# Drawings of Complete Multipartite Graphs up to Triangle Flips


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## Abstract

For a drawing of a labeled graph, the rotation of a vertex or crossing is the cyclic order of its incident edges, represented by the labels of their other endpoints. The extended rotation system (ERS) of the drawing is the collection of the rotations of all vertices and crossings. A drawing is simple if each pair of edges has at most one common point. Gioan's Theorem states that for any two simple drawings of the complete graph $K_n$ with the same crossing edge pairs, one drawing can be transformed into the other by a sequence of triangle flips (a.k.a. Reidemeister moves of Type 3). This operation refers to the act of moving one edge of a triangular cell formed by three pairwise crossing edges over the opposite crossing of the cell, via a local transformation.

We investigate to what extent Gioan-type theorems can be obtained for wider classes of graphs. A necessary (but in general not sufficient) condition for two drawings of a graph to be transformable into each other by a sequence of triangle flips is that they have the same ERS. As our main result, we show that for the large class of complete multipartite graphs, this necessary condition is in fact also sufficient. We present two different proofs of this result, one of which is shorter, while the other one yields a polynomial time algorithm for which the number of needed triangle flips for graphs on $n$ vertices is bounded by $O(n^{16})$. The latter proof uses a Carathéodory-type theorem for simple drawings of complete multipartite graphs, which we believe to be of independent interest.

Moreover, we show that our Gioan-type theorem for complete multipartite graphs is essentially tight in the following sense: For the complete bipartite graph $K_{m,n}$ minus two edges and $K_{m,n}$ plus one edge for any $m, n \geq 4$, as well as $K_n$ minus a 4-cycle for any $n \geq 5$, there exist two simple drawings with the same ERS that cannot be transformed into each other using triangle flips. So having the same ERS does not remain sufficient when removing or adding very few edges.



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## 1 Introduction

Gioan's Theorem states that any two simple drawings of the complete graph $K_n$ in which the same pairs of edges cross can be transformed into each other (up to strong isomorphism) via a sequence of triangle flips. Informally, a triangle flip is the act of moving one edge of a triangular cell formed by three pairwise crossing edges over the opposite crossing of the cell; see Figure 1 for an illustration of this operation and Section 2 for the formal definition.

**Figure 1** A sketch of a triangle flip.

Gioan's Theorem can be seen as a generalization of results on pseudolines by Ringel [29] from 1955 and Roudneff [30] from 1988 to simple drawings of $K_n$. Gioan's conference paper [15] from 2005 contained a proof sketch only. A full proof was first published in 2017 by Arroyo, McQuillan, Richter, and Salazar [4], who also coined the name "Gioan's Theorem". In 2021, Schaefer [31] generalized Gioan's Theorem to slightly sparser graphs, namely, simple drawings of $K_n$ minus any non-perfect matching. A full version of Gioan's proof [16] finally appeared in 2022.

A priori it is not clear how to generalize Gioan's Theorem beyond Schaefer's result. For transforming drawings of general graphs via triangle flips, it is not sufficient to only have the same crossing edge pairs. We should also consider the rotation of a vertex or edge crossing, which is defined as the cyclic order of emanating edges. For example, Figure 2 shows two simple drawings of the complete bipartite graph $K_{3,3}$ with the same crossing edge pairs and the same rotations of vertices, but different rotations of the crossings involving $b_1r_3$. Observe that triangle flips do not change the rotations of crossings or vertices. A take-away from this observation is that for a Gioan-type theorem to hold, the rotations of all crossings and vertices must be the same in both drawings. A concept capturing exactly this necessity is the extended rotation system. The extended rotation system (ERS) of a drawing of a graph is the collection of the rotations of all vertices and crossings. In this light, one of the contributions of Gioan's Theorem is that for drawings of the complete graph, having the same crossing edge pairs is equivalent to having the same ERS (up to global inversion) [15, 16]. This fact has been first stated by Gioan [15]; the first published proofs are by Kynčl [22, 23]. An analogous statement for $K_n$ minus any non-perfect matching has been shown by Schaefer [31]. For complete multipartite graphs, this equivalence does not hold; see again Figure 2.

As our main result, we show that having the same ERS is sufficient to transform simple drawings of complete multipartite graphs into each other via triangle flips. We thus obtain a Gioan-type theorem for a large class of graphs that includes the before studied graphs, namely complete graphs [4, 15, 16, 31] and complete graphs minus a non-perfect matching [31].

▶ **Theorem 1.** *Let $D_1$ and $D_2$ be two simple drawings of a complete multipartite graph on the sphere $\mathcal{S}^2$ with the same ERS. Then there is a sequence of triangle flips that transforms $D_1$ into $D_2$.*

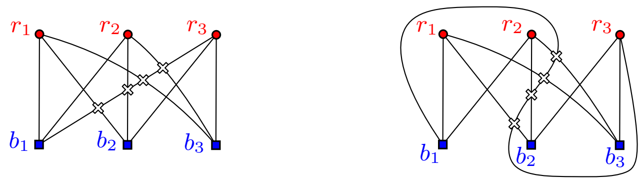


**Figure 2** Two simple drawings of $K_{3,3}$ with the same crossing edge pairs and same rotations at all vertices but different rotations at all crossings involving the edge $b_1r_3$ and hence different ERSs.

We also show that Theorem 1 is essentially tight in the sense that having the same ERS does not remain sufficient when removing or adding very few edges.

▶ **Theorem 2.** *For any $m, n \geq 3$ and $K_{m,n}$ minus any two edges, there exist two simple drawings with the same ERS that cannot be transformed into each other using triangle flips. The same holds for any $n \geq 5$ and $K_n$ minus any four-cycle $C_4$, as well as for any $m \geq 4, n \geq 1$ and $K_{m,n}$ plus one edge between vertices in the bipartition class of size $m$.*

The first part of Theorem 2 implies that an analogue to Schaefer's generalization of Gioan's Theorem for $K_n$ minus a non-perfect matching cannot be achieved for complete bipartite graphs, not even for $K_{m,n}$ minus a matching of size two. Note that $K_{m,n}$ with $m \geq 4$ and $n \geq 1$ is a subgraph of $K_{n+m}$ minus a 4-cycle. Hence, the second part of Theorem 2 implies that – perhaps counterintuitively – the set of graphs for which a Gioan-type theorem holds is not closed under adding edges. From the proof of Theorem 2 it follows that Theorem 1 cannot be extended to any graph that contains a $K_5$ minus a four-cycle $C_4$ or a $K_{3,2}$ minus two edges incident to the same vertex of the smaller partition class, as an induced subgraph.

