Non-Crossing Hamiltonian Paths and Cycles in Output-Polynomial Time

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Abstract

We show that, for planar point sets, the number of non-crossing Hamiltonian paths is polynomially bounded in the number of non-crossing paths, and the number of non-crossing Hamiltonian cycles (polygonalizations) is polynomially bounded in the number of surrounding cycles. As a consequence, we can list the non-crossing Hamiltonian paths or the polygonalizations, in time polynomial in the output size, by filtering the output of simple backtracking algorithms for non-crossing paths or surrounding cycles respectively. To prove these results we relate the numbers of non-crossing structures to two easily-computed parameters of the point set: the minimum number of points whose removal results in a collinear set, and the number of points interior to the convex hull. These relations also lead to polynomial-time approximation algorithms for the numbers of structures of all four types, accurate to within a constant factor of the logarithm of these numbers.

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1 Introduction

In how many ways can we "connect the dots", turning planar points into the vertices of a simple polygon? Despite heavy study, the answer is still unclear. Steinhaus proved in the 1960s that general-position points always have at least one polygonalization [32], but the condition is too strong: non-collinearity suffices. Points in convex position have only one polygonalization; other inputs can have exponentially many, but with upper and lower bounds that are far from matching [16, 31]. The complexity of counting polygonalizations is unknown [13, 22, 24]. Although all polygonalizations can be listed in singly-exponential time [35, 37], it was unknown (prior to our work) how to list them more quickly when there are few, for instance in polynomial time per output or in time polynomial in the output size.

Our main result is that both polygonalizations and a closely related structure, non-crossing Hamiltonian paths, can be listed in time polynomial in the output size by simple backtracking algorithms. These algorithms search spaces of easier-to-list structures, the *surrounding polygons* [37] and non-crossing paths, respectively. To prove these new results, following our work on monotone parameters of point sets [12], we relate the numbers of all four types of structures (polygonalizations, surrounding polygons, non-crossing Hamiltonian paths, and non-crossing paths) to two easily-computed parameters that depend only on the order-type of the points: the smallest number of points whose removal results in a collinear subset, and the number of points interior to the convex hull. These relations imply that the number of polygonalizations is at most polynomial in the number of surrounding polygons, and that the number of non-crossing Hamiltonian paths is at most polynomial in the number of non-crossing paths. Therefore, an algorithm for the easier-to-list structures will take output-polynomial time when its output is filtered to generate only the harder-to-list structures. Our methods also provide a polynomial-time approximation algorithm for counting these structures, obtaining a constant approximation ratio with respect to the logarithm of the count.

We do not calculate explicitly the exponents of the polynomials relating these numbers of structures, as our upper bounds are quite imprecise. Instead, in a final section, we describe point sets whose exponential numbers of non-crossing structures can be calculated precisely. These examples demonstrate that the exponent of paths in terms of Hamiltonian paths can be at least $\log_2 3 \approx 1.585$ and that the exponent of surrounding polygons in terms of polygonalizations can be at least $\log_2\big((3+\sqrt{5})/2\big) \approx 1.388$.

Output-polynomial time bounds, such as we prove, are not as good as a time bound that multiplies the output size by a polynomial of the input size, and even less good than polynomial delay for each output structure. Nevertheless, our results represent a significant improvement on previously known time bounds, which are singly exponential even for inputs whose output size is subexponential.

Because of space limitations, we omit many proofs from this proceedings version. A full version of this paper is available at arXiv:2303.00147.

1.1 Related work

A *planar straight line graph*, with given points as vertices, consists of line segments having the points as endpoints, with no line segment passing through a given point and no two line segments intersecting except at a shared endpoint. When the form of the resulting graph is constrained, one obtains non-crossing structures of various types, including triangulations (non-crossing maximal planar graphs), non-crossing spanning trees, and polygonalizations (non-crossing Hamiltonian cycles). Extensive research in discrete and computational geometry has sought upper and lower bounds for numbers of non-crossing structures, and studied algorithmic problems of counting or listing these structures for a given point set, or of finding a non-crossing structure that is optimal for some objective function [2–4,8,11,15,16,19,22,29–31]. Many problems of this sort have algorithms for listing all graphs, with polynomial time per output, based on systems of local moves that link the state space into a connected structure [1,5,7,20,33,35–37].

However, for polygonalizations and non-crossing Hamiltonian paths no such structure is known, and natural systems of local moves that change two or three edges at a time are known not to connect all polygonalizations [18]. Certain generalizations of polygonalizations have state spaces connected by local moves [9,37], but this does not directly yield fast algorithms for polygonalizations, because of the many non-polygonalizations in these state spaces. As a 2011 survey of Welzl summarizes, "Basically nothing is known for related algorithmic questions (determining the number of simple polygonizations for a given point set, enumerating all simple polygonizations)" [34]. The shortest polygonalization is the NP-hard Euclidean traveling salesperson tour [28], and several other optimal polygonalizations are also NP-hard [10,14]; the complexity of the longest polygonalization is another unknown [11].

Past work on non-crossing Hamiltonian paths includes approximation to the longest path [3,11] and the existence of properly colored paths for colored point sets [6]. The number of non-crossing Hamiltonian paths can range from one, for collinear points, to exponentially large; for instance, $n \geq 2$ points in convex position have exactly $n2^{n-3}$ paths [27].

