

An Extension Theorem for Signotopes

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Abstract

In 1926, Levi showed that, for every pseudoline arrangement \mathcal{A} and two points in the plane, \mathcal{A} can be extended by a pseudoline which contains the two prescribed points. Later extendability was studied for arrangements of pseudohyperplanes in higher dimensions. While the extendability of an arrangement of proper hyperplanes in \mathbb{R}^d with a hyperplane containing d prescribed points is trivial, Richter-Gebert found an arrangement of pseudoplanes in \mathbb{R}^3 which cannot be extended with a pseudoplane containing two particular prescribed points.

In this article, we investigate the extendability of signotopes, which are a combinatorial structure encoding a rich subclass of pseudohyperplane arrangements. Our main result is that signotopes of odd rank are extendable in the sense that for two prescribed crossing points we can add an element containing them. Moreover, we conjecture that in all even ranks $r \geq 4$ there exist signotopes which are not extendable for two prescribed points. Our conjecture is supported by examples in ranks 4, 6, 8, 10, and 12 that were found with a SAT based approach.

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1 Introduction

Given a family of hyperplanes \mathcal{H} in \mathbb{R}^d , any d points in \mathbb{R}^d , not all on a common hyperplane of \mathcal{H} , define a hyperplane which is distinct from the hyperplanes in \mathcal{H} . For dimension $d = 2$, Levi [14] proved in his pioneering article on pseudoline arrangements that the fundamental extendability of line arrangements also applies to the more general setting of pseudoline arrangements. A *pseudoline* is a Jordan curve in the Euclidean plane such that its removal from the plane results in two unbounded components, and a *pseudoline arrangement* is a family of pseudolines such that each pair of pseudolines intersects in exactly one point, where the two curves cross properly.



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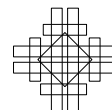
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► **Theorem 1.1** (Levi’s extension lemma for pseudoline arrangements [14]). *Given an arrangement \mathcal{A} of pseudolines and two points in \mathbb{R}^2 , not on a common pseudoline of \mathcal{A} . Then \mathcal{A} can be extended by an additional pseudoline which passes through the two prescribed points.*

Several proofs for Levi’s extension lemma are known today (besides [14], see also [1, 9, 19]) and generalizations to higher dimensions have been studied in the context of oriented matroids, which by the representation theorem of Folkman and Lawrence [10] have representations as projective pseudohyperplane arrangements. For more about oriented matroids, see [8].

Goodman and Pollack [12] presented an arrangement of 8 pseudoplanes in \mathbb{R}^3 and a selection of three points such that there is no extension of the arrangement with a pseudoplane containing the points. Richter-Gebert [18] then investigated a weaker version with only two prescribed points in dimension 3 such that the extending pseudohyperplane contains these two points. He found an example of a rank 4 oriented matroid on 8 elements such that there is no one element extension with an element containing the two prescribed cocircuits. With the representation theorem this implies that even the weaker extendability with two prescribed points does not hold. The existence of an extension theorem or of counterexamples in higher dimensions/ranks remains open.

We present a proof of Levi’s extension lemma in a purely combinatorial setting and show that the proof can be adapted to work for higher dimensions. We represent the geometry by r -signotopes and prove extendability in even dimensions d , that is, when the rank $r = d + 1$ is odd; see Theorem 1.2. Surprisingly, there are non-extendable examples in rank 4, 6, 8, 10, and 12. We conjecture that there is no extension theorem for any even rank $r \geq 4$; see Conjecture 1.4.

Signotopes are in close relation to higher Bruhat orders which were introduced by Manin and Schechtman [15] and further studied in [21]. In rank 3, signotopes correspond to pseudoline arrangements in the plane [9]. In higher ranks they are a subclass of pseudohyperplane arrangements.

Before we formulate our extension theorem for r -signotopes, we introduce some notation and discuss the relation between pseudoline arrangements and 3-signotopes (in Section 1.1). This leads to a reformulation of Levi’s extension lemma which will be investigated in the context of signotopes of odd rank in Section 1.2.

1.1 Signotopes

Signotopes are a combinatorial structure generalizing permutations and *simple* pseudoline arrangements (i.e., no three pseudolines cross in a common point). For $r \geq 1$ a *signotope* of rank r (short: r -signotope) on n elements is a mapping σ from r -element subsets (r -subsets) of $[n]$ to $+$ or $-$, i.e., $\sigma : \binom{[n]}{r} \rightarrow \{+, -\}$ such that for every $(r + 1)$ -subset $X = \{x_1, \dots, x_{r+1}\}$ (r -packet) of $[n]$ with $x_1 < x_2 < \dots < x_{r+1}$ there is at most one sign change in the sequence

$$\sigma(X \setminus \{x_1\}), \sigma(X \setminus \{x_2\}), \dots, \sigma(X \setminus \{x_{r+1}\}).$$

Note that this sequence lists the signs of all induced r -subsets of X in reverse lexicographic order. For 3-signotopes, the following 8 sign patterns on 4-subsets are allowed:

$$++++, +++-, ++--, +---, ----, ---+, --++, -++++.$$

For sake of readability, we write $X = (x_1, \dots, x_t)$ to denote a t -subset of $[n]$ with sorted elements $x_1 < x_2 < \dots < x_t$. For such an X we denote by $X_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_t)$

the set without x_j . With the convention $- < +$, the condition about sign changes in r -signotopes can be written as a monotonicity condition for r -packets $X = (x_1, \dots, x_{r+1})$:

$$\sigma(X_1) \leq \sigma(X_2) \leq \dots \leq \sigma(X_{r+1}) \quad \text{or} \quad \sigma(X_1) \geq \sigma(X_2) \geq \dots \geq \sigma(X_{r+1}).$$

