

# Space Complexity of Perfect Matching in Bounded Genus Bipartite Graphs

Samir Datta<sup>1</sup>, Raghav Kulkarni<sup>2</sup>, Raghunath Tewari<sup>3</sup>, and N. Variyam Vinodchandran<sup>4</sup>

- 1 **Chennai Mathematical Institute**  
Chennai, India  
sdatta@cmi.ac.in
- 2 **University of Chicago**  
Chicago, USA  
raghav@cs.uchicago.edu
- 3 **University of Nebraska-Lincoln**  
Lincoln, USA  
rtewari@cse.unl.edu
- 4 **University of Nebraska-Lincoln**  
Lincoln, USA  
vinod@cse.unl.edu

---

## Abstract

We investigate the space complexity of certain perfect matching problems over bipartite graphs embedded on surfaces of constant genus (orientable or non-orientable). We show that the problems of deciding whether such graphs have (1) a perfect matching or not and (2) a unique perfect matching or not, are in the logspace complexity class SPL. Since SPL is contained in the logspace counting classes  $\oplus L$  (in fact in  $\text{Mod}_k L$  for all  $k \geq 2$ ),  $C=L$ , and PL, our upper bound places the above-mentioned matching problems in these counting classes as well. We also show that the search version, computing a perfect matching, for this class of graphs is in  $\text{FL}^{\text{SPL}}$ . Our results extend the same upper bounds for these problems over bipartite planar graphs known earlier.

As our main technical result, we design a logspace computable and polynomially bounded weight function which isolates a minimum weight perfect matching in bipartite graphs embedded on surfaces of constant genus. We use results from algebraic topology for proving the correctness of the weight function.

**1998 ACM Subject Classification** Computational Complexity

**Keywords and phrases** perfect matching, bounded genus graphs, isolation problem

**Digital Object Identifier** 10.4230/LIPIcs.STACS.2011.579

## 1 Introduction

The *perfect matching* problem and its variations are one of the most well-studied problems in theoretical computer science. Research in understanding the inherent complexity of computational problems related to matching has led to important results and techniques in complexity theory and elsewhere in theoretical computer science. However, even after decades of research, the exact complexity of many problems related to matching is not yet completely understood.

We investigate the *space complexity* of certain well studied perfect matching problems over bipartite graphs. We prove new uniform space complexity upper bounds on these problems for *graphs embedded on surfaces of constant genus*. We prove our upper bounds



© S. Datta, R. Kulkarni, R. Tewari, N.V. Vinodchandran;  
licensed under Creative Commons License NC-ND  
28th Symposium on Theoretical Aspects of Computer Science (STACS'11).  
Editors: Thomas Schwentick, Christoph Dürr; pp. 579–590  
Leibniz International Proceedings in Informatics

**LIPICs** Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



SYMPOSIUM  
ON THEORETICAL  
ASPECTS  
OF COMPUTER  
SCIENCE

by solving the technical problem of ‘deterministically isolating’ a perfect matching for this class of graphs.

Distinguishing a single solution out of a set of solutions is a basic algorithmic problem with many applications. The *Isolation Lemma* due to Mulmuley, Vazirani, and Vazirani provides a general randomized solution to this problem. Let  $\mathcal{F}$  be a non-empty set system on  $U = \{1, \dots, n\}$ . The Isolation Lemma says, for a random weight function on  $U$  (bounded by  $n^{O(1)}$ ), with high probability there is a *unique* set in  $\mathcal{F}$  of minimum weight [14]. This lemma was originally used to give an elegant RNC algorithm for constructing a maximum matching (by isolating a minimum weight perfect matching) in general graphs. Since its discovery, the Isolation Lemma has found many applications, mostly in discovering new randomized or non-uniform upper bounds, via isolating minimum weight solutions [14, 15, 8, 1]. Clearly, derandomizing the Isolation Lemma in sufficient generality will improve these upper bounds to their deterministic counterparts and hence will be a major result. Unfortunately, recently it is shown that such a derandomization will imply certain circuit lower bounds and hence is a difficult task [3].

Can we bypass the Isolation Lemma altogether and deterministically isolate minimum weight solutions in specific situations? Recent results illustrate that one may be able to use the structure of specific computational problems under consideration to achieve non-trivial deterministic isolation. In [4], the authors used the structure of directed paths in planar graphs to prescribe a simple weight function that is computable deterministically in logarithmic space with respect to which the minimum weight directed path between any two vertices is unique. In [6], the authors isolated a perfect matching in planar bipartite graphs. In this paper we extend the deterministic isolation technique of [6] to isolate a minimum weight perfect matching in bipartite graphs embedded on constant genus surfaces.

## Our Contribution

Let  $G$  be a bipartite graph with a weight function  $w$  on its edges. For an even cycle  $C = e_1 e_2 \cdots e_{2k}$ , the circulation of  $C$  with respect to  $w$  is the sum  $\sum_{i=1}^{2k} (-1)^i w(e_i)$ . The main technical contribution of the present paper can be stated (semi-formally) as follows.

**Main Technical Result.** There is a logspace matching preserving reduction  $f$ , and a logspace computable and polynomially bounded weight function  $w$ , so that given a bipartite graph  $G$  with a combinatorial embedding on a surface of constant genus, the circulation of any simple cycle in  $f(G)$  with respect to  $w$  is non-zero. (This implies that the minimum weight perfect matching in  $f(G)$  is unique [6]).

We use this result to establish (using known techniques) the following new upper bounds. Refer to the next section for definitions.

**New Upper Bounds.** For bipartite graphs, combinatorially embedded on surfaces of constant genus the problems DECISION-BPM and UNIQUE-BPM are in SPL, and the problem SEARCH-BPM is in FL<sup>SPL</sup>.

