Geometric Realizations of the 3D Associahedron

Satyan L. Devadoss
Department of Mathematics, University of San Diego
San Diego, CA 92129, USA
devadoss@sandiego.edu

Daniel D. Johnson
Department of Computer Science, Harvey Mudd College
Claremont, CA 91711, USA
ddjohnson@hmc.edu

Justin Lee
Department of Computer Science, Boston University
Boston, MA 02215, USA
jbwlee@bu.edu

Jackson Warley
Department of Computer Science, University of Massachusetts
Amherst, MA 01003, USA
jwarley@cs.umass.edu

Abstract

The associahedron is a convex polytope whose 1-skeleton is isomorphic to the flip graph of a convex polygon. There exists an elegant geometric realization of the associahedron, using the remarkable theory of secondary polytopes, based on the geometry of the underlying polygon. We present an interactive application that visualizes this correspondence in the 3D case.

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1 The associahedron

Let us begin with a formal definition of the associahedron; details are given in [13].

▶ **Theorem.** *Let $A(n)$ be the poset of sets of non-crossing diagonals of a convex polygon P with n vertices, ordered so that for any $a, a' \in A(n)$, we have $a \prec a'$ if a is obtained from a' by adding a new diagonal to P. The* associahedron *$\mathcal{K}_{n-1}$ is a simple convex polytope of dimension $n-3$ whose face poset is isomorphic to $A(n)$.*

By construction, each vertex of $\mathcal{K}_{n-1}$ corresponds to a triangulation of P, whereas each facet of $\mathcal{K}_{n-1}$ corresponds to a diagonal of P. Just as each triangulation T of P uses exactly $n-3$ diagonals, each vertex of $\mathcal{K}_{n-1}$ is incident to $n-3$ facets (making it a simple polytope), corresponding to the diagonals used in T. Furthermore, each edge of $\mathcal{K}_{n-1}$ connecting two vertices corresponds to an edge flip between the respective triangulations.

The associahedron was constructed independently by Haiman (unpublished) and Lee [13], though Stasheff had defined the underlying abstract object twenty years prior in his work on

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associativity in homotopy theory [17]. The famous *Catalan numbers* enumerate the vertices, with over one hundred different combinatorial and geometric interpretations available. The beauty of this polytope is the multitude of settings in which it makes an appearance, including root systems [5], real algebraic geometry [6], computational geometry [8], phylogenetics [1], string theory [9], J-holomorphic curves [14], and hypergeometric functions [18].

Rather than just a combinatorial framework, $\mathcal{K}_n$ also has a rich *geometric* perspective. There are numerous realizations of the associahedron, obtained by taking the convex hull of the coordinates of its vertices. In a beautiful recent paper, Ceballos, Santos, and Ziegler classify all such realizations into three main groups [4]: The first one is based on *generalized permutohedra*, spearheaded by Postnikov, and constructs the associahedron from truncations of a simplex [16]. The second is based on *cluster complexes* of Coxeter type A_n, arising from the work of Fomin and Zelevinsky [5]. The final realization follows the classical work on *secondary polytopes* by Gelfand, Kapranov, and Zelevinsky [10].

The first two of these constructions have the property that $\mathcal{K}_n$ has exactly $n-2$ pairs of parallel facets. Indeed, Hohlweg and Lange [11] and Santos [4] showed that both of these realizations are particular cases of exponentially many different constructions of the associahedron. The third (secondary polytope) realization, denoted as GKZ, has a radically different flavor. It not only yields a polytope that never has parallel facets, but results in a continuous deformation of the associahedron obtained from a continuous deformation of its underlying polygon. We focus on an interactive visualization of this wonderful realization.

2 The GKZ realization

This realization provides a beautiful blend of combinatorics, geometry, and the convex hull; see [10] for the general theory and [7, Chapter 3] for a readable version. Let P be a planar convex geometric polygon with vertices $p_1, \ldots, p_n$. For a triangulation T of P, let

$$\psi_T(p_i) \;=\; \sum_{p_i \in \Delta \in T} \operatorname{area}(\Delta)$$

be the sum of the areas of all triangles Δ incident to the vertex p_i. In other words, the value $\psi_T(p_i)$ associated to each vertex p_i of the polygon is the sum of the areas of the triangles incident to p_i. Thus, to each triangulation of the convex polygon P, we obtain an *area vector*

$$\Phi(T) \;=\; (\psi_T(p_1), \ldots, \psi_T(p_n)) \in \mathbb{R}^n \,.$$

The number of potentially different area vectors associated to a given convex polygon with n vertices is the Catalan number. The following result comes from [10]:

▶ **Theorem.** *If P is a convex polygon with n vertices, the convex hull of the area vectors in $\mathbb{R}^n$ of all triangulations of P results in a geometric realization of the associahedron $\mathcal{K}_{n-1}$.*

What is remarkable about this result is that, although the convex hulls of the area vectors of different convex polygons are geometrically distinct, they output the same combinatorial structure for *any* convex polygon with n vertices.

