Real Numbers Equally Compressible in Every Base

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Abstract

This work solves an open question in finite-state compressibility posed by Lutz and Mayordomo [20] about compressibility of real numbers in different bases.

Finite-state compressibility, or equivalently, finite-state dimension, quantifies the asymptotic *lower density of information* in an infinite sequence.

Absolutely normal numbers, being finite-state incompressible in every base of expansion, are precisely those numbers which have finite-state dimension equal to 1 in every base. At the other extreme, for example, every rational number has finite-state dimension equal to 0 in every base.

Generalizing this, Lutz and Mayordomo in [20] (see also Lutz [19]) posed the question: are there numbers which have absolute positive finite-state dimension strictly between 0 and 1 - equivalently, is there a real number ξ and a compressibility ratio $s \in (0, 1)$ such that for every base b, the compressibility ratio of the base-b expansion of ξ is precisely s? It is conceivable that there is no such number. Indeed, some works explore "zero-one" laws for other feasible dimensions [11] - *i.e.* sequences with certain properties either have feasible dimension 0 or 1, taking no value strictly in between.

However, we answer the question of Lutz and Mayordomo affirmatively by proving a more general result. We show that given any sequence of rational numbers $\langle q_b \rangle_{b=2}^{\infty}$, we can *explicitly construct* a single number ξ such that for any base b, the finite-state dimension/compression ratio of ξ in base-b is q_b. As a special case, this result implies the existence of absolutely dimensioned numbers for *any* given rational dimension between 0 and 1, as posed by Lutz and Mayordomo.

In our construction, we combine ideas from Wolfgang Schmidt's construction of absolutely normal numbers from [23], results regarding low discrepancy sequences and several new estimates related to exponential sums.

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1 Introduction

Finite-state compressibility is the lower asymptotic ratio of compression achievable on an infinite string using information-lossless finite-state compressors [28, 25]. Finite-state dimension was originally defined by Dai, Lathrop, Lutz and Mayordomo in [10] using finite-state s-gales, as a finite-state analogue of Hausdorff dimension [18, 2]. Surprisingly, these notions are equivalent [10, 7]. They also have several equivalent characterizations in terms of automatic Kolmogorov complexity [16], finite-state predictors [13], block-entropy rates [13], etc. establishing their mathematical robustness.

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In this work, we solve an open question posed by Lutz and Mayordomo [20] about the existence of numbers whose finite-state compression ratio does not depend on the base of expansion, where the compression ratio is neither 0 nor 1. We explain the context behind this question below.

In [24], Schnorr and Stimm show the celebrated result that finite-state incompressibility in base-b is equivalent to Borel normality in base-b, establishing a deep connection between information theory and metric number theory. Absolutely normal numbers [1, 4, 20, 22, 23], being finite-state incompressible in every base-b, are precisely the set of numbers whose finite-state dimension is 1 in every base. Almost every number in $[0, 1]$ is absolutely normal (see [6]). Several explicit constructions absolutely normal numbers are known [23, 1, 20].

At the other extreme, there are numbers which have finite-state dimension 0 in every base. For example, any rational number in $[0, 1]$ has eventually periodic expansion in any base b and hence, finite-state dimension 0 in any base of expansion.

But, no method for even proving the existence of numbers with absolute finite-state dimension strictly between 0 and 1 are known at the time of writing this paper. Lutz and Mayordomo in [20] ask the following question:

> *Does there exist a real number ξ whose finite-state dimension/compressibility, $\dim^b_{FS}(\xi)$, does not depend on the choice of base b, and such that $0 < \dim^b_{FS}(\xi) < 1$?*

This question was originally posed by Jack Lutz during the session "Alan Turing in the twenty-first century: normal numbers, randomness, and finite automata" in CCR 2012 at the Isaac Newton Institute of Mathematical Sciences, University of Cambridge [19]. In this work, we answer the question in the affirmative, by proving a stronger result: *Given any list $\langle q_b \rangle_{b=1}^{\infty}$ of rationals in $(0, 1]$ (respecting the natural equivalence between bases b), there exists an explicitly constructible number ξ such that $\dim^b_{FS}(\xi) = q_b$ for any base b.* In the special case when every q_b is equal, this provides an affirmative answer (and an explicitly constructible example) to the above question posed by Lutz and Mayordomo.

We now state our main result. Two positive integer bases r and s are said to be equivalent, denoted $r \sim s$, if there are $m, n \in \mathbb{N}$ such that $r^m = s^n$. (Formal definitions follow in Section 2). If $r \sim s$, then we can verify that for any $\xi \in [0, 1]$, $\dim^r_{FS}(\xi) = \dim^s_{FS}(\xi)$. Our main result is the following.

▶ **Theorem 1.** *Let $\langle q_b \rangle_{b=1}^{\infty}$ be a sequence of rationals in $(0, 1]$ such that for any r and s, if $r \sim s$, then $q_r = q_s$. Then, there exists a $\xi \in [0, 1]$ such that for any base b, $\dim^b_{FS}(\xi) = q_b$.*

When $q_b = q$ for every base b, for some $q \in \mathbb{Q} \cap [0, 1]$, we have the following corollary.

▶ **Corollary 2.** *Let $q \in [0, 1] \cap \mathbb{Q}$. Then, there exists a $\xi \in [0, 1]$ such that $\dim^b_{FS}(\xi) = q$ for every base $b \geq 2$.*

For $q \in (0, 1]$, the existence of ξ in the above corollary follows from Theorem 1. For $q = 0$, any rational number ξ in $[0, 1]$ satisfies the required conclusion. Therefore, this corollary provides a positive answer to the question posed by Lutz and Mayordomo in [20].

Explicit constructions of numbers with specific compressibility ratios often use combinatorial techniques (for example, see [8]). Combinatorial constructions are often simpler to understand. However, in our work, we control multiple bases and dimensions in each base. Since this implies working with different alphabets simultaneously, a combinatorial approach to the solution is not easy. We take an approach which involves exponential sums, which allows us to handle the construction requirements successfully.

In this construction, we modify and combine techniques from Wolfgang Schmidt's construction of absolutely normal numbers in [23] along with results regarding low discrepancy sequences [12, 21] and several new estimates. Schmidt's method has been generalized to construct numbers exhibiting specific kinds of normality (or non-normality) in different bases (see for example [3]). Another interesting such generalization is [5] where Schmidt's method is adapted to demonstrate the pairwise independence of discrepancy functions for non-equivalent bases. Our construction is yet another generalization of Schmidt's method which yields numbers having prescribed rates of information in different bases. While the construction in [5] yields absolutely normal numbers with controlled oscillation of discrepancy rates in different bases, we use results regarding low discrepancy sequences and Schmidt's method to construct numbers with controlled oscillation of block entropy rates in different bases (which are not normal in base b unless $q_b = 1$).

The construction consists of multiple stages. In every stage we arrange a *controlled oscillation* of block entropies in a particular base b between q_b and 1, by fixing digits in base b using multiple *low discrepancy* strings. Meanwhile we *stabilize* the entropies in other non-equivalent bases around 1, using generalizations of bounds from [23] and estimates relating exponential averages to block entropies. We also ensure that the block entropies in different bases do not *fall too much* during the transition between consecutive stages. This is accomplished by using estimates involving low discrepancy strings and lower bounds based on the concavity of the Shannon entropy function ([9]).

The rest of the paper is organized as follows. After the preliminaries in Section 2, we outline the construction and the *requirements* that it should satisfy in Section 3. We also prove that if the requirements hold, then Theorem 1 follows. In Section 4, we develop some important technical tools required for the construction. In Section 5 we describe our stage-wise construction in detail. Finally, in Section 6, we *verify* that the construction in Section 5 satisfies all the *requirements* given in Section 3.

2 Preliminaries

2.1 Basic definitions and notation

We use Σ to denote any finite alphabet. For any natural number $b > 1$, Σ_b denotes the alphabet $\{0, 1, 2, \dots b-1\}$. Observe that for any $b_1 \leq b_2$, we have $\Sigma_{b_1} \subseteq \Sigma_{b_2}$. For any finite alphabet Σ, we use Σ^* to represent the set of finite binary strings and Σ^∞ to represent the set of infinite sequences in alphabet Σ. We use capital letters X, Y to denote infinite strings in any finite alphabet Σ_b. We use small letters w, z to represent finite strings in any finite alphabet Σ_b. For any $X = X_1X_2X_3\dots$ from Σ^∞, we use X_i^{i+n} to denote the substring $X_iX_{i+1}\dots X_{i+n}$ and for any $w \in \Sigma^*$, w_i^{i+n} to represent $w_iw_{i+1}\dots w_{i+n}$.

Greek letters α, β and γ are used to represent constants which are either absolute or dependent on some of the parameters which are relevant in the context of their use. Small letters x, y and the Greek letter ξ are used to denote general real numbers in $[0, 1]$. Small letters b, r and s are used for representing integer bases of expansion greater than or equal to 2. Now, we define the occurrence (sliding) probabilities of finite strings.

▶ **Definition 3.** *We define the* occurrence count *of $z \in \Sigma^*$ in $w \in \Sigma^*$ denoted $N(z, w)$, as $N(z, w) = |\{i \in [1, |w| - |z| + 1] : w_i^{i+|s|-1} = z\}|$. The* occurrence probability *of z in w denoted $P(z, w)$, is defined as $P(z, w) = N(z, w)/(|w| - |z| + 1)$.*

The following is the definition of the base-b finite-state dimension of a sequence ξ using the block entropy characterization of finite-state dimension given in [7], which we use in this work instead of the original definition using s-gales from [10].

