Realizations of Indecomposable Persistence Modules of Arbitrarily Large Dimension

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Abstract

While persistent homology has taken strides towards becoming a widespread tool for data analysis, multidimensional persistence has proven more difficult to apply. One reason is the serious drawback of no longer having a concise and complete descriptor analogous to the persistence diagrams of the former. We propose a simple algebraic construction to illustrate the existence of infinite families of indecomposable persistence modules over regular grids of sufficient size. On top of providing a constructive proof of representation infinite type, we also provide realizations by topological spaces and Vietoris-Rips filtrations, showing that they can actually appear in real data and are not the product of degeneracies.

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1 Introduction

Recently, persistent homology [8] has established itself as the flagship tool of topological data analysis. It provides the persistence diagrams, an easy to compute and understand compact summary of topological features in various scales in a filtration. Fields where it has been successfully applied include materials science [12, 14], neuroscience [10, 13], genetics [7] or medicine [6, 17].

Over a filtration, persistence diagrams can be used because the persistence module can be uniquely decomposed into indecomposable modules which are intervals. To these intervals, we associate the lifespans of topological features. However, considering persistence modules over more general underlying structures, indecomposables are no longer intervals and can be more complicated.

As an example, in multidimensional persistence [5] over the commutative grid, the representation category is no longer representation finite. In other words, the number of possible indecomposables is infinite for a large enough finite commutative grid. The

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minimal size for a commutative grid to be representation infinite is relatively small. For two dimensional grids, we need a size of at least 2×5 or 3×3, as the $2 \times n$ grid is representation finite for $n \leq 4$ [9]. When considering three dimensional grids, it is enough to have a $2 \times 2 \times 2$ grid. For a particular finite dimensional persistence module, it is true that it can be uniquely decomposed into a direct sum of indecomposables, but we cannot list all the possible indecomposables a priori.

From another point of view, recent progress in software [15] has made practical application of multidimensional persistence more accessible. However, they approach this by computing incomplete invariants instead of indecomposable decompositions. This has the advantage of being easier to compute than the full decomposition and easier to visualize.

In this work, we aim to provide more intuition for the structure of some indecomposable modules. First, we provide algebraic constructions of some infinite families of indecomposable modules over all representation infinite grids.

Next, we tackle the problem of realizing these constructions. Any module over a commutative grid can realized as the k-th persistent homology module of a simplicial complex for any $k > 0$. This result was first claimed in [5] and proved in [11] with a construction which is straightforward algebraically but difficult to visualize.

Here, we realize our infinite families of indecomposable modules using a more visual construction. In all our algebraic constructions, our infinite families depend on a dimension parameter d and a parameter $\lambda \in K$, which we set to $\lambda = 0$ for our topological constructions. Our topological constructions are formed by putting together d copies of a repeating simple pattern. Finally we provide a Vietoris-Rips construction for our realization of the 2×5 case. Moreover, we show that this construction is stable with respect to small perturbations.

A direct corollary of our result is a constructive proof of the representation infinite type of the grids we consider. It also provides insight into one possible topological origin of these algebraic complications. Given that the construction is stable with respect to noise and appears through a relatively simple configuration, we argue that these kinds of complicated indecomposables may appear when applying multidimensional persistence to real data. Therefore these structures cannot be ignored and we hope that our construction provides insight into a source of indecomposability.

2 Background

We start with a quick overview of the necessary background. First we recall some basic definitions from the representation theory of bound quivers and detail how we check the indecomposability of representations. More details can be found in [2], for example. In the second part, we explain the block matrix formalism [1] we use to simplify some computations. We assume some familiarity with algebraic topology [16], in particular homology.

2.1 Representations of bound quivers

A quiver is a directed graph. In this work, we consider only quivers with a finite number of vertices and arrows and no cycles. A particular example is the linear quiver with n vertices. Let $n \in \mathbb{N}$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_{n-1})$ be a sequence of symbols $\tau_i = f$ or $\tau_i = b$. Below, the two-headed arrow $\longleftrightarrow$ stands for either an arrow pointing to the right or left. The quiver $\mathbb{A}_n(\tau)$ is the quiver with n vertices and $n-1$ arrows, where the i^{th} arrow points to the right if $\tau_i = f$ and to the left otherwise: $\bullet \longleftrightarrow \bullet \longleftrightarrow \ldots \longleftrightarrow \bullet$. In the case that all arrows are pointing forwards, we use the notation $\vec{\mathbb{A}}_n = \mathbb{A}_n(f \ldots f)$.

Throughout this work, let K be a field. A representation V of a quiver Q is a collection $V = (V_i, V_\alpha)$ where V_i is a finite dimensional K-vector space for each vertex i of Q, and each *internal map* $V_\alpha : V_i \to V_j$ is a linear map for each arrow α from i to j in Q.

A homomorphism from V to W, both representations of the same quiver Q, is a collection of linear maps $\{f_i : V_i \to W_i\}$ ranging over vertices i of Q such that $W_\alpha f_i = f_j V_\alpha$ for each arrow α from i to j. The set of all homomorphisms from V to W is the K-vector space $\mathrm{Hom}(V, W)$. The endomorphism ring of a representation V is $\mathrm{End}(V) = \mathrm{Hom}(V, V)$.

General quivers do not impose any constraints on the internal maps of their representations. However, we will mostly consider the case of commutative quivers, a special kind of quiver bound by relations. In particular, representations of a commutative quiver are required to satisfy the property that they form a commutative diagram. For more details see [2]. In the sequel, we shall take Q to mean either a quiver or a commutative quiver, depending on context, and $\mathrm{rep}\, Q$ its category of representations.

