The Sum of Squares in Polycubes

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Abstract

We give several ways to derive and express classic summation problems in terms of polycubes. We visualize them with 3D printed models. The video is here: `http://go.ncsu.edu/sum_of_squares`.

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1 Introduction

In 1960, Martin Gardner popularized a generalization of dominoes called polyominoes [3]. These shapes, made from squares glued together along edges, led to many computational problems associated with their enumeration including higher-dimensional variants called *polycubes* [6, 2]. In his book on polyominoes, Solomon Golomb describes many puzzle problems based on the idea of reconfiguring a set polyominoes into different shapes [4]. In this video, we show how polycube reconfiguration problems lead to closed form solutions to classic summation problems.

2 A Closed Form for the Sum of Squares

In his work *On Spirals*[1], Archimedes used tangents of a spiral to solve the hardest computational problems of his day, trisecting an angle and squaring the circle. This treatise also includes a formula for the sum of squares. In modern notation, the identity said

$$3\sum_{k=1}^{n} k^2 = n^2(n+1) + \sum_{k=1}^{n} k \tag{1}$$

Equivalently,

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n(n+1)(n+1/2). \tag{2}$$

In Chapter 2 of their book *Concrete Mathematics* [5], Graham, Knuth, and Patashnik give seven different ways to derive this formula for the sum of squares analytically. Below, we show four more ways one could derive this sum from manipulating polycubes.

3 A Proof Without Words

In 1984, in Mathematics Magazine, Man-Keung Siu published a proof without words of the sum of squares formula by representing the sum of squares as a pyramid [8]. Each layer is a square. Let $\mathrm{pyr}(n)$ denote the sum of squares.

$$\mathrm{pyr}(n) = \sum_{k=1}^{n} k^2.$$

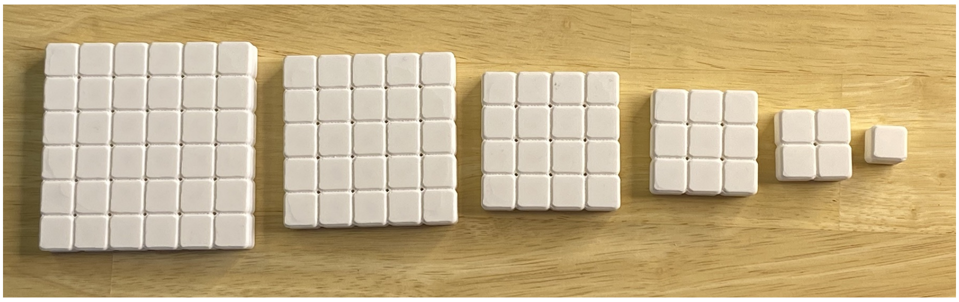

Figure 1 The sum of squares can be viewed as a collection of polycubes.

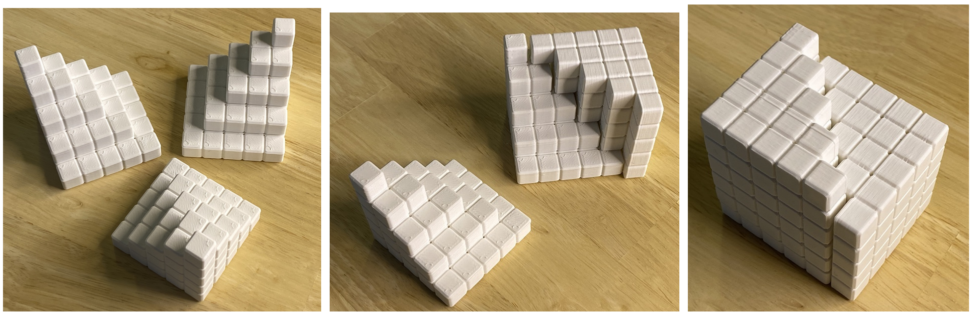

Figure 2 The photo shows a 3D printed version of Siu's proof without words.

Three pyramids can be put together to make a cube, and the leftover pieces give the low-order terms.

Specifically, there is one extra $n \times n$ square and a (discrete) triangle. Let $\mathrm{tri}(n)$ denote the size of the triangle, i.e.,

$$\mathrm{tri}(n) = \sum_{k=1}^{n} k = \binom{n+1}{2}.$$

So, if one believes the picture, the result is

$$3\mathrm{pyr}(n) = \mathrm{cube}(n) + \mathrm{square}(n) + \mathrm{tri}(n) = n^3 + n^2 + \binom{n+1}{2}.$$

More recently, Siu's picture was posted to Math Stack Exchange[1] and by far, the most upvoted comment was one that said they didn't think the picture was convincing on it own, but maybe would be more believable with a physical model. The concern is that it depends on believing that there are no holes in the interior of the cube and that everything indeed, fits perfectly together. This comment was the motivation for 3D printing physical models.

