

Improved NP-Hardness of Approximation for Orthogonality Dimension and Minrank

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Abstract

The orthogonality dimension of a graph G over \mathbb{R} is the smallest integer k for which one can assign a nonzero k -dimensional real vector to each vertex of G , such that every two adjacent vertices receive orthogonal vectors. We prove that for every sufficiently large integer k , it is NP-hard to decide whether the orthogonality dimension of a given graph over \mathbb{R} is at most k or at least $2^{(1-o(1)) \cdot k/2}$. We further prove such hardness results for the orthogonality dimension over finite fields as well as for the closely related minrank parameter, which is motivated by the index coding problem in information theory. This in particular implies that it is NP-hard to approximate these graph quantities to within any constant factor. Previously, the hardness of approximation was known to hold either assuming certain variants of the Unique Games Conjecture or for approximation factors smaller than $3/2$. The proofs involve the concept of line digraphs and bounds on their orthogonality dimension and on the minrank of their complement.

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1 Introduction

A graph G is said to be k -colorable if its vertices can be colored by k colors such that every two adjacent vertices receive distinct colors. The chromatic number of G , denoted by $\chi(G)$, is the smallest integer k for which G is k -colorable. As a fundamental and popular graph quantity, the chromatic number has received a considerable amount of attention in the literature from a computational perspective, as described below.

The problem of deciding whether a graph G satisfies $\chi(G) \leq 3$ is one of the classical twenty-one NP-complete problems presented by Karp [22] in 1972. Khanna, Linial, and Safra [24] proved that it is NP-hard to distinguish between graphs G that satisfy $\chi(G) \leq 3$ from those satisfying $\chi(G) \geq 5$. This result, combined with the approach of Garey and Johnson [11] and with a result of Stahl [33], implies that for every $k \geq 6$, it is NP-hard to decide whether a graph G satisfies $\chi(G) \leq k$ or $\chi(G) \geq 2k - 2$. Brakensiek and Guruswami [5] proved that for every $k \geq 3$, it is NP-hard to distinguish between the cases $\chi(G) \leq k$ and $\chi(G) \geq 2k - 1$, and the $2k - 1$ bound was further improved to $2k$ by Barto, Bulín, Krokhin, and Opršal [3]. For large values of k , it was shown by Khot [25] that it is NP-hard to decide whether a graph G satisfies $\chi(G) \leq k$ or $\chi(G) \geq k^{\Omega(\log k)}$, and the latter condition



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was strengthened to $\chi(G) \geq 2^{k^{1/3}}$ by Huang [20]. A substantial improvement was recently obtained by Wrochna and Živný [34], who proved that for every $k \geq 4$, it is NP-hard to decide whether a given graph G satisfies $\chi(G) \leq k$ or $\chi(G) \geq \binom{k}{\lfloor k/2 \rfloor}$. The proof of this result combined the hardness result of [20] with the construction of line digraphs [16] and with a result of Poljak and Rödl [31]. Note that under certain variants of the Unique Games Conjecture, stronger hardness results are known to hold, namely, hardness of deciding whether a given graph G satisfies $\chi(G) \leq k_1$ or $\chi(G) \geq k_2$ for all integers $k_2 > k_1 \geq 3$ [9] (see also [10]).

The present paper studies the computational complexity of algebraic variants of the chromatic number of graphs. A k -dimensional orthogonal representation of a graph $G = (V, E)$ over a field \mathbb{F} is an assignment of a vector $u_v \in \mathbb{F}^k$ with $\langle u_v, u_v \rangle \neq 0$ to each vertex $v \in V$, such that for every two adjacent vertices v and v' it holds that $\langle u_v, u_{v'} \rangle = 0$. Here, for two vectors $x, y \in \mathbb{F}^k$, we consider the standard inner product defined by $\langle x, y \rangle = \sum_{i=1}^k x_i y_i$ with operations over \mathbb{F} . The orthogonality dimension of G over \mathbb{F} , denoted by $\bar{\xi}_{\mathbb{F}}(G)$, is the smallest integer k for which G admits a k -dimensional orthogonal representation over \mathbb{F} (see Remark 4). It can be easily seen that for every graph G and for every field \mathbb{F} , it holds that $\bar{\xi}_{\mathbb{F}}(G) \leq \chi(G)$. In addition, if \mathbb{F} is a fixed finite field or the real field \mathbb{R} , it further holds that $\bar{\xi}_{\mathbb{F}}(G) \geq \Omega(\log \chi(G))$. Both bounds are known to be tight in the worst case (see Claim 8 and [28, Chapter 10]). The study of orthogonal representations and orthogonality dimension was initiated in the seminal work of Lovász [27] on the ϑ -function and has found applications in various areas, e.g., information theory [27], graph theory [29], and quantum communication complexity [8, Chapter 8.5].

The interest in the hardness of determining the orthogonality dimension of graphs dates back to a paper of Lovász, Saks, and Schrijver [29], where it was noted that the problem seems difficult. The aforementioned relations between the chromatic number and the orthogonality dimension yield that hardness of deciding whether a graph G satisfies $\chi(G) \leq k_1$ or $\chi(G) \geq k_2$ implies the hardness of deciding whether it satisfies $\bar{\xi}_{\mathbb{F}}(G) \leq k_1$ or $\bar{\xi}_{\mathbb{F}}(G) \geq \Omega(\log k_2)$, provided that \mathbb{F} is a finite field or \mathbb{R} . It therefore follows from [9] that assuming certain variants of the Unique Games Conjecture, it is hard to decide whether a graph G satisfies $\bar{\xi}_{\mathbb{F}}(G) \leq k_1$ or $\bar{\xi}_{\mathbb{F}}(G) \geq k_2$ for all integers $k_2 > k_1 \geq 3$. This reasoning, however, does not yield NP-hardness results for the orthogonality dimension (without additional complexity assumptions), even using the strongest known NP-hardness results of the chromatic number. Yet, a result of Peeters [30] implies that for every field \mathbb{F} , it is NP-hard to decide if a given graph G satisfies $\bar{\xi}_{\mathbb{F}}(G) \leq 3$, hence it is NP-hard to approximate the orthogonality dimension of a graph over \mathbb{F} to within any factor smaller than $4/3$. Over the reals, the hardness of approximation for the orthogonality dimension was recently extended in [12] to any factor smaller than $3/2$.

