Elimination Distances, Blocking Sets, and Kernels for Vertex Cover

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Abstract

The Vertex Cover problem plays an essential role in the study of polynomial kernelization in parameterized complexity, i.e., the study of provable and efficient preprocessing for NP-hard problems. Motivated by the great variety of positive and negative results for kernelization for Vertex Cover subject to different parameters and graph classes, we seek to unify and generalize them using so-called blocking sets. A blocking set is a set of vertices such that no optimal vertex cover contains all vertices in the blocking set, and the study of minimal blocking sets played implicit and explicit roles in many existing results.

We show that in the most-studied setting, parameterized by the size of a deletion set to a specified graph class $\mathcal{C}$, bounded minimal blocking set size is necessary but not sufficient to get a polynomial kernelization. Under mild technical assumptions, bounded minimal blocking set size is showed to allow an essentially tight efficient reduction in the number of connected components.

We then determine the exact maximum size of minimal blocking sets for graphs of bounded elimination distance to any hereditary class $\mathcal{C}$, including the case of graphs of bounded treedepth. We get similar but not tight bounds for certain non-hereditary classes $\mathcal{C}$, including the class $\mathcal{C}_{\text{LP}}$ of graphs where integral and fractional vertex cover size coincide. These bounds allow us to derive polynomial kernels for Vertex Cover parameterized by the size of a deletion set to graphs of bounded elimination distance to, e.g., forest, bipartite, or $\mathcal{C}_{\text{LP}}$ graphs.

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1 Introduction

In the Vertex Cover problem we are given an undirected graph $G = (V, E)$ and an integer k; the question is whether there exists a set $S \subseteq V$ of at most k vertices such that each edge of G is incident with a vertex of S, or, in other words, such that $G - S$ is an independent set. Despite Vertex Cover being NP-complete, it is known that there are efficient preprocessing algorithms that reduce any instance (G, k) to an equivalent instance with $\mathcal{O}(k^2)$ or even at most $2k$ vertices (and size polynomial in k). The existence or non-existence of such preprocessing routines for NP-hard problems has been studied intensively in the field of parameterized complexity under the term *polynomial kernelization*,[1] and Vertex Cover has turned out to be one of the most fruitful research subjects with a variety of upper and (conditional) lower bounds subject to different parameters (see, e.g., [2]).

In the present work, we seek to unify and generalize existing results by using so-called *blocking sets*. A blocking set in a graph $G = (V, E)$ is any set $Y \subseteq V$ that is not a subset of any minimum cardinality vertex cover of G, e.g., the set V itself is always a blocking set. Of particular interest are *minimal* blocking sets, i.e., those that are minimal under set inclusion. Several graph classes have constant upper bounds on the size of minimal blocking sets, e.g., in any forest (or even in any bipartite graph) every minimal blocking set has size at most two. On the other hand, even restrictive classes like outerplanar graphs have unbounded minimal blocking set size, i.e., for each d there is a graph in the class with a minimal blocking set of size greater than d. As a final example, cliques are the unique graphs for which V is the only (minimal) blocking set because all optimal vertex covers have form $V \setminus \{v\}$ for any $v \in V$; in particular, any graph class containing all cliques has unbounded minimal blocking set size.

For ease of reading, the introduction focuses mostly on hereditary graph classes, i.e., those closed under vertex deletion; the later sections give more general results modulo suitable technical conditions, where appropriate. Apart from that, throughout the paper, we restrict study to graph classes $\mathcal{C}$ that are *robust*, that is, they are closed under disjoint union and under deletion of connected components. In other words, a graph G is in $\mathcal{C}$ if and only if all of its connected components belong to $\mathcal{C}$. Most graph classes studied in the context of kernels for Vertex Cover are robust, and a large number of them are also hereditary. A particular non-hereditary graph class of interest for us is the class $\mathcal{C}_{\text{LP}}$ of graphs whose minimum vertex cover size equals the minimum size of a fractional vertex cover (denoted LP as it is also the optimum solution value for the vertex cover LP relaxation).

1.1 Blocking sets and kernels for Vertex Cover

Most known polynomial kernelizations for Vertex Cover are for parameterization by the vertex deletion distance to some fixed hereditary graph class $\mathcal{C}$ that is also robust, e.g., for $\mathcal{C}$ being the class of forests [13], graphs of maximum degree one or two [18], pseudoforests [8] (each component has at most one cycle), bipartite graphs [17], d-quasi forests/bipartite graphs [10] (at most d vertex deletions per component away from being a forest/bipartite), cluster graphs of bounded clique size [18], or graphs of bounded treedepth [2]. Concretely, the input is of form (G, k, X), asking whether G has a vertex cover of size at most k, where $X \subseteq V$ such that $G - X \in \mathcal{C}$; the size $\ell = |X|$ of the *modulator* X is the parameter. Blocking sets have been implicitly or explicitly used for most of these results and we point out that all the mentioned classes have bounded minimal blocking set size.

As our first result, we show that this is not a coincidence: If $\mathcal{C}$ is closed under disjoint union (or, more strongly, if $\mathcal{C}$ is robust) then bounded size of minimal blocking sets in graphs of $\mathcal{C}$ is necessary for a polynomial kernel to exist (Section 3.1). Moreover, the maximum size of minimal blocking sets in $\mathcal{C}$ yields a lower bound for the possible kernel size.

▶ **Theorem 1.** *Let $\mathcal{C}$ be a graph class that is closed under disjoint union. If $\mathcal{C}$ contains any graph with a minimal blocking set of size d then Vertex Cover parameterized by the size of a modulator X to $\mathcal{C}$ does not have a kernelization of size $\mathcal{O}(|X|^{d-\varepsilon})$ for any $\varepsilon > 0$ unless* NP $\subseteq$ coNP/poly *and the polynomial hierarchy collapses.*

[1] A polynomial kernelization is an efficient algorithm that given any instance with parameter value ℓ returns an equivalent instance of size polynomial in ℓ. In the initial Vertex Cover example we have parameter $\ell = k$. For more information on the topic, see [4, 7]

To the best of our knowledge, this theorem captures all known kernel lower bounds for Vertex Cover parameterized by deletion distance to any union-closed graph class $\mathcal{C}$, e.g., ruling out polynomial kernels for $\mathcal{C}$ being the class of mock forests (each vertex is in at most one cycle) [8], outerplanar graphs [12], or any class containing all cliques [1]; and getting kernel size lower bounds for graphs of bounded treedepth [2] or cluster graphs of bounded clique size [18]. To get lower bounds of this type, it now suffices to prove (or observe) that $\mathcal{C}$ has large or even unbounded minimal blocking set size.

