Performance laws of large heterogeneous cellular networks

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***Abstract*—We propose a model for heterogeneous cellular networks assuming a space-time Poisson process of call arrivals, independently marked by data volumes, and served by different types of base stations (having different transmission powers) represented by the superposition of independent Poisson processes on the plane. Each station applies a processor sharing policy to serve users arriving in its vicinity, modeled by the Voronoi cell perturbed by some random signal propagation effects (shadowing). Users' peak service rates depend on their signal-to-interference-and-noise ratios (SINR) with respect to the serving station. The mutual-dependence of the cells (due to the extra-cell interference) is captured via some system of cell-load equations impacting the spatial distribution of the SINR. We use this model to study in a semi-analytic way (involving only static simulations, with the temporal evolution handled by the queuing theoretic results) network performance metrics (cell loads, mean number of users) and the quality of service perceived by the users (mean throughput) served by different types of base stations. Our goal is to identify macroscopic laws regarding these performance metrics, involving averaging both over time and the network geometry. The reveled laws are validated against real field measurement in an operational network.**

***Index Terms*—Het-Nets; traffic demand; user-throughput; cell-load; processor sharing; Little's law; Poisson point process; typical-cell; queuing theory; Palm theory; measurements**

I. Introduction

Wireless cellular networks are constantly evolving to cope with the accelerating increase of the traffic demand. The technology progressed from 3G enhancement with HSDPA (High-Speed Downlink Packet Access) to 4G with LTE (Long Term Evolution). The networks become also more dense and more heterogeneous; i.e. new base stations (BS) of different types are added. In particular, operators introduce *micro* BS, which transmit with smaller powers than the original ones (called *macro* BS) in order to cope with local increase of the traffic demand (hotspots). The reasons for using smaller transmitting powers is to avoid a harmful increase of interference and reduce energy consumption as well as human exposure to the electromagnetic radiation. The deployment of micro BS is expected to increase significantly in the nearest future.

Usage of different tiers of BS (as the micro and macro stations) with variable transmission powers as well as antenna gains, height etc, makes cellular networks heterogeneous. Besides, even the macro tiers in commercial cellular networks are never perfectly regular: the locations of BS is usually far from being perfectly hexagonal, because of various deployment constraints. Irregularity of the spatial patterns of BS is usually more pronounced in dense urban environments. Physical irregularity of the urban environment (shadowing) induces additional variability of radio conditions. Irregularity and heterogeneity of cellular networks implies a spatial disparity of base station performance metrics and quality of service (QoS) parameters observed by users in different cells of the network. This represents a challenge for the network operators, in particular in the context of the network dimensioning. How to describe and analyze the performance of a large, irregular, heterogeneous network? Which tier in the given network disposes larger capacity margins? Is it the macro tier since its BS transmit with larger powers or the micro tier, whose BS serve smaller zones? The goal of this paper is to propose a model, validated with respect to real field measurements in an operational network, which can help answering these questions.

Our objective faces us with the following important aspects of the modeling problem: (i) capturing the static but irregular and heterogeneous network geometry, (ii) considering the dynamic user service process at individual network BS (cells), and last but not least (iii) taking into account the dependence between these service processes. This latter dependence is due to the fact that the extra-cell interference makes the service of a given cell depend on the "activity" of other cells in the network. Historically, geometric (i) and dynamic (ii) aspects are usually addressed separately on the ground of stochastic geometry and queueing theory, respectively.

Cellular network models based on the planar Poisson point process have been shown recently to give tractable expressions for many characteristics built from the powers of different BS received at one given location, as e.g. the signal-to-interference-and noise ratio(s) (SINR) of the, so-called, typical user. They describe potential resources of the network (peak bit-rates, spectral or energy efficiency etc) but not yet its real performance when several users have to share these resources. On the other hand, various classical queueing models can be tailored to represent the dynamic resource sharing at one or several BS (as e.g. loss models for constant bit-rates services and processor sharing queues for variable bit-rates services).

Our model considered in this paper combines the stochastic-geometric approach with the queueing one to represent the network in its spacial irregularity and temporal evolution. It assumes the usual *multi-tier Poisson model for BS locations* with shadowing and the *space-time Poisson process of call arrivals* independently marked by data volumes. Each station applies a processor sharing policy to serve users which receive its signal as the strongest one, with the peak service rates depending on the respective SINR. The mutual-dependence of cell performance (iii) is captured via a system of *cell-load (fixed point) equations*. By the load we mean the ratio of the actual traffic demand to its critical value, which can be interpreted, when it is smaller than one, as the busy probability in the classical processor sharing queue. The cell load equations make the load of a given station dependent on the busy probabilities (hence loads) of other stations, by taking them as weighting factors of the interference induced by these stations. Given network realization, this decouples the temporal (processor-sharing) queueing processes of different cells, allowing us to use the classical results to evaluate their steady state characteristics (which depend on the network geometry). We (numerically) solve the cell-load fixed point problem calculating loads and other characteristics of the individual cells. Appropriate spatial

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(network) averaging of these characteristics, expressed using the formalism of the typical cell offered by Palm theory of point processes, provides useful macroscopic description of the network performance.

