Disjoint Faces in Drawings of the Complete Graph and Topological Heilbronn Problems

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Abstract

Given a complete simple topological graph G, a k-face generated by G is the open bounded region enclosed by the edges of a non-self-intersecting k-cycle in G. Interestingly, there are complete simple topological graphs with the property that every odd face it generates contains the origin. In this paper, we show that every complete n-vertex simple topological graph generates at least $\Omega(n^{1/3})$ pairwise disjoint 4-faces. As an immediate corollary, every complete simple topological graph on n vertices drawn in the unit square generates a 4-face with area at most $O(n^{-1/3})$. Finally, we investigate a $\mathbb{Z}_2$ variant of Heilbronn's triangle problem for not necessarily simple complete topological graphs.

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1 Introduction

A *topological graph* is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by non-self-intersecting arcs connecting the corresponding points. The arcs are not allowed to pass through vertices different from their endpoints, and if two edges share an interior point, then they must properly cross at that point in common. A topological graph is *simple* if every pair of its edges intersect at most once, either at a common endpoint or at a proper crossing point. A topological graph is called *plane* if there are no two crossing edges. If the edges are drawn as straight-line segments, then the graph is said to be *geometric*. Simple topological graphs have been extensively studied [1, 22, 13, 5, 8], and are sometimes referred to as simple drawings [8, 2]. In this paper, we study the crossing pattern of the faces generated by a simple topological graph.

If $\gamma \subset \mathbb{R}^2$ is a *Jordan curve* (i.e. non-self-intersecting closed curve), then by the Jordan curve theorem, $\mathbb{R}^2 \setminus \gamma$ has two connected components one of which is bounded. For any Jordan curve $\gamma \subset \mathbb{R}^2$, we refer to the bounded open region of $\mathbb{R}^2 \setminus \gamma$ given by the Jordan curve theorem as the *face inside* of γ. We refer to the *area* of γ as the area of the face inside of γ, which we denote by area(γ).

It is known that every complete simple topological graph G of n vertices contains many non-self-intersecting k-cycles, for $k = (\log n)^{1/4-o(1)}$ (e.g. see [13, 22, 14, 12]). A *k-face* generated by G is the face inside of a non-self-intersecting k-cycle in G. For simplicity, we

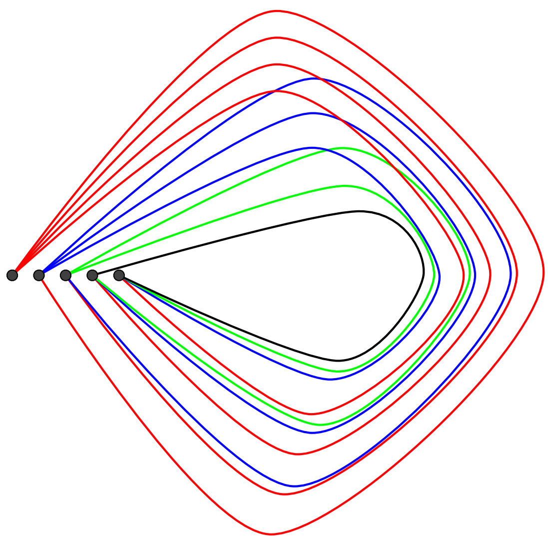


Figure 1 The complete twisted graph on 5 vertices.

say that a k-face is *in* G, if G generates it, and we call it an *odd* (even) face if k is odd (even). Let us remark that a k-face in G may contain other vertices and edges from G. Moreover, notice that if G is simple then every 3-cycle in G must be non-self-intersecting, so for convenience, we call 3-faces *triangles*.

Surprisingly, one cannot guarantee two disjoint 3-faces in complete simple topological graphs. In the next section, we will show that the well-known construction due to Harborth and Mengerson [7], known as the twisted graph and depicted in Figure 1, shows the following.

▶ **Proposition 1.** *For every $n \geq 1$, there exists a complete n-vertex simple topological graph such that every odd face it generates contains the origin.*

See Figure 1. However, the main result in this paper shows that we can guarantee many pairwise disjoint 4-faces.

▶ **Theorem 2.** *Every n-vertex complete simple topological graph generates at least $\Omega(n^{1/3})$ pairwise disjoint 4-faces.*

We apply the results mentioned above to a topological variant of Heilbronn's triangle problem. Over 70 years ago, Heilbronn asked: What is the smallest $h(n)$ such that any set of n points in the unit square spans a triangle whose area is at most $h(n)$? A simple triangulation argument shows that $h(n) \leq O(\frac{1}{n})$. This was improved several times by Roth and Schmidt [19, 16, 17, 18, 21], and currently, the best known upper bound is $\frac{1}{n^{8/7-o(1)}}$ due to Komlós, Pintz, and Szemerédi [9]. Heilbronn conjectured that $h(n) = \Theta(\frac{1}{n^2})$, which was later disproved by Komlós, Pintz, and Szemerédi [10], who showed that $h(n) \geq \Omega(\frac{\log n}{n^2})$. Erdős [4] conjectured that this new bound is asymptotically tight.

Here, we study Heilbronn's problem for topological graphs. A simple variant of Proposition 1 shows that one cannot guarantee a small triangle in a complete simple topological graph drawn in the unit square.

▶ **Proposition 3.** *For every $n \geq 1$ and $\varepsilon > 0$, there exists a complete n-vertex simple topological graph in the unit square such that every odd face it generates has area at least $1 - \varepsilon$.*

On the other hand, as an immediate corollary to Theorem 2, we have the following.

▶ **Corollary 4.** *Every n-vertex complete simple topological graph drawn in the unit square generates a 4-face with area at most $O(\frac{1}{n^{1/3}})$.*

In the other direction, a construction due to Lefmann [11] shows that the complete n-vertex geometric graph can be drawn in the unit square such that every 4-face has area at least $\Omega\left(\frac{\log^{1/2} n}{n^{3/2}}\right)$. It would be interesting to see if one can improve this bound for simple topological graphs. Lastly, let us mention that Heilbronn's triangle problem has been studied for k-gons, and we refer the interested reader to [11] for more results.

Our paper is organized as follows. In Section 2, we establish Propositions 1 and 3. In Section 3, we establish a lemma on finding 4-faces in complete simple topological graph. In Section 4, we use this lemma to prove Theorem 2. Finally in Section 5, we consider Heilbronn's triangle problem for (not necessarily simple) topological graphs.