▶ **Definition 4** ([10, 7]). *For a given base b and a block length l, we define the l-length* block entropy *over $w \in \Sigma_b^*$ as follows.*

$$H_l^b(w) = -(l\log(b))^{-1} \quad \sum\nolimits_{z\in\Sigma_b^l} \quad P(z,w)\log P(z,w).$$

Let $\xi \in [0,1]$ and let $X \in \Sigma_b^\infty$ represent the base-b expansion of ξ (for numbers having two base-b expansions, let X denote any one of these). The base-b finite-state dimension *of ξ, denoted $\dim_{FS}^b(\xi)$, is defined by*

$$\dim_{FS}^b(\xi) = \inf_l \liminf_{n\to\infty} H_l^b(X_1^n).$$

Numbers of the form k/b^n for some $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ have two base-b expansions. But their finite-state dimension is equal to 0 irrespective of the infinite sequence chosen as the base-b expansion of x. Therefore, the base-b finite-state dimension is well-defined for every $\xi \in [0,1]$.

▶ Remark. The fact that $\dim_{FS}(x)$ is equivalent to the lower finite-state compressibility of x using lossless finite-state compressors, follows from the results in [28] and [10].

For every $x \in \mathbb{R}$, let $e(x) = e^{2\pi i x}$. For $x \in \mathbb{R}$ and $d > 0$, let $B_d(x)$ denote the open neighborhood of x having radius d, i.e, $B_d(x) = \{y \in \mathbb{R} : |y - x| < d\}$. For any $w \in \Sigma_b^*$, let $v_b(w)$ denote the rational number,$v_b(w) = \sum_{i=1}^{|w|} w_i b^{-i}$. Therefore, the interval $I_w^b = [v_b(w), v_b(w) + b^{-|w|})$ denotes the set of all numbers in $[0,1]$ whose base-b expansion begins with the string w. The *characteristic function* χ_w of a string $w \in \Sigma_b^*$ is defined as, $\chi_w(x) = 1$ if $x \in I_w^b$ and $\chi_w(x) = 0$ otherwise.

The following is a well-known Fourier series approximation for characteristic functions of cylinder sets (see [22, 15]) which acts as the basic *bridge* between the combinatorial and analytic approaches.

▶ **Lemma 5** ([22, 15]). *For any string $w \in \Sigma_b^*$, $\delta > 0$ and $i \in \{1,2\}$, there exists coefficients C_t^i satisfying $\left|C_t^i\right| \le \frac{2}{t^2\delta}$, such that for every real number x,*

$$b^{-|w|} - \delta + \sum_{t\in\mathbb{Z}\setminus\{0\}} C_t^1 e(tx) \quad \le \quad \chi_w(x) \quad \le \quad b^{-|w|} + \delta + \sum_{t\in\mathbb{Z}\setminus\{0\}} C_t^2 e(tx).$$

2.2 Low discrepancy sequences

Let $X_1X_2X_3\ldots$ be the sequence in Σ_b^∞ representing the base-b expansion of $x \in [0,1]$. For any real interval $(\alpha_1,\alpha_2) \subseteq [0,1]$ let $R^b(x,n,\alpha_1,\alpha_2)$ be defined as follows.

$$R^b(x,n,\alpha_1,\alpha_2) = \left|\frac{|\{1 \le i \le n \mid 0.X_iX_{i+1}X_{i+2}\cdots \in (\alpha_1,\alpha_2)\}|}{n} - (\alpha_2-\alpha_1)\right|$$

The *discrepancy function* $D_n^b(x)$ is defined to be the supremum over all $(\alpha_1,\alpha_2) \subseteq [0,1]$ of $R^b(x,n,\alpha_1,\alpha_2)$ ([12], [21]). The following theorem regarding the low discrepancy of almost every real number follows from the results in [12] and [21].

▶ **Theorem 6** ([12, 21]). *For any base b, there exists a constant C_b such that for almost every x, $\limsup_{n\to\infty} \frac{nD_n^b(x)}{\sqrt{n\log\log n}} < C_b$.*

The following lemma is a corollary of Theorem 6.

▶ **Lemma 7.** *For any base b, there exists a constant C_b such that for any $\epsilon > 0$, there exists $N_b(\epsilon)$ such that outside a set of measure at most ϵ, for any $\alpha_1 < \alpha_2$ and $n \geq N_b(\epsilon)$, we have $R^b(x, n, \alpha_1, \alpha_2)$ is strictly less than $C_b \cdot \frac{\sqrt{\log\log n}}{\sqrt{n}}$.*

The following is a corollary of the above lemma that we need in our construction.

▶ **Corollary 8.** *For any base b, there exists a constant C_b such that for any $\epsilon > 0$, there exists $N_b(\epsilon)$ satisfying the following: for any $n' > N_b(\epsilon)$, every string w of length n' except at most $\epsilon \cdot b^{n'}$ of them is such that given any string z of length at most $n' - N_b(\epsilon) + 1$ and n ranging from $N_b(\epsilon)$ to $n' - |z| + 1$, we have*

$$\left| \frac{N(z, w_1^{n+|z|-1})}{n} - \frac{1}{b^{|z|}} \right| < C_b \cdot \frac{\sqrt{\log\log n}}{\sqrt{n}}. \tag{1}$$

For any base b and $\epsilon = \frac{1}{2}$, let the collection of all strings of length n' satisfying inequality (1) be referred to as $\mathcal{G}_b^{n'}$. Corollary 8 therefore says that for any length $n' > N_b(1/2)$, $|\mathcal{G}_b^{n'}|$ is at least $b^{n'}/2$.

2.3 Schmidt's construction method ([23])

The individual *steps* of the construction in the proof of Theorem 1 are based on the construction method used by Schmidt in his construction of absolutely normal numbers [23]. As far as possible we use Schmidt's notation from [23] in this paper. We give a brief account of Schmidt's method below.

Let $u(1), u(2), u(3), \ldots$ be any sequence of natural numbers. This sequence represents the bases in which we *fix* the digits in each step of our construction. Now, as in [23], define, $\langle m \rangle = \lceil e^{\sqrt{m}} + 2 \cdot u(1) \cdot m^3 \rceil$ and $\langle m; r \rangle = \lceil \langle m \rangle / \log(r) \rceil$. Also define, $a_m = \langle m; u(m) \rangle$ and $b_m = \langle m+1; u(m) \rangle$. Notice that for any m, if $u(m) = u(m+1)$, we have $b_m = a_{m+1}$. For convenience of notation, we define $\langle 0 \rangle = 0$. Let p be a function from $\mathbb{N}$ to $\mathbb{N}$ such that for every $m \geq 1$, $p(u(m)) \leq u(m)$. The exact function p we use in our construction is defined in section 3. For any $m \geq 1$ and positive real number λ, let $g_m(\lambda)$ denote the smallest natural number such that, $g_m(\lambda)u(m)^{-a_m} \geq \lambda$. Now, define $\eta_m(\lambda) = g_m(\lambda)u(m)^{-a_m}$. We define $\sigma_m(\lambda)$ to be the set of numbers,

$$\eta_m(\lambda) + c_{a_m+1}^{u(m)} u(m)^{-(a_m+1)} + c_{a_m+2}^{u(m)} u(m)^{-(a_m+2)} + \cdots + c_{b_m-2}^{u(m)} u(m)^{-(b_m-2)}$$

with coefficients $c_i^{u(m)}$ taking values from the set $\{0, 1, \ldots u(m) - 1\}$. We define $\sigma_m^*(\lambda)$ to be the set in which the coefficients $c_i^{u(m)}$ takes values from $\{0, 1, \ldots p(u(m)) - 1\}$. Let $\xi_0 = 0$ and let $\xi_1, \xi_2, \xi_3 \ldots$ be a sequence of real numbers such that, $\xi_m \in \sigma_m(\xi_{m-1})$ or $\xi_m \in \sigma_m^*(\xi_{m-1})$.

Since $\xi_m \geq \xi_{m-1}$, it follows that there exists $\xi \in [0, 1]$ such that $\lim_{m\to\infty} \xi_m = \xi$. Furthermore, digits $c_{a_m+1}^{u(m)} c_{a_m+2}^{u(m)} \ldots c_{b_m-2}^{u(m)}$ appears in positions $a_m + 1$ to $b_m - 2$ in the base-$u(m)$ expansion of ξ. This follows as a consequence of the following lemma.

▶ **Lemma 9** ([23]). $|\xi - \xi_m| < u(m)^{-(b_m-2)}$.

We refer to the limit ξ as the *constructed number*. The number ξ is uniquely determined by the choice of the sequence $\langle u(m) \rangle$ and the function p. The exact function p we use in our construction is given in section 3 while the sequence $\langle u(m) \rangle$ is defined in a stage-wise manner in sections 3 and 5.

Adapting the technique in [23], we use the following function A_m while choosing ξ_m from $\sigma_m(\xi_{m-1})$ or $\sigma^*_m(\xi_{m-1})$ in the construction of the required number ξ,

$$A_m(x) = \sum\nolimits_{\substack{t=-m \\ t\neq 0}}^{m} \quad \sum\nolimits_{\substack{h=1 \\ u(h)\not\sim u(m)}}^{m} \left| \sum\nolimits_{j=\langle m;u(h)\rangle+1}^{\langle m+1;u(h)\rangle} e(u(h)^{j-1}tx) \right|^2 .$$

In our construction in each step, ξ_m is chosen according to either of the following criteria:

1. (Criterion 1.) ξ_m is any element of $\sigma^*_m(\xi_{m-1})$ such that $c^{u(m)}_{a_m+1}c^{u(m)}_{a_m+2}\dots c^{u(m)}_{b_m-2} \in \mathcal{G}^{b_m-a_m+2}_{p(u(m))}$ with the minimum A_m value among all elements of $\sigma^*_m(\xi_{m-1})$ satisfying this condition.
2. (Criterion 2.) ξ_m is any element of $\sigma_m(\xi_{m-1})$ such that $c^{u(m)}_{a_m+1}c^{u(m)}_{a_m+2}\dots c^{u(m)}_{b_m-2} \in \mathcal{G}^{b_m-a_m+2}_{u(m)}$ with the minimum A_m value among elements of $\sigma_m(\xi_{m-1})$ satisfying this condition.