The surrounding polygons that we use for our polygonalization algorithm are another class of planar straight line graphs: they are the simple polygons that use a subset of the input points as vertices, and contain all of them. Yamanaka et al. [37] introduced these polygons, and showed how to list them in polynomial time per polygon, from which they derived a singly-exponential time bound and polynomial space bound for listing polygonalizations. Despite this progress they were unable to obtain an output-sensitive time bound for polygonalizations. It is their algorithm that we follow here, with a new output-sensitive analysis. Yamanaka et al. also prove that, in the worst-case exponential bounds for numbers of polygonalizations and surrounding polygons, the bases of the exponentials are within 1 of each other; however, this analysis does not imply our stronger result, that on arbitrary instances the numbers of these two types of polygons are bounded by polynomials of each other. The examples from our final section confirm theoretically the empirical results of Yamanaka et al. that polygonalizations and surrounding polygons can grow at different rates.

2 Two simple backtracking algorithms

In this section we outline two simple algorithms (one a standard backtracking search, and the other the reverse search algorithm of Yamanaka et al. [37]) for listing non-crossing paths and surrounding cycles. These known algorithms will be the ones we use to prove our new time bounds for listing non-crossing Hamiltonian paths and polygonalizations. We state these bounds as theorems in this section, and defer the proofs to later sections.

▶ **Definition 1.** *Define a* non-crossing path *for a set of points S to be a non-self-intersecting polygonal curve P that passes through a subset of points of S, has endpoints in S, and turns only at points of S. Define the* vertices *of a non-crossing path to be the set $P \cap S$, counting a point of S as a vertex even when P passes straight through that point. Define a* non-crossing path sequence *to be the sequence of vertices in a non-crossing path of a given set of points in the plane, including also one-vertex sequences and the empty sequence. Let $|S|$ denote the number of points in S, let $\#\text{PATH}(S)$ denote the number of non-crossing paths with vertices in S, and let $\#\text{HAM}(S)$ denote the number of non-crossing Hamiltonian paths of S.*

Each non-crossing path corresponds to two sequences (one for each end-to-end order in which its vertices can be placed). We can form a rooted tree with the non-crossing path sequences as its nodes, in which the empty sequence is the root, by defining the parent of any other sequence to be the subsequence obtained by removing its last vertex. The non-crossing path sequences can be listed in polynomial time per sequence by performing a depth-first search of this tree. With a little care in listing the children of each node quickly, we can reduce this to linear time per sequence:

▶ **Subroutine 2** (listing children of a non-crossing path sequence)**.** To find the children of a non-crossing path sequence σ, ending at point p:

- Apply the simple stack-based linear time algorithm of Lee to determine the visibility polygon V of the final vertex of the path, the region of the plane within which that vertex can be connected to another point by a segment that does not cross the existing path [21].
- Let U be the set of points not in σ, and find the radial ordering of U around p. When there are ties, keep only the closest of the tied points to p.
- Merge the radial orderings of U and V to determine, for each point u in U, the edge of V that is crossed by a ray from p through u.
- Whenever the merge finds a point u that is closer to p than the corresponding edge of V, make a child sequence by concatenating u to the end of σ.

The radially-sorted lists of all points around each of the given point can be precomputed and stored in $O(|S|^2)$ time; essentially, this is the same as the problem of constructing and storing an arrangement of lines dual to the points. With this precomputation, all steps of this algorithm take time $O(|S|)$.

▶ **Subroutine 3** (listing all non-crossing paths)**.** To list all non-crossing paths of a given set of points, perform a depth-first search of the tree of non-crossing path sequences, using Subroutine 2 to list the children of each node in the tree. For each node that the search reaches, output it as a path whenever the starting vertex of the sequence has a smaller index than the ending vertex, so that we only output each non-crossing path once. For a point set S, we spend $O(|S|^2)$ preprocessing time and $O(|S|)$ time per path. The number of paths is always $\Omega(|S|^2)$, even for collinear point sets, because a Hamiltonian path always exists and contains that many paths within it, so the preprocessing time is dominated by the per-path time and the total time is $O(|S| \cdot \#\textsc{path}(S))$.

With these subroutines in hand, we can list all non-crossing Hamiltonian paths by the following very simple algorithm.

▶ **Algorithm 4** (listing non-crossing Hamiltonian paths)**.** To list all non-crossing Hamiltonian paths in a point set S:
- List all non-crossing paths by Subroutine 3.
- Whenever a path uses all points of S, output it as a non-crossing Hamiltonian path.

▶ **Theorem 5.** *For a point set S (not assumed to be in general position), Algorithm 4 takes time $(|S| \cdot \#\textsc{ham}(S))^{O(1)}$ to list all non-crossing Hamiltonian paths.*

Proof. This follows from Theorem 20, later in this paper, which states that

$$\#\textsc{path}(S) = \big(|S| \cdot \#\textsc{ham}(S)\big)^{O(1)}. \qquad ◀$$

A very similar tree search can also be used for polygonalizations, instead of Hamiltonian paths.