It is natural to ask whether the converse holds, i.e., whether a bound on the minimal blocking set size directly implies the existence of a polynomial kernelization. Unfortunately, we show that this does not hold in a strong sense: There is a class $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have minimal blocking sets of size one, but there is no polynomial kernelization (Section 3.2). More strongly, solving Vertex Cover on $\mathcal{C}$ is not in $\mathsf{RP} \supseteq \mathsf{P}$ unless $\mathsf{NP} = \mathsf{RP}$.

▶ **Theorem 2.** *There exists a graph class $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have minimal blocking set size one and such that Vertex Cover on $\mathcal{C}$ is not solvable in polynomial time, unless* $\mathsf{NP} = \mathsf{RP}$.

In light of this result, one could ask what further assumptions on $\mathcal{C}$, apart from the necessity of bounded minimal blocking set size, are required to allow for polynomial kernels. Clearly, polynomial-time solvability of Vertex Cover on the class $\mathcal{C}$ is necessary and (as we implicitly showed) not implied by $\mathcal{C}$ having bounded blocking set size. If, slightly stronger, we require that blocking sets in graphs of $\mathcal{C}$ can be efficiently recognized[2] then we show that there is an efficient algorithm that reduces the number of components of $G - X$ for any instance (G, k, X) of Vertex Cover parameterized by deletion distance to $\mathcal{C}$ to $\mathcal{O}(|X|^d)$ (Section 3.3). This is a standard opening step for kernelization and can be followed up by shrinking and bounding the size of those components. Note that this requires that deletion of any component yields a graph in $\mathcal{C}$ (e.g., implied by $\mathcal{C}$ being robust), which here is covered already by $\mathcal{C}$ being hereditary.

▶ **Theorem 3.** *Let $\mathcal{C}$ be any hereditary graph class with minimal blocking set size d on which Vertex Cover can be solved in polynomial time. There is an efficient algorithm that given (G, k, X) such that $G - X \in \mathcal{C}$ returns an equivalent instance (G', k', X) such that $G' - X \in \mathcal{C}$ has at most $\mathcal{O}(|X|^d)$ connected components.*

We point out that the number $\mathcal{O}(|X|^d)$ of components is essentially tight (assuming that $\mathsf{NP} \nsubseteq \mathsf{coNP/poly}$) because the lower bound underlying Theorem 1 creates instances where components have a constant $c = c(d)$ many vertices. Reducing to $\mathcal{O}(|X|^{d-\varepsilon})$ components, for any $\varepsilon > 0$, would violate the kernel size lower bound.

1.2 Minimal blocking set size relative to elimination distances

Recently, Bougeret and Sau [2] presented a polynomial kernelization for Vertex Cover parameterized by the size of a modulator X such that $G - X$ has treedepth at most d; here d is a fixed constant and the degree of the polynomial in the kernel size depends exponentially on d. To get the kernelization, they prove (in different but equivalent terms) that the size of any minimal blocking set in a graph of treedepth d is at most 2^d, and they give a lower bound of 2^{d-3}. As our first result here, we determine the exact maximum size of minimal blocking sets in graphs of treedepth d (see below, and see Section 4 for all these results).

[2] This condition holds for all hereditary $\mathcal{C}$ on which Vertex Cover can be solved in polynomial time: Given $G = (V, E)$ and $Y \subseteq V$ it suffices to compute solutions for G and $G - Y$. Clearly, the set Y is a blocking set if and only if $\mathrm{OPT}(G) < \mathrm{OPT}(G - Y) + |Y|$.

Bulian and Dawar [3] introduced the notion of elimination distance to a class $\mathcal{C}$, generalizing treedepth, which corresponds to elimination to the empty graph (see Section 2 for a formal definition). This is defined in the same way as treedepth except that all graphs from $\mathcal{C}$ get value 0 rather than just the empty graph. (Note that here it is convenient that $\mathcal{C}$ is robust because the definition assigns value 0 to disjoint unions of graphs from $\mathcal{C}$.) Intuitively, elimination distance to $\mathcal{C}$ can be pictured as having a tree-like deletion of vertices (as for treedepth) but being allowed to stop when the remaining connected components belong to $\mathcal{C}$ (rather than continuing to the empty graph). For hereditary $\mathcal{C}$, we determine the exact maximum size of minimal blocking sets in graphs of elimination distance at most d to $\mathcal{C}$, denoted $\beta_{\mathcal{C}}(d)$, depending on the maximum minimal blocking set size $b_{\mathcal{C}}$ in the class $\mathcal{C}$.

▶ **Theorem 4.** *Let $\mathcal{C}$ be a robust hereditary graph class where $b_{\mathcal{C}}$ is bounded. For every integer $d \geq 1$ it holds that*

$$\beta_{\mathcal{C}}(d) = \begin{cases} 2^{d-1} + 1 & \text{, if } b_{\mathcal{C}} = 1, \\ (b_{\mathcal{C}} - 1)2^d + 1 & \text{, if } b_{\mathcal{C}} \geq 2. \end{cases}$$

The lower bound holds as well for any non-hereditary class $\mathcal{C}$ but we only get a slightly weaker upper bound for such classes (and also require a further technical condition called f-robustness). In particular, we get such an upper bound for the class $\mathcal{C}_{\mathrm{LP}}$ mentioned above. Note that if $\mathcal{C}$ has unbounded minimal blocking set size then the same is true for graphs of any bounded elimination distance to $\mathcal{C}$ (irrespective of $\mathcal{C}$ being hereditary or not).

The bound for graphs of treedepth at most d is included in the theorem by using that having treedepth at most d is equivalent to having elimination distance at most $d-1$ to the class of independent sets (i.e., graphs of treedepth one), for which all minimal blocking sets have size 1. Concretely, for treedepth d we get $\beta(d) = 1$ for $d = 1$ and $\beta(d) = 2^{d-2} + 1$ for $d \geq 2$.

1.3 Some consequences for kernels for Vertex Cover

Our bounds for the minimal blocking set size relative to elimination distances allow us to generalize and combine previous polynomial kernelization results for VERTEX COVER. We state this explicitly for elimination distances to hereditary graph classes.