The function A_m in Schmidt's paper [23] is defined so that the exponential sums are *minimized* over a sequence of bases $\langle r_i\rangle$ while *fixing digits* (using only 0's and 1's) in another sequence of bases $\langle s_i\rangle$. This ensures that the constructed number ξ is normal in all bases from $\langle r_i\rangle$ while it is not even simply normal in all bases from $\langle s_i\rangle$. The function A_m we use in our construction is defined so that the exponential sums are *minimized* in all bases $u(h) \not\sim u(m)$ for h ranging from 1 to m while *fixing digits* in base $u(m)$ during step number m. Such a definition of A_m is necessary because we need to ensure that the block entropy rate in every base b reaches sufficiently *close* to q_b infinitely often (and it remains *close* to 1 when it is away from q_b). Since the definition of finite-state definition is based on a lim inf, this modified construction yields a number ξ having the required finite-state dimension q_b in every base b. We demonstrate this in sections 3 and 5.

In Lemma 13, we show an upper bound for the function A_m defined above, which is similar to the upper bound in Lemma 7 from [23] for the original definition of A_m. The validity of this upper bound enables us to use Schmidt's technique in our construction with the modified definition of A_m.

3 Overview of the Proof of Theorem 1

We need to construct a number ξ having finite-state dimension equal to q_b in base b for every b. For every $b \geq 2$, let e_b and d_b be natural numbers such that $q_b = e_b/d_b$ in the lowest terms. Below we demonstrate the construction of a number ξ with dimension q_b in base b^{d_b} for every b. This number has dimension equal to q_b in base b for every $b \geq 2$.

Let $\langle r_k\rangle_{k=1}^{\infty}$ be any sequence of natural numbers greater than or equal to 2 such that every equivalence class of numbers (according to the relation $\sim$) has a unique representative in the sequence, which appears infinitely often. Furthermore, we require that $r_1 = 2$ and no consecutive elements in the sequence are equal.

It is straightforward to construct sequences satisfying the above conditions. Given the sequence $\langle q_b\rangle_{b=1}^{\infty}$, define the function $p : \mathbb{N} \to \mathbb{N}$ as $p(b) = \lfloor b^{q_b} \rfloor$. For every $k \geq 1$, define $v(k) = r_k^{d_{r_k}}$ and $v^*(k) = p(v(k))$. From the defining property of the sequence $\langle q_b\rangle_{b=1}^{\infty}$, we get that for any k, $q_{r_k^{d_{r_k}}} = q_{r_k}$. Therefore, $v^*(k) = r_k^{e_{r_k}}$.

Let the sequence $\langle u(m)\rangle$ be initially empty and let $\xi_0 = 0$. At every *step* $m \geq 1$ in the construction, we fix the m^{th} value of the sequence $\langle u(m)\rangle$ and choose ξ_m from $\sigma_m(\xi_{m-1})$ or $\sigma^*_m(\xi_{m-1})$. The whole construction is divided into a sequence of *stages* such that each individual stage comprises of two different *substages*. Each of the *substages* consist of multiple consecutive *steps*.

Suppose by stage $k-1$, $u(1)$, $u(2)$, $\dots$, $u(n_{k-1})$ have been determined. Then in the k^{th} stage, we set $u(n_{k-1}+1) = u(n_{k-1}+2) = \cdots = u(n_k) = v(k)$, and fix the digits in the base $v(k)$ expansion of ξ. Hence, $u(1) = v(1) = r_1^{d_{r_1}} = 2^{d_2}$. Within the first substage, at step m, we choose ξ_m from $\sigma^*_m(\xi_{m-1})$ according to Criterion 1. Within the second substage, at step m, we choose ξ_m from $\sigma_m(\xi_{m-1})$ according to Criterion 2.

For any $k \geq 1$, let $X(k)$ denote the infinite sequence in alphabet $\Sigma_{v(k)}$ representing the base-$v(k)$ expansion of ξ. Let P_k^1 denote the final step number contained within the first substage of stage k. Similarly, let P_k^2 denote the final step number contained within the second substage of stage k. For convenience, let $P_0^1 = P_0^2 = 0$. Define $I_1^1 = 1$ and $I_k^1 = \langle P_{k-1}^2 + 1; v(k) \rangle + 1$ for $k > 1$. Now, I_k^1 denotes the index of the initial digit in $X(k)$ fixed during stage k. Also, $F_k^1 = \langle P_k^1 + 1; v(k) \rangle$ denotes the index of the final digit in $X(k)$ fixed during the first substage of stage k. Let $I_k^2 = \langle P_k^1 + 1; v(k) \rangle + 1$ denote the index of the initial digit in $X(k)$ fixed during the second substage of stage k. Finally, let $F_k^2 = \langle P_k^2 + 1; v(k) \rangle$ denote the index of the final digit in $X(k)$ fixed during the second substage of stage k.

The number of individual steps in the stages and substages are ensured to be large enough so that the constructed ξ satisfies the following *requirements*. For every $k \geq 1$, we have the following *end of substage* requirements:

1. $\mathcal{F}_k$: $|H_l^{v(k)}(X(k)_1^n) - q_{r_k}| \leq 2^{-k}$ for every $l \leq k$ when $n = F_k^1$.
2. $\mathcal{S}_{k,1}$: $|H_l^{v(k)}(X(k)_1^n) - 1| \leq 2^{-(k+1)}$ for every $l \leq k$ when $n = F_k^2$.
3. $\mathcal{S}_{k,2}$: If there exists $k' < k$ such that $v(k') = v(k+1)$, then $|H_l^{v(k+1)}(X(k+1)_1^n) - 1| \leq 2^{-k}$ for every $l \leq k$ when $n = \langle P_k^2 + 1; v(k+1) \rangle$.

Requirement $\mathcal{F}_k$ ensures that the block entropies of ξ in base-$v(k)$ are *close* to q_{r_k} by the end of the first substage of stage k. Similarly, $\mathcal{S}_{k,1}$ ensures that $H_l^{v(k)}$ are *close* to 1 by the end of the second substage of stage k. $\mathcal{S}_{k,2}$ ensures that the block entropies of ξ in base-$v(k+1)$ are *close* to 1 before the start of stage $k+1$ (provided that $v(k+1)$ has appeared as $v(k')$ for some $k' < k$).

For every $k > 1$, the following requirement specifies the *stability of non-equivalent base entropies*:

4. $\mathcal{R}_k$: For any $k' < k$ such that $v(k') \not\sim v(k)$, $|H_l^{v(k')}(X(k')_1^n) - 1| \leq 2^{-(k'+1)}$ for every $l \leq k'$ when $\langle P_{k-1}^2 + 1; v(k') \rangle + 1 \leq n \leq \langle P_k^2 + 1; v(k') \rangle$.

Requirement $\mathcal{R}_k$ ensures that the block entropies of ξ in any base $v(k')$ for $k' < k$ which is not equivalent to $v(k)$ remains *stable* around 1 *throughout the course* of stage k.

Two particularly important requirements we need to enforce for every k are the *transition requirements*:

5. $\mathcal{T}_{k,1}$: $H_l^{v(k)}(X(k)_1^n) \geq q_{r_k} - 2^{-(k-1)}$ for every $l \leq k$ when $F_k^1 \leq n \leq F_k^2$.
6. $\mathcal{T}_{k,2}$: If there exists $k' < k$ such that $v(k') = v(k+1)$, then $H_l^{v(k+1)}(X(k+1)_1^n) \geq q_{r_{k+1}} - 2^{-(k-1)}$ for every $l \leq k$ when $I_{k+1}^1 \leq n \leq F_{k+1}^1$.

$\mathcal{T}_{k,1}$ ensures that the base-$v(k)$ entropies do not *fall much below* q_{r_k} during the transition between the first and second substage of stage k. Similarly, $\mathcal{T}_{k,2}$ ensures that the base-$v(k+1)$ entropies do not *fall much below* $q_{r_{k+1}}$ during the transition between stages k and $k+1$.

We now show that if we satisfy the above requirements, then Theorem 1 follows (It will then suffice to show that these requirements are met by our construction).

Proof of Theorem 1. Assume that the construction satisfies the requirements $\mathcal{F}_k$, $\mathcal{S}_{k,1}$, $\mathcal{S}_{k,2}$, $\mathcal{T}_{k,1}$ and $\mathcal{T}_{k,2}$ for every $k \geq 1$ and $\mathcal{R}_k$ for $k > 1$. Let b be an arbitrary base of expansion. Let $\bar{k}$ be the smallest number such that $r_{\bar{k}} \sim b$. Such a *representative* $r_{\bar{k}}$ exists for any b due to the condition imposed on the sequence $\langle r_k \rangle_{k=1}^{\infty}$ at the start of this section. Let $X(\bar{k}) \in \Sigma_{v(\bar{k})}^{\infty}$ denote the base-$v(\bar{k})$ expansion of ξ. In order to prove Theorem 1, it is enough to show that $\inf_l \liminf_{n\to\infty} H_l^{v(\bar{k})}(X(\bar{k})_1^n) = q_{r_{\bar{k}}}$. This implies that, $\dim_{FS}^b(\xi) = \dim_{FS}^{v(\bar{k})}(\xi) = \inf_l \liminf_{n\to\infty} H_l^{v(\bar{k})}(X(\bar{k})_1^n) = q_{r_{\bar{k}}} = q_b$. Here we used the fact that the sequence of rational dimensions is such that $q_r = q_s$ if $r \sim s$. We show that for every $l \geq 1$, $\liminf_{n\to\infty} H_l^{v(\bar{k})}(X(\bar{k})_1^n) = q_{r_{\bar{k}}}$.

Fix any length l. Let k' be any large enough number such that $k' > \max\{\bar{k}, l\}$ and $r_{k'} = r_{\bar{k}}$ (therefore $v(k') = v(\bar{k})$). In the rest of the argument by referring to the index in $X(\bar{k})$ at the *start of stage* k we mean the index $n = \langle P_{k-1}^2 + 1; v(\bar{k})\rangle + 1$, and the index at the *end of stage* k refers to the index $n = \langle P_k^2; v(\bar{k})\rangle$.