We present two different proofs of Theorem 1. Our first proof uses a similar approach as the proof of Gioan's Theorem by Schaefer [31]. His proof heavily relies on a (plane) spanning star as a basis for transforming one drawing into the other. While plane spanning stars exist in any simple drawing of $K_n$, also minus a non-perfect matching, this is in general not the case for complete multipartite graphs. However, any simple drawing of a complete multipartite graph $G$ contains a plane spanning tree [2]. We show that for drawings of $G$ with the same ERS, such a plane spanning tree can be used for transforming one drawing into the other. The resulting proof is shorter and probably more elegant than the second proof. But it does not directly yield a polynomial time transformation algorithm, as it is still an open question [2] whether a plane spanning tree can be found in polynomial time.

Our second proof yields a polynomial time algorithm for the transformation. It uses a similar approach as the proof of Gioan's Theorem by Arroyo, McQuillan, Richter, and Salazar [4]. Several ingredients of their proof are known properties of drawings of complete graphs or follow directly from such properties, while it was unknown whether analogous statements hold for drawings of other graphs. Hence, for our proof we discover a number of useful, fundamental properties of simple drawings of complete multipartite graphs. For example, we establish a Carathéodory-type theorem for them.

The classic Carathéodory Theorem states that if a point $p \in \mathbb{R}^2$ lies in the convex hull of a set $A \subset \mathbb{R}^2$ of $n \geq 3$ points, then there exists a triangle spanned by points of $A$ that contains $p$. In the terminology of drawings, if a point $p$ lies in a bounded cell of a straight-line drawing $D$ of $K_n$ in $\mathbb{R}^2$, then there exists a 3-cycle $C$ in $D$ so that $p$ lies in the bounded cell of $C$. This statement has been generalized to simple (not necessarily straight-line) drawings of $K_n$ [6, 7]. However, it clearly does not generalize to arbitrary (non-complete) graphs; consider for example a simple drawing of a path with self-intersections that forms a bounded cell. A natural question is, for which classes of graphs this statement, or a variation of it, holds. We show that it holds for complete multipartite graphs if in addition to 3-cycles – which might not exist in those graphs – we also allow 4-cycles to contain $p$.

▶ **Theorem 3** (Carathéodory-type theorem for simple drawings of complete multipartite graphs)**.** *Let $D$ be a simple drawing of a complete multipartite graph $G$ in the plane. For every point $p$ in a bounded cell of $D$, there exists a cycle $C$ of length three or four in $D$ such that $p$ is contained in a bounded cell of $C$. This statement is tight in the sense that it may not hold for $G$ minus one edge.*

**Number of triangle flips.** Schaefer [31, Remark 3.3] showed that for $K_n$, polynomially many triangle flips are sufficient and gave an upper bound of $O(n^{20})$ for the number of required flips. Using a different approach in our second proof of Theorem 1, we show an upper bound of $O(n^{16})$ triangle flips for complete multipartite graphs on $n$ vertices. We further present drawings which, regardless of the approach, require at least $\Omega(n^6)$ triangle flips.

**Motivation and related work.** Originally, rotation systems were invented to investigate embeddings of graphs on higher-genus surfaces [17]. Nowadays they are widely used to represent drawings of graphs in the plane and to derive their structural properties. Gioan's Theorem implies that for simple drawings of complete graphs, the set of crossing pairs of edges determines the drawing's ERS. Conversely, for drawings of complete graphs, the rotation system determines which pairs of edges cross [22, 27]. These relations are crucial in the study of simple drawings of complete graphs, their generation and enumeration [1, 22, 24].

For non-complete graphs, the literature on rotation systems for simple drawings is rather sparse. Besides the recent work of Schaefer [31], we are only aware of work by Cardinal and Felsner [8], who investigate the realization of complete bipartite graphs as outer drawings. The main reason why there are no further results on rotation systems beyond drawings of complete graphs is the lack of known properties in these cases. Our work contributes towards the generalization of rotation systems to drawings of wider graph classes, not only by the main statement but also due to the structural results obtained along the way.

We note that rotation systems of drawings also play a role in a wider context. For example, they are crucial in a recent breakthrough result devising an algorithm for the subpolynomial approximation of the crossing number for non-simple drawings of general graphs [10].

The study of triangle flips has a long history in several different contexts. In addition to the mentioned work on Gioan's Theorem [4, 15, 16, 31], this in particular includes work on arrangements of pseudolines [14, 29, 30, 32], knot theory [3, 20, 21, 25, 28, 35, 36], as well as on transforming curves on compact oriented surfaces [9].

**Outline.** In Section 2, we mainly state definitions, introduce notation, and give a characterization of complete multipartite graphs. In Sections 3 and 4 we sketch the proofs of the Carathéodory-type Theorem 3 and Theorem 2, respectively. Section 5 is devoted to proving Theorem 1, where the first proof is given nearly fully, and the second one is shortly sketched to explain the algorithm. In Section 6 we present bounds on the required number of triangle flips derived from the second proof. We conclude the paper with open questions in Section 7.

## 2 Definitions and preliminaries

A graph $G = (V, E)$ is *multipartite* if its vertex set $V$ can be partitioned into $k$ nonempty subsets $V_1, \ldots, V_k$, for some $k \in \mathbb{N}$, such that each $V_i$, for $i \in \{1, \ldots, k\}$, induces an independent set in $G$, that is, no two vertices in $V_i$ are adjacent. A *complete multipartite* *graph* $G = (V, E)$ contains *all* edges outside of the independent sets, that is, we have $E = \{v_iv_j : v_i \in V_i \ \wedge \ v_j \in V_j \ \wedge \ 1 \leq i < j \leq k\}$. For a multiset $\{n_1, \ldots, n_k\}$ of natural numbers, there is a unique (up to isomorphism) complete multipartite graph $K_{n_1,\ldots,n_k}$ with $|V_j| = n_j$, for all $j \in \{1, \ldots, k\}$. Note that both the empty graph on $n$ vertices (with $k = 1$ and $n_1 = n$) and the complete graph $K_n$ (with $k = n$ and $n_1 = \cdots = n_k = 1$) are complete multipartite graphs. We also have the following useful characterization, whose proof is an easy graph-theoretic exercise.