▶ **Theorem 5.** *Let $\mathcal{C}$ be a hereditary and robust graph class for which $b_{\mathcal{C}}$ is bounded, such that VERTEX COVER has a (randomized) polynomial kernelization parameterized by the size of a modulator to $\mathcal{C}$. Then VERTEX COVER also has a (randomized) polynomial kernelization parameterized by the size of a modulator to graphs of bounded elimination distance to $\mathcal{C}$.*

As an example, this combines known polynomial kernels relative (to the size of) modulators to a forest [13] resp. to graphs of bounded treedepth [2] to polynomial kernels relative to a modulator to graphs of bounded forest elimination distance. Similarly, the randomized polynomial kernel for VERTEX COVER parameterized by a modulator to bipartite graphs is generalized to a modulator to graphs of bounded bipartite elimination distance. The approach to this result (also in the non-hereditary case) uses our bounds for minimal blocking set size relative to elimination distances and, apart from that, is inspired by the result of Bougeret and Sau [2]. Intuitively, these kernels are obtained by suitable reductions to the known kernelizable cases, and thus carry over their properties (e.g., being deterministic or randomized).

As an explicit example for the non-hereditary case, we state a new kernelization result relative to the size of a modulator to the class of graphs of bounded elimination distance to $\mathcal{C}_{\mathrm{LP}}$, i.e., bounded elimination distance to graphs where optimum vertex cover size equals optimum fractional vertex cover size.

▶ **Theorem 6.** *Vertex Cover admits a randomized polynomial kernel parameterized by the size of a modulator to graphs that have bounded elimination distance to $\mathcal{C}_{\mathrm{LP}}$.*

This subsumes several polynomial kernels for Vertex Cover (except for their size bounds).

2 Preliminaries

For $n \in \mathbb{N}$ we use $[n]$ to denote $\{1, \ldots, n\}$. We use standard graph notation [6]. We say that a graph class $\mathcal{C}$ is *robust* if $\mathcal{C}$ is closed under disjoint union and deletion of connected components. A set $S \subseteq V(G)$ is a *vertex cover* of a graph G, if for each edge $e \in E(G)$, at least one of its endpoints is contained in S. We will use $\mathrm{OPT}(G)$ to denote the size of a minimum vertex cover of G. The linear program relaxation for Vertex Cover for G is

$$\mathrm{LP}(G) = \min \left\{ \sum_{v \in V(G)} x_v \mid \forall \{u,v\} \in E(G) : x_u + x_v \geq 1 \wedge \forall v \in V(G) : 0 \leq x_v \leq 1 \right\}.$$

It is well known that there is always an optimal feasible solution such that $x_v \in \{0, \frac{1}{2}, 1\}$ for all $v \in V(G)$. We call a solution for which this holds a *half-integral* solution.

▶ **Definition 7.** *Let $\mathcal{C}$ be a graph class and let G be a graph. We define the* elimination distance to $\mathcal{C}$ *as*

$$\mathrm{ed}_{\mathcal{C}}(G) := \begin{cases} 0 & \text{if } G \in \mathcal{C}, \\ \min_{v \in V(G)} \mathrm{ed}_{\mathcal{C}}(G - \{v\}) + 1 & \text{if } G \notin \mathcal{C} \text{ and } G \text{ is connected}, \\ \max_{i \in [t]} \mathrm{ed}_{\mathcal{C}}(G_i) & \text{if } G \text{ consists of connected components } G_1, \ldots, G_t. \end{cases}$$

The treedepth *of a graph G is simply its elimination distance to the empty graph.*

Let $\mathcal{C}$ be a graph class, let G be a graph and let $X \subseteq V(G)$. We say X is a $\mathcal{C}$*-modulator* if $G - X \in \mathcal{C}$. We say that X is a $(\mathcal{C}, d)$*-modulator* if $\mathrm{ed}_{\mathcal{C}}(G - X) \leq d$. When considering Vertex Cover parameterized by the size of a $\mathcal{C}$-modulator or a $(\mathcal{C}, d)$-modulator, we will assume that this modulator is given on input. As such, inputs to the problem are triplets (G, k, X) such that X is a modulator and the problem is to decide whether G has a vertex cover of size at most k.

For a graph class $\mathcal{C}$, let $\mathcal{C} + c$ be the graph class consisting of all graphs that have a $\mathcal{C}$-modulator of size at most c, i.e., $\mathcal{C} + c := \{G \mid \exists X \subseteq V(G), |X| \leq c \colon G - X \in \mathcal{C}\}$.

▶ **Definition 8.** *Let G be a graph and let $Y \subseteq V(G)$ be a subset of its vertices. We say that Y is a* blocking set *in G if there exists no vertex cover S of G such that $Y \subseteq S$ and $|S| = \mathrm{OPT}(G)$. In other words, there is no optimal vertex cover of G that contains Y. A blocking set Y is* minimal *if no strict subset of Y is also a blocking set.*

Let G be a graph, we use $\beta(G)$ to denote the size of the largest minimal blocking set in G. For a graph class $\mathcal{C}$, let $b_{\mathcal{C}} := \max_{G \in \mathcal{C}} \beta(G)$, let $b_{\mathcal{C}} := \infty$ if the minimal blocking set size of graphs in this graph class can be arbitrarily large. Define $\beta_{\mathcal{C}}(d) := \max\{\beta(G) \mid \mathrm{ed}_{\mathcal{C}}(G) \leq d\}$.

3 Relation between minimal blocking sets and polynomial kernels

In this section, we show relations between the size of minimal blocking sets in $\mathcal{C}$ and kernelization bounds for Vertex Cover parameterized by a $\mathcal{C}$-modulator.

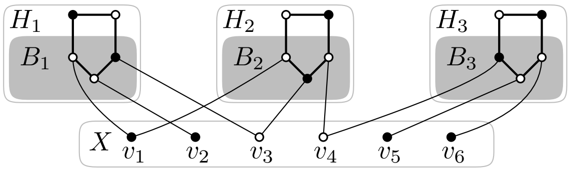

Figure 1 The Vertex Cover instance G' obtained in the proof of Theorem 1, corresponding to instance $(\mathcal{F}, 2)$ with $\mathcal{F} = \{\{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_4, v_5, v_6\}\}$. Since $\mathcal{F}$ has a hitting set of size 2, G' has a vertex cover of size $2 + 3 \cdot 3$, indicated in white.