Since the requirement $\mathcal{F}_{k'}$ is met, after the first substage of stage k', $H_l^{v(\bar{k})} = H_l^{v(k')}$ is inside $B_{2^{-k'}}(q_{r_{\bar{k}}}) = B_{2^{-k'}}(q_{r_{k'}})$. Since the requirement $\mathcal{S}_{k',1}$ is met, by the end of stage k', $H_l^{v(\bar{k})}$ moves to $B_{2^{-k'}}(1)$ such that at any index n during this transition, $H_l^{v(\bar{k})}(X(\bar{k})_1^n) \geq q_{r_{\bar{k}}} - 2^{-(k'-1)}$. This inequality follows from the fact that $\mathcal{T}_{k',1}$ is satisfied. We know that $\langle r_k\rangle_{k=1}^{\infty}$, satisfies $r_{k'} \neq r_{k'+1}$ (and therefore $v(k') \neq v(k'+1)$). Since $\mathcal{R}_{k'+1}$ is satisfied, during the transition to the next stage and during the course of the next stage, $H_l^{v(\bar{k})}$ remains inside $B_{2^{-(k'+1)}}(1)$. Furthermore, $H_l^{v(\bar{k})}$ remains inside $B_{2^{-(k'+1)}}(1)$ until stage k'' where k'' is the smallest number such that $k'' > k'$ and $r_{k''} = r_{k'} = r_{\bar{k}}$. This follows from the fact that $\mathcal{R}_{k'+i}$ is satisfied for every $i < k'' - k'$. Since, $\mathcal{S}_{k''-1,2}$ is satisfied and $v(\bar{k}) = v(k') = v(k'') = v(k'' - 1 + 1)$, by the end of the second substage of stage $k'' - 1$, $H_l^{v(\bar{k})}$ is inside $B_{2^{-k''}}(1)$. During stage k'', $H_l^{v(\bar{k})}$ starts being inside $B_{2^{-(k'')}}(1)$ and moves to $B_{2^{-k''}}(q_{r_{\bar{k}}})$ (since $\mathcal{F}_{k''}$ is met). During this transition, since $\mathcal{T}_{k''-1,2}$ is met and $v(k'') = v(k''-1+1) = v(k')$ for $k' < k'' - 1$, it follows that $H_l^{v(\bar{k})}(X(k)_1^n) > q_{r_{\bar{k}}} - 2^{-(k''-2)}$. Therefore, $H_l^{v(\bar{k})}$ remains above $q_{r_{\bar{k}}} - 2^{-(k''-2)}$. Since k' was arbitrary, the above observations together imply that, $\liminf_{n\to\infty} H_l^{v(\bar{k})}(X(k)_1^n) = q_{r_{\bar{k}}}$. This completes the proof of Theorem 1. ◀

Hence, the proof of Theorem 1 is complete if we show the construction of a number ξ satisfying all the above requirements. We demonstrate the construction of ξ and verify that all the requirements are satisfied, in the following sections.

4 Technical Lemmas for the Main Construction

We require two main technical lemmas for the main construction in the proof of Theorem 1. In order to state the first lemma, we require the following generalization of Lemma 5 from [23].

▶ **Lemma 10.** *Consider any two bases r and s. Let K and l be natural numbers such that $\ell \geq s^K$. Then there exists a constant $\alpha(r,s)$ depending only on r and s such that,*

$$\sum_{n=0}^{N-1} \prod_{i=K+1}^{\infty} \left| \frac{\sin(p(s)\pi r^n \ell / s^i)}{p(s)\sin(\pi r^n \ell / s^i)} \right| \leq 2 \cdot N^{1-\alpha(r,s)}.$$

Lemma 5 from [23] is a special case of the above lemma when $p(s) = 2$. We assume that for any r and s, the constant $\alpha(r,s)$ in the above lemma is at most $1/2$ and $\alpha(r,s) = \alpha(s,r)$.

Proof of Lemma 10. The function, $f(x) = |\sin(p(s)\pi x)/p(s)\sin(\pi x)|$ has the limit 1 as $x \to 0$ and $f(x)$ takes values strictly less then 1 when $|x| < 1$. The first property follows from the fact that $\lim\limits_{x\to 0} \sin(x)/x = 1$ and the second fact follows from the inequality $|\sin(nx)| < n|\sin(x)|$, which is easily proved using induction on n when $|x| < \pi$. If x has an obedient digit pair [23] (i.e, a pair of digits in base s that are not both equal to 0 or both equal to $s-1$) then,

$$\left| \frac{\sin(p(s)\pi x/s^i)}{p(s)\sin(\pi x/s^i)} \right| \leq \left| \frac{\sin(p(s)\pi/s^2)}{p(s)\sin(\pi/s^2)} \right| < \gamma_1 < 1.$$

The constant γ_1 above depends only on s (since the function p is fixed). Let $z_K(x)$ denote the number of obedient digit pairs $c_{i+1}c_i$ in x for $i \geq K$ (where $c_{\lceil \log_s(x) \rceil} \dots c_2 c_1$ is the representation of x in base s). Lemma 4 from [23] implies that if $l \geq s^K$, then among the numbers $\ell, \ell r, \ell r^2 \dots \ell r^{N-1}$, there are at most $N^{1-a_{14}}$ numbers such that z_K is smaller than $a_{15} \log N$ (where a_{14} and a_{15} are constants depending only on r and s and independent of ℓ and K). If for some n, ℓr^n has z_K greater than $a_{15} \log N$, then,

$$\prod_{i=K+1}^{\infty} \left| \frac{\sin(p(s)\pi r^n \ell / s^i)}{p(s) \sin(\pi r^n \ell / s^i)} \right| \leq \gamma_1^{a_{15} \log N} = N^{1-\gamma_2}$$

for some $\gamma_2 \in (0,1)$ dependent only on r and s. Now, using the above bound along with Lemma 4 from [23], we get that,

$$\sum_{n=0}^{N-1} \prod_{i=K+1}^{\infty} \left| \frac{\sin(p(s)\pi r^n \ell / s^i)}{p(s) \sin(\pi r^n \ell / s^i)} \right| \leq N^{1-a_{14}} + N^{1-\gamma_2} \leq 2 \cdot N^{1-\alpha(r,s)}$$

for some $\alpha(r,s) \in (0,1)$ dependent only on r and s. ◀

Now, we define the notion of good sequences of natural numbers.

▶ **Definition 11** (Good sequences of natural numbers). *A sequence $\langle u(m) \rangle_{m=1}^{\infty}$ is a* good sequence *of natural numbers if the following conditions are satisfied:*

1. $\prod_{i=b_m-1}^{\infty} \left| \frac{\sin(p(u(m))\pi/2^{i+1})}{p(u(m))\sin(\pi/2^{i+1})} \right| \geq \prod_{i=1}^{\infty} \left| \cos(\pi/2^{i+1}) \right|$ *for every $m > 1$.*
2. $\beta_m \geq \beta_1 \frac{1}{\sqrt[4]{m}}$ *for every $m \geq 1$ where*
$$\beta_m = \min\left(\{\alpha(u(i),u(j)) : 1 \leq i \leq j \leq m \text{ such that } u(i) \not\sim u(j)\} \cup \{1/2\}\right).$$
3. $u(m) \leq u(1)m$ *for every $m \geq 1$.*
4. *For any $m \geq 1$, if there exists any $m' < m$ such that $u(m') \neq u(m)$, then $b_m - a_m \geq \max\{N_{u(m)}(1/2), N_{p(u(m))}(1/2)\}$, where the constants on the right are from Corollary 8.*

From condition 2 and the fact that there does not exist any $u(j) \not\sim u(1)$ with $j \leq 1$, it follows that $\beta_1 = 1/2$. For every i, we use the notation β'_i to denote $\beta_i/2$. The existence of good sequences follows from following lemma and the fact that b_m is increasing in m.

▶ **Lemma 12.** *For any $n \in \mathbb{N}$, the infinite product $\prod_{i=1}^{\infty} \left| \frac{\sin(n\pi/2^{i+1})}{n \sin(\pi/2^{i+1})} \right|$ is convergent.*

Since b_m is strictly increasing in m, any *sufficiently delayed* sequence of natural numbers is a good sequence. Therefore, it is straightforward to verify that given any subset S of $\mathbb{N}$ there is a good sequence $\langle u(m) \rangle$ containing exactly the elements of S such that every element of S occurs infinitely many times in $\langle u(m) \rangle$. This observation is important in our construction.

The first technical lemma is a generalization of the bound on exponential sums given in Lemma 7 from [23].

▶ **Lemma 13.** *Let $\langle u(m) \rangle_{m=1}^{\infty}$ be any good sequence of bases greater than or equal to 2. Let ξ be the real number that is obtained as the limit of $\langle \xi_m \rangle_{m=1}^{\infty}$ where each ξ_m is chosen according to Criterion 1 or 2. Then, there exists a constant δ depending only on $u(1)$ such that for every $m \geq 1$, $A_m(\xi) \leq \delta m^2(\langle m+1 \rangle - \langle m \rangle)^{2-\beta_m}$.*

The proof of Lemma 13 is similar to that of Lemma 7 from [23], except for the following important differences. We *fix* digits in step m using strings from the subset $\mathcal{G}_{p(u(m))}^{b_m-a_m+2}$ of $\Sigma_{p(u(m))}^{b_m-a_m+2}$. Nevertheless, the upper bounds from [23] are true in our setting up to a multiplicative factor of 2. This follows using the Markov's inequality since $|\mathcal{G}_{p(u(m))}^{b_m-a_m+2}| \geq |\Sigma_{p(u(m))}^{b_m-a_m+2}|/2$ (the argument is similar in the case with $u(m)$ instead of $p(u(m))$). We fix digits in every step m using the larger alphabet $\Sigma_{p(u(m))}$ instead of Σ_2 used in [23]. So, we require upper bounds on products of terms of the form $|\sin(p(u(m))x)/p(u(m))\sin(x)|$ instead of $|\cos(x)|$ as in the proof of Lemma 7 from [23]. These bounds are obtained from Lemma 10 and condition 1 in Definition 11 (since $\langle u(m)\rangle$ is a good sequence). The constant δ depends only on $u(1)$ as a consequence of the *growth control* imposed on the sequence $\langle u(m)\rangle$ in conditions 3 and 4 of Definition 11.

