Reversible Pebble Games and the Relation Between Tree-Like and General Resolution Space

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Abstract

We show a new connection between the space measure in tree-like resolution and the reversible pebble game in graphs. Using this connection, we provide several formula classes for which there is a logarithmic factor separation between the space complexity measure in tree-like and general resolution. We show that these separations are not far from optimal by proving upper bounds for tree-like resolution space in terms of general resolution clause and variable space. In particular we show that for any formula F, its tree-like resolution space is upper bounded by $\mathrm{space}(\pi)\log\big(\mathrm{time}(\pi)\big)$, where π is any general resolution refutation of F. This holds considering as $\mathrm{space}(\pi)$ the clause space of the refutation as well as considering its variable space. For the concrete case of Tseitin formulas, we are able to improve this bound to the optimal bound $\mathrm{space}(\pi)\log n$, where n is the number of vertices of the corresponding graph.

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1 Introduction

Resolution is one of the best-studied systems for refuting unsatisfiable propositional formulas. This is due to its theoretical simplicity, as well as its practical importance since it is the proof system at the root of many modern SAT solvers. Several complexity measures for the analysis of resolution refutations have been used in the last decades. In this paper, we will mainly concentrate on space bounds, which measure the amount of memory that is needed in a resolution refutation. Intuitively, the clause space (CS) measures the number of clauses required simultaneously in a refutation, while the variable space (VS) measures the maximum number of distinct variables kept simultaneously in memory during this process. Experimental results have shown that space measures for resolution correlate well with the hardness of refuting unsatisfiable formulas with SAT solvers in practice [2, 18].

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Tree-like resolution is a restricted kind of resolution that is especially important since the original DPLL algorithm [13, 12] on which many SAT solvers are based, is equivalent to this restriction of the resolution system. Contrary to general resolution, in tree-like resolution, if a clause is needed more than once in a refutation, it has to be rederived each time. It is known that general resolution can be exponentially more efficient than tree-like resolution in terms of length (number of clauses in a refutation) [8, 3]. In [3], the authors give an almost optimal separation between general and tree-like resolution. They show that for each natural number n, there are unsatisfiable formulas in $\mathrm{O}(n)$ variables that have resolution refutations of length L, linear in n, but for which any tree-like resolution refutation of the formula requires length $\exp\big(\Omega(\frac{L}{\log L})\big)$. They also give an almost matching upper bound of $\exp\big(\mathrm{O}\big(\frac{L \log\log L}{\log L}\big)\big)$ for the tree-like resolution length of any formula that can be refuted in length L by general resolution.

In this paper we study space separations between general and tree-like resolution. Space separations are much more modest than the ones for length. It is known from [15] that all space measures considered here for a formula with n variables are between constant and $n+2$. Also, it is not hard to see that variable space coincides in general and tree-like resolution. Therefore, we only consider the clause space measure for the case of tree-like resolution. The first space separation between general and tree-like resolution was given in [16]. There, a family of formulas $(F_n)_{n=1}^{\infty}$ was presented which require tree-resolution clause space s_n but have a general resolution refutation in clause space $c \cdot s_n$, for some constant $c < 1$. More recently, in [18], a family of formulas $(F_n)_{n=1}^{\infty}$ is presented with $\mathrm{O}(n)$ variables that can be refuted by general resolution in constant clause space but requires $\Theta(\log n)$ tree-like resolution space, thus showing that both measures are fundamentally different.

In this paper, we present a systematic study of tree-like resolution space providing upper bounds for this measure, which show that the logarithmic factor in the separation of [18] as well as in other separations provided here are basically optimal. Our main tools are several versions of pebbling games played on graphs, which have been extensively used in the past for analysing different computation models and in particular for analysing proof systems (see [20] for an excellent survey). We formally define these games in the preliminaries. Intuitively, the idea of the pebble game is to measure the number of pebbles needed by a single player in order to place a pebble on the sink of a directed acyclic graph following certain rules. In the standard game, pebbles can only be placed on a vertex if it is a source or if all its direct predecessors already have a pebble, but they can be removed at any time. In the reversible pebble game, pebbles can only be placed or removed from a vertex if all the direct predecessors of the vertex contain a pebble. Based on the pebble game, a class of contradictory formulas, called pebbling formulas, was introduced in [6]. These formulas have been extremely useful for analysing several proof systems. The reason for this is that some of the pebbling properties of the underlying graphs can be translated into parameters for the complexity of their corresponding pebbling contradictions. Known results of pebbling can therefore be translated into proof complexity results.

Our main contribution is a new connection between tree-like resolution clause space and the reversible pebble game. We show that for any graph G, the tree-like resolution space of a (certain kind of) pebbling contradiction of the graph is at least the reversible pebbling number of G and at most twice this number. More interestingly, we show that for *any* unsatisfiable CNF formula F, the tree-like resolution clause space of a refutation of F is at most the reversible pebbling number of any refutation graph of F, not necessarily a tree-like refutation. This result adds one more connection to the rich set of interrelations between pebbling and resolution [20]. A central tool in the proofs of these results is the Raz–McKenzie game [23], a two-player game on graphs, and the fact that this game is equivalent to reversible pebbling in a precise sense [10]. The clause space measure for any formula can be exactly characterised in terms of the black pebble game on a refutation graph of the formula [15]. We find the fact that tree-like clause space is upper bounded by the reversible pebble game quite surprising.

Using these bound and known results on reversible pebbling [11, 30], we show in Section 3 that there are families of pebbling formulas $(F_n)_{n=1}^{\infty}$ with $\mathrm{O}(n)$ variables, that have general clause space $\mathrm{O}(s)$ and tree-like resolution space $\Omega(s\log n)$ for any function s smaller than $n^{1/2-\varepsilon}$. This separation (as well as the one in [18]) is almost optimal since we also show that for any pebbling formula F, its tree-like clause space is at most $\min_{\mathcal{P}}\big(\mathsf{space}(\mathcal{P})\cdot\log\mathsf{time}(\mathcal{P})\big)$, where $\mathcal{P}$ is a black pebbling of the underlying graph of F. This means that for graphs of size n where the smallest black pebbling space is achieved in a one-shot pebbling strategy, that is, a strategy in which every vertex in the graph is pebbled at most once, the $\log n$ factor in the separation is optimal and the only room for improvement is with graph families in which the space-optimal black pebbling is not one-shot. It is possible that for one such family, the $\log n$ separation factor can be improved to a $\log\mathsf{time}(\mathcal{P})$ factor. We provide, however, for the first time a family of graphs for which the minimum pebbling space is obtained in a strategy that is not one-shot, but for which the clause space separation between general and tree-like resolution is also only a $\log n$ factor. We conjecture that this is optimal, and that this separation cannot be improved for other graph classes. This question is closely related to proving optimal upper bounds for reversible pebbling in terms of black pebbling. Another motivation for providing this new graph family is to increase the set of examples of formulas with concrete resolution space bounds that can be used for the testing of SAT solvers, as done for example in [18].

In Section 4, we prove upper bounds on the tree-like clause space for *any* unsatisfiable CNF formula F in terms of the variable space and clause space for general resolution of the formula. We use the *amortised space measures* for resolution introduced by Razborov in [24], that penalise configurational proofs for being unreasonably long. In his paper he defined the notations $\mathrm{VS}^*(F\vdash\Box) := \min_{\pi:F\vdash\Box}\big(\mathrm{VS}(\pi)\cdot\log\mathrm{L}(\pi)\big)$ and $\mathrm{CS}^*(F\vdash\Box) := \min_{\pi:F\vdash\Box}\big(\mathrm{CS}(\pi)\cdot\log\mathrm{L}(\pi)\big)$, where $\mathrm{L}(\pi)$ is the length of the configurational proof π. We show the upper bounds $\text{Tree-CS}(F\vdash\Box)\le\mathrm{VS}^*(F\vdash\Box)+2$ and $\text{Tree-CS}(F\vdash\Box)\le\mathrm{CS}^*(F\vdash\Box)+2$. The first inequality is especially interesting since it shows that clause space can be meaningfully bounded in terms of variable space, a question posed by Razborov in [24]: $\mathrm{CS}(F\vdash\Box)\le\mathrm{VS}^*(F\vdash\Box)+2$. Again, from the separations in Sections 3 and 5, the only room for improvement in this upper bounds is to decrease the $\log\mathrm{L}(\pi)$ factor to a $\log n$ factor, where n is the size of the formula F.

