Covering and Partitioning of Split, Chain and Cographs with Isometric Paths

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Abstract

Given a graph G, an *isometric path cover* of a graph is a set of isometric paths that collectively contain all vertices of G. An isometric path cover $\mathcal{C}$ of a graph G is also an *isometric path partition* if no vertex lies in two paths in $\mathcal{C}$. Given a graph G, and an integer k, the objective of Isometric Path Cover (resp. Isometric Path Partition) is to decide whether G has an isometric path cover (resp. partition) of cardinality k.

In this paper, we show that Isometric Path Partition is NP-complete even on split graphs, *i.e.* graphs whose vertex set can be partitioned into a clique and an independent set. In contrast, we show that both Isometric Path Cover and Isometric Path Partition admit polynomial time algorithms on cographs (graphs with no induced P_4) and chain graphs (bipartite graphs with no induced $2K_2$).

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1 Introduction and results

Finding paths in graphs is of fundamental interest to the algorithmic graph theory community. An example is the Path Cover where the objective is to decide whether the vertex set of a graph can be covered by at most k paths. This problem is NP-hard even if $k = 1$ which is equivalent to Hamiltonian Path. Recently, researchers have studied the problem of covering graphs with *isometric paths* i.e. shortest path between its end-vertices. Given a graph G, an *isometric path cover* of a graph is a set of isometric paths that collectively contain all vertices of G. An isometric path cover $\mathcal{C}$ of a graph G is also an *isometric path partition* if no vertex lies in two paths in $\mathcal{C}$. Given a graph G, and an integer k, the objective of Isometric Path Cover (resp. Isometric Path Partition) is to decide whether G has an isometric path cover (resp. partition) of cardinality at most k. Besides algorithmic graph theory, Isometric Path Cover has been studied in different contexts like machine learning [14], combinatorial games [1] etc.

Despite being a natural covering problem, the algorithmic complexity of Isometric Path Cover has thus far remained mostly unexplored. Isometric Path Cover and Isometric Path Partition have been proven to be NP-hard in *chordal* graphs (i.e. graphs without induced cycles of order four or higher) and bipartite graphs of diameter 4 [5, 8]. The

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Parameterized complexity of Isometric Path Cover has been studied with respect to various parameters like solution size, neighbourhood diversity, etc [7, 8]. Approximability of Isometric Path Cover has also been studied [4, 5]. It was shown that Isometric Path Cover admits constant factor approximation algorithms on graph classes like chordal graphs and more generally on graphs with bounded treelength and bounded hyperbolicity, outerstring graphs, universally-signable graphs and more generally on (theta, pyrmaid, prism)-free graphs [4, 5]. It was also shown that Isometric Path Cover admits $O(\log n)$-approximation algorithm on general graphs [14]. On the other hand, polynomial time solvability of Isometric Path Cover is only known for special graph classes such as block graphs [12] which is a subclass of chordal graphs. This motivated us to study the computational complexities of Isometric Path Cover and Isometric Path Partition on *split* graphs which is a popular subclass of chordal graphs.

A graph is a *split* graph if the vertex set can be partitioned into a clique C and an independent set I. In this paper, we prove that the Isometric Path Partition problem remains NP-hard on split graphs answering an open question in the literature [5]

▶ **Theorem 1.** *Isometric Path Partition is* NP*-hard on split graphs.*

Our reduction techniques deviate significantly from the known ones which typically reduce the problem of partitioning a graph into induced P_k (for appropriately chosen k, also known as Induced P_k Partition) by adding few vertices of large degree, so that a path in the reduced graph is isometric if and only if it was an induced P_k in the original graph. However, one difficulty in applying this technique is that the complexity of Induced P_k Partition for $k \in \{3, 4\}$ on split graphs is not known. The problems of partitioning split graphs into (non-induced) P_3's can be solved in polynomial time [15] by reducing it to finding a maximum matching in some auxiliary graph.

We reduce from the NP-complete problem 3-XSAT [13] where the input is a CNF formula that has exactly 3 positive literals in every clause and every variable occurs exactly 3 times, and the objective is to decide if there is an assignment that satisfies exactly one literal from every clause. As a biproduct of our proof, we also get the following corollary. Isometric $P_{\geq t}$ Partition denotes the problem of partitioning the vertex set of a graph into isometric paths with at least t vertices.

▶ **Corollary 2.** *For $t \leq 3$, Isometric $P_{\geq t}$ Partition is* NP*-hard on split graphs.*

We note that the computational complexity of Isometric Path Cover on split graphs remains open. Theorem 1 motivates the study of Isometric Path Partition on natural subclasses of split graphs. *Threshold* graphs, the class of split graphs without an induced P_4, is one of the well-studied subclass of split graphs [11]. We show that both Isometric Path Cover and Isometric Path Partition admit polynomial-time algorithms on threshold graphs. In fact, we prove the following more general result. Cographs were introduced in [6] and are exactly the class of graphs with no induced P_4.

▶ **Theorem 3.** *Isometric Path Cover and Isometric Path Partition admit polynomial time algorithms on cographs.*

Our algorithm for Isometric Path Partition on cographs is based on dynamic programming on *cotrees* [6], which also characterise cographs. The class of cographs contains the class of threshold graphs, which are exactly the class of split graphs whose vertices in the independent set can be linearly ordered under inclusion of their open neighbourhoods. We note that designing algorithms for cographs is a popular direction of research in algorithmic graph theory [2, 3, 9, 10].

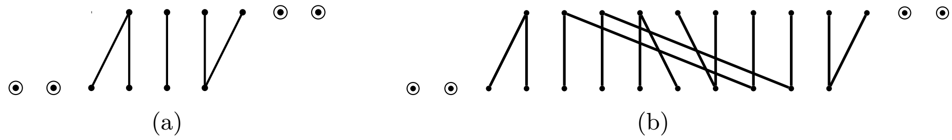

Figure 1 Normal forms of optimal solutions for Isometric Path Partition on chain graphs. Connected components or marked vertices indicate isometric paths in the solution.

Motivated by the success on cographs we then consider the class of *chain* graphs, the bipartite analogue of threshold graphs. A chain graph is a bipartite graph such that the vertices of each color class can be linearly ordered under inclusion of their open neighbourhoods. We show the following result:

▶ **Theorem 4.** *Isometric Path Cover and Isometric Path Partition admit polynomial time algorithms on chain graphs.*

Note that a path in a chain graph or a cograph is an isometric path if and only if it is induced. Due to Theorems 3 and 4, we have the following corollary.