2 The complete twisted graph

The *complete twisted graph* on n vertices is a complete simple topological graph with vertices labelled 1 to n which we will draw on the horizontal axis from left to right, with the property that two edges intersect if their indices are nested, i.e., edges (i, j) and (k, ℓ), with $i < j$, $k < \ell$, intersect if and only if $i < k < \ell < j$ or $k < i < j < \ell$. See Figure 1. The complete twisted graph was introduced by Harborth and Mengerson [7] as an example of a complete simple topological graph with no subgraph that is weakly isomorphic[1] to the complete convex geometric graph on five vertices. See also [13, 22] for more applications.

▶ **Proposition 5.** *There exists a common point in the interior of all the odd faces generated by the complete twisted graph. Moreover, for every $\varepsilon > 0$, the complete twisted graph can be drawn in the unit square such that every odd face has area at least $1 - \varepsilon$.*

We will need the following claim, which is essentially equivalent to the Jordan curve theorem for piecewise smooth curves. In what follows, a *ray* is a straight, semi-infinite arc.

▶ **Lemma 6.** *Let γ be a piecewise smooth Jordan closed curve in $\mathbb{R}^2$. Let $\vec{v}$ be a direction such that every line parallel to $\vec{v}$ intersects γ in a finite number of points. Then $\mathbb{R}^2 \setminus \gamma$ has two path-connected components, one bounded and one unbounded. Let p be a point not on γ. Then p is in the bounded region of $\mathbb{R}^2 \setminus \gamma$ if and only if the ray α emanating from p in direction $\vec{v}$ properly crosses γ in an odd number of points.*

By Sard's lemma (see for instance [6]), given a smooth curve γ, almost every $\vec{v}$ satisfies the assumption, and in the proof below, the directions that satisfy this assumption will be referred to as *generic*. A differentiable geometry proof of Lemma 6 in the case of γ smooth can be found in [6] (see exercise 12 in Chapter 2.5). The proof we give below is an adaptation of a well-known elementary proof for the case when γ is piecewise linear (polygons), which can be found in Chapter 5.3 of [3] for instance.

[1] Two simple topological graphs G and H are *weakly isomorphic* if there is an incidence preserving bijection between G and H such that two edges of G cross if and only if the corresponding edges in H cross as well.

Figure 2 Parity of proper crossings under a perturbation of the arc, a local picture.

Proof of Lemma 6. Partition γ into a finite number of smooth arcs. If two points p and p' are connected by a segment that doesn't intersect γ, and the rays α and α' have direction $\vec{v}$ and emanate from p and p' respectively, then they satisfy $|\gamma \cap \alpha| (\text{mod } 2) = |\gamma \cap \alpha'| (\text{mod } 2)$. Indeed, as we move α to α', the only moments where the number of intersections between α and γ might change is when the tangent of γ is parallel to $\vec{v}$, or when we pass a singular point, in which, the ray locally leaves the two smooth arcs of γ on the same side. In these cases, the number of proper intersections between α and γ changes by two. Proper crossings can only appear or disappear in pairs when α is perturbed parallel to itself. See Figure 2. By a similar argument, changing $\vec{v}$ for fixed p does not change the parity of the number of intersections.

In what follows, we denote by $w_2(p, \gamma)$ the parity of the number of intersections between γ and any ray α emanating from p in a generic direction. Notice that if p and q are connected by a piecewise linear arc that avoids γ, by the aforementioned argument for each straight segment of the arc, we obtain $w_2(p, \gamma) = w_2(q, \gamma)$.

Consider for every pair of points p and q in the same path-connected component of $\mathbb{R}^2 \setminus \gamma$, two rays α_p and α_q that emanate in the generic direction $\vec{v}$ from p and q respectively. Then modify α_p and α_q by stopping each ray just before the first proper crossing it has with γ. We then extend α_p to a piecewise linear arc by following γ very closely without ever intersecting γ. Since γ is piecewise smooth, if p and q are in the same path-connected component, then the extension of α_p can be chosen so that it eventually reaches the end point of the segment α_q. This is a piecewise linear arc connecting p and q that avoids γ. Furthermore, two points that lie near γ and on opposite sides γ, have different parity so we can conclude that the two path-connected components of $\mathbb{R}^2 \setminus \gamma$ can be identified with the two possible values of $w_2(., \gamma)$. Finally, observe that for any point p sufficiently far from γ there exists a ray that doesn't intersect γ, hence $w_2(p, \gamma) = 0$, and we can conclude that a point is in the bounded component of $\mathbb{R}^2 \setminus \gamma$ if and only if $w_2(p, \gamma) = 1$. ◀

Proof of Proposition 5. Consider the complete n-vertex twisted graph such vertex v_i is placed at $(i, 0)$. See Figure 1. Let $p = (n + 1, 0)$ and consider a ray emanating out of p that passes just above the vertices. This ray intersects each edge of the twisted drawing exactly once. Hence, for any non-self intersecting odd cycle γ in G, $w_2(p, \gamma) = 1$. By Lemma 6, p lies in the face of γ. To upgrade this drawing so that each odd face has large area, we can apply a homeomorphism ϕ to the plane such that the drawing lies in the unit square, all the vertices cluster around the origin, and each face that contains $\phi(p)$ has area at least $1 - \varepsilon$. ◀

3 Finding a 4-face inside a large face

In this section, we establish several lemmas that will be used in the proof of Theorem 2. First, let us clarify some terminology. Given a *planar* graph H drawn in the plane with no crossing edges, the components of the complement of H are called the *faces* of H. Let G be a complete simple topological graph and let T be a triangle in G. We let $V(T)$ denote the set of vertices of the 3-cycle in G that generates T. We say that T is *incident* to vertex $v \in V(G)$, if $v \in V(T)$. We say that triangle T is *empty*, if there is no vertex from G that lies in T. We will repeatedly use the following lemma due to Ruiz-Vargas (see also [5]).

▶ **Lemma 7** ([20]). *Let G be a complete simple topological graph and H be a connected plane subgraph of G with at least two vertices. Let v be a vertex of G that is not in H, and let F be the face of H that contains v. Then there exist two edges of G emanating out of v to the boundary of F such that their interior lies complete inside of F.*

If the plane subgraph $H \subset G$ in Lemma 7 contains a single edge incident to vertex v, then by deleting this edge and applying Lemma 7 to v and the remaining plane subgraph, we obtain the following.