▶ **Lemma 4.** *A graph $G = (V, E)$ is complete multipartite if and only if for every edge $uv \in E$ and every vertex $w \in V \setminus \{u, v\}$ we have $uw \in E$ or $vw \in E$ (or both).*

**Drawings.** A *drawing* $\gamma$ of a graph $G = (V, E)$ is a geometric representation of $G$ by points and curves on an oriented surface $\mathcal{S}$. More precisely, every vertex $v$ of $G$ is mapped to a point $\gamma_v$ on $\mathcal{S}$ and every edge $uv$ of $G$ is mapped to a simple (that is, continuous and not self-intersecting) curve $\gamma_{uv}$ on $\mathcal{S}$ with endpoints $\gamma_u$ and $\gamma_v$, such that: (1) any two vertices are mapped to distinct points ($\gamma_u = \gamma_v \Longrightarrow u = v$, for all $u, v \in V$), (2) no vertex is mapped to the relative interior of an edge ($\gamma_{uv} \cap \gamma_w = \emptyset$, for all $uv \in E$ and $w \in V \setminus \{u, v\}$), and (3) every pair of curves $\gamma_e, \gamma_f$, for $e \neq f$, intersects in at most finitely many points, each of which is either a common endpoint or a proper, transversal crossing.

In this paper, we consider drawings on the sphere $\mathcal{S}^2$, except for a few places – specified explicitly – where we consider drawings in the plane $\mathbb{R}^2$. All our graphs and drawings are labeled. Hence, we often identify vertices and edges with their geometric representation in a drawing. Any subgraph $H$ of $G$ induces a *subdrawing* $\gamma[H]$ that is obtained by restricting $\gamma$ to the vertices and edges of $H$. For a graph $F$, an $F$*-subdrawing* of $\gamma$ is a subdrawing $\gamma[H]$ that is induced by some subgraph $H$ of $G$ that is isomorphic to $F$. A drawing partitions $\mathcal{S}$ into vertices (*endpoints*) and *crossings* of the curves $\{\gamma_e : e \in E\}$, *edge fragments* (the connected components of the curves $\{\gamma_e : e \in E\}$ after removing all vertices and crossings), and *cells* (the connected components of $\mathcal{S}$ after removing all vertices, crossings, and edge fragments). For a cell $C$ we denote by $\partial C$ the *boundary* of $C$. A cell that is bounded by exactly three edge fragments is called a *tricell*.

The class of drawings of a graph is vast and for many purposes too rich to be directly useful. To begin with, it is not clear in general how to represent a drawing using a finite amount of space. Two natural approaches to address this concern are to (1) further restrict the class of drawings or (2) study drawings on a much coarser level, up to some notion of isomorphism. In this work, we use a combination of both of these approaches.

**Simple drawings.** An example for the first approach are *straight-line drawings* in the Euclidean plane (also known as *geometric graphs*), where the geometry of an edge is uniquely determined by the location of its endpoints; see the *Handbook of Discrete and Computational Geometry* [34, Chapter 10] and references therein. In this work, we consider a more general class of drawings, which appear in the literature as *simple drawings* [11], *good drawings* [5, 12], *topological graphs* [26], *simple topological graphs* [22], and even just as *drawings* [18]. In a simple drawing, every pair of edges has at most one point in common, either a common endpoint or a proper crossing. Additionally, we may assume that no three edges meet at a common point. Simple drawings are a combinatorial/topological generalization of straight-line drawings. If the graph $G$ has $n$ vertices, then every simple drawing of $G$ has $O(n^4)$ crossings, edge fragments, and cells. Simple drawings are also important for crossing minimization because all crossing-minimal drawings are simple [33].


**Figure 3** Two drawings of $K_{3,3}$ that have same ERS but are not strongly isomorphic (because $ux$ crosses $vy$ and $wz$ in different order). The shaded tricell is an invertible triangle.

**Strong isomorphism.** An example for the second approach is the notion of strong isomorphism for drawings, defined as follows. Two drawings $\gamma$ and $\eta$ of a graph $G = (V, E)$ are *strongly isomorphic*, denoted by $\gamma \cong \eta$, if there exists an orientation-preserving homeomorphism[1] of $\mathcal{S}$ that maps $\gamma$ to $\eta$, that is, $\gamma_v \mapsto \eta_v$, for all $v \in V$, and $\gamma_e \mapsto \eta_e$, for all $e \in E$. A combinatorial formulation, which is equivalent for connected drawings, can be obtained as follows [22]: (1) the same pairs of edges cross (this is called *weak isomorphism*); (2) the order of crossings along each edge is the same; and (3) at each vertex and crossing the *rotation*, that is, the clockwise circular order of incident edges, is the same (see next paragraph for more details). The notion of strong isomorphism encapsulates basically everything that can be said about a drawing from a topological or combinatorial point of view: the order of edges around vertices and cells, which pairs of edges cross, and in which order the crossings appear along an edge. For our purposes, we consider strongly isomorphic drawings to be equivalent.

**Extended rotation systems.** A coarser notion of equivalence can be obtained by requiring two drawings to have the same *rotation system*, which is the collection of the rotations of all vertices. Property (3) in the above-mentioned combinatorial description uses a slightly stronger notion of equivalence, where also the rotations at crossings are the same in both drawings. More formally, the *rotation of a crossing* $\chi$ is the clockwise cyclic order of the four vertices of the crossing edge pair which is induced by the cyclic order of edge fragments around $\chi$. (In other words, the rotation of a crossing $\chi$ is the rotation of an additional degree-4 vertex $v_\chi$ obtained by splitting the crossing edge pair at $\chi$ and replacing $\chi$ by $v_\chi$.) The *extended rotation system* (ERS) of a drawing is the collection of rotations of all vertices and crossings. Any two strongly isomorphic drawings have the same ERS [22]. But the converse is not true in general, as the example in Figure 3 demonstrates.