We make crucial use of the following lemma regarding the *uniform normality* of the infinite sequences constructed by choosing successive set of digits from $\mathcal{G}_j^{b_m-a_m-2}$, in the proof of Theorem 1.

▶ **Lemma 14.** *Let b be any base and $j \leq b$. For any finite string $w \in \Sigma_j^*$, $\epsilon > 0$, there exists an integer $L'_{b,j}(w,\epsilon)$ satisfying the following property. If X is any infinite sequence in Σ_j^∞ such that, $X_{\langle m;b\rangle+1}X_{\langle m;b\rangle+2}\dots X_{\langle m+1;b\rangle-2} \in \mathcal{G}_j^{\langle m+1;b\rangle-\langle m;b\rangle-2}$ for every $m > 0$ and if $T \geq 0$ is any non-negative integer, then for all $n \geq \langle T;b\rangle + L'_{b,j}(w,\epsilon)$, $|P(X_{\langle T;b\rangle+1}^n, w) - j^{-|w|}| \leq \epsilon$.*

Note especially that $L'_{b,j}(w,\epsilon)$ is a constant which depends only on w and ϵ. This constant is independent of the infinite sequence X and the *starting block number* T. When $T = 0$, we have $\langle 0;b\rangle = 0$ and hence the above statement asserts that, $|N(w,X_1^n)/n - j^{-|w|}| \leq \epsilon$ for all $n \geq L'_{b,j}(w,\epsilon)$.

Proof Sketch of Lemma 14. We equivalently prove the existence of a number $L'_{b,j}(w,\epsilon)$ such that

$$\left|\frac{N(w,X_{\langle T;b\rangle+1}^n)}{n-\langle T;b\rangle-|w|+1} - \frac{1}{j^{|w|}}\right| \leq \epsilon$$

for all $n \geq \langle T;b\rangle + L'_{b,j}(w,\epsilon)$. To abbreviate the expressions in the proof, let $\hat{\ell}_w = |w| - 1$. . For any $m > 0$, if $\langle m+1;b\rangle - \langle m;b\rangle - 2 \geq N_b(1/2) + N_1^j + \hat{\ell}_w$, then from Corollary 8 it follows that

$$\left|\frac{N(w,X_{\langle m;b\rangle+1}^{\langle m+1;b\rangle-2})}{\langle m+1;b\rangle-\langle m;b\rangle-\hat{\ell}_w} - \frac{1}{j^{|w|}}\right| \leq \epsilon. \tag{2}$$

Since $\langle m+1;b\rangle - \langle m;b\rangle$ is increasing in m, $\langle m+1;b\rangle - \langle m;b\rangle - 2$ is greater than $N_b(1/2) + N_1^j + \hat{\ell}_w$ for all but finitely many m. Let M_2 denote the smallest integer such that $\langle m+1;b\rangle - \langle m;b\rangle - 2 \geq N_b(1/2) + N_1^j + \hat{\ell}_w$ for every $m \geq M_2$. Consider any $n \geq \langle M_2;b\rangle$. Let M' be the (unique) integer such that $\langle M';b\rangle \leq n < \langle M'+1;b\rangle$. We have $M' \geq M_2$. Let N_T denote the following quantity.

$$N_T = \left|\frac{N(w,X_{\langle T;b\rangle+1}^n)}{n-\langle T;b\rangle-\hat{\ell}_w+2} - \frac{1}{j^{|w|}}\right|.$$

We first consider the case when $n = \langle M';b\rangle$ for some M'. Then,

$$N_T \leq \frac{S(w,\epsilon)}{n - \langle T;b\rangle - \hat{\ell}_w + 2} \sum_{m=T}^{M_2-1} \frac{\langle m+1;b\rangle - \langle m;b\rangle - \hat{\ell}_w}{S(w,\epsilon)} \left| \frac{N(w, X_{\langle m;b\rangle+1}^{\langle m+1;b\rangle-2})}{\langle m+1;b\rangle - \langle m;b\rangle - \hat{\ell}_w} - \frac{1}{j^{|w|}} \right|$$
$$+ \frac{S'(w,\epsilon)}{n - \langle T;b\rangle - \hat{\ell}_w + 2} \sum_{m=M_2}^{M'-1} \frac{\langle m+1;b\rangle - \langle m;b\rangle - \hat{\ell}_w}{S'(w,\epsilon)} \left| \frac{N(w, X_{\langle m;b\rangle+1}^{\langle m+1;b\rangle-2})}{\langle m+1;b\rangle - \langle m;b\rangle - \hat{\ell}_w} - \frac{1}{j^{|w|}} \right|$$
$$+ \left(1 - \frac{S(w,\epsilon) + S'(w,\epsilon)}{n - \langle T;b\rangle - \hat{\ell}_w + 2}\right)\left(1 + \frac{1}{j^{|w|}}\right)$$

where, $S(w,\epsilon) = \sum_{T \leq m < M_2} \langle m+1;b\rangle - \langle m;b\rangle - \hat{\ell}_w$ and $S'(w,\epsilon) = \sum_{M_2 \leq m < M'} \langle m+1;b\rangle - \langle m;b\rangle - \hat{\ell}_w$. From the definition of M_2 and the properties of $N_b(1/2)$ and N_1^j, the upper bound in (2) is true for every $m \geq M_2$. Therefore, the second term in the upper bound for N_T above is at most ϵ. The sum of the other two terms are shown to be at most 3ϵ for large enough values of n. Hence there exists a large enough number $L'_{b,j}(w,\epsilon)$ such that for any $n \geq \langle T;b\rangle + L'_{b,j}(w,\epsilon)$ satisfying $n = \langle M';b\rangle$ for some M', we have $N_T \leq 4\epsilon$. The case when n is between $\langle M';b\rangle$ and $\langle M'+1;b\rangle$ for some M' follows using similar arguments. ◀

The following is an immediate corollary of the above lemma.

▶ **Corollary 15.** *Let b be any base and $j \leq b$. For any $k > 0$ and $\epsilon > 0$, there exists an integer $L_{b,j}(k,\epsilon)$ satisfying the following property. If X is any infinite sequence in Σ_j^∞ such that,$X_{\langle m;b\rangle+1}X_{\langle m;b\rangle+2}\dots X_{\langle m+1;b\rangle-2} \in \mathcal{G}_j^{\langle m+1;b\rangle-\langle m;b\rangle-2}$ for every $m > 0$ and if $T \geq 0$ is any non-negative integer, then for every $w \in \Sigma^*$ with $|w| \leq k$ and all $n \geq \langle T;b\rangle + L_{b,j}(k,\epsilon)$, the inequality in Lemma 14 holds.*

5 Main construction

The construction for proving Theorem 1 works by stages. In the k^{th} stage, we *fix* digits in base $v(k)$ by fixing elements of the sequence $\langle u(m)\rangle$ to be $v(k)$. Each stage consists of two substages, which have several steps. In both the substages, we fix $u(m) = v(k)$ for sufficiently large number of steps m. During the first substage, we choose ξ_m from $\sigma_m^*(\xi_{m-1})$ according to Criterion 1 and during the second substage, we choose ξ_m from $\sigma_m(\xi_{m-1})$ according to Criterion 2.

We ensure that $\langle u(m)\rangle$ is a good sequence of natural numbers. During stage 1, $u(1)$ is set to 2^{d_2}, since $r_1 = 2$. All the conditions in the definition of a good sequence are trivially satisfied at this stage. Further checks are performed at the end of the second substage of every stage k to ensure that on transitioning to stage $k+1$, where $u(m)$ is set to $v(k+1)$, none of the conditions in Definition 11 are being violated.

The *duration* of the substages are controlled carefully so that all the requirements given in Section 3 are satisfied. We describe the construction of the two substages of the k^{th} stage in the subsections below.

5.1 First substage of the k^{th} stage

For any $\epsilon > 0$, there exists a constant $\delta_k(\epsilon)$ satisfying the following: If μ_1 and μ_2 are probability distributions over $\Sigma_{v(k)}^l$ for any $l \leq k$ such that $|\mu_1(w) - \mu_2(w)| \leq \delta_k(\epsilon)$ for every $w \in \Sigma_{v(k)}^l$, then $|H(\mu_1) - H(\mu_2)| \leq \epsilon$. The existence of $\delta_k(\epsilon)$ follows from the uniform continuity of the Shannon entropy function (see [14], [9]).

In the first substage of stage k we set $u(m) = v(k)$ for sufficiently large number of m's and choose ξ_m from $\sigma^*_m(\xi_{m-1})$ according to Criterion 1. As in Section 3, let P^1_k denote the index of the last step in the first substage of stage k. Recall that $F^1_k = \langle P^1_k + 1; v(k)\rangle$ denotes the index of the final digit in $X(k)$ fixed during the first substage of stage k. We make P^1_k large enough so that the following conditions are satisfied:

1. $|H^{v(k)}_l(X(k)^n_1) - q_{r_k}| \leq 2^{-k}$ for every $l \leq k$ when $n = F^1_k$.
2. For every $m \geq P^1_k$,

$$b_m - a_m \geq \frac{L_{v(k),v(k)}(k, \delta_k(2^{-k})/2) + 2k}{\min\{\delta_k(2^{-k}), 2^{-k}\}/2} + k. \tag{3}$$

The constant $L_{v(k),v(k)}(k, \delta_k(2^{-k})/2)$ in condition 2 is from Corollary 15. Recall that $v^*(k) = r_k^{e_{r_k}}$. Condition 1 is satisfied for large enough P^1_k because the occurrence probability of any finite string w in alphabet $\{0, 1, \ldots, v^*(k) - 1\}$ converges to $v^*(k)^{-|w|}$ on choosing ξ_m from $\sigma^*_m(\xi_{m-1})$ according to Criterion 1 for sufficiently large number of m's. This follows as a consequence of Lemma 14. Since $b_m - a_m \geq (e^{\sqrt{m+1}} - e^{\sqrt{m}} + m^2)/\log(v(k)) - 1$ and the right hand side of (3) is a constant depending only on k, condition 2 is satisfied for all sufficiently large m.

