') = \theta(\lambda)$. Moreover, we can notice that c is a safe temporal constraint of $\lambda'(D) = (V, C')$ and $\lambda((V, C \setminus \{c\}) = (V, C' \setminus \{c\})$. From Proposition 19, it follows that $\lambda'(D)$ is consistent. Hence, $\lambda' \in \mathsf{LCR}(D)$. Consequently, we have $\mathcal{I}_\omega(D) \leq \omega(\lambda')$ and $\mathcal{I}_\theta(D) \leq \theta(\lambda')$. From this and the fact that $\omega(\lambda') = \omega(\lambda) = \mathcal{I}_\omega((V, C \setminus \{c\}))$ and $\theta(\lambda') = \theta(\lambda) = \mathcal{I}_\theta((V, C \setminus \{c\}))$, we can can assert that $\mathcal{I}_\omega(D) \leq \mathcal{I}_\omega((V, C \setminus \{c\}))$ and $\mathcal{I}_\theta(D) \leq \mathcal{I}_\theta((V, C \setminus \{c\}))$. We can conclude that $\mathcal{I}_\omega(D) = \mathcal{I}_\omega((V, C \setminus \{c\}))$ and $\mathcal{I}_\theta(D) = \mathcal{I}_\theta((V, C \setminus \{c\}))$. ◀

▶ **Proposition 21.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ do not satisfy PCD.*

Proof. Let $D = (V = \{x_1, x_2, x_3\}, C = \{c_1, c_2, c_3\})$ be the DTP defined by:

- $c_1 = x_1 - x_2 \in [5, 5]$,
- $c_2 = x_1 - x_2 \in [15, 15]$,
- $c_3 = x_1 - x_2 \in [10, 10]$.

D is an inconsistent DTP. Moreover, we have $\mathsf{MIS}(D) = \{\{c_1, c_2\}, \{c_1, c_3\}, \{c_2, c_3\}\}$ and $\mathsf{Free}(D) = \emptyset$. On the other hand, we have $\mathcal{I}_\omega(D) = 10$ and $\mathcal{I}_\theta(D) = 5$. Now, by considering the DTP $(V, C \setminus \{c_3\})$ we have $\mathcal{I}_\omega((V, C \setminus \{c_3\})) = 10 = \mathcal{I}_\omega(D)$ and $\mathcal{I}_\theta((V, C \setminus \{c_3\})) = 5 = \mathcal{I}_\theta(D)$. From this example we can assert that $\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ do not satisfy the postulate PCD. ◀

▶ **Proposition 22.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfy SDI.*

Proof. Let $D = (V, C)$ and $D' = (V', C')$ be two DTPs such that $V \cap V' = \emptyset$ and D' is consistent. Since $\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfy Mono we know that $\mathcal{I}_\omega(D \cup D') \geq \mathcal{I}_\omega(D)$ and $\mathcal{I}_\theta(D \cup D') \geq \mathcal{I}_\theta(D)$. Now, let us prove that $\mathcal{I}_\omega(D \cup D') \leq \mathcal{I}_\omega(D)$ and $\mathcal{I}_\theta(D \cup D') \leq \mathcal{I}_\theta(D)$. Let λ be a local c-relaxation of D belonging to $\mathsf{LCR}(D)$ and let λ' be the local c-transformation of $D \cup D'$ defined by $\lambda'(c) = \lambda(c)$ for all $c \in C$ and $\lambda'(c) = c$ for all $c \in C'$. Clearly, $\lambda'(c)$ is a local c-relaxation of $D \cup D'$. Also, we have $\omega(\lambda) = \omega(\lambda')$ and $\theta(\lambda) = \theta(\lambda')$. Moreover, we can show that $\lambda'(D \cup D') = \lambda(D) \cup D'$. Since $\mathsf{vars}(\lambda(D)) = \mathsf{vars}(D) = V$, we have $\mathsf{vars}(\lambda(D)) \cap \mathsf{vars}(D') = \emptyset$. From all this and the fact that $\lambda(D)$ and D' are two consistent DTPs we can assert that $\lambda'(D \cup D') = \lambda(D) \cup D'$ is a consistent DTP. It follows that λ' belongs to $\mathsf{LCR}(D \cup D')$. Hence, $\mathcal{I}_\omega(D \cup D') \leq \omega(\lambda') = \omega(\lambda)$ and $\mathcal{I}_\theta(D \cup D') \leq \theta(\lambda') = \theta(\lambda)$. Now, suppose that λ is such that $\omega(\lambda) = min\{\omega(\lambda'') : \lambda'' \in \mathsf{LCR}(D)\} = \mathcal{I}_\omega(D)$ (resp. such that $\theta(\lambda) = min\{\theta(\lambda'') : \lambda'' \in \mathsf{LCR}(D)\} = \mathcal{I}_\theta(D)$). With this additional property about λ we can deduce that $\mathcal{I}_\omega(D \cup D') \leq \mathcal{I}_\omega(D)$ (resp. $\mathcal{I}_\theta(D \cup D') \leq \mathcal{I}_\theta(D)$). From all this, we can conclude that $\mathcal{I}_\omega(D \cup D') = \mathcal{I}_\omega(D)$ and $\mathcal{I}_\theta(D \cup D') = \mathcal{I}_\theta(D)$. ◀