4 Pyramids and Tetrahedra

A different way to derive a sum of squares formula is to observe that a pyramid is the sum of two tetrahedra. A discrete tetrahedron is a sum of triangles:

$$\mathrm{tet}(n) = \sum_{k=1}^{n} \binom{k+1}{2} = \binom{k+2}{3}.$$

[1] `http://math.stackexchange.com/a/48152/301977`

The closed form can be checked by applying induction and Pascal's identity. It can also be understood as counting the number of ways to choose integers a, b, c, d such that $a+b+c+d = n-1$. These sums correspond to the barycentric coordinates of the integer points of a tetrahedron embedded in $\mathbb{R}^4$ with vertices at $n-1$ times the standard basis vectors. The same intuition explains why $\mathrm{tri}(n) = \binom{n+1}{2}$ and implies that the discrete d-simplex has size

$$s_d(n) := \sum_{k=1}^{n} s_{d-1}(k) = \binom{n+d-1}{d}.$$

Splitting the pyramid into two tetrahedra as in Figure 4, we get that

$$\mathrm{pyr}(n) = \mathrm{tet}(n) + \mathrm{tet}(n-1).$$

Equivalently,

$$\sum_{k=1}^{n} k^2 = \binom{n+2}{3} + \binom{n+1}{3},$$

which is another way to express the closed forms in (1) and (2).

Breaking the pyramid into two tetrahedra allows them to be rearranged into another shape with the same volume. This new shape can be stacked with two other pyramids to form a (discrete) triangular prism with base $\mathrm{tri}(n)$ and height $2n+1$.

Thus, we get another equivalent construction as shown in Figure 5.

$$3\mathrm{pyr}(n) = \mathrm{tri}(n)(2n+1) = \binom{n+1}{2}(2n+1).$$

5 Four Pyramids

If we put four pyramids together, they make a new pyramid shape that is not as steep. Each layer is a square, but we only get even squares. Separating this pyramid into two tetrahedra results in a sum of odd triangles plus a sum of even triangles. Rearranging the triangles results in a regular tetrahedron that is twice as tall.

That is,

$$4\mathrm{pyr}(n) = \mathrm{tet}(2n),$$

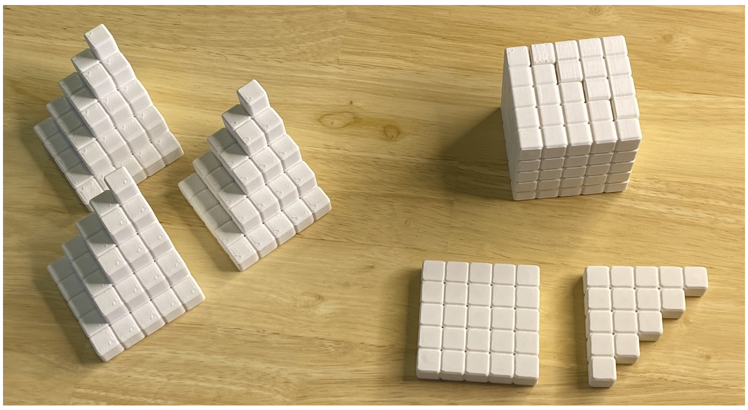

Figure 3 The three pyramids are equal to a cube, a square, and a triangle.

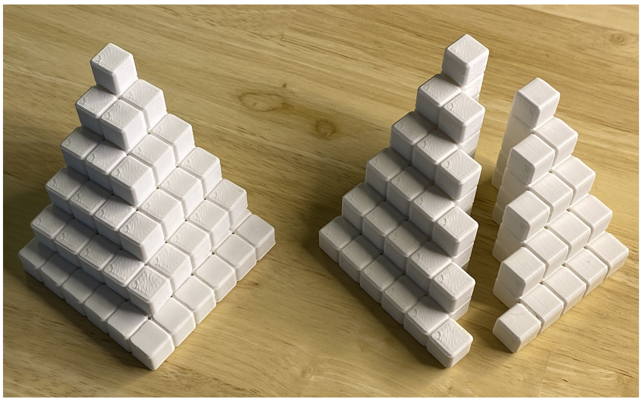

Figure 4 The pyramid can be divided into two tetrahedra.

and thus,

$$\sum_{k=1}^{n} k^2 = \frac{1}{4}\binom{2n+2}{3}.$$

This is yet another way to express the same closed form.

6 A Sum of Cubes

Writing a sum of squares as a sum of two tetrahedra follows from the fact that each square is a sum of two triangles. To extend this idea to a sum of squares, we might decompose a cube into six (discrete) tetrahedra. In this case, we would use the following construction of a cube.

$$\mathrm{cube}(n) = \mathrm{tet}(n) + 4\mathrm{tet}(n-1) + \mathrm{tet}(n-2).$$

It follows that

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} (\mathrm{tet}(k) + 4\mathrm{tet}(k-1) + \mathrm{tet}(k-2)) \\ &= \sum_{k=1}^{n} \left(\binom{k+2}{3} + 4\binom{k+1}{3} + \binom{k}{3}\right) \\ &= \binom{n+3}{4} + 4\binom{n+2}{4} + \binom{n+1}{4}.\end{aligned}$$

However, this is not the most popular way to express the sum of cubes, because it can be written more simply as

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2.$$

In other words, it is the product of two triangles in $\mathbb{R}^4$. Perhaps the reader will find a way to visualize this 4-dimensional object as polycubes. An open source library is available to generate the models printed for this project[7]. It is based on the OpenSCAD software, which uses the CGAL library [9] to perform solid geometry constructions.