The language of representation theory was introduced to persistence in [4], where zigzag persistence modules were considered as representations of $\mathbb{A}_n(\tau)$. From now, we use the term persistence module over Q interchangeably with a representation of Q.

Any pair of representations $V = (V_i, V_\alpha)$ and $W = (W_i, W_\alpha)$ of a quiver Q has a direct sum $V \oplus W = (V_i \oplus W_i, V_\alpha \oplus W_\alpha)$ which is also a representation of Q. A representation V is said to be indecomposable if $V \cong W \oplus W'$ implies that either W or W' is the zero representation. We are concerned with the indecomposability of representations and use the following property relating the endomorphism ring with indecomposability.

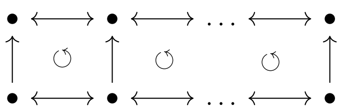
▶ **Definition 1.** Let R be a ring with unity. R is said to be *local* if $0 \neq 1$ in R and for each $x \in R$, x or $1 - x$ is invertible.

▶ **Lemma 2** (Corollary 4.8 of [2])**.** *Let V be a representation of a (bound) quiver Q.*
1. *If $\mathrm{End}\, V$ is local then V is indecomposable.*
2. *If V is finite dimensional and indecomposable, then $\mathrm{End}\, V$ is local.*

2.2 Block matrix formalism

Our first construction will be for a particular family of bound quivers called commutative ladders, which are the commutative grids of size $2 \times n$.

▶ **Definition 3.** The commutative ladder of length n with orientation τ, denoted $CL_n(\tau)$ is



which is a quiver with two copies of $\mathbb{A}_n(\tau)$ with the same orientation τ for the top and bottom rows, and bound by all commutativity relations.

Let us review the block matrix formalism for persistence modules on commutative ladders $CL_n(\tau)$ introduced in [1]. We denote by $\mathrm{arr}(\mathrm{rep}\, \mathbb{A}_n(\tau))$ the *arrow category* (also known as the *morphism category*) of $\mathrm{rep}\, \mathbb{A}_n(\tau)$, which is formed by the morphisms of $\mathrm{rep}\, \mathbb{A}_n(\tau)$ as objects. The following proposition allows us to identify representations of the commutative ladder $CL_n(\tau)$ with morphisms between representations of $\mathbb{A}_n(\tau)$. Since the structure of the latter is well-understood, we use this to simplify some computations.

▶ **Lemma 4.** *Let τ be an orientation of length n. There is an isomorphism of K-categories $F : \mathrm{rep}\, CL_n(\tau) \cong \mathrm{arr}(\mathrm{rep}\, \mathbb{A}_n(\tau))$.*

Proof. Given $M \in \operatorname{rep} CL_n(\tau)$, the bottom row (denoted M_1) of M and the top row (denoted M_2) of M are representations of $\mathbb{A}_n(\tau)$. By the commutativity relations imposed on M, the internal maps of M pointing upwards defines a morphism $\phi : M_1 \to M_2$ in $\operatorname{rep} \mathbb{A}_n(\tau)$. The functor F maps M to this morphism and admits an obvious inverse. ◀

Each morphism $(\phi : U \to V) \in \operatorname{arr}(\operatorname{rep} \mathbb{A}_n(\tau))$ has representations of $\mathbb{A}_n(\tau)$ as domain and codomain. As such, they each can be decomposed into interval representations. Thus, ϕ is isomorphic to some

$$\Phi : \bigoplus_{1 \le a \le b \le n} \mathbb{I}[a,b]^{m_{a,b}} \to \bigoplus_{1 \le c \le d \le n} \mathbb{I}[c,d]^{m'_{c,d}}.$$

Relative to these decompositions, Φ can be written in a block matrix form $\Phi = [\Phi^{c:d}_{a:b}]$ where each block is defined by composition with the corresponding inclusion and projection

$$\Phi^{c:d}_{a:b} : \ \mathbb{I}[a,b]^{m_{a,b}} \xrightarrow{\iota} \bigoplus_{1 \le a \le b \le n} \mathbb{I}[a,b]^{m_{a,b}} \xrightarrow{\Phi} \bigoplus_{1 \le c \le d \le n} \mathbb{I}[c,d]^{m'_{c,d}} \xrightarrow{\pi} \mathbb{I}[c,d]^{m'_{c,d}}.$$

Next, we analyze these blocks by looking at the homomorphism spaces between intervals.

▶ **Definition 5.** The relation $\trianglerighteq$ is the relation on the set of interval representations of $\mathbb{A}_n(\tau)$, $\{\mathbb{I}[b,d] : 1 \le b \le d \le n\}$, such that $\mathbb{I}[a,b] \trianglerighteq \mathbb{I}[c,d]$ if and only if $\operatorname{Hom}(\mathbb{I}[a,b], \mathbb{I}[c,d])$ is nonzero.