**Crossing triangles.** In fact, the only difference between the two drawings in Figure 3 with respect to strong isomorphism stems from the tricell formed by the triple $ux, vy, wz$ of pairwise crossing edges, which is shaded gray in the figure: In the left drawing, this cell lies to the right of the oriented edge $ux$, whereas in the right drawing, it lies to the left of $ux$. Given a simple drawing, a tricell $\Delta$ in the subdrawing of three pairwise crossing edges $e_1, e_2, e_3$ is called a *crossing triangle*; the three edges $e_1, e_2, e_3$ are said to *span* $\Delta$. Note that every edge triple in a simple drawing spans at most one crossing triangle. The following lemma shows that the crossing triangles are well-defined for complete multipartite graphs. It follows from the proof of Theorem 1, but can also be shown directly (and with a much shorter proof).

[1] Strong isomorphism can also be defined for unlabeled drawings; then a mapping for the vertex sets is needed. The homeomorphism is sometimes not required to be orientation-preserving; then, e.g., mirror-images of drawings are also considered to be strongly isomorphic.

▶ **Lemma 5.** *In every simple drawing of a complete multipartite graph, the set of edge triples that span crossing triangles is uniquely determined by the ERS.*

**Invertible triangles and triangle flips.** To formally define the triangle flip operation, globally fix an orientation $\pi$ of the edges of the abstract graph $G$. This orientation can be arbitrary, but once we fix the graph, we also fix its orientation. With this orientation $\pi$, we can assign every crossing triangle a parity as follows. The *parity* of a crossing triangle $\Delta$ in a drawing is the parity (odd or even) of the number of bounding edges of $\Delta$ such that $\Delta$ lies to the left of the edge (when going along the edge according to its orientation). See Figure 3 for two drawings with even (left) and odd (right) parity of the crossing triangle. A crossing triangle $\Delta$ in a drawing $\gamma$ is *invertible* if there exists another simple drawing $\gamma' \neq \gamma$ of the same graph $G$ with the same edge orientation $\pi$ and with the same ERS in which $\Delta$ appears with the opposite parity. We will show that any invertible triangle in a drawing of a complete multipartite graph is empty of vertices.

Locally redrawing the edges of an empty crossing triangle and thereby changing its parity is an elementary operation to transform a given drawing, say, the one in Figure 3 (left), into a new drawing, such as the one in Figure 3 (right). Up to strong isomorphism, there is a unique way for the redrawing. This operation is referred to as *triangle flip* [4], *triangle mutation* [15], *slide move* [31], *homotopy move* [9, 20], or *Reidemeister move of Type 3*, where the latter name has been extensively used[2] in knot theory [3, 21, 25, 28, 35, 36].

**Triangle flip graphs.** Based on the triangle flip as an elementary operation, we can define a meta graph whose vertices are drawings and whose edges correspond to triangle flips. We fix a graph $G$ and consider all simple drawings of $G$ on $\mathcal{S}$ up to strong isomorphism; these are the vertices of the triangle flip graph $\mathcal{T}(G)$. Any two such drawings $\gamma, \eta$ are connected by an edge in $\mathcal{T}(G)$ if $\eta$ can be obtained from $\gamma$ by a single triangle flip. As triangle flips are reversible, edges are symmetric. So we consider $\mathcal{T}(G)$ as an undirected graph.

Observe that a triangle flip does not change the rotation of any vertex or crossing, only the order of crossings along the edges changes. Therefore only drawings that have the same ERS can be in the same component of $\mathcal{T}(G)$. In general, the flip graph $\mathcal{T}(G)$ may be disconnected. Consider, for instance, the two drawings of a path depicted in in Figure 4. As neither drawing contains any crossing triangle, both are isolated vertices in $\mathcal{T}(G)$.


**Figure 4** Two drawings of a path with the same ERS, but the order of crossings along the edge $cd$ differs, thus, the drawings are not strongly isomorphic. Neither drawing contains any tricell to flip.

## 3 A Carathéodory-type theorem for complete multipartite graphs

This section is devoted to a proof outline of the Carathéodory-type Theorem 3. The corresponding statement for simple drawings of $K_n$, which is a direct generalization of the

[2] albeit in the context of knots also an above/below relationship among the curves is relevant

classic theorem for convex sets in $\mathbb{R}^2$, was shown by Balko, Fulek, and Kynčl [6]. A simpler proof was given later by Bergold, Felsner, Scheucher, Schröder, and Steiner [7], whose proof idea we follow.

**Sketch of Proof.** If $G$ is empty or a star $K_{1,n}$, then the statement is vacuously true. So we assume that $G$ is neither, and thus every pair of distinct vertices $u, v \in V$ with $uv \notin E$ has at least two distinct common neighbors. By studying a minimal counter-example we prove Theorem 3 by contradiction. To that aim, we consider a simple drawing $D$ of $G$ and a point $p$, such that the following holds: (1) $p$ is in a bounded cell of $D$, (2) $p$ is not contained in a bounded cell of any induced $C_i$-subdrawing of $D$, for $i \in \{3,4\}$, and (3) when removing any vertex from $D$, the point $p$ lies in the unbounded cell.

Let $a$ be a vertex of $G$, and let $O$ be the smallest set of edges incident to $a$ such that removal of all edges of $O$ from $D$ puts $p$ into the unbounded cell of the resulting drawing $D^-$. Then in $D^-$ one can draw a simple curve $P$ from $p$ to the interior of the unbounded cell of $D$ so that $P$ does not intersect any vertex or edge of $D^-$. Subject to this constraint, we select $P$ to minimize the number of crossings with edges of $D$. We show that we can assume every edge in $O$ crosses $P$ exactly once. Finally we consider an edge $ab \in O$, which crosses $P$ in a point $p_{ab}$, and analyze two cases depending on whether $ab$ crosses another edge between $a$ and $p_{ab}$ or not. We show that in both cases, $p$ is contained in a bounded cell of an induced $C_i$-subdrawing of $D$, for $i \in \{3,4\}$.

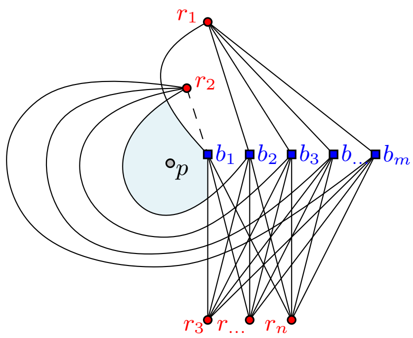


**Figure 5** Drawing of $K_{m,n}$ minus one edge ($r_2b_1$, drawn dashed), based on Figure 6. The point $p$ lies in a bounded cell, but in no $C_i$, for $i \in \{3,4,5\}$.