Finally, in Section 5, we give optimal separations for the space in tree-like resolution for the class of Tseitin formulas. We show that for any graph G with n vertices and odd marking χ, the inequalities $\text{Tree-CS}\big(\mathrm{Ts}(G,\chi)\vdash\Box\big)\le\mathrm{CS}\big(\mathrm{Ts}(G,\chi)\vdash\Box\big)\cdot\log n+2$ and $\text{Tree-CS}\big(\mathrm{Ts}(G,\chi)\vdash\Box\big)\le\mathrm{VS}\big(\mathrm{Ts}(G,\chi)\vdash\Box\big)\cdot\log n+2$ hold, thus improving the upper bound from the previous sections from logarithmic in the resolution length down to a $\log n$ factor. We also provide a class of formulas with a matching space separation showing that this is optimal.

2 Preliminaries

For a positive integer n we let $[n] := \{1,2,\ldots,n\}$. The base of all logarithms in this paper is 2. The *size of a graph* is the number of vertices of it. Given a directed acyclic graph (*DAG*) $G=(V,E)$, we say that a vertex u is a *direct predecessor* of a vertex v, if there is a directed edge from u to v. We denote by $\mathrm{pred}_G(v)$ the *set of all direct predecessors* of v in G. The *maximal in-degree* of a graph G is defined to be $\max_{v \in V} |\mathrm{pred}_G(v)|$. A vertex in a DAG with no incoming edges is called a *source* and a vertex with no outgoing edges is called a *sink*.

2.1 Pebble Games

Black pebbling was first mentioned implicitly in [21]. Note, that there exist several variants of the pebble game in the literature. In this paper, we focus on the variant without *sliding* and requiring the sink of the graph to be pebbled at the end. For differences between these variants, we refer to the survey [20], from which we borrowed most of our notation. For the following definitions, let $G = (V, E)$ be a DAG with a unique sink vertex z.

▶ **Definition 1** (Black pebble game). *The* black pebble game *on G is the following one-player game: At any time i of the game, we have a* pebble configuration $\mathbb{P}_i$, *where $\mathbb{P}_i \subseteq V$ is the set of black pebbles. A pebble configuration $\mathbb{P}_{i-1}$ can be changed to $\mathbb{P}_i$ by applying exactly one of the following rules:*

Black pebble placement on v: *If all direct predecessors of an empty vertex v have pebbles on them, a black pebble may be placed on v. More formally, letting $\mathbb{P}_i = \mathbb{P}_{i-1} \cup \{v\}$ is allowed if $v \notin \mathbb{P}_{i-1}$ and* $\mathrm{pred}_G(v) \subseteq \mathbb{P}_{i-1}$. *In particular, a black pebble can always be placed on an empty source vertex s, since* $\mathrm{pred}_G(s) = \varnothing$.

Black pebble removal from v: *A black pebble may be removed from any vertex at any time. Formally, if $v \in \mathbb{P}_{i-1}$, then we can set $\mathbb{P}_i = \mathbb{P}_{i-1} \setminus \{v\}$.*

A black pebbling *of G is a sequence of pebble configurations $\mathcal{P} = (\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_t)$ such that $\mathbb{P}_0 = \varnothing$, $\mathbb{P}_t = \{z\}$, and for all $i \in [t]$ it holds that $\mathbb{P}_i$ can be obtained from $\mathbb{P}_{i-1}$ by applying exactly one of the above-stated rules. It is* one-shot *if each $v \in V$ is pebbled at most once.*

Finally, we mention the reversible pebble game introduced in [7]. In the reversible pebble game, the moves performed in reverse order should also constitute a legal black pebbling, which means that the rules for pebble placements and removals have to become symmetric.

▶ **Definition 2** (Reversible pebble game). *The* reversible pebble game *on G is the following one-player game: At any time i of the game, we have a pebble configuration $\mathbb{P}_i \subseteq V$. A pebble configuration $\mathbb{P}_{i-1}$ can be changed to $\mathbb{P}_i$ by applying exactly one of the following rules:*

Pebble placement on v: *Letting $\mathbb{P}_i = \mathbb{P}_{i-1} \cup \{v\}$ is allowed if $v \notin \mathbb{P}_{i-1}$ and* $\mathrm{pred}_G(v) \subseteq \mathbb{P}_{i-1}$. *In particular, a pebble can always be placed on an empty source vertex.*

Reversible pebble removal from v: *Letting $\mathbb{P}_i = \mathbb{P}_{i-1} \setminus \{v\}$ is allowed if $v \in \mathbb{P}_{i-1}$ and* $\mathrm{pred}_G(v) \subseteq \mathbb{P}_{i-1}$. *In particular, a pebble can always be removed from a source vertex.*

A reversible pebbling *of G is a sequence of pebble configurations $\mathcal{P} = (\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_t)$ such that $\mathbb{P}_0 = \varnothing$, $\mathbb{P}_t = \{z\}$, and for all $i \in [t]$ it holds that $\mathbb{P}_i$ can be obtained from $\mathbb{P}_{i-1}$ by applying exactly one of the above-stated rules.*

▶ **Definition 3** (Time, space, and price of pebblings). *The* time *of a pebbling $\mathcal{P} = (\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_t)$ is* $\mathsf{time}(\mathcal{P}) := t$ *and the* space *of it is* $\mathsf{space}(\mathcal{P}) := \max_{i \in [t]} |\mathbb{P}_i|$. *The* black pebbling price *(or* number*) of G, denoted by* $\mathsf{Black}(G)$, *is the minimum space of any black pebbling of G, whereas the* reversible pebbling price *of G, which we will denote by* $\mathsf{Rev}(G)$, *is the minimum space of any reversible pebbling of G.*

2.2 Resolution

A *literal* over a Boolean variable x is either x itself (also denoted as x^1) or its negation $\overline{x}$ (also denoted as x^0). A *clause* $C = a_1 \vee \cdots \vee a_\ell$ is a (possibly empty) disjunction of literals a_i over pairwise disjoint variables. The *set of variables occurring in a clause C* will be denoted by

$\mathrm{Vars}(C)$. A clause C is called *unit* if $|\mathrm{Vars}(C)| = 1$. We let $\Box$ denote the contradictory *empty clause* (the clause without any literals). A *CNF formula* $F = C_1 \wedge \cdots \wedge C_m$ is a conjunction of clauses. It is often advantageous to think of clauses and CNF formulas as sets. The notion of the set of variables in a clause is extended to CNF formulas by taking unions. A CNF formula is a k*-CNF*, if all clauses in it have at most k variables. An *assignment/restriction* α for a CNF formula F is a function that maps some subset of $\mathrm{Vars}(F)$ to $\{0, 1\}$. It is applied to F, which we denote by $F\restriction_\alpha$, in the usual way (see e. g. [6, 25]). We denote the *empty assignment* with $\varnothing$.

The standard definition of a *resolution derivation* of a clause D from a CNF formula F (denoted by $\pi : F \vdash D$) is an ordered sequence of clauses $\pi = (C_1, \ldots, C_t)$ such that $C_t = D$, and each clause C_i, for $i \in [t]$, is either an *axiom clause* $C_i \in F$, or is derived from clauses C_j and C_k, with $j, k < i$, by the *resolution rule* $\frac{B\vee x \quad C\vee \overline{x}}{B\vee C}$. In the resolution rule, we call $B \vee x$ and $C \vee \overline{x}$ the *parents* and $B \vee C$ the *resolvent*. A derivation $\pi : F \vdash \Box$ of the empty clause from an unsatisfiable CNF formula F is called *refutation*. Note, that resolution is a sound and complete proof system for unsatisfiable formulas in CNF.

To study space in resolution, we consider the following definitions of the resolution proof system from [15, 1].