▶ **Lemma 8.** *Let G be a complete simple topological graph and H be a connected plane subgraph of G with at least two vertices. Let v be a vertex of H with degree one, and let F be the face of H whose boundary contains v. Then there exist an edge of G emanating out of v to the boundary of F such that its interior lies complete inside of F.*

We will also need the following lemma, which is a simple consequence of Lemma 7.

▶ **Lemma 9.** *Let G be a complete simple topological graph on four vertices, and let T be a triangle in G with a vertex $v \in V(G)$ inside of it. Then G generates a 4-face that lies inside of the triangle T.*

Lastly, we will need following key lemma, which can be considered as a generalization of Lemma 9. Given a plane graph H and a face F in H, the *size* of F, denoted by $|F|$, is the total length of the closed walk(s) in H bounding the face F. Given two vertices u, v along the boundary of F, the *distance* between u and v is the length of the shortest walk from u to v along the boundary of F.

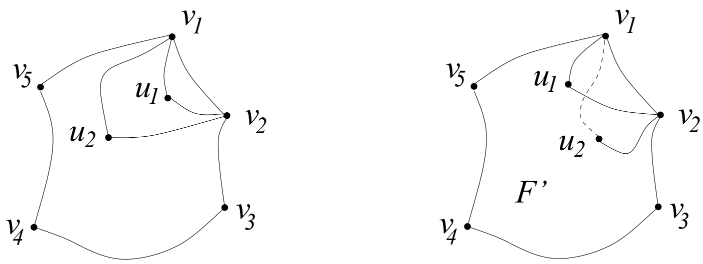
▶ **Lemma 10.** *Let $k \geq 5$ and G be a complete simple topological graph and H be a connected plane subgraph of G with minimum degree two. Let F be a face of H such that $|F| = k$ and F contains at least $6(k-4)$ vertices of G in its interior. Then G generates a 4-face that lies inside of F.*

Proof. We proceed by induction on k, the size of F. For the base case $k = 5$, since H has minimum degree two, the boundary of F must be a simple 5-cycle. Let $v_1, \ldots, v_5$ be the vertices along the boundary of F appearing in clockwise order. Let $u_1, \ldots, u_6$ be the vertices of G in the interior of F. By applying Lemma 7 to u_i and the plane graph H, we obtain two edges emanating out of u_i to the boundary of F, whose interior lies completely inside of F. If the endpoints of these edges have distance more than one along the boundary of F, then we have generated a 4-face inside of F and we are done. Therefore, we can assume that for each u_i, the two edges emanating out of it obtained from Lemma 7 have endpoints at distance one (consecutive) along the boundary of F.

Since $|F| = 5$, by the pigeonhole principle, there are two vertices, say u_1 and u_2, such that the two edges emanating out of u_1 and u_2 obtained from Lemma 7 go to the same two consecutive vertices, say v_1, v_2. If these 4 edges are non-crossing, then we obtain a triangle with a vertex inside of it. See Figure 3a. By Lemma 9, we obtain a 4-face inside of F and we are done. Therefore, without loss of generality, we can assume that edges u_2v_1 and u_1v_2 cross.

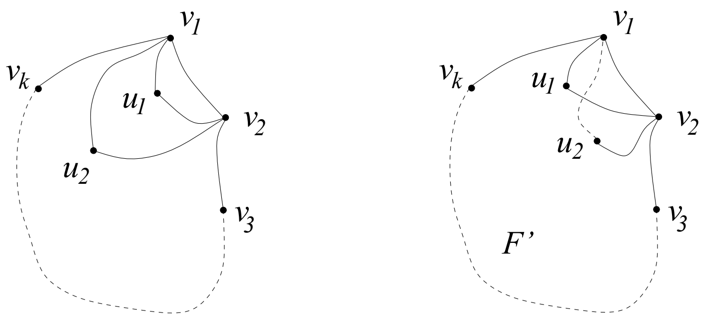
Let $H' = H \cup \{u_1v_1, u_1v_2, u_2v_2\}$, and let F' be the face such that u_2 lies on the boundary of F'. See Figure 3b. Since u_2 has degree one in H', we apply Lemma 8 to u_2 and H' to obtain an edge u_2v_i emanating out of u_2 to the boundary of F', whose interior lies in F'.

If $v_i = v_3$, then we obtain a 4-face inside of F by following the sequence of vertices (v_3, u_2, v_1, v_2) in G. If $v_i = v_4$, then we obtain a 4-face inside of F by following the sequence vertices (v_4, u_2, v_2, v_3) in G. If $v_i = v_5$, then we obtain the 4-face inside of F by following sequence vertices (v_5, v_1, v_2, u_2) in G. Finally, if $v_i = u_1$, then by following the sequence vertices (u_2, u_1, v_1, v_2) in G, we obtain a 4-face inside of F.



(a) Non-crossing edges. **(b)** Plane subgraph H'.

Figure 3 Finding a 4-face inside a 5-face.



(a) Non-crossing edges. **(b)** Plane subgraph H'.

Figure 4 Finding a 4-face inside a face of size k.

For the inductive step, assume that the statement holds for all $k' < k$. Let F be a face of H such that $|F| = k$, and let $(v_1, v_2, \ldots, v_k, v_1)$ be the closed walk(s) along the entire boundary of F. Set $t = 6(k-4)$, and let $u_1, \ldots, u_t$ be vertices of G that lie in the interior of F. For each u_i, we apply Lemma 7, with respect to H, to obtain two edges emanating out of u_i to the boundary of F, such that their interior lies inside of F. The proof now falls into the following cases.

Case 1. Suppose there is a u_i such that the two edges emanating out of u_i obtained from Lemma 7 have endpoints at distance two along the boundary of F. Then we have created a 4-face inside of F and we are done.

Case 2. Suppose there is a vertex u_i such that the two edges emanating out of u_i obtained from Lemma 7 have endpoints at distance at least 3. Then these two edges emanating out of u_i partition F into two parts, F_s and F_r, such that $|F_s| = s, |F_r| = r$, $5 \leq s, r \leq k-1$ and $s + r = k + 4$. By the pigeonhole principle, G has at least $6(s-4)$ vertices inside of F_s or $6(r-4)$ vertices inside F_r. Indeed, otherwise the total number of vertices inside of F (including vertex u_i) is at most

$$6(s-4) - 1 + 6(r-4) - 1 + 1 = 6(k-4) - 1,$$

contradiction. Hence, we can apply induction to F_s or F_r to obtain a 4-face inside of F and we are done.