▶ **Definition 6.** *Yamanaka et al. [37] define a* surrounding polygon *of a point set S to be a simple polygon having a subset of the points as its vertices, surrounding all of the vertices. These include the polygonalizations (in which the subset is all of the points) as well as other polygons; in particular, the convex hull is always a surrounding polygon. Define $\#\textsc{surround}(S)$ to be the number of surrounding polygons in S, and $\#\textsc{poly}(S)$ to be the number of polygonalizations in S.*

Yamanaka et al. define a tree structure on surrounding polygons in which the root is the convex hull, and the parent of any surrounding polygon is obtained by removing one vertex (in a canonically chosen way) from its cyclic sequence of vertices. It follows from a version of the two-ears theorem for polygons (often credited to G. H. Meisters, but used earlier by Max Dehn [17, 23]) that every polygon that is not the convex hull has a parent.

▶ **Subroutine 7** (listing surrounding polygons)**.** As Yamanaka et al. [37] describe, the surrounding polygons of point set S can be listed in time $|S|^{O(1)}$ per polygon, and space $O(|S|)$, by a depth-first search of the tree of polygons. The algorithm uses the method of *reverse search* [5] to perform the depth-first search while only maintaining the identity of a bounded number of tree nodes.

Again, we can use this to list all polygonalizations, as was already done by Yamanaka et al. [37].

▶ **Algorithm 8** (listing polygonalizations)**.** To list all polygonalizations of a point set S:
- List all surrounding polygons by Subroutine 7.
- Whenever a polygon uses all points of S output it as a polygonalization

Yamanaka et al. analyzed this algorithm as having singly-exponential time, but we instead prove that it is output-sensitive:

▶ **Theorem 9.** *For a point set S (not assumed to be in general position), Algorithm 8 takes time $\left(|S| \cdot \#\textsc{poly}(S)\right)^{O(1)}$ to list all polygonalizations.*

Proof. This follows from Theorem 27, later in this paper, which states that

$$\#\textsc{surround}(S) = \#\textsc{poly}(S)^{O(1)}.$$ ◀

3 Counting paths

The main result of this section is to prove a polynomial relation between the number of non-crossing paths and the number of non-crossing Hamiltonian paths, in any point set. This result combines a lower bound on the non-crossing Hamiltonian paths of a point set S, and an upper bound on the number of non-crossing paths, both expressed as a function of the following quantity.

▶ **Definition 10.** *Following [12], define $\textsc{offline}(S)$ to be the smallest k such that removing k points from S leaves a collinear subset.*

It will be convenient to have the following standard bound on logarithms of binomial coefficients. We use log to denote the natural logarithm, but this choice of base makes no difference to the following lemma, because a change of base would only change the logarithm by a constant factor.

▶ **Lemma 11.** *For integers k and n with $k \le n/2$,*

$$\log \binom{n}{k} = \Theta\left(k \log \frac{n}{k}\right).$$

Proof. By applying Stirling's formula, taking logarithms, and omitting terms that are $O(\log n)$, we obtain

$$\log \binom{n}{k} \approx \log \frac{n^n}{k^k (n-k)^{n-k}} = k \log \frac{n}{k} + (n-k) \log \frac{n}{n-k}.$$

This expression is symmetric in k and $n-k$, and when $k \le n/2$, the first term of this approximation is the larger of the two. ◀

3.1 Upper bound

We begin our bounds on non-crossing paths with the simplest one to prove, the upper bound. It is known that the number of paths is at most singly-exponential in the number n of vertices; we prove a tighter bound depending exponentially on the smaller number $\textsc{offline}(S)$ and only polynomially on n.

▶ **Lemma 12.** *Let S be a set of n points, with $\textsc{offline}(S) = k$. Then*

$$\log \#\textsc{path}(S) = O\left(\log n + k \left(\log \frac{n}{k+1}\right)\right).$$

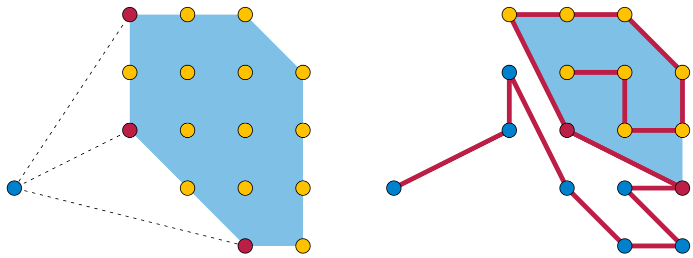

Figure 1 Left: A point p (blue), a point set S (red and yellow), and the visible vertices of S from p (the three red vertices). Note that points of S that lie within convex hull edges are not counted as vertices. Right: A maximal visible-vertex path (red edges) for $S \cup \{p\}$ starting from p, showing for one of its steps the hull (light blue) and visible vertices (red) of the remaining points.

Proof. We describe a method for encoding a non-crossing path using this many bits of information, so that each path is uniquely described by this encoding. For an encoding with b bits of information, the number of paths can be at most 2^b and the logarithm of this number is $O(b)$. Let K be a subset of S, with $|K| = k$, such that $S \setminus K$ lies on a line L, and let P be any given non-crossing path in S. Let $\ell = |L \cap S|$; because k is the minimum number of points that can be removed to form a collinear set, no point of K can lie on L, and $\ell = n - k$. To describe P, we combine the following pieces of information:

- The set Q of points of $L \cap S$ that belong to P, but for which zero or one of their neighbors in the path belong to L. Each such point is one of the two neighbors of a point in K, or one of the two ends of P, so $|Q| \le 2k + 2$. Q can be encoded by specifying its size and the subset of $L \cap S$ of that size, out of $\binom{\ell}{|Q|}$ possibilities, so by Lemma 11 the number of bits needed to specify it is $O\big(\log n + k + k \log(n/k)\big)$. (Both the $\log n$ term and the $+1$ in the statement of the lemma are included to handle the case when $k = 0$ but $|Q| > 0$. Lemma 11 applies only when $k \le n/2$ but for larger k the bound to be proven is superlinear and the result is immediate.)
- For each point in Q, a specification of whether it has a neighbor in L, and if so in which direction. This takes $O(k)$ bits of information.
- The induced subgraph $P[K \cup Q]$, a linear forest using only the points in $K \cup Q$, and omitting the edges of P that lie entirely within L. As with any type of planar straight-line graph, the number of linear forests on $O(k)$ points is singly exponential in k [30], so $P[K]$ can be encoded with $O(k)$ bits of information.