It is well-known that every arrangement of pseudolines is isomorphic to an arrangement of x -monotone pseudolines [11]. In such a representation, we label the pseudolines from top to bottom on the left by $1, \dots, n$. Since two pseudolines cross exactly once, the pseudolines appear in reversed order on the right. Now the corresponding 3-signotope σ is obtained as follows: The sign of $\sigma(a, b, c)$ for $a < b < c$ indicates the orientation of the triangle formed by the pseudolines a, b, c (see Figure 1). If the crossing of a and c is below b , it is $\sigma(a, b, c) = +$ and if the crossing of a and c is above b , it is $\sigma(a, b, c) = -$. In the following we identify the crossings with the elements which cross, i.e. for 3-signotopes crossings are subsets of size 2. The 3-signotope σ gives information about the partial order of the crossings from left to right. If $\sigma(a, b, c) = +$ it holds $ab \prec ac \prec bc$ and if $\sigma(a, b, c) = -$ it is $bc \prec ac \prec ab$.



■ **Figure 1** Connection between pseudoline arrangements and 3-signotopes.

Felsner and Weil [9] showed that rank 3 signotopes are in bijection with simple pseudoline arrangements in \mathbb{R}^2 with a fixed top cell. For $r \geq 4$, r -signotopes correspond to special pseudohyperplane arrangements in \mathbb{R}^{r-1} , i.e., they are a subclass of oriented matroids of rank r . A geometric representation of r -signotopes in the plane is presented in [16] (see also [3] for the rank 3 case).

1.2 An extension theorem for signotopes

In Levi’s extension lemma for pseudoline arrangements, each of the two prescribed points can either lie in a cell of the arrangement, on one pseudoline, or be the crossing point of two pseudolines. To formulate an extension lemma in terms of 3-signotopes we restrict our considerations to simple pseudoline arrangements and to crossing points as prescribed points. Since the extending pseudoline passes through the two prescribed crossing points, the extension yields a non-simple arrangement. However, by perturbing the extending pseudoline at the non-simple crossing points, we arrive at a simple arrangement, see Figure 2.



■ **Figure 2** Perturbing an extending pseudoline at the two non-simple crossing points.

A perturbation at a prescribed crossing together with the new inserted pseudoline yields a *triangular cell* incident to the crossing. This cell is bounded by the two pseudolines

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defining the crossing and the extending pseudoline. Triangular cells play an important role in the study of pseudoline arrangements, since it is possible to change the orientation of a triangle by moving one of the pseudolines over the crossing of the two others. Such a local perturbation is called a *triangle flip*, it does not change the orientation of any other triangle in the arrangement. The triangular cells of the arrangement represented by a 3-signotope σ are in one to one correspondence with 3-subsets such that if we change the sign of this 3-subset in σ we obtain a new signotope σ' . We call such a 3-subset a *fliple*. The notion of fliples generalizes to higher ranks. In an r -signotope σ on $[n]$, an r -subset $X \subseteq [n]$ is a *fliple* if both assignments $+$ and $-$ to $\sigma(X)$ result in a signotope. It is worth noting that fliples in signotopes are the analogon of mutations in oriented matroids. While every signotope contains at least two fliples [9], it remains a central open problem in combinatorial geometry to decide whether every uniform oriented matroid contains a mutation [8, Chapter 7.3].

Let \mathcal{A} be an arrangement of pseudolines, which are labeled $1, \dots, n$ from top to bottom on the left. When applying Levi's extension lemma to extend \mathcal{A} the left endpoint of the extending line ℓ will be between two consecutive endpoints of pseudolines of \mathcal{A} . To re-establish the properties of the labeling, we have to set the label of ℓ accordingly and increase the label of every pseudoline that starts below ℓ by one. To cope with this relabeling-issue in terms of signotopes, we introduce the following notion. For $k \in [n]$ and a subset X of $[n]$, we define

$$X \downarrow_k = \{x \mid x \in X, x < k\} \cup \{x - 1 \mid x \in X, x > k\}.$$

Note that the cardinality of X and $X \downarrow_k$ is the same if and only if $k \notin X$. For an r -signotope σ on $[n]$, we define the *deletion* of an element $k \in [n]$ as $\sigma \downarrow_k$ by

$$\sigma \downarrow_k(X \downarrow_k) := \sigma(X)$$

for all r -sets $X \subseteq [n]$ with $k \notin X$. This is an r -signotope on $[n - 1]$ because each r -packet has been an r -packet for σ .

► **Definition.** An r -signotope σ on $[n]$ is t -extendable if for all pairwise disjoint $(r - 1)$ -subsets $I_1, \dots, I_t \in \binom{[n]}{r-1}$, there exists $k \in [n + 1]$ and an r -signotope σ^* on $[n + 1]$ with fliples I_1^*, \dots, I_t^* such that $\sigma^* \downarrow_k = \sigma$, and $I_j^* \downarrow_k = I_j$ for all $j = 1, \dots, t$. Hence the element k extends σ to σ^* .

Note that a t -extendable r -signotope on $n \geq (r - 1)t$ elements is clearly $(t - 1)$ -extendable. While the 1-extendability is a simple exercise¹, the first interesting part is the 2-extendability, which we investigate in this paper.

► **Theorem 1.2** (An extension theorem for signotopes of odd rank). *For every odd rank $r \geq 3$, every r -signotope is 2-extendable.*

Surprisingly, our proof of Theorem 1.2 (see Section 3) generalizes to the more general setting, where the $(r - 1)$ -subsets I and J , which are fliples in the extension, intersect.