▶ **Definition 4** (Configuration-style resolution). *A* resolution refutation $\pi : F \vdash \Box$ *of an unsatisfiable CNF formula F is an ordered sequence of* memory configurations *(sets of clauses) $\pi = (\mathbb{M}_0, \ldots, \mathbb{M}_t)$ such that $\mathbb{M}_0 = \varnothing$, $\Box \in \mathbb{M}_t$ and for each $i \in [t]$, the configuration $\mathbb{M}_i$ is obtained from $\mathbb{M}_{i-1}$ by applying exactly one of the following rules:*

Axiom Download: $\mathbb{M}_i = \mathbb{M}_{i-1} \cup \{C\}$ *for some axiom clause $C \in F$.*

Erasure: $\mathbb{M}_i = \mathbb{M}_{i-1} \setminus \{C\}$ *for some $C \in \mathbb{M}_{i-1}$.*

Inference: $\mathbb{M}_i = \mathbb{M}_{i-1} \cup \{D\}$ *for some resolvent D inferred from $C_1, C_2 \in \mathbb{M}_i$ by the resolution rule.*

The proof π is said to be tree-like*, if we replace the inference rule with the following rule [15]:*

Tree-like Inference: $\mathbb{M}_i = \big(\mathbb{M}_{i-1} \cup \{D\}\big) \setminus \{C_1, C_2\}$ *for some resolvent D inferred from $C_1, C_2 \in \mathbb{M}_i$ by the resolution rule, i. e., we delete both parent clauses immediately.*

To every configurational refutation π we can associate a *refutation-DAG* G_π, with the clauses of the refutation labelling the vertices of the DAG and with edges from the parents to the resolvent for each application of the resolution rule. There might be several different derivations of a clause C during the course of the refutation, but if so, we can label each occurrence of C with a timestamp when it was derived and keep track of which copy of C is used where (cf. [20]). Using this representation, if π is tree-like, then G_π is a tree.

▶ **Definition 5** (Complexity measures for resolution). *The* length *of a resolution refutation $\pi = (\mathbb{M}_0, \ldots, \mathbb{M}_t)$ is defined to be $\mathrm{L}(\pi) := t$.*

The clause space *of a memory configuration $\mathbb{M}$ is defined as $\mathrm{CS}(\mathbb{M}) := |\mathbb{M}|$, i. e., the number of clauses in $\mathbb{M}$. The* variable space *of a memory configuration $\mathbb{M}$ is defined as $\mathrm{VS}(\mathbb{M}) := \big|\bigcup_{C\in\mathbb{M}} \mathrm{Vars}(C)\big|$, i.e., the number of distinct variables mentioned in $\mathbb{M}$.*

The clause space (variable space) of a refutation $\pi = (\mathbb{M}_0, \ldots, \mathbb{M}_t)$ is defined by $\mathrm{CS}(\pi) := \max_{i\in[t]} \mathrm{CS}(\mathbb{M}_i)$ and $\mathrm{VS}(\pi) := \max_{i\in[t]} \mathrm{VS}(\mathbb{M}_i)$, respectively.

Taking the minimum over all refutations of a formula F, we define $\mathrm{L}(F \vdash \Box) := \min_{\pi:F\vdash\Box} \mathrm{L}(\pi)$, $\mathrm{CS}(F \vdash \Box) := \min_{\pi:F\vdash\Box} \mathrm{CS}(\pi)$ and $\mathrm{VS}(F \vdash \Box) := \min_{\pi:F\vdash\Box} \mathrm{VS}(\pi)$ as the length, clause space and variable space of refuting F in resolution, respectively. We define $\text{Tree-CS}(F \vdash \Box) := \min_{\pi':F\vdash\Box} \mathrm{CS}(\pi')$, where the minimum is taken over all tree-like refutations π' of the formula F.

▶ **Proposition 6** ([15]). *Let F be an unsatisfiable formula. Then it holds* $\mathrm{CS}(F \vdash \Box) = \min_{\pi : F \vdash \Box} \mathsf{Black}(G_\pi)$.

Razborov introduced *amortised space measures* for resolution in [24], that penalise configurational proofs for being unreasonably long.

▶ **Definition 7** (Amortised space measures for resolution). *The* amortised clause space *(*amortised variable space*) of a resolution refutation π is defined by* $\mathrm{CS}^*(\pi) := \mathrm{CS}(\pi) \cdot \log \mathrm{L}(\pi)$ *and* $\mathrm{VS}^*(\pi) := \mathrm{VS}(\pi) \cdot \log \mathrm{L}(\pi)$*, respectively.*

Taking the minimum over all refutations of a formula F, we define $\mathrm{CS}^*(F \vdash \Box) := \min_{\pi : F \vdash \Box} \mathrm{CS}^*(\pi)$ *and* $\mathrm{VS}^*(F \vdash \Box) := \min_{\pi : F \vdash \Box} \mathrm{VS}^*(\pi)$.

2.3 Formula Families

Pebbling Formulas and Their XORification

In the last years, there has been renewed interest in pebbling in the context of proof complexity. This is so, because pebbling results can be partially translated into proof complexity results by studying so-called pebbling formulas [6, 5]. These are unsatisfiable CNF formulas encoding the pebble game played on a DAG G. We define them next.

▶ **Definition 8** (Pebbling formulas). *Let $G = (V, E)$ be a DAG with a set of sources $S \subseteq V$ and a unique sink z. We identify every vertex $v \in V$ with a Boolean variable v. The* pebbling contradiction *over G, denoted* Peb_G*, is the conjunction of the following clauses:*

- *for all sources $s \in S$, a unit clause s,* *(*source axioms*)*
- *for all non-source vertices v, the clause* $\bigvee_{u \in \mathrm{pred}_G(v)} \overline{u} \vee v$, *(*pebbling axioms*)*
- *for the unique sink z, the unit clause $\overline{z}$.* *(*sink axiom*)*

Often, it turns out, that the formulas in Definition 8 are a bit too easy to refute. A good way to make them slightly harder is to substitute some suitable Boolean function $f(x_1, \ldots, x_d)$ of arity d for each variable x and expand the result into CNF. This general case is discussed in [20]. We restrict ourselves to the special case of the second degree XORification.

For notational convenience, we assume that the formula F we are trying to make harder only has variables x, y, z, et cetera, without subscripts, so that $x_1, x_2, y_1, y_2, z_1, z_2$, et cetera, are new variables not occurring in F.

▶ **Definition 9** (Substitution formulas, [4]). *For a positive literal x define the* XORification *of x to be* $x[\oplus_2] := \{x_1 \vee x_2, \overline{x_1} \vee \overline{x_2}\}$. *For a negative literal $\overline{y}$, the XORification is* $\overline{y}[\oplus_2] := \{y_1 \vee \overline{y_2}, \overline{y_1} \vee y_2\}$. *The XORification of a clause $C = a_1 \vee \cdots \vee a_k$ is the CNF formula*

$$C[\oplus_2] := \bigwedge_{C_1 \in a_1[\oplus_2]} \cdots \bigwedge_{C_k \in a_k[\oplus_2]} (C_1 \vee \cdots \vee C_k)$$

and the XORification of a CNF formula F is $F[\oplus_2] := \bigwedge_{C \in F} C[\oplus_2]$.

▶ Remark 10 ([4]). If G has n vertices and maximal in-degree ℓ, then $\mathrm{Peb}_G[\oplus_2]$ is an unsatisfiable $2(\ell + 1)$-CNF formula with at most $2^{\ell+1} \cdot n$ clauses over $2n$ variables.

Tseitin Formulas

Tseitin formulas encode the combinatorial principle that for all graphs the sum of the degrees of the vertices is *even*. This class of formulas was introduced in [28] and has been extremely useful for the analysis of proof systems.

▶ **Definition 11** (Tseitin formulas). *Let $G = (V, E)$ be a connected undirected graph and let $\chi\colon V \to \{0,1\}$ be a* marking *of the vertices of G. A marking χ is called* odd *if it satisfies the property $\sum_{v\in V} \chi(v) \equiv 1 \pmod 2$ otherwise it is called* even. *Associate to every edge $e \in E$ a propositional variable e. The CNF formula $\mathrm{PARITY}_{v,\chi(v)}$ states that the parity of the values of the edges that have vertex v as endpoint coincides with $\chi(v)$, i. e.,*

$$\mathrm{PARITY}_{v,\chi(v)} := \bigwedge \left\{ \bigvee_{e\ni v} e^{a(e)} \ :\ a(e) \in \{0,1\},\ \textit{such that} \bigoplus_{e\ni v} (a(e) \oplus 1) \not\equiv \chi(v) \right\}.$$

Then, the Tseitin formula *associated to the graph G and the marking χ is the CNF formula defined by $\mathrm{Ts}(G, \chi) := \bigwedge_{v\in V} \mathrm{PARITY}_{v,\chi(v)}$.*

For a partial truth assignment α, applying α to $\mathrm{Ts}(G, \chi)$ corresponds to the following simplification of the underlying graph: Setting a variable $e = \{u, v\}$ to 0 corresponds to deleting the edge e in the graph, and setting it to 1 corresponds to deleting the edge from the graph and toggling the value of $\chi(u)$ and $\chi(v)$ in G. We denote by $G{\restriction_\alpha}$ and by $\chi{\restriction_\alpha}$ the remaining graph and marking after applying α according to this process.