3.1 Polynomial kernel implies a bound on the minimal blocking set size

In this section we prove Theorem 1, showing that if $\mathcal{C}$ is a graph class where minimal blocking sets can have size d, then this gives a kernelization lower bound for Vertex Cover when parameterized by the size of a $\mathcal{C}$-modulator. This generalizes most existing lower bounds for Vertex Cover when parameterized by the size of a modulator to $\mathcal{C}$ for some graph class $\mathcal{C}$ [8, 12, 14, 18]. Under the assumption that $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$, the theorem shows that having bounded blocking set size is necessary to obtain a polynomial kernel in the following sense. For a graph class $\mathcal{C}$ closed under disjoint union, for which Vertex Cover parameterized by a modulator to $\mathcal{C}$ admits a polynomial kernel of size $\mathcal{O}(k^d)$, it must hold that $b_{\mathcal{C}} \leq d$.

▶ **Theorem 1** (★[3]). *Let $\mathcal{C}$ be a graph class that is closed under disjoint union. If $\mathcal{C}$ contains any graph with a minimal blocking set of size d then Vertex Cover parameterized by the size of a modulator X to $\mathcal{C}$ does not have a kernelization of size $\mathcal{O}(|X|^{d-\varepsilon})$ for any $\varepsilon > 0$ unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$ and the polynomial hierarchy collapses.*

Proof sketch. The lower bound is obtained by a linear-parameter transformation from d-Hitting Set. An input to this problem consists of a set family $\mathcal{F}$ over a universe U and integer k where every set in $\mathcal{F}$ has size exactly d. The problem is to decide whether $\mathcal{F}$ has a hitting set of size at most k. A hitting set is a set $X \subseteq U$ such that for every set $S \in \mathcal{F}$ we have $S \cap X \neq \emptyset$. The lower bound will then follow from the fact that for $d \geq 2$, d-Hitting Set parameterized by the universe size n does not have a kernel of size $\mathcal{O}(n^{d-\varepsilon})$ for any $\varepsilon > 0$ unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$ [5, Theorem 2].

Suppose we are given an instance $(\mathcal{F}, k)$ for d-Hitting Set. Choose $H \in \mathcal{C}$ with a minimal blocking set of size d. We construct G' for Vertex Cover starting with vertex set $X := U$ (note $G' - X \in \mathcal{C}$), and adding a copy H_j of graph H for each $S_j \in \mathcal{F}$. The vertices in X that correspond to vertices in S_j are now each connected to a distinct vertex in the size-d minimal blocking set of H_j. This ensures that for any vertex cover S' of G', if none of the vertices from S_j is contained in S', then $|S' \cap H_j| > \mathrm{OPT}(H_j)$. We can use this to show that there exists a minimum vertex cover S' of G' such that $S' \cap X$ corresponds to a hitting set of $\mathcal{F}$. A sketch of the constructed instance $(G', k' := k + m \cdot \mathrm{OPT}(H))$ for Vertex Cover is shown in Figure 1. ◀

3.2 Bounded minimal blocking set size is not sufficient

Now that it is clear that, proving that a graph class has bounded blocking set size is essential towards obtaining a polynomial kernel for Vertex Cover parameterized by the size of a modulator to this graph class, one may wonder whether this condition is also sufficient. It

[3] For statements marked with a (★), the (full) proof can be found in the full version of the paper, which is available at `https://arxiv.org/abs/1905.03631`.

turns out that it is *not*. We show that there exists a graph class $\mathcal{C}$ for which all minimal blocking sets have size 1, for which VERTEX COVER is not solvable in polynomial time unless NP = RP. This implies that VERTEX COVER is unlikely to be FPT when parameterized by the size of a $\mathcal{C}$-modulator, immediately implying that it does not have a polynomial kernel when parameterized by a $\mathcal{C}$-modulator.

▶ **Theorem 2.** *There exists a graph class $\mathcal{C}$ such that all graphs in $\mathcal{C}$ have minimal blocking set size one and such that* VERTEX COVER *on $\mathcal{C}$ is not solvable in polynomial time, unless* NP = RP.

Proof. It is known that the UNIQUE-SAT problem cannot be solved in polynomial-time unless NP = RP [19, Corollary 1.2]. An input to UNIQUE-SAT is a CNF-formula $\mathcal{F}$ that has either exactly one satisfying solution or is unsatisfiable. The problem is to decide whether $\mathcal{F}$ is satisfiable. It can be shown that the same result holds for UNIQUE-3-SAT [15, Example 26.7], where the input formula is further restricted to be in 3-CNF. We show that the following polynomial-time reduction from UNIQUE-3-SAT to VERTEX COVER exists.

▷ Claim 9 (★). There is a polynomial-time reduction from UNIQUE-3-SAT to VERTEX COVER, that given a formula $\mathcal{F}$, outputs an instance (G, k) for VERTEX COVER such that:

- If $\mathcal{F}$ has exactly one satisfying assignment, then G has a unique minimum vertex cover of size k.
- If $\mathcal{F}$ is unsatisfiable, then G has a unique minimum vertex cover of size $k + 1$.

To conclude the proof, let $\mathcal{C}$ be the graph class consisting of all graphs that are obtained via the reduction given in the claim above, when starting from a formula $\mathcal{F}$ that has zero or one satisfying assignments. As such, solving VERTEX COVER on $\mathcal{C}$ in polynomial time corresponds to solving UNIQUE-3-SAT in polynomial time, implying NP = RP. Since any graph in $\mathcal{C}$ has exactly one minimum vertex cover, we obtain that indeed $b_{\mathcal{C}} = 1$, as any vertex that is not part of the minimum vertex cover forms a (minimal) blocking set. ◀

Graphs in the graph class $\mathcal{C}$ constructed in the proof of Theorem 2 are always connected, since they are the complement of a disconnected graph. As such, $\mathcal{C}$ is closed under removing connected components. However, $\mathcal{C}$ is not robust because it is not closed under disjoint union. We can however define $\mathcal{C}'$ to contain all graphs for which all connected components lie in $\mathcal{C}$. Observe that $\mathcal{C}'$ is robust, but that $b_{\mathcal{C}'} = 1$ and VERTEX COVER is not solvable in polynomial time on $\mathcal{C}' \supseteq \mathcal{C}$ unless RP = NP.