► **Corollary 1.3.** *Let σ be an r -signotope on $[n]$, I and J be two $(r - 1)$ -subsets of $[n]$ such that $|I \cap J| + r$ is odd. Then σ is extendable to an r -signotope σ^* on $[n + 1]$ with fliples I^*, J^* and an extending element $k \in [n + 1]$ such that $\sigma^* \downarrow_k = \sigma$, and $I^* \downarrow_k = I$, and $J^* \downarrow_k = J$.*

¹ For the sake of completeness, we give a proof of 1-extendability in Corollary 3.2 which uses more evolved techniques.

Despite the restrictions to simple arrangements and crossing points as prescribed points we can derive Levi's extension lemma (Theorem 1.1) in its full generality with little extra work from Theorem 1.2. Details are deferred to Section 5.

The statement of Theorem 1.2 applies only to signotopes of odd rank. This is not just a defect of our proof because signotopes in even rank indeed behave differently. For ranks $r = 4, 6, 8, 10, 12$ we found signotopes on $n = 2r$ elements, which are not 2-extendable. The examples and the source code to verify their correctness are available as supplemental data [4]; see Section 4 for details. Based on these examples, we dare to conjecture:

► **Conjecture 1.4** (No extension theorem for signotopes of even rank). *For every even rank $r \geq 4$, there is an r -signotope which is not 2-extendable.*

1.3 Signotopes as a rich subclass of oriented matroids

It is well known that the number of oriented matroids of rank r on n elements is $2^{\Theta(n^{r-1})}$ [8, Corollary 7.4.3]. As shown by Balko [2, Theorem 3], r -signotopes are a rich subclass of oriented matroids of rank r ; see the full version [5] for a shorter proof of the following proposition.

► **Proposition 1.5** (Balko [2]). *For $r \geq 3$, the number of r -signotopes on $[n]$ is $2^{\Theta(n^{r-1})}$.*

In ranks 1 and 2 there are 2^n and $n!$ signotopes on $[n]$, respectively. Rank 1 signotopes are mappings from $[n]$ to $\{+, -\}$ without any additional property and 2-signotopes are permutations. For rank $r \geq 3$, the precise number of r -signotopes on $[n]$ has been computed for small values of r and n ; see A6245 (rank 3) and A60595 to A60601 (rank 4 to rank 10) on the OEIS [17].

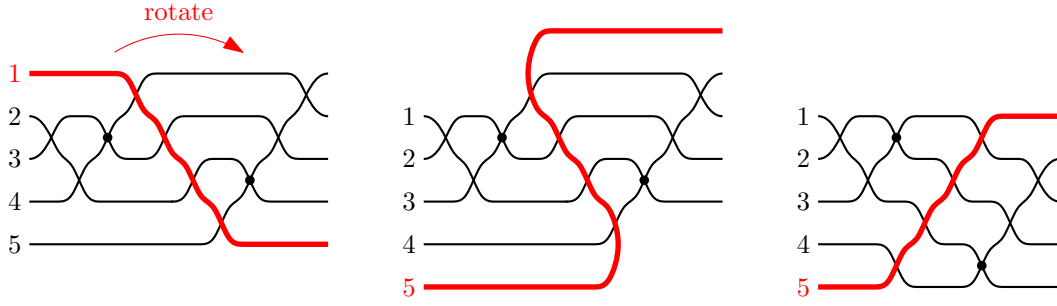
2 Preliminaries

We now prepare for the proof of Theorem 1.2. As discussed in Section 1.1, signotopes of rank 3 can be represented by an arrangement with x -monotone pseudolines. The order of the crossings from left to right gives a partial order on the 2-subsets. In general, r -signotopes can be represented by a sweepable arrangement of pseudohyperplanes in \mathbb{R}^{r-1} , which similarly allows to define a partial order on $(r-1)$ -subsets which correspond to the crossings of $r-1$ elements. This partial order is combinatorially defined as follows. For an r -signotope σ and every r -subset $X = (x_1, \dots, x_r)$ define:

$$\begin{aligned} X_1 \succ X_2 \succ \dots \succ X_r & \quad \text{if } \sigma(x_1, \dots, x_r) = +, & \quad \text{and} \\ X_1 \prec X_2 \prec \dots \prec X_r & \quad \text{if } \sigma(x_1, \dots, x_r) = -. \end{aligned}$$

Recall that we use the convention $x_1 \leq \dots \leq x_r$ and $X_i = X \setminus \{x_i\}$. By taking the transitive closure of all relations obtained from r -subsets, we obtain a partial order on the $(r-1)$ -subsets corresponding to σ [9, Lemma 10].

If we rotate an arrangement of pseudolines, i.e., we choose another unbounded cell as the top cell, we get a pseudoline arrangement with the same cell structure. However, the signotope does not stay the same. If we rotate only a single pseudoline, then the orientation of the triangle spanned by 3 pseudolines stays the same if and only if the rotated pseudoline is not involved (see for example the triangle spanned by $\{2, 3, 4\}$ in the left-hand side arrangement, resp. $\{1, 2, 3\}$ in the right-hand side arrangement in Figure 3). When rotating *clockwise*, the first element of σ becomes the last one in the rotated signotope σ_{rot} . In terms of the



■ **Figure 3** An illustration of a clockwise rotation of pseudolines. The rotated pseudoline is highlighted red.

3-signotope σ the signs of the rotated signotope σ_{rot} are: $\sigma_{\text{rot}}(a, b, c) = \sigma(a + 1, b + 1, c + 1)$ if $c \neq n$ and $\sigma_{\text{rot}}(a, b, n) = -\sigma(1, a + 1, b + 1)$.