▶ **Fact 12** ([28, 29, 15]). *Let χ be an odd marking of of a connected graph G and e an edge in G that, when deleted divides G in two connected components G_1 and G_2. Then for $i \in \{1, 2\}$ there is a partial assignment α_i of variable e so that $\chi{\restriction_{\alpha_i}}$ is an odd marking of G_i.*

2.4 Combinatorial Games for Tree-Like Clause Space in Resolution

Important tools for our results are two two-player combinatorial games. The Prover-Delayer game is played on *formulas* and was introduced in [22] in order to prove lower bounds for tree-like resolution length. Later it was shown in [16] that the game exactly characterises tree-like resolution space. The Raz–McKenzie game is played on *DAGs* and was introduced in [23] as a tool for studying the depth complexity of decision trees for search problems.

▶ **Definition 13** (Prover-Delayer game, [22, 16, 3]). *The* Prover-Delayer game *is a game between two players, called* Prover *(he), and* Delayer *(she), played on an unsatisfiable CNF formula F. The game is played in rounds. Each round starts with Prover querying the value of a variable. Delayer can give one of three answers: 0, 1, or $*$. If 0 or 1 is chosen by Delayer, no points are scored by her and the queried variable is set to the chosen bit. If Delayer answers $*$, then Prover gets to decide the value of that variable, and Delayer scores one point. The game finishes when any clause in F has been falsified by the partial assignment constructed this way. If this is not the case, the next round begins. The aim of Delayer is to win as many points as possible, while Prover aims to minimise this quantity.*

▶ **Definition 14** (Game value of the Prover-Delayer game). *Let F be an unsatisfiable CNF formula. The game value of the Prover-Delayer game played on F, denoted by $\mathsf{PD}(F)$, is the greatest number of points Delayer can score on F against an optimal strategy of Prover.*

The Prover-Delayer game exactly characterises the tree-like clause space of a formula. The constant term of the original result in [16, Theorem 2.2] was slightly modified to match our definitions of clause space and the pebble game (without sliding).

▶ **Theorem 15** ([16]). *If F is an unsatisfiable CNF formula,* $\text{Tree-CS}(F \vdash \Box) = \mathsf{PD}(F) + 2$.

▶ **Definition 16** (Raz–McKenzie game). *The* Raz–McKenzie game *is played on a single-sink DAG G by two players,* Pebbler *and* Colourer. *The game is played in rounds. In the first round, Pebbler places a pebble on the sink and Colourer colours it red. In all subsequent* *rounds, Pebbler places a pebble on an arbitrary empty vertex of G and Colourer colours this new pebble either red or blue. The game ends when there is a vertex with a red pebble that is either a source vertex or all its direct predecessors in the graph have blue pebbles.*

▶ **Definition 17** (Raz–McKenzie price). *The* Raz–McKenzie price $\mathsf{R\text{-}Mc}(G)$ *of a single sink DAG G is the smallest number r such that Pebbler has a strategy to make the game end in at most r rounds against an optimal strategy of Colourer.*

▶ **Theorem 18** ([10]). *For any single-sink DAG G we have* $\mathsf{R\text{-}Mc}(G) = \mathsf{Rev}(G)$.

3 Separations Between Tree-Like and General Resolution Space for Pebbling Formulas Using the Raz–McKenzie Game

We will now establish a connection between tree-like clause space in resolution and the Raz–McKenzie price. We simplify the proof by following the intuition behind the game and identify the colour blue with 1 and the colour red with 0.

▶ **Theorem 19.** *For any single-sink DAG G it holds*

$$\mathsf{R\text{-}Mc}(G) + 2 \leq \text{Tree-CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) \leq 2 \cdot \mathsf{R\text{-}Mc}(G) + 2.$$

Proof. Let G be a fixed DAG with a unique sink. We prove that $\mathsf{R\text{-}Mc}(G) \leq \mathsf{PD}(\mathrm{Peb}_G[\oplus_2])$ and $\mathsf{PD}(\mathrm{Peb}_G[\oplus_2]) \leq 2 \cdot \mathsf{R\text{-}Mc}(G)$. The result then follows from Theorem 15. We first show the inequality $\mathsf{PD}(\mathrm{Peb}_G[\oplus_2]) \leq 2 \cdot \mathsf{R\text{-}Mc}(G) =: 2r$ by giving a strategy for Prover, such that Delayer can score at most $2r$ points. Prover basically simulates the strategy of Pebbler in the Raz–McKenzie game: If Pebbler pebbles a vertex v of G, Prover will query the variables v_1 and v_2 of $\mathrm{Peb}_G[\oplus_2]$ in this order. The Raz–McKenzie game ends after at most r rounds. We will argue, that the Prover-Delayer game also ends after at most $2r$ queries. Thus, Delayer only gets a chance to score $2r$ points (if a variable pair gets queried for the first time, she can always answer $*$; only the second variable of the pair matters due to the XORification). In case the second variable of a pair gets queried, the best choice Delayer has is to follow the strategy of Colourer (Colourer is following an optimal strategy, thus, if Delayer had a better answer, this would correspond to a better answer for Colourer) and to ensure that $v_1 \oplus v_2$ is true under her constructed assignment if v is coloured 1; and false if v is coloured 0. At the end of the Raz–McKenzie game either a source vertex s in G is coloured 0, or a vertex v of G is coloured 0, while all its direct predecessors are coloured 1. In the first case, the source s being coloured 0 leads to the falsification of the corresponding source axiom $s[\oplus_2]$ by Delayer. In the second case, Delayer will falsify a clause of the corresponding pebbling axiom $\big(\bigwedge_{u \in \mathrm{pred}_G(v)} \overline{u} \vee v\big)[\oplus_2]$.

Next, we show the inequality $\mathsf{PD}(\mathrm{Peb}_G[\oplus_2]) \geq \mathsf{R\text{-}Mc}(G) =: r$ by giving a strategy for Delayer, such that under any strategy of Prover, she scores at least r points. By Definition 17, there is a strategy of Colourer, such that Pebbler has to pebble r vertices to end the game. Delayer will essentially copy this strategy: The first time a variable pair gets queried, she can answer $*$. The second time, she can copy the response of Colourer. Thus, she scores at least r points. ◀

From the equivalence between the Raz–McKenzie game and reversible pebbling we get:

▶ **Corollary 20.** *It holds* $\mathsf{Rev}(G) + 2 \leq \text{Tree-CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) \leq 2 \cdot \mathsf{Rev}(G) + 2$ *for all DAGs G with a unique sink.*

The result that for any DAG G it holds $\mathrm{CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) = \mathrm{O}\big(\mathsf{Black}(G)\big)$ is considered as folklore (the idea behind it is that the pebbling formula can be resolved following the order in which the vertices of the graph are being pebbled). Combining this fact with Corollary 20, it follows that for any graph G with a gap between its black and reversible pebbling prices, the same separation can be obtained between the general and tree-like clause space of the corresponding pebbling formula. We mention some examples for which such a separation is known:

- The *path graphs*. Consider P_n to be a directed path with n vertices. Bennett [7] noticed, that these graphs provide a separation between black and reversible pebbling, proving that $\mathsf{Rev}(P_n) = \lceil \log n \rceil$. It was shown in [18], using a direct proof, that $\mathrm{CS}(\mathrm{Peb}_{P_n}[\oplus_2] \vdash \Box) = \mathrm{O}(1)$ while $\text{Tree-CS}(\mathrm{Peb}_{P_n}[\oplus_2] \vdash \Box) = \Theta(\log n)$.
- The *road graphs* from [11] provide a class of graphs for which the black pebbling price is non-constant and the reversible pebbling number is larger by a logarithmic factor.