3.3 Reducing the number of components outside the modulator

As mentioned in the previous subsections, bounded blocking set size is necessary to obtain polynomial kernels for VERTEX COVER. Many papers that give polynomial kernels for VERTEX COVER parameterized by the size of a $\mathcal{C}$-modulator showed that their graph class $\mathcal{C}$ has bounded blocking set size, see for example [2, 8, 10, 13, 18]. Some of them used the blocking set size of class $\mathcal{C}$ to bound the number of connected components. More precisely, given an instance (G, k, X) of VERTEX COVER with $G - X \in \mathcal{C}$ they showed that one can reduce the number of connected components of $G - X$ to $\mathcal{O}(|X|^{b_{\mathcal{C}}+1})$. We will show that one can reduce the number of connected components of $G - X$ to $|X|^{b_{\mathcal{C}}}$, as a first step towards proving Theorem 3. Here we assume that the class $\mathcal{C}$ is robust in order to guarantee that deletion of connected components of $G - X$ again results in a graph of $\mathcal{C}$. At the end of this section, we discuss suitable conditions so that this component reduction can be done efficiently.

Let (G, k, X) be an instance of Vertex Cover parameterized by the size of a $\mathcal{C}$-modulator. First, we define the set $\mathcal{X} = \{Z \subseteq X \mid Z$ is an independent set in G and $1 \leq |Z| \leq b_{\mathcal{C}}\}$ as the collection of *chunks* of X. The intuition of defining the set $\mathcal{X}$ of chunks is to find sets in the modulator X for which at least one vertex must be contained in any optimum vertex cover of G. The concept of chunks was first introduced by Jansen and Bodlaender [13].

To reduce the number of connected components of $G - X$, we use the following result due to Hopcroft and Karp [11] which computes a certain crown-like structure in a bipartite graph. The second part of the theorem is not standard (but well known).

▶ **Theorem 10** ([11]). *Let G be an undirected bipartite graph with bipartition V_1 and V_2, on n vertices and m edges. Then we can find a maximum matching of G in time $\mathcal{O}(m\sqrt{n})$. Furthermore, in time $\mathcal{O}(m\sqrt{n})$ we can find either a maximum matching that saturates V_1 or a set $Z \subseteq V_1$ such that $|N_G(Z)| < |Z|$ and such that there exists a maximum matching M of $G - N_G[Z]$ that saturates $V_1 \setminus Z$.*

We construct a bipartite graph G_B to which we will apply Theorem 10 to find a set of connected components in $G - X$ that can be safely removed from G. We denote the set of connected components in $G - X$ by $\mathcal{F}$. The two parts of the bipartite graph G_B are the set $\mathcal{X}$ of chunks and the set $\mathcal{F}$ of connected components in $G - X$. More precisely, for every chunk $Z \in \mathcal{X}$ and for every connected component $H \in \mathcal{F}$ we add a vertex to the bipartite graph. To simplify notation we denote the vertex of G_B that corresponds to a connected component $H \in \mathcal{F}$ resp. a chunk $Z \in \mathcal{X}$ by H resp. Z. We add an edge between a vertex $H \in \mathcal{F}$ and a vertex $Z \in \mathcal{X}$ when $N_G(Z) \cap V(H)$ is a blocking set in H, i.e., when $\mathrm{OPT}(H - N_G(Z)) + |N_G(Z) \cap V(H)| > \mathrm{OPT}(H)$. Observe that G_B can only be constructed in polynomial time, if this condition can be tested in polynomial time, which is possible under some additional assumptions on $\mathcal{C}$, as we will see later.

It follows from Theorem 10 that there exists either a maximum matching M of G_B that saturates $\mathcal{X}$ or a set $\mathcal{X}' \subseteq \mathcal{X}$ such that $|N_{G_B}(\mathcal{X}')| < |\mathcal{X}'|$ and such that there exists a maximum matching M of $G_B - N_{G_B}[\mathcal{X}']$ that saturates $\hat{\mathcal{X}} = \mathcal{X} \setminus \mathcal{X}'$. If there exists a maximum matching M of G_B that saturates $\mathcal{X}$ then let $\mathcal{X}' = \emptyset$ and let $\hat{\mathcal{X}} = \mathcal{X}$. Let $\mathcal{F}_D = \mathcal{F} \setminus (N_{G_B}(\mathcal{X}') \cup V(M))$ be the set of connected components in $\mathcal{F}$ that are neither in the neighborhood of $\mathcal{X}'$ nor endpoint of a matching edge of M.

▶ **Reduction Rule 1.** Delete all connected components in $\mathcal{F}_D$ from G and decrease the size of k by $\mathrm{OPT}(\mathcal{F}_D)$ the size of an optimum vertex cover in $\mathcal{F}_D$.

Observe that Reduction Rule 1 deletes also all connected components $H \in \mathcal{F}$ which have the property that for all sets $Z \in \mathcal{X}$ it holds that $N(Z) \cap V(H)$ is not a blocking set of H because these connected components correspond to isolated vertices in the bipartite graph G_B. To show the correctness of Reduction Rule 1 we will use the following lemma which guarantees us the existence of certain optimum vertex covers of G.

▶ **Lemma 11** (★). *There exists an optimum vertex cover S of G with $S \cap Z \neq \emptyset$ for all $Z \in \hat{\mathcal{X}}$.*

Now, we show the correctness of Reduction Rule 1 using Lemma 11. Let $(\widetilde{G}, \widetilde{k}, X)$ be the reduced instance, i.e., $\widetilde{G} = G - \mathcal{F}_D$ and $\widetilde{k} = k - \mathrm{OPT}(\mathcal{F}_D)$. Obviously, if (G, k, X) is a yes-instance then $(\widetilde{G}, \widetilde{k}, X)$ is a yes-instance. For the other direction, assume that $(\widetilde{G}, \widetilde{k}, X)$ is a yes-instance. Observe that M is also a matching in $\widetilde{G}_B$ that saturates $\hat{\mathcal{X}}$ because we delete no connected component that is an endpoint of a matching edge. Furthermore, it holds that either $\hat{\mathcal{X}} = \mathcal{X}$ or $|N_{\widetilde{G}_B}(\mathcal{X}')| < |\mathcal{X}'|$ because we delete no connected component that corresponds to a vertex in $|N_{\widetilde{G}_B}(\mathcal{X}')|$. Thus, it follows from Lemma 11 that there exists an optimum vertex cover $\widetilde{S}$ of $\widetilde{G}$ with $\widetilde{S} \cap Z \neq \emptyset$ for all sets $Z \in \hat{\mathcal{X}}$. Note that every connected component $H \in \mathcal{F}_D$ is only adjacent to vertices in $\hat{\mathcal{X}}$ in G_B. Since every set $Z \in \hat{\mathcal{X}}$ has a non-empty intersection with the set $\widetilde{S}$, it holds that there exists an optimum vertex cover S_H of H which contains the set $N_G(X \setminus \widetilde{S}) \cap V(H)$. Let S be the set that results from adding for each connected component $H \in \mathcal{F}_D$ the optimum vertex cover S_H to the set $\widetilde{S}$. By construction, it holds that S is a vertex cover of G of size $|\hat{S}|+\mathrm{OPT}(\mathcal{F}_D) \leq \hat{k}+\mathrm{OPT}(\mathcal{F}_D) = k$. This proves that (G, k, X) is also a yes-instance. Overall, we showed that Reduction Rule 1 is safe.