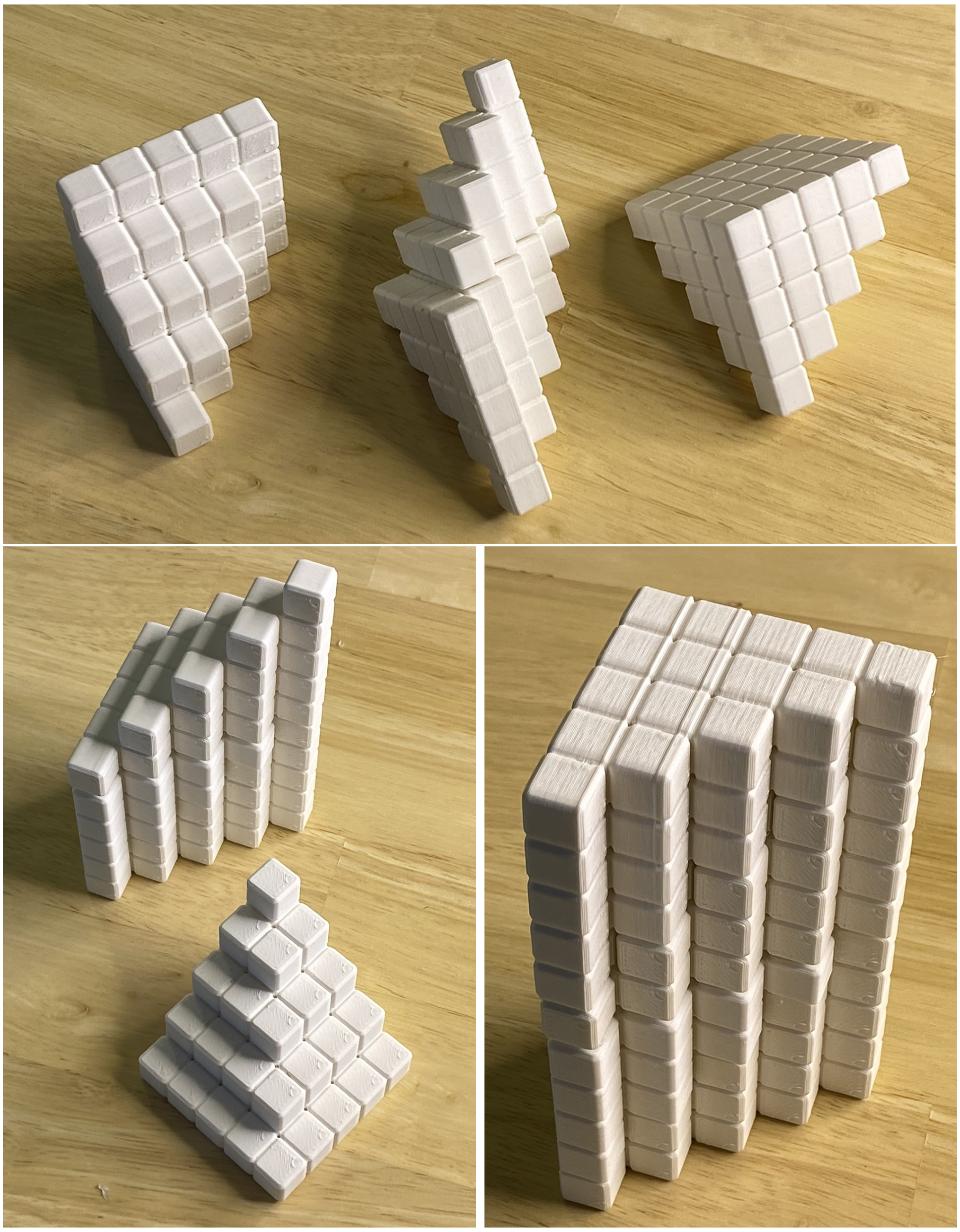

Figure 5 A triangular prism is constructed from three pyramids.

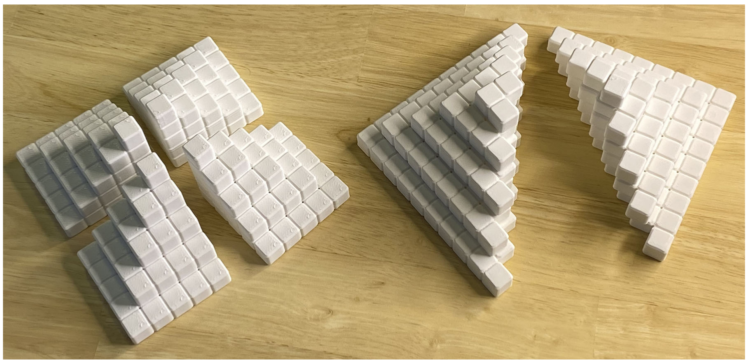

Figure 6 Four pyramids come together to make a pyramid of even squares.

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