▶ **Proposition 23.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfy WSub.*

Proof. Let $D = (V, C)$ be a DTP and two temporal constraints c, c' such that $c \notin C$ and c subsumes c'. Let us denote by D' (resp. by D'') the DTP $(V \cup \mathsf{vars}(c), C \cup \{c\})$ (resp. the DTP $(V \cup \mathsf{vars}(c'), C \cup \{c'\})$). Firstly, note that in the case where $c' \in C$, we have by mono that $\mathcal{I}_\omega(D') \geq \mathcal{I}_\omega(D'')$ and $\mathcal{I}_\theta(D') \geq \mathcal{I}_\theta(D'')$ since $D' = D'' \cup (\mathsf{vars}(c), \{c\})$. In the sequel, we will suppose that $c' \notin C$. Let λ' be a local c-relaxation belonging to $\mathsf{LCR}(D')$ such that for each $c'' \in C \cup \{c\}$, we have $|\{l = (x - y \in I) \in \mathsf{Lit}(c'') : I \neq (\lambda'(c''))(l)\}| \leq 1$. Note that from Proposition 4, this last assumption is not restrictive. In the case where $\lambda'(c) = c$ we define the local c-transformation λ'' of D'' by $\lambda''(c'') = \lambda'(c'')$ for all $c'' \in C$ and for all $l \in \mathsf{Lit}(c')$, $(\lambda''(c'))(l) = l$. In the case where $\lambda'(c) \neq c$, let $l' = (x - y \in I')$ be the temporal literal of c such that $(\lambda'(c))(l') \neq l'$ and let $l'' = (x - y \in I'')$ one temporal literal of c' such that $I' \subseteq I''$. For this case, we define the local c-transformation λ'' of D'' by $\lambda''(c'') = \lambda'(c'')$ for all $c'' \in C$, $(\lambda''(c'))(l')$ is defined by the smallest interval of $\mathcal{I}^{\mathbb{Z}}$ including $(\lambda'(c))(l') \cup I''$ and for all $l \in c'$ such that $l \neq l'$, $(\lambda''(c'))(l) = l$. Whatever the considered case and the definition of λ'' we can show that λ'' is a local c-relaxation of D'' such that $\omega(\lambda') \geq \omega(\lambda'')$, $\theta(\lambda') \geq \theta(\lambda'')$ such that $\lambda''(D'')$ is consistent (since any solution of $\lambda'(D')$ can be extended to a solution $\lambda''(D'')$). It follows that $\lambda'' \in \mathsf{LCR}(D'')$, $\omega(\lambda') \geq \omega(\lambda'') \geq \mathcal{I}_\omega(D'')$ and $\omega(\lambda') \geq \omega(\lambda'') \geq \mathcal{I}_\theta(D'')$. Now suppose that λ' is such that $\omega(\lambda') = \mathcal{I}_\omega(D')$ (resp. $\theta(\lambda') = \mathcal{I}_\theta(D')$). With this additional assumption we can deduce that $\mathcal{I}_\omega(D') \geq \mathcal{I}_\omega(D'')$ and $\mathcal{I}_\theta(D') \geq \mathcal{I}_\theta(D'')$. ◀

▶ **Proposition 24.** *$\mathcal{I}_\omega$ does not satisfy* Sub*.*

Proof. Let $D = (V = \{x_1, x_2, x_3\}, C = \{c_1, c_2\})$ be the DTP defined by:

- $c_1 = x_1 - x_2 \in [0, 0]$,
- $c_2 = x_1 - x_2 \in [1, 1]$,
- $c_3 = x_1 - x_2 \in [4, 4]$.