The above approach is validated by estimating the model parameters from the real field measurements of a given operational network and comparing the macroscopic network performance characteristics calculated using this model to the performance of the real network.

The remaining part of the paper is organized as follows: In Section I-A we briefly present the related work. Our model is introduced in Section II and studied in Section III. Numerical results validating our approach are presented in Section IV.

A. Related work

There are several "pure" simulation tools developed for the performance evaluation of cellular networks such as those developed by the industrial contributors to 3GPP (*3rd Generation Partnership Project*) [1], TelematicsLab LTE-Sim [2], University of Vien LTE simulator [3, 4] and LENA tool [5, 6] of CTTC. They do not unnecessarily allow to identify the macroscopic laws regarding network performance metrics.

A possible analytical approach to this problem is based on the information theoretic characterization of the individual link performance; cf e.g. [7, 8], in conjunction with a queueing theoretic modeling and analysis of the user traffic cf. e.g. [9–14]. These works are usually focused on some particular aspects of the network and do not consider a large, irregular, heterogeneous, multi-cell scenario.

Stochastic geometric approach [15] to wireless communication networks consist in taking spatial averages over node (emitter, receiver) locations. It was first shown in [16] to give analytically tractable expressions for the typical-user characteristics in Poisson models of cellular networks, with the Poisson assumption being justified by representing highly irregular base station deployments in urban areas [17] or mimicking strong log-normal shadowing [18, 19], or both. Expressions for the SINR coverage in multi-tier network models were developed in [20–23]. Several extensions of this initial model are reported in [24]. The concept of equivalence of heterogeneous networks (from the point of view of its typical user), which we use in the present paper, was recently formulated in [25], but previously used in several works e.g. in [23, 26–28].

The fixed-point cell-load equation was postulated independently in [14] and [29] to capture the dependence of processor sharing queues modeling performance of individual BS, in the context of regular hexagonal and fixed deterministic network models, respectively. Our present paper, combining stochastic geometry with queueing theory complements [30], where a homogeneous network is considered, and [31] where the distribution of the QoS metrics in the heterogeneous network has been studied by simulation. A network dimensioning methodology based on this approach was recently proposed in [32].

II. Model description

In this section we describe the components of our model.

A. Network geometry

1) Multi-tier network of BS: We consider a multi-tier cellular network consisting of J types (tiers) of BS characterized by different transmitting powers P_j, $j = 1, \ldots, J$. Locations of BS are modeled by independent homogeneous Poisson point processes Φ_j on the plane, of intensity λ_j stations per km^2. Let $\Phi = \{X_n\}$ be the superposition of $\Phi_1, \ldots, \Phi_J$ (capturing the locations of all BS of the network). Denote by $Z_n \in \{1, \ldots, J\}$ the type of BS $X_n \in \Phi$ (i.e., the index of the tier it belongs to). It is known that Φ is a Poisson point process of intensity parameter $\lambda = \sum_{j=1}^{J} \lambda_j$ and Z_n form independent, identically distributed (i.i.d) marks of Φ with $\mathbb{P}(Z_n = j) = \lambda_j/\lambda$.

2) Propagation effects: The propagation loss is modeled by a deterministic *path-loss function* $l(x) = (K|x|)^{\beta}$, where $K > 0$ and $\beta > 2$ are given constants, and some random propagation effects. We split these effects into two categories conventionally called *(fast) fading* and *shadowing*. The former will be accounted in the model at the link-layer (in the peak bit-rate function cf. (34)). The latter impacts the choice of the serving BS and thus needs to be considered together with the network geometry. To this regard we assume that the shadowing between a given station $X_n \in \Phi$ and all locations y on the plane is modeled by some positive valued stochastic process $\mathbf{S}_n(y - X_n)$. We assume that the processes $\mathbf{S}_n(\cdot)$ are i.i.d. marks of Φ. [1] Moreover we assume that $\mathbf{S}_1(y)$ are identically distributed across y, but do not make any assumption regarding the dependence of $\mathbf{S}_n(y)$ across y.

Thus the inverse of the power averaged over fast fading, received at y from BS X_n, denoted by $L_{X_n}(y) = L_n(y)$ which we call (slightly abusing the terminology) the propagation-loss from this station is given by

$$L_{X_n}(y) = \frac{l(|y - X_n|)}{P_{Z_n}\mathbf{S}_n(y - X_n)}. \tag{1}$$

In what follows, we will often simplify the notation writing $L_X(\cdot)$ for the propagation-loss of BS $X \in \Phi$.