Case 3. Assume for each u_i, the two edges emanating out of u_i obtained from Lemma 7 have endpoints that have distance one along the boundary of F (consecutive vertices along F). Since $t = 6(k-4) > k$, by the pigeonhole principle, there are two vertices, say u_1 and u_2, such that the two edges emanating out of u_1 and u_2 obtained from Lemma 7 go to the same two vertices, say v_1, v_2. If these four edges are noncrossing, then we have a triangle with a vertex inside. By Lemma 9, we obtain a 4-face inside of F and we are done. See Figure 4a. Therefore, without loss of generality, we can assume that edges u_1v_2 an u_2v_1 cross.

Let $H' = H \cup \{u_1v_1, u_1v_2, u_2v_2\}$, which implies that u_2 has degree one in H'. Let F' be the face that contains u_2 on its boundary. See Figure 4b. We apply Lemma 8 to u_2 and the plane graph H', to obtain an edge u_2v_i whose interior lies inside of F' and v_i lies on the boundary of F'. If $v_i = v_3$, then we obtain a 4-face inside of F by following the sequence of vertices (u_2, v_1, v_2, v_3) in G. If $v_i = u_1$, then we obtain a 4-face inside of F by following the sequence of vertices (u_2, u_1, v_1, v_2) in G. If $v_i = v_k$, then again, we obtain a 4-face inside of F by following the sequence of vertices (u_2, v_k, v_1, v_2) in G.

Finally, if $v_i \neq v_k, u_1, v_3$, then at least one of $u_2v_2 \cup u_2v_i$ or $u_2v_1 \cup u_2v_i$ partitions F into two parts, F_s and F_r, such that $|F_s| = s$, $|F_r| = r$, where $5 \leq s, r \leq k-1$ and $s + r = k + 4$. By following the arguments in Case 2, we can apply induction on F_s or F_r to obtain a 4-face inside of F. This completes the proof. ◀

4 Pairwise disjoint 4-faces in simple drawings

In this section, we prove Theorem 2. Roughly speaking, we follow the arguments of Fulek and Ruiz-Vargas [5] by constructing a large planar subgraph $H \subset G$ using Lemma 7. Then, by combining the pigeonhole principle with Dilworth's theorem, H will contain either

1. a planar $K_{2,t}$ for t large, or
2. many nested triangles, or
3. many interior disjoint triangles.

Here, *large* and *many* means $\Omega(n^{\frac{1}{3}})$. In the first case, it is easy to find many pairwise disjoint 4-faces. In the second case, we use Lemma 10 to find them. In the last case however, the set of interior disjoint triangles may not give rise to many pairwise disjoint 4-faces, as it is possible that the triangles are empty. In order to rectify this, we carefully construct our planar subgraph H using Lemma 11 below. We now flesh out the details of the proof.

Proof of Theorem 2. Let $G = (V, E)$ be a complete n-vertex simple topological graph. We can assume that $n \geq 40$ since otherwise the statement is trivial. Notice that the edges of G divide the plane into several cells (regions), one of which is unbounded. We can assume that there is a vertex $v_0 \in V$ such that v_0 lies on the boundary of the unbounded cell. Indeed, otherwise we can project G onto a sphere, then choose an arbitrary vertex v_0 and then project G back to the plane such that v_0 lies on the boundary of the unbounded cell. Moreover, the new drawing is isomorphic to the original one as topological graphs.

Consider the topological edges emanating out from v_0 in clockwise order, and label their endpoints $v_1, \ldots, v_{n-1}$. For convenience, we write $v_i \prec v_j$ if $i < j$. Given subsets $U, W \subset \{v_1, \ldots, v_{n-1}\}$, we write $U \prec W$ if $u \prec w$ for all $u \in U$ and $w \in W$. We start by partitioning our vertex set

$$\mathcal{P} : V(G) = V_0 \cup V_1 \cup \cdots \cup V_{\lfloor \frac{n-1}{5} \rfloor},$$

such that for $j < \lfloor \frac{n-1}{5} \rfloor$, we have

$$V_j = \{v_{5j+1}, v_{5j+2}, v_{5j+3}, v_{5j+4}, v_{5j+5}\},$$

and $\left| V_{\lfloor \frac{n-1}{5} \rfloor} \right| \leq 5$. Let $H \subset G$ be a plane subgraph of G, and let T, T' be two triangles in H that are incident to v_0. We say that T and T' are *adjacent* if $V(T) = \{v_0, v_i, v_j\}$

and $V(T') = \{v_0, v_j, v_k\}$ such that $v_i \prec v_j \prec v_k$, and the edges v_0v_i, v_0v_j, v_0v_k appear consecutively in clockwise order among the edges emanating out of v_0 in H (not in G). See Figures 6c and 7a for an example.

In what follows, we will construct a plane subgraph $H \subset G$ so that, at each step, we use Lemma 7 to add at least one edge within the vertex set $\{v_1, \ldots, v_{n-1}\}$. The goal at each step is to add an edge without creating any empty triangles incident to v_0. If we are forced to create such an empty triangle, we then create another triangle incident to v_0 that is adjacent to it, so that we obtain a 4-face. We now give the details of this process.

▶ **Lemma 11.** *For each $i \in \{0, 1, \ldots, \lfloor n/12 \rfloor\}$, there is a plane subgraph $H_i \subset G$ such that $V(H_i) = V(G)$ and H_i satisfies the following properties.*

1. *H_i has at least i edges with both endpoints in the vertex set $\{v_1, \ldots, v_{n-1}\}$.*
2. *The number of parts $V_j \in \mathcal{P}$ with the property that each vertex in V_j has degree one in H_i is at least $\lfloor (n-1)/5 \rfloor - 2i$.*
3. *If the vertex set $\{v_0, v_k, v_\ell\}$ induces an empty triangle T in H_i, then both vertices v_k, v_ℓ must lie in the same part $V_j \in \mathcal{P}$ and $\ell = k + 1$. Moreover, given that such an empty triangle T exists, there must be another triangle T' adjacent to T in H_i, such that $V(T') = \{v_0, v_t, v_{t'}\}$ and $v_t, v_{t'} \in V_j$.*
4. *If the edge v_0v_t is not in H_i, then v_t is an isolated vertex in H_i.*

Proof. We start by setting H_0 as the plane subgraph of G consisting of all edges emanating out of v_0. Clearly, H_0 satisfies the properties above. For $i < n/12$, having obtained H_i with the properties described above, we obtain H_{i+1} as follows.