3 Visualizing the 3D associahedron

We restrict our attention to $n = 6$, in which P is a planar convex hexagon with vertices $p_1, \ldots, p_6$. Thus, P has 14 distinct triangulations $T_1, \ldots, T_{14}$, and the theorem above ensures that the convex hull of the set of area vectors $\Phi(T_i)$ yields a realization $\mathfrak{K}$ of K_5 in $\mathbb{R}^6$. Since

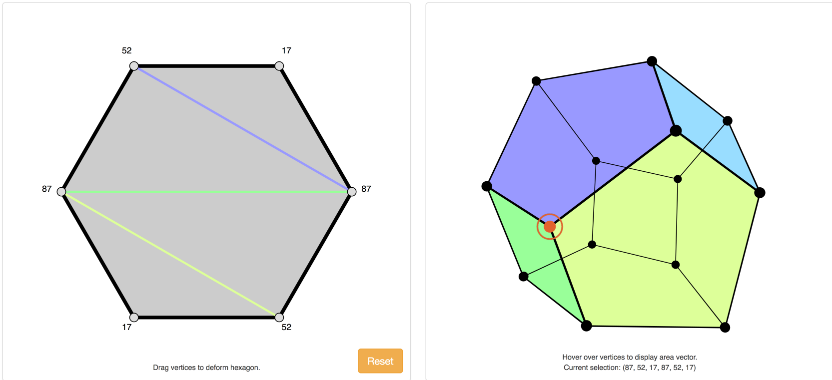

Figure 1 The interface of the web application. On the left, the convex hexagon, with one triangulation shown. On the right, the GKZ realization $\mathfrak{K}$ of $\mathcal{K}_5$, with the vertex corresponding to the triangulation highlighted in orange.

the GKZ realization lies in a 3-dimensional subspace $S \subseteq \mathbb{R}^6$, choosing an orthonormal basis $\mathcal{B}$ for S allows us to embed $\mathfrak{K}$ in $\mathbb{R}^3$ while preserving its geometric structure. We simply apply the Gram-Schmidt process to the vectors $\Phi(T_1), \Phi(T_2), \ldots, \Phi(T_{14})$, where the triangulations are ordered arbitrarily.

We provide an application that calculates the area vectors corresponding to each triangulation of P, and uses the Gram-Schmidt process to obtain an orthonormal basis for S. Our visualization application is implemented in HTML5 and JavaScript; it runs client-side in the web browser and can be accessed at `http://www.hexahedria.com/associahedron/`. Due to finite numerical precision, there may be small errors that cause our computed vertices of $\mathfrak{K}$ to lie slightly outside of the desired subspace S. To prevent this, we "snap" vectors with magnitude smaller than a numerical tolerance factor epsilon to zero, and project the vertices into the subspace spanned by the first three nonzero basis vectors. The program leverages several open source libraries: D3 [2] and Three.js [3] for graphics, numeric.js [15] for arithmetic, and PolyK [12] for simple polygon operations such as testing convexity.

The interface includes two side-by-side visuals: one of a triangulated convex hexagon P (left side), and one of the corresponding GKZ embedding $\mathfrak{K}$ (right side). The three diagonals of one triangulation T of P are displayed, and each vertex p_i of P is labeled with

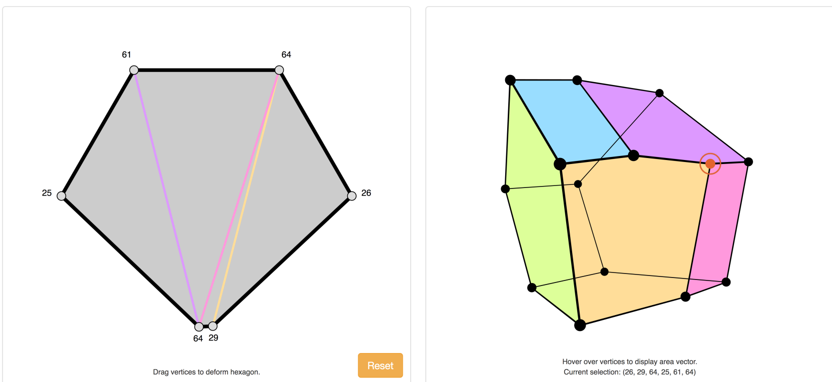

Figure 2 A deformed hexagon and its corresponding deformed associahedral realization.

the area $\psi_T(p_i)$ of the triangles incident to it in the triangulation. On the left side, users can deform the hexagon by manipulating its vertices. On the right side, users can rotate the associahedron and change the active triangulation by hovering over a different vertex. Moreover, each diagonal of the hexagon is color-coded to match the corresponding facet of the associahedron. As the user moves a vertex in the hexagon view, the areas of the triangles change and the polytope deforms accordingly. Indeed, under extreme deformations of the underlying polygon, highly interesting geometric representations appear, many in degenerate configurations.

Although our application does not allow nonconvex deformations of the underlying polygon, the mathematical theory is known [8]. The resulting "associahedral" structure will not be a convex polytope but a *polytopal complex*. This complex is not only (topologically) contractible, but possesses a geometric realization based on the theory of secondary polytopes.

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