SPL is a logspace complexity class that was first studied by Allender, Reinhardt, and Zhou [1]. This is the class of problems reducible to the determinant with the promise that the determinant is either 0 or 1. In [1], the authors show, using a non-uniform version of Isolation Lemma, that perfect matching problem for general graphs is in a ‘non-uniform’ version of SPL. In [6], using the above-mentioned deterministic isolation, the authors show that for planar bipartite graphs, DECISION-BPM is in fact in SPL (uniformly). Recently, Hoang showed that for graphs with polynomially many matchings, perfect matchings and

many related matching problems are in SPL [9]. SPL is contained in logspace counting classes such as  $\text{Mod}_k\text{L}$  for all  $k \geq 2$  (in particular in  $\oplus\text{L}$ ), PL, and  $\text{C=L}$ , which are in turn contained in  $\text{NC}^2$ . Thus the upper bound of SPL that we prove implies that the problems DECISION-BPM and UNIQUE-BPM for the class of graphs we study are in these logspace counting classes as well.

The techniques that we use in this paper can also be used to isolate directed paths in graphs on constant genus surfaces. This shows that the reachability problem for this class of graphs can be decided in the unambiguous class UL, extending the results of [4]. But this upper bound is already known since recently Kynčl and Vyskočil show that reachability for bounded genus graphs logspace reduces to reachability in planar graphs [11].

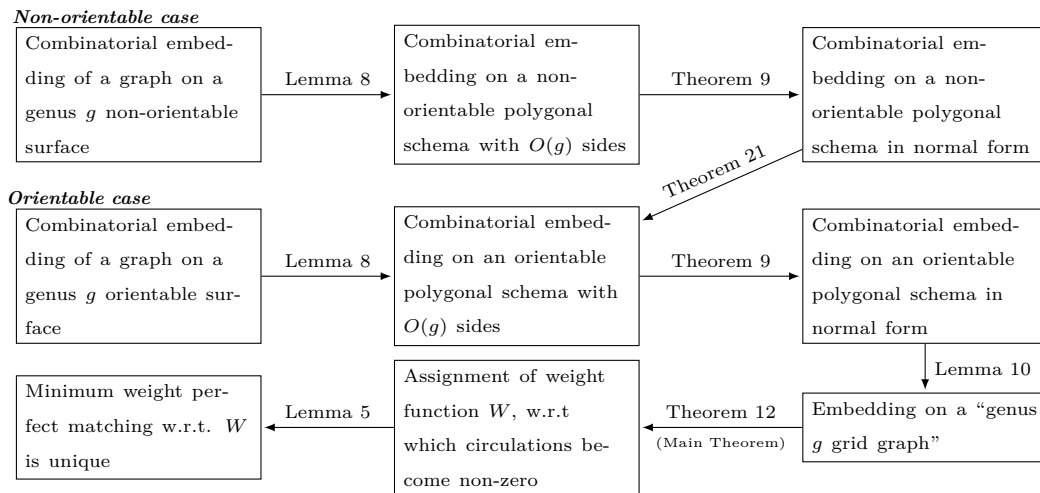
Matching problems over graphs of low genus have been of interest to researchers, mainly from a parallel complexity viewpoint. The matching problems that we consider in this paper are known to be in NC. In particular in [10], the authors present an  $\text{NC}^2$  algorithm for computing a perfect matching for bipartite graphs on surfaces of  $O(\log n)$  genus (readers can also find an account of known parallel complexity upper bounds for matching problems over various classes of graphs in their paper). However, the space complexity of matching problems for graphs of low genus has not been investigated before. The present paper takes a step in this direction.

**Proof Outline.** We assume that the graph  $G$  is presented as a combinatorial embedding on a surface (orientable or non-orientable) of genus  $g$ , where  $g$  is a constant. This is a standard assumption when dealing with graphs on surfaces, since it is NP-complete to check whether a graph has genus  $\leq g$  [16]. We first give a sequence of two reductions to get, from  $G$ , a graph  $G'$  with an embedding on a genus  $g$  ‘polygonal schema in normal form’. These two reductions work for both orientable and non-orientable cases. At this point we take care of the non-orientable case by reducing it to the orientable case. Once we have the embedding on an orientable polygonal schema in normal form, we further reduce  $G'$  to  $G''$  where  $G''$  is embedded on a constant genus ‘grid graph’. These reductions are matching preserving, bipartiteness preserving and computable in logspace. Finally, for  $G''$ , we prescribe a set of  $4g + 1$  weight functions,  $\mathcal{W} = \{w_i\}_{1 \leq i \leq 4g+1}$ , so that for any cycle  $C$  in  $G''$ , there is a weight function  $w_i \in \mathcal{W}$  with respect to which the circulation of  $C$  is non-zero. Since  $g$  is constant, we can take a linear combination of the elements in  $\mathcal{W}$ , for example  $\sum_{w_i \in \mathcal{W}} w_i \times (n^c)^i$  (where  $n$  is the number of vertices in the grid) for some fixed constant  $c$  (say  $c = 4$ ), to get a single weight function with respect which the circulation of any cycle is non-zero.

The intuition behind these weight functions is as follows (for some of the definitions, refer to later sections). The set  $\mathcal{W}$  is a disjoint union  $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \{w\}$  of the sets of weight functions  $\mathcal{W}_1$ ,  $\mathcal{W}_2$ , and  $\{w\}$ . Consider a graph  $G$  embedded on a fundamental polygon with  $2g$  sides. There are two types cycles in  $G$ : *surface separating* and *surface non-separating*. A basic theorem from algebraic topology implies that a surface non-separating cycle will intersect at least one of the sides of the polygon an odd number of times. This leads to  $2g$  weight functions in  $\mathcal{W}_1$  to take care of all the surface non-separating cycles. There are two types of surface separating cycles: (a) ones which completely lie inside the polygon and (b) the ones which cross some boundary. Cycles of type (a) behave exactly like cycles in the plane so the weight function  $w$  designed for planar graphs works (from [6]). For dealing with cycles of type (b), we first prove that if such a cycle intersects a boundary, it should alternate between ‘coming in’ and ‘going out’. This leads to  $2g$  weight functions in  $\mathcal{W}_2$  which handle all type (b) cycles.