▶ **Corollary 5.** *Induced Path Partition and Induced Path Cover admit polynomial time algorithms on cographs and chain graphs.*

Our algorithm for Isometric Path Partition on chain graphs is based on the observation that any (optimal) solution can be converted into one of three *normal forms*, two of which are illustrated in Figure 1 and where the induced paths can be thought of (roughly) as following the ordering of the vertices according to the neighbourhood inclusion relation. Obtaining the normal forms is also the main challenge behind the algorithm as it turned out to be surprising difficult to find the right order of operations required to transform any solution into one of these normal forms without sacrificing previously made progress. Once the normal form is obtained, however, Isometric Path Partition can be solved via a simple brute-force algorithm that guesses the important positions of a solution in normal form. Finally, a simple reduction from Isometric Path Cover to Isometric Path Partition on chain graphs then allows us to obtain the polynomial-time algorithm for Isometric Path Cover on chain graphs.

Structure of the paper. In Section 2 we introduce some definitions. In Section 3 and Section 4 we prove Theorems 3 and 4 respectively. In Section 5 we prove Theorem 1.

Statements whose full proofs are omitted due to space constraints can be found in the full version.

2 Preliminaries

All graphs considered here are finite, simple, and undirected. That is, a *graph* $G = (V, E)$ consists of a finite set V of *vertices* and a set $E \subseteq V^{(2)}$ of *edges*, where $V^{(2)}$ is the set of 2-element subsets of V. An edge $\{u, v\}$ will also be written as uv. The graph $H = (U, F)$ is a *subgraph* of $G = (V, E)$ if $U \subseteq V$ and $F \subseteq E$. The subgraph H is *induced* if $F = E \cap U^{(2)}$, denoted as $H = G[U]$. The *neighbourhood* $N_G(v)$, or simply $N(v)$ if G is clear from the context, of a vertex $v \in V$ in graph G is $\{u \mid uv \in E\}$. Moreover, for a subset V' of vertices, we denote by $N_G(V')$ the set $\bigcup_{v \in V'} N_G(v)$.

A *path* in G is a subgraph $P = (U, F)$ with $U = \{u_i \mid 0 \leq i \leq l\}$ and $F = \{u_{i-1}u_i \mid 1 \leq i \leq l\}$. We usually refer to P by the sequence $(u_0, u_1, u_2, \ldots, u_l)$. The *length* of this path is l, the number of edges in P.

The *distance* between two vertices u and v of G is the length of a shortest path between u and v. It is denoted by $d_G(u, v)$. A path $P = (u, \ldots, v)$ is *isometric* if $d_G(u, v) = d_P(u, v)$. Every isometric path is an induced path, and induced paths of length at most two are isometric.

3 Algorithms for Chain Graphs

Here, we provide our algorithms for Isometric Path Partition and Isometric Path Cover on chain graphs. A bipartite graph $G = (T, B, E)$ is a *chain graph* if, for every pair of vertices $v, u \in T$ or $v, u \in B$, we have $N(v) \subseteq N(u)$ or $N(v) \supseteq N(u)$, respectively; see also [16]. This implies an ordering $<$ of the vertices in T and the vertices in B such that $u < v$ if $N_G(u) \subseteq N_G(v)$. For convinience and by resolving ties arbitrarily, we will assume that $<$ is a total ordering on T and on B.

The main step of our proofingredient and the main challenge for the algorithms is to show that any solution to Isometric Path Partition on a chain graph can be transformed into an equally-sized solution that follows a certain normal form (Section 3.1). Assuming this normal form, then essentially allows us to solve Isometric Path Partition in polynomial-time via a brute-force approach (Section 3.2). Finally, our polynomial-time algorithm for Isometric Path Cover then uses a simple reduction from Isometric Path Cover to Isometric Path Partition (Section 3.2).

3.1 Normal Form

In this subsection we introduce our normal form for solutions to Isometric Path Partition on chain graphs and show that there is always an optimal solution adhering to this normal form. We start by introducing some important notation.

Let $G = (T, B, E)$ be a chain graph and let $\mathcal{P}$ be a set of isometric paths. A pattern can be thought of as a string over the symbols $_{\circ}$, Λ, I, V, N, and $^{\circ}$, where it is possible for symbols to overlap with each other; such as in NΛ or NV, which consists of two overlapping symbols N and Λ respectively N and V. We also allow special symbols to indicate that a symbol is repeated arbitrary many times, e.g., $\dot{_{\circ}}$, $\dot{\Lambda}$, $\dot{V}$, and $\dot{^{\circ}}$ represent arbitrary many (non-crossing) $_{\circ}$'s, Λ's, V's, and $^{\circ}$'s, respectively. Moreover, NNN represents arbitrary many N's that are pairwise in pattern NNN, e.g., NNNNNNNN is an element of NNN.

We say that $\mathcal{P}$ *has pattern* α if the drawing of all paths in $\mathcal{P}$, *i.e.*, the drawing obtained by drawing all vertices in T from right to left in the order $<$ on top of all vertices in B drawn according to $<$ from left to right, looks like α. For instance, if $P \in \mathcal{P}$ is a single path, then P has pattern $_{\circ}$, Λ, I, V, N, or $^{\circ}$, if $|V(P) \cap T| = 0$ and $|V(P) \cap B| = 1$, $|V(P) \cap T| = 1$ and $|V(P) \cap B| = 2$, $|V(P) \cap T| = 1$ and $|V(P) \cap B| = 1$, $|V(P) \cap T| = 2$ and $|V(P) \cap B| = 1$, $|V(P) \cap T| = 2$ and $|V(P) \cap B| = 2$, or $|V(P) \cap T| = 1$ and $|V(P) \cap B| = 0$, respectively. Alternatively, we also say that P *is an* α*-path*, if P has pattern α. We denote by $\mathcal{P}_\alpha$ the set of all α-paths in $\mathcal{P}$ and we denote by $\mathcal{P}_\circ$ the set $\mathcal{P}_{_\circ} \cup \mathcal{P}_{^\circ}$. Observe that since G is a chain graph every path in $\mathcal{P}$ is either a $_{\circ}$-path, a Λ-path, an I-path, a V-path, a N-path, or a $^{\circ}$-path.

We say that a path $P \in \mathcal{P}$ is *before (or to the left)* of a path $P' \in \mathcal{P}$ if $b < b'$ for every $b \in V(P) \cap B$ and $b' \in V(P') \cap B$ and moreover $t > t'$ for every $t \in V(P) \cap T$ and $t' \in V(P') \cap T$. We say that P is *after (or to the right of* P'*)* if P' is before P. We say that P and P' *cross* if P is neither before nor after P'.

The main aim of this section is to show the following theorem.

▶ **Theorem 6.** *Let $G = (T, B, E)$ be a chain graph. G has an optimal isometric path partition that has one of the following patterns:*

(1) ∘*Λ*V*°*, *i.e., any number of* ∘*-paths, followed by any number of* Λ*-paths, followed by any number of* V*-paths, followed by any number of* °*-paths,*

(2) ∘*Λ*IV*°*, *i.e., same as (1) but with exactly one* I*-path in the center,*

(3) ∘*Λ*[ΛVN]*V*°*, *i.e., same as (2) but with the* I*-path replaced by any number of* Λ*-paths,* V*-paths, and* N*-paths such that every pair of those paths has one of the following patterns* ΛΛ, VV, ΛV, [NΛ], [NV], *or* [NN].