Then P may be recovered by combining the induced subgraph $P[K \cup Q]$ with segments of L starting and ending at points of Q and continuing in the specified direction from each of these points. All pieces of this encoding add up to the stated bound on the number of bits needed to encode the entire path. ◀

3.2 Visible-vertex paths

Our lower bounds will greedily construct paths such that, at each step, the remaining unused points have a convex hull that is uncrossed by the current path and is visible from the endpoint of the current path.

▶ **Definition 13.** *Define a* visible vertex *of a finite point set S, from a point p that is not in the convex hull of S, to be a vertex q of the convex hull of S such that the open line segment pq is disjoint from the convex hull of S (Figure 1, left). We do not consider points interior to edges of the convex hull to be vertices, and in particular they are not visible vertices, even when segment pq is disjoint from the hull.*

▶ **Observation 14.** *Every nonempty finite point set S and point p not in the convex hull of S has at least one visible vertex of S from p. If $S \cup \{p\}$ is not collinear, there are at least two visible vertices.*

Define a *visible-vertex path* to be a polygonal chain formed by a greedy algorithm that builds a sequence of vertices, beginning with any vertex of the convex hull, by repeatedly appending to the sequence a visible vertex of the remaining points not yet in the sequence, as viewed from the last point of the sequence (Figure 1, right). A *maximal visible-vertex path* is a visible-vertex path that uses all of the points in a given point set S; this name is justified by the following lemma.

▶ **Lemma 15.** *Every visible-vertex path is a non-crossing path. Every maximal visible-vertex path is a non-crossing Hamiltonian path. Every non-maximal visible-vertex path can be extended to a longer visible-vertex path.*

Proof. Maximal visible-vertex paths are Hamiltonian, by definition: they are defined to be paths that use all the points. Because each step in a visible-vertex path is defined locally, without respect to the earlier parts of the path, the ability to extend every non-maximal path follows immediately from Observation 14.

It remains to prove that the resulting paths are non-crossing. For every segment pq of the path, the next segment is incident to q and therefore cannot cross pq. All segments after the next segment lie within the convex hull of the remaining points after q. Since p and q are vertices of their respective convex hulls, the convex hull of the points after them in the path does not contain them. Moreover, segment pq does not cross this hull, because if it did then q would not be visible from p. Thus, these later segments are disjoint (as closed line segments) from the closed line segment pq and so cannot cross pq nor even pass through q. Therefore, pq cannot intersect any later segment. For each pair of segments in the path, a crossing is ruled out by applying this argument to the earlier of the two segments in the path, so no crossing can exist. ◀

▶ **Lemma 16.** *In every finite point set S, every two vertices p and q of the convex hull of S form the endpoints of a non-crossing Hamiltonian path.*

Proof. Form a maximal visible-vertex path beginning at p, at each step choosing a visible vertex that is not q when this is possible. By Observation 14, the path will have two points to choose between (one of which is not q) until the remaining points become collinear. Once they do, all remaining points that are not q will lie between q and the current end of the path, so q cannot be included in the path until it is the last point remaining. ◀

3.3 Lower bound for far-from-collinear sets

To lower-bound the number of non-crossing Hamiltonian paths in a point set S, as a function of $\textsc{offline}(S)$, we divide into two cases: the case where $\textsc{offline}(S)$ is large (at least proportional to a constant fraction of $|S|$) and the case where it is small. The following lemma is valid for all non-collinear S, but is only tight (within a constant factor of the logarithm) in the large case.

▶ **Lemma 17.** *For every point set S that does not lie on a single line, the number of non-crossing Hamiltonian paths is at least $\frac{3}{2} \cdot 2^{\textsc{offline}(S)}$.*

Proof. We form a tree T of visible-vertex paths, in which the root is the empty sequence of vertices and the parent of any nonempty path is obtained by removing its last vertex (Figure 2). By Lemma 15, the leaves of T are non-crossing Hamiltonian paths. The root