▶ **Lemma 6** (Lemma 1 of [1]). *Let $\mathbb{I}[a,b], \mathbb{I}[c,d]$ be interval representations of $\mathbb{A}_n(\tau)$.*

1. *The dimension of $\operatorname{Hom}(\mathbb{I}[a,b], \mathbb{I}[c,d])$ as a K-vector space is either 0 or 1.*
2. *There exists a canonical basis $\{f^{c:d}_{a:b}\}$ for each nonzero $\operatorname{Hom}(\mathbb{I}[a,b], \mathbb{I}[c,d])$ such that*

$$(f^{c:d}_{a:b})_i = \begin{cases} 1_K : K \to K, & \text{if } i \in [a,b] \cap [c,d] \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By the commutativity requirement on morphisms between representations, a nonzero morphism $g = \{g_i\} \in \operatorname{Hom}(\mathbb{I}[a,b], \mathbb{I}[c,d])$, if it exists, is completely determined by one of its internal morphisms, $g_j \in \operatorname{Hom}(K,K)$ for some fixed j in $[a,b] \cap [c,d]$. Since $\operatorname{Hom}(K,K)$ is of dimension 1, part 1 follows. The $f^{c:d}_{a:b}$ in part 2 is the g determined by $g_j = 1_K$. ◀

Each block $\Phi^{c:d}_{a:b} : \mathbb{I}[a,b]^{m_{a,b}} \to \mathbb{I}[c,d]^{m'_{c,d}}$ can be written in a matrix form where each entry is a morphism in $\operatorname{Hom}(\mathbb{I}[a,b], \mathbb{I}[c,d])$. Lemma 6 allows us to factor the common basis element of each entry and rewrite $\Phi^{c:d}_{a:b}$ using a K-matrix $M^{c:d}_{a:b}$ of size $m'_{c,d} \times m_{a,b}$:

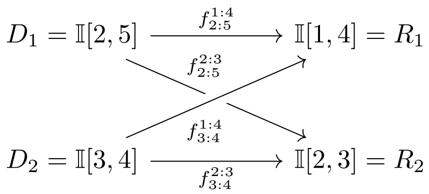
$$\Phi^{c:d}_{a:b} = \begin{cases} M^{c:d}_{a:b} f^{c:d}_{a:b}, & \text{if } \mathbb{I}[a,b] \trianglerighteq \mathbb{I}[c,d], \\ 0, & \text{otherwise} \end{cases}$$

3 Algebraic construction

Let us provide the construction for the commutative ladders of length 5. Details are provided for $\operatorname{rep} CL_5(ffff)$ using the formalism introduced for $\operatorname{arr}(\operatorname{rep} \vec{\mathbb{A}}_5)$. Slight adaptations to the construction make it work for any orientation τ.

3.1 Construction of a family of representations

Define the interval representations $D_1 = \mathbb{I}[2,5]$, $D_2 = \mathbb{I}[3,4]$, $R_1 = \mathbb{I}[1,4]$, and $R_2 = \mathbb{I}[2,3]$ of $\vec{\mathbb{A}}_5$. Note that for each of the four choices of ordered pairs (D_i, R_j), there exists a nonzero morphism $D_i \to R_j$, while no nonzero morphism exists between D_1 and D_2, and between R_1 and R_2. The directed graph with vertices given by the chosen intervals and arcs $x \to y$ defined by $x \trianglerighteq y$ is the complete bipartite directed graph $\vec{K}_{2,2}$.



Such a configuration is crucial to ensure that all four blocks can be nonzero in $\phi(d,\lambda)$ defined below, and to ensure indecomposability. Insight on how we find it is provided in the full version. The second ingredient we use is the Jordan cell $J_d(\lambda)$. Given $d \geq 1$ and $\lambda \in K$, $J_d(\lambda)$ is the matrix with value λ on the diagonal, 1 on the superdiagonal, and 0 elsewhere.

$$J_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

With this, we are ready to define an arrow $\phi(d,\lambda)$ and a representation $M(d,\lambda)$, which are identified to each other using Proposition 4 as $F(M(d,\lambda)) = \phi(d,\lambda)$.

▶ **Definition 7.** Let $d \geq 1$ and $\lambda \in K$.

1. We define the arrow $\phi(d,\lambda) \in \operatorname{arr}(\operatorname{rep} \vec{\mathbb{A}}_5)$ $\phi(d,\lambda) : \mathbb{I}[3,4]^d \oplus \mathbb{I}[2,5]^d \to \mathbb{I}[1,4]^d \oplus \mathbb{I}[2,3]^d$ by the matrix form $\phi(d,\lambda) = \begin{bmatrix} I f^{1:4}_{3:4} & I f^{1:4}_{2:5} \\ I f^{2:3}_{3:4} & J_d(\lambda) f^{2:3}_{2:5} \end{bmatrix}$ where I is the $d \times d$ identity matrix.
2. We also define the representation $M(d,\lambda) \in \operatorname{rep} CL_5(ffff)$ by

$$M(d,\lambda) : \quad \begin{array}{ccccccccc} K^d & \xrightarrow{\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]} & K^{2d} & \xrightarrow{\text{id}} & K^{2d} & \xrightarrow{[I\ 0]} & K^d & \longrightarrow & 0 \\ \uparrow & & \uparrow{\scriptstyle \left[\begin{smallmatrix} I \\ J_d(\lambda) \end{smallmatrix}\right]} & & \uparrow{\scriptstyle \left[\begin{smallmatrix} I & I \\ I & J_d(\lambda) \end{smallmatrix}\right]} & & \uparrow{\scriptstyle [I\ I]} & & \uparrow \\ 0 & \longrightarrow & K^d & \xrightarrow[\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]]{} & K^{2d} & \xrightarrow[\text{id}]{} & K^{2d} & \xrightarrow[{[0\ I]}]{} & K^d \end{array}$$

3.2 Indecomposability

We now show that the representations constructed above are indeed indecomposable and are pairwise non-isomorphic.