Figure 1 gives a pictorial view of the components involved in the proof of our main technical result.

The rest of the paper is organized as follows. In Section 2 we give the necessary definitions and state results from earlier work, that we use in this paper. In Section 3 we give matching preserving, logspace reductions from a combinatorial embedding of the graph on a surface of genus  $g$ , to a grid embedding. Due to space constraints we omit the proof of the reductions (for more details regarding the proofs please refer to the ECCC version of this paper [7]). In Section 4 we state and prove our upper bounds assuming a grid embedding. In Section 5 we reduce the non-orientable case to the orientable one.



■ **Figure 1** Outline of the steps. Note that all reductions are matching preserving and logspace computable.

## 2 Preliminaries

### 2.1 Topological graph theory

We introduce the necessary terminology from algebraic topology. For a more comprehensive understanding of this topic, refer to any standard algebraic topology book such as [12].

A *2-manifold* is a topological space such that every point has an open neighborhood homeomorphic to  $\mathbb{R}^2$  and two distinct points have disjoint neighborhoods. A 2-manifold is often called a *surface*. The *genus* of a surface  $\Gamma$  is the maximum number  $g$ , such that there are  $g$  cycles  $C_1, C_2, \dots, C_g$  on  $\Gamma$ , with  $C_i \cap C_j = \emptyset$  for all  $i, j$  and  $\Gamma \setminus (C_1 \cup C_2 \cup \dots \cup C_g)$  is connected. A surface is called *orientable* if it has two distinct sides, else it is called *non-orientable*. A cycle  $C$  in  $\Gamma$  is said to be *non-separating* if there exists a path between any two points in  $\Gamma \setminus C$ , else it is called *separating*.

A *polygonal schema* of a surface  $\Gamma$ , is a polygon with  $2g'$  directed sides, such that the sides of the polygon are partitioned into  $g'$  classes, each class containing exactly two sides and gluing the two sides of each equivalence class gives the surface  $\Gamma$  (upto homeomorphism). A side in the  $i$ th equivalence class is labelled  $\sigma_i$  or  $\bar{\sigma}_i$  depending on whether it is directed clockwise or anti-clockwise respectively. The *partner* of a side  $\sigma$  is the other side in its equivalence class. By an abuse of notation, we shall sometimes refer to the symbol of a side's partner, as the partner of the symbol. Frequently we will denote a polygonal schema as a linear ordering of its sides moving in a clockwise direction, denoted by  $X$ . For a polygonal schema  $X$ , we shall refer to any polygonal schema which is a cyclic permutation, or a reversal

of the symbols, or a complementation ( $\sigma$  mapped to  $\bar{\sigma}$  and vice versa) of the symbols, as being the same as  $X$ . A polygonal schema is called orientable (resp. non-orientable) if the corresponding surface is orientable (resp. non-orientable).

► **Definition 1.** An orientable polygonal schema is said to be in *normal form* if it is in one of the following forms:

$$\sigma_1\tau_1\bar{\sigma}_1\bar{\tau}_1\sigma_2\tau_2\bar{\sigma}_2\bar{\tau}_2\dots\sigma_m\tau_m\bar{\sigma}_m\bar{\tau}_m \tag{2.1}$$

$$\sigma\bar{\sigma} \tag{2.2}$$

A non-orientable polygonal schema is said to be in normal form if it is of one of the following forms:

$$\sigma\sigma X \tag{2.3}$$

$$\sigma\tau\bar{\sigma}\tau X \tag{2.4}$$

where,  $X$  is a string representing an orientable schema in normal form (i.e. like Form 2.1 or 2.2 above) or possibly an empty string.

We denote the polygonal schema in the normal form of a surface  $\Gamma$  as  $\Lambda(\Gamma)$ . We will refer to two orientable symbols  $\sigma, \tau$  which form the following contiguous substring:  $\sigma\tau\bar{\sigma}\bar{\tau}$  as being clustered together while a non-orientable symbol  $\sigma$  which occurs like  $\sigma\sigma$  as a contiguous substring is said to form a *pair*. Thus, in the first and third normal forms above all symbols are clustered. The first normal form represents a connected sum of torii and the third of a projective plane and torii. In the fourth normal form all but one of the orientable symbols are clustered while the only non-orientable symbol is sort of clustered with the other orientable symbol. This form represents a connected sum of a Klein Bottle and torii. The second normal form represents a sphere.

We next introduce the concept of  $\mathbb{Z}_2$ -homology. Given a 2-manifold  $\Gamma$ , a *1-cycle* is a closed curve in  $\Gamma$ . The set of 1-cycles forms an Abelian group, denoted as  $\mathcal{C}_1(\Gamma)$ , under the *symmetric difference* operation,  $\Delta$ . Two 1-cycles  $C_1, C_2$  are said to be homologically equivalent if  $C_1\Delta C_2$  forms the boundary of some region in  $\Gamma$ . Observe that this is an equivalence relation. Then the *first homology group* of  $\Gamma$ ,  $H_1(\Gamma)$ , is the set of equivalence classes of 1-cycles. In other words, if  $\mathcal{B}_1(\Gamma)$  is defined to be the subset of  $\mathcal{C}_1(\Gamma)$  that are homologically equivalent to the empty set, then  $H_1(\Gamma) = \mathcal{C}_1(\Gamma)/\mathcal{B}_1(\Gamma)$ . If  $\Gamma$  is a genus  $g$  surface then  $H_1(\Gamma)$  is generated by a system of  $2g$  1-cycles, having only one point in common, and whose complement is homeomorphic to a topological disk. Such a disk is also referred to as the *fundamental polygon* of  $\Gamma$ .