In general, we define the *clockwise rotated* signotope σ_{rot} of a given r -signotope σ as:

$$\sigma_{\text{rot}}(x_1, \dots, x_r) = \begin{cases} -\sigma(1, x_1 + 1, \dots, x_{r-1} + 1) & \text{if } x_1 < x_2 < \dots < x_r = n, \\ \sigma(x_1 + 1, \dots, x_r + 1) & \text{if } x_1 < x_2 < \dots < x_r < n. \end{cases}$$

Here we use the usual convention $-\cdot + = -$ and $-\cdot - = +$. To keep track of the index shift caused by a clockwise rotation, we define $x_{\text{rot}} = x - 1$ if $x \neq 1$ and $1_{\text{rot}} = n$, and

$$X_{\text{rot}} = \{x_{\text{rot}} : x \in X\} = \begin{cases} (x_1 - 1, x_2 - 1, \dots, x_k - 1) & \text{if } x_1 > 1; \\ (x_2 - 1, \dots, x_k - 1, n) & \text{if } x_1 = 1 \end{cases}$$

for any subset $X = (x_1, \dots, x_k)$ of $[n]$ with $x_1 < \dots < x_k$. Note that this allows us to write $\sigma_{\text{rot}}(X_{\text{rot}}) = \sigma(X)$ if $1 \notin X$ and $\sigma_{\text{rot}}(X_{\text{rot}}) = -\sigma(X)$ if $1 \in X$.

As the following lemmas show, this is indeed an r -signotope, which moreover has essentially the same flips. The proofs and further properties are deferred to the full version [5].

► **Lemma 2.1.** *Let σ be an r -signotope on $[n]$. Then σ_{rot} is an r -signotope on $[n]$.*

► **Lemma 2.2.** *Let σ be an r -signotope and let F be a fliple of σ . Then F_{rot} is a fliple in the clockwise rotated signotope σ_{rot} .*

3 Extension theorem for signotopes

In this section, we give a proof for the extension theorem for signotopes of odd rank. The central ingredient of our proof is as follows. If σ is an r -signotope on $[n]$ and the prescribed two $(r - 1)$ -sets I and J are incomparable in the partial order associated with σ (see Section 2), then σ is extendable by a “last” element $n + 1$ such that $I \cup \{n + 1\}$ and $J \cup \{n + 1\}$ are fliples in the extension. Figure 2 gives an illustration for the rank 3 case. More abstractly we can extend the signotope when there is a down-set in the partial order on $(r - 1)$ -sets which has I and J as maximal elements. A *down-set* of a partial order $(\mathcal{P}, <)$ is a subset $\mathcal{D} \subseteq \mathcal{P}$ such that for all $p \in \mathcal{P}$ and $d \in \mathcal{D}$ with $p \preceq d$ it holds $p \in \mathcal{D}$. Similarly, an *up-set* is a subset $\mathcal{U} \subseteq \mathcal{P}$ such that for all $p \in \mathcal{P}$ and $u \in \mathcal{U}$ with $p \succeq u$ it holds $p \in \mathcal{U}$.

► **Proposition 3.1** (Extension for incomparable elements). *Let $(\mathcal{P}, <)$ be the partial order on $(r - 1)$ -sets corresponding to an r -signotope σ on $[n]$. For every down-set $\mathcal{D} \subseteq \mathcal{P}$ there exists an r -signotope σ^* on $[n + 1]$ such that all r -subsets of the form $M \cup \{n + 1\}$, where M is a maximal element of \mathcal{D} , are fliples of σ^* and $\sigma^*_{\downarrow n+1} = \sigma$.*

Proof. Define the extended r -signotope σ^* on $[n + 1]$ as follows:

$$\sigma^*(x_1, \dots, x_r) = \begin{cases} \sigma(x_1, \dots, x_r) & \text{if } x_1, \dots, x_r \in [n]; \\ + & \text{if } x_r = n + 1 \text{ and } \{x_1, \dots, x_{r-1}\} \in \mathcal{D}; \\ - & \text{if } x_r = n + 1 \text{ and } \{x_1, \dots, x_{r-1}\} \notin \mathcal{D}. \end{cases}$$

First we show that σ^* is an r -signotope on $[n + 1]$. Consider an r -packet $X = (x_1, \dots, x_{r+1})$. We need to show that the sequence

$$\sigma^*(X_1), \sigma^*(X_2), \dots, \sigma^*(X_{r+1})$$

has at most one sign change.

If $x_{r+1} \leq n$, then all signs on the considered r -subsets are the same as for σ . Since σ is an r -signotope, there is at most one sign change in the sequence.

In the other case, we have $x_{r+1} = n + 1$. For all $j \leq r$ we have $n + 1 \in X_j$. Furthermore, $\sigma^*(X_{r+1}) = \sigma(X_{r+1})$ because $n + 1 \notin X_{r+1}$. We consider two cases. First, if $\sigma(X_{r+1}) = +$ we have by definition of the partial order

$$X \setminus \{x_{r+1}, x_i\} \succ X \setminus \{x_{r+1}, x_j\} \quad \text{for } i < j.$$

By the property of a down-set this means that, whenever $X \setminus \{x_{r+1}, x_i\} \in \mathcal{D}$, we also have $X \setminus \{x_{r+1}, x_j\} \in \mathcal{D}$ for $i < j$. Let i^* be the smallest integer such that $X \setminus \{x_{r+1}, x_{i^*}\} \in \mathcal{D}$. Then by definition of σ^* we have $\sigma^*(X_j) = -$ for all $j < i^*$ and $\sigma^*(X_j) = +$ for all $j \geq i^*$.

Similar arguments apply if $\sigma(X_{r+1}) = -$. Then we have

$$X \setminus \{x_{r+1}, x_i\} \prec X \setminus \{x_{r+1}, x_j\} \quad \text{for } i < j.$$

This time let i^* be the smallest integer such that $X \setminus \{x_{r+1}, x_{i^*}\} \notin \mathcal{D}$. Then by definition of σ^* we have $\sigma^*(X_j) = +$ for all $j \leq i^*$ and $\sigma^*(X_j) = -$ for all $j > i^*$.