To see that the theorem may not hold if we remove one edge from $G$, consider the simple drawing of $K_{m,n}$, $m, n \geq 2$, depicted in Figure 5. When removing the edge $b_1r_2$, the point $p$ still lies in a bounded cell, but any cycle that encloses $p$ has at least six vertices. ◀

## 4 Theorem 1 is essentially tight

Theorem 2 implies that Theorem 1 is essentially tight: The removal or addition of very few edges may yield a graph for which the theorem does not hold. This implies that the class of graphs for which this Gioan-type theorem holds is not closed under the operation of taking (non-induced) subgraphs or supergraphs. We sketch the proof of Theorem 2 by depicting the drawings we use to show tightness.

Each of Figures 6–9 contains two simple drawings of a graph with the same ERS. In all of them, the crossing order along $b_1r_1$ differs between the two drawings. This order cannot be changed via triangle flips because the edges crossing $b_1r_1$ in different orders are pairwise non-crossing. Figures 6 and 7 cover the case of $K_{m,n}$ minus two adjacent or disjoint edges, Figure 8 is an extension of Figure 6 to $K_m$ minus a 4-cycle, and Figure 9 shows subdrawings of Figure 8 that form a $K_{m-1,n+1}$ plus one edge.

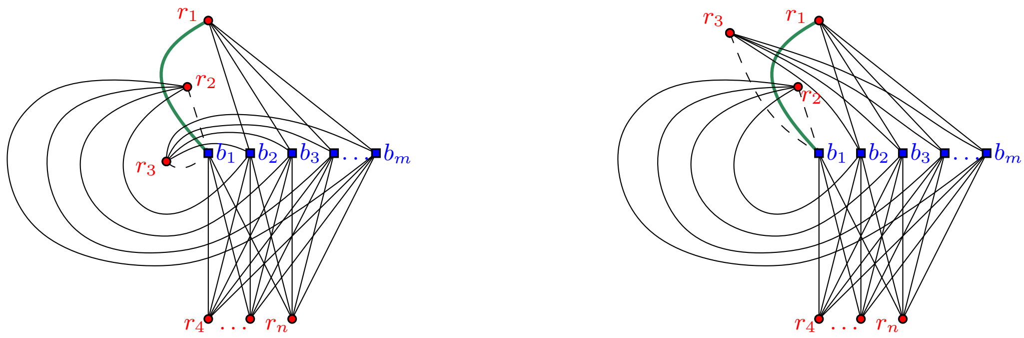


**Figure 6** Two drawings of $K_{m,n}$ minus two adjacent edges $b_1r_2$ and $b_1r_3$ (drawn as dashed lines) that have the same ERS but cannot be transformed into each other via triangle flips.

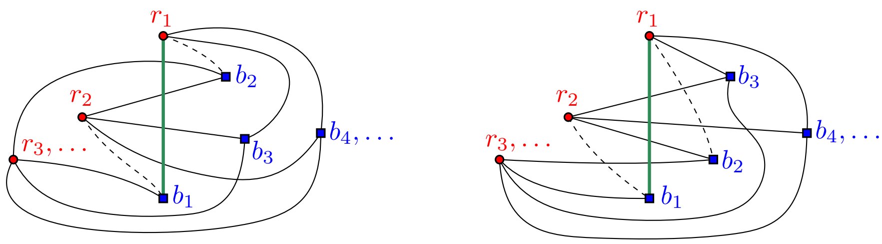


**Figure 7** Two drawings of $K_{m,n}$ minus two independent edges $b_2r_1$ and $b_1r_2$ (drawn dashed) that have the same ERS but cannot be transformed into each other via triangle flips.

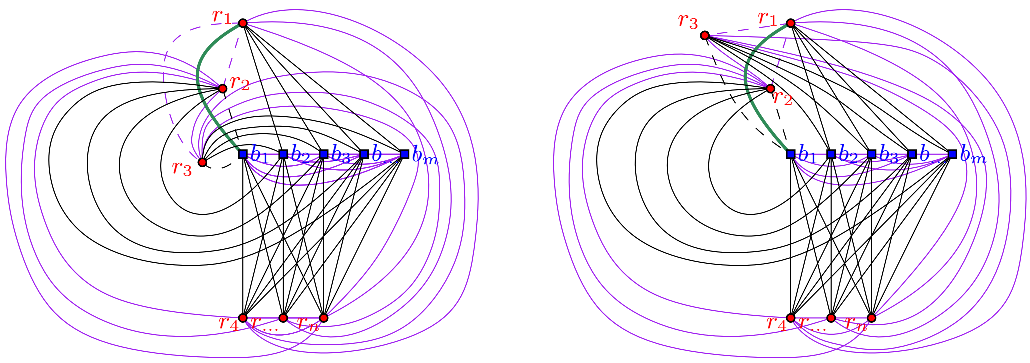


**Figure 8** Two drawings of $K_m$ minus a 4-cycle (drawn dashed) that have the same ERS, but cannot be transformed into each other via triangle flips.

We remark that also two simple drawings with the same ERS that cannot be transformed into each other via triangle flips exist for any graph that contains (1) a $K_5$ minus a 4-cycle, or (2) a $K_{2,3}$ minus two edges sharing a vertex in the bipartition class of cardinality two (where the list of induced subgraphs is not exhaustive). This can be shown by choosing appropriate subdrawings in the construction from Figure 8.

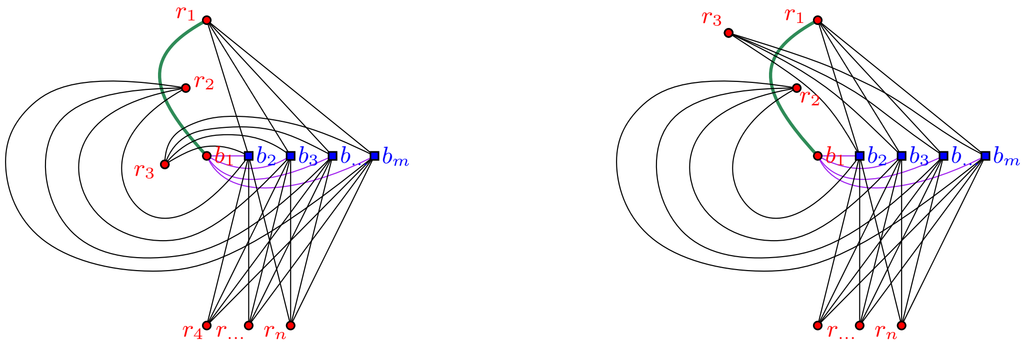


**Figure 9** Two drawings of $K_{m-1,n+1}$ plus one edge $(b_1 r_1)$ that cannot be transformed into each other via triangle flips.