An undirected graph  $G$  is said to be embedded on a surface  $\Gamma$  if it can be drawn on  $\Gamma$  so that no two edges cross. We assume that the graph is given with a *combinatorial embedding* on a surface of constant genus. Refer to the book by Mohar and Thomassen [13] for details. The genus of a graph  $G$  is the minimum number  $g$  such that  $G$  has an embedding on a surface of genus  $g$ . We shall also refer to such an embedding as the *minimal embedding* of  $G$ . A genus  $g$  graph is said to be orientable (non-orientable) if the surface is orientable (non-orientable). A *2-cell embedding* of a graph is a combinatorial embedding of the graph on a surface such that every face is homeomorphic to the disk. Note that a minimal embedding of a graph is always a 2-cell embedding but the converse is not true. For our purposes it is enough to assume a 2-cell embedding of the given graph.

► **Definition 2.** The *polygonal schema of a graph  $G$*  is a combinatorial embedding given on the polygonal schema of some surface  $\Gamma$  together with the ordered set of vertices on each

side of the polygon. Formally it is a tuple  $(\phi, \mathcal{S})$ , where  $\phi$  is a cyclic ordering of the edges around a vertex (also known as the *rotation system* of  $G$ ) and  $\mathcal{S} = (S_1, S_2, \dots, S_{2g})$  is the cyclic ordering of the directed sides of the polygon. Each  $S_i$  is an ordered sequence of the vertices, from the tail to the head of the side  $S_i$ . Moreover every  $S_i$  is paired with some other side, say  $S_i^{-1}$  in  $\mathcal{S}$ , such that the  $j$ th vertex of  $S_i$  (say from the tail of  $S_i$ ) is the same as the  $j$ th vertex of  $S_i^{-1}$  (from the tail of  $S_i^{-1}$ ).

## 2.2 Complexity Theory

For a nondeterministic machine  $M$ , let  $acc_M(x)$  and  $rej_M(x)$  denote the number of accepting computations and the number of rejecting computations respectively on an input  $x$ . Denote  $gap_M(x) = acc_M(x) - rej_M(x)$ .

► **Definition 3.** A language  $L$  is in SPL if there exists a logspace bounded nondeterministic machine  $M$  so that for all inputs  $x$ ,  $gap_M(x) \in \{0, 1\}$  and  $x \in L$  if and only if  $gap_M(x) = 1$ .  $FL^{SPL}$  is the class of functions computed by a logspace machine with an SPL oracle.  $UL$  is the class of languages  $L$ , decided by a nondeterministic logspace machine (say  $M$ ), such that for every string in  $L$ ,  $M$  has exactly one accepting path and for a string not in  $L$ ,  $M$  has no accepting path.

Alternatively, we can define SPL as the class of problems logspace reducible to the problem of checking whether the determinant of a matrix is 0 or not under the promise that the determinant is either 0 or 1. For definitions of other complexity classes refer to any standard textbooks such as [2, 17]. All reductions discussed in this paper are logspace reductions.

Given an undirected graph  $G = (V, E)$ , a *matching*  $M$  is a subset of  $E$  such that no two edges in  $M$  have a vertex in common. A *maximum matching* is a matching of maximum cardinality.  $M$  is said to be a *perfect matching* if every vertex is an endpoint of some edge in  $M$ .

► **Definition 4.** We define the following computational problems related to matching:

- DECISION-BPM : Given a bipartite graph  $G$ , checking if  $G$  has a perfect matching.
- SEARCH-BPM: Given a bipartite graph  $G$ , constructing a perfect matching, if one exists.
- UNIQUE-BPM: Given a bipartite graph  $G$ , checking if  $G$  has a unique perfect matching.

## 2.3 Necessary Prior Results

► **Lemma 5** ([6]). *For any bipartite graph  $G$  and a weight function  $w$ , if all circulations of  $G$  are non-zero, then  $G$  has a unique minimum weight perfect matching.*

► **Lemma 6** ([1]). *For any weighted graph  $G$  assume that the minimum weight perfect matching in  $G$  is unique and also for any subset of edges  $E' \subseteq E$ , the minimum weight perfect matching in  $G \setminus E'$  is also unique. Then deciding if  $G$  has a perfect matching is in SPL. Moreover, computing the perfect matching (in case it exists) is in  $FL^{SPL}$ .*

## 3 Embedding on a Grid

We define  $K\text{-ORI-GG}$  to be the class of genus  $g$  graphs such that: for every  $G \in K\text{-ORI-GG}$ ,  $G$  is a grid graph embedded on a grid of size  $2m \times 2m$ . We assume that the distance between adjacent horizontal (and similarly vertical) vertices is of unit length. The entire boundary of the grid is divided into  $4g$  segments, and each segment has even length, for some constant  $g$ . The  $4g$  segments are labeled as  $(S_1, S_2, S'_1, S'_2, \dots, S_{2i-1}, S_{2i}, S'_{2i-1}, S'_{2i},$

$\dots, S_{2g-1}, S_{2g}, S'_{2g-1}, S'_{2g}$ ), together with a direction, namely,  $S_i$  is directed from counter-clockwise and  $S'_i$  is directed from clockwise for each  $i \in [2g]$ . The  $j$ th vertex on a segment  $S_i$  is the  $j$ th vertex on the border of the grid, starting from the tail of the segment  $S_i$  and going along the direction of the segment. Finally the segments  $S_i$  and  $S'_i$  are glued to each other for each  $i \in [2g]$  in the same direction. In other words, the  $j$ th vertex on segment  $S_i$  is the same as the  $j$ th vertex on segment  $S'_i$ . Also there are no edges along the boundary of the grid. In Theorem 7 we show that it is enough to consider graphs in K-ORI-GG.