5.2 Second substage of the k^{th} stage

In the second substage, we set $u(m) = v(k)$ in every step m and choose ξ_m from $\sigma_m(\xi_{m-1})$ according to Criterion 2. In order to describe the construction of the second substage, we need the following technical lemmas.

▶ **Lemma 16.** *Let b be an arbitrary base and $\epsilon > 0$. Let $\langle c_i\rangle_{i=1}^{\infty}$ be any non-increasing sequence of real numbers in $[0, 1]$ such that $c_1 = 1/4$ and $c_i \geq c_1/\sqrt[4]{i}$. Let δ be the constant from Lemma 13. Then, there exists a large enough number $M(\epsilon, b)$ depending only on ϵ and b satisfying the following. For $m \geq M(\epsilon, b)$ and any $l \leq \langle m + 1; b\rangle - \langle m; b\rangle$,*

$$(\langle m; b\rangle + l)^{-1}\left(\delta m \sum\nolimits_{i=1}^{m-1} (\langle i + 1\rangle - \langle i\rangle)^{1-c_i} + l\right) \leq \epsilon.$$

Proof Sketch. We consider the case when $l = 0$. In this case, we have,

$$\delta m \sum_{i=1}^{m-1} (\langle i + 1\rangle - \langle i\rangle)^{1-c_i} \leq \delta m \sum_{i=1}^{m-1} (\langle i + 1\rangle - \langle i\rangle)^{1-c_m} \leq \delta m^2 \langle m\rangle^{1-c_m}.$$

The second inequality follows due to the Hölder's inequality with $p = 1/(1-c_m)$ and $q = 1/c_m$. Therefore,

$$\frac{1}{\langle m; b\rangle}\left(\delta m \sum_{i=1}^{m-1} (\langle i + 1\rangle - \langle i\rangle)^{1-c_i}\right) \leq \delta \log b \frac{m^2}{\langle m\rangle^{c_m}} \leq \delta \log b \frac{m^2}{e^{\sqrt{m}c_m}}.$$

We have $c_m \geq \frac{c_1}{\sqrt[4]{m}} = \frac{1}{4\sqrt[4]{m}}$. Since $m^2 = o(e^{\sqrt[4]{m}/4})$, the right hand side term above converges to 0 for large enough m. The speed of convergence of this term to 0 is independent of the sequence $\langle c_i\rangle$, and hence given any $\epsilon > 0$, there exists a large enough number $M'(\epsilon, b)$ such that for every $m \geq M'(\epsilon, b)$,

$$\frac{1}{\langle m; b\rangle}\left(\delta m \sum_{i=1}^{m-1} (\langle i + 1\rangle - \langle i\rangle)^{1-c_i}\right) \leq \epsilon.$$

The case when $l \neq 0$ follows using similar arguments by proving that for any $\epsilon > 0$, there exists a large enough number $M''(\epsilon, b)$ such that for any $m \geq M''(\epsilon, b)$ and $l \leq \langle m+1; b\rangle - \langle m; b\rangle$,

$$\frac{\langle m+1; b\rangle - \langle m; b\rangle}{\langle m; b\rangle + l} < \epsilon.$$ ◀

▶ **Lemma 17.** *Let b be an arbitrary base, k be any natural number and $\epsilon > 0$. There exists a large enough integer $T(\epsilon, b, k)$ and a positive real number $\gamma(\epsilon, b, k)$ satisfying the following. Let x is any real number in $[0, 1]$ having base-b expansion $X \in \Sigma_b^\infty$. If $|\frac{1}{n}\sum_{i=1}^{n} e(tb^{(j-1)}x)| < \gamma(\epsilon, b, k)$ for every t with $|t| \leq T(\epsilon, b, k)$, then, $\left|H_l^b(X_1^n) - 1\right| < \epsilon$ for every $l \leq k$.*

Proof Sketch. Consider any $w \in \Sigma^*$. Using the Fourier expansion of characteristic functions of cylinder sets given in Lemma 5 ([15]), it is shown that

$$\left|\frac{N(w, X_1^n)}{n} - \frac{1}{b^{|w|}}\right| < \frac{\epsilon}{2} + \sum\nolimits_{t \in \mathbb{Z}\setminus\{0\}} \frac{8}{t^2\epsilon} \frac{|\sum_{j=1}^{n} e(tb^{(j-1)}x)|}{n}$$

where the constant δ is set to $\epsilon/2$. Terms with $t \geq T'(\epsilon) = 64/\epsilon^2$ contribute at most $\epsilon/4$ to the above sum. If the exponential averages for every non-zero parameter t between $-T'(\epsilon)$ and $T'(\epsilon)$ are less than $\gamma'(\epsilon) = \epsilon^2/32 \cdot T'(\epsilon)$, then the sum of the remaining terms is also less than $\epsilon/4$. Therefore $|N(w, X_1^n)/n - b^{-|w|}| < \epsilon$. The constants $T(\epsilon, b, k)$ and $\gamma(\epsilon, b, k)$ are obtained from T' and γ' using the continuity of the Shannon entropy function ([14], [9]). ◀

As in Section 3, let P_k^2 denote the denote the index of the last step in the second substage of stage k. We make P_k^2 large enough so that the following conditions are satisfied at the end of the substage:

1. (Entropy rates in base $v(k)$ are close to 1) $|H_l^{v(k)}(X(k)_1^n) - 1| \leq 2^{-(k+1)}$ for every $l \leq k$ when $n = F_k^2$.
2. (Exponential averages for base $v(k)$ are small) For every t with $|t| \leq T(2^{-(k+1)}, v(k), k)$,

$$\left|(F_k^2)^{-1}\sum\nolimits_{i=1}^{F_k^2} e(tv(k)^{(j-1)}\xi)\right| < \gamma(2^{-(k+1)}, v(k), k)/2. \tag{4}$$

   where T and γ are the constants from Lemma 17.
3. $P_k^2 \geq \max\{M(\gamma(2^{-(k+1)}, v(k), k)/2, v(k)), T(2^{-(k+1)}, v(k), k)\}$, where M is the constant from Lemma 16.
4. (Entropy rates in non-equivalent bases are close to 1) If there exists $k' < k$ such that $v(k') = v(k+1)$, then $|H_l^{v(k+1)}(X(k+1)_1^n) - 1| \leq 2^{-k}$ for every $l \leq k$ when $n = \langle P_k^2 + 1; v(k+1)\rangle$.
5. For every $m \geq P_k^2$,

$$b_m - a_m \geq \frac{L_{v(k+1),v^*(k+1)}(k, \delta_{k+1}(2^{-k})/2) + 2k}{\min\{\delta_k(2^{-k}), \delta_{k+1}(2^{-k}), 2^{-k}\}/2} + k \tag{5}$$

6. The sequence $\langle u'(m)\rangle_{m=1}^{\infty}$ defined such that $u'(m) = u(m)$ for every $m \leq P_k^2$ and $u'(m) = v(k+1)$ for $m \geq P_k^2 + 1$, is a good sequence.

The constant $L_{v(k+1),v^*(k+1)}(k, \delta_{k+1}(2^{-k})/2)$ in condition 5 is from Corollary 15. Condition 1 is satisfied for large enough P_k^2 because the occurrence probability of any finite string $w \in \Sigma_{v(k)}^*$ converges to $v(k)^{-|w|}$ on choosing ξ_m from $\sigma_m(\xi_{m-1})$ according to Criterion 2 for sufficiently large number of m's, as a consequence of Lemma 14. For any t, on extending the second substage by increasing P_k^2, the corresponding exponential averages in (4) converges to 0 as a consequence of the Weyl Criterion for normality (see [27, 17]) and Lemma 14. Therefore, condition 2 is satisfied for large enough values of P_k^2. Since $b_m - a_m \geq (e^{\sqrt{m+1}} - e^{\sqrt{m}} + m^2)/\log(v(k)) - 1$ and the right hand side of (5) is a constant depending only on k, condition 5 is satisfied for all sufficiently large m.

It is easily verified from the definition of a good sequence that for large enough P_k^2, on setting $u(P_k + 1) = v(k+1)$ the sequence $\langle u(m)\rangle_{m=1}^{P_k^2+1}$ satisfies all the conditions in the definition of good sequences (Definition 11). On extending the sequence from this value of $P_k^2 + 1$ onwards, by setting $u(m) = v(k+1)$ for every $k \geq P_k^2 + 2$, none of the conditions in Definition 11 are violated. Therefore, condition 6 is satisfied for all large enough values of P_k^2. During stage 1, all the conditions in the definition of a good sequence are trivially satisfied. Therefore, the validity of condition 6 at the end of every second substage, inductively ensures that the constructed sequence $\langle u(m)\rangle$ is a good sequence.

Proving that condition 4 holds for large enough values of P_k^2, requires an argument using Lemmas 16 and 17.

▶ **Lemma 18.** *Condition 4 in the construction is true for all large enough values of P_k^2.*

Proof Sketch. From the statement of condition 4, we have $v(k') = v(k+1)$ for some $k' < k$. Recall that the sequence $\langle r_k\rangle_{k=1}^{\infty}$ satisfies $r_k \neq r_{k+1}$. Therefore, we get $v(k) \neq v(k+1)$ and hence $v(k') \not\sim v(k)$. Consider any index m of $\langle u(m)\rangle$ that is set during the second substage of stage k. Since ξ_m was chosen from $\sigma_m(\xi_{m-1})$ using Criterion 2, from Lemma 13, we get that for any non-zero t with $|t| < m$,

$$\left|\sum_{j=\langle m;v(k+1)\rangle+1}^{\langle m+1;v(k+1)\rangle} e(v(k+1)^{j-1}t\xi)\right| \leq \delta m(\langle m+1\rangle - \langle m\rangle)^{1-\beta'_m}.$$

Now, the convergence of base $v(k+1)$ entropies to 1 for large enough P_k^2 is obtained by using Lemma 16 (for an appropriately defined sequence $\langle c_i\rangle$) followed by an application of Lemma 17. ◀

6 Verification

In this section we prove that all the requirements given in section 3 are satisfied by the construction in section 5. The following proofs along with the argument provided at the end of Section 3 complete the proof of Theorem 1.