▶ **Theorem 8.** *Let $d \geq 1$ and $\lambda, \lambda' \in K$.*

1. *$M(d,\lambda)$ is indecomposable.*
2. *If $\lambda \neq \lambda'$ then $M(d,\lambda) \ncong M(d,\lambda')$.*

Proof. We check that $\operatorname{End} M(d,\lambda)$ is local. By Proposition 4, $\operatorname{End} M(d,\lambda) \cong \operatorname{End} \phi(d,\lambda)$. Letting (g_0, g_1) be an endomorphism of $\phi(d,\lambda)$, the diagram

$$\begin{array}{ccc} \mathbb{I}[3,4]^d \oplus \mathbb{I}[2,5]^d & \xrightarrow{\phi(d,\lambda)} & \mathbb{I}[1,4]^d \oplus \mathbb{I}[2,3]^d \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} \\ \mathbb{I}[3,4]^d \oplus \mathbb{I}[2,5]^d & \xrightarrow{\phi(d,\lambda)} & \mathbb{I}[1,4]^d \oplus \mathbb{I}[2,3]^d \end{array} \tag{1}$$

commutes. Then, g_0 and g_1 in matrix form with respect to the decompositions are

$$g_0 = \begin{bmatrix} Af_{3:4}^{3:4} & 0 \\ 0 & Bf_{2:5}^{2:5} \end{bmatrix} \text{ and } g_1 = \begin{bmatrix} Cf_{1:4}^{1:4} & 0 \\ 0 & Df_{2:3}^{2:3} \end{bmatrix}$$

where A, B, C, D are K-matrices of size $d \times d$. Since there are no nonzero morphisms from $\mathbb{I}[2,5]$ to $\mathbb{I}[3,4]$, nor vice versa, the off-diagonal entries of g_0 are 0. Likewise, there are no nonzero morphisms between $\mathbb{I}[1,4]$ and $\mathbb{I}[2,3]$ so the off-diagonal entries of g_1 are 0.

From the commutativity of Eq. (1), we get the equality

$$\begin{bmatrix} Af_{3:4}^{1:4} & Af_{2:5}^{1:4} \\ Bf_{3:4}^{2:3} & BJ_d(\lambda)f_{2:5}^{2:3} \end{bmatrix} = \begin{bmatrix} Cf_{3:4}^{1:4} & Df_{2:5}^{1:4} \\ Cf_{3:4}^{2:3} & J_d(\lambda)Df_{2:5}^{2:3} \end{bmatrix}$$

which implies that $A = B = C = D$ and $AJ_d(\lambda) = J_d(\lambda)A$ as K-matrices since the morphisms $f_{a:b}^{c:d}$ appearing above are nonzero. We infer that

$$\operatorname{End}\phi(d,\lambda) \cong \{A \in K^{d\times d} \mid AJ_d(\lambda) = J_d(\lambda)A\}.$$

A direct computation shows that A is a member of the above ring if and only if A is upper triangular Toeplitz:

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_{d-1} & a_d \\ 0 & a_1 & a_2 & \dots & a_{d-1} \\ & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_1 & a_2 \\ 0 & \dots & 0 & 0 & a_1 \end{bmatrix}.$$

The matrix A is invertible if and only if a_1 is nonzero, and so for any A, either A or $I - A$ is invertible. Thus, $\operatorname{End}\phi(d,\lambda)$ is local and $M(d,\lambda)$ is indecomposable.

By a similar computation, $M(d,\lambda) \cong M(d,\lambda')$ implies that there is some invertible matrix A such that $AJ_d(\lambda) = J_d(\lambda')A$ which is impossible when $\lambda \neq \lambda'$. ◀

Proposition 8 together with the observation that if $d \neq d'$ then $M(d,\lambda) \not\cong M(d',\lambda')$ provides an easy proof for the following corollary when $\tau = ffff$. Moreover, the method above can be used to produce similar examples for any orientation τ on length $n = 5$ by finding a similar configuration isomorphic to $\vec{K}_{2,2}$ from the intervals and arcs determined by $\trianglerighteq$. As a result, we get a constructive proof of the following result as a corollary.

▶ **Corollary 9.** *For any $n \geq 5$ and orientation τ, the commutative ladder $CL_n(\tau)$ is representation infinite.*

4 Realizations

We give realizations of our indecomposable persistence modules for $\lambda = 0$, first relying purely on topological spaces and then using a geometric Vietoris-Rips construction.

4.1 Topological construction

Given $d \geq 1$, we build a diagram $\mathbb{S}(d)$ of topological spaces and inclusions. The spaces in the middle column take the form of a sandal consisting of a planar sole and a set of d straps. Other spaces are either missing some edges or have some faces filled in. Figure 1 presents the complete realization.