► **Theorem 7.** *Given a 2-cell embedding of a graph  $G$  of constant genus, there is a logspace transducer that constructs a graph  $G' \in \text{K-ORI-GG}$ , such that, there is a perfect matching in  $G$  iff there is a perfect matching in  $G'$ . Moreover, given a perfect matching  $M'$  in  $G'$ , in logspace one can construct a perfect matching  $M$  in  $G$ .*

We divide the construction in Theorem 7 in an iterative manner starting from a 2-cell embedding. Applying Lemma 8 we first get an embedding on the polygonal schema of the graph. Then we normalize the obtained polygonal schema by applying Theorem 9. Finally we give an embedding of the graph on a grid by applying Lemma 10.

► **Lemma 8.** *Given the combinatorial embedding of a constant genus graph we can find a polygonal schema for the graph in logspace.*

► **Theorem 9.** *Given a combinatorial embedding of constant genus, say  $g$  (which is positive or otherwise), for a graph  $G$ , in logspace we can find a polygonal schema for the graph in normal form. of genus  $O(|g|)$  in magnitude, and also the corresponding combinatorial embedding.*

Let K-GON-BI be the class of constant genus, bipartite graphs along with an embedding given on the polygonal schema in normal form of the surface in which the graph has an embedding. Moreover, for every graph in this class, no edge has both its end points incident on the boundary of the polygon.

► **Lemma 10.** *If  $G$  is an orientable graph in K-GON-BI, then one can get a logspace, matching-preserving reduction from  $G$  to a graph  $H \in \text{K-ORI-GG}$*

## 4 New Upper Bounds

In this section we establish new upper bounds on the space complexity of certain matching problems on bipartite constant genus graphs, embedded on a ‘genus  $g$  grid’.

► **Definition 11.** If  $C$  is a cycle in  $G$ , we denote the circulation of  $C$  with respect to a weight function  $w$  as  $\text{circ}_w(C)$ . For any subset  $E' \subseteq C$ ,  $\text{circ}_w(E')$  is the value of the circulation restricted to the edges of  $E'$ . An example of a cycle on a grid is given in Figure 2.

► **Theorem 12 (Main Theorem).** *Given a graph  $G \in \text{K-ORI-GG}$ , there exists a logspace computable and polynomially bounded weight function  $W : E(G) \rightarrow \mathbb{Z}$ , such that for any cycle  $C \in G$ ,  $\text{circ}_W(C) \neq 0$ .*

► **Theorem 13.** *For a graph embedded on a constant genus surface,*

- (a) DECISION-BPM is in SPL,
- (b) SEARCH-BPM is in  $\text{FL}^{\text{SPL}}$  and
- (c) UNIQUE-BPM is in SPL.

**Proof.** As a result of Theorem 7, we can assume that our input graph  $G \in \text{K-ORI-GG}$ . Using Theorem 12 and Lemma 5 we get a logspace computable weight function  $W$ , such that the minimum weight perfect matching in  $G$  with respect to  $W$  is unique. Moreover, for any subset  $E' \subseteq E$ , Theorem 12 is valid for the subgraph  $G \setminus E'$  also, with respect to the same weight function  $W$ . Now (a) and (b) follows from Lemma 6. Checking for uniqueness can be done by first computing a perfect matching, then deleting an edge from the matching and rechecking to see if a perfect matching exists in the new graph. If it does, then  $G$  did not have a unique perfect matching, else it did. Note that Theorem 12 is valid for any graph formed by deletion of edges of  $G$ . ◀

Theorem 12 also gives an alternative proof of directed graph reachability for constant genus graphs.

► **Theorem 14** ([4, 11]). *Directed graph reachability for constant genus graphs is in UL.*

The proof of Theorem 14 follows from Lemma 15 and [4]. We adapt Lemma 15 from [6].

► **Lemma 15.** *There exist a logspace computable weight function that assigns polynomially bounded weights to the edges of a directed graph such that: (a) the weights are skew symmetric, i.e.,  $w(u,v) = -w(v,u)$ , and (b) the sum of weights along any (simple) directed cycle is non-zero.*

► **Lemma 16.** *In any class of graphs closed under the subdivision of edges, Theorem 12 implies the hypothesis of Lemma 15.*

#### 4.1 Proof of Main Theorem

**Proof of Theorem 12.** For a graph  $G \in \text{K-ORI-GG}$ , we define  $W$  is a linear combination of the following  $4g + 1$  weight functions defined below. This is possible in logspace since  $g$  is constant.

Define  $4g + 1$  weight functions as follows:

- For each  $i \in [2g]$ ,

$$w_i(e) = \begin{cases} 1 & \text{if } e \text{ lies on the segment } S_i \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

- For each  $i \in [2g]$ ,

$$w'_i(e) = \begin{cases} j & \text{if } e \text{ lies on the segment } S_i \text{ at index } j \text{ from the head of } S_i \text{ and } j \text{ is odd} \\ -j & \text{if } e \text{ lies on the segment } S_i \text{ at index } j \text{ from the head of } S_i \text{ and } j \text{ is even} \\ 0 & \text{otherwise} \end{cases} \quad (4.2)$$

-

$$w''(e) = \begin{cases} (-1)^{i+j}(i-1) & \text{if } e \text{ is the } j\text{th horizontal edge from left, lying in row } i \\ & \text{from bottom, and not lying on the boundary} \\ 0 & \text{otherwise} \end{cases} \quad (4.3)$$