▶ **Lemma 19.** *For all $k \geq 1$, the requirements $\mathcal{F}_k$, $\mathcal{S}_{k,1}$ and $\mathcal{S}_{k,2}$ are met by the construction.*

Proof. $\mathcal{F}_k$ follows from the validity of condition 1 from section 5.1 at the end of the first substage of every stage k. From the validity of condition 1 from section 5.2 at the end of the second substage of every stage k, we get that $|H_l^{v(k)}(X(k)_1^n) - 1| \leq 2^{-(k+1)}$ for every $l \leq k$ when $n = F_k^2$. Therefore, $\mathcal{S}_{k,1}$ is satisfied for every $k \geq 1$. $\mathcal{S}_{k,2}$ follows directly from the validity of condition 4 from section 5.2 at the end of the second substage of every stage k. ◀

▶ **Lemma 20.** *For every $k > 1$, the requirement $\mathcal{R}_k$ is satisfied by the construction.*

Proof Sketch. Let k' be any stage number below k such that $v(k') \not\sim v(k)$. Without loss of generality, let us assume that there does not exist any k'' between k' and k such that $v(k'') = v(k')$. Since the equivalence class of base $v(k')$ has a unique representative in the sequence $\langle r_k\rangle_{k=1}^{\infty}$, we also get that $v(k'') \not\sim v(k')$ for any k'' between k' and k. Let m denote any index such that $P_{k'}^2 + 1 \leq m \leq P_k^2$. Using condition 3 in Section 5.2 and Lemma 13, we obtain,

$$\left|\sum_{j=\langle m;v(k')\rangle+1}^{\langle m+1;v(k')\rangle} e(v(k')^{j-1}t\xi)\right| \leq \delta m(\langle m+1\rangle - \langle m\rangle)^{1-\beta'_m}.$$

for every t with $|t| \le T(2^{-(k'+1)}, v(k'), k')$. Since condition 2 from Section 5.2 is valid at the end of the second substage of stage k', we have,

$$\left|(F^2_{k'})^{-1}\sum_{i=1}^{F^2_{k'}} e(tv(k')^{(j-1)}\xi)\right| < \gamma(2^{-(k'+1)}, v(k'), k')/2$$

for every t with $|t| \le T(2^{-(k'+1)}, v(k'), k')$. By combining the above inequalities and Lemma 16 (for an appropriately defined sequence $\langle c_i\rangle$), precise bounds are obtained for the exponential averages in base $v(k')$ along stage k. The proof of the lemma now follows by invoking Lemma 17 using these new bounds. ◀

▶ **Lemma 21.** *For every $k \ge 1$, the requirement $\mathcal{T}_{k,2}$ is met by the construction.*

Proof Sketch. We assume that there exists $k' < k$ such that $v(k') = v(k+1)$. Otherwise, $\mathcal{T}_{k,2}$ is vacuously satisfied. Let $F^2_k = \langle P^2_k; v(k+1)\rangle$ denote the index of the final digit fixed during the second substage of stage k in the base $v(k+1)$ expansion of ξ. Let $n \ge F^2_k$ denote the index of any digit that is fixed during the first substage of stage $k+1$ in the base $v(k+1)$ expansion of ξ. For every $w \in \Sigma^*_s$, let $\mathbb{P}^{k+1}(w, j_1, j_2)$ denote the fraction of $|w|$-length blocks containing w among the digits in the base-$v(k+1)$ expansion of ξ with indices in the range j_1 to j_2. Then, $\mathbb{P}^{k+1}(w, 1, n)$ is equal to

$$\frac{F^2_k - |w| + 1}{n - |w| + 1}\mathbb{P}^{k+1}(w, 1, F^2_k) + \frac{n - F^2_k - |w| + 1}{n - |w| + 1}\mathbb{P}^{k+1}(w, F^2_k + 1, n) + o(1/n). \tag{6}$$

The above error term is negligible for $n \ge F^2_k$. If $n - F^2_k \le L_{v(k+1),v^*(k+1)}(k, \delta_{k+1}(2^{-k})/2)$ then second term above is easily verified to be negligible using inequality (5). Also, since there exists $k' < k$ such that $v(k') = v(k+1)$ and the requirement $\mathcal{S}_{k,2}$ is satisfied, we get,

$$\left|(l \log v(k+1))^{-1} \quad H(\mathbb{P}^{k+1}_l(\cdot, 1, F^2_k)) - 1\right| \le 2^{-k}. \tag{7}$$

Hence, the required conclusion follows. Otherwise, let $n - F^2_k > L_{v(k+1),v^*(k+1)}(k, \delta_{k+1}(2^{-k})/2)$. Observe that $\log v^*(k+1)/\log v(k+1) = q_{r_{k+1}}$. From Lemma 14 and the definition of $\delta_{k+1}(2^{-k})$ we obtain that,

$$\left|(l \log v(k+1))^{-1} H(\mathbb{P}^{k+1}_l(\cdot, F^2_k + 1, n)) - q_{r_{k+1}}\right| \le 2^{-k}. \tag{8}$$

for any $l \le k$. Now, from the concavity of the Shannon entropy function (see [9]) and (6) we get that for any $l \le k$,

$$H(\mathbb{P}^{k+1}_l(\cdot, 1, n)) \ge \frac{F^2_k - l + 1}{n - l + 1} H(\mathbb{P}^{k+1}_l(\cdot, 1, F^2_k)) + \frac{n - F^2_k - l + 1}{n - l + 1} H(\mathbb{P}^{k+1}_l(\cdot, F^2_k + 1, n)).$$

Combining inequalities (7) and (8) with the above lower bound, we get,

$$\frac{1}{l \log v(k+1)} H(\mathbb{P}^{k+1}_l(\cdot, 1, n)) \ge \lambda_{l,k} \times 1 + (1 - \lambda_{l,k}) \times q_{r_{k+1}} - \frac{1}{2^{k-1}} \ge q_{r_k} - \frac{1}{2^{k-1}}$$

where $\lambda_{l,k} = (F^2_k - l + 1)/(n - l + 1)$. Therefore, the requirement $\mathcal{T}_{k,2}$ is satisfied for every $k \ge 1$. ◀

▶ **Lemma 22.** *For every $k \ge 1$, the requirement $\mathcal{T}_{k,1}$ is met by the construction.*

Lemma 22 is proved using the same techniques as in the proof of Lemma 21. The fact that the requirement $\mathcal{F}_k$ is satisfied (as shown in Lemma 19) is used along with inequality (3) and the *concavity* of the Shannon entropy, to ensure that the block entropies in base $v(k)$ does not fall below $q_{r_k} - 2^{-(k-1)}$ during the transition between the first and second substages of stage k.

7 Discussion and Open Questions

It is open whether our main results are true in the setting of finite-state strong dimension ([2, 7]). In particular: *Does there exist an absolutely strong dimensioned number with finite-state strong dimension strictly between* 0 *and* 1*?*. The strong dimension of ξ is 1. It is unclear how to modify our construction to bound the limit superior of the block entropies away from 1. This is because while we control the block entropies in base $v(k)$ during stage k, the block entropies in all bases with $k' < k$ and $v(k') \not\sim v(k)$ are converging to 1 (since the exponential averages in these bases are *getting smaller* after every individual step within stage k). This behavior seems to be an essential feature of constructions based on Schmidt's method [23]. Hence, the question regarding *absolute strong dimension*, if such a number exists, may require new construction techniques that are capable of stabilizing block entropies in non-equivalent bases simultaneously around values strictly between 0 and 1.

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A Proofs from Section 2

A.1 Proofs from Subsection 2.2

Proof of Lemma 7. Theorem 6 implies that for almost every x, there exists a minimum number N_x such that,

$$R^b(x, n, \alpha_1, \alpha_2) < C_b \cdot \frac{\sqrt{\log\log N}}{\sqrt{n}}$$

for any $\alpha_1 < \alpha_2$ and every $n \geq N_x$. For every $i \in \mathbb{N}$, define $U_i = \{x : N_x \geq i\}$. U_i's are a monotonically decreasing sequence of sets such that $\mu(\bigcap_{i\in\mathbb{N}} U_i) = 0$. Using the continuity of measure from above (see [6]), there exists a large enough $N_b(\epsilon)$ such that $\mu(U_i) < \epsilon$ for every $i \geq N_b(\epsilon)$. This completes the proof of the lemma. ◀

Proof of Corollary 8. For any $z \in \Sigma^*$, $I^b_z = [v_b(z), v_b(z) + b^{-|z|})$ represents the interval containing exactly those real numbers in $[0, 1]$ whose base-b expansion starts with the string z. Setting $\alpha_1 = v_b(z)$ and $\alpha_2 = v_b(z)+b^{-|z|}$, it follows from Lemma 7 that for $x = 0.X_1X_2X_3\ldots$ outside a set of measure at most ϵ,

$$\left| \frac{N(z, X_1^{n+|z|-1})}{n} - \frac{1}{b^{|z|}} \right| < C_b \frac{\sqrt{\log\log n}}{\sqrt{n}} \tag{9}$$

for $n' \geq N_b(\epsilon)$, string z of length less than $n' - N_b(\epsilon) + 1$ and n ranging from $N_b(\epsilon)$ to $n' - |z| + 1$. This implies that the number of strings of length $n' \geq N_b(\epsilon)$ such that,

$$\left| \frac{N(z, X_1^{n+|z|-1})}{n} - \frac{1}{b^{|z|}} \right| \geq C_b \frac{\sqrt{\log\log n}}{\sqrt{n}} \tag{10}$$

for some string z of length less than $n' - N_b(\epsilon) + 1$ and some n between $N_b(\epsilon)$ to $n' - |z| + 1$ is at most $\epsilon \cdot b^n$. If this is not the case, then the union of the cylinder sets corresponding to the strings violating (9) is a set of measure greater than ϵ in which each element violates (9). Since, this leads to a contradiction, the number of strings of length $n' \geq N_b(\epsilon)$ such that (10) holds, is at most $\epsilon \cdot b^n$. This completes the proof of the corollary. ◀

A.2 Proofs from Subsection 2.3

Proof of Lemma 9. For every m, $\xi_m \geq \xi_{m-1}$ and thus we have,

$$\begin{aligned}|\xi_m - \xi_{m-1}| &= \xi_m - \xi_{m-1} \\ &= \xi_m - \eta_m(\xi_{m-1}) + \eta_m(\xi_{m-1}) - \xi_{m-1} \\ &\leq \frac{1}{u(m)^{a_m}} + \frac{1}{u(m)^{a_m}} = \frac{2}{u(m)^{a_m}}.\end{aligned}$$

Then,

$$\sum_{i=m+1}^{\infty} \frac{1}{u(i)^{a_i}} \leq \frac{1}{e^{\langle m+1 \rangle}}\left(1 + \frac{1}{e^2} + \frac{1}{e^4} + \dots\right) < \frac{3}{2}\frac{1}{e^{\langle m+1 \rangle}} < \frac{3}{2}\frac{1}{u(m)^{b_m - 1}} \leq \frac{1}{2}\frac{1}{u(m)^{b_m-2}}.$$

Therefore, the limit ξ satisfies,

$$0 \leq \xi - \xi_m = |\xi - \xi_m| < \frac{1}{u(m)^{b_m - 2}}.$$

◀

B Proofs from Section 4

In order to prove Lemma 12, we need the following inequality.