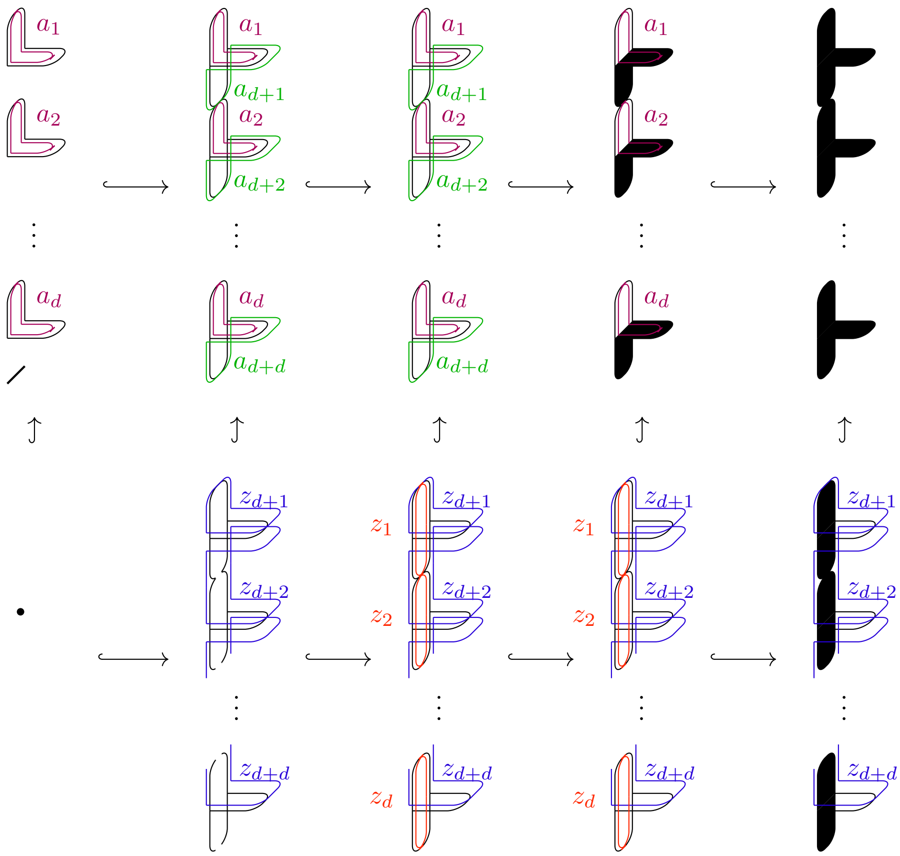


Figure 1 Diagram of spaces; and representatives for homology bases (in color).

The resulting diagram of spaces has maps that are all inclusions, and therefore all squares commute. Using the singular homology functor with coefficient in field K, we obtain a representation $H_1(\mathbb{S}(d))$ of $CL_5(ffff)$.

▶ **Theorem 10.**

$H_1(\mathbb{S}(d)) \cong M(d, 0)$

Proof. Relative to the choice of bases indicated in Fig. 1, the induced maps have the same matrix forms as the matrices in $M(d, 0)$. ◀

4.2 Vietoris-Rips construction

Next, we use the well-known Vietoris-Rips construction to build simplicial complexes having the suitable topology. Recall that the Vietoris-Rips complex $V(P, r)$ is the clique complex of the set of all edges that can be formed from points $p \in P$ with length less than $2r$. Note that, for $r \leq r'$, $V(P, r) \subset V(P, r')$, and hence we have a filtration.

In our construction, we provide two different points sets P_ℓ and P_u corresponding to the lower and upper rows. The point sets P_ℓ and P_u are built by assembling what we call *tiles* in a regular pattern. The union (possibly sharing common points along the edges) of translations and reflections of these tiles defines the global point sets.

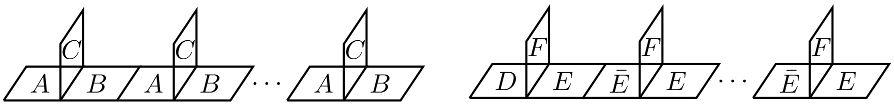


Figure 2 Assembling of the tiles with the upper row on the left.

We define three types of tiles, labelled A, B, C for the upper row and three, D, E, F for the lower row. The tile denoted $\bar{E}$ is obtained by reflection of tile E and we call it the reversed E tile. These tiles are arranged as in Fig. 2 to obtain P_u and P_ℓ, respectively. Note that the lower row presents an asymmetry as the D tile is only used once. Details on the content of the tiles and additional figures can be found in the full version.

We then choose five radius parameters $r_1, \dots, r_5$ so that the diagram

$$\begin{array}{ccccccccc} |V(P_u,r_1)| & \longrightarrow & |V(P_u,r_2)| & \longrightarrow & |V(P_u,r_3)| & \longrightarrow & |V(P_u,r_4)| & \longrightarrow & |V(P_u,r_5)| \\ {\scriptstyle f_1}\uparrow & & {\scriptstyle f_2}\uparrow & & {\scriptstyle f_3}\uparrow & & {\scriptstyle f_4}\uparrow & & {\scriptstyle f_5}\uparrow \\ |V(P_\ell,r_1)| & \longrightarrow & |V(P_\ell,r_2)| & \longrightarrow & |V(P_\ell,r_3)| & \longrightarrow & |V(P_\ell,r_4)| & \longrightarrow & |V(P_\ell,r_5)| \end{array}$$

of topological spaces and continuous maps has the same homology as $\mathbb{S}(d)$, where $|V(P,r)|$ denotes underlying topological space. The vertical maps are "obvious" ones that give the same induced maps as the inclusions in Figure 1. Details for the spaces and maps are provided in the full version. The resulting diagram is displayed in Figure 3. For clarity of the illustration, we omit some extra edges and triangles not affecting homology.