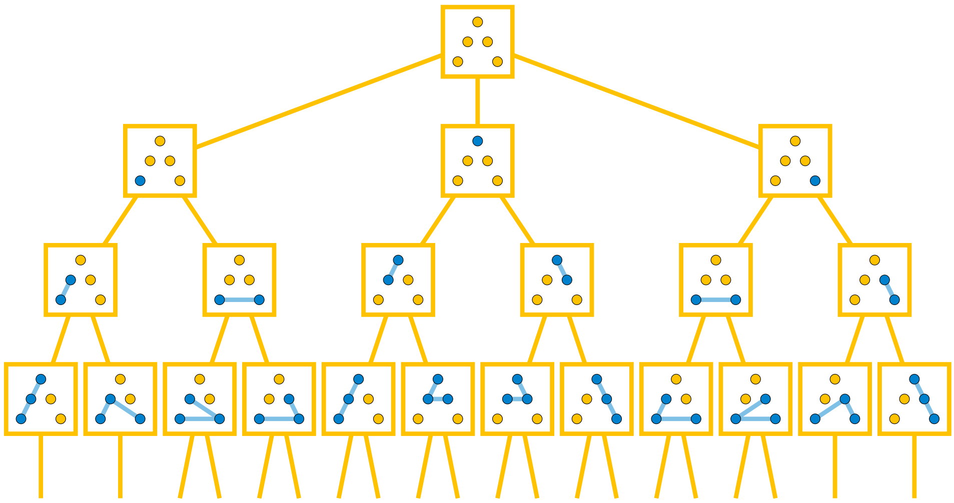

Figure 2 A tree of the visible-vertex paths in a point set, where the parent of each path is obtained by removing its last point. This point set has $\textsc{offline}(S) = 2$. The root and the next two levels of nodes each have multiple children, but some nodes on the last level shown in the figure have only one child, because the last point on the path is collinear with all remaining points.

node of T has at least three children (one for each convex hull vertex). In the nodes of T at distance at most $\textsc{offline}(S)$ from the root, the last point in the path represented by each node is not collinear with the remaining points, by the definition of $\textsc{offline}$. It follows by Observation 14 that these nodes have at least two choices of visible vertices and therefore that they have at least two children.

As a tree in which the root node has at least three children and the nodes in the next $\textsc{offline}(S)$ levels have at least two children, T has at least $3 \cdot 2^{\textsc{offline}(S)}$ leaves. Each leaf is a non-crossing Hamiltonian path, and each non-crossing Hamiltonian path can come from at most two leaves (one for each endpoint, if both endpoints are convex hull vertices). Therefore, there are at least $\frac{3}{2} \cdot 2^{\textsc{offline}(S)}$ non-crossing Hamiltonian paths. ◀

3.4 Lower bound for near-collinear sets

Although the next lemma covers only a special class of point sets, it is the key to our lower bounds for the case where $\textsc{offline}(S)$ is small.

▶ **Lemma 18.** *Let S be a set of points with $|S| = n$, such that ℓ points of S lie on a line L and the remaining points all lie on the same side of the line. Then*

$$\#\textsc{ham}(S) \geq \binom{n - \lceil \ell/2 \rceil}{\lfloor \ell/2 \rfloor}.$$

Proof. We may assume without loss of generality that $\ell \geq 2$, for otherwise the lemma states only that there exists a single non-crossing Hamiltonian path, known to be true. For convenience consider an orientation of the plane in which L is horizontal and S lies in the closed halfspace above L. We will prove the lemma by constructing many non-crossing Hamiltonian paths in which the points of $L \cap S$ appear in left-to-right order, and no point in L has two neighbors in $S \setminus L$. For any such path, define its *signature* to be the binary sequence with a 1-bit for points in L and a 0-bit for points in $S \setminus L$ (Figure 3). It has length

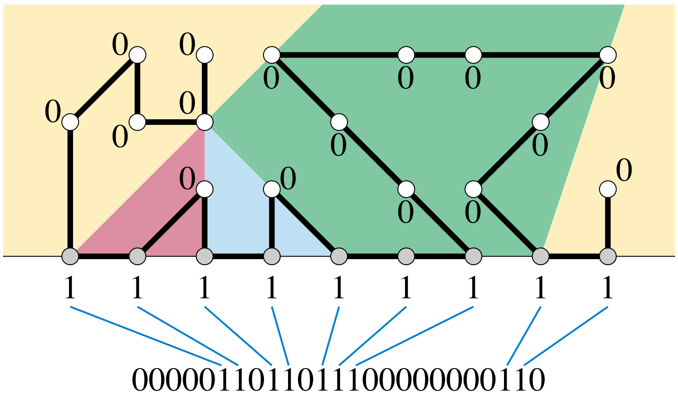

Figure 3 Partition of the halfspace above L into convex subsets, a non-crossing Hamiltonian path respecting the partition, and its signature, a 010-avoiding binary sequence.

n, with ℓ 1-bits, and does not have any three consecutive bits in the pattern 010. The number of 010-avoiding sequences with n bits and ℓ 1-bits is [25]

$$\sum_{j=0}^{n-\ell}(-1)^j\binom{n-\ell-1}{j}\binom{|n-2j|}{\ell-j}\geq\binom{n-\lceil\ell/2\rceil}{\lfloor\ell/2\rfloor},$$

where for the simpler formula on the right hand side of the inequality the 1-bits are grouped into $\lfloor\ell/2\rfloor$ pairs (with one group of three if ℓ is odd) and we count only the sequences in which each pair or triple appears consecutively. (By the assumption that $\ell\geq 2$, at least one such grouping is possible.) As we argue in the remainder of the proof, every 010-avoiding sequence is the signature of at least one non-crossing Hamiltonian path, so this lower bound on the number of 010-avoiding sequences also provides a lower bound on the number of non-crossing Hamiltonian paths.