Fix a part $V_j \in \mathcal{P}$ such that each vertex in V_j has degree one in H_i and $|V_j| = 5$. For simplicity, set $u_i = v_{5j+i}$, for $i \in \{1, \ldots, 5\}$, which implies $V_j = \{u_1, u_2, u_3, u_4, u_5\}$. Since

$$\left\lfloor \frac{n-1}{5} \right\rfloor - 2i \geq \frac{n-1}{5} - \frac{n}{6} > 1,$$

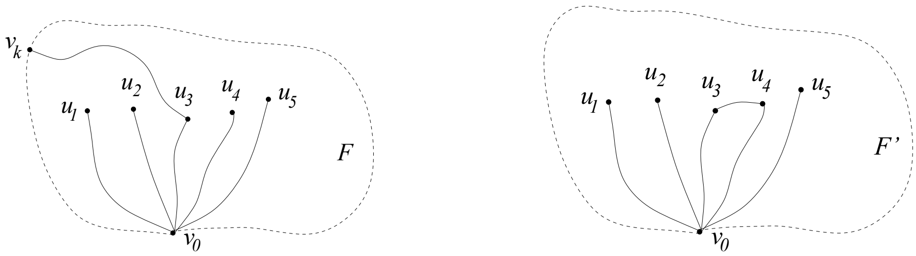
such a part $V_j \in \mathcal{P}$ exists. Clearly, all vertices in V_j lie on the boundary of a face F in the plane graph H_i. We then apply Lemma 8 to the plane graph H_i and the vertex u_3, and obtain edge u_3v_k, whose interior lies within F and v_k is on the boundary of F. We now consider the following cases.

Case 1. Suppose $v_k \neq u_2, u_4$. See Figure 5a. We then set $H_{i+1} = H_i \cup \{u_3v_k\}$. Clearly, H_{i+1} does not contain two crossing edges. Moreover, the number of edges in H_{i+1} within the vertex set $\{v_1, \ldots, v_{n-1}\}$ is at least $i + 1$. Also, the only vertices that no longer have degree one in H_{i+1} are u_3 and v_k. Hence, the number of parts $V_\ell \in \mathcal{P}$ with the property that all vertices in V_ℓ have degree one in H_{i+1} is at least

$$\left\lfloor \frac{n-5}{2} \right\rfloor - 2i - 2 = \left\lfloor \frac{n-5}{2} \right\rfloor - 2(i+1).$$

Since $v_k \neq u_2, u_4$, no empty triangles incident to v_0 were created. Also, no edge emanating out of v_0 was deleted from H_i. Thus, H_{i+1} satisfies the conditions described above.

Case 2. Suppose $v_k = u_2$ or $v_k = u_4$. Without loss of generality, we can assume $v_k = u_4$, since otherwise a symmetric argument would follow. If there is a vertex of G inside the triangle $T = \{v_0, u_3, u_4\}$, then we set $H_{i+1} = H_i \cup \{u_3u_4\}$. By the same arguments as above, H_{i+1} satisfies the properties described above.



(a) $v_k \neq u_2, u_4$. **(b)** $v_k = u_4$.

Figure 5 Cases 1 and 2 in Lemma 11.

Hence, we can assume that the triangle T, where $V(T) = \{v_0, u_3, u_4\}$, is empty in G. Set $H' = H_i \cup \{u_3u_4\}$, and let F' be the face in H' whose boundary contains u_2. See Figure 5b. We apply Lemma 8 to H' and u_2 and obtain another edge u_2v_ℓ whose interior lies inside F'. If $v_\ell = u_3$, then we set $H_{i+1} = H_i \cup \{u_3u_4, u_2u_3\}$, which implies that the empty triangle T is adjacent to triangle T', where $V(T') = \{v_0, u_2, u_3\}$. Clearly, H_{i+1} has at least $i+2 > i+1$ edges within the vertex set $\{v_1, \dots, v_{n-1}\}$. The number of parts $V_\ell \in \mathcal{P}$ with the property that all vertices in V_ℓ have degree one in H_{i+1} is at least

$$\left\lfloor \frac{n-5}{2} \right\rfloor - 2i - 1 > \left\lfloor \frac{n-5}{2} \right\rfloor - 2(i+1).$$

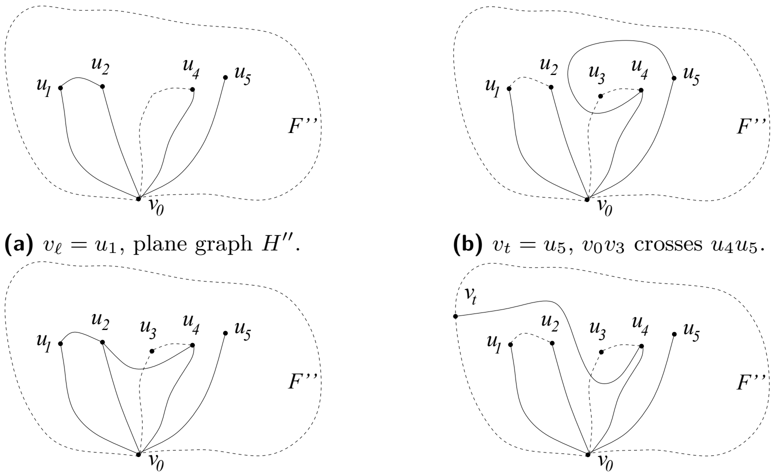
If $v_\ell \neq u_1, u_3$, then edge u_2v_ℓ does not create any empty triangles incident to v_0 and we set $H_{i+1} = H_i \cup \{u_2v_\ell\}$. By the same argument as above, H_{i+1} satisfies the desired properties.

Finally, let us consider the case that $v_\ell = u_1$. If the triangle T' is not empty, where $V(T') = \{v_0, u_1, u_2\}$, we set $H_{i+1} = H_i \cup \{u_1u_2\}$ and we are done by the arguments above. Therefore, we can assume that the triangle T' is also empty.

Let $H'' = (H_i \cup \{u_1u_2\}) \setminus \{u_3\}$. Let F'' be the face whose boundary contains u_4 in H''. See Figure 6a. We apply Lemma 8 to H'' and the vertex u_4 to obtain edge u_4v_t whose interior lies inside F''. We now examine $H_i \cup \{u_1u_2, u_3u_4, u_4v_t\}$. The proof now falls into the following cases.