3) Service zones, SINR and peak bit-rates: We assume that each (potential) user located at y on the plane is served by the BS offering the strongest received power among all the BS in the network. Thus, the zone served by BS $X \in \Phi$, denoted by $V(X)$, which we keep calling *cell* of X (even if random shadowing makes it need not to be a polygon or even a connected set) is given by

$$V(X) = \left\{y \in \mathbb{R}^2 : L_X(y) \le L_Y(y) \text{ for all } Y \in \Phi\right\} \tag{2}$$

We define the (downlink) SINR at location $y \in V(X)$ (with respect to the serving BS $X \in \Phi$) as follows

$$\mathrm{SINR}(y, \Phi) := \frac{1/L_X(y)}{N + \sum_{Y \in \Phi \setminus \{X\}} \varphi_Y / L_Y(y)}, \tag{3}$$

where N is the noise power and the *activity factors* $\varphi_Y \in [0, 1]$ account (in a way that will be made specific in Section II-D) for the activity of stations $Y \in \Phi$. In general, we assume that φ_Y are additional (not necessarily independent) marks of the point process Φ, possibly dependent on tiers and shadowing of all BS.

We assume that the *(peak) bit-rate* at location y, defined as the number of bits per second a user located at y can download when served alone by its BS, is some function $R(\mathrm{SINR})$ of the SINR. Our general analysis presented in Section III does not depend on any particular form of this function. A specific expression will be assumed for the numerical results in Section IV.

B. Network users

1) User-arrival process: We consider variable bit-rate (VBR) traffic; i.e., users arrive to the network and require to transmit some volumes of data at bit-rates induced by the network. We

[1] The assumption that all types of base stations have the same distribution of the shadowing can be easily relaxed.

assume a *homogeneous time-space Poisson point process of user arrivals* of intensity γ arrivals per second per km^2. This means that the time between two successive arrivals in a given zone of surface S is exponentially distributed with parameter $\gamma \times S$, and all users arriving to this zone take their locations independently and uniformly. The time-space process of user arrivals is independently marked by random, identically distributed volumes of data the users want to download from their respective serving BS. These volumes are arbitrarily distributed and have mean $1/\mu$ bits.

The above arrival process induces the *traffic demand per surface unit* $\rho = \gamma/\mu$ expressed in bits per second per km^2. The *traffic demand in the cell of BS* $X \in \Phi$ equals

$$\rho\left(X\right) = \rho\left|V\left(X\right)\right|, \tag{4}$$

where $|A|$ denotes the surface of the set A; $\rho(X)$ is expressed in bits per second.

2) Processor-sharing service policy: We shall assume that the BS allocates an equal fraction of its resources to all users it serves at a given time. Thus, when there are k users in a cell, each user obtains a bit-rate equal to its peak bit-rate divided by k. More explicitly, if a base station located at X serves k users located at $y_1, y_2, \ldots, y_k \in V\left(X\right)$ then the bit-rates of these users are equal to $R\left(\mathrm{SINR}\left(y_j, \Phi\right)\right)/k$, $j = 1, 2, \ldots, k$, respectively. Users having completed their service (download of the requested volumes) leave the system.

C. Time-averaged cell characteristics

Given the network realization (including the shadowing and the cell activity factors), the performance of each cell $V(X)$ of $X \in \Phi$ corresponds to a (spatial version of the) processor sharing-queue. More specifically, due to complete independence property of the Poisson process of arrivals, the temporal dynamics of these queses are independent. Thus, we can use the classical queuing-theoretic results regarding processor-sharing queues to describe the time-averaged (steady-state) characteristic of all individual cells. Besides the traffic demand $\rho(X)$ already specified in Section II-B1, these characteristics are: the critical traffic $\rho_c(X)$, cell load $\theta(X)$, mean number of users $N(X)$, average user throughout $r(X)$, busy (non-idling) probability $p(X)$. In what follows we present these characteristics in a form tailored to our wireless context; cf [10]. All these characteristics can be seen as further, general (non-independent) marks of the point process Φ and depend also on BS types, their activity factors and shadowing processes.

1) Critical traffic: The processor-sharing queue of the base station $X \in \Phi$ is stable if and only if its traffic demand $\rho(X)$ is smaller than the critical value which is the harmonic mean of the peak bit-rates over the cell; cf [14]

$$\rho_{\mathrm{c}}\left(X\right) := \left|V\left(X\right)\right| \left(\int_{V(X)} R^{-1}\left(\mathrm{SINR}\left(y, \Phi\right)\right) dy\right)^{-1}. \tag{5}$$

2) Cell load: We define it as the ratio between the (actual) cell traffic demand and its critical value

$$\theta\left(X\right) := \frac{\rho\left(X\right)}{\rho_{\mathrm{c}}\left(X\right)} = \int_{V(X)} \rho R^{-1}\left(\mathrm{SINR}\left(y, \Phi\right)\right) dy\,. \tag{6}$$