For a given 010-avoiding sequence σ, let n_i denote the length of the ith non-empty block of consecutive 0-bits in σ. We will partition the halfplane above L into convex sets C_i, each containing n_i points of $S\setminus L$, by a greedy process that maintains a convex subset of the halfplane containing the remaining points to be partitioned. Initially the convex subset is the entire halfplane and the remaining points are $S\setminus L$. On the ith step (for any i other than the last one), let p_i be the point of $S\cap L$ that corresponds to the 1-bit of σ following the ith block of 0-bits. Sort the remaining points of $S\setminus L$ radially around p_i (in left to right order with respect to L) breaking ties in favor of closer points to p_i, and let q_i be the point in position n_i of this sorted order. Draw line p_iq_i, separating C_i on its left from a remaining convex subset on its right. Assign the n_i points up to q_i in the radial sorted order to set C_i, and leave the remaining points (possibly including farther points on line p_iq_i) unassigned. In the final step of this construction, assign all remaining points to the final remaining convex region. For instance, in Figure 3, the leftmost set C_1 (yellow) is separated from the rest of the halfplane by line p_1q_1. Here p_1 is the leftmost point of L, and q_1 is the fifth point in the radial ordering around p_1. Point q_1 lies on the boundary of four convex regions but is assigned to the first, C_1. Line p_1q_1 also contains another point of S, farther from p_1, which is assigned to C_4 (green).

Once this partition into convex sets has been determined, use Lemma 16 to find a non-crossing Hamiltonian path within each convex set C_i that starts and ends at its (one or two) points on L, and connect these paths in sequence by segments of L to form a non-crossing Hamiltonian path for all of S, with the given 010-avoiding sequence σ as its signature. ◀

3.5 Putting the bounds together

Combining the two different lower bounds into a single formula, we have:

▶ **Lemma 19.** *Let S be a set of n points with* $\textsc{offline}(S) = k$. *Then*

$$\log \#\textsc{ham}(S) = \Omega\left(k\left(\log \frac{n}{k+1}\right)\right).$$

Proof. For any k, the logarithm of the number of non-crossing Hamiltonian paths is $\Omega(k)$ by Lemma 17. For $k \geq n/3$, $\log(n/k) = O(1)$, so for this range of k, this $\Omega(k)$ bound is equivalent to the bound stated in the theorem.

For smaller values of k, let K be any set of k points whose removal from S leaves a collinear set of size $\ell = n - k = \Omega(n)$, belonging to a line L. Partition K into the two subsets K_1 and K_2 on the two sides of L, with $|K_1| \geq |K_2|$; let $|K_1| = k' \geq k/2$. Let p be the first point of $S \cap L$, in the sorted sequence of the points along this line, let $\ell' = \ell - 1$ be the number of remaining points in $S \cap L$, and let $n' = \ell' + k'$. By Lemma 18, the number of non-crossing Hamiltonian paths of $S \setminus K_2$ that start at p is at least

$$\binom{n' - \lceil \ell'/2 \rceil}{\lfloor \ell'/2 \rfloor} = \binom{\lfloor \ell'/2 \rfloor + k'}{k'}.$$

Each such path can be extended to a non-crossing Hamiltonian path of all of S by concatenating any non-crossing path through p and the points of K_2. By the assumption that $k < n/3$, the bottom term of the right binomial coefficient is at most half the top term, allowing us to apply Lemma 11. By this lemma, and the facts that $k' = \Theta(k)$ and $\ell' = \Theta(n)$, the logarithm of this binomial coefficient is $\Omega\big(k \log(n/k)\big)$ as stated. ◀

Although the upper bound of Lemma 12 and the lower bound of Lemma 19 are not quite the same, we can combine them to achieve a constant factor approximation to the logarithm of the number of non-crossing paths, or Hamiltonian paths.

▶ **Theorem 20.** *For a given point set S,*

$$\#\textsc{path}(S) = \big(|S| \cdot \#\textsc{ham}(S)\big)^{O(1)}.$$

Proof. Taking logs of both sides, it is equivalent to write that

$$\log \#\textsc{path}(S) = O(\log |S| + \log \#\textsc{ham}(S)).$$

This follows immediately from Lemma 19 and Lemma 12, according to which $\log \#\textsc{ham}(S)$ is lower-bounded and $\log \#\textsc{path}(S)$ upper-bounded (respectively) to within constant factors by formulas that differ from each other only in an additive $\log |S|$ term. ◀

4 Counting cycles

For counting both surrounding cycles and polygonalizations of general-position point sets, in place of $\textsc{offline}(S)$ (which the general-position assumption makes trivial) we use the following parameter:

▶ **Definition 21.** *Let* $\textsc{inhull}(S)$ *denote the number of points of S that are interior to the convex hull of S.*

For counting cycles and polygonalizations of point sets that are not assumed to be in general position, we will use a combined analysis in terms of both OFFLINE and INHULL.

4.1 Omitted lemmas

Our bounds for surrounding cycles and polygonalizations follow similar arguments to our bounds for paths. We defer many details and all proofs to the full version of this paper because of space limitations.