▶ **Theorem 21** ([11]). *For any function $s(n) = \mathrm{O}\big(n^{1/2-\varepsilon}\big)$ with $0 < \varepsilon < \frac{1}{2}$ constant there is a family of DAGs $(G_n)_{n=1}^{\infty}$ of size $\Theta(n)$ with a single sink and maximal in-degree 2 such that $\mathsf{Black}(G_n) = \mathrm{O}\big(s(n)\big)$ and $\mathsf{Rev}(G_n) = \Omega\big(s(n)\log n\big)$.*

▶ **Corollary 22.** *For any function $s(n) = \mathrm{O}\big(n^{1/2-\varepsilon}\big)$ with $0 < \varepsilon < \frac{1}{2}$ constant there is a family of pebbling formulas $(\mathrm{Peb}_{G_n}[\oplus_2])_{n=1}^{\infty}$ with $\Theta(n)$ variables such that $\mathrm{CS}(\mathrm{Peb}_{G_n}[\oplus_2] \vdash \Box) = \mathrm{O}\big(s(n)\big)$ and $\text{Tree-CS}(\mathrm{Peb}_{G_n}[\oplus_2] \vdash \Box) = \Omega\big(s(n)\log n\big)$.*

The logarithmic factor in the number of vertices is almost the largest separation that can be obtained using this method since it is known that the reversible pebbling price can be upper bounded in terms of black pebbling space and time:

▶ **Theorem 23** ([19]). *If a DAG G has a black pebbling of time t and space s, the graph G has a reversible pebbling price of at most $s\lceil \log t \rceil$.*

By virtue of this result and Corollary 20 we obtain:

▶ **Corollary 24.** *For any DAG G with a unique sink vertex it holds*

$$\text{Tree-CS}(\mathrm{Peb}_G[\oplus_2] \vdash \Box) = \mathrm{O}\left(\min_{\mathcal{P}}\big(\mathsf{space}(\mathcal{P}) \cdot \log \mathsf{time}(\mathcal{P})\big)\right),$$

where the minimum is taken over all black pebblings $\mathcal{P}$ of G.

This shows that the given separations cannot be improved for graphs for which the minimum black pebbling space is obtained with a one-shot strategy as it is the case for the path and road graphs, since the pebbling time for such a strategy is n. We present the first graph class for which the best pebbling strategy is not one-shot with a separation between black and reversible pebbling space. We do not obtain, however, any better separation than the $\log n$ factor obtained in the previous examples. We conjecture that this is in fact optimal. Our graphs $\hat{G}(c,k)$ are simplified versions of the original Carlson–Savage graphs [9]. Another adaptation of the original graphs is the family $\Gamma(c,r)$ studied in [20], for which an upper bound on the reversible pebble price was recently shown in [14]. We have simplified the graphs, eliminating the original pyramids since we are not analysing the black-white pebbling price, but our lower bound on reversible pebbling can be adapted to the original graphs or those in the family $\big(\Gamma(c,r)\big)_{c,r=1}^{\infty}$.

▶ **Definition 25** (Simplified Carlson–Savage graphs). *The class of DAGs $\big(G(c,k)\big)_{c,k=1}^{\infty}$ with parameters $c,k \geq 1$ is inductively defined in k. The base case $G(c,1)$ is the graph with one source node connected to c sink nodes. The graph $G(c,k+1)$ is composed of the graph*

$G(c,k)$ and c spines. A spine *is just a path of length $2c^2k$. The last node of each of the spines is a sink for $G(c,k+1)$. A spine is divided into $2ck$* sections *of c consecutive vertices each. For each section and for each i with $1 \leq i \leq c$, there is an edge from the i-th sink of $G(c,k)$ to the i-th vertex in the section. In order to have single sink graphs, for $k \geq 2$ we also define $\hat{G}(c,k)$ exactly as $G(c,k)$ but with just one spine at the k-th level (all other levels have c spines). The last vertex of this spine is the only sink of $\hat{G}(c,k)$. For all c, the graph $\hat{G}(c,1)$ consists of just one edge.*

▶ **Lemma 26.** *The following claims hold:*
(i) *$\hat{G}(c,k)$ has $\Theta(c^3k^2)$ vertices,*
(ii) $\mathsf{Black}\big(\hat{G}(c,k)\big) \leq k+1$ *for any $c,k \geq 1$, while*
(iii) $\mathsf{Rev}\big(\hat{G}(c,k)\big) \geq \min\big\{c,(k-1)\log c + \log(k!)\big\}$ *for any $c,k \geq 1$.*

Proof. The first part follows easily by inductive counting.

For part (ii) of the lemma, we show inductively over k that any sink of $G(c,k)$ can be pebbled using $k+1$ pebbles. The result follows since $\hat{G}(c,k)$ is a subgraph of $G(c,k)$. The claim is trivial for $k=1$. For bigger values of k, the first vertex in any of the spines in $G(c,k)$ can be pebbled by placing a pebble on the corresponding sink of $G(c,k-1)$, removing all the pebbles except this one, and then pebbling the first vertex in the spine. The following strategy can be used for any other vertex v in the spine once its direct predecessor in the spine is pebbled: remove all the pebbles in the graph except the one on the direct spine predecessor of v, pebble the sink connected to v in $G(c,k-1)$, remove all the pebbles except the 2 on the direct predecessors of v, and then place a pebble on v. For this, by the induction hypothesis, at most $k+1$ pebbles are needed.

Part (iii) is more involved. We use the equivalence between reversible pebbling and the Raz–McKenzie game and show, also by induction over k, that the number of rounds to finish a game on $\hat{G}(c,k)$ starting from a configuration in which less than c vertices have been coloured blue, and no vertex in the unique spine of $\hat{G}(c,k)$ (except the sink) is coloured, is at least $\min\big\{c,(k-1)\log c + \log(k!)\big\}$. We give a strategy for Colourer obtaining this bound on the number of rounds. The base case is trivial. For $k \geq 2$, initially the only vertex coloured red is the unique sink of $\hat{G}(c,k)$. Let us denote the unique spine from $\hat{G}(c,k)$ as the *k-spine.* The game is divided in k stages (starting at stage k and finishing at stage 1). Stage k finishes when there is a blue vertex in the k-spine at a distance less than $2c$ from a red vertex. In stage k, if Colourer gives the colour red to a vertex v, this vertex has to be in the k-spine. If some vertex in $G(c,k-1)$ is queried by Pebbler, Colourer always answers with the blue colour. Because of this, the game cannot finish before the end of stage k. For simplicity we may assume that the first vertex of the k-spine has been coloured blue (for free, this can only make the strategy of Colourer harder), also for the clarity of exposition let us say that the k-spine is directed from left to right. The strategy of Colourer on the k-spine is to keep the gap between the rightmost blue vertex a (initially the initial node of the spine) and the leftmost red vertex b (initially the sink) as large as possible. That is, for any queried vertex v in the k-spine, if v lies at the left of a, it is coloured blue, if it is at the right of b it is coloured red and otherwise (i. e., if v is between a and b) if the distance from a to v is smaller that or equal to the distance from v to b, then v is coloured blue, otherwise it is coloured red. This strategy is followed by Colourer as long as the gap between a and b is at least $2c$. Once it is smaller than $2c$, stage k ends. If at this moment at least c vertices have been queried, there have been at least c rounds and the result follows. Otherwise there has to be a spine in $G(c,k-1)$ without any coloured vertex on it (there are c spines). Let us call t the sink of this spine and t' its rightmost uncoloured successor in the k-spine. We can suppose that at this moment Colourer colours (for free) t, t' as well as all uncoloured vertices to the right of t' in the k-spine with colour red, and all the uncoloured vertices to the left of t' in the k-spine with blue. Again this only makes the strategy of Colourer harder since we are not counting these rounds. But now the game has been reduced to the instance of the graph $\hat{G}(c, k-1)$ containing the sink t. The number of rounds in stage k is at least $\log(\frac{2c^2k}{2c}) = \log c + \log k$ (this would happen with a binary search strategy of Pebbler on the k-spine). If in all the stages less than c vertices are queried, by induction, the rounds to finish the game on $\hat{G}(c, k-1)$ are at least $(k-2)\log c + \log\big((k-1)!\big)$. Adding these rounds to those from stage k we get the result. ◀

▶ **Theorem 27.** *For any function $s(n) = \Theta\big(n^{1/5-\varepsilon}\big)$ with $0 < \varepsilon < \frac{1}{5}$ constant there is a family of pebbling formulas $(\mathrm{Peb}_{G_n}[\oplus_2])_{n=1}^{\infty}$ with $\mathrm{O}(n)$ variables such that $\mathrm{CS}(\mathrm{Peb}_{G_n}[\oplus_2] \vdash \Box) = \mathrm{O}\big(s(n)\big)$ and $\text{Tree-CS}(\mathrm{Peb}_{G_n}[\oplus_2] \vdash \Box) = \Omega\big(s(n)\log n\big)$, and the best strategy for pebbling the graphs G_n is not one-shot.*

Proof. We show that for any such function s there is a graph family $\big(\hat{G}\big(c(n), \lceil s(n)\rceil\big)\big)_{n=1}^{\infty}$ with the corresponding gap between its black and reversible pebbling prices. The result follows from Corollary 20.