3) Mean number of users: The mean number of users in the steady state of the processor sharing queue at BS $X \in \Phi$ can be expressed as

$$N\left(X\right) := \begin{cases} \dfrac{\theta\left(X\right)}{1-\theta\left(X\right)} & \text{if } \theta(X) < 1 \\ \infty & \text{otherwise}\,. \end{cases} \tag{7}$$

4) User throughput: is defined as the ratio between the mean volume request $1/\mu$ and the mean typical-user service time in the cell X. By the Little's law it can be expressed as

$$r\left(X\right) := \frac{\rho(X)}{N(X)}\,. \tag{8}$$

5) Busy probability: The probability that the BS $X \in \Phi$ is not idling (serves at least one user) in the steady state is equal to

$$p\left(X\right) = \min\left(\theta\left(X\right), 1\right). \tag{9}$$

It is easy to see that all the above characteristics (marks) of the BS $X \in \Phi$ can be expressed using the traffic demand $\rho(X)$ and the cell load $\theta(X)$ in the following order

$$\rho_c(X) = \frac{\rho(X)}{\theta(X)}\,, \tag{10}$$

$$r(X) = \max(\rho_{\mathrm{c}}\left(X\right) - \rho\left(X\right), 0)\,, \tag{11}$$

$$N\left(X\right) = \frac{\rho\left(X\right)}{r\left(X\right)}\,. \tag{12}$$

D. Spatial inter-dependence of cells — cell load equations

The individual cell characteristics described in the previous section depend on the location of all base stations, shadowing realizations but also on the cell activity factors φ_X, $X \in \Phi$, introduced in Section II-A3 to weight the extra cell interference in the SINR expression, and which have been arbitrary numbers between 0 and 1 up to now. These factors suppose to account for the fact that BS might not transmit with their respective maximal powers P_j depending on the BS types $j = 1, \ldots, J$ all the time.

It is quite natural to think that BS transmit only when they serve at least one user. [2] Taking this fact into account in an exact way requires introducing in the denominator of (3) the indicators that a given station $Y \in \Phi$ at a given time is not idling. This, in consequence, would lead to the probabilistic dependence of the service process at different cell, thus revoking the explicit expressions for their characteristics presented in Section II-C and the model becomes non-tractable. [3] For this reason, we take into account whether Y is idling or not in a simpler way, multiplying its maximal transmitted power by the *probability* $p(Y)$ that it is busy in the steady state. In other words, in the SINR expression (3) we take $\varphi_Y = p(Y)$ where $p(Y)$ is given by (9); i.e.,

$$\mathrm{SINR}\left(y, \Phi\right) = \frac{\frac{1}{L_X(y)}}{N + \sum_{Y \in \Phi\setminus\{X\}} \frac{\min(\theta(Y),1)}{L_Y(y)}}\,. \tag{13}$$

We call this model *(load)-weighted interference model*. Clearly this assumption means that $\theta(X)$ cannot be calculated independently for all cells but rather are solutions of the following fixed point problem, which we call *cell load equations*

$$\theta\left(X\right) = \rho \int_{V(X)} R^{-1}\left(\frac{\frac{1}{L_X(y)}}{N + \sum_{Y \in \Phi\setminus\{X\}} \frac{\min(\theta(Y),1)}{L_Y(y)}}\right) dy\,. \tag{14}$$

This is a system of equations which needs to be solved for $\{\theta\left(X\right)\}_{X \in \Phi}$ given network and shadowing realization. In the remaining part of this paper we assume that such a solution

[2] Analysis of more sophisticated power control schemes is beyond the scope of this paper.

[3] We are even not aware of any result regarding the stability of such a family of dependent queues.

exists and is unique. [4] The other characteristics of each cell are then deduced from the cell load and traffic demands using the relations described in Section II-C.

III. Model analysis

We begin our analysis by recalling some useful results regarding the Poisson network model. Next, in Section III-B we present our main results and in Section III-C postulate some simplified approach inspired by these results.

A. Preliminaries: typical and zero-cell of the multi-tier network

We briefly recall here the notions of the typical and zero-cell, usually considered for the Voronoi tessellation and here regarding our network cells. Both objects will play their respective roles in the remaining part of the paper.

We denote by $\mathbb{P}$ the probability corresponding to the stationary distribution of our model as described in Section II.

1) The typical cell: This is a mathematical formalization of a cell whose BS is "arbitrarily chosen" from the set of all stations, without any bias towards its characteristics, in particular its type and the cell size. The formalization is made on the ground of Palm theory, where the typical cell $V(0)$ is this of the BS $X_0 = 0$ located at the origin under the *Palm probability* $\mathbb{P}^0$. By the Slivnyak's theorem the Palm distribution of the Poisson process corresponds to the homogeneous (stationary) one, with the "extra" point $X_0 = 0$ added at the origin. In the case of i.i.d. marked Poisson process, as in our case, this extra point gets and independent copy of the mark, with the original mark distribution.