D is an inconsistent DTP. Moreover, we have $\mathcal{I}_\omega(V, C) = 4$. Now consider the constraint $c_4 = x_1 - x_2 \in [4, 5]$. Clearly, c_3 subsumes c_4. On the other hand, we have $\mathcal{I}_\omega(V, C \cup \{c_4\}) = \mathcal{I}_\omega(V, C) = 4$ and $\mathcal{I}_\omega(V, C \cup \{c_4\}) = \mathcal{I}_\omega(V, \{c_1, c_2, c_3, c_4\}) = 7$. From this example we can assert that $\mathcal{I}_\omega$ does not satisfy the postulate Sub. ◀

▶ **Proposition 25.** *Let $D = (V, C)$ be a DTP and two temporal constraints c, c' such that $c \in C$ and c subsumes c'. We have $\mathcal{I}_\theta(D) = \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$.*

Proof. In the case where $c' \in C$, the property is obvious, in the sequel we will suppose that $c' \notin C$. By Mono (Proposition 18) we know that $\mathcal{I}_\theta(D) \leq \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. We will show that $\mathcal{I}_\theta(D) \geq \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. Let λ be a local c-relaxation of $\mathsf{LCR}(D)$ such that $\theta(\lambda) = min\{\theta(\lambda'') : \lambda'' \in \mathsf{LCR}(D)\} = \mathcal{I}_\theta(D)$. Moreover we suppose that for each $c'' \in C \cup \{c\}$, we have $|\{l = (x - y \in I) \in \mathsf{Lit}(c'') : I \neq (\lambda(c''))(l)\}| \leq 1$. Note that, from Proposition 4, this last assumption is not restrictive. We define the local c-relaxation λ' of $(V \cup \mathsf{vars}(c'), C \cup \{c'\})$ in the following way. In the case where $|\{l = (x - y \in I) \in \mathsf{Lit}(c) : I \neq (\lambda(c))(l)\}| = 0$ (Case 1), $(\lambda'(c''))(l) = (\lambda(c''))(l)$ for all $c'' \in C$ and $l \in \mathsf{Lit}(c'')$. Moreover, $(\lambda'(c'))(l) = I$ for all $l = (x - y \in I) \in \mathsf{Lit}(c')$. In the case where $|\{l = (x - y \in I) \in \mathsf{Lit}(c) : I \neq (\lambda(c))(l)\}| = 1$ (Case 2), let $l' = (x' - y' \in I')$ be the literal of c such that $(\lambda(c))(l' = (x' - y' \in I')) \neq I'$ and $l'' = (x' - y' \in I'')$ be a literal of c' such that $I' \subseteq I''$. For this case, λ' is defined by $(\lambda'(c''))(l) = (\lambda(c''))(l)$ for all $c'' \in C$ and $l \in \mathsf{Lit}(c'')$, $(\lambda'(c'))(l) = I$ for all $l = (x - y \in I) \in \mathsf{Lit}(c') \setminus \{l''\}$ and $(\lambda'(c'))(l'')$ is defined by the smallest interval of $\mathcal{I}^{\mathbb{Z}}$ including $\lambda(c))(l') \cup I''$. Whatever the definition of λ' we have $\theta(\lambda') = \theta(\lambda)$ and $\lambda' \in \mathsf{LCR}((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. It follows that $\theta(\lambda) \geq \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. Consequently, $\mathcal{I}_\theta(D) \geq \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. ◀

▶ **Proposition 26.** *$\mathcal{I}_\theta$ satisfies* Sub*.*

Proof. Let $D = (V, C)$ be a DTP and two temporal constraints c, c' such that c subsumes c'. We have two cases that arise: $c \notin C$ or $c \in C$. By considering the case $c \notin C$, from WSub (Proposition 23), we know that $\mathcal{I}_\theta((V \cup \mathsf{vars}(c), C \cup \{c\})) \geq \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. Now, consider the case $c \in C$. For this case $\mathcal{I}_\theta((V \cup \mathsf{vars}(c), C \cup \{c\})) = \mathcal{I}_\theta((V, C))$. Moreover, from Proposition 25, we can assert that $\mathcal{I}_\theta((V, C)) = \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$. From all this, we can conclude that $\mathcal{I}_\theta((V \cup \mathsf{vars}(c), C \cup \{c\})) \geq \mathcal{I}_\theta((V \cup \mathsf{vars}(c'), C \cup \{c'\}))$ and that $\mathcal{I}_\theta$ satisfies Sub. ◀