Another algebraic quantity of graphs is the minrank parameter that was introduced in 1981 by Haemers [15] in the study of the Shannon capacity of graphs. The minrank parameter was used in [14, 15] to answer questions of Lovász [27] and was later applied by Alon [1], with a different formulation, to disprove a conjecture of Shannon [32]. The minrank of a graph G over a field \mathbb{F} , denoted by $\text{minrk}_{\mathbb{F}}(G)$, is closely related to the orthogonality dimension of the complement graph \bar{G} over \mathbb{F} and satisfies $\text{minrk}_{\mathbb{F}}(G) \leq \bar{\xi}_{\mathbb{F}}(\bar{G})$. The difference between the two quantities comes, roughly speaking, from the fact that the definition of minrank involves the notion of orthogonal bi-representations rather than orthogonal representations (for the precise definitions, see Section 2.1). The study of the minrank parameter is motivated by various applications in information theory and in theoretical computer science. A prominent one is the well-studied index coding problem, for which the minrank parameter perfectly characterizes the optimal length of its linear solutions, as was shown by Bar-Yossef, Birk, Jayram, and Kol [2] (see Section 2.2).

Similarly to the situation of the orthogonality dimension, it was proved in [30] that for every field \mathbb{F} , it is NP-hard to decide if a given graph G satisfies $\text{minrk}_{\mathbb{F}}(G) \leq 3$. It was further shown by Dau, Skachek, and Chee [7] that it is NP-hard to decide whether a given digraph G satisfies $\text{minrk}_{\mathbb{F}_2}(G) \leq 2$. Note that for (undirected) graphs, the minrank over any field is at most 2 if and only if the complement graph is bipartite, a property that can be checked in polynomial time. Motivated by the computational aspects of the index coding problem, Langberg and Sprintson [26] related the minrank of a graph to the chromatic number of its complement and derived from [9] that assuming certain variants of the Unique Games Conjecture, it is hard to decide whether a given graph G satisfies $\text{minrk}_{\mathbb{F}}(G) \leq k_1$ or $\text{minrk}_{\mathbb{F}}(G) \geq k_2$, provided that $k_2 > k_1 \geq 3$ and that \mathbb{F} is a finite field. Similar hardness results were obtained in [26] for additional settings of the index coding problem, including the general (non-linear) index coding problem over a constant-size alphabet.

1.1 Our Contribution

This paper provides improved NP-hardness of approximation results for the orthogonality dimension and for the minrank parameter over various fields. We start with the following result, which is concerned with the orthogonality dimension over the reals.

► **Theorem 1.** *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $f(k) = 2^{(1-o(1)) \cdot k/2}$ such that for every sufficiently large integer k , it is NP-hard to decide whether a given graph G satisfies $\bar{\xi}_{\mathbb{R}}(G) \leq k$ or $\bar{\xi}_{\mathbb{R}}(G) \geq f(k)$.*

Theorem 1 implies that it is NP-hard to approximate the orthogonality dimension of a graph over the reals to within any constant factor. Previously, such NP-hardness result was known to hold only for approximation factors smaller than $3/2$ [12].

We proceed with the following result, which is concerned with the orthogonality dimension and the minrank parameter over finite fields.

► **Theorem 2.** *For every finite field \mathbb{F} , there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $f(k) = 2^{(1-o(1)) \cdot k/2}$ such that for every sufficiently large integer k , the following holds.*

1. *It is NP-hard to decide whether a graph G satisfies $\bar{\xi}_{\mathbb{F}}(G) \leq k$ or $\bar{\xi}_{\mathbb{F}}(G) \geq f(k)$.*
2. *It is NP-hard to decide whether a graph G satisfies $\text{minrk}_{\mathbb{F}}(G) \leq k$ or $\text{minrk}_{\mathbb{F}}(G) \geq f(k)$.*

Theorem 2 implies that over any finite field, it is NP-hard to approximate the orthogonality dimension and the minrank of a graph to within any constant factor. Let us stress that this hardness result relies solely on the assumption $\text{P} \neq \text{NP}$ rather than on stronger complexity assumptions and thus settles a question raised in [26]. Prior to this work, it was known that it is NP-hard to approximate the minrank of graphs to within any factor smaller than $4/3$ [30] and the minrank of digraphs over \mathbb{F}_2 to within any factor smaller than $3/2$ [7].

A central component of the proofs of Theorems 1 and 2 is the notion of line digraphs, introduced in [16], that was first used in the context of hardness of approximation by Wrochna and Živný [34] (see also [13]). It was shown in [17, 31] that the chromatic number of any graph is exponential in the chromatic number of its line digraph. This result was iteratively applied by the authors of [34] to improve the NP-hardness of the chromatic number from the k vs. $2^{k^{1/3}}$ gap of [20] to their k vs. $\binom{k}{\lfloor k/2 \rfloor}$ gap. The main technical contribution of the present work lies in analyzing the orthogonality dimension of line digraphs and the minrank parameter of their complement. We actually show that on line digraphs, these graph parameters are quadratically related to the chromatic number (see Theorems 15, 17, and 20). This allows us to derive our hardness results from the hardness of the chromatic number given in [34], where the obtained gaps are only quadratically weaker. We further discuss some limitations of our approach, involving an analogue of Sperner's theorem for subspaces due to Kalai [21].