Note that in our network the probability that an "arbitrarily chosen" BS is of type j, $j = 1, \ldots, J$, is equal to λ_j/λ. More formally,

$$\mathbb{P}^0(Z_0 = j) = \lambda_j/\lambda\,. \tag{15}$$

We remark, that the typical cell does not have any physical existence in a given network. It is rather a useful mathematical tool, in the sense that the mathematical expectations under $\mathbb{E}^0$ of the typical cell $V(0)$ characteristics (as the cell traffic demand $\rho(0)$, cell load $\theta(0)$, etc) can be interpreted as network-averages of the (already time-averaged) cell performance metrics. For example the network-averaged traffic demand per cell, considering all cells or only cells of type $j = 1, \ldots, J$, equal, respectively

$$\bar{\rho} := \mathbb{E}^0[\rho(0)] = \lim_{|A|\to\infty} \frac{1}{\Phi\,(A)} \sum_{X\in\Phi\cap A} \rho\,(X)\;, \tag{16}$$

$$\bar{\rho}_j := \mathbb{E}^0[\rho(0)\,|\,Z_0 = j] = \lim_{|A|\to\infty} \frac{1}{\Phi_j\,(A)} \sum_{X\in\Phi_j\cap A} \rho\,(X)\;. \tag{17}$$

where A denotes a disc centered at the origin, of radius increasing to infinity. The convergence is $\mathbb{P}$-almost sure and follows from the ergodic theorem for point processes (see [33, Theorem 13.4.III]). We define similarly the network-average load (overall and per cell type)

$$\bar{\theta} := \mathbb{E}^0[\theta(0)]\,, \tag{18}$$

$$\bar{\theta}_j := \mathbb{E}^0[\theta(0)\,|\,Z_0 = j] \quad j = 1, \ldots, J. \tag{19}$$

[4]Note that the mapping in the right-hand-side of (14) is increasing in all $\theta(Y)$, $Y \in \Phi$ provided function R is increasing. Using this property it is easy to see that successive iterations of this mapping started off $\theta(Y) \equiv 0$ on one hand side and off $\theta(Y) = 1$ (full interference model) on the other side, converge to a minimal and maximal solution of (14), respectively. The uniqueness of the solution (in the Poisson or more general) network is an interesting theoretical question, which is however beyond the scope of this paper. A very similar problem (with finite number of stations and a discrete traffic demand) is considered in [29], where the uniqueness of the solution is proved.

The convergence analogue to (16), (17) holds for each of the previously considered local characteristics. However (at least for Poisson network) it is not customary to consider directly $\mathbb{E}^0[N(0)]$ since the (almost sure) existence of some (even arbitrarily small) fraction of BS X which are not stable (with $\rho(X) \geq \rho_c(X)$, hence $N(X) = \infty$) makes $\mathbb{E}^0[N(0)] = \infty$. [5]

Also, as we will explain in what follows, $\mathbb{E}^0[r(0)]$ does not have a natural interpretation. In particular it *cannot* be interpreted as the mean user throughput.

2) Zero cell: This is the cell (of the stationary distributed network) that covers the origin 0 of the plane, which plays the role of an arbitrarily fixed location. The characteristics of the zero-cell correspond to the characteristics of the cell which serves the typical user. Clearly this is a size-biased choice and indeed the zero cell has different distributional characteristics from the typical cell. Let us denote by X^* the location of the BS serving the zero-cell and its type by Z^*.

We will recall now a useful result regarding multi-tier networks, from which we will derive the distribution of Z^*; cf [25, Lemma 1].

Lemma 1: *Assume that* $\mathbb{E}\left[S^{2/\beta}\right] < \infty$. *Then* $\hat{\Phi} = \{(L_n = L_n(0), Z_n)\}_n$ *is a Poisson point process on* $[0,\infty) \times \{1, \ldots, J\}$ *with intensity measure*

$$\Lambda\left((0,t]\times\{j\}\right) := \mathbb{E}[\#\{n : L_n \leq t, Z_n = j\}] = a_j t^{2/\beta}\,, \tag{20}$$

$t \geq 0$, $j = 1, \ldots, J$, *where*

$$a_j := \frac{\pi\mathbb{E}\left[S^{2/\beta}\right]}{K^2}\lambda_j P_j^{2/\beta}\,. \tag{21}$$

Remark 2: The form (20) of the intensity measure Λ of $\hat{\Phi}$ allows us to conclude that the point process $\{L_n(0)\}_n$ of propagation-loss values (between all base stations and the origin) is a Poisson point process of intensity $\Lambda((0,t]\times\{1,\ldots,J\}) = at^{2/\beta}$, where

$$a := \sum_{j=1}^{J} a_j\,. \tag{22}$$

Moreover, the types Z_n of the BS corresponding to the respective propagation-loss values L_n constitute i.i.d. marking of this latter process of propagation-loss values, with the probability that an arbitrarily chosen propagation-loss comes from a BS of type j having probability a_j/a. In particular, for the serving station (offering the smallest propagation-loss) we have

$$\mathbb{P}\{\,Z^* = j\,\} = a_j/a\,.^{6} \tag{23}$$

Our second remark on the result of Lemma 1 regards an equivalent way of generating the Poisson point process of intensity (20).