## 5 A Gioan-type theorem for complete multipartite graphs

In this section, we present our two proofs of Theorem 1 and include a short algorithmic discussion of the second one.

### 5.1 First proof of Theorem 1

For our first proof of Theorem 1, we use the same general approach as Schaefer [31]. To closely follow the lines of Schaefer, we also use homeomorphisms in this proof.

**Proof.** Let $G$ be a complete multipartite graph, and let $D_1$ and $D_2$ be two simple drawings of $G$ on $\mathcal{S}^2$ with the same ERS. Let $R = \{r_1, r_2, \dots, r_n\}$ be a maximal independent set in $G$ and let $B = \{b_1, b_2, \dots, b_m\}$ denote the set of the remaining vertices. Note that the graph on the vertex set $R \cup B$ together with all edges with an endpoint in $R$ and one in $B$ forms a complete bipartite graph $K_{n,m}$, and the set $R$ is an independent set in $G$ while $B$ might not necessarily be an independent set.

By [2], the subdrawing of $D_1$ spanned by this $K_{n,m}$ contains a spanning tree $T$ which is drawn crossing-free in this subdrawing and hence also in $D_1$. As $D_1$ and $D_2$ have the same crossing edge pairs, $T$ is drawn crossing-free in $D_2$ as well. Since the rotation systems of $D_1$ and $D_2$ are the same by assumption, the drawings of $T$ in $D_1$ and $D_2$ are homeomorphic. Thus there exists a drawing $D :\cong D_1$ with the following properties.

1. The drawing of $T$ is the same for both drawings $D$ and $D_2$, implying that also the vertex locations are the same in both drawings.
2. Considering the set of the vertices and edges of $D$ and $D_2$ together as the *combined drawing* of $D$ and $D_2$, we denote the cyclical order of edges in $D$ and $D_2$ emanating from a vertex as *combined rotation* at that vertex. For each edge $e$ of $G$ (not in $T$) and each vertex $v$ of $e$, the two drawings of $e$ are consecutive in the combined rotation at $v$.
3. For each edge $e$ of $G$, the two drawings of $e$ are either identical or have only finitely many points in common (two are its endpoints and the others are proper crossings).

Our goal is to change $D$ via triangle flips (and orientation-preserving homeomorphisms) until we obtain $D = D_2$. Since the vertex locations in both drawings are the same, we can speak about two drawings of an edge, one in $D$, and one in $D_2$, being the same or not. As in Schaefer's proof, we iteratively reduce the number of edges that are drawn differently in $D$ and $D_2$. Let $E_=$ be the set of edges whose drawings in $D$ and $D_2$ are the same. Initially, $E_=$ contains at least all edges of $T$. If $E_=$ contains all edges of $G$ then we are done.

So suppose that this is not the case and consider an edge $e$ that is drawn differently in $D$ and $D_2$. Let $e_1$ and $e_2$ denote the curves representing $e$ in $D$ and $D_2$, respectively. Since $D$ and $D_2$ have the same ERS, $e_1$ and $e_2$ cross the same edges of $T$ and they do so with the same crossing rotations. Moreover, the following lemma implies that they also cross those edges in the same order. The lemma can be proven relying on Lemma 4 and using a case distinction for drawings with six vertices.

▶ **Lemma 6.** *Let $D$ be a simple drawing of a complete multipartite graph $G$ on $\mathcal{S}^2$ and let $vw$ be an edge of $G$. Then for any pair of adjacent or disjoint edges crossed by $vw$, the ERS of $D$ determines the order in which $vw$ crosses them.*

Hence $e_1$ and $e_2$ are equivalent with respect to the drawing of $T$ (which is the same in $D$ and $D_2$), that is, $e_1$ has the same sequence of directed crossings with $T$ as $e_2$. Let $\Gamma = e_1 \cup e_2$ be the (not necessarily simple) closed curve formed by $e_1$ and $e_2$. A *lens* in $\Gamma$ is a cell of $\Gamma$ whose boundary is formed by exactly two edge fragments of $\Gamma$, where one is from $e_1$ and one is from $e_2$. Next, consider the drawing $D_T$ of $T$ plus the drawings $e_1$ and $e_2$ of $e$. A lens of $\Gamma$ is called *empty* if it contains no vertices of $T$ (and hence also no vertices of $G$) in its interior. With the next lemma, we show that $\Gamma$ forms an *empty lens*. This lemma is a special case of a result of Hass and Scott on intersecting curves on surfaces [19, Lemma 3.1], which is also known as the *bigon criterion* [13, Section 1.2.4]. Schaefer [31, Lemma 3.2] gives an elementary proof in the planar (or spherical) case when the plane spanning tree $T$ is a star. However, he only uses that the star is a spanning subdrawing that is crossing-free and that $e_1$ and $e_2$ are equivalent with respect to the star. Thus, we can follow the proof line by line to obtain the result for any plane spanning tree $T$.

▶ **Lemma 7** ([13, 19, 31])**.** *Let $D_1$ and $D_2$ be two simple drawings of a graph on $\mathcal{S}^2$ that contain the same crossing-free drawing $D_T$ of a spanning tree $T$ as a subdrawing. Let $e$ be an edge for which the drawings $e_1$ and $e_2$ differ, but are equivalent with respect to $D_T$. Then $\Gamma = e_1 \cup e_2$ forms an empty lens.*

Let $L$ be an empty lens of $\Gamma$, which is formed by the edge fragments $\gamma_1$ of $e_1$ and $\gamma_2$ of $e_2$, respectively. Each of the two points of $\gamma_1 \cap \gamma_2$ is either an endpoint or a crossing between $e_1$ and $e_2$. Recall that, in the combined drawing of $D$ and $D_2$, $e_1$ and $e_2$ are consecutive in the combined rotation at each of their endpoints. Hence, independent of whether the points of $\gamma_1 \cap \gamma_2$ are crossings or endpoints, $\gamma_2$ is what Schaefer calls a "*homotopic detour* of $\gamma_1$ on $e_1$". We next need his detour lemma, which we restate here using slightly different terminology (and for drawings on the sphere instead of in the plane).