Case 2.a. Suppose edge u_4v_t crosses edge v_0u_3. If $v_t = u_5$, then $\{v_0, u_4, u_5\}$ induces a non-empty triangle in G, so we set $H_{i+1} = H_i \cup \{u_4u_5\} \setminus \{v_0v_3\}$. Then u_3 is an isolated vertex in H_{i+1} and we did not create any empty triangles incident to v_0, and we are done. See Figure 6b. If $v_t = u_2$, then we set $H_{i+1} = H_i \cup \{u_1u_2, u_2u_4\} \setminus \{v_0u_3\}$. Then the empty triangle on $\{v_0, u_1, u_2\}$ is adjacent to the triangle on $\{v_0, u_2, u_4\}$ in H_{i+1}, u_3 is an isolated vertex, and we are done. See Figure 6c. Otherwise, if $v_t \neq u_2, u_5$, we set $H_{i+1} = (H_i \cup \{u_4v_t\}) \setminus \{v_0u_3\}$. Then u_3 is an isolated vertex, we do not create any empty triangles incident to v_0, and we are done. See Figure 6d.

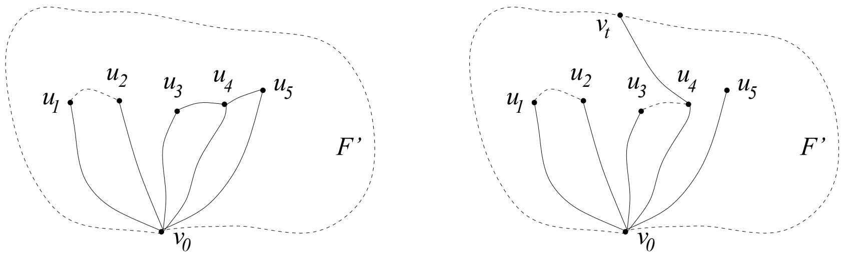
Case 2.b. Suppose edges v_0u_3 and u_4v_t do not cross. If $v_t = u_5$, then we set $H_{i+1} = H_i \cup \{u_3u_4, u_4u_5\}$, and the empty triangle on the vertex set $\{v_0, u_3, u_4\}$ is adjacent to the triangle on $\{v_0, u_4, u_5\}$, and we are done. See Figure 7a. If $v_t \neq u_5$, then we set $H_{i+1} = H_i \cup \{u_4v_t\}$. Since we do not create any empty triangles incident to v_0, we are done. See Figure 7b. This completes the proof of the statement. ◀



(a) $v_\ell = u_1$, plane graph H''. **(b)** $v_t = u_5$, v_0v_3 crosses u_4u_5.

(c) $v_t = u_2$ **(d)** $v_t \neq u_2, u_5$

Figure 6 Cases 2.a in Lemma 11. Edge u_4v_t crosses v_0v_3.



(a) $v_t = u_5$ **(b)** $v_t \neq u_5$

Figure 7 Case 2.b in Lemma 11. Edge u_4v_t does not cross v_0v_3.

Set $H = H_{\lfloor n/12 \rfloor}$. We now will use the plane graph H and the vertices of G to find many pairwise disjoint 4-faces. If there is a vertex $v_j \in \{v_1, \dots, v_{n-1}\}$ with degree at least $n^{1/3}$ in H, then together with v_0, we have a plane drawing of $K_{2,\lfloor n^{1/3} \rfloor}$. Indeed, recall that in H, every vertex is either connected to v_0, or an isolated vertex. This gives rise to $\Omega(n^{1/3})$ pairwise disjoint 4-faces and we are done.

Hence, we can assume that every vertex $v_i \in \{v_1, \dots, v_{n-1}\}$ has degree at most $n^{1/3}$. Since there are at least $n/12$ edges induced on the vertex set $\{v_1, \dots, v_{n-1}\}$ in the plane graph H, there is a plane matching M on $\{v_1, \dots, v_{n-1}\}$ of size at least $n^{2/3}/16$. Notice that there is a natural partial ordering $\prec^*$ on M. Given two edges $v_iv_j, v_kv_\ell \in M$, we write $v_kv_\ell \prec^* v_iv_j$ if $v_i \prec v_k \prec v_\ell \prec v_j$. By Dilworth's theorem, M contains either a chain or antichain of length at least $n^{1/3}/4$ with respect to the partial ordering $\prec^*$. The proof now falls into two cases.

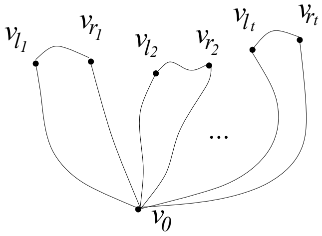
Case 1. Suppose we have an antichain M' of size $n^{1/3}/4$. Let

$$M' = \{v_{\ell_1}v_{r_1}, v_{\ell_2}v_{r_2}, \dots, v_{\ell_t}v_{r_t}\},$$

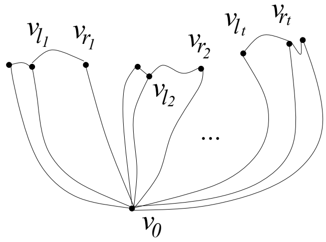
where $t = n^{1/3}/4$ and $\ell_i < r_i$ for all i. Since H is a plane drawing, and every non-isolated vertex is connected to v_0, we have

$$\{v_{\ell_1}, v_{r_1}\} \prec \{v_{\ell_2}, v_{r_2}\} \prec \cdots \prec \{v_{\ell_t}, v_{r_t}\}.$$

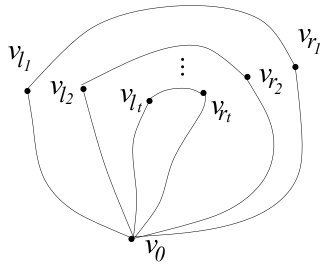
See Figure 8a. If at least half of the edges in M' give rise to a non-empty triangle incident to v_0, then we apply Lemma 9 to each such triangle to obtain $\Omega(n^{1/3})$ pairwise disjoint 4-faces. Hence, we can assume at least half of these triangles are empty. By construction of H, each such empty triangle has another triangle adjacent to it. Since the three edges emanating out of v_0 of two adjacent triangles must be consecutive in H (by definition), this corresponds to $\Omega(n^{1/3})$ pairwise disjoint 4-faces. See Figure 8b.