We finally show that our approach might be useful for proving hardness results for the general (non-linear) index coding problem over a constant-size alphabet, for which no NP-hardness result is currently known. It was shown by Langberg and Sprintson [26] that for an instance of the index coding problem represented by a graph G , the length of an optimal solution is at most $\chi(\overline{G})$ and at least $\Omega(\log \log \chi(\overline{G}))$. It thus follows that an NP-hardness result for the chromatic number with a double-exponential gap would imply an NP-hardness result for the general index coding problem. However, no such NP-hardness result is currently known for the chromatic number without relying on further complexity assumptions. To tackle this issue, we study the index coding problem on instances which are complement of line digraphs (see Theorem 22). As a consequence of our results, we obtain that the NP-hardness of the general index coding problem can be derived from an NP-hardness result of the chromatic number with only a single-exponential gap, not that far from the best known gap given in [34]. For a precise statement, see Theorem 29.

1.2 Related Work

We gather here several related results from the literature.

- A result of Zuckerman [35] asserts that for any $\varepsilon > 0$, it is NP-hard to approximate the chromatic number of a graph on n vertices to within a factor of $n^{1-\varepsilon}$. It would be interesting to figure out if such hardness result holds for the orthogonality dimension and for the minrank parameter. The present paper, however, focuses on the hardness of gap problems with constant thresholds, independent of the number of vertices.
- As mentioned earlier, Peeters [30] proved that for every field \mathbb{F} , it is NP-hard to decide if the minrank (or the orthogonality dimension) of a given graph is at most 3. We note that for finite fields, this can also be derived from a result of Hell and Nešetřil [19].
- For the chromatic number of hypergraphs, the gaps for which NP-hardness is known to hold are much stronger than for graphs. For example, it was shown in [4] that for some $\delta > 0$, it is NP-hard to decide if a given 4-uniform hypergraph G on n vertices satisfies $\chi(G) \leq 2$ or $\chi(G) \geq \log^\delta n$. An analogue result for the orthogonality dimension of hypergraphs over \mathbb{R} was proved in [18].
- On the algorithmic side, a long line of work has explored the number of colors that an efficient algorithm needs for properly coloring a given k -colorable graph, where $k \geq 3$ is a fixed constant. For example, there exists a polynomial-time algorithm that on a given 3-colorable graph with n vertices uses $O(n^{0.19996})$ colors [23]. Algorithms of this nature exist for the graph parameters studied in this work as well. Indeed, there exists a polynomial-time algorithm that given a graph G on n vertices with $\bar{\xi}_{\mathbb{R}}(G) \leq 3$ finds a proper coloring of G with $O(n^{0.2413})$ colors [18]. Further, there exists a polynomial-time algorithm that given a graph G on n vertices with $\text{minrk}_{\mathbb{F}_2}(\overline{G}) \leq 3$ finds a proper coloring of G with $O(n^{0.2574})$ colors [6]. Note that the colorings obtained by these two algorithms provide, respectively, orthogonal and bi-orthogonal representations for the input graph G (see Claim 8).

1.3 Outline

The rest of the paper is organized as follows. In Section 2, we collect several definitions and results that will be used throughout this paper. In Section 3, we study the underlying graphs of line digraphs and their behavior with respect to the orthogonality dimension, the minrank parameter, and the index coding problem. We also discuss some limitations of our approach. Finally, in Section 4, we prove our hardness results and complete the proofs of Theorems 1 and 2. Some of the proofs are omitted and can be found in the full version of the paper.

2 Preliminaries

Throughout the paper, undirected graphs are referred to as graphs, and directed graphs are referred to as digraphs. All the considered graphs and digraphs are simple, and all the logarithms are in base 2 unless otherwise specified. For an integer n , we use the notation $[n] = \{1, 2, \dots, n\}$.

2.1 Orthogonality Dimension and Minrank

The orthogonality dimension of a graph is defined as follows (see, e.g., [28, Chapter 11]).

► **Definition 3** (Orthogonality Dimension). *A k -dimensional orthogonal representation of a graph $G = (V, E)$ over a field \mathbb{F} is an assignment of a vector $u_v \in \mathbb{F}^k$ with $\langle u_v, u_v \rangle \neq 0$ to each vertex $v \in V$, such that $\langle u_v, u_{v'} \rangle = 0$ whenever v and v' are adjacent vertices in G . Here, for two vectors $x, y \in \mathbb{F}^k$, we let $\langle x, y \rangle = \sum_{i=1}^k x_i y_i$ denote the standard inner product of x and y over \mathbb{F} . The orthogonality dimension of a graph G over a field \mathbb{F} , denoted by $\bar{\xi}_{\mathbb{F}}(G)$, is the smallest integer k for which there exists a k -dimensional orthogonal representation of G over \mathbb{F} .*

► **Remark 4.** We note that orthogonal representations are sometimes defined in the literature such that the vectors associated with *non-adjacent* vertices are required to be orthogonal, that is, as orthogonal representations of the complement graph. While we find it more convenient to use the other definition in this paper, one can view the notation $\bar{\xi}_{\mathbb{F}}(G)$ as standing for $\xi_{\mathbb{F}}(\bar{G})$, i.e., the orthogonality dimension of the complement graph. The same holds for the notion of orthogonal bi-representations, given in Definition 6.

The minrank parameter, introduced in [15], is defined as follows.

► **Definition 5** (Minrank). *Let $G = (V, E)$ be a digraph on the vertex set $V = [n]$, and let \mathbb{F} be a field. We say that a matrix $M \in \mathbb{F}^{n \times n}$ represents G if $M_{i,i} \neq 0$ for every $i \in V$, and $M_{i,j} = 0$ for every distinct vertices $i, j \in V$ such that $(i, j) \notin E$. The minrank of G over \mathbb{F} is defined as $\text{minrk}_{\mathbb{F}}(G) = \min\{\text{rank}_{\mathbb{F}}(M) \mid M \text{ represents } G \text{ over } \mathbb{F}\}$. The definition is naturally extended to graphs by replacing every edge with two oppositely directed edges.*

We next describe an alternative definition due to Peeters [30] for the minrank of graphs. This requires the following extension of orthogonal representations, called orthogonal bi-representations.