Towards showing Theorem 6, we will introduce various intermediate normal forms and we will then show that a solution can be transformed step-by-step into more and more restrictive normal forms. Let $G = (T, B, E)$ be a chain graph and $\mathcal{P}$ be an isometric path partition of G. Most of our normal forms are based on the correct relationships between pairs of paths in $\mathcal{P}$. In particular, we say that two distinct paths P and P' in $\mathcal{P}$ are in *normal position* if either:

(1) If P or P' is a ∘-path, then $\{P, P'\}$ has pattern ∘∘, ∘Λ, ∘V, ∘N, or ∘°. In other words, all ∘-paths are to the left of all other paths in $\mathcal{P}$.

(2) If P or P' is a °-path, then $\{P, P'\}$ has pattern ∘°, Λ°, V°, N°, or °°. In other words, all °-paths are to the right of all other paths in $\mathcal{P}$.

(3) If P is a I-path and P' is a Λ-path (V-path), then $\{P, P'\}$ has pattern ΛI (IV). In other words, all I-paths are to the right of all Λ-paths and to the left of all V-paths in $\mathcal{P}$.

(4) If both P and P' are Λ-paths, then $\{P, P'\}$ has pattern ΛΛ. In other words, no two Λ-paths in $\mathcal{P}$ cross each other.

(5) If both P and P' are V-paths, then $\{P, P'\}$ has pattern VV. In other words, no two V-paths in $\mathcal{P}$ cross each other.

(6) If both P and P' are N-paths, then $\{P, P'\}$ has pattern [NN].

(7) If P is a N-path and P' is a Λ-path (V-path), then $\{P, P'\}$ has pattern ΛN or [NΛ] (NV or [NV]).

(8) If P is a Λ-path and P' is a V-path, then $\{P, P'\}$ has pattern ΛV.

We say that $\mathcal{P}$ satisfies (i), $i \in [8]$, if condition (i) holds for every two distinct paths P and P' in $\mathcal{P}$. We are now ready to define our normal forms.

▶ **Definition 7.** *Let $\mathcal{P}$ be an isometric path partition of a chain graph G.*

- *We say that $\mathcal{P}$ is* NI-minimal *if either $\mathcal{P}_{\mathsf{N}} = \emptyset$ and $|\mathcal{P}_{\mathsf{I}}| \leq 1$ or $\mathcal{P}_{\mathsf{N}} \neq \emptyset$ and $\mathcal{P}_{\mathsf{I}} = \emptyset$.*
- *We say that $\mathcal{P}$ is in* S-normal form *if $\mathcal{P}$ is NI-minimal and $\mathcal{P}$ satisfies (1) and (2).*
- *We say that $\mathcal{P}$ is in* I-normal form, A-normal form, V-normal form, *or* N-normal form *if $\mathcal{P}$ satisfies (3), (4), (5), or (6), respectively.*
- *We say that $\mathcal{P}$ is in* mixed normal form *if $\mathcal{P}$ is in S-normal form and additionally $\mathcal{P}$ satisfies (7) and (8).*
- *We say that $\mathcal{P}$ is in* pair-normal form *if $\mathcal{P}$ is in mixed normal form, in I-normal form, in A-normal form, in V-normal form, and in N-normal form.*

All normal forms given above are purely defined in terms of restrictions on the relationships between pairs of paths in an isometric path partition. However, to obtain a normal form for the pattern given in Theorem 6 (3), we need to additionally put restrictions on the patterns allowed for triples of isometric paths. We say that $\mathcal{P}$ is in *AN-normal form* if there are no distinct $P_A \in \mathcal{P}_{\Lambda}$, $P_N, P'_N \in \mathcal{P}_{\mathsf{N}}$ such that $\{P_N, P_A\}$ has pattern [NΛ] and $\{P'_N, P_A\}$ has pattern ΛN. In other words, for every $P_A \in \mathcal{P}_{\Lambda}$ it holds that either $\{P_A, P_N\}$ has pattern ΛN for every $P_N \in \mathcal{P}_{\mathsf{N}}$ or $\{P_A, P_N\}$ has pattern [NΛ] for every $P_N \in \mathcal{P}_{\mathsf{N}}$. Similarily, we say that $\mathcal{P}$ is in

NV-normal form if and there are no distinct $P_V \in \mathcal{P}_{\mathsf{V}}$, $P_N, P'_N \in \mathcal{P}_{\mathsf{N}}$ such that $\{P_N, P_V\}$ has pattern N̸V and $\{P'_N, P_V\}$ has pattern NV. Finally, we say that $\mathcal{P}$ is in *normal form* if $\mathcal{P}$ is in pair-normal form, AN-normal form, and in NV-normal form.

Observe that any isometric path partition in normal form has one of the three patterns given in Theorem 6. To show Theorem 6, it therefore suffices to show the following.

▶ **Theorem 8** ($\star$)**.** *Let $G = (T, B, E)$ be a chain graph. There is an optimal isometric path partition of G in normal form.*

The main challenge for proving Theorem 8, whose proof turned out to be surprisingly difficult, is to define the right operations and to apply them in the right order to obtain more and more restricted normal forms while maintaining the progress already made. In particular, after defining the required operations, we will show how to obtain normal form in the following order: S-normal form, mixed normal form, I-normal form, A-normal form, N-normal form, V-normal form, AN-normal form, and finally NV-normal form.

We say that a pattern α implies a pattern β (denoted by $\alpha \to \beta$) if for any chain graph that contains a set $\mathcal{P}$ of pairwise disjoint isometric paths with pattern α there is a set $\mathcal{P}'$ of pairwise disjoint isometric paths with pattern β such that $V(\mathcal{P}) = V(\mathcal{P}')$. (Here $V(\mathcal{P})$ is the set of vertices covered by the paths in $\mathcal{P}$.) For a set Δ of patterns we write $\Delta \to \beta$ if $\delta \to \beta$ for every $\delta \in \Delta$.

For a pattern α, we denote by $\alpha^{\curvearrowright}$ the pattern obtained from α after rotating α by 180° degrees around the center of the pattern. For instance, Λ̸N$^{\curvearrowright}$ = N̸V, ΛN$^{\curvearrowright}$ = NV, ΛV$^{\curvearrowright}$ = ΛV, ΛI$^{\curvearrowright}$ = IV, and $\alpha = \alpha^{\curvearrowright\curvearrowright}$. Moreover, for a set of patterns Δ, we define $\Delta^{\curvearrowright}$ as $\{\, \delta^{\curvearrowright} \mid \delta \in \Delta \,\}$. The following observation is very useful to reduce the number of cases that have to be considered in the proofs and follows easily from the symmetry of chain graphs w.r.t. rotation by 180° degree around the center.