(a) Anti-chain of size t.



(b) Disjoint 4-faces.



(c) Chain of length t

Figure 8 Large antichain and chain.

Case 2. Suppose we have a chain $M' \subset M$ of size $n^{1/3}/4$. Hence,

$$M' = \{v_{\ell_1}v_{r_1}, v_{\ell_2}v_{r_2}, \ldots, v_{\ell_t}v_{r_t}\},$$

where $t = n^{1/3}/4$ and we have

$$v_{\ell_t}v_{r_t} \prec^* \cdots \prec^* v_{\ell_2}v_{r_2} \prec^* v_{\ell_1}v_{r_1}.$$

See Figure 8c. Set $M'' \subset M'$ such that $M'' = \{v_{\ell_{7j}}v_{r_{7j}}\}_j$. Hence, $|M''| \geq \Omega(n^{1/3})$. Let us consider edges $v_{\ell_7}v_{r_7}$ and $v_{\ell_{14}}v_{r_{14}}$ from M'', and the region F enclosed by the six edges.

$$v_{\ell_7}v_{r_7}, v_{\ell_{14}}v_{r_{14}}, v_0v_{\ell_7}, v_0v_{\ell_{14}}, v_0v_{r_7}, v_0v_{r_{14}}.$$

See Figure 9. Let H' be the plane subgraph on the vertex set $\{v_0, v_{\ell_{14}}, v_{r_{14}}, v_{r_7}, v_{\ell_7}\}$ and the six edges listed above. By construction of M'', we know that there are at least 12 vertices of $V(G)$ inside F. Since $|F| = 6$, we can apply Lemma 10 to find a 4-face inside of F. By repeating this argument for each consecutive pair of edges in the matching M'' with respect to the partial order $\prec^*$, we obtain $\Omega(n^{1/3})$ pairwise disjoint 4-faces. ◀

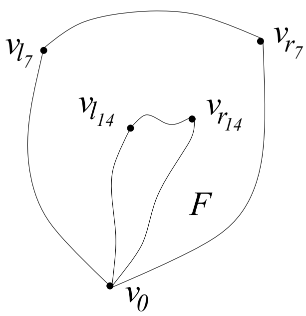


Figure 9 Face F of size 6.

5 $\mathbb{Z}_2$-cycles in topological graphs

Now we pass to a variant of Heilbronn's triangle problem for not necessarily simple topological graphs. Specifically, if γ is piecewise smooth closed curve with transverse self intersections, then one can consider Lemma 6, from Section 2, as a definition of the $\mathbb{Z}_2$-inside of γ. That is, p is in the interior of γ if any arc with one endpoint at p and the other outside a large disk containing γ, intersects γ an odd number of times at proper crossings.

Does every complete topological graph drawn inside the unit square contain a cycle whose $\mathbb{Z}_2$-inside has small area? More generally, we will consider this question for the group of $\mathbb{Z}_2$-cycles instead of graph cycles. The result of this section is a negative answer to this question. Using a simple probabilistic construction, we show that there are complete topological graphs in the unit square in which every cycle has constant area.

5.1 Chain complexes

Let us recall the basic objects of cellular homology, refer to [15] for a gentle introduction. If X is a cell complex, for each i, the group of i-th chain group, denoted by $C_i(X, \mathbb{Z}_2)$ is the group of formal linear combinations of the i-dimensional cells. An element of $C_i(X, \mathbb{Z}_2)$ has the form $\sum_{\sigma \in F_i(X)} a_\sigma \sigma$, where σ is an element of F_i, the set of i-dimensional cells, and a_σ is an element of $\mathbb{Z}_2$, the field with two elements. The boundary operator is a linear map $\partial C_i(X, \mathbb{Z}_2) \to C_{i-1}(X, \mathbb{Z}_2)$, which can be succinctly described using a pair of basis, one for $C_i(X, \mathbb{Z}_2)$ and one for $C_{i-1}(X, \mathbb{Z}_2)$ which have an element for each cell, then the boundary map of a cell σ is the linear combinations of the $(i-1)$-cells that are incident to σ. The kernel of the boundary operator is the group of cycles $Z_i(X, \mathbb{Z}_2)$ and its image is the group of boundaries $B_{i-1}(X, \mathbb{Z}_2)$, the quotient group $Z_i(X, \mathbb{Z}_2)/B_i(X, \mathbb{Z}_2)$ is the i-th cellular homology group $H_i(X, \mathbb{Z}_2)$. In the following we will use that a two dimensional disk has trivial homology. This is the case because homology is invariant under homotopy equivalences and a disk can be contracted to a point which can be modeled with a cell complex that has no higher dimensional cells.

Consider K_n as a simplicial complex, in other words, $F_1(K_n)$ is the set of edges and $F_0(K_n)$ is the set of vertices of the complete graph.

In this case the boundary $\partial\colon C_1(K_n, \mathbb{Z}_2) \to C_0(K_n, \mathbb{Z}_2)$ is defined as follows: if $e = (i, j)$ is an edge, then the chain $1e$ is mapped to $\partial(e) = 1i + 1j$. The kernel of ∂ is the group of 1-cycles of K_n, $Z_1(K_n) := \ker \partial$. Elements in $Z_1(K_n)$ can be identified with (possibly disjoint) graphs in which every vertex has even degree.

Let us consider the planar graph induced by G by introducing a vertex at every intersection between two edges, and let $\hat{G}$ be the cell decomposition of the smallest closed topological disk that contains G. More precisely, every intersection between edges of G is a vertex of $\hat{G}$ (including the vertices of G). Two consecutive intersections along an edge of G share an edge in $\hat{G}$. The regions of $\mathbb{R}^2 \setminus G$ are the 2 dimensional cells of $\hat{G}$. Consider the chain groups $C_i(\hat{G}, \mathbb{Z}_2)$, and observe that for $i = 0, 1$ there exists linear maps $f_i\colon C_i(K_n, \mathbb{Z}_2) \to C_i(\hat{G}, \mathbb{Z}_2)$. For example, for a given edge $e \in E(K_n)$, $f_1(e)$ is the linear combination of the edges in $\hat{G}$ that support the arc representing e, and similarly, for the vertices.