▶ **Lemma 22.** *Let S be a set of n points, with* $\textsc{inhull}(S) = h$. *Then*

$$\log \#\textsc{surround}(S) = O\left(h\left(\log\frac{n}{h}+1\right)\right).$$

▶ **Corollary 23.** *Let S be a set of n points with* $\min(\textsc{offline}(S), \textsc{inhull}(S)) = m$. *Then*

$$\log \#\textsc{surround}(S) = O\left(m\left(\log\frac{n}{m}+1\right)\right).$$

▶ **Lemma 24.** *Let S be a set of n points for which at most $|S|/7$ points lie on any line, and at most $|S|/7$ points lie on the convex hull. Then the number of polygonalizations of S is at least singly exponential in S.*

▶ **Lemma 25.** *Let S be a set of points with $|S| = n$, such that h points of S lie on its convex hull (either as vertices or within its edges). Then*

$$\#\textsc{ham}(S) \geq \binom{\lfloor h/4\rfloor + \lceil (n-h)/2\rceil - 1}{\lceil (n-h)/2\rceil}$$

4.2 Putting the bounds together

Combining our lower bounds into a single formula, we have:

▶ **Lemma 26.** *Let S be a set of n points with* $\min(\textsc{offline}(S), \textsc{inhull}(S)) = m$. *Then*

$$\log \#\textsc{poly}(S) = \Omega\left(m\left(\log\frac{n}{m}+1\right)\right).$$

Proof. We consider the following cases:

- If $\textsc{inhull}(S) \leq 6n/7$, the result follows from Lemma 25. In particular this applies when the largest subset of collinear points in S has size $\geq n/7$ and is part of the convex hull.
- If the largest subset of collinear points in S has size $\geq n/7$ but is not part of the convex hull, then let L be the line through this subset. Because L does not lie on the convex hull, $S \setminus L$ includes points on both sides of L, with at least $m/2$ points in one of these two halfplanes. By Lemma 18 the number of Hamiltonian paths through the points in L and the points in this halfplane, starting and ending at the two extreme points of L, meets or exceeds the lower bound in the statement of this lemma. Each of these Hamiltonian paths can be completed to a polygonalization through the points in the other halfplane bounded by L, by Lemma 16.
- In the remaining case, the largest subset of collinear points in S has size $< n/7$ and $\textsc{inhull}(S) \geq 6n/7$. In this case, $m = \Omega(n)$ and the bound of the lemma reduces to $\Omega(n)$. The result follows from Lemma 24.

Since all cases have at least the number of polygonalizations stated, the bound holds.

For a bound of $\Omega(i)$, apply Lemma 24, and for $\Omega(i \log n/i)$ when $i \leq n/2$, apply Lemma 25, in both cases using Lemma 11 to estimate the logarithm of the binomial coefficient. ◀

From the fact that the upper bound of Corollary 23 and the lower bound of Lemma 26 have exactly the same form, we obtain a constant factor approximation to the logarithm of the number of surrounding cycles and of polygonalizations. For our bound on the complexity of the algorithm for listing polygonalizations we need it in the following form:

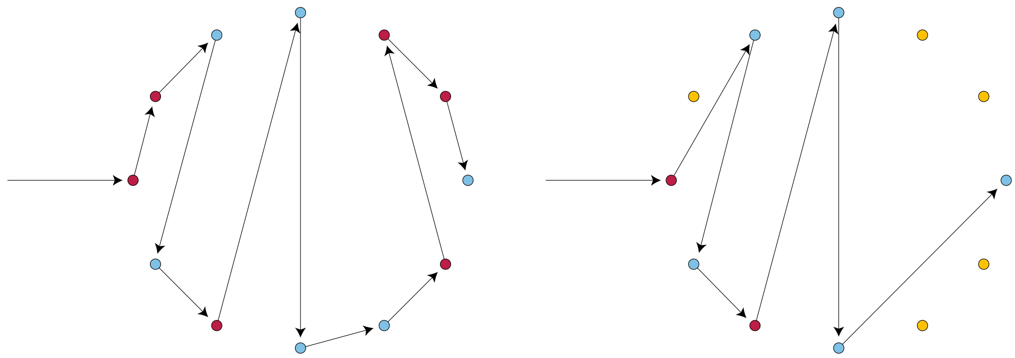

Figure 4 Coloring-based arguments for the number of non-crossing paths and Hamiltonian paths of a point set in convex position. Each 2-coloring of the points and choice of starting point determines a non-crossing Hamiltonian path in which the colors determine the direction of the next step; each 3-coloring and choice of starting point determines a path, skipping vertices of one of the colors.

▶ **Theorem 27.** *For a given point set S,*

$$\#\textsc{surround}(S) = \#\textsc{poly}(S)^{O(1)}.$$

5 Nonlinearity

In this section we investigate the exponent of the polynomial bounds on non-crossing paths as a function of non-crossing Hamiltonian paths, and on surrounding cycles as a function of polygonalizations. In both cases we show that the exponent is bounded away from one, for inputs in which the number of non-crossing configurations is exponential. This implies, in particular, that the backtracking algorithms that we investigate for Hamiltonian paths and polygonalizations can in some instances be forced to take an amount of time per output that is exponential in the input size, despite being polynomial in the output size.

The construction for paths and Hamiltonian paths is very simple:

▶ **Theorem 28.** *There exist sets S of points for which*

$$\#\textsc{path}(S) \geq \#\textsc{ham}(S)^{\log_2 3 - o(1)} \approx \#\textsc{ham}(S)^{1.585}$$

and moreover for which both the number of non-crossing paths and the number of non-crossing Hamiltonian paths is exponential in $|S|$.