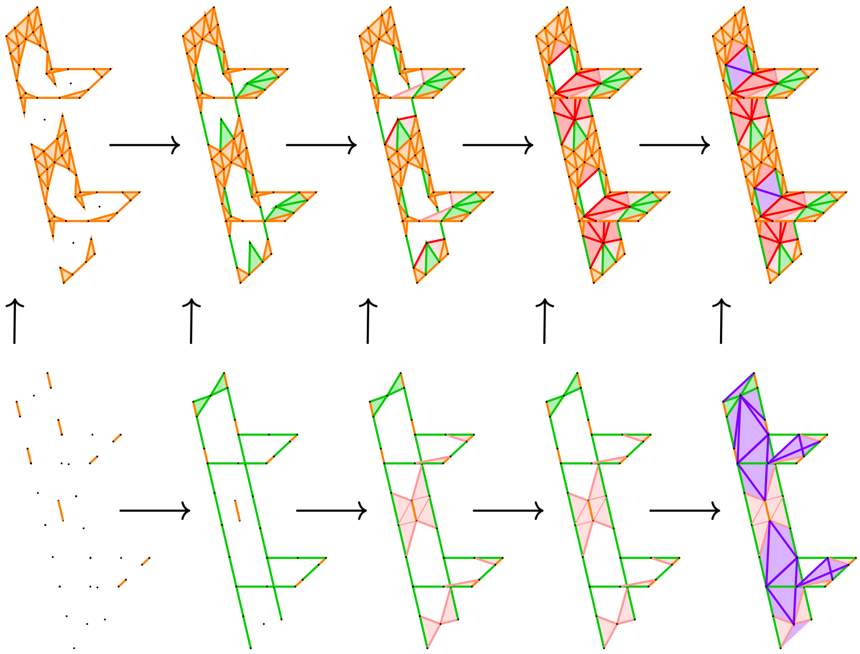


Figure 3 Complete realization, $d = 2$ case

Importantly, our construction does not rely on degeneracy. With P_ℓ and P_u fixed, we can freely choose the radius parameters r_i within intervals (x_i, y_i) of non-zero width. Dually, small perturbations of the point sets do not change the homology of the construction. Being even more restrictive, we can find similar intervals $I_i \subset (x_i, y_i)$ with positive width where there are no changes in the filtration. We get the following.

▶ **Lemma 11.** *Let ρ be the minimum of the diameters of I_i, and fix each r_i to be the center of I_i, for $i = 1, \ldots, 5$. Replacing each point of our input by a point within a ball of radius $\frac{\rho}{2}$ around it does not change the topology.*

Proof. By construction, for any i, there are no edges of length l such that $2r_i - \rho < l < 2r_i + \rho$.

We now replace every point of P by a point located within distance $\frac{\rho}{2}$. Note that the pairwise distance after this addition of noise are modified by at most ρ. Therefore, the complexes for radii r_i are unaffected as no pairwise distance can cross that threshold. ◀

5 Other elementary grids

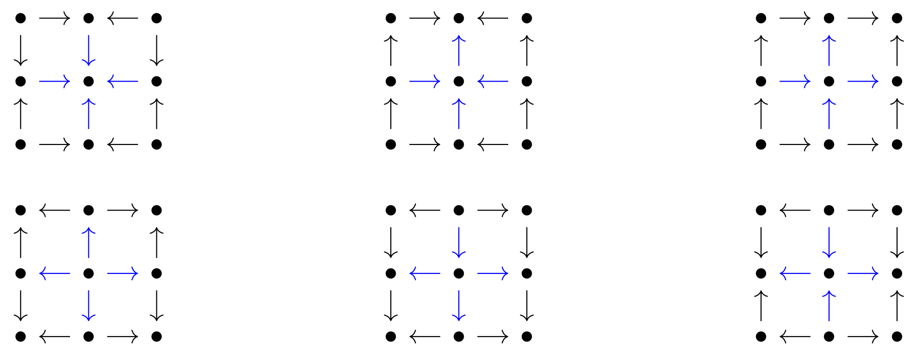
Previously we considered the 2×5 grid. We now consider the two other elementary representation infinite grids, the 3×3 grid and the $2 \times 2 \times 2$ cube. With this, we will have constructions for all possible representation infinite commutative grids, via embedding.

5.1 Three by three grid

The Euclidean quiver of type $\tilde{D}_4$ is representation infinite for any orientation of the arrows.

$$\tilde{D}_4 : \quad \begin{array}{ccccc} & & 1 & & \\ & & | & & \\ 2 & — & 0 & — & 4 \\ & & | & & \\ & & 3 & & \end{array}$$

We build infinite family of indecomposables for the $\tilde{D}_4$-type quiver with appropriate orientation and complete it to be indecomposable representations of a 3×3 grid. Up to symmetries, there are six different orientations of the 3×3 grid. We classify them according to the resulting orientation of the central $\tilde{D}_4$, shown in Fig. 4.



(a) $\tilde{D}_4$ same direction **(b)** 3 in, 1 out; and dual **(c)** 2 in, 2 out

Figure 4 Configurations for the 3×3 grid.