▶ **Lemma 23.** *For every $n \geq 2$ and x with $|x| < 1$,*

$$\frac{\sin(nx)}{n\sin(x)} \geq 1 - \frac{(n^2-1)x^2}{6}.$$

Proof of Lemma 23. We prove the statement using induction on n. Consider the base case when $n = 2$. We have,

$$\frac{\sin(2x)}{2\sin(x)} = \cos(x) \geq 1 - \frac{x^2}{2} = 1 - \frac{(2^2-1)x^2}{6}.$$

The inequality $\cos(x) \geq 1 - x^2/2$ follows easily from the Taylor series expansion of $\cos(x)$ since $|x| < 1$.

Assume that the statement in the conclusion holds for arbitrary n. We now show that this implies the required conclusion for $n + 1$. We have,

$$\begin{aligned}\frac{\sin((n+1)x)}{(n+1)\sin(x)} &= \frac{\sin(nx)\cos(x) + \cos(nx)\sin(x)}{(n+1)\sin(x)} \\ &= \frac{\sin(nx)}{n\sin(x)} \cdot \frac{n}{n+1} \cdot \cos(x) + \frac{1}{n+1} \cdot \cos(nx).\end{aligned}$$

Using the induction hypothesis and the inequality $\cos(x) \geq 1 - x^2/2$, we get,

$$\begin{aligned}\frac{\sin((n+1)x)}{(n+1)\sin(x)} &\geq \left(1 - \frac{(n^2-1)x^2}{6}\right) \cdot \frac{n}{n+1} \cdot \left(1 - \frac{x^2}{2}\right) + \frac{1}{n+1} \cdot \left(1 - \frac{n^2x^2}{2}\right) \\ &> \frac{n}{n+1} - \frac{1}{n+1} \cdot \frac{(n^3 - n + 3n)x^2}{6} + \frac{1}{n+1} - \frac{1}{n+1} \cdot \frac{n^2x^2}{2} \\ &= 1 - \frac{1}{n+1} \cdot \frac{(n^3 + 3n^2 + 2n)x^2}{6} \\ &= 1 - \frac{1}{n+1} \frac{(n+1)(n^2 + 2n)x^2}{6} \\ &= 1 - \frac{(n^2 + 2n)x^2}{6} \\ &= 1 - \frac{((n+1)^2 - 1)x^2}{6}\end{aligned}$$

The lemma now follows due to induction. ◀

Now, we prove Lemma 12.

Proof of Lemma 12. From Lemma 23, we get that,

$$\left|\frac{\sin(nx)}{n\sin(x)} - 1\right| \leq \frac{(n^2-1)x^2}{6}.$$

Now,

$$\left|\frac{\sin(nx)}{n\sin(x)} - 1\right| \leq \frac{(n^2-1)x^2}{6}.$$

This implies that,

$$\sum_{i=1}^{\infty} \left|\frac{\sin(n\pi/2^{i+1})}{n\sin(\pi/2^{i+1})} - 1\right| < \infty.$$

Now, using Proposition 3.1 from [26], it follows that the infinite product is convergent. The convergence of $\prod_{i=1}^{\infty} \left|\cos(\pi/2^{i+1})\right|$ also follows from this argument since,

$$\cos\left(\pi/2^{i+1}\right) = \frac{\sin\left(2\pi/2^{i+1}\right)}{2\sin\left(\pi/2^{i+1}\right)}.$$ ◀

C Proofs from Section 6

Full Proof of Lemma 20. Let k' be any stage number below k such that $v(k') \not\sim v(k)$. Without loss of generality, let us assume that there does not exist any k'' between k' and k such that $v(k'') = v(k')$. Since the equivalence class of base $v(k')$ has a unique representative in the sequence $\langle r_k \rangle$, we also get that $v(k'') \not\sim v(k')$ for any k'' between k' and k. Since condition 2 from Section 5.2 is valid at the end of the second substage of stage k', we have,

$$\left|\frac{1}{F_{k'}^2} \sum_{i=1}^{F_{k'}^2} e(tv(k')^{(j-1)}\xi)\right| < \frac{\gamma(2^{-(k'+1)}, v(k'), k')}{2} \tag{11}$$

for every t with $|t| \leq T(2^{-(k'+1)}, v(k'), k')$. From the validity of condition 3 from Section 5.2 at the end of the second substage of stage k', we have,

$$P^2_{k'} \geq \max\{M(\gamma(2^{-(k'+1)}, v(k'), k')/2, v(k')), T(2^{-(k'+1)}, v(k'), k')\}.$$

Let m denote any index such that $P^2_{k'} + 1 \leq m \leq P^2_k$. From Lemma 13, we get that,

$$\sum_{\substack{t=-m \\ t\neq 0}}^{m} \sum_{\substack{h=1 \\ u(h)\not\sim u(m)}}^{m} \left| \sum_{j=\langle m;u(h)\rangle+1}^{\langle m+1;u(h)\rangle} e(u(h)^{j-1}t\xi) \right|^2 \leq \delta m^2(\langle m+1\rangle - \langle m\rangle)^{2-\beta_m}$$

for every t with $|t| \leq m$. Since for every $k'' \in [k'+1, k]$, $v(k'') \not\sim v(k')$, from the above we obtain,

$$\left| \sum_{j=\langle m;v(k')\rangle+1}^{\langle m+1;v(k')\rangle} e(v(k')^{j-1}t\xi) \right| \leq \delta m(\langle m+1\rangle - \langle m\rangle)^{1-\beta'_m}.$$

Define the sequence $\langle c_i\rangle_{i=1}^{\infty}$ as $c_i = \beta'_i$. It is easily verified that $\langle c_i\rangle$ is a non-increasing sequence satisfying $c_1 = 1/4$ and $c_i \geq c_1/\sqrt[4]{i}$ for every $i \geq 1$. Consider any m satisfying $P^2_{k'} + 1 \leq m \leq P^2_k$ and $l \leq \langle m+1; v(k')\rangle - \langle m; v(k')\rangle$. Now, for any t with $|t| \leq T(2^{-(k'+1)}, v(k'), k') \leq m$, we have,

$$\begin{aligned}
&\frac{1}{\langle m; v(k')\rangle + l} \left| \sum_{j=1}^{\langle m;v(k')\rangle+l} e(v(k')^{j-1}t\xi) \right| \\
&= \frac{1}{\langle m; v(k')\rangle + l} \left| \sum_{i=1}^{\langle P^2_{k'};v(k')\rangle} e(tv(k')^{(j-1)}\xi) \right| \\
&+ \frac{1}{\langle m; v(k')\rangle + l} \left| \sum_{j=\langle P^2_{k'};v(k')\rangle+1}^{\langle m;v(k')\rangle+l} e(v(k')^{j-1}t\xi) \right| \\
&= \frac{1}{\langle m; v(k')\rangle + l} \left| \sum_{i=1}^{F^2_{k'}} e(tv(k')^{(j-1)}\xi) \right| + \frac{1}{\langle m; v(k')\rangle + l} \left| \sum_{j=\langle P^2_{k'};v(k')\rangle+1}^{\langle m;v(k')\rangle+l} e(v(k')^{j-1}t\xi) \right| \\
&\leq \left| \frac{1}{F^2_{k'}} \sum_{i=1}^{F^2_{k'}} e(tv(k')^{(j-1)}\xi) \right| + \frac{\sum_{i=P^2_{k'}}^{m-1} (\langle i+1\rangle - \langle i\rangle)^{1-\beta'_i} + l}{\langle m; v(k')\rangle + l} \\
&\leq \frac{\gamma(2^{-(k'+1)}, v(k'), k')}{2} + \frac{\sum_{i=1}^{m-1} (\langle i+1\rangle - \langle i\rangle)^{1-c_i} + l}{\langle m; v(k')\rangle + l} \\
&\leq \frac{\gamma(2^{-(k'+1)}, v(k'), k')}{2} + \frac{\gamma(2^{-(k'+1)}, v(k'), k')}{2} \\
&= \gamma(2^{-(k'+1)}, v(k'), k').
\end{aligned}$$

The second last inequality above follows from Lemma 16 since,

$$m \geq P^2_{k'} \geq M(\gamma(2^{-(k'+1)}, v(k'), k')/2, v(k')).$$

Finally, from the above inequalities and Lemma 17 we obtain that, $|H_l^{v(k')}(X(k')_1^n) - 1| \leq 2^{-(k'+1)}$ for every $l \leq k'$ when $\langle P^2_{k-1} + 1; v(k')\rangle + 1 \leq n \leq \langle P^2_k + 1; v(k')\rangle$. ◀