▶ **Observation 9.** *Let α and β be patterns. Then $\alpha \to \beta$ implies $\alpha^{\curvearrowright} \to \beta^{\curvearrowright}$.*

We say that a pattern α is *valid* if there is a set of pairwise disjoint isometric paths in some chain graph with pattern α. For a pattern α, we denote by $[\alpha]$, the set of all valid patterns that can be obtained from α after reordering the endpoints of the lines on the top and/or on the bottom of the pattern α in an arbitrary manner. For instance, [N∘] = {N∘, N̸∘, ∘N} and [ΛI] = {Λ\, /Ⲭ, Ⲭ, Ⲭ, Ⲭ\, /Λ}. Note that [N] does not contain any pattern where the lines of the N cross each other because those patterns are not valid.

The following auxiliary lemma provides the most important production rules on patterns that we will use to transform an arbitrary solution into a solution in normal form.

▶ **Lemma 10** ($\star$)**.** *The following holds:*

(a) [∘∘] → ∘∘ *and* [°°] → °°,
(b) [∘°] → ∘°,
(c) [∘I] → ∘I *and* [I°] → I°,
(d) [∘Λ] → ∘Λ *and* [V°] → V°,
(e) [Λ°] ∪ [II] → Λ° *and* [∘V] ∪ [II] → ∘V,
(f) [ΛI] ∪ [∘N] \ {∘N} → ΛI *and* [IV] ∪ [N°] \ {N°} → IV,
(g) [ΛΛ] → ΛΛ *and* [VV] → VV,
(h) [ΛV] ∪ [NI] → ΛV,
(i) [ΛN] \ {ΛN} → Λ̸N *and* [NV] \ {NV} → N̸V,

Sketch. (a) and (b) follow immediately because $[{}_{\circ\circ}] = \{{}_{\circ\circ}\}$, $[{}^{\circ\circ}] = \{{}^{\circ\circ}\}$, and $[{}_{\circ}{}^{\circ}] = \{{}_{\circ}{}^{\circ}\}$.

Let $G = (T, B, E)$ be a chain graph and let P and P' be two isometric paths in G. Let $(V(P) \cup V(P')) \cap T = \{t_1, \dots, t_c\}$ and $(V(P) \cup V(P')) \cap B = \{b_1, \dots, b_d\}$ such that $t_1 > \dots > t_c$ and $b_1 < \dots < b_d$.

Towards showing (e), assume that $\{P, P'\}$ has pattern α for some $\alpha \in [\mathsf{\Lambda}^{\circ}] \cup [\mathsf{II}]$. Then, $c = 2$, $d = 2$ and b_1 must have degree at least 1 in G. Therefore, because $N_G(t_2) \subseteq N_G(t_1)$, it follows that $b_1t_1 \in E(G)$, which in turn implies that $b_2t_1 \in E(G)$ since $N_G(b_1) \subseteq N_G(b_2)$. Therefore, $\{(b_1, t_1, b_2), (t_2)\}$ is a pair of isometric paths with pattern $\mathsf{\Lambda}^{\circ}$, as required. Because of Observation 9, we also obtain $[{}_{\circ}\mathsf{V}] \cup [\mathsf{II}] \to {}_{\circ}\mathsf{V}$. The remaining cases can be shown in an analogous manner. ◀

The remainder of this subsection is about showing that there are optimal solutions that have more and more restrictive normal forms. As an illustrative example we show next how to obtain a mixed normal form from a solution in S-normal form. We start by stating the lemma required to obtain a S-normal form.

▶ **Lemma 11** ($\star$). *Let $G = (T, B, E)$ be a chain graph. There is an optimal isometric path partition $\mathcal{P}$ which is in S-normal form.*

Building upon S-normal form, the following lemma now shows that we can achieve mixed normal form. The proof of the lemma is based on an exhaustive application of certain production rules from Lemma 10 combined with a potential function approach for showing that this process terminates.

▶ **Lemma 12** ($\star$). *Let $G = (T, B, E)$ be a chain graph. There is an optimal isometric path partition of G in mixed normal form.*

Sketch. Let $G = (T, B, E)$ be a chain graph and let $\mathcal{P}$ be an optimal isometric path partition of G. Because of Lemma 11, we can assume that $\mathcal{P}$ is in S-normal form.

Let $\mathcal{P}'$ be the isometric path partition of G obtained from $\mathcal{P}$ after exhaustively applying the following transformation rules from Lemma 10:

(1) $[\mathsf{\Lambda V}] \to \mathsf{\Lambda V}$,

(2) $[\mathsf{\Lambda N}] \setminus \{\mathsf{\Lambda N}\} \to \mathsf{\Lambda\!\!N}$, and

(3) $[\mathsf{NV}] \setminus \{\mathsf{NV}\} \to \mathsf{N\!\!V}$,

Observe that if $\mathcal{P}'$ exists, then it trivially satisfies all the claims made in the statement of the lemma. It therefore suffices to show the existence of $\mathcal{P}'$ or in other words that the above rules can only be applied finitely often to $\mathcal{P}$.

Towards showing this we will define a potential function Φ that assigns a two dimensional vector of natural numbers to every isometric path partition of G. For the definition of Φ, we need the following additional notation. For two vertices u and v of G such that either $u, v \in B$ or $u, v \in T$, we denote by $[u, v]_G$ the set $\{u, v\} \cup \{\, w \mid u < w < v \,\}$ of vertices of G.

For a path $P = (p_1, p_2, p_3, p_4) \in \mathcal{P}_{\mathsf{N}}$, we denote by $W_G(P)$ the integer $|[p_1, p_3]_G| + |[p_2, p_4]_G|$. For a path $P = (p_1, p_2, p_3) \in \mathcal{P}_{\mathsf{\Lambda}}$, we denote by $L_G(P)$ the integer $|[f_b, p_1]_G| + |[f_b, p_3]_G| + |[f_t, p_2]_G|$, where f_b is the smallest vertex in B and f_t is the largest vertex in T w.r.t. $<$.

For a path $P = (p_1, p_2, p_3) \in \mathcal{P}_{\mathsf{V}}$, we denote by $R_G(P)$ the integer $|[l_t, p_1]_G| + |[l_b, p_2]_G| + |[l_t, p_3]_G|$, where l_t is the smallest vertex in T and l_b is the largest vertex in B w.r.t. $<$.

For a isometric path partition $\mathcal{P}^\star$ of G, we define the first and second component of $\Phi(\mathcal{P}^\star)$ as follows.

$$\Phi(\mathcal{P}^\star)[1] = \sum_{P \in \mathcal{P}^\star_{\mathsf{N}}} W_G(P) \qquad \Phi(\mathcal{P}^\star)[2] = -\Big(\sum_{P \in \mathcal{P}^\star_{\mathsf{\Lambda}}} L_G(P)\Big) - \Big(\sum_{P \in \mathcal{P}^\star_{\mathsf{V}}} R_G(P)\Big)$$

Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two isometric path partitions of G such that $\mathcal{P}^2$ is obtained from $\mathcal{P}^1$ by applying exactly one of the operations given in (1)–(3). We claim that $\Phi(\mathcal{P}^2) > \Phi(\mathcal{P}^1)$, where $>$ is the lexicographical ordering among two dimensional integer vectors. Because Φ is finite, *i.e.*, $(0, -|B||\mathcal{P}^\star_{\mathsf{\Lambda}} \cup \mathcal{P}^\star_{\mathsf{V}}|) \leq \Phi(\mathcal{P}^\star) \leq (|B||\mathcal{P}^\star_{\mathsf{N}}|, 0)$, this then shows that $\mathcal{P}'$ is well defined.