► **Definition 6.** *A k -dimensional orthogonal bi-representation of a graph $G = (V, E)$ over a field \mathbb{F} is an assignment of a pair of vectors $(u_v, w_v) \in \mathbb{F}^k \times \mathbb{F}^k$ with $\langle u_v, w_v \rangle \neq 0$ to each vertex $v \in V$, such that $\langle u_v, w_{v'} \rangle = \langle u_{v'}, w_v \rangle = 0$ whenever v and v' are adjacent in G .*

The following proposition follows directly from Definitions 5 and 6 combined with the fact that for every matrix $M \in \mathbb{F}^{n \times n}$, $\text{rank}_{\mathbb{F}}(M)$ is the smallest integer k for which M can be written as $M = M_1^T \cdot M_2$ for two matrices $M_1, M_2 \in \mathbb{F}^{k \times n}$.

► **Proposition 7** ([30]). *For every field \mathbb{F} and for every graph G , $\text{minrk}_{\mathbb{F}}(\bar{G})$ is the smallest integer k for which there exists a k -dimensional orthogonal bi-representation of G over \mathbb{F} .*

The following claim summarizes some known relations between the studied graph parameters.

▷ **Claim 8.** For every field \mathbb{F} and for every graph G , it holds that $\text{minrk}_{\mathbb{F}}(\bar{G}) \leq \bar{\xi}_{\mathbb{F}}(G) \leq \chi(G)$. In addition, if \mathbb{F} is finite, then $\text{minrk}_{\mathbb{F}}(\bar{G}) \geq \log_{|\mathbb{F}|} \chi(G)$.

We finally recall that a homomorphism from a graph $G_1 = (V_1, E_1)$ to a graph $G_2 = (V_2, E_2)$ is a function $g : V_1 \rightarrow V_2$ such that for every two vertices $x, y \in V_1$ with $\{x, y\} \in E_1$, it holds that $\{g(x), g(y)\} \in E_2$. Observe that if there exists a homomorphism from G_1 to G_2 then we have $\chi(G_1) \leq \chi(G_2)$, and for every field \mathbb{F} , we have $\bar{\xi}_{\mathbb{F}}(G_1) \leq \bar{\xi}_{\mathbb{F}}(G_2)$ and $\text{minrk}_{\mathbb{F}}(G_1) \leq \text{minrk}_{\mathbb{F}}(G_2)$.

2.2 Index Coding

The index coding problem, introduced in [2], is concerned with economical strategies for broadcasting information to n receivers in a way that enables each of them to retrieve its own message, a symbol from some given alphabet Σ . For this purpose, each receiver is allowed to use some prior side information that consists of a subset of the messages required by the other receivers. The side information map is naturally represented by a digraph on $[n]$, which includes an edge (i, j) if the i th receiver knows the message required by the j th receiver. The objective is to minimize the length of the transmitted information. For simplicity, we consider here the case of symmetric side information maps, represented by graphs rather than by digraphs. The formal definition follows.

► **Definition 9 (Index Coding).** *Let G be a graph on the vertex set $[n]$, and let Σ be an alphabet. An index code for G over Σ of length k is an encoding function $E : \Sigma^n \rightarrow \Sigma^k$ such that for every $i \in [n]$, there exists a decoding function $g_i : \Sigma^{k+|N_G(i)|} \rightarrow \Sigma$, such that for every $x \in \Sigma^n$, it holds that $g_i(E(x), x|_{N_G(i)}) = x_i$. Here, $N_G(i)$ stands for the set of vertices in G adjacent to the vertex i , and $x|_{N_G(i)}$ stands for the restriction of x to the indices of $N_G(i)$. If Σ is a field \mathbb{F} and the encoding function E is linear over \mathbb{F} , then we say that the index code is linear over \mathbb{F} .*

Bar-Yossef et al. [2] showed that the minrank parameter characterizes the length of optimal solutions to the index coding problem in the linear setting.

► **Proposition 10 ([2]).** *For every field \mathbb{F} and for every graph G , the minimal length of a linear index code for G over \mathbb{F} is $\text{minrk}_{\mathbb{F}}(G)$.*

3 Line Digraphs

In 1960, Harary and Norman [16] introduced the concept of line digraphs, defined as follows.

► **Definition 11 (Line Digraph).** *For a digraph $G = (V, E)$, the line digraph of G , denoted by δG , is the digraph on the vertex set E that includes a directed edge from a vertex (x, y) to a vertex (z, w) whenever $y = z$.*

Definition 11 is naturally extended to graphs G by replacing every edge of G with two oppositely directed edges. Note that in this case, the number of vertices in δG is twice the number of edges in G . We will frequently consider the underlying graph of the digraph δG , i.e., the graph obtained from δG by ignoring the directions of the edges.

The following result of Poljak and Rödl [31], which strengthens a previous result of Harner and Entringer [17], shows that the chromatic number of a graph G precisely determines the chromatic number of the underlying graph of δG . The statement of the result uses the function $b : \mathbb{N} \rightarrow \mathbb{N}$ defined by $b(n) = \binom{n}{\lfloor n/2 \rfloor}$.

► **Theorem 12 ([17, 31]).** *Let G be a graph, and let H be the underlying graph of the digraph δG . Then, $\chi(H) = \min\{n \mid \chi(G) \leq b(n)\}$.*

Using the fact that $b(n) \sim \frac{2^n}{\sqrt{\pi n/2}}$, Theorem 12 implies that the chromatic number of G is exponential in the chromatic number of H . Our goal in this section is to relate the chromatic number of G to other graph parameters of H , namely, the orthogonality dimension, the minrank of the complement, and the optimal length of an index code for the complement.