Given any such space function $s(n) = \Theta\big(n^{1/5-\varepsilon}\big)$ with $0 < \varepsilon < \frac{1}{5}$ constant, we define $c(n) := \lceil s(n) \cdot \log n\rceil$. This allows us to consider the graphs $\hat{G}\big(c(n), \lceil s(n)\rceil\big)$. By Lemma 26 (i), this graph has $\mathrm{O}\big(c(n)^3 \cdot \lceil s(n)\rceil^2\big) = \mathrm{O}\big(s(n)^5 \cdot \log^3 n\big) = \mathrm{O}\big(n^{1-5\varepsilon} \cdot \log^3 n\big) = \mathrm{O}(n)$ vertices. By Lemma 26 (ii), the graph has a black pebbling number upper bounded by $\lceil s(n)\rceil + 1 = \mathrm{O}(s(n))$. It only remains to show, that the reversible pebbling number of the graph is asymptotically lower bounded by $s(n)\log n$. For this, we consider two cases.

Case 1: $\min\Big\{c(n), \big(s(n)-1\big)\log c(n) + \log\big(s(n)!\big)\Big\} = c(n)$. In this case, Lemma 26 (iii) implies, that the reversible pebbling number of the graph is lower bounded by $c(n)$, which, by definition, is greater than or equal to $s(n)\log n$.

Case 2: $\min\Big\{c(n), \big(s(n)-1\big)\log c(n) + \log\big(s(n)!\big)\Big\} = \big(s(n)-1\big)\log c(n) + \log\big(s(n)!\big)$. In this case, one can notice, that already the first term, i. e., $\big(s(n)-1\big)\log c(n)$ is in

$$\begin{aligned}\Omega\Big(\big(s(n)-1\big)\log\big(s(n)\log n\big)\Big) &= \Omega\Big(\big(s(n)-1\big)\log\big(s(n)\big) + \big(s(n)-1\big)\log\log n\Big)\\ &= \Omega\Big(\big(s(n)-1\big)\big(1/5-\varepsilon\big)\log n + \big(s(n)-1\big)\log\log n\Big) = \Omega\big(s(n)\log n\big).\end{aligned}$$

◀

4 Upper Bounds for Tree-CS for General Formulas

Next, we provide generalisations of Corollary 24 for general formulas.

▶ **Theorem 28.** *For any unsatisfiable formula F it holds*

$$\text{Tree-CS}(F \vdash \Box) \le \mathrm{VS}^*(F \vdash \Box) + 2 = \min_{\pi : F \vdash \Box}\big(\mathrm{VS}(\pi)\cdot\log \mathrm{L}(\pi)\big) + 2.$$

Proof of Theorem 28. Consider a configurational refutation $\pi = (\mathbb{M}_0, \ldots, \mathbb{M}_t)$ of F. Let α be the current partial assignment constructed in the Prover-Delayer game played on the formula F. At the beginning we have $\alpha = \varnothing$. We give a strategy for Prover that allows him to finish the game with at most $\mathrm{VS}(\pi)\cdot\log \mathrm{L}(\pi)$ points scored by Delayer regardless of her answers. The strategy of Prover proceeds in *bisection steps* indexed with k. Prover keeps as an invariant in these steps an interval $I_k = [a_k, b_k] \subseteq [0, t]$ such that $\pi_{[a_k,b_k]}\restriction_\alpha := \big(\mathbb{M}_{a_k}\restriction_\alpha, \ldots, \mathbb{M}_{b_k}\restriction_\alpha\big)$ is a configurational refutation of $F\restriction_\alpha$ for all k. Initially, $I_0 := [0,t]$, thus $\pi_{[0,t]}\restriction_\varnothing = \pi$ is obviously a refutation of $F\restriction_\varnothing = F$. In each bisection step, Prover starts querying the variables present in the configuration $\mathbb{M}_{m_k}$, with $m_k = \lfloor \frac{a_k+b_k}{2} \rfloor$, that have not been assigned yet, in any order. If Delayer answers $*$ to some variable, Prover will assign 0 to it (actually, Prover could assign any value). In this way α is extended to all the variables in the configuration $\mathbb{M}_{m_k}$. Prover then proceeds according to the following cases:

(i) If after the assignment to the queried variables, a clause in the configuration $\mathbb{M}_{m_k}$ is falsified, Prover continues with the upper half of the proof (i. e., he sets $I_{k+1} = [a_{k+1}, b_{k+1}] := [a_k, m_k]$) and proceeds with the next bisection step.

(ii) If after the assignment to the queried variables, all the clauses in $\mathbb{M}_{m_k}$ are satisfied, Prover continues with the lower half of the proof (i. e., he sets $I_{k+1} = [a_{k+1}, b_{k+1}] := [m_k, b_k]$) and proceeds with the next bisection step.

Prover queries at most $\mathrm{VS}(\pi)$ variables in each bisection step. It remains to show that the invariant is indeed kept and that Prover wins the game by following this strategy.

First, we show inductively, that the invariant is kept. In case (i) this is true by following the Resolution Restriction Lemma (see e.g. [25]) because $\mathbb{M}_{b_{k+1}}\restriction_\alpha = \mathbb{M}_{m_k}\restriction_\alpha$ contains the empty clause and thus $(\mathbb{M}_{a_{k+1}}\restriction_\alpha, \ldots, \mathbb{M}_{m_k}\restriction_\alpha)$ is a configurational refutation of $F\restriction_\alpha$. In case (ii) we have $\mathbb{M}_{m_k}\restriction_\alpha = \varnothing$ and $\mathbb{M}_{b_k}\restriction_\alpha \ni \Box$ by the induction hypothesis, yet π was a refutation for F. Hence, for $i \in (a_k, b_k)$ the axioms contained in the memory configurations $\mathbb{M}_i\restriction_\alpha$ must be downloaded from $F\restriction_\alpha$. Thus, $(\mathbb{M}_{a_{k+1}}\restriction_\alpha, \ldots, \mathbb{M}_{b_{k+1}}\restriction_\alpha)$ is a legal refutation of $F\restriction_\alpha$.

Prover has to win the game since for every k, the formula $F\restriction_\alpha$ has a configurational refutation, namely $\pi_{I_k}\restriction_\alpha$, of length upper bounded by $\frac{1}{2}\mathrm{L}(\pi_{I_{k-1}})$. The strategy proceeds until $F\restriction_\alpha$ has a configurational refutation of length 1. Then, $\Box \in F\restriction_\alpha$. In other words, the constructed assignment α falsifies a clause in F and Prover wins the game.

Summarising, Prover queries at most $\mathrm{VS}(\pi)$ variables in each bisection step and since there are at most $\lceil \log \mathrm{L}(\pi) \rceil$ configurations that get queried, Prover in total queries at most $\mathrm{VS}(\pi) \cdot \log \mathrm{L}(\pi)$ variables. Theorem 15 yields the desired inequality. ◀

We prove now that Theorem 28 also works for clause space. For this, we show that the tree-like clause space of a formula F is always upper bounded by the reversible pebble game played on a refutation of F. Note, that the minimum in the theorem is taken over all possible refutations of F, not only over the tree-like ones. The inequality in Theorem 29 works only in one direction. For example the formula with a clause with n negated variables and n unit clauses containing one of the variables each, has constant tree-resolution space while the reversible pebbling price for any refutation graph is at least $\log n$.