Let M be a maximal element of the down-set \mathcal{D} . By the analysis above it follows that $M \cup \{n + 1\}$ is adjacent to a sign change in each packet in which it is contained. Hence it is a fliple. ◀

From this proposition it follows that for all $r \geq 2$ all r -signotopes are 1-extendable. Moreover the 1-extension contains the extending element at the last position.

▶ **Corollary 3.2** (1-extendability). *For $r \geq 2$ let σ be an r -signotope on $[n]$ and I an $(r - 1)$ -subset. Then there is an extending r -signotope σ^* on $[n + 1]$ elements such that $I \cup \{n + 1\}$ is a fliple and $\sigma^* \downarrow_{n+1} = \sigma$.*

The following two propositions show that, for odd rank, we can always find a rotation of the corresponding signotope such that the two prescribed $(r - 1)$ -subsets are incomparable. We can then use Proposition 3.1 to define an extension.

▶ **Proposition 3.3.** *Let σ be an r -signotope on $[n]$. For two $(r - 1)$ -subsets I, J with $I \prec J$ and $1 \notin I \cap J$, it holds I_{rot} and J_{rot} are incomparable in \prec_{rot} or $I_{\text{rot}} \prec_{\text{rot}} J_{\text{rot}}$.*

The proof of Proposition 3.3 needs some more structural properties of the partial order and its interaction with the rotation. The details are deferred to the full version [5].

▶ **Proposition 3.4.** *Let $r \geq 3$ be an odd integer, let σ be an r -signotope on $[n]$ and let I, J be two disjoint $(r - 1)$ -subsets. After at most $n - 1$ clockwise rotations, σ, I , and J are transformed into σ', I' , and J' , resp., such that I' and J' are incomparable in the partial order \prec' corresponding to σ' .*

Proof. Assume I and J are comparable in the partial order \prec corresponding to the r -signotope σ with $I \prec J$, otherwise we are done. We show that after n clockwise rotations, all signs of σ are reversed. Hence the partial order \prec' corresponding to the (possible multiple times) rotated signotope σ' is the reversed relation to \prec .

The sign of an r -subset (z_1, \dots, z_r) changes from $+$ to $-$ or vice versa if and only if the rotated element is contained in (z_1, \dots, z_r) , i.e., if we rotate z_1 . Hence after rotating n times in total every z_i was rotated and thus the sign of an r -subset changes exactly r times. Since r is odd, the sign after rotating n times is opposite. The obtained signotope σ' is the reverse of the original signotope σ and the corresponding partial order is also reversed.

Furthermore we cannot reverse the order of two disjoint $(r - 1)$ -sets in one rotation as shown in Proposition 3.3. Hence there will be a moment where the two disjoint sets are incomparable. \blacktriangleleft

Although the following lemma is trivial in the setting of pseudoline arrangements, we need to prove it in the context of general r -signotopes. We show that the extension of a rotated signotope when rotated back does contain the original signotope. To show this we need to investigate the interaction between the rotation and deletion of elements.

► Lemma 3.5. *Let σ be an r -signotope on $[n]$ and $x \in [n]$. Then it is $\sigma_{\text{rot}\downarrow n} = \sigma_{\downarrow 1}$ and $\sigma_{\text{rot}\downarrow x_{\text{rot}}} = (\sigma_{\downarrow x})_{\text{rot}}$ for $x \neq 1$.*

Proof. Because of the index shift it does not matter whether we delete the first element or we rotate σ such that in the first element becomes the last and delete the last element in this rotated signotope. Hence the first part $\sigma_{\text{rot}\downarrow n} = \sigma_{\downarrow 1}$ holds.

Now assume $x \neq 1$ which implies $x_{\text{rot}} \neq n$. Both mappings are r -signotopes on $[n - 1]$. We need to check whether they map to the same signs. Let X be an r -subset of $[n - 1]$ and let X^* be an r -subset of $[n]$ with $x_{\text{rot}} \notin X^*$ and $X^*_{\downarrow x_{\text{rot}}} = X$. We obtain

$$\sigma_{\text{rot}\downarrow x_{\text{rot}}}(X) = \sigma_{\text{rot}\downarrow x_{\text{rot}}}(X^*_{\downarrow x_{\text{rot}}}) = \sigma_{\text{rot}}(X^*).$$

We will now rewrite the term to get the statement. Recall that rotating an r -signotope on n elements exactly $2n$ times results in the original signotope. Hence rotating $2n - 1$ times corresponds to a counterclockwise rotation, i.e., the inverse operation of a clockwise rotation. We denote this counterclockwise rotation by $\text{rot}(-1)$. Since $x_{\text{rot}} \notin X^*$, we have $x \notin (X^*)_{\text{rot}(-1)}$. By definition it is

$$\sigma_{\text{rot}}(X^*) = \varepsilon \cdot \sigma((X^*)_{\text{rot}(-1)}) = \varepsilon \cdot \sigma_{\downarrow x}(((X^*)_{\text{rot}(-1)})_{\downarrow x}) = \varepsilon \cdot \sigma_{\downarrow x}(X_{\text{rot}(-1)}) = (\sigma_{\downarrow x})_{\text{rot}}(X),$$

where the sign $\varepsilon = +$ (resp. $\varepsilon = -$) if $n \notin X^*$ (resp. $n \in X^*$). Note that $n \in X^*$ is equivalent to $1 \in X_{\text{rot}(-1)}$ for $x \neq 1$. This completes the proof of the lemma. \blacktriangleleft

With Proposition 3.1, Proposition 3.4 and Lemma 3.5 we are now ready to prove Theorem 1.2.