Remark 3: Consider a homogeneous Poisson network of intensity λ, in which all stations emit with the same power

$$P = \left(\sum_{j=1}^{J} \frac{\lambda_j}{\lambda} P_j^{2/\beta}\right)^{\beta/2} \tag{24}$$

and assume the same model of the propagation-loss with shadowing as described in Section II-A2. Let us "artificially"

[5]For a well dimensioned network one does not expect unstable cells. For a perfect hexagonal network model Φ without shadowing *all* cells are stable or unstable depending on the value of the per-surface traffic demand ρ. For an (infinite) homogeneous Poisson model Φ, for arbitrarily small ρ there exists a non-zero fraction of BS $X \in \Phi$, which are non-stable. This fraction is very small for reasonable ρ, allowing to use Poisson to study QoS metrics which, unlike $\mathbf{E}^0[N(0)]$, are not "sensitive" to this artifact.

[6]Interpreting (15) and (23) we can say that an arbitrarily chosen BS is of type j with probability λ_j/λ, while an arbitrarily chosen propagation-loss (measured at the origin) comes from a BS of type j with probability a_j/a.

(without altering the power P) mark these BS by randomly, independently selecting a mark $j = 1 \ldots, J$ for each station with probability a_j/a. A direct calculation shows that the marked propagation-loss process observed in this homogeneous network by a user located at the origin, analogue to $\tilde{\Phi}$, has the same intensity measure Λ given by (20). Consequently, the distribution of all user/network characteristics, which are functionals of the marked propagation-loss process $\hat{\Phi}$ can be equivalently calculated using this *equivalent homogeneous* model.

B. Global network performance metrics

The objective of this section is to express pertinent, global network characteristics and relate them to mean throughput of the typical user of the network.

1) Traffic and load per cell: The mean traffic demand and load of the typical cell, globally and per cell type, can be expressed as follows.

Proposition 4: *We have for the traffic demand*

$$\bar{\rho} = \frac{\rho}{\lambda}$$
$$\bar{\rho}_j = \bar{\rho}\frac{P_j^{2/\beta}}{P^{2/\beta}}, \quad j = 1, \ldots, J, \tag{25}$$

where P is the "equivalent network" power given by (24).

Proof: We have

$$\bar{\rho} = \mathbb{E}^0\left[\rho\left(0\right)\right] = \rho\mathbb{E}^0\left[\left|V\left(0\right)\right|\right] = \frac{\rho}{\lambda},$$

where the second equality is due to (4) and the last one follows from the inverse formula of Palm calculus [34, Theorem 4.2.1] (which may be extended to the case where the cell associated to each BS is not necessarily the Voronoi cell; the only requirement is that the user located at 0 belongs to a unique cell almost surely). Similarly,

$$\bar{\rho}_j = \rho\mathbb{E}^0\left[\left|V\left(0\right)\right| \, Z_0 = j\right] = \rho\frac{\mathbb{E}^0\left[\left|V\left(0\right)\right| \times \mathbf{1}\{Z_0 = j\}\right]}{\mathbb{P}^0\left(Z_0 = j\right)}$$
$$= \frac{\rho}{\lambda}\frac{\mathbb{P}\left(Z^* = j\right)}{\mathbb{P}^0\left(Z_0 = j\right)} = \bar{\rho}\frac{a_j/a}{\lambda_j/\lambda} = \bar{\rho}\frac{P_j^{2/\beta}}{P^{2/\beta}},$$

where the third equality follows again from the inverse formula of Palm calculus and the two remaining ones from (15), (23) and (21), (22), respectively. ∎

Proposition 5: *We have for the cell load*

$$\bar{\theta} = \frac{\rho}{\lambda}\mathbb{E}\left[R^{-1}\left(\mathrm{SINR}\left(0, \Phi\right)\right)\right], \tag{26}$$

$$\bar{\theta}_j = \bar{\theta}\frac{P_j^{2/\beta}}{P^{2/\beta}}, \quad j = 1, \ldots, J, \tag{27}$$

where P is given by (24).