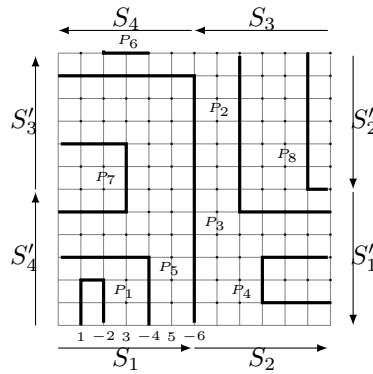


Figure 2 Example of a cycle on the grid that crosses each segment an even number of times with the weights  $w'_1$

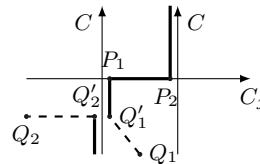


Figure 3 Construction of a path from  $Q_1$  to  $Q_2$  in  $\Gamma \setminus C$  (the dotted path is a path between  $Q_1$  and  $Q'_1$  (resp. between  $Q_2$  and  $Q'_2$ )).

Note that if  $e$  does not lie on the boundary of the grid then  $w''(e)$  is same as the weight function defined in [6].

Let  $C$  be a simple cycle in  $G$ . If  $C$  does not intersect any of the boundary segments, then  $C$  does not have any edge on the boundary since there are no edges along the boundary by definition of  $K$ -ORI-GG. Therefore  $\text{circ}_{w''}(C) \neq 0$  by [6]. Now suppose there exists a segment  $S_i$ , such that  $C$  crosses  $S_i$  an odd number of times. Then  $\text{circ}_{w_i}(C) \neq 0$ . Otherwise  $C$  crosses each segment an even number of times. Now without loss of generality, assume  $C$  intersects segment  $S_1$ . Let  $E_1^C$  be the set of edges of  $C$  that intersect  $S_1$ . Note that  $\text{circ}_{w'_1}(C) = \text{circ}_{w'_1}(E_1^C)$ . By Lemma 18 it follows that the edges of  $E_1^C$ , alternate between going out and coming into the grid. Then using Lemma 19 we get that  $\text{circ}_{w'_1}(E_1^C) \neq 0$  and thus  $\text{circ}_{w'_1}(C) \neq 0$ . (See below for Lemma 18 and 19) ◀

To establish Lemma 18 we use an argument (Lemma 17) from homology theory. For two cycles (directed or undirected)  $C_1$  and  $C_2$ , let  $I(C_1, C_2)$  denote the number of times  $C_1$  and  $C_2$  cross each other (that is one of them goes from the left to the right side of the other, or vice versa).

Next we adapt the following Lemma from Cabello and Mohar [5]. Here we assume we are given an orientable surface (Cabello and Mohar gives a proof for a graph on a surface).

► **Lemma 17** ([5]). *Given a genus  $g$  orientable, surface  $\Gamma$ , let  $\mathcal{C} = \{C_i\}_{i \in [2g]}$  be a set of cycles that generate the first homology group  $H_1(\Gamma)$ . A cycle  $C$  in  $\Gamma$  is non-separating if and only if there is some cycle  $C_i \in \mathcal{C}$  such that  $I(C, C_i) \equiv 1 \pmod{2}$ .*

**Proof.** Let  $\tilde{C}$  be some cycle in  $\Gamma$ . We can write  $\tilde{C} = \sum_{i \in [2g]} t_i C_i$  since  $\mathcal{C}$  generates  $H_1(\Gamma)$ . Define  $I_{\tilde{C}}(C) = \sum_{i \in [2g]} t_i I(C, C_i) \pmod{2}$ . One can verify that  $I_{\tilde{C}} : \mathcal{C}_1(\Gamma) \rightarrow \mathbb{Z}_2$  is a group homomorphism. Now since  $\mathcal{B}_1(\Gamma)$  is a normal subgroup of  $\mathcal{B}_1(\Gamma)$ ,  $I_{\tilde{C}}$  induces a homomorphism from  $H_1(\Gamma)$  to  $\mathbb{Z}_2$ .

Any cycle is separating if and only if it is homologous to the empty set. Therefore if  $C$  is separating, then  $C \in \mathcal{B}_1(\Gamma)$  and thus every homomorphism from  $H_1(\Gamma)$  to  $\mathbb{Z}_2$  maps it to 0. Hence for every  $i \in [2g]$ ,  $I(C, C_i) \equiv I_{C_i}(C) = 0$ .

Suppose  $C$  is non-separating. One can construct a cycle  $C'$  on  $\Gamma$ , that intersects  $C$  exactly once. Let  $C' = \sum_{i \in [2g]} t'_i C_i$ . Now  $1 \equiv I_{C'}(C) \equiv \sum_{i \in [2g]} t'_i I(C, C_i) \pmod{2}$ . This implies that there exists  $i \in [2g]$  such that  $I(C, C_i) \equiv 1 \pmod{2}$ . ◀

► **Lemma 18.** *Let  $C$  be a simple directed cycle on a genus  $g$  orientable surface  $\Gamma$  and let  $\mathcal{C} = \{C_i\}_{i \in [2g]}$  be a system of  $2g$  directed cycles on  $\Gamma$ , having exactly one point in common and  $\Gamma \setminus \mathcal{C}$  is the fundamental polygon, say  $\Gamma'$ . If  $I(C, C_i)$  is even for all  $i \in [2g]$  then for all  $j \in [2g]$ ,  $C$  alternates between going from left to right and from right to left of the cycle  $C_j$  in the direction of  $C_j$  (if  $C$  crosses  $C_j$  at all).*

**Proof.** Suppose there exists a  $j \in [2g]$  such that  $C$  does not alternate being going from left to right and from right to left with respect to  $C_j$ . Thus if we consider the ordered set of points where  $C$  intersects  $C_j$ , ordered in the direction of  $C_j$ , there are two consecutive points (say  $P_1$  and  $P_2$ ) such that at both these points  $C$  crosses  $C_j$  in the same direction.