3.1 Proof of Theorem 1.2

Let σ be an r -signotope on $[n]$ and let I, J be a pair of disjoint $(r - 1)$ -subsets. By Proposition 3.4 there exists $k \in \{0, \dots, n - 1\}$ such that the k -fold rotated $(r - 1)$ -subsets $I_{\text{rot}(k)}, J_{\text{rot}(k)}$ are incomparable in the k -fold rotated signotope $\sigma_{\text{rot}(k)}$.

To extend the signotope $\sigma_{\text{rot}(k)}$, we use the down-set \mathcal{D} consisting of $I_{\text{rot}(k)}, J_{\text{rot}(k)}$, and all $(r - 1)$ -subsets below. In this down-set $I_{\text{rot}(k)}$ and $J_{\text{rot}(k)}$ are maximal elements since they are incomparable. Hence we can apply Proposition 3.1 in order to add a new element at

position $n + 1$ in the rotated signotope $\sigma_{\text{rot}(k)}$ such that $I_{\text{rot}(k)} \cup \{n + 1\}$ and $J_{\text{rot}(k)} \cup \{n + 1\}$ are flipes. The extended signotope is denoted by $\sigma_{\text{rot}(k)}^*$ and fulfills $\sigma_{\text{rot}(k)}^* \downarrow_{n+1} = \sigma_{\text{rot}(k)}$.

Finally, we need to find a rotation of $\sigma_{\text{rot}(k)}^*$ which contains the original signotope σ . For this we perform $k + 1$ counterclockwise rotations (or equivalently, $2n + 1 - k$ clockwise rotations) and denote the so-obtained signotope by σ^* . Note that we perform $k + 1$ counterclockwise rotations since the newly added element needs to be rotated and the k -fold clockwise rotation needs to be undone. After $k + 1$ counterclockwise rotations, the added element $n + 1$ in $\sigma_{\text{rot}(k)}^*$ becomes the element $k + 1$ in σ^* . It remains to show that $\sigma^* \downarrow_{k+1} = \sigma$.

After the first counterclockwise rotation, the added element $n + 1$ in $\sigma_{\text{rot}(k)}^*$ becomes the first element 1 in $(\sigma_{\text{rot}(k)}^*)_{\text{rot}(-1)}$. By Lemma 3.5 it holds $((\sigma_{\text{rot}(k)}^*)_{\text{rot}(-1)}) \downarrow_1 = (\sigma_{\text{rot}(k)}^*) \downarrow_{n+1} = \sigma_{\text{rot}(k)}$. After additional k counterclockwise rotations, the added element $n + 1$ in $\sigma_{\text{rot}(k)}^*$ becomes the element $k + 1$ in σ^* . Furthermore $I \cup \{k + 1\}$ and $J \cup \{k + 1\}$ are flipes of σ^* by Lemma 2.2. Since we do not rotate the extending element, applying the second part of Lemma 3.5 multiple times shows $((\sigma_{\text{rot}(k)}^*)_{\text{rot}(-1)}) \downarrow_1 = (\sigma^* \downarrow_{k+1})_{\text{rot}(k)}$. Together with the previous equation this shows $\sigma_{\text{rot}(k)} = (\sigma^* \downarrow_{k+1})_{\text{rot}(k)}$, which further implies $\sigma = \sigma^* \downarrow_{k+1}$. Hence we obtain the signotope σ when deleting $k + 1$ from σ^* . This completes the proof of Theorem 1.2.

3.2 Proof of Corollary 1.3

To prove Corollary 1.3, we proceed similar as in the proof of Theorem 1.2. By Proposition 3.1, it suffices to show that after some rotations the $(r - 1)$ -subsets corresponding to I and J are incomparable.

Let $s = |I \cap J|$. Since Theorem 1.2 covers the case $s = 0$, we may assume $s \geq 1$. We consider the following two cases.

First, assume that r is odd and s is even. For odd rank r , we have already seen that after n rotations, the signotope is reversed and hence the corresponding partial order is reversed. For even s , the relation between I and J is reversed s times (whenever we rotate one element $x \in I \cap J$). These are the only s times where we reverse the order in one single rotation. Since s is even and the order is reversed after n rotations, the corresponding $(r - 1)$ -subsets must be incomparable in between.

Second, assume that r is even and s is odd. For even rank r , the n -fold rotation leaves the signotope unchanged and hence also the partial orders are the same. Since s is odd, we reverse the orientation of I and J exactly s times in a single rotation step. Hence they must be incomparable in between.

The statement now follows from Proposition 3.1 and Lemma 3.5 similar as in the proof of Theorem 3.1. This completes the proof of Corollary 1.3.

4 Examples in even rank: SAT attack and properties

Since the proof for the extension theorem (Theorem 1.2) applies only for odd ranks, we had to investigate even ranks in a different manner. For rank 4, we used computer assistance to enumerate all signotopes and then tested 2-extendability for each signotope. On 6 and 7 elements all 4-signotopes are 2-extendable. On 8 elements we found non-2-extendable 4-signotopes. For both, the enumeration and the 2-extendability test, we modeled SAT instances which were then solved using the python interfaces `pycosat` [20] and `pysat` [13] to run the SAT solver `picosat`, version 965, [6] and `cadical`, version 1.0.3 [7], respectively.

Using this two-level-SAT approach we managed to find the first examples of 4-signotopes which are not 2-extendable. In order to keep symmetries and similarities of our nicely structured example of rank 4, we restricted our search space to examples in rank r on

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$2r$ elements. While for rank 4 all signotopes on 8 elements can be enumerated within a few seconds, the complete enumeration in higher ranks is unpractical as the number of r -signotopes on $2r$ elements grows faster than doubly exponential in r (cf. Proposition 1.5). Hence, to be able to approach higher ranks, we further analyzed the structure of our non-2-extendable rank 4 examples together with an analyze of the already found rank 6 examples. These made it possible to find a recursive construction. See Section 4.3 for more details.