Proof. We may take S to be in convex position. If a set S in convex position has n points, it has $n2^{n-3}$ non-crossing Hamiltonian paths [27], but

$$\frac{n}{4}(3^{n-1} + 3)$$

non-crossing paths in total, considering a single vertex to count as a path of length zero.

Both bounds can be proven by a simple coloring argument (Figure 4). For non-crossing Hamiltonian paths, consider the 2^n ways of coloring the points red and blue, and n choices of where to start. For each choice, follow a path that, at each red point, steps clockwise to the next available vertex, and at each blue point steps counterclockwise. Each non-crossing Hamiltonian path is found for exactly eight choices: the path can start at either end, and the final two vertex colors are irrelevant. Thus, the number of paths is $n2^n/8$.

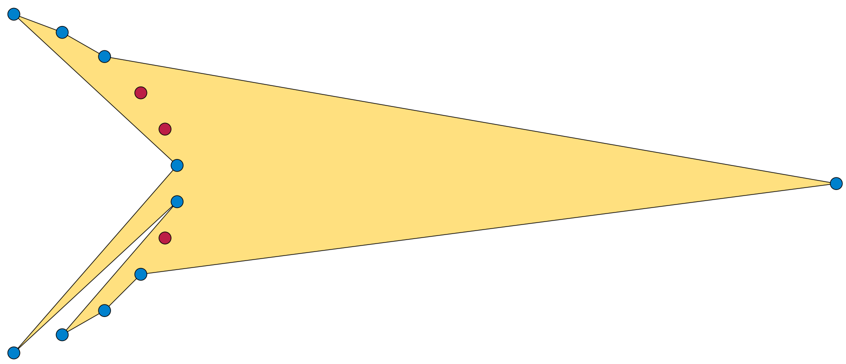

Figure 5 Points formed by replacing an edge of a triangle by a convex chain of points within the triangle, and a surrounding polygon (yellow) for these points. Points can be omitted as polygon vertices (red) only when they lie between the two neighbors of the apex (the rightmost point of the figure).

For the bound on non-crossing paths, consider instead the 3^n ways of coloring the points red, blue, and yellow, and skip the yellow points both in choosing a starting point and in considering which vertices are available in subsequent steps of the path. Cyclically permuting the colors in a coloring groups the 3^n colorings into 3^{n-1} orbits, each of which has a total of $2n$ red or blue starting points, so there are $2n \cdot 3^{n-1}$ choices. Again, each path is found for exactly eight choices, except for the single-vertex paths, which are found for only two choices. The total number of paths is obtained by dividing $2n \cdot 3^{n-1}$ by eight, and adjusting for the number of single-vertex paths. ◀

For an analogous separation between surrounding polygons and polygonalizations, we replace one edge of a triangle by a convex chain of $n-2$ edges, within the triangle (Figure 5). The polygonalizations of this point set are obtained from non-crossing Hamiltonian paths through the points of the convex chain, with both ends of the path connected to the one point that does not belong to the convex chain (which we call the *apex*). The edges to the apex cannot cross any other edge, so all Hamiltonian paths lead to polygonalizations in this way. Therefore, by the same formula already used above, the number of polygonalizations is exactly $(n-1)2^{n-4}$.

A surrounding polygon for these points must have the apex as a vertex (because it lies on the convex hull). It must also include as vertices all of the points of the convex chain that lie outside the two neighbors of the apex. The points of the convex chain that lie between the two neighbors of the apex may either be vertices of the surrounding polygon, or omitted; if omitted, they will automatically be surrounded. We may parameterize a surrounding polygon by three non-negative numbers: a, the number of *outer points* of the convex chain that lie outside the two neighbors of the apex, b, the number of *inner points* that lie between these two neighbors and are vertices of the polygon, and c, the number of *omitted points* that are not vertices of the surrounding polygon. Necessarily, $a+b+c=n-3$, because the points that are counted by these three numbers include all points except the apex and its two neighbors.

Once a, b, and c have been chosen, the surrounding polygon itself may be determined by three more choices: how to partition the outer a points into left and right subsets, with $a+1$ possibilities, how to alternate between outer and inner points along the polygon, with $\binom{a+b}{a}$ possibilities, and how to partition the points between the two neighbors of the apex into inner points and omitted points, with $\binom{b+c}{b}$ possibilities. Therefore, the total number of surrounding polygons of this point set is

$$\sum_{a+b+c=n-3} (a+1)\binom{a+b}{a}\binom{b+c}{b}.$$

These numbers, for $n = 3, 4, 5, \ldots$, are

$$1, 4, 13, 40, 120, 354, 1031, 2972, 8495, 24110, \ldots,$$

a sequence having the generating function $(1-2x)/(1-3x+x^2)^2$ and growing proportionally to $n(\varphi+1)^n$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio [26]. This example proves:

▶ **Theorem 29.** *There exist sets S of points for which*

$$\#\textsc{surround}(S) \geq \#\textsc{poly}(S)^{\log_2(\varphi+1)-o(1)} \approx \#\textsc{poly}(S)^{1.388}$$

and moreover for which both the number of surrounding polygons and the number of polygonalizations is exponential in $|S|$.

6 Conclusions

We have developed simple output-sensitive algorithms for listing all non-crossing Hamiltonian paths and all polygonalizations for a point set. However, their dependence on the output size is polynomial, not linear. It would be of interest to find alternative algorithms with a better dependence on the output size, as well as more accurate approximations for the numbers of non-crossing structures.

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