Let  $Q_1$  and  $Q_2$  be two points in  $\Gamma \setminus C$ . We will show that there exists a path in  $\Gamma \setminus C$  between  $Q_1$  and  $Q_2$ . Consider the shortest path from  $Q_1$  to  $C$ . Let  $Q'_1$  be the point on this path that is as close to  $C$  as possible, without lying on  $C$ . Similarly define a point  $Q'_2$  corresponding to  $Q_2$ . Note that it is sufficient for us to construct a path between  $Q'_1$  and  $Q'_2$  in  $\Gamma \setminus C$ . If both  $Q'_1$  and  $Q'_2$  locally lie on the same side of  $C$ , then we get a path from  $Q'_1$  to  $Q'_2$  not intersecting  $C$ , by traversing along the boundary of  $C$ . Now suppose  $Q'_1$  and  $Q'_2$  lie on opposite sides (w.l.o.g. assume that  $Q'_1$  lies on the right side) of  $C$ . From  $Q'_1$  start traversing the cycle until you reach cycle  $C_j$  (point  $P_1$  in Figure 3). Continue along cycle  $C_j$  towards the adjacent intersection point of  $C$  and  $C_j$ , going as close to  $C$  as possible, without intersecting it (point  $P_2$  in Figure 3). Essentially this corresponds to switching from one side of  $C$  to the other side without intersecting it. Next traverse along  $C$  to reach  $Q'_2$ . Thus we have a path from  $Q'_1$  to  $Q'_2$  in  $\Gamma \setminus C$ . We give an example of this traversal in Figure 3. This implies that  $C$  is non-separating.

It is well known that  $\mathcal{C}$  forms a generating set of  $H_1(\Gamma)$ , the first homology group of the surface. Now from Lemma 17 it follows that  $I(C, C_l) \equiv 1 \pmod{2}$  for some  $l \in [2g]$ , which is a contradiction. ◀

► **Lemma 19.** *Let  $G$  be a graph in K-ORI-GG with  $C$  being a simple cycle in  $G$  and  $E_1^C$  being the set of edges of  $C$  that intersect segment  $S_1$ . Assume  $|E_1^C|$  is even and the edges in  $E_1^C$  alternate between going out and coming into the grid. Let  $i_1 < i_2 < \dots < i_{2p-1} < i_{2p}$  be the distinct indices on  $S_1$  where  $C$  intersects it. Then,  $|\text{circ}_{w'_1}(E_1^C)| = |\sum_{k=1}^p (i_{2k} - i_{2k-1})|$  and thus non-zero unless  $E_1^C$  is empty.*

**Proof.** Let  $e_j = (u_j, v_j)$  for  $j \in [2p]$  be the  $2p$  edges of  $G$  incident on the segment  $S_1$ . Assume without loss of generality that the vertices  $v_j$ 's lie on  $S_1$ . Assign an orientation to  $C$  such that  $e_1$  is directed from  $u_1$  to  $v_1$ . Also assume that  $i_1$  is even and the circulation gives a positive sign to the edge  $e_1$ . Therefore  $\text{circ}_{w'_1}(\{e_1\}) = -i_1$ .

Now consider any edge  $e_j$  such that  $j$  is even. By Lemma 18, the edge enters the segment  $S_1$  (i.e., the head of the edge with respect to the assigned orientation is incident on  $S_1$ ). Suppose  $i_j$  is odd. Then consider the following cycle  $C'$  formed by tracing  $C$  from  $u_j$  to  $u_1$ , without the edges  $e_1$  and  $e_j$  and then moving along the segment  $S_1$  back to  $u_j$ . Since  $i_j$  is odd therefore the latter part of  $C'$  has odd length. Note that  $C'$  need not be a simple cycle. By Lemma 20,  $|C'|$  is even, therefore the part of  $C'$  from  $u_1$  to  $u_j$  also has odd length. This

implies that the circulation gives a positive sign to the edge  $e_j$ . Therefore,  $\text{circ}_{w'_1}(\{e_j\}) = i_j$ . Similarly, if  $i_j$  is odd, then the part of  $C'$  from  $u_1$  to  $u_j$  will have even length. Thus the circulation gives a negative sign to the edge  $e_j$  and therefore  $\text{circ}_{w'_1}(\{e_j\}) = -(-i_j) = i_j$ .

If  $j$  is odd, the above argument can be applied to show that  $\text{circ}_{w'_1}(\{e_j\}) = -i_j$ . Therefore we have,  $\text{circ}_{w'_1}(E_1^C) = \sum_{k=1}^p (i_{2k} - i_{2k-1})$ .

Now removing the assumptions at the beginning of this proof would show that the LHS and RHS of the above equation is true modulo absolute value as required. ◀

To prove Lemma 19 we need to argue that any graph in a “genus  $g$  grid” is bipartite and thus any cycle will have even length. Lemma 20 establishes this fact.

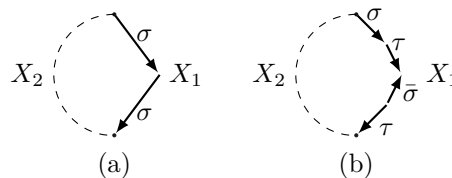
► **Lemma 20.** *Any graph  $G \in \text{K-ORI-GG}$  is bipartite.*

It is interesting to note here that similar method does not show that bipartite matching in non-orientable constant genus graphs is in SPL. The reason is that Lemma 18 crucially uses the fact that the surface is orientable. In fact, one can easily come with counterexample to the Lemma if the surface is non-orientable.