With the observed properties as additional constraints, we further restricted the search space so that only “reasonable” candidates were enumerated. Under these restrictions, we managed to find examples for rank 6, 8, 10, and 12 which are not 2-extendable.

4.1 SAT model for enumeration

To encode r -signotope on n elements, we proceed as following. We use Boolean variables S_X for every $X \in \binom{[n]}{r}$ to indicate whether $\sigma(X) = +$. To ensure that these variables model a valid signotope, we add constraints which ensure that for every r -packet $Y = \{y_1, \dots, y_{r+1}\} \in \binom{[n]}{r+1}$ there is at most one sign-change in the sequence

$$\sigma(Y_1), \dots, \sigma(Y_{r+1}).$$

More precisely, since there are exactly $2r + 2$ possible assignment of this sequence, we introduce auxiliary variables $T_{Y,t}$ for $t \in \{1, \dots, 2r + 2\}$ to indicate which of the assignments applies.²

Next we introduce auxiliary variables $F_{X,Y}$ for every r -packet $Y \in \binom{[n]}{r+1}$ and every r -tuple $X \in \binom{[n]}{r}$ to indicate whether X is a fliple when σ is restricted to Y . This is done in a similar fashion as for the S_X variables. Using the $F_{X,Y}$ variables, we can assert the variables $F_X = \bigvee_{Y \in \binom{[n]}{r+1}: X \subset Y} F_{X,Y}$ for every $X \in \binom{[n]}{r}$ to indicate whether X forms a fliple. Last but not least, we introduce variables $L_{X,k}$ to indicate whether X is the k -th fliple. This will allow us to enumerate only configurations with a prescribed number of fliples.

4.2 SAT model for testing 2-extendability

We are now ready to formulate a SAT instance to decide whether a given signotope σ on $[n]$ and given disjoint $(r - 1)$ -tuples I, J can be extended by an additional element $n + 1$ such that $I \cup \{n + 1\}$ and $J \cup \{n + 1\}$ are fliples in the extension σ^* . This is sufficient to test extendability since whenever there is an extension, there is a rotation such that the signotope is extendable by an element at the last position. As in Section 4.1, we create a SAT instance to find an $(n + 1)$ -element signotope but we add constraints to fix σ and to assert that $I \cup \{n + 1\}$ and $J \cup \{n + 1\}$ are fliples in σ^* .

For a given signotope σ on elements $[n]$ we can now iterate over all disjoint $(r - 1)$ -tuples I, J and test whether there is a rotation of σ and I, J such that in the extension σ^* by the element $n + 1$ the r -tuples $I \cup \{n + 1\}$ and $J \cup \{n + 1\}$ are fliples. If for some I, J no such rotations exists, we have certified that σ is not 2-extendable.

² Alternatively one can assert $\neg S_{Y_i} \vee S_{Y_j} \vee \neg S_{Y_k}$ and $S_{Y_i} \vee \neg S_{Y_j} \vee S_{Y_k}$ for every Y and $1 \leq i < j < k \leq r + 1$. Even though this approach does not require auxiliary variables to indicate the types of $(r + 1)$ -tuples, we need these auxiliary variables to assign the variables for fliples later anyhow.

4.3 Structure of the examples supporting Conjecture 1.4

In order to find the first witnessing examples for Conjecture 1.4 in rank 4, we used the two-step SAT approach as described in Sections 4.1 and 4.2. To make investigations in higher ranks, we had to get a better understanding of the examples found in rank 4. Hence we filtered those with regularities and symmetries to come up with a generalization of the observed properties and analyzed their structure. Our aim was to find a relation between examples in different ranks, for example using projection and deletion arguments. For this we investigated the structure of our rank 4 examples together with some already found rank 6 examples.

One of the first and crucial observations was that there exist signotopes such that for every choice of even indices $I \subset E_r := \{2, 4, \dots, 2r\}$ and every choice of odd indices $J \subset O_r := \{1, 3, \dots, 2r - 1\}$ there is no such extension. In fact, for such examples it is sufficient to check $I = \{2, 4, \dots, 2r - 2\}$ and $J = \{1, 3, \dots, 2r - 3\}$ to verify the non-2-extendability. This observation not only allowed us to restrict the search space, but also to speed up the extendability-test by a factor of $\Theta(r^2)$ since not all pairs of $(r - 1)$ -tuples I, J need to be tested.

While we came up with further observations one by one over the time, we here give a summary of all the properties, which we desire from the examples in rank r with $n = 2r$ elements. In the following we denote by $X = (x_1, x_2, \dots, x_r)$ an r -tuple and use the notation $(-)^i = +$ if i is even and $(-)^i = -$ if i is odd.

- (a) $\sigma = \sigma_{\text{rot}(4)}$, where $\sigma_{\text{rot}(4)}$ is obtained by the 4-fold rotation of σ .
- (b) $\sigma(2, 4, \dots, 2r) = -$ and $\sigma(1, 3, \dots, 2r - 1) = +$.
- (c) If there is only one even or only one odd element in X , then the sign $\sigma(X)$ depends only on the position of that element in X . More specifically: If $e = x_i$ is the only even element in X , then $\sigma(X) = (-)^i$. If $o = x_i$ is the only odd element in X , then it is $\sigma(X) = (-)^{i+1}$.
- (d) If $x_1, \dots, x_i \in E_r$ and $x_{i+1}, \dots, x_r \in O_r$ with $2 \leq i \leq r - 2$, then the sign is $\sigma(X) = (-)^{i+1}$.
- (e) Let $x_1, \dots, x_i \in O_r$ and $x_{i+1}, \dots, x_r \in E_r$ for $2 \leq i < r - 2$. If $x_r < 2r$, then $\sigma(X) = -$. If $x_j = 2j$ for all $j = i + 1, \dots, r$, then $\sigma(X) = +$.