▶ **Proposition 27.** *$\mathcal{I}_\theta$ does not satisfy VIA and SA.*

Proof. Let $D = (V = \{x_1, x_2, x_3, x_4\}, C = \{c_1, c_2, c_3, c_4\})$ be the DTP defined by:

- $c_1 = x_1 - x_2 \in [0, 0]$,
- $c_2 = x_1 - x_2 \in [1, 1]$,
- $c_3 = x_3 - x_4 \in [0, 0]$,
- $c_4 = x_3 - x_4 \in [1, 1]$.

Consider the two DTPs $D = (V = \{x_1, x_2\}, C = \{c_1, c_2\})$ and $D' = (V' = \{x_1, x_2\}, C' = \{c_3, c_4\})$. We have $V \cap V' = \emptyset$ and $C \cap C' = \emptyset$. Moreover, $\mathcal{I}_\theta(D \cup D') = 1$, $\mathcal{I}_\theta(D) = 1$ and $\mathcal{I}_\theta(D') = 1$. From this example we can assert that $\mathcal{I}_\theta$ do not satisfy the postulates VIA and SA. ◀

▶ **Proposition 28.** *$\mathcal{I}_\omega$ satisfies SA.*

Proof. Let $D = (V, C)$ and $D' = (V', C')$ be two DTPs such that $C \cap C' = \emptyset$. Let λ'' be a local c-relaxation of $\mathsf{LCR}(D \cup D')$ such that $\omega(\lambda'') = min\{\omega(\lambda''') : \lambda''' \in \mathsf{LCR}(D \cup D')\} = \mathcal{I}_\omega(D \cup D')$. We define the local c-relaxation λ (resp. λ') of D (resp. of D') by $\lambda(c) = \lambda''(c)$ (resp. $\lambda'(c) = \lambda''(c)$) for all $c \in C$ (resp. for all $c' \in C'$). Clearly, since $C \cap C' = \emptyset$ we have $\omega(\lambda'') = \omega(\lambda) + \omega(\lambda')$. Moreover we have can show that $\lambda \in \mathsf{LCR}(D)$ and $\lambda' \in \mathsf{LCR}(D')$. Hence, $\omega(\lambda) \geq \mathcal{I}_\omega(D)$ and $\omega(\lambda') \geq \mathcal{I}_\omega(D')$. It results that $\mathcal{I}_\omega(D \cup D') = \omega(\lambda'') = \omega(\lambda) + \omega(\lambda') \geq \mathcal{I}_\omega(D) + \mathcal{I}_\omega(D')$. ◀

▶ **Proposition 29.** *$\mathcal{I}_\omega$ satisfies VIA.*

Proof. Let $D = (V, C)$ and $D' = (V', C')$ be two DTPs such that $V \cap V' = \emptyset$. As $V \cap V' = \emptyset$ we have $C \cap C' = \emptyset$. From SA (Proposition 28) we know that that $\mathcal{I}_\omega(D \cup D') \geq \mathcal{I}_\omega(D) + \mathcal{I}_\omega(D')$. Let us prove that $\mathcal{I}_\omega(D \cup D') \leq \mathcal{I}_\omega(D) + \mathcal{I}_\omega(D')$. Let λ be a local c-relaxation of $\mathsf{LCR}(D)$ such that $\omega(\lambda) = min\{\omega(\lambda''') : \lambda''' \in \mathsf{LCR}(D)\} = \mathcal{I}_\omega(D)$ and let λ' be a local c-relaxation of $\mathsf{LCR}(D')$ such that $\omega(\lambda') = min\{\omega(\lambda''') : \lambda''' \in \mathsf{LCR}(D')\} = \mathcal{I}_\omega(D')$. Let the local c-transformation λ'' defined by $\lambda''(c)$ $\lambda(c)$ for all $c \in C$ and $\lambda''(c)$ $\lambda'(c)$ for all $c \in C'$. We can show that $\omega(\lambda'') = \omega(\lambda) + \omega(\lambda')$ (since $C \cap C' = \emptyset$) and $\lambda'' \in \mathsf{LCR}(D \cup D')$. This last belonging comes from the fact that $\lambda''(D \cup D') = \lambda(D) \cup \lambda'(D')$, $V \cap V' = \emptyset$, $\lambda(D)$ is a consistent DTP and $\lambda'(D')$ is a consistent DTP. It results that $\mathcal{I}_\omega(D \cup D') \leq \omega(\lambda'') = \omega(\lambda) + \omega(\lambda') = \mathcal{I}_\omega(D) + \mathcal{I}_\omega(D')$. From all this, we can conclude that $\mathcal{I}_\omega(D \cup D') = \mathcal{I}_\omega(D) + \mathcal{I}_\omega(D')$. ◀