3.1 Orthogonality Dimension

For a field \mathbb{F} , an integer n , and a subspace U of \mathbb{F}^n , we denote by U^\perp the subspace of \mathbb{F}^n that consists of the vectors that are orthogonal to U over \mathbb{F} , i.e.,

$$U^\perp = \{w \in \mathbb{F}^n \mid \langle w, u \rangle = 0 \text{ for every } u \in U\}.$$

Consider the following family of graphs.

► **Definition 13.** For a field \mathbb{F} and an integer n , let $S_1(\mathbb{F}, n)$ denote the graph whose vertices are all the subspaces of \mathbb{F}^n , where two distinct subspaces U_1 and U_2 are adjacent if there exists a vector $w \in \mathbb{F}^n$ with $\langle w, w \rangle \neq 0$ that satisfies $w \in U_1 \cap U_2^\perp$ and, in addition, there exists a vector $w' \in \mathbb{F}^n$ with $\langle w', w' \rangle \neq 0$ that satisfies $w' \in U_2 \cap U_1^\perp$.

In words, two subspaces of \mathbb{F}^n are adjacent in the graph $S_1(\mathbb{F}, n)$ if each of them includes a non-self-orthogonal vector that is orthogonal to the entire other subspace. Note that for an infinite field \mathbb{F} and for $n \geq 2$, the vertex set of $S_1(\mathbb{F}, n)$ is infinite.

We argue that the chromatic number of a graph G can be used to estimate the orthogonality dimension of the underlying graph H of its line digraph δG . First, recall that by Theorem 12, the chromatic number of H is logarithmic in $\chi(G)$. This implies, using Claim 8, that the orthogonality dimension of H over any field is at most logarithmic in $\chi(G)$. For a lower bound on the orthogonality dimension of H , we need the following lemma that involves the chromatic numbers of the graphs $S_1(\mathbb{F}, n)$.

► **Lemma 14.** Let \mathbb{F} be a field, let G be a graph, let H be the underlying graph of the digraph δG , and put $n = \bar{\xi}_{\mathbb{F}}(H)$. Then, $\chi(G) \leq \chi(S_1(\mathbb{F}, n))$.

Proof. Put $G = (V_G, E_G)$ and $H = (V_H, E_H)$. The assumption $n = \bar{\xi}_{\mathbb{F}}(H)$ implies that there exists an n -dimensional orthogonal representation of H over \mathbb{F} , that is, an assignment of a vector $u_v \in \mathbb{F}^n$ with $\langle u_v, u_v \rangle \neq 0$ to each vertex $v \in V_H$, such that $\langle u_v, u_{v'} \rangle = 0$ whenever v and v' are adjacent in H . Recall that the vertices of H , just as the vertices of δG , are the ordered pairs (x, y) of adjacent vertices x, y in G .

For every vertex $y \in V_G$, let U_y denote the subspace spanned by the vectors of the given orthogonal representation that are associated with the vertices of H whose tail is y , namely,

$$U_y = \text{span}(\{u_v \mid v = (x, y) \text{ for some } x \in V_G\}).$$

Note that U_y is a subspace of \mathbb{F}^n , and thus a vertex of $S_1(\mathbb{F}, n)$.

Consider the function that maps every vertex $y \in V_G$ of G to the vertex U_y of $S_1(\mathbb{F}, n)$. We claim that this function forms a homomorphism from G to $S_1(\mathbb{F}, n)$. To see this, let $x, y \in V_G$ be adjacent vertices in G , and consider the vector $w = u_{(x, y)}$ assigned by the given orthogonal representation to the vertex (x, y) of H . By the definition of an orthogonal representation, it holds that $\langle w, w \rangle \neq 0$. Since (x, y) is a vertex of H whose tail is y , it follows that $w \in U_y$. Further, every vertex of H of the form (x', x) for some $x' \in V_G$ is adjacent in H to (x, y) , hence it holds that $\langle u_{(x', x)}, w \rangle = 0$. Since the subspace U_x is spanned by those vectors $u_{(x', x)}$, we obtain that w is orthogonal to the entire subspace U_x . It thus follows

that the vector w satisfies $\langle w, w \rangle \neq 0$ and $w \in U_y \cap U_x^\perp$. By symmetry, there also exists a vector $w' \in \mathbb{F}^n$ satisfying $\langle w', w' \rangle \neq 0$ and $w' \in U_x \cap U_y^\perp$, hence the subspaces U_x and U_y are adjacent vertices in $S_1(\mathbb{F}, n)$. We conclude that the above function is a homomorphism from G to $S_1(\mathbb{F}, n)$, hence the chromatic numbers of these graphs satisfy $\chi(G) \leq \chi(S_1(\mathbb{F}, n))$, as required. ◀

In order to derive useful bounds from Lemma 14, we need upper bounds on the chromatic numbers of the graphs $S_1(\mathbb{F}, n)$. Every vertex of $S_1(\mathbb{F}, n)$ is a subspace of \mathbb{F}^n and thus can be represented by a basis that generates it. For a finite field \mathbb{F} of size q , the number of possible bases does not exceed q^{n^2} , which obviously yields that $\chi(S_1(\mathbb{F}, n)) \leq q^{n^2}$. While this simple bound suffices for proving our hardness results for the orthogonality dimension over finite fields, we note that the number of vertices in $S_1(\mathbb{F}, n)$ is in fact $q^{(1+o(1)) \cdot n^2/4}$, where the $o(1)$ term tends to 0 when n tends to infinity.¹

We conclude this discussion with the following theorem.