▶ **Theorem 29.** *For any unsatisfiable formula F with n variables it holds*

$$\text{Tree-CS}(F \vdash \Box) \le \min_{\pi: F \vdash \Box} \mathsf{Rev}(G_\pi) + 2, \quad \textit{and}$$

$$\min_{\pi: F \vdash \Box} \mathsf{Rev}(G_\pi) \le \text{Tree-CS}(F \vdash \Box)\big(\lceil \log n \rceil + 1\big).$$

Proof. Let F be an unsatisfiable formula with n variables.

For proving the first inequality, let π be a resolution refutation of F with a refutation-graph G_π and $\mathsf{Rev}(G_\pi) =: k$. We will use Theorem 15, as well as Theorem 18 applied to G_π: It suffices to give a strategy for Prover in the Prover-Delayer game played on F under which he has to pay at most k points. Prover basically simulates the strategy of Pebbler in the Raz–McKenzie game played on G_π, which coincides with reversible pebbling. By doing so, a partial assignment α falsifying an initial clause of F will be produced. The game is divided in stages. Initially the partial assignment is the empty assignment. In each stage, if Pebbler chooses a clause $C \in V(G_\pi)$, Prover queries the variables in C not yet assigned by α one by one, extending the partial assignment α with the answers of Delayer, until either:

(i) the clause C is satisfied or falsified by α, or

(ii) a variable x in C is given value $*$ by Delayer.

In case (i), Prover moves to the next stage, simulating the strategy of Pebbler assuming Colourer has given clause C the colour $C\restriction_\alpha$. In case (ii), Prover extends α by assigning x with the value that satisfies C and moves to the next stage, simulating the strategy of Pebbler, assuming Colourer has given clause C the colour 1. The game is played until α falsifies a clause in F. After at most k stages the Raz–McKenzie games finishes and therefore Delayer can score at most k points. It is only left to show that at the end of the game a clause in F is falsified by α. When the Raz–McKenzie game finishes, either a source in G_π is assigned colour 0 by Colourer, or a vertex with all its direct predecessors being coloured 1 is coloured 0. Since α defines Colourer's answers, the first situation corresponds to α falsifying a clause in F. The second situation is not possible since for any partial assignment α it cannot be that α satisfies two parent clauses in a resolution proof, while falsifying their resolvent.

For the proof of the second inequality, let $k := \text{Tree-CS}(F \vdash \Box)$. By Proposition 6, we know that there is a refutation π of F whose underlying graph G_π is a tree with black pebbling price k. We can suppose that the refutation is *regular*, that is, in every path from the empty clause to a clause in F in the refutation tree, each variable is resolved at most once [15, Theorem 5.1]. This implies that the depth of the tree is at most n. For any node v in the refutation tree let T_v be the subtree of G_π rooted at v. For the sake of convenience, we refer to $\mathsf{Black}(T_v)$ as the *pebbling number of* v.

We show by induction on κ that for any vertex v in G_π, if $\mathsf{Black}(T_v) = \kappa$ then there is a strategy for Pebbler in the Raz–McKenzie game on T_v with most $\kappa(\lceil \log n \rceil + 1)$ rounds. For the base case $\kappa = 1$, the vertex v must be a leaf node and the game needs only one round. For $\kappa > 1$, the game starts, according to the rules, by Pebbler querying the root v of the subtree and Colourer answering 0. We consider two cases, depending on whether for both predecessors v_1 and v_2 of v in G_π, $\mathsf{Black}(T_{v_1}) = \mathsf{Black}(T_{v_2}) = \kappa - 1$ or not. In the former case, Pebbler queries one of them, say v_1. If the answer is 0, he continues on T_{v_1} and otherwise continues on T_{v_2}. By induction, the number of rounds in this case is at most $2 + (\kappa - 1)(\lceil \log n \rceil + 1) \leq \kappa(\lceil \log n \rceil + 1)$. In case, it is not true, that $\mathsf{Black}(T_{v_1}) = \mathsf{Black}(T_{v_2}) = \kappa - 1$, since $\mathsf{Black}(T_v) = \kappa$, and G_π is a tree, one of the trees T_{v_1} or T_{v_2} leading to v must have pebbling number κ and the other one must have pebbling number smaller than κ. Pebbler considers the path of nodes starting at v and going towards the leaves, having all the nodes in the path pebbling number κ, until a node u is reached, for which both predecessors have pebbling number $\kappa - 1$. Such a node u must exist because G_π is a tree. Let u_1 be one of the predecessors of u. The length of the path from v to u_1 is at most n since the refutation is regular. Pebbler queries the vertices in the path between v and u_1 with binary search, until a vertex t is found that is coloured with colour 0 by Colourer, while its predecessor in the path $v \rightsquigarrow u_1$ has been coloured 1. At this point, Pebbler continues playing the game on the tree rooted at the uncoloured predecessor of t. It is also possible that all the queried nodes in the path from v to u_1 (including u_1) are coloured 0 by Colourer. In this case Pebbler continues with T_{u_1}. In all situations at most $1 + \lceil \log n \rceil$ vertices have been queried and the game has been reduced to a subgraph with smaller pebbling number. ◀

▶ **Corollary 30.** *For any unsatisfiable formula F it holds*

$$\text{Tree-CS}(F \vdash \Box) \leq \text{CS}^*(F \vdash \Box) + 2 = \min_{\pi : F \vdash \Box} \big(\text{CS}(\pi) \cdot \log \text{L}(\pi)\big) + 2.$$

Proof. By Theorem 23, $\min_{\pi:F\vdash\Box}\mathsf{Rev}(G_\pi)+2 \leq \min_{\mathcal{P}}\big(\mathsf{space}(\mathcal{P})\cdot\log\mathsf{time}(\mathcal{P})\big)+2$, where the minimum is taken over all black pebblings $\mathcal{P}$ of G_π. The result follows with (a slight adaption of) Proposition 6 since every black pebbling $\mathcal{P}$ of G_π defines a configurational refutation of F with clause space equal to $\mathsf{space}(\mathcal{P})$ and length $\mathsf{time}(\mathcal{P})$. ◀

5 Optimal Separations for Tseitin Formulas

In this section, we prove optimal separations between tree-like clause space and variable space, as well as clause space in the context of Tseitin formulas. This complements the relations between clause space and variable space of Tseitin formulas recently given in [17].

▶ **Theorem 31.** *For any connected graph G with n vertices and odd marking χ we have*

$$\begin{aligned}&\text{Tree-CS}\big(\text{Ts}(G,\chi)\vdash\Box\big) \leq \text{CS}\big(\text{Ts}(G,\chi)\vdash\Box\big)\cdot\log n+2, \quad \textit{and}\\ &\text{Tree-CS}\big(\text{Ts}(G,\chi)\vdash\Box\big) \leq \text{VS}\big(\text{Ts}(G,\chi)\vdash\Box\big)\cdot\log n+2.\end{aligned}$$

Proof. The proof is based on the one for the lower bound for CS of Tseitin formulas from [27]. Let $G=(V,E)$ be a connected graph with n vertices, χ an odd marking, and $\pi=(\mathbb{M}_0,\ldots,\mathbb{M}_t)$ a refutation of $\text{Ts}(G,\chi)$ with $\text{CS}(\pi)=:k$. We use π to give a strategy for Prover in the Prover-Delayer game for which he has to pay at most $k\log n$ points. We say that a partial assignment α of some of the variables in $\text{Ts}(G,\chi)$ is *non-splitting* if after applying α to the formula, the resulting graph still has a connected component with an odd marking (*odd component*) of size at least $\lceil\frac{|V|}{2}\rceil$, and the rest are components with even markings. Consider the last configuration $\mathbb{M}_s$ in π for which there is a partial assignment α fulfilling:

(i) α simultaneously satisfies all clauses in $\mathbb{M}_s$, and
(ii) α is non-splitting.

This stage must exist since before the initial step the empty truth assignment is trivially a non-splitting partial assignment satisfying the clauses in $\mathbb{M}_0=\varnothing$. At the end, the last configuration $\mathbb{M}_t$ in the refutation contains the empty clause which cannot be satisfied by any assignment. Thus, stage s must exist in between.