▶ **Theorem 12 (★).** *Let (G, k, X) be an instance of* Vertex Cover *parameterized by the size of a $\mathcal{C}$-modulator that is reduced under Reduction Rule 1. The graph $G - X$ has at most $|X|^{b_\mathcal{C}}$ connected components.*

To use Theorem 12 to prove that we can efficiently reduce the number of connected components in $G - X$ when X is a $\mathcal{C}$-modulator, we need to show that, under certain assumptions, Reduction Rule 1 can be applied in polynomial time. We start by providing two sufficient conditions in the next lemma.

▶ **Lemma 13 (★).** *If $b_\mathcal{C}$ is bounded, if* Vertex Cover *is solvable in polynomial time on graphs of class $\mathcal{C}$ and if we can verify in polynomial time whether a given set Y is a blocking set in a graph of class $\mathcal{C}$ then we can apply Reduction Rule 1 in polynomial time.*

We continue by providing two cases that satisfy the preconditions for the lemma above, such that Reduction Rule 1 can be applied in polynomial time on these graph classes.

First of all, we consider the case that graph class $\mathcal{C}$ is hereditary. In this case, being solvable in polynomial time on the class $\mathcal{C}$ is sufficient to also be able to verify whether a given subset of the vertices is a blocking set, thus allowing us to apply Reduction Rule 1 in polynomial time. As mentioned in Subsection 3.3 we also need that $b_\mathcal{C}$ is bounded. Overall, we assume that $\mathcal{C}$ is a hereditary graph class on which Vertex Cover is polynomial-time solvable and where $b_\mathcal{C}$ is bounded.

▶ **Lemma 14 (★).** *Let $\mathcal{C}$ be any hereditary graph class on which* Vertex Cover *can be solved in polynomial time and where $b_\mathcal{C}$ is bounded. Then Reduction Rule 1 can be applied in polynomial time.*

Theorem 3 (restated below) now follows directly from Theorem 12 and Lemmas 13 and 14.

▶ **Theorem 3.** *Let $\mathcal{C}$ be any hereditary graph class with minimal blocking set size d on which* Vertex Cover *can be solved in polynomial time. There is an efficient algorithm that given (G, k, X) such that $G - X \in \mathcal{C}$ returns an equivalent instance (G', k', X) such that $G' - X \in \mathcal{C}$ has at most $\mathcal{O}(|X|^d)$ connected components.*

We can actually further generalize Theorem 3 to some non-hereditary graph classes. However, we have more problems to show that Lemma 13 holds for non-hereditary graph classes, because after deleting vertices from a graph G that is contained in a non-hereditary graph class $\mathcal{C}$ we do not know whether the resulting graph still belongs to the graph class $\mathcal{C}$. As such we need the additional assumption that Vertex Cover is also polynomial-time solvable on graph class $\mathcal{C} + 1$. This additional assumption is not unreasonable, when our goal is to obtain a kernelization algorithm for Vertex Cover. In fact, in order to obtain any kernel for Vertex Cover parameterized by the size of a modulator to $\mathcal{C}$ it is necessary to assume that the problem is FPT. From this, it immediately follows that we can solve Vertex Cover in polynomial time on $\mathcal{C} + 1$.

▶ **Theorem 15** (★). *Let $\mathcal{C}$ be any robust graph class with minimal blocking set size d on which* Vertex Cover *can be solved in polynomial time. Furthermore, assume that* Vertex Cover *can be solved in polynomial time on graphs of graph class $\mathcal{C}+1$. There is an efficient algorithm that given (G, k, X) such that $G - X \in \mathcal{C}$ returns an equivalent instance (G', k', X) such that $G' - X \in \mathcal{C}$ has at most $\mathcal{O}(|X|^d)$ connected components.*

4 Minimal blocking sets in graphs of bounded elimination distance

As seen in the previous section, minimal blocking sets play an important role for Vertex Cover kernelization. In this section we try to combine different structural parameters by considering the minimal blocking set size of graphs that have elimination distance d to some graph class $\mathcal{C}$ that has bounded minimal blocking set size. We prove the following theorem.

▶ **Theorem 4** (★). *Let $\mathcal{C}$ be a robust hereditary graph class where $b_{\mathcal{C}}$ is bounded. For every integer $d \geq 1$ it holds that*

$$\beta_{\mathcal{C}}(d) = \begin{cases} 2^{d-1} + 1 & \text{, if } b_{\mathcal{C}} = 1, \\ (b_{\mathcal{C}} - 1)2^d + 1 & \text{, if } b_{\mathcal{C}} \geq 2. \end{cases}$$

Proving the theorem consists of proving both the upper and the lower bound. The upper bound for $\beta_{\mathcal{C}}(d)$ given above only holds when $\mathcal{C}$ is hereditary. When $\mathcal{C}$ is not hereditary, we obtain an upper bound when $\mathcal{C}$ satisfies the following additional property:

▶ **Definition 16.** *We say that a graph class $\mathcal{C}$ is f-*robust*, if $b_{\mathcal{C}+c} \leq f(c) = f(b_{\mathcal{C}}, c)$ for a computable function f.*

▶ **Theorem 17** (★). *Let $\mathcal{C}$ be an f-robust and robust graph class where $b_{\mathcal{C}}$ is bounded, and let $d \geq 0$. It holds that*

$$\beta_{\mathcal{C}+c}(d) \leq \left(\sum_{i=0}^{d} \binom{d}{i} f(c+i) \right) - 2^d + 1.$$