We fix the vector spaces to be K^{2d} at the central vertex and K^d elsewhere. Note that at least two arrows of $\tilde{D}_4$ will be pointing in the same direction relative to the central vertex. On two of these arrows, we assign matrices $[\,I\;0\,]$ and $[\,0\;I\,]$ or $\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ depending on their orientation. The remaining two arrows are assigned $[\,I\;J_d(\lambda)\,]$ or its transpose, and $[\,I\;I\,]$ or its transpose. For example, with three arrows pointing in and one out, we can have:

$$\begin{array}{ccccc} & & K^d & & \\ & & \Big\uparrow {\scriptstyle [\, I \;\; J_d(\lambda) \,]} & & \\ K^d & \xrightarrow{\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]} & K^{2d} & \xleftarrow[\left[\begin{smallmatrix} I \\ I \end{smallmatrix}\right]]{} & K^d \\ & & {\scriptstyle \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]} \Big\uparrow & & \\ & & K^d & & \end{array} .$$

The proof of indecomposability is again by computation of the endomorphism ring. Let $f = (f_0, \ldots, f_4)$ be an endomorphism. In matrix form, $f_0 = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : K^{2d} \to K^{2d}$, where 0 is the central vertex. Without loss of generality, we assume that the pair of arrows pointing in the same direction and assigned matrices $\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ (or $[\, I \; 0 \,]$ and $[\, 0 \; I \,]$), start from (or point towards) vertices 1 and 2, respectively. From commutativity requirement for endomorphisms, $f_0 = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$. Suppose that the arrow assigned $[\, I \; I \,]$ (or its transpose) points to (or starts from) vertex 3. The commutativity requirement with f_3 then requires $f_1 = f_3 = f_2$. Final commutativity requirement then forces $f_1 = f_2 = f_3 = f_4$ and $f_1 J_d(\lambda) = J_d(\lambda) f_1$. Thus, the endomorphism ring is local.

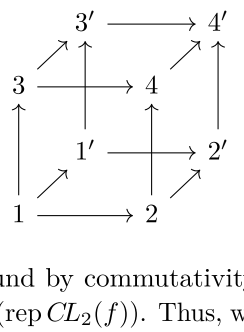
Completing the example into the 3×3 grid is easy. In the squares where the representation $\tilde{D}_4$ provides a nonzero composition of maps, we use that composition on one arrow and the identity on the other arrow. Otherwise we simply use a 0 vector space and 0 maps.

$$\begin{array}{ccccc} K^d & \xrightarrow{I} & K^d & \xleftarrow{I} & K^d \\ {\scriptstyle [\, I \;\; J_d(\lambda) \,] \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = I} \Big\uparrow & & \Big\uparrow {\scriptstyle [\, I \;\; J_d(\lambda) \,]} & & \Big\uparrow {\scriptstyle J_d(\lambda+1) = [\, I \;\; J_d(\lambda) \,] \left[\begin{smallmatrix} I \\ I \end{smallmatrix}\right]} \\ K^d & \xrightarrow{\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]} & K^{2d} & \xleftarrow[\left[\begin{smallmatrix} I \\ I \end{smallmatrix}\right]]{} & K^d \\ \Big\uparrow & & {\scriptstyle \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]} \Big\uparrow & & \Big\uparrow \\ 0 & \longrightarrow & K^d & \longleftarrow & 0 \end{array}$$

This algebraic construction can then be realized through a diagram of topological spaces and inclusions using our previous *sandal* construction. Details are provided in the full version.

5.2 Commutative cube

The commutative cube C is defined to be the quiver



bound by commutativity relations. Similar to Lemma 4, it can be checked that $\operatorname{rep} C \cong \operatorname{arr}(\operatorname{rep} CL_2(f))$. Thus, we write a representation of C as a morphism between representations of $CL_2(f)$ by taking the morphism from the front face to the back face.

In $\operatorname{rep}(CL_2(f))$, an analogue of Lemma 6 does not hold. In particular, the indecomposables

I_1, I_2 in $\mathrm{rep}(CL_2(f))$ given by:

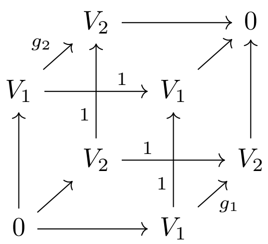
$$I_1 : \begin{array}{ccc} K & \xrightarrow{1} & K \\ \uparrow & & \uparrow{\scriptstyle 1} \\ 0 & \longrightarrow & K \end{array} \quad \text{and } I_2 : \begin{array}{ccc} K & \longrightarrow & 0 \\ {\scriptstyle 1}\uparrow & & \uparrow \\ K & \xrightarrow{1} & K \end{array}$$

have $\dim \mathrm{Hom}(I_1, I_2) = 2$. The vector space $\mathrm{Hom}(I_1, I_2)$ can be given the basis $\{f_2, f_3\}$, where f_2 is the identity on the lower right corner and zero elsewhere, and f_3 is the identity on the upper left corner and zero elsewhere.

Intuitively, we see that this is related to representations of the *Kronecker quiver* Q_2 : $1 \rightrightarrows 2$ by thinking about the two arrows as the linearly independent f_2, f_3. This statement can be made precise by the following.

▶ **Theorem 12.** *There is a fully faithful K-functor $\theta : \mathrm{rep}\, Q_2 \to \mathrm{rep}\, C$ that preserves indecomposability and isomorphism classes, where θ takes a representation $V :\ V_1 \underset{g_2}{\overset{g_1}{\rightrightarrows}} V_2$ to*

$\theta(V)$:



and a morphism $\phi = (\phi_1, \phi_2) : V \to W$ to $(0, \phi_1, \phi_1, \phi_1, \phi_2, \phi_2, \phi_2, 0) : \theta(V) \to \theta(W)$, where these maps are specified for the vertices in the order $1, \ldots, 4, 1', \ldots, 4'$.