The step from s to $s+1$ was no deletion step (otherwise this would be a contradiction to the maximality of s). The only new clause in $\mathbb{M}_{s+1}$ must be an axiom C of $\text{Ts}(G,\chi)$ since any other clause that could be added to the list of clauses in memory at stage $s+1$ would be a resolvent of two clauses from stage s, but in this case any partial assignment satisfying the clauses at stage s would also satisfy those at $s+1$. For some vertex v in G, this axiom clause C introduced at stage $s+1$ belongs to the formula $\text{PARITY}_{v,\chi(v)}$. Let α be a partial assignment of minimal size satisfying the conditions at stage s. It is possible to extend α to satisfy the clause $C\restriction_\alpha$ since v either belongs to an even component in $(G\restriction_\alpha,\chi\restriction_\alpha)$ or to the large odd component in this graph and therefore $C\restriction_\alpha\neq\Box$. Because of this, vertex v must belong to the unique odd component since otherwise α could be extended in a non-splitting way.

Let $C\restriction_\alpha=(\ell_1,\ldots,\ell_m)$, $m\geq 1$, where the ℓ_i's are literals corresponding to the edges with endpoint v in $G\restriction_\alpha$. Observe that deleting any of these edges e_i in $C\restriction_\alpha$ cuts the connected component of v in two pieces because otherwise assigning any value to the corresponding edge would not modify the size of the connected components in $C\restriction_\alpha$ and there would be a non-splitting way to extend α to e satisfying C. Also, any component remaining after assigning all the literals in $C\restriction_\alpha$ must have size at most $\lfloor\frac{|V|}{2}\rfloor$ since otherwise there would be a way to extend α satisfying C and producing an odd marking for the largest such component (Fact 12), and this extension would be non-splitting.

The strategy of Prover is to query the variables assigned in α thus paying at most $k-1$ points and obtaining a partial assignment γ from Delayer. If at this point one of the connected components of size at most $\lfloor \frac{|V|}{2} \rfloor$ is odd then Prover moves to this component and starts playing the game on it. Otherwise Prover queries the variables in $C\restriction_\gamma$ one by one. If for a variable e_i Delayer answers with $*$, Prover just has to assign e_i so that the smallest of the two components that appear in $G\restriction_\gamma$ after assigning e_i is odd (not necessarily satisfying $C\restriction_\gamma$). This is always possible because of Fact 12. If no $*$ is answered, Prover queries the next variable until no variable in $C\restriction_\gamma$ is left. Let γ' be the assignment obtained this way. Either γ' falsifies C and the game ends, or it satisfies the clause and in this case all the components (odd or even) remaining after applying γ' have size at most $\lfloor \frac{|V|}{2} \rfloor$. In every case, after applying γ' Prover wins the game or there is an odd connected component of size at most half as large as the initial graph. The original problem has been reduced to another in a graph with at most $\frac{n}{2}$ many vertices. Also Prover has to pay at most k for obtaining γ'. After repeating this process at most $\log n$ times, an initial clause is falsified.

The second part of the theorem is a little simpler and follows by considering a configurational proof π of variable space k. Everything in the above proof works in the same way, observing that the partial assignment α satisfying all clauses in memory at stage s, when extended to all the variables in the new clause at stage $s+1$ needs to assign at most k variables (all those included in the configuration) and is either splitting or falsifies the axiom. Observe that this implies that in every configurational proof there is a point in which every assignment to the variables in the configuration is spliting. ◀

Next, we show, that the upper bounds in Theorem 31 are tight by proving that there is a family of Tseitin formulas that provide matching lower bounds. These are formulas corresponding to grid graphs with constant width, which can be considered as the Tseitin version of path graphs.

▶ **Definition 32** (Grid graphs). *For a natural number $\ell \geq 1$, the* grid graphs $G_{2\times\ell}$ *are given by the vertex set $V(G_{2\times\ell}) := [2] \times [\ell]$ and the edge set*

$$E(G_{2\times\ell}) := \Big\{ \big\{(i,j),(i',j')\big\} \,:\, i,i' \in [2],\ j,j' \in [\ell],\ \text{and}\ |i-i'|+|j-j'|=1 \Big\}.$$

▶ **Theorem 33.** *For the family of Tseitin formulas $\big(\mathrm{Ts}(G_{2\times\ell},\chi_\ell)\big)_{\ell=1}^{\infty}$ with $3\ell-2$ variables it holds* $\text{Tree-CS}\big(\mathrm{Ts}(G_\ell,\chi_\ell)\vdash\Box\big) = \Theta(\log \ell)$*, and* $\mathrm{CS}\big(\mathrm{Ts}(G_\ell,\chi_\ell)\vdash\Box\big) = \mathrm{O}(1)$*, as well as* $\mathrm{VS}\big(\mathrm{Ts}(G_\ell,\chi_\ell)\vdash\Box\big) = \mathrm{O}(1)$.

Proof. To show the lower bound on tree-like clause space with Theorem 15, we give a strategy for Delayer such that he scores $\Omega(\log \ell)$ points playing on $G_{2\times\ell}$. In the following, for a subgraph G' of $G_{2\times\ell}$, we define $\mathrm{Block}(G') := \max\big\{b \in \mathbb{N} \,:\, \text{there is a subgraph of } G' \text{ that is isomorphic to } G_{2\times b}\big\}$. The strategy of Delayer is as follows:

(a) If an edge e in an even component is queried, Delayer should answer according to some assignment satisfying this component.

(b) If an edge e in an odd component is queried, Delayer proceed as follows:

  (i) If the deletion of e does not increase the number of connected components in G, Delayer should answer $*$.

  (ii) If the deletion of e cuts the graph and both endpoints of e are separated in different connected components, Delayer should answer in a way, that from these two components, the component G' with largest $\mathrm{Block}(G')$ receives the odd marking.

At the beginning of the Prover-Delayer game we have $\mathrm{Block}(G_{2\times\ell}) = \ell$. After each assignment of a variable in the game we have $\mathrm{Block}(G') \geq \lfloor \frac{1}{2}\,\mathrm{Block}(G) \rfloor$, where we let G denote the underlying graph before the assignment and G' the graph after the assignment: Notice, that rule (b)(ii) guarantees that the component with the largest Block-value always receives an odd marking. If Delayer plays according to this strategy, we must have $\mathrm{Block}(G) = 0$ at the beginning of some round. This means that the Block-value, starting the game with $G_{2\times\ell}$, has to change at least $\Omega(\log \ell)$ times before the game can end. It is easy to see, that if the Block-value changes in a step, the number of connected components does not increase in this step. According to rule (b) (i), Delayer has answered $*$ in this round and has scored a point.

For the second part, consider the variables (edges) ordered (from left to right) with $\big\{(1,j),(2,j)\big\} \prec \big\{(1,j),(1,j+1)\big\} \prec \big\{(2,j),(2,j+1)\big\}$ and edges with lower j defined to be smaller (with respect to $\prec$) than those with higher j for $1 \leq j \leq \ell - 1$, and consider a resolution refutation completely resolving the variables in decreasing order (from right to left). That is, the clauses containing variable $\big\{(1,\ell),(2,\ell)\big\}$ will be first resolved with all clauses containing this variable in negated form (in case it is possible to resolve), and so on. Since the graph has degree at most 3, there is a small number of clauses containing this variable. Also observe that after resolving the last three variables in the ordering in this way, the set of derived clauses plus the initial clauses contain a subset of clauses encoding the formula $\mathrm{Ts}(G_{2\times(\ell-1)}, \chi')$ for some odd marking χ'. The set of newly derived clauses in this subset has constant size, and the number of clauses in all the resolution configurations until this point is also constant. Continuing in this order with the complete resolution of all the variables, we obtain a refutation of $\mathrm{Ts}(G_{2\times\ell}, \chi)$ with constant clause and variable space. ◀

6 Conclusions and Open Problems

By introducing a new connection between tree-like resolution space and the reversible pebble game, we have studied the relation between tree-like space and space measures for general resolution, obtaining almost optimal separations between these measures. We conjecture that these separations are optimal and that in fact, the $\log\big(\mathrm{time}(\pi)\big)$ factors in the upper bounds of Theorems 23 and 28 and Corollaries 24 and 30 can be improved to a $\log n$ factor (n being the number of graph vertices or formula size, depending on the setting). We have been able to prove this for the restricted case of the Tseitin contradictions.

We have seen that a source for obtaining space separations between tree-like and general resolution are graph classes with a gap between their reversible and black pebbling prices and we have provided a new class of such graphs. An interesting question is whether there exists a graph class with such a separation for a space function larger than $n^{1/2}$.

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