▶ **Lemma 8** (detour lemma [31, Lemma 2.1])**.** *Let $\gamma_2$ be a homotopic detour of the arc $\gamma_1$ on the edge $e_1$ in a simple drawing of a graph. Let $F$ be the set of edges which cross $\gamma_2$ at least twice. Then we can apply a sequence of triangle flips and homeomorphisms of the sphere $\mathcal{S}^2$ so that in the resulting drawing, $\gamma_1$ is routed arbitrarily close to $\gamma_2$, without intersecting it. The triangle flips and homeomorphisms only affect a small open neighborhood of the region bounded by $\gamma_1 \cup \gamma_2$, and only edges in $F$ and the $\gamma_1$ part of $e_1$ are redrawn.*

Note that the set $F$ of edges that are affected by the transformation is disjoint from $E_=$, because any edge of $E_=$ is identical in $D$ and $D_2$ and hence intersects $\gamma_2$ at most once.

If at least one of the points of $\gamma_1 \cap \gamma_2$ is a crossing, then after applying the detour lemma, we can redraw $e_1$ (via a homeomorphism) to have at least one fewer crossing with $e_2$ and repeat the process of applying Lemmas 7 and 8 with the redrawn edge.

If none of the points of $\gamma_1 \cap \gamma_2$ is a crossing, then $e_1 \cup e_2$ is a simple closed curve and $\gamma_1 = e_2$ is a homotopic detour of $\gamma_2 = e_1$. Hence, after one final application of Lemma 8, we can redraw $e_1$ to be identical to $e_2$. With this step, $e_2$ is added to $E_=$ and we have reduced the number of edges differing between $D$ and $D_2$ by one.

Repeating this process for the remaining differing edges we obtain two identical drawings. Omitting the homeomorphisms, the process yields a sequence of triangle flips for transforming $D_1$ into $D_2$ (up to strong isomorphism), which completes the proof of the theorem. ◀

## 5.2 Second Proof of Theorem 1

Our second proof of Theorem 1, which we briefly outline here, uses the same general framework as the proof of Gioan's Theorem by Arroyo, McQuillan, Richter, and Salazar [4].

**Sketch of Proof.** We consider two simple drawings $D_1$ and $D_2$ of a complete (multipartite) graph $G = (V, E)$ with the same ERS, and one of them, say $D := D_1$, is iteratively transformed to become "more similar" to the other. Similarity is measured using a subgraph $X$ of $G$ for which we demand as an invariant that the induced subdrawings $D[X]$ and $D_2[X]$ are strongly isomorphic. In each iteration, we will add one edge to $X$ and then perform a sequence of triangle flips in $D$ so as to reestablish the invariant.

Initially, we establish the invariant in the following way. As in the first proof, we consider an independent set $R \subseteq V$ of vertices such that $G$ contains a complete bipartite subgraph between $R$ and $B := V \setminus R$. If $G$ is complete, then $R$ contains a single vertex only; in general, it may contain several vertices. We then pick one vertex $r_0 \in R$ and start by taking $X$ to be the maximal induced substar of $G$ centered at $r_0$ (which includes all vertices of $B$). Then the invariant holds because both drawings have the same rotation system by assumption.

We then consider the (possibly) remaining vertices of $R$ in an arbitrary order. Let $r \in R$ be the next vertex to be considered. First, we show that the position of $r$ in the induced – strongly isomorphic, by the invariant – subdrawings $D[X]$ and $D_2[X]$ is consistent, that is, the vertex $r$ lies in the same (according to isomorphism) face of these drawings. (The proof of this statement uses the Carathéodory-type Theorem 3.)

We add the edges incident to $r$ one by one to $X$. When adding an edge $rb$ to $X$ to obtain $X' = X \cup \{rb\}$, the drawings $D[X']$ and $D_2[X']$ may not be strongly isomorphic because the edge $rb$ may cross other edges in a different order in both drawings. We consider a sort of overlay $O$ of both drawings $D[X']$ and $D_2[X']$, in which the two versions of $rb$ together form a closed curve $\Gamma$ with $O(|V(X')|^4)$ self-crossings, where $|V(X')|$ is the number of vertices of $X'$. In $\Gamma$, we can identify a nice substructure, which we refer to as a *free lens*, and show that it always exists. A lens in $\Gamma$ is *free* if it does not contain any vertex of $O$; it may contain edge crossings, though. Each such edge crossing corresponds to an invertible triangle in $D$. Invertible triangles are empty of vertices not only of the vertices in $X$ but also of the (possibly) not yet considered vertices of $R$. Hence, the edges of $D$ that cross an invertible triangle $\Delta$ behave similarly to a collection of pseudolines inside $\Delta$, except that not all pairs need to cross. Let $m$ be the number of edges that cross $\Delta$. Using a classic sweeping algorithm by Hershberger and Snoeyink [32, Lemma 3.1], all $m$ edges can be "swept" out of $\Delta$ via triangle flips in $D$, where the total number of flips is bounded by $O(m^3)$. After these flips, $\Delta$ has become a crossing triangle and can be flipped in $D$. Processing all invertible triangles inside a selected free lens in this fashion effectively destroys this lens. And after iteratively destroying all free lenses, the resulting drawing $D[X']$ is strongly isomorphic to $D_2[X']$.

After all vertices in $R$ and the complete bipartite subgraph of $G$ between $R$ and $B$ have been added to $X$, we add the remaining edges (the ones with both endpoints in $B$) in exactly the same fashion as described above. ◀

While the outline of the above proof mostly follows the one for $K_n$ [4], its core challenges lie in the proofs of several statements, whose analogues are known for $K_n$ but not for complete multipartite graphs. Among others, these include the arguments about the existence of a free lens and that invertible triangles are empty.

**Algorithmic complexity.** The above proof yields an algorithm that can be implemented using standard computational geometry data structures. Its runtime is polynomial in the size of the input and the number of performed triangle flips.

## 6 On the number of triangle flips

The *flip distance* between two different drawings of a complete multipartite graph with the same ERS is the minimum number of triangle flips that are required to transform one drawing into the other. This section is devoted to obtain bounds on the flip distance.