Furthermore, we fix the following set of 8 fliples for rank 4.

$$F_4 = \{(1, 3, 5, 7), (2, 4, 6, 8), (2, 3, 7, 8), (1, 3, 4, 8), \\ (1, 2, 4, 7), (3, 5, 6, 8), (4, 5, 7, 8), (3, 4, 6, 7)\}$$

Together with the 4-fold symmetry it is sufficient to mention only some of them:

$$\widehat{F}_4 = \{(1, 3, 5, 7), (2, 4, 6, 8), (4, 5, 7, 8), (3, 4, 6, 7), (1, 2, 4, 7)\}$$

In rank 4, there are only four signs which are not determined by the above properties:

$$(1, 3, 4, 8), \quad (4, 5, 7, 8), \quad (2, 3, 7, 8), \quad (3, 4, 6, 7)$$

By the 4-fold symmetry, the assignment of $(1, 3, 4, 8)$ also determines the sign of $(4, 5, 7, 8)$, and vice versa. The third and fourth tuple have a similar interaction. Hence, there are precisely 4 signotopes in rank 4 which fulfill the above properties. We fix one of the four configurations (the choice does not play a role) and refer to it as σ_4 in the following.

In order to find examples in higher ranks, we use the following property.

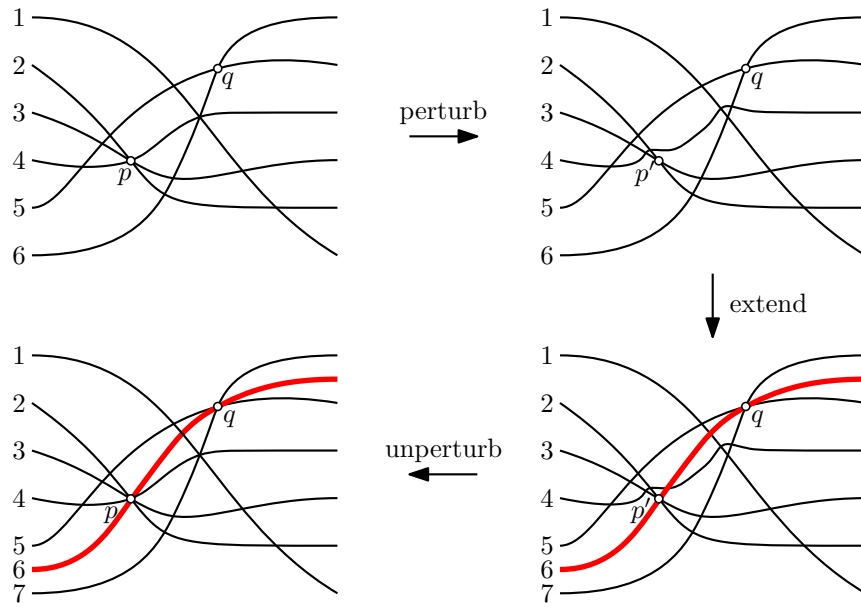


Figure 4 An illustration how Theorem 1.2 implies Levi’s extension lemma (Theorem 1.1). When perturbing the top-left arrangement, the multi-crossing point p (the intersection of 2, 3, and 4) is split into three simple crossing points, including the point p' (the intersection of 2 and 3). After the extension, we again contract these three crossing points to one multi-crossing point.

- (f) Let σ_{r-2} be an example of rank $r - 2$ on $2r - 4$ elements. For an r -tuple $X \subseteq [2r]$ with $1, 3 \notin X$ and $2, 4 \in X$, we define the sign

$$\sigma_r(X) = \sigma_{r-2}(X \downarrow_{1,2,3,4}),$$

where $X \downarrow_{1,2,3,4} = (((X \downarrow_4) \downarrow_3) \downarrow_2) \downarrow_1$ denotes the $(r - 2)$ tuple on $[2r - 4]$. Note that $X \downarrow_{1,2,3,4}$ is obtained by deleting the elements 2 and 4 from X and a further index shift by -2 caused by deleting³ 1 and 3, which are not contained in X .

Altogether, if we start with one example from rank 4 and recursively construct examples in higher ranks with the desired properties and further prescribe $(r/2)^2 + (r/2) + 2$ flips for rank r , it finally turned out that there is a unique example in each of the ranks $r = 6, 8, 10, 12$. All examples and the source code to verify their correctness are available as supplemental data [4].

In the future we hope to find an argument for the non-2-extendability based on the described properties and construct an infinite family of examples. Even though we conjecture that there is an infinite family, we want to clarify that we found examples in rank 4 and 6 which do not have the above properties and hence the assumptions might also be too strong.

5 Theorem 1.2 implies Levi’s extension lemma (Theorem 1.1)

As outlined in Section 1.2, it is sufficient to prove Levi’s extension lemma for simple arrangements of pseudolines and for crossing points as prescribed points. Given a non-simple arrangement, we can slightly perturb the multiple crossing points (as depicted in Figure 2)

³ Inspired by the language of oriented matroids, such an operation might be called *contraction*.

to obtain a simple arrangement. We obtain simplicial cells instead of the multiple crossings. This simple arrangement can then be extended, and each of the multiple crossing points of the original arrangement can again be obtained by contracting the simplicial cells to a point. Whenever a prescribed point lies on a pseudosegment or inside a cell, we can extend the arrangement through an adjacent crossing. By perturbing the extending pseudoline, we can ensure that the pseudoline passes through the originally prescribed point.

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