Proof. That θ is a K-functor is easy to check. By the definition, $\theta(\phi) = 0$ implies that $\phi = 0$. Thus, θ is faithful. To see that θ is full, let V be as above, $W :\ W_1 \underset{g_2'}{\overset{g_1'}{\rightrightarrows}} W_2$ and

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{1'}, \alpha_{2'}, \alpha_{3'}, \alpha_{4'}) : \theta(V) \to \theta(W)$$

be a morphism. Then, $\alpha_1 : 0 \to 0$ and $\alpha_{4'} : 0 \to 0$ are zero maps by the forms of $\theta(V)$ and $\theta(W)$. The commutativity requirements for morphisms imply that $\alpha_2 = \alpha_3 = \alpha_4$ and $\alpha_{2'} = \alpha_{3'} = \alpha_{4'}$, and furthermore, $\alpha_{3'} g_2 = g_2' \alpha_3$ and $\alpha_{2'} g_1 = g_1' \alpha_2$. Thus, $(\alpha_2, \alpha_{2'})$ is a morphism from V to W such that $\theta((\alpha_2, \alpha_{2'})) = \alpha$.

Let $V \in \mathrm{rep}\, Q_2$ be indecomposable. By part 2 of Lemma 2, $\mathrm{End}\, V$ is local. By the above result that θ is fully faithful, $\mathrm{End}\, \theta(V) \cong \mathrm{End}\, V$, and so $\mathrm{End}\, \theta(V)$ is local. Therefore, $\theta(V) \in \mathrm{rep}\, C$ is indecomposable by part 1 of Lemma 2.

If $\alpha : \theta(V) \to \theta(W)$ is an isomorphism with inverse β then $(\alpha_2, \alpha_{2'}) : V \to W$ is an isomorphism with inverse $(\beta_2, \beta_{2'})$. Thus, if $\theta(V) \cong \theta(W)$, then $V \cong W$. This shows that θ preserves isomorphism classes. ◀

The quiver Q_2 is representation infinite, and its indecomposable representations are well-known. See for example Proposition 1.6 of [3]. In particular, one example of an infinite family is the regular indecomposables $R_n(\lambda) :\ K^n \underset{J_n(\lambda)}{\overset{I}{\rightrightarrows}} K^n$ for $n \geq 0$ and $\lambda \in K$. The following corollary is immediate from Theorem 12.

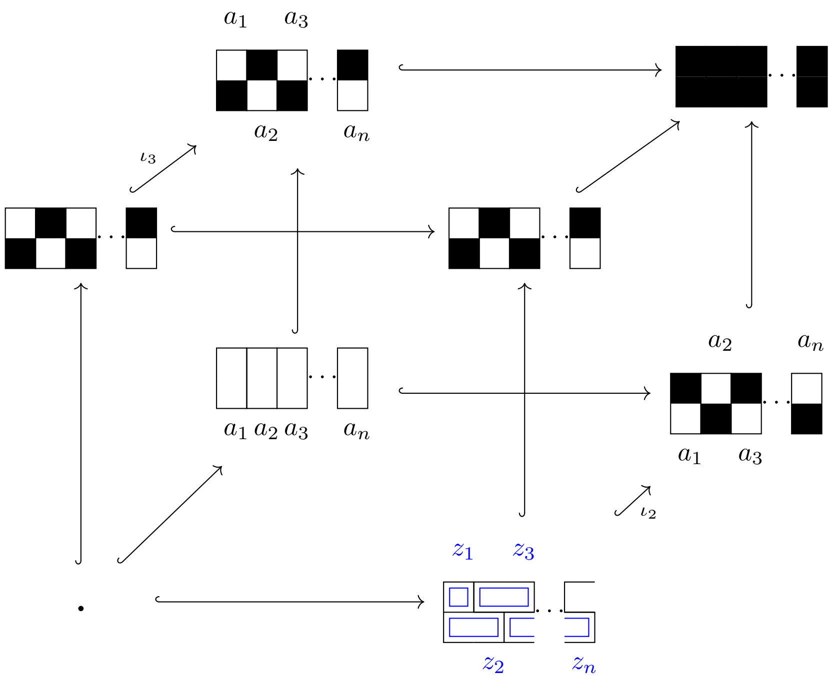
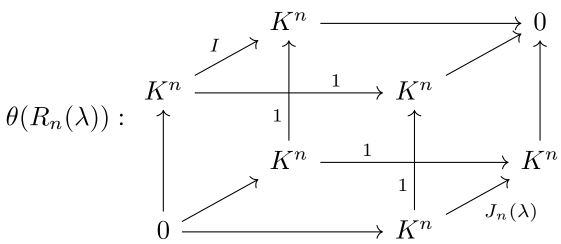


Figure 5 Topological realization of $\theta(R_n(0))$.

▶ **Corollary 13.** *The commutative cube C is representation infinite.*

By the above arguments, $\theta(R_n(\lambda))$:



are indecomposable and pairwise non-isomorphic for $n \geq 0$. We give a topological realization for $\theta(R_n(0))$ in Fig. 5. In the back face, we have n half filled-in strips arranged side by side horizontally in an alternating pattern (Fig. 5 shows the case n even). Using the lower left corner, we are able to flip the pattern. Coming from the front face, and with the given choice of basis, the induced map $H_1(\iota_2)$ is $J_n(0)$ while $H_1(\iota_3)$ is the identity I.

6 Discussion

We have illustrated constructions of infinite families of indecomposable persistence modules together with topological realizations over the small commutative grids. By embedding, this provides constructions for all possible representation infinite commutative grids.

In addition to our families of indecomposables, other parametrized families might be of interest. More generally, for representation tame commutative grids, could we realize parametrized families that generate all indecomposables?

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