For an upper bound, Schaefer [31, Remark 3.3] showed that any two simple drawings of $K_n$ with the same rotation system can be transformed into each other with at most $O(n^{20})$ triangle flips. Using our second proof of Theorem 1, we can obtain an upper bound of $O(n^{16})$ on the flip distance between two simple drawings of any complete multipartite graph with $n$ vertices and the same ERS (and thus also for such drawings of $K_n$).

▶ **Theorem 9.** *Let $D_1$ and $D_2$ be two simple drawings of a complete multipartite graph $G$ on $\mathcal{S}^2$ with $n$ vertices and with the same ERS. Then $D_1$ can be transformed into $D_2$ via a sequence of $O(n^{16})$ triangle flips, obtained via the algorithm in the second proof of Theorem 1.*

**Proof.** We analyze the number of flips performed through the second proof of Theorem 1. Recall that in this proof, we iteratively consider the edges of $G$. We perform flips in a drawing $D$ (initially set to $D_1$) so that the subdrawings of $D$ and $D_2$ induced by the already considered edges become (strongly) isomorphic.

When considering a new edge $e$, we imagine to add both versions of it (the one from $D$ and the one from $D_2$) to the already isomorphic subdrawing $X$ of $D$ and $D_2$. In the full version, we show that this can be done such that in the combined drawing, the two copies of $e$ have $O(|V(X)|^4) = O(n^4)$ crossings, where $|V(X)|$ is the number of vertices of $X$.

Let $C$ be the closed curve formed by the two copies of $e$. In order to transform $D$ to make the drawing of $e$ in $D$ isomorphic to the one in $D_2$, we iteratively resolve a free lens of $C$. At every iteration, we reduce the number of crossings of $C$, except for the very last iteration (i.e, for the very last lens). Hence, the number of lenses we need to resolve when processing $e$ is bounded by $O(n^4)$ as well. To resolve a free lens, we need to flip all inverted triangles in this lens that have $e$ as an edge, of which there are at most $O(n^4)$ many. For one inverted triangle $\Delta$ intersected by $m = O(n^2)$ edges, this can be done with $O(m^3) = O(n^6)$ flips. Hence resolving one free lens can be achieved with $O(n^4) \cdot O(n^6) = O(n^{10})$ flips.

Repeating this for all lenses of $C$ and for each of the $O(n^2)$ edges of $G$, we obtain an upper bound of $O(n^2) \cdot O(n^4) \cdot O(n^{10}) = O(n^{16})$ for the total number of triangle flips. ◀

▶ **Theorem 10.** *Let $G$ be a multipartite graph $G$ with $n$ vertices that contains two vertex-disjoint subgraphs each forming a $K_{m,m}$ for some $m = \Theta(n)$. Then $G$ admits two drawings $D_1$ and $D_2$ with the same ERS that have flip distance $\Omega(n^6)$.*

**Proof idea.** To transform the two drawings of $K_n$ in Figure 10 into each other, each of the $\Theta(n^2)$ edges $b_i d_j$ needs to be moved over the $\Theta(n^4)$ crossings formed by edges $a_k c_\ell$, yielding the $\Omega(n^6)$ lower bound. An according example of two drawings of a $K_{m,m}$ can be obtained by disregarding all edges $a_i b_j$ and $c_i d_j$. ◀

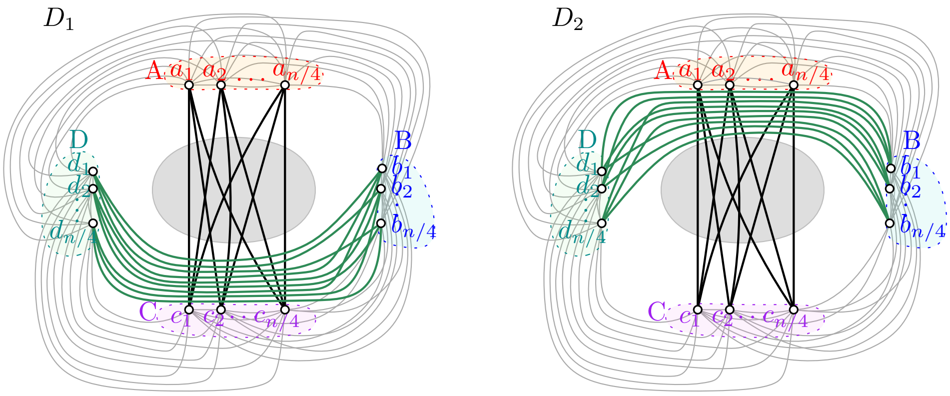


**Figure 10** Two simple drawings of $K_n$ with the same ERS whose flip distance is $\Omega(n^6)$.

## 7 Conclusion & open questions

We have shown that Gioan's Theorem holds for complete multipartite graphs (Theorem 1), extending previous results [4, 15, 16, 31]. Further, we have shown that the class of graphs for which an analogue statement holds is not closed under addition or removal of edges (Theorem 2). We also provide several obstructions such that Gioan's Theorem does not hold for any graph that contains any of these obstructions as a substructure. However, the list of obstructions is probably incomplete. A full characterization of graphs for which a Gioan-type statement for drawings with the same ERS holds remains open.

▶ **Question 1.** *Can we completely characterize all graphs for which a Gioan-type theorem holds for drawings with the same ERS?*

Further, having the same ERS is not the only necessary condition for a Gioan-type statement to hold. Another example of such a condition is that incident or disjoint edges must have the same crossing orders over all drawings. The constructions in the proof of Theorem 2 rely on violating this condition.

▶ **Question 2.** *Can we characterize all graphs for which a Gioan-type theorem holds for classes of drawings which fulfill (subsets of) obviously necessary conditions?*

In Section 3, we have proven a Carathéodory-type theorem for simple drawings of complete multipartite graphs with the same ERS (Theorem 3). It would be interesting to know for which further classes of graphs a similar statement is true.

Naturally, we would also like to narrow or even close the gap between the lower bound of $\Omega(n^6)$ and the upper bound of $O(n^{16})$ for the flip distance, obtained in Section 6.

▶ **Question 3.** *What is the worst case flip distance between two simple drawings of a complete multipartite graph on $n$ vertices with a given ERS?*

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