► **Theorem 15.** *Let \mathbb{F} be a finite field of size q , let G be a graph, and let H be the underlying graph of the digraph δG . Then, it holds that*

$$\bar{\xi}_{\mathbb{F}}(H) \geq \sqrt{\log_q \chi(G)}.$$

Proof. Put $n = \bar{\xi}_{\mathbb{F}}(H)$, and apply Lemma 14 to obtain that $\chi(G) \leq \chi(S_1(\mathbb{F}, n)) \leq q^{n^2}$. By rearranging, the proof is completed. ◀

3.1.1 The Chromatic Number of $S_1(\mathbb{R}, n)$

For the real field \mathbb{R} and for $n \geq 2$, the vertex set of the graph $S_1(\mathbb{R}, n)$ is infinite, and yet, its chromatic number is finite. To see this, let us firstly observe a simple upper bound of 2^{3^n} . To each vertex of $S_1(\mathbb{R}, n)$, i.e., a subspace U of \mathbb{R}^n , assign the subset of $\{0, \pm 1\}^n$ that consists of all the sign vectors of the vectors of U . This assignment forms a proper coloring of the graph, because for adjacent vertices U and V there exists a nonzero vector $w \in U$ that is orthogonal to V , hence the sign vector of w belongs to the set of sign vectors of U but does not belong to the one of V (because the inner product of two vectors with the same nonzero sign vector is positive). Since the number of subsets of $\{0, \pm 1\}^n$ is 2^{3^n} , it follows that $\chi(S_1(\mathbb{R}, n)) \leq 2^{3^n}$.

The above double-exponential bound is not sufficient for deriving NP-hardness of approximation results for the orthogonality dimension over \mathbb{R} from the currently known NP-hardness results of the chromatic number. We therefore need the following lemma that provides an exponentially better bound which is suitable for our purposes. For a vector $w \in \mathbb{R}^n$, we use here the notation $\|w\| = \sqrt{\langle w, w \rangle}$ for the Euclidean norm of w .

► **Lemma 16.** *For every integer n , it holds that $\chi(S_1(\mathbb{R}, n)) \leq (2n + 1)^{n^2}$.*

Proof. We define a coloring of the vertices of the graph $S_1(\mathbb{R}, n)$ as follows. For every vertex of $S_1(\mathbb{R}, n)$, i.e., a subspace U of \mathbb{R}^n , let (u_1, \dots, u_k) be an arbitrary orthonormal basis of U where $k \leq n$, and assign U to the color $c(U) = (u'_1, \dots, u'_k)$ where u'_i is a vector obtained from u_i by rounding each of its values to a closest integer multiple of $\frac{1}{2n}$. Note that for every $i \in [k]$, the vectors u_i and u'_i differ in every coordinate by no more than $\frac{1}{2n}$ in absolute value.

¹ To see this, observe that the number of k -dimensional subspaces of \mathbb{F}^n is precisely $\prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}$ and that every term in this product lies in $[q^{n-k-1}, q^{n-k+1}]$. Hence, the total number of subspaces of \mathbb{F}^n is at least $\sum_{k=0}^n q^{(n-k-1)k}$ and at most $\sum_{k=0}^n q^{(n-k+1)k}$. It follows that the number of subspaces of \mathbb{F}^n is $q^{(1+o(1)) \cdot n^2/4}$.

We claim that c is a proper coloring of $S_1(\mathbb{R}, n)$. To see this, let U and V be adjacent vertices in the graph. If $\dim(U) \neq \dim(V)$ then it clearly holds that $c(U) \neq c(V)$. So suppose that the dimensions of U and V are equal, and put $k = \dim(U) = \dim(V)$. Denote the orthonormal bases associated with U and V by (u_1, \dots, u_k) and (v_1, \dots, v_k) respectively, and let $c(U) = (u'_1, \dots, u'_k)$ and $c(V) = (v'_1, \dots, v'_k)$ be their colors. Our goal is to show that $c(U) \neq c(V)$.

Assume for the sake of contradiction that $c(U) = c(V)$, that is, $u'_i = v'_i$ for every $i \in [k]$. This implies that for every $i \in [k]$, the vectors u_i and v_i differ in each coordinate by no more than $\frac{1}{n}$ in absolute value, hence

$$\|u_i - v_i\| \leq \sqrt{n \cdot \frac{1}{n^2}} = \frac{1}{\sqrt{n}}. \tag{1}$$

Since U and V are adjacent in the graph $S_1(\mathbb{R}, n)$, by scaling, there exists a unit vector $u \in U \cap V^\perp$. Write $u = \sum_{i \in [k]} \alpha_i \cdot u_i$ for coefficients $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. Since the given basis of U is orthonormal, it follows that $\sum_{i \in [k]} \alpha_i^2 = \|u\|^2 = 1$. Now, consider the vector $v = \sum_{i \in [k]} \alpha_i \cdot v_i$, and observe that v is a unit vector that belongs to the subspace V . Observe further that

$$\begin{aligned} \|u - v\| &= \left\| \sum_{i \in [k]} \alpha_i \cdot (u_i - v_i) \right\| \leq \sum_{i \in [k]} |\alpha_i| \cdot \|u_i - v_i\| \\ &\leq \left(\sum_{i \in [k]} \alpha_i^2 \right)^{1/2} \cdot \left(\sum_{i \in [k]} \|u_i - v_i\|^2 \right)^{1/2} \leq 1, \end{aligned} \tag{2}$$

where the first inequality follows from the triangle inequality, the second from the Cauchy-Schwarz inequality, and the third from (1) using $k \leq n$. However, u and v are orthogonal unit vectors, and as such, the distance between them satisfies $\|u - v\| = \sqrt{2}$. This yields a contradiction to (2), hence $c(U) \neq c(V)$.

To complete the proof, we observe that the number of colors used by the proper coloring c does not exceed $(2n + 1)^{n^2}$. Indeed, every color can be represented by an $n \times n$ matrix whose values are of the form $\frac{a}{n}$ for integers $-n \leq a \leq n$ (where the matrix associated with a subspace of dimension k consists of the rounded k column vectors concatenated with $n - k$ columns of zeros). Since the number of those matrices is bounded by $(2n + 1)^{n^2}$, we are done. ◀

We derive the following theorem.