It suffices to show the claim for the cases that $\mathcal{P}^2$ is obtained from $\mathcal{P}^1$ using one of the operations (1)–(3). We show the case for operation (1) as an illustration and leave operations (2) and (3) for the full version. If $\mathcal{P}^2$ is obtained from $\mathcal{P}^1$ by applying operation (1) to two paths $\mathsf{P}_A \in \mathcal{P}^1_{\mathsf{\Lambda}}$ and $P_V \in \mathcal{P}^1_{\mathsf{V}}$, then $\Phi(\mathcal{P}^2)[1] = \Phi(\mathcal{P}^1)[1]$ and $\Phi(\mathcal{P}^2)[2] > \Phi(\mathcal{P}^1)[2]$ because no path in $\mathcal{P}^1$ other than P_A and P_V is changed and moreover because $\mathsf{\Lambda V}$ is the unique pattern that maximizes $\Phi(\{P, P'\})$ for any two paths $P \in \mathcal{P}^1_{\mathsf{\Lambda}}$ and $P' \in \mathcal{P}^1_{\mathsf{V}}$. ◀

Surprisingly, even after reaching mixed normal form, we are still rather far away from our final normal form. In particular, we will go through the following normal forms (in that order): I-normal form, A-normal form, N-normal form, V-normal form, AN-normal form, and finally NV-normal form ($\star$).

3.2 The Algorithms

Having developed our normal forms, we are now ready to show Theorem 4. We start by showing the result for Isometric Path Partition.

▶ **Lemma 13** ($\star$)**.** *Isometric Path Partition admits a polynomial-time algorithm on chain graphs.*

Sketch. Let $G = (T, B, E)$ be a chain graph. Theorem 6 implies that there is an optimal isometric path partition of G having pattern $\overset{*}{\circ}\overset{*}{\mathsf{\Lambda}}\overset{*}{\mathsf{V}}\overset{*}{\circ}$, $\overset{*}{\circ}\overset{*}{\mathsf{\Lambda}}\mathsf{I}\overset{*}{\mathsf{V}}\overset{*}{\circ}$, or $\overset{*}{\circ}\overset{*}{\mathsf{\Lambda}}\mathsf{\ulcorner\!\!-\!\!I}\overset{*}{\mathsf{\Lambda}}\overset{*}{\mathsf{V}}\mathsf{I\!\!-\!\!\lrcorner}\overset{*}{\mathsf{V}}\overset{*}{\circ}$. It therefore suffices to show that we can compute an optimal solution having any of these patterns in polynomial-time. It suffices to provide the proof for the most involved of those patterns, *i.e.*, the pattern $\overset{*}{\circ}\overset{*}{\mathsf{\Lambda}}\mathsf{\ulcorner\!\!-\!\!I}\overset{*}{\mathsf{\Lambda}}\overset{*}{\mathsf{V}}\mathsf{I\!\!-\!\!\lrcorner}\overset{*}{\mathsf{V}}\overset{*}{\circ}$, as the proof of the other two patterns is analogous. Hence, let $\mathcal{P}$ be an isometric path partition of G having pattern $\overset{*}{\circ}\overset{*}{\mathsf{\Lambda}}\mathsf{\ulcorner\!\!-\!\!I}\overset{*}{\mathsf{\Lambda}}\overset{*}{\mathsf{V}}\mathsf{I\!\!-\!\!\lrcorner}\overset{*}{\mathsf{V}}\overset{*}{\circ}$, then $\mathcal{P}$ can be defined by the following 6 numbers:

- the number p_1 of $\circ$-paths,
- the number p_2 of $\mathsf{\Lambda}$-paths that are to the left of any N-path,
- the number p_3 of N-paths,
- the number p_4 of $\mathsf{\Lambda}$-paths that are inside all N-paths,
- the number p_5 of V-paths that are inside all N-paths, and
- the number p_6 of V-paths that are to the right of any N-path.

We say that a tuple $U = (p_1, \ldots, p_6)$ of those six numbers is *valid* if there is an isometric path partition of G with pattern $\overset{*}{\circ}\overset{*}{\mathsf{\Lambda}}\mathsf{\ulcorner\!\!-\!\!I}\overset{*}{\mathsf{\Lambda}}\overset{*}{\mathsf{V}}\mathsf{I\!\!-\!\!\lrcorner}\overset{*}{\mathsf{V}}\overset{*}{\circ}$ with the number of paths as given by U. It is now straightforward to show that the validity of any tuple U can be verified in time $\mathcal{O}(|V(G)|)$ by checking the existence and non-existence of certain edges in G.

Putting everything together, we can solve Isometric Path Partition in time $\mathcal{O}(|V(G)|^7)$ by enumerating all of the at most $|V(G)|^6$ tuples U in time $\mathcal{O}(|V(G)|^6)$, testing their validity in time $\mathcal{O}(|V(G)|)$, and then returning the solution that corresponds to a valid tuple U minimizing $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + (|T| - p_2 - 2p_3 - p_4 - 2p_5 - 2p_6)$. ◀

We will now show that Isometric Path Cover on chain graphs can be reduced to Isometric Path Partition on chain graphs with the help of the following lemma.

▶ **Lemma 14** ($\star$). *Let $G = (T, B, E)$ be a chain graph, $t \in T$ be a vertex such that $t > t'$ for all $t' \in T \setminus \{t\}$, and $b \in B$ be a vertex such that $b' < b$ for all $b' \in B \setminus \{b\}$. Then, there is an optimal isometric path cover $\mathcal{P}$ of G such that all vertices in $V(G) \setminus \{t, b\}$ appear in exactly one path in $\mathcal{P}$.*

Now to solve Isometric Path Cover on a chain graph $G = (T, B, E)$ we do the following. Fix two vertices $t \in T$ and $b \in B$ such that $t > t'$ for all $t' \in T \setminus \{t\}$, and $b' < b$ for all $b' \in B \setminus \{b\}$. From Lemma 14, we know that there is an optimal isometric path cover $\mathcal{C}$ such that no vertex in $V(G) \setminus \{t, b\}$ belongs to more than one path. Now we guess the numbers n_t and n_b such that the number of paths in $\mathcal{C}$ that contain t and b are n_t and n_b, respectively, and create a new graph G' by adding $n_t - 1$ copies $a_1, \ldots, a_{n_t-1}$ of t, and $n_b - 1$ copies $c_1, \ldots, c_{n_b-1}$ of b. Moreover, each vertex in $\{a_1, \ldots, a_{n_t-1}\}$ is adjacent to all the neighbours of t and each vertex in $\{c_1, \ldots, c_{n_b-1}\}$ is adjacent to all the neighbours of b. Then, clearly there is an isometric path partition $\mathcal{P}$ in G' of size $|\mathcal{C}|$. Also, any isometric path partition $\mathcal{Q}$ in G' can be converted into an isometric path cover in G of size $|\mathcal{Q}|$. Hence, we solve Isometric Path Partition on G' and output accordingly.