The lower bound given by Theorem 4 however extends to any robust graph class $\mathcal{C}$. The lower bound is proven by constructing for each graph class $\mathcal{C}$ where $b_{\mathcal{C}}$ is bounded and each integer $d \geq 1$ a graph G with $\mathrm{ed}_{\mathcal{C}}(G) = d$ that contains a minimal blocking set of size at least $2^{d-1} + 1$ when $b_{\mathcal{C}} = 1$, and of size at least $(b_{\mathcal{C}} - 1)2^d + 1$ when $b_{\mathcal{C}} \geq 2$. Since $\beta_{\mathcal{C}}(d) = \max\{\beta(G) \mid \mathrm{ed}_{\mathcal{C}}(G) \leq d\}$ this gives the desired result. This result is obtained by showing that, given a graph H, we can obtain a graph H' such that $\mathrm{ed}_{\mathcal{C}}(H') \leq \mathrm{ed}_{\mathcal{C}}(H) + 1$ and $\beta(H') \geq 2\beta(H) - 1$. For $b_{\mathcal{C}} = 1$ we need an additional construction that, given a graph H, allows us to obtain a graph H' with $\mathrm{ed}_{\mathcal{C}}(H') \leq \mathrm{ed}_{\mathcal{C}}(H) + 1$ and $\beta(H') \geq \beta(H) + 1$. A depiction of both constructions is shown in Figure 2.[4]

5 Kernelization results

In this section, we will combine the results from Sections 3.3 and 4 to obtain polynomial kernelizations for Vertex Cover parameterized by a $\mathcal{C}$-modulator or a $(\mathcal{C}, d)$-modulator. In Section 3.3 we have seen necessary assumptions on a graph class $\mathcal{C}$ such that Reduction Rule 1 can be applied efficiently. We can show that for hereditary graph classes, for which Vertex Cover is solvable in polynomial time, Vertex Cover can also be solved efficiently on graphs for which $\mathrm{ed}_{\mathcal{C}}(G)$ is bounded. From that, we then obtain the following.

[4] For a proof of the correctness of this construction, refer to Section 4.1 in the full version of the paper.

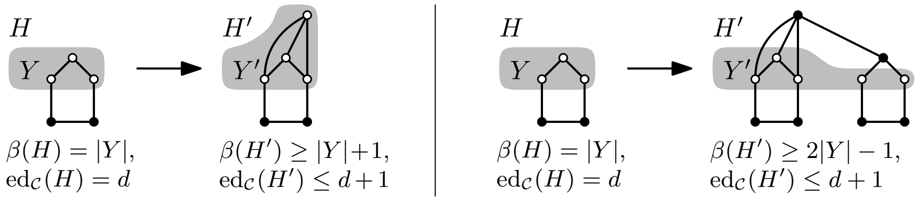

Figure 2 This figure depicts the construction used to prove the lower bound on $\beta_{\mathcal{C}}(d)$ in Theorem 4. White vertices form minimal blocking sets.

▶ **Corollary 18** (★). *Reduction Rule 1 is applicable in polynomial time on graphs G with a given $(\mathcal{C}, d)$-modulator X, where $\mathcal{C}$ is a hereditary graph class on which* VERTEX COVER *is solvable in polynomial time, and where $b_{\mathcal{C}}$ is bounded.*

For non-hereditary graph classes $\mathcal{C}$ we also need that VERTEX COVER is solvable in polynomial time on graph class $\mathcal{C} + c$ with c constant. For the next corollary, observe that in particular $\beta_{\mathcal{C}}(d)$ is bounded if $\mathcal{C}$ is known to be either hereditary or f-robust, by Theorems 4 and 17.

▶ **Corollary 19** (★). *Reduction Rule 1 is applicable in polynomial time on graphs G with a given $(\mathcal{C}, d)$-modulator X, where $\mathcal{C}$ is a robust graph class with the properties that $\beta_{\mathcal{C}}(d)$ is bounded for any constant d and that* VERTEX COVER *is polynomial-time solvable on graph class $\mathcal{C} + c$ for constant c.*

5.1 General results

In this section, we show that VERTEX COVER parameterized by the size of a $(\mathcal{C}, d)$-modulator has a polynomial kernel when the graph class $\mathcal{C}$ fulfills some additional properties. The assumptions that $\beta_{\mathcal{C}}(d)$ is bounded and that VERTEX COVER is polynomial-time solvable on the considered graph class are necessary, if these assumptions fail a polynomial kernel is unlikely to exist. The same holds for the assumption that VERTEX COVER parameterized by a $\mathcal{C}$-modulator has a polynomial kernel. We additionally require that $\mathcal{C}$ is a robust graph class that is either hereditary, or has the property that VERTEX COVER is polynomial-time solvable on $\mathcal{C} + c$. These assumptions will ensure that our reduction rule can be applied in polynomial time.

▶ **Lemma 20** (★). *Let $\mathcal{C}$ be a robust graph class for which $\beta_{\mathcal{C}}(d)$ is bounded and on which* VERTEX COVER *is polynomial-time solvable, such that $\mathcal{C}$ is hereditary or* VERTEX COVER *is polynomial-time solvable on $\mathcal{C} + c$ for all constants c.*

Suppose VERTEX COVER *parameterized by the size of a $\mathcal{C}$-modulator $\hat{X}$ has a (randomized) polynomial kernel with $g(|\hat{X}|)$ vertices. Then* VERTEX COVER *parameterized by the size of a $(\mathcal{C}, d)$-modulator X has a (randomized) polynomial kernel with $\mathcal{O}(g(|X|^b))$ vertices, where $b = \prod_{i=1}^{d} \beta_{\mathcal{C}}(i)$.*

The kernel is obtained by induction, by transforming an instance (G, k, X) of VERTEX COVER parameterized by $(\mathcal{C}, d)$-modulator to an instance parameterized by $(\mathcal{C}, d-1)$-modulator. This is done by first reducing the number of connected components in $G - X$ using Reduction Rule 1, and then adding the root of the treedepth decomposition of each connected component of $G - X$ to the modulator. This method for kernelization is similar to the kernelization for VERTEX COVER parameterized by the size of a d-treedepth modulator (see [2]). One difference is that we do not introduce hyper-edges.