► **Theorem 17.** *There exists a constant $c > 0$, such that for every graph G with $\chi(G) \geq 3$, the underlying graph H of the digraph δG satisfies*

$$\bar{\xi}_{\mathbb{R}}(H) \geq c \cdot \sqrt{\frac{\log \chi(G)}{\log \log \chi(G)}}.$$

Proof. Put $n = \bar{\xi}_{\mathbb{R}}(H)$, and combine Lemma 14 with Lemma 16 to obtain that

$$\chi(G) \leq \chi(S_1(\mathbb{R}, n)) \leq (2n + 1)^{n^2},$$

which yields the desired bound. ◀

A discussion on the clique numbers of the graphs $S_1(\mathbb{F}, n)$ can be found in the full version of the paper.

3.2 Minrank

As in the previous section, we start with a definition of a family of graphs.

► **Definition 18.** For a field \mathbb{F} and an integer n , let $S_2(\mathbb{F}, n)$ denote the graph whose vertices are all the pairs of subspaces of \mathbb{F}^n , where two distinct pairs (U_1, W_1) and (U_2, W_2) are adjacent if there exist two vectors $u, w \in \mathbb{F}^n$ with $\langle u, w \rangle \neq 0$ such that $u \in U_1 \cap W_2^\perp$ and $w \in W_1 \cap U_2^\perp$ and, in addition, there exist two vectors $u, w \in \mathbb{F}^n$ with $\langle u, w \rangle \neq 0$ such that $u \in U_2 \cap W_1^\perp$ and $w \in W_2 \cap U_1^\perp$.

We next argue that the chromatic number of a graph G can be used to estimate the minrank of the complement of the underlying graph of its line digraph δG . This is established using the following lemma that involves the chromatic numbers of the graphs $S_2(\mathbb{F}, n)$. Its proof resembles that of Lemma 14 and can be found in the full version of the paper.

► **Lemma 19.** Let \mathbb{F} be a field, let G be a graph, let H be the underlying graph of the digraph δG , and put $n = \text{minrk}_{\mathbb{F}}(\overline{H})$. Then, $\chi(G) \leq \chi(S_2(\mathbb{F}, n))$.

We derive the following theorem.

► **Theorem 20.** Let \mathbb{F} be a finite field of size q , let G be a graph, and let H be the underlying graph of the digraph δG . Then, it holds that

$$\text{minrk}_{\mathbb{F}}(\overline{H}) \geq \sqrt{\frac{1}{2} \cdot \log_q \chi(G)}.$$

Proof. Put $n = \text{minrk}_{\mathbb{F}}(\overline{H})$, and apply Lemma 19 to obtain that

$$\chi(G) \leq \chi(S_2(\mathbb{F}, n)) \leq q^{2n^2},$$

where the second inequality holds because the number of vertices in $S_2(\mathbb{F}, n)$ does not exceed q^{2n^2} . By rearranging, the proof is completed. ◀

3.2.1 The Chromatic Number of $S_2(\mathbb{R}, n)$

We next consider the problem of determining the chromatic numbers of the graphs $S_2(\mathbb{R}, n)$. The following theorem shows that these graphs cannot be properly colored using a finite number of colors, in contrast to the graphs $S_1(\mathbb{R}, n)$ addressed in Lemma 16. Its proof can be found in the full version of the paper.

► **Theorem 21.** For every integer $n \geq 3$, it holds that $\chi(S_2(\mathbb{R}, n)) = \infty$.

3.3 Index Coding

For the general (non-linear) index coding problem, we provide the following result (recall Definition 9). Its proof can be found in the full version of the paper.

► **Theorem 22.** Let Σ be an alphabet of size at least 2, let G be a graph, and let H be the underlying graph of the digraph δG . If there exists an index code for \overline{H} over Σ of length k , then $\chi(G) \leq 2^{|\Sigma|^k}$.

4 Hardness Results

In this section, we prove our hardness results for the orthogonality dimension and for minrank. We also suggest a potential avenue for proving hardness results for the general index coding problem over a constant-size alphabet.

The starting point of our hardness proofs is the following theorem of Wrochna and Živný [34]. Recall that the function $b : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $b(n) = \binom{n}{\lfloor n/2 \rfloor}$.

► **Theorem 23** ([34]). *For every integer $k \geq 4$, it is NP-hard to decide whether a given graph G satisfies $\chi(G) \leq k$ or $\chi(G) \geq b(k)$.*

Our hardness results for the orthogonality dimension and the minrank parameter over finite fields are given by the following theorem, which confirms Theorem 2.

► **Theorem 24.** *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $f(k) = (1 - o(1)) \cdot \sqrt{b(k)}$ such that for every finite field \mathbb{F} and for every sufficiently large integer k , the following holds.*

1. *It is NP-hard to decide whether a given graph G satisfies*

$$\bar{\xi}_{\mathbb{F}}(G) \leq k \quad \text{or} \quad \bar{\xi}_{\mathbb{F}}(G) \geq \frac{1}{\sqrt{\log |\mathbb{F}|}} \cdot f(k).$$

2. *It is NP-hard to decide whether a given graph G satisfies*

$$\text{minrk}_{\mathbb{F}}(G) \leq k \quad \text{or} \quad \text{minrk}_{\mathbb{F}}(G) \geq \frac{1}{\sqrt{2 \cdot \log |\mathbb{F}|}} \cdot f(k).$$

Proof. Fix a finite field \mathbb{F} of size q . We start by proving the first item of the theorem. For an integer $k \geq 4$, consider the problem of deciding whether a given graph G satisfies

$$\chi(G) \leq b(k) \quad \text{or} \quad \chi(G) \geq b(b(k)),$$

whose NP-hardness follows from Theorem 23. To obtain our hardness result on the orthogonality dimension over \mathbb{F} , we reduce from this problem. Consider the reduction that given an input graph G produces and outputs the underlying graph H of the digraph δG . This reduction can clearly be implemented in polynomial time (in fact, in logarithmic space).