4 Algorithms on Cographs

Here we design polynomial time algorithms for Isometric Path Partition and Isometric Path Cover on cographs. Complement-reducible graphs (or *cographs* for short) were introduced in [6]. To define the class we use the operations *union* $\oplus$ and *join* $\otimes$ for graphs $G = (V, E)$ and $H = (U, F)$ with $V \cap U = \emptyset$, i.e., $G \oplus H = (V \cup U, E \cup F)$ and $G \otimes H = (V \cup U, E \cup F \cup \{vu \mid v \in V, u \in U\})$.

▶ **Definition 15.** *The cographs can be defined recursively:*
- *K_1 is a cograph.*
- *If G and H are cographs then so are $G \oplus H$ and $G \otimes H$.*
- *There are no other cographs.*

A *cotree* T of a graph $G = (V, E)$ can be defined as a rooted binary tree where the leaves are the vertices in V and the inner nodes are marked with $\oplus$ and $\otimes$ such that two vertices $u, v \in V$ are adjacent if and only if their least common ancestor in T is marked by $\otimes$. Then we say that G has a cotree.

▶ **Theorem 16** ([6]). *For every graph G the following conditions are equivalent.*
1. *G is a cograph.*
2. *G does not contain P_4 as induced subgraph.*
3. *G has a cotree.*

The following observations follow from the definition of cotrees.

▶ **Observation 17.** *Let T be a cotree of a cograph G. Let (v_1, v_2, v_3) be an induced path in G. Then, there is a node t labelled $\otimes$ in the tree such that the least common ancestor of v_1 and v_2, as well as v_2 and v_3 is t.*

▶ **Observation 18.** *Let T be a cotree of a cograph G. For a node t in T, let T_t be the subtree of T rooted at t. For a node t, if G_t is the cograph that has the cotree T_t, then G_t is the subgraph of G induced on $V(G_t)$. This implies that for any $u, v \in V(G_t)$, if $(u, v) \notin E(G_t)$, then $(u, v) \notin E(G)$.*

Now, we discuss an algorithm for ISOMETRIC PATH PARTITION on cographs. For a cograph $G = (V, E)$ let $Q(G)$ denote the set of quadruples $(\ddot{p}, p_1, p_2, p_3)$ such that V has a partition $\mathcal{S}$ into $\ddot{p} + p_1 + p_2 + p_3$ subsets $S \subseteq V$ such that

- for $i \in \{1, 2, 3\}$, exactly p_i sets $S \in \mathcal{S}$ induce a graph $G[S]$ isomorphic to P_i, which is a path on i vertices, and
- for the remaining $\ddot{p}$ subsets $S \in \mathcal{S}$ the subgraph $G[S]$ is isomorphic to $2P_1$, which is an edgeless graph on two vertices.

Theorem 16 implies that each partition of a cograph into isometric paths consists of P_1s, P_2s and P_3s only.

▶ **Lemma 19.** *For a cograph G on n vertices we can compute $Q(G)$ in time $\mathcal{O}(n^{10})$.*

Proof. A cotree T of G can be computed in linear time [6]. We compute $Q(G)$ as follows:

$$\begin{aligned}
Q(K_1) &= \{(0, 1, 0, 0)\} \\
Q(G' \oplus H) &= \{(\ddot{p} + \ddot{q} + r, p_1 + q_1 - 2r, p_2 + q_2, p_3 + q_3) \mid 0 \leq r \leq \min\{p_1, q_1\}, \\
&\qquad (\ddot{p}, p_1, p_2, p_3) \in Q(G'), (\ddot{q}, q_1, q_2, q_3) \in Q(H)\} \\
Q(G' \otimes H) &= \{\big((\ddot{p} - k) + (\ddot{q} - l), (p_1 - i - l) + (q_1 - i - k), p_2 + q_2 + i, \\
&\qquad (p_3 + k) + (q_3 + l)\big) \mid 0 \leq i, 0 \leq k \leq \ddot{p}, 0 \leq i + k \leq q_1, 0 \leq l \leq \ddot{q}, \\
&\qquad 0 \leq i + l \leq p_1, (\ddot{p}, p_1, p_2, p_3) \in Q(G'), (\ddot{q}, q_1, q_2, q_3) \in Q(H)\}
\end{aligned}$$

The equation for $\oplus$ holds because $G' \oplus H$ does not contain any edges between G' and H. Every path in $G' \oplus H$ is either a path in G' or in H. We can count a P_1 in G' and a P_1 in H as one $2P_1$ in the first coordinate or as two P_1 in the second. For $\otimes$ we can create i paths P_2 from a P_1 in G and a P_1 in H, k paths P_3 from one $2P_1$ in G and one P_1 in H, and l paths P_3 from one P_1 in G and one $2P_1$ in H. Using induction on the nodes of the cotree and the Observations 17 and 18, we prove that the above recursive formulae are correct.

Note that for any cograph G', the number of quadruples in $Q(G')$ is at most n^3, because, by knowing specific values for $\ddot{p}$, p_2 and p_3, we can determine p_1 through the equation $2\ddot{p} + p_1 + 2p_2 + 3p_3 = |V(G')|$. The cotree T has $2n - 1$ nodes. Now we estimate the time to compute $Q(G' \otimes H)$, which is asymptotically larger than that of $Q(G' \oplus H)$. For each $(\ddot{p}, p_1, p_2, p_3) \in Q(G')$ and $(\ddot{q}, q_1, q_2, q_3) \in Q(H)$, we run over three values k, l, i, each of them is upper bounded by n, and constructs tuples in $Q(G' \otimes H)$. The number of choices for k, l, i is at most n^3. Thus, for each each $(\ddot{p}, p_1, p_2, p_3) \in Q(G')$ and $(\ddot{q}, q_1, q_2, q_3) \in Q(H)$, we will be taking $O(n^3)$ time. Since the cardinalities of $Q(G')$ and $Q(H)$ are upper bounded by n^3 each, the total running time to compute $Q(G' \otimes H)$ is $O(n^9)$. Since the cotree has $2n - 1$ nodes, the total running time of our algorithm is $O(n^{10})$. ◀

▶ **Lemma 20.** *ISOMETRIC PATH PARTITION can be solved in $\mathcal{O}(n^{10})$ time on cographs.*

Proof. The recurrence for Q leads to a dynamic programming algorithm computing the isometric path partition number of a cograph G, which is $\min\{p_1 + p_2 + p_3 \mid (0, p_1, p_2, p_3) \in Q(G)\}$. ◀

Next, we explain how to use the values in $Q(G)$ to solve ISOMETRIC PATH COVER on connected cographs.