Observe that in the above lemma statement, when Vertex Cover parameterized by a $\mathcal{C}$ modulator allows a polynomial kernel, the fact that Vertex Cover is solvable in polynomial time on graphs from $\mathcal{C} + c$ is immediate: since the problem has a polynomial kernel, it must be FPT in the parameter. Since in this case the size of a $\mathcal{C}$-modulator is c, which is constant, the result follows.

In the above lemma statement, we assume that $\beta_{\mathcal{C}}(d)$ is bounded to obtain the kernelization. We observe that for hereditary graph classes, this assumption is not needed, it follows from our results in Theorem 4 that it suffices to bound $b_{\mathcal{C}}$. Furthermore, a bound on $b_{\mathcal{C}}$ often comes naturally: if Vertex Cover parameterized by a $\mathcal{C}$-modulator has a polynomial kernel, it follows from Theorem 1 that, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$, there must exist a constant d such that $b_{\mathcal{C}} \leq d$.

▶ **Theorem 5 (★).** *Let $\mathcal{C}$ be a hereditary and robust graph class for which $b_{\mathcal{C}}$ is bounded, such that Vertex Cover has a (randomized) polynomial kernelization parameterized by the size of a modulator to $\mathcal{C}$. Then Vertex Cover also has a (randomized) polynomial kernelization parameterized by the size of a modulator to graphs of bounded elimination distance to $\mathcal{C}$.*

Similarly, for non-hereditary graph classes, it suffices if $\mathcal{C}$ is f-robust to obtain a polynomial kernel. The size of the kernel depends on f.

▶ **Corollary 21 (★).** *Let $\mathcal{C}$ be a robust and f-robust graph class for which $b_{\mathcal{C}}$ is bounded and for which Vertex Cover parameterized by the size of a $\mathcal{C}$ modulator $\hat{X}$ has a (randomized) polynomial kernel. Then Vertex Cover parameterized by the size of a $(\mathcal{C}, d)$-modulator has a (randomized) polynomial kernel.*

5.2 Kernel for modulator to bounded $\mathcal{C}_{\mathrm{LP}}$ elimination distance

In this section, we show how Theorem 6 follows from the general results in the previous section, to have an explicit example for a non-hereditary base class $\mathcal{C}$. That is, we show how to get a randomized polynomial kernel for Vertex Cover parameterized by the size of a modulator X such that $G - X$ has bounded elimination distance to the non-hereditary class $\mathcal{C}_{\mathrm{LP}}$ of graphs where integral and fractional vertex cover size coincide. Towards proving this result, we show the following relation between the value of $\ell = \mathrm{OPT}(G) - \mathrm{LP}(G)$ and the size of a $\mathcal{C}_{\mathrm{LP}}$-modulator in G.

▶ **Lemma 22 (★).** *Let G be a graph, and let $\ell = \mathrm{OPT}(G) - \mathrm{LP}(G)$. There exists a vertex set $X \subseteq V(G)$ of size at most 2ℓ such that $\mathrm{OPT}(G - X) = \mathrm{LP}(G - X)$.*

We can show that $\mathcal{C}_{\mathrm{LP}}$ is f-robust[5] with $f(c) = f(b_{\mathcal{C}}, c) = 2c + b_{\mathcal{C}_{\mathrm{LP}}} = 2c + 2$. Using Theorem 17 and Lemma 20 we can now generalize the kernelization for Vertex Cover parameterized by the size of a d-treedepth modulator and parameterized by the difference between an optimum vertex cover and an optimum LP solution using the size of a $(\mathcal{C}_{\mathrm{LP}}, d)$-modulator as the parameter. The following theorem subsumes Theorem 6.

▶ **Theorem 23 (★).** *An optimum $(\mathcal{C}_{\mathrm{LP}}, d)$-modulator of a graph G has at most the size of a d-treedepth modulator of G and at most twice the size of $\mathrm{OPT}(G) - \mathrm{LP}(G)$. Furthermore, Vertex Cover parameterized by the size of a $(\mathcal{C}_{\mathrm{LP}}, d)$-modulator admits a randomized polynomial kernel.*

[5] This is proven in the full version at the end of Section 4.2.

6 Conclusion

In the first part (Section 3) we have showed that bounded minimal blocking set size in $\mathcal{C}$ is necessary but not sufficient to get a polynomial kernel for Vertex Cover when parameterized by the size of a modulator X to a robust (or at least union-closed) class $\mathcal{C}$. We then showed that bounded minimal blocking set size suffices to efficiently reduce the number of components of $G - X$ assuming that $\mathcal{C}$ is robust (so deletion of components lets $G - X$ stay in $\mathcal{C}$) and that we can efficiently compute optimum vertex covers and test blocking sets in graphs of $\mathcal{C}$. The obtained bound of $\mathcal{O}(|X|^{b_\mathcal{C}})$ components is likely optimal because it matches the size lower bound proved earlier, which requires only components of constant size.

In the second part we first proved bounds for the minimal blocking set size relative to elimination distances to classes $\mathcal{C}$, motivated by the bounds that Bougeret and Sau [2] obtained relative to treedepth (Section 4). We obtain the exact value for all hereditary classes $\mathcal{C}$ and slightly weaker upper bounds for certain non-hereditary classes $\mathcal{C}$. This enabled new polynomial kernelization results for Vertex Cover that effectively replace (the size of) a modulator to a class $\mathcal{C}$ to modulators to graphs of bounded elimination distance to $\mathcal{C}$, e.g., when $\mathcal{C}$ is the class of forests, bipartite graphs, or C_{LP} (where integral and fraction vertex cover size coincide).

As future work it would be great to get a similar kernelization result when parameterized by the size of a modulator to bounded elimination distance to the graph class $\mathcal{C}_{2\mathrm{LP}-\mathrm{MM}}$ where $\mathrm{OPT} = 2\mathrm{LP} - \mathrm{MM}$ (i.e., minimum vertex cover size equals two times fractional cost minus size of a maximum matching, cf. [9]), which relates to the randomized kernelization for the corresponding above guarantee parameterization [16]. This would essentially subsume and generalize all currently known polynomial kernelizations for Vertex Cover (to which we came close with the result for bounded elimination distance to $\mathcal{C}_{\mathrm{LP}}$). It would also be nice to have tight bounds for the maximum size of minimal blocking sets in the non-hereditary case, and to get such bounds with fewest possible technical assumptions.

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