To prove the correctness of the reduction, we analyze the orthogonality dimension of H over \mathbb{F} . If G is a YES instance, that is, $\chi(G) \leq b(k)$, then by combining Claim 8 with Theorem 12, it follows that

$$\bar{\xi}_{\mathbb{F}}(H) \leq \chi(H) \leq k.$$

If G is a NO instance, that is, $\chi(G) \geq b(b(k))$, then by Theorem 15, it follows that

$$\bar{\xi}_{\mathbb{F}}(H) \geq \sqrt{\log_q \chi(G)} \geq \sqrt{\log_q b(b(k))} = \frac{1-o(1)}{\sqrt{\log q}} \cdot \sqrt{b(k)},$$

where the $o(1)$ term tends to 0 when k tends to infinity. Note that we have used here the fact that $b(n) = \Theta(2^n/\sqrt{n})$. By letting k be any sufficiently large integer, the proof of the first item of the theorem is completed.

The proof of the second item of the theorem is similar. To avoid repetitions, we briefly mention the needed changes in the proof. First, to obtain a hardness result for the minrank parameter, the reduction has to output the complement \bar{H} of the graph H rather than H itself. Second, in the analysis of the NO instances, one has to apply Theorem 20 instead of Theorem 15 to obtain that

$$\text{minrk}_{\mathbb{F}}(\bar{H}) \geq \sqrt{\frac{1}{2} \cdot \log_q \chi(G)} \geq \sqrt{\frac{1}{2} \cdot \log_q b(b(k))} = \frac{1-o(1)}{\sqrt{2 \cdot \log q}} \cdot \sqrt{b(k)}.$$

This completes the proof of the theorem. ◀

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As an immediate corollary of Theorem 24, we obtain the following.

► **Corollary 25.** *For every finite field \mathbb{F} , the following holds.*

1. *It is NP-hard to approximate $\bar{\xi}_{\mathbb{F}}(G)$ for a given graph G to within any constant factor.*
2. *It is NP-hard to approximate $\text{minrk}_{\mathbb{F}}(G)$ for a given graph G to within any constant factor.*

We next prove a hardness result for the orthogonality dimension over the reals, confirming Theorem 1.

► **Theorem 26.** *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $f(k) = \Theta(\sqrt{b(k)/k})$ such that for every sufficiently large integer k , it is NP-hard to decide whether a given graph G satisfies*

$$\bar{\xi}_{\mathbb{R}}(G) \leq k \quad \text{or} \quad \bar{\xi}_{\mathbb{R}}(G) \geq f(k).$$

Proof. As in the proof of Theorem 24, for an integer $k \geq 4$, we reduce from the problem of deciding whether a given graph G satisfies

$$\chi(G) \leq b(k) \quad \text{or} \quad \chi(G) \geq b(b(k)),$$

whose NP-hardness follows from Theorem 23. Consider the polynomial-time reduction that given an input graph G produces and outputs the underlying graph H of the digraph δG .

To prove the correctness of the reduction, we analyze the orthogonality dimension of H over \mathbb{R} . If G is a YES instance, that is, $\chi(G) \leq b(k)$, then by combining Claim 8 with Theorem 12, it follows that

$$\bar{\xi}_{\mathbb{R}}(H) \leq \chi(H) \leq k.$$

If G is a NO instance, that is, $\chi(G) \geq b(b(k))$, then by Theorem 17 combined with the fact that $b(n) = \Theta(2^n/\sqrt{n})$, it follows that

$$\bar{\xi}_{\mathbb{R}}(H) \geq c \cdot \sqrt{\frac{\log b(b(k))}{\log \log b(b(k))}} = \Theta\left(\sqrt{\frac{b(k)}{k}}\right),$$

where c is an absolute positive constant. This completes the proof of the theorem. ◀

As an immediate corollary of Theorem 26, we obtain the following.

► **Corollary 27.** *It is NP-hard to approximate $\bar{\xi}_{\mathbb{R}}(G)$ for a given graph G to within any constant factor.*

We end this section with a statement that might be useful for proving NP-hardness results for the general index coding problem. Consider the following definition.

► **Definition 28.** *For an alphabet Σ and for two integers $k_1 < k_2$, let $\text{Index-Coding}_{\Sigma}(k_1, k_2)$ denote the problem of deciding whether the minimal length of an index code for a given graph G over Σ is at most k_1 or at least k_2 .*

We prove the following result.

► **Theorem 29.** *Let Σ be an alphabet of size at least 2, and let k_1, k_2 be two integers. Then, there exists a polynomial-time reduction from the problem of deciding whether a given graph G satisfies $\chi(G) \leq b(k_1)$ or $\chi(G) \geq k_2$ to $\text{Index-Coding}_{\Sigma}(k_1, \log_{|\Sigma|} \log k_2)$.*

Proof. Consider the polynomial-time reduction that given an input graph G produces the underlying graph H of the digraph δG and outputs its complement \overline{H} . For correctness, suppose first that G is a YES instance, that is, $\chi(G) \leq b(k_1)$. Then, by combining Claim 8 with Theorem 12, it follows that $\text{minrk}_{\mathbb{F}_2}(\overline{H}) \leq \chi(H) \leq k_1$. By Proposition 10, it further follows that there exists a linear index code for \overline{H} over \mathbb{F}_2 of length k_1 . In particular, using $|\Sigma| \geq 2$, there exists an index code for \overline{H} over the alphabet Σ of length k_1 . Suppose next that G is a NO instance, that is, $\chi(G) \geq k_2$. By Theorem 22, it follows that the length of any index code for \overline{H} over Σ is at least $\log_{|\Sigma|} \log k_2$, so we are done. ◀

Theorem 29 implies that in order to prove the NP-hardness of the general index coding problem over some finite alphabet Σ of size at least 2, it suffices to prove for some integer k that it is NP-hard to decide whether a given graph G satisfies $\chi(G) \leq b(k)$ or $\chi(G) > 2^{|\Sigma|^k}$.

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