▶ **Lemma 21.** *Let G be a connected cograph. Let $Q(G)$ be the set defined before in this section. Then, the cardinality of an optimal isometric path cover of G is*

$$\min\{\ddot{p} + p_1 + p_2 + p_3 \mid (\ddot{p}, p_1, p_2, p_3) \in Q(G)\}.$$

Proof. Let $(\ddot{p}, p_1, p_2, p_3) \in Q(G)$. We claim that there is an isometric path cover of G of size $\ddot{p} + p_1 + p_2 + p_3$. Since $(\ddot{p}, p_1, p_2, p_3) \in Q(G)$, we know that $V(G)$ has a partition into $\ddot{p} + p_1 + p_2 + p_3$ subsets $S \subseteq V(G)$ of which,

(a) for $i \in \{1, 2, 3\}$, exactly p_i induced $G[S]$ is isomorphic to P_i, and

(b) for the remaining $\ddot{p}$ subsets we have $G[S]$ isomorphic to $2P_1$.

Since G is a connected cograph, in Case (b), there is an induced path of length 2 with the vertices in S being its end vertices. So these paths along with the paths in Case (a) forms an isometric path cover of G of size $\ddot{p} + p_1 + p_2 + p_3$.

Now we prove the reverse direction. Without loss of generality we assume that there is an optimal isometric path cover $\mathcal{P}$ of G where for each path $P \in \mathcal{P}$, its end vertices appear only in one path which is P. Let $\ell = |\mathcal{P}|$. Now we claim that there is a tuple $(\ddot{p}, p_1, p_2, p_3) \in Q(G)$ such that $\ddot{p}+p_1+p_2+p_3 = \ell$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$, where $\mathcal{P}_j$ contains all paths on j vertices from $\mathcal{P}$. Let $\mathcal{Q}_1 = \{V(P) \colon P \in \mathcal{P}_1\}$ and $\mathcal{Q}_2 = \{V(P) \colon P \in \mathcal{P}_2\}$. Now we construct $\ddot{\mathcal{Q}}$ and $\mathcal{Q}_3$. Initially, we set $\ddot{\mathcal{Q}} := \emptyset$ and $\mathcal{Q}_3 := \emptyset$. Consider a vertex z such that z is an intermediate vertex in a path in $\mathcal{P}_3$. Suppose z is present in n_z paths. Because of our assumption of $\mathcal{P}$, we know that z is the intermediate vertex in all those n_v paths in $\mathcal{P}_3$. Let $R_1, R_2, \ldots, R_{n_z}$ be these paths. Now let $S_{z,i}$ be the set containing the end vertices of R_i for all $i \in [n_z - 1]$ and $S_{z,n_z} = V(R_{n_z})$. Now we set $\ddot{\mathcal{Q}} := \ddot{\mathcal{Q}} \cup \{S_{z,i} : i \in [n_z - 1]\}$ and $\mathcal{Q}_3 := \mathcal{Q}_3 \cup \{S_{i,n_z}\}$. We do this procedure for each z such that it is an intermediate vertex of a path in $\mathcal{P}_3$. Let $\ddot{p} = |\ddot{\mathcal{Q}}|$ and $p_i = |\mathcal{Q}_i|$ for all $i \in \{1, 2, 3\}$. It is easy to see that $\ddot{p} + p_1 + p_2 + p_3 = \ell$ and the above construction of $\ddot{\mathcal{Q}}$ and $\mathcal{Q}_3$ implies that $(\ddot{p}, p_1, p_2, p_3) \in Q(G)$. ◀

Lemma 21 implies that Isometric Path Cover can be solved in time $O(n^{10})$ on cographs.

5 Hardness on Split Graphs

Here, we show that Isometric Path Partition is already NP-hard on split graphs.

▶ **Theorem 22** (⋆). *Isometric Path Partition is NP-hard on split graphs.*

Sketch. We provide a polynomial-time reduction from the NP-complete 3-XSAT problem [13, Lemma 5], where given a propositional formula Φ in CNF such that every clause of Φ has exactly 3 positive literals and every variable occurs exactly in 3 clauses of Φ, the task is to decide whether there is an assignment of the variables of Φ that satisfies exactly one literal from every clause. Let Φ be the given instance of 3-XSAT. We will construct a split graph $G = (T, B, E)$ where T is the independent set and B is the clique such that Φ has an assignment satisfying exactly one literal from every clause if and only if G has isometric path partition $\mathcal{P}$ of size at most $32n^2 - 4n$, where n is a number of clauses (and also the number of variables) of Φ. We shall frequently refer to B as "bottom" and T as "top".

Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be the set of clauses of Φ and let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be the set of variables of Φ. G is constructed from three types of gadgets defined as follows and illustrated in Figure 2. For every variable x_a, G contains the gadget $G_{\mathcal{X}}(x_a)$ defined as follows. Let C_i, C_j and C_k with $1 \leq i < j < k \leq n$ be the 3 clauses that contain x_a. For every $b \in \{i, j, k\}$, $G_{\mathcal{X}}(x_a)$ has the vertices $x_{a,b}$, $\overline{x_{a,b}}$, $x^{\otimes}_{a,b}$ and $x^{\odot}_{a,b}$ with $x_{a,b}, \overline{x_{a,b}} \in B$ and $x^{\otimes}_{a,b}, x^{\odot}_{a,b} \in T$ and the edges $x_{a,b}x^{\otimes}_{a,b}$ and $\overline{x_{a,b}}x^{\odot}_{a,b}$. Additionally, $G_{\mathcal{X}}(x_a)$ contains the edges $\overline{x_{a,i}}x^{\otimes}_{a,j}$, $\overline{x_{a,j}}x^{\otimes}_{a,k}$, and $\overline{x_{a,k}}x^{\otimes}_{a,i}$. Intuitively, the gadget is used to ensure that all occurrences of the variable x_a are assigned in the same manner, which is achieved by forcing that any solution for G can cover all vertices in the gadget in only two manners, which are illustrated in Figure 2.