It is not hard to see that this chain map induces a well defined map between cycle groups, $f_1\colon Z_1(K_n) \to Z_1(\hat{G})$. Now, since the homology group $H_1(\hat{G}, \mathbb{Z}_2)$ is trivial, for any cycle $z \in Z_1(\hat{G})$ there exists a 2 chain $c \in C_2(\hat{G}, \mathbb{Z}_2)$ such that $\partial(c) = z$. On the other hand, if some other chain $c' \neq c$ satisfied $\partial c' = z$, then $\partial(c + c') = 0$, hence $c + c'$ would be a two dimensional cycle, but since there are no 3 dimensional faces, this would imply that the homology group $H_2(\hat{G}, \mathbb{Z}_2) \neq 0$, which is absurd. So there exists a unique chain c such that $\partial c = z$, and the interior of its support corresponds to the set of points $\{p \in \mathbb{R}^2 : w_2(p, z) = 1\}$.

Figure 10 A possible edge in the construction of proposition 12 before re-scaling and perturbing.

5.2 A topological graph without $\mathbb{Z}_2$-cycles of small area

▶ **Proposition 12.** *There exists a drawing of the complete graph inside* $[0,1]^2$ *such that the* $\mathbb{Z}_2$*-inside of every* $\mathbb{Z}_2$*-cycle of the complete graph has area at least* $\frac{1}{4}$.

We begin describing a random construction. Consider a rectangle of size $m \times 1$, with corners at $\{(0,0),(0,1),(m,0),(m,1)\}$ where m will be a large number with respect to n that we will define later on. We perform the area analysis for this drawing, but notice that by applying the linear transformation $(x,y) \to (\frac{x}{m}, y)$, we can transform it back to the unit square.

We place all the points in general position on a small neighbourhood of the lower corner (0,0) of the rectangle. The drawing will be random and at the end it will be perturbed by an arbitrary small amount so that it is in general position. To refer to this small perturbation we use the word "near" in the description below. Notice that one could perturb each edge so that it stays piecewise linear or one could smooth each edge, as long as areas of cycles do not change too much and every intersection is a proper crossing (in the language of differentiable topology this corresponds to the curves being transverse and in PL topology to general position).

Each edge will go all the way to near $(m,0)$ and come back near $(0,0)$. Choose two vertices i,j, the edge $e=(i,j)$ will be represented by an arc that begins at the vertex i and is a concatenation of almost vertical and almost horizontal arcs. More precisely, for each $k \in \{0,1,2,\dots m-1\}$ assume that we have constructed a path $\alpha_{ij}(k)$ that begins at i (near $(0,0)$) and ends at (k,Y_k) with $Y_k \in \{0,1\}$, let Y_{k+1} be a Bernoulli random variable with probability $\frac{1}{2}$, and extend the arc $\alpha_{ij}(k)$ by concatenating it with the segment $\{(k+t,y_k) : t \in [0,1]\}$ if $Y_{k+1}=Y_k$, and by the concatenation of the segments $\{(k,t) : t \in [0,1]\}$ followed by $\{(k+t,Y_{k+1}) : t \in [0,1]\}$ if $Y_{k+1} \neq Y_k$.

When we reach $x=m$, if $y=1$, we concatenate it to $(m,0)$. In both cases $y=0,1$, we end the arc by concatenating all the way back to the vertex j near $(0,0)$ with a long near horizontal arc close to the x-axis. Finally, we perturb what we have constructed a very small amount so that the intersections between any two such edges is a finite set of points where they cross properly, and we re-scale the x-axis so that the whole picture is contained in the unit square.

Proof of Proposition 12. We work with the rectangle and make some observations about the re-scaling and perturbing at the end of the proof. Using the random construction described above, to compute the expected area of a cycle z, consider a point p in the interior of the rectangle, say that p has coordinates $(k+\frac{1}{2}, \frac{1}{2})$, and consider the horizontal segment of a fixed edge e that joins (k,Y_k) with $(k+1,Y_{k+1})$. The vertical ray emanating up from p, intersects this edge with probability $\frac{1}{2}$. Conditioned on all the other edges of a cycle z containing e, the square $\{(k,0),(k,1),(k+1,1),(k+1,0)\}$ is $\mathbb{Z}_2$-inside z with probability $\frac{1}{2}$ and $\mathbb{Z}_2$-outside z with probability $\frac{1}{2}$. This implies that the area of every cycle is a sum of m independent Bernoulli random variables with probability $\frac{1}{2}$, so $\mathbb{E}[\text{area}(z)] \geq \frac{m}{2}$, and Chernoff bound yields:

$$\Pr(\text{area}(z) < \frac{m}{3}) \leq e^{-\frac{m}{128}}.$$

There are exactly $2^{\binom{n-1}{2}} - 1$ non-zero elements in $Z_1(K_n)$, while the areas of two different cycles z and z' that share some edge are dependent random variables, if we let $m = 64n^2$, the union bound yields

$$\Pr(\exists z \in Z_1(K_n), \text{area}(z) < \frac{m}{3}) \leq 2^{-\frac{n}{2}}$$

Since this probability is strictly smaller than 1, there exists some drawing such that the area of every cycle is at least $\frac{m}{3}$, which after perturbing and re-scaling by $\frac{1}{m}$, corresponds to all cycles having area at least $\frac{1}{3} - \epsilon$, for any given $\epsilon > 0$. ◀

▶ Remark 13. If we only cared about graph cycles, i.e. connected subgraphs of K_n in which each vertex has degree two, then it is enough to take $m = O(n \log n)$.

▶ Remark 14. There is nothing special about $\frac{1}{4}$ or about $\frac{1}{3}$, at the cost of making m larger, we can force all cycles to have area at least $\frac{1}{2} - \epsilon$ for any $\epsilon > 0$. It is easy to see that for any complete topological graph there exists $z \in Z_1(G)$ with $\text{area}(z) \leq \frac{1}{2}$.

In the aforementioned construction, for two fixed edges e, e', and a fixed integer i, there is a constant probability that e and e' cross near the vertical line at $\{(i, x) : x \in \mathbb{R}^1\}$, hence the expected number of crossings is $\Omega(n^6)$

▶ **Problem 15.** *For a fixed k, is there a function $\epsilon_k(n)$ with $\epsilon_k(n) \to 0$ when $n \to \infty$, such that for every drawing of K_n in which every pair of edges intersect at most k times, we can find a cycle of area at most $\epsilon_k(n)$?*

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