▶ **Proposition 30.** *$\mathcal{I}_\omega$ and $\mathcal{I}_\theta$ satisfies SI.*

Proof. Let $D = (V, C)$ be a DTP and $k \in \mathbb{Z}$. Let λ be a local c-relaxation of $\mathsf{LCR}(D)$ such that $\omega(\lambda) = min\{\omega(\lambda'') : \lambda'' \in \mathsf{LCR}(D)\} = \mathcal{I}_\omega(D)$ and let λ' be a local c-relaxation of $\mathsf{LCR}(D)$ such that $\theta(\lambda') = min\{\theta(\lambda'') : \lambda'' \in \mathsf{LCR}(D)\} = \mathcal{I}_\theta(D)$. From λ and λ' we define the two local c-relaxations of $D \oplus k$ λ'' and λ''' in the following way: $(\lambda''(c))(l) = (\lambda(c))(l) \oplus k$ and $(\lambda'''(c))(l) = (\lambda'(c))(l) \oplus k$ for all $c \in C$ and $l \in \mathsf{Lit}(c)$. We can show that $\omega(\lambda) = \omega(\lambda'')$ and $\theta(\lambda') = \theta(\lambda''')$. We can also show that $\lambda''(D \oplus k)$ and $\lambda'''(D \oplus k)$ are consistent. It follows that λ'' and λ''' belong to $\mathsf{LCR}(D \oplus k)$. It results that $\mathcal{I}_\omega(D) = \omega(\lambda) = \omega(\lambda'') \geq \mathcal{I}_\omega(D \oplus k)$ and $\mathcal{I}_\theta(D) = \theta(\lambda') = \theta(\lambda''') \geq \mathcal{I}_\theta(D \oplus k)$.

Now, let us prove that $\mathcal{I}_\omega(D) \leq \mathcal{I}_\omega(D \oplus k)$ and $\mathcal{I}_\theta(D) \leq \mathcal{I}_\theta(D \oplus k)$. Let λ be a local c-relaxation of $\mathsf{LCR}(D \oplus k)$ such that $\omega(\lambda) = min\{\omega(\lambda'') : \lambda'' \in \mathsf{LCR}(D \oplus k)\} = \mathcal{I}_\omega(D \oplus k)$ and let λ' be a local c-relaxation of $D \oplus k$ such that $\theta(\lambda') = min\{\theta(\lambda'') : \lambda'' \in \mathsf{LCR}(D \oplus k)\} =$

$\mathcal{I}_\theta(D \oplus k)$. From λ and λ' we define the two c-relaxation of D λ'' and λ''' in the following way: $(\lambda''(c))(l) = (\lambda(c))(l) \oplus (-k)$ and $(\lambda'''(c))(l) = (\lambda'(c))(l) \oplus (-k)$ for all $c \in C$ and $l \in \mathsf{Lit}(c)$. We can show that $\omega(\lambda) = \omega(\lambda'')$ and $\theta(\lambda') = \theta(\lambda''')$. We can also show that $\lambda''(D)$ and $\lambda'''(D)$ are consistent. It follows that λ'' and λ''' belong to $\mathsf{LCR}(D)$. It results that $\mathcal{I}_\omega(D \oplus k) = \omega(\lambda) = \omega(\lambda'') \geq \mathcal{I}_\omega(D)$ and $\mathcal{I}_\theta(D \oplus k) = \theta(\lambda') = \theta(\lambda''') \geq \mathcal{I}_\theta(D)$.
From all this, we can conclude that $\mathcal{I}_\omega(D \oplus k) = \mathcal{I}_\omega(D)$ and $\mathcal{I}_\theta(D \oplus k) = \mathcal{I}_\theta(D)$. ◀