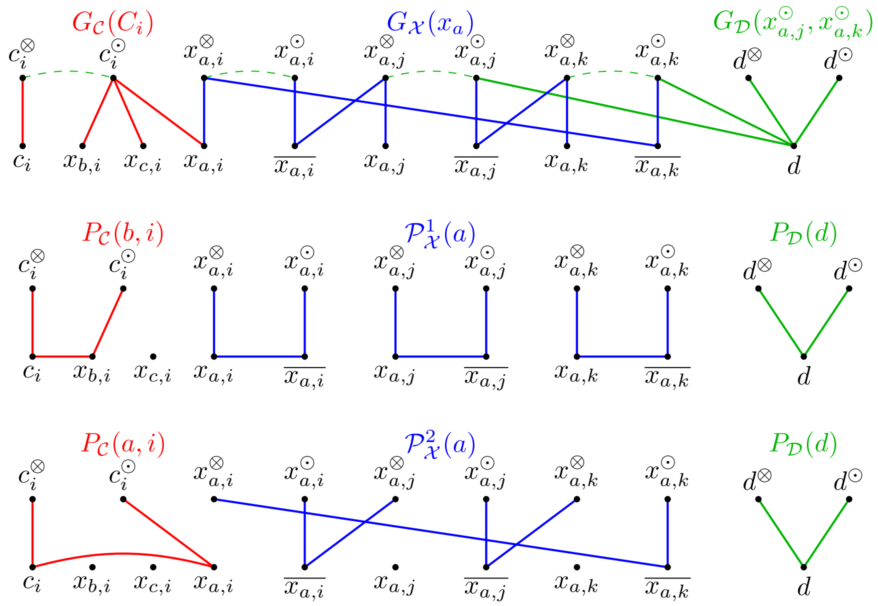

Figure 2 Illustration of the gadgets $G_{\mathcal{C}}(C_i)$, $G_{\mathcal{X}}(x_a)$, and $G_{\mathcal{D}}(x^{\odot}_{a,j}, x^{\odot}_{a,k})$ used in the proof of Theorem 1. The colors of the edges indicate, which gadget they belong to, *i.e.*, red, blue, and green edges are part of $G_{\mathcal{C}}(C_i)$, $G_{\mathcal{X}}(x_a)$, and $G_{\mathcal{D}}(x^{\odot}_{a,j}, x^{\odot}_{a,k})$, respectively. The top figure provides the edges that are part of each gadget (without the edges that are part of the clique on B). The dashed green edges indicate pairs of twin vertices, *i.e.*, the only pairs on the top that can be used as endpoints of isometric paths of length 3. The center and the bottom figure together provide the two possible configurations for how the vertices of the gadgets can be cover in an isometric path partition, which is also descripted in (1), (3), and (4). The center figure illustrates the case that the variable x_a is set to 0 and does not satisfy the clause C_i and the bottom figure illustrates the opposite case.

For every clause $C_i \in \mathcal{C}$, G contains the gadget $G_{\mathcal{C}}(C_i)$. The gadget $G_{\mathcal{C}}(C_i)$ consists of 1 new bottom vertex c_i, 2 new top vertices $c^{\otimes}_i$ and $c^{\odot}_i$, and the edges $c_i c^{\otimes}_i$ and $x_{a,i} c^{\odot}_i$ for every $x_a \in C_i$. This gadget $G_{\mathcal{C}}(C_i)$ will ensure that every clause in Φ is satisfied by exactly one literal. Given two distinct top vertices u and v, the last gadget $G_{\mathcal{D}}(u,v)$, which we call the *destroyer gadget*, has 1 bottom vertex $d_{u,v}$, 2 top vertices $d^{\otimes}_{u,v}$ and $d^{\odot}_{u,v}$ and the edges $d_{u,v} d^{\otimes}_{u,v}$, $d_{u,v} d^{\odot}_{u,v}$, $d_{u,v} u$ and $d_{u,v} v$. The purpose of the destroyer gadget is to ensure that the two top vertices u and v can not occur together in a path of length 3.

We are now ready to define the graph G. Let $G' = \bigcup_{x \in \mathcal{X}} G_{\mathcal{X}}(x) \cup \bigcup_{C \in \mathcal{C}} G_{\mathcal{C}}(C)$. Note that $|T(G')| = 6n + 2n = 8n$. We say that two top vertices v and v' from $T(G')$ are twins, if there exists $u \in B(G')$ such that $v = u^{\otimes}$ and $v' = u^{\odot}$. Then, G is the union of G' and an instance of the destroyer gadget $G_{\mathcal{D}}(u,v)$ for every two distinct top vertices u and v of $T(G')$ that are not twins. Let $B_{\mathcal{D}}$ be the set of all bottom vertices from all of the destroyer gadgets. Note that $|T(G)| = |T(G')|(|T(G')| - 2) + |T(G')| = 64n^2 - 8n$.

Consider an isometric path partition $\mathcal{P}$ of G with $|\mathcal{P}| = \frac{|T(G)|}{2} = 32n^2 - 4n$. The correctness of the reduction can now be shown rather straightforwardly from the following properties ($\star$):

(1) For every $d \in B_{\mathcal{D}}$, the path $P_{\mathcal{D}}(d) = (d^{\otimes}, d, d^{\odot})$ is included in $\mathcal{P}$.
(2) If a path $P \in \mathcal{P}$ is of length 3, then its endpoints are twins.
(3) For every $i \in [n]$ there exists $a \in [n]$ such that $P_{\mathcal{C}}(a,i) = (c^{\otimes}_i, c_i, x_{a,i}, c^{\odot})$ is in $\mathcal{P}$.

(4) For any $a, i, j, k \in [n]$ with x_a belonging to $C_i \cap C_j \cap C_k$ and $i < j < k$, precisely one of the following scenarios occurs:

(a) The set of paths $\mathcal{P}^1_{\mathcal{X}}(a) = \{ (x^{\otimes}_{a,\ell}, x_{a,\ell}, \overline{x_{a,\ell}}, x^{\odot}_{a,\ell}) \mid \ell \in \{i, j, k\} \}$ is a subset of $\mathcal{P}$, or

(b) $\mathcal{P}^2_{\mathcal{X}}(a) = \{ (x^{\otimes}_{a,i}, \overline{x_{a,k}}, x^{\odot}_{a,k}), (x^{\otimes}_{a,j}, \overline{x_{a,i}}, x^{\odot}_{a,i}), (x^{\otimes}_{a,k}, \overline{x_{a,j}}, x^{\odot}_{a,j}) \}$ is a subset of $\mathcal{P}$.

(5) For every $a, i \in [n]$ and $P \in \mathcal{P}$, such that $x_a \in C_i$ and $x_{a,i} \in V(P)$, $P = P_{\mathcal{C}}(a, i)$ or $P \in \mathcal{P}^1_{\mathcal{X}}(a)$. ◀

6 Conclusion

In this paper we proved that Isometric Path Partition remains NP-hard on split graphs. We also showed that both Isometric Path Cover and Isometric Path Partition admit polynomial time algorithms on chain graphs and cographs. Algorithms faster than the ones provided in this paper would be interesting. Another direction of research is to look for other graph classes where Isometric Path Cover and Isometric Path Partition admit polynomial time algorithms. Graph classes like bipartite permutation graphs, proper interval graphs, strongly chordal split graphs, *etc.* are natural candidates. The computational complexities of Isometric Path Cover and Induced Path Cover on split graphs remain open.

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