Proof: Denote $g\left(y, \Phi\right) = R^{-1}\left(\mathrm{SINR}\left(y, \Phi\right)\right)$. In the same lines as the proof of Proposition 4, by the inverse formula of Palm calculus $\bar{\theta} = \mathbb{E}^0\left[\theta\left(0\right)\right] = \frac{\rho}{\lambda}\mathbb{E}\left[g\left(0, \Phi\right)\right]$. Similarly

$$\bar{\theta}_j = \mathbb{E}^0\left[\theta\left(0\right) \mid Z_0 = j\right]$$
$$= \rho\mathbb{E}^0\left[\int_{V(0)} g\left(y, \Phi\right)\mathbf{1}\{Z_0 = j\}\, dy\right]/\mathbb{P}^0\left(Z_0 = j\right)$$
$$= \frac{\rho}{\lambda}\mathbb{E}\left[g\left(0, \Phi\right)\mathbf{1}\{Z^* = j\}\right]/\mathbb{P}^0\left(Z_0 = j\right)$$
$$= \frac{\rho}{\lambda}\mathbb{E}\left[g\left(0, \Phi\right)\right]\frac{a_j/a}{\lambda_j/\lambda},$$

where the third equality follows from the inverse formula of Palm calculus, and the fourth equality from the independent marking of the propagation-loss process by the BS types; cf Remark 2. ∎

2) Number of users per cell and mean user throughput: For the reasons already explained at the end of Section III-A1 it is more convenient to average the number of users per cell in the stable part of the network. To this regard we define the network-averaged number of users per *stable* cell as

$$\bar{N} := \mathbb{E}^0\left[N\left(0\right)\mathbf{1}\{\theta\left(0\right) < 1\}\right]$$

and similarly for each cell tier $j = 1, \ldots, J$

$$\bar{N}_j := \mathbb{E}^0\left[N\left(0\right)\mathbf{1}\{\theta\left(0\right) < 1\}\left|Z_0 = j\right.\right].$$

Note that the mean traffic demand $\bar{\rho}$, load $\bar{\theta}$ and number of users $\bar{N}$ per (stable) cell characterize network performance from the point of view of its typical (or averaged) cell. We move now to a typical user performance metric that is its mean throughput. This latter QoS metric is traditionally (in queueing theory) defined as the mean data volume requested by the typical user to the mean service duration of the typical user. In what follows we apply this definition (already retained at the local, cell level in Section II-C4) globally to the whole network, filtering out the impact of unstable cells.

Denote by $\mathcal{S}_j$ the union of stable cells of type $j = 1, \ldots, J$; that is $\mathcal{S}_j = \bigcup_{X \in \Phi_j : \theta(X) < 1} V\left(X\right)$ and $\mathcal{S} = \bigcup_{j=1}^{J} \mathcal{S}_j$. Let $\pi^{\mathcal{S}}$ ($\pi_j^{\mathcal{S}}$) be the probability the typical user is served in a stable cell (of type $j = 1, \ldots, J$)

$$\pi^{\mathcal{S}} = \mathbb{P}\left(\theta\left(X^*\right) < 1\right)$$
$$\pi_j^{\mathcal{S}} = \mathbb{P}\left(\theta\left(X^*\right) < 1 \,|\, Z^* = j\right), \quad j = 1, \ldots, J,$$

where (recall) X^* is the BS whose cell covers the origin and Z^* is its type. Note that $\pi_j^{\mathcal{S}} = a/a_j\mathbf{E}[\mathbf{1}\{0 \in \mathcal{S}_j\}]$ and thus it can be related to the *volume fraction* of the stable part of the network served by tier j and similarly for $\pi^{\mathcal{S}} = \mathbf{E}[\mathbf{1}\{0 \in \mathcal{S}\}]$.

We define the *(global) mean user throughput* as

$$\bar{r} := \lim_{|A| \to \infty} \frac{1/\mu}{\text{mean call duration in } A \cap \mathcal{S}}$$

and for each cell type $j = 1, \ldots, J$,

$$\bar{r}_j := \lim_{|A| \to \infty} \frac{1/\mu}{\text{mean call duration in } A \cap \mathcal{S}_j},$$

where A denotes a disc centered at the origin of radius increasing to infinity. These limits exist almost surely by the ergodic theorem; cf [33, Theorem 13.4.III]. Here is our main result regarding this mean user QoS. It can be seen as a consequence of a spatial version of the Little's law.