## 5 Reducing the non-orientable case to the orientable case

Let  $G$  be a bipartite graph embedded on a genus  $g$  non-orientable surface. As a result of Theorem 9 we can assume that we are given a combinatorial embedding (say  $\Pi$ ) of  $G$  on a (non-orientable) polygonal schema, say  $\Lambda(\Gamma)$ , in the normal form with  $2g'$  sides. (Here  $g'$  is a function of  $g$ .)

Let  $Y = (X_1, X_2)$  be the cyclic ordering of the labels of the sides of  $\Lambda(\Gamma)$ , where  $X_2$  is the ‘orientable part’ and  $X_1$  is the ‘non-orientable part’. More precisely, for the polygonal schema in the normal form, we have:  $X_1$  is either  $(\sigma, \sigma)$  (thus corresponds to the projective plane) or it is  $(\sigma, \tau, \bar{\sigma}, \tau)$  (thus corresponds to the Klein bottle). See Figure 4.



■ **Figure 4** (a)  $\Lambda(\Gamma)$  when the surface is a sum of an orientable surface and the projective plane. (b)  $\Lambda(\Gamma)$  when the surface is a sum of an orientable surface and the Klein bottle.

Now let  $G$  be a bipartite graph embedded on a non-orientable polygonal schema  $\Lambda(\Gamma)$  with  $2g'$  sides. We will construct a graph  $G'$  embedded on an *orientable* polygonal schema with  $4g' - 2$  sides such that  $G$  has a perfect matching iff  $G'$  has a perfect matching. Moreover, given a perfect matching in  $G'$  one can retrieve in logspace a perfect matching in  $G$ . This is illustrated in Theorem 21.

► **Theorem 21.** *Let  $G$  be a bipartite graph given with its embedding on a non-orientable polygonal schema in normal form  $\Lambda(\Gamma)$ , with  $2g'$  sides as above. One can construct in logspace, another graph  $G'$  together with its embedding on the polygonal schema of an orientable surface  $\Gamma'$  of genus  $4g' - 2$  such that:  $G$  has a perfect matching iff  $G'$  has a perfect matching. Moreover, given a perfect matching in  $G'$ , one can construct in logspace a perfect matching in  $G$ .*

Thus we see that the non-orientable case can be reduced to the orientable case. The resulting polygonal schema need not be in the normal form. Once again we apply Theorem 9 to get a combinatorial embedding on a polygonal schema in the normal form.

## Acknowledgment

The third author would like to thank Prof. Mark Brittenham from the Mathematics department at the University of Nebraska-Lincoln, for numerous discussions that they had and for providing valuable insight into topics in algebraic topology. We would also like to thank the anonymous referees for their valuable comments and suggestions which helped in improving the overall presentation of the paper.

---

## References

- 1 Eric Allender, Klaus Reinhardt, and Shiyu Zhou. Isolation, matching, and counting: Uniform and nonuniform upper bounds. *Journal of Computer and System Sciences*, 59:164–181, 1999.
- 2 Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 1 edition, 2009.
- 3 V. Arvind and Partha Mukhopadhyay. Derandomizing the isolation lemma and lower bounds for circuit size. In *Proceedings of RANDOM '08*, pages 276–289, 2008.
- 4 Chris Bourke, Raghunath Tewari, and N. V. Vinodchandran. Directed planar reachability is in unambiguous log-space. *ACM Trans. Comput. Theory*, 1(1):1–17, 2009.
- 5 Sergio Cabello and Bojan Mohar. Finding shortest non-separating and non-contractible cycles for topologically embedded graphs. *Discrete Comput. Geom.*, 37(2):213–235, 2007.
- 6 Samir Datta, Raghav Kulkarni, and Sambuddha Roy. Deterministically isolating a perfect matching in bipartite planar graphs. *Theory of Computing Systems*, 47(3):737–757, 2010.
- 7 Samir Datta, Raghav Kulkarni, Raghunath Tewari, and N. V. Vinodchandran. Space complexity of perfect matching in bounded genus bipartite graphs. Technical Report TR10-079, Electronic Colloquium on Computational Complexity, 2010.
- 8 Anna Gal and Avi Wigderson. Boolean complexity classes vs. their arithmetic analogs. *Random Structures and Algorithms*, 9:1–13, 1996.
- 9 Thanh Minh Hoang. On the matching problem for special graph classes. In *IEEE Conference on Computational Complexity*, pages 139–150, 2010.
- 10 Raghav Kulkarni, Meena Mahajan, and Kasturi R. Varadarajan. Some perfect matchings and perfect half-integral matchings in NC. *Chicago Journal of Theoretical Computer Science*, 2008(4), September 2008.
- 11 Jan Kynčl and Tomáš Vyskočil. Logspace reduction of directed reachability for bounded genus graphs to the planar case. *ACM Trans. Comput. Theory*, 1(3):1–11, 2010.
- 12 William S. Massey. *A Basic Course in Algebraic Topology*. Springer-Verlag, 1991.
- 13 Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces*. John Hopkins University Press, 2001.
- 14 Ketan Mulmuley, Umesh Vazirani, and Vijay Vazirani. Matching is as easy as matrix inversion. *Combinatorica*, 7:105–113, 1987.
- 15 Klaus Reinhardt and Eric Allender. Making nondeterminism unambiguous. *SIAM Journal of Computing*, 29:1118–1131, 2000. An earlier version appeared in FOCS 1997, pp. 244–253.
- 16 C. Thomassen. The graph genus problem is np-complete. *J. Algorithms*, 10(4):568–576, 1989.
- 17 Heribert Vollmer. *Introduction to Circuit Complexity - A Uniform Approach*. Springer-Verlag, 1999.