Proposition 6: *We have for the mean user throughput*

$$\bar{r} = \frac{\bar{\rho}}{\bar{N}}\pi^{\mathcal{S}}$$
$$\bar{r}_j = \frac{\bar{\rho}_j}{\bar{N}_j}\pi_j^{\mathcal{S}}, \quad j = 1, \ldots, J \tag{28}$$

Proof: Let $W_j = \bigcup_{X \in A \cap \mathcal{S}_j} V\left(X\right)$. Consider call arrivals and departures to W_j. By Little's law

$$N^{W_j} = \gamma\left|W_j\right|T^{W_j},$$

where T^{W_j} is the mean call duration in W_j and N^{W_j} is the steady-state mean number of users in W_j. Thus mean user throughput, with users restricted to W_j, equals

$$\frac{1/\mu}{T^{W_j}} = \frac{\rho\left|W_j\right|}{N^{W_j}} = \rho\frac{\sum_{X \in A \cap \Phi}\left|V\left(X\right)\right|\mathbf{1}\{\theta\left(X\right) < 1, X \in \Phi_j\}}{\sum_{X \in A \cap \Phi} N\left(X\right)\mathbf{1}\{\theta\left(X\right) < 1, X \in \Phi_j\}}.$$

Letting $|A| \to \infty$, it follows from the ergodic theorem that

$$\bar{r}_j = \rho\frac{\mathbb{E}^0\left[\left|V\left(0\right)\right|\mathbf{1}\{\theta\left(0\right) < 1, Z_0 = j\}\right]}{\mathbb{E}^0\left[N\left(0\right)\mathbf{1}\{\theta\left(0\right) < 1, Z_0 = j\}\right]}.$$

By the inverse formula of Palm calculus

$$\mathbb{E}^0\left[\left|V\left(0\right)\right|\mathbf{1}\{\theta\left(0\right) < 1, Z_0 = j\}\right] = \frac{1}{\lambda}\mathbb{P}\left(\theta\left(X^*\right) < 1, Z^* = j\right)$$

and consequently

$$\bar{r}_j = \frac{\rho}{\lambda}\frac{\mathbb{P}\left(\theta\left(X^*\right)<1, Z^*=j\right)}{\mathbb{P}^0\left(Z_0=j\right)\bar{N}_j} = \frac{\bar{\rho}_j}{\bar{N}_j}\mathbb{P}\left(\theta\left(X^*\right)<1|Z^*=j\right) = \frac{\bar{\rho}_j}{\bar{N}_j}\pi_j\,.$$

The expression for $\bar{r}$ may be proved in the same lines as above. ■

The mean number of users per stable cell and the mean user throughput in the stable part of the network do not admit explicit analytic expressions. We calculate these expressions by Monte-Carlo simulation of the respective expectations with respect to the distribution of the Poisson network model. We call this semi-analytic approach the *typical cell* approach.

C. Mean cell approach

We will propose now a more heuristic approach, in which we try to capture the performance of the heterogeneous network considering J simple M/G/1 processor sharing queues related to each other via their cell loads, which solve a simplified version of the cell load equation.

Recall that in the original approach, in the cell load fixed point equation (14) we have an unknown cell loads $\theta(X)$ for each cell of the network. Recall also that knowing all these cell loads and the cell traffic demands (which depend directly on the cell surfaces) we can calculate all other cell characteristics. We will consider now a simpler "mean" cell load fixed point equation in which all cells of a given type $j=1,\ldots,J$ share the same constant unknown $\tilde{\theta}_j$. [7] Specifically, in analogy to (27), we assume that the new unknowns $\tilde{\theta}_j$ are related to each other by

$$\tilde{\theta}_j = \tilde{\theta}\frac{P_j^{2/\beta}}{P^{2/\beta}}, \quad j=1,\ldots,J \tag{29}$$

where P is given by (24) and $\tilde{\theta}$ solves the following equation

$$\tilde{\theta} = \frac{\rho}{\lambda}\mathbb{E}\left[R^{-1}\left(\frac{\frac{1}{L_{X^*}(0)}}{N+\tilde{\theta}\sum_{j=1}^{J}\frac{P_j^{2/\beta}}{P^{2/\beta}}\sum_{Y\in\Phi_j\setminus\{X^*\}}\frac{1}{L_Y(0)}}\right)\right] \tag{30}$$

The mean fixed point cell load equations boils down hence to an equation in one variable $\tilde{\theta}$. Note that the argument of R^{-1} in (30) is a functional of the marked path-loss process $\hat{\Phi}$ and thus the expectation in this expression can be evaluated using the equivalent homogeneous model described in Remark 3.

By the *mean cell of type* $j=1,\ldots,J$ we understand a (virtual) processing sharing queue with the traffic demand

$$\tilde{\rho}_j := \bar{\rho}_j = \frac{\rho}{\lambda}\frac{P_j^{2/\beta}}{P^{2/\beta}}$$

and the traffic load $\tilde{\theta}_j$ given by (29), where $\tilde{\theta}$ is the solution of (30). The remaining mean cell characteristics (the critical load, user throughput and the number of users) are related to these two "primary" characteristics in analogy to (10), (11) and (12) via

$$\tilde{\rho_c}_j := \frac{\tilde{\rho}_j}{\tilde{\theta}_j}\,, \tag{31}$$

$$\tilde{r}_j := \max(\tilde{\rho_c}_j - \tilde{\rho}_j, 0)\,, \tag{32}$$

$$\tilde{N}_j := \frac{\tilde{\rho}_j}{\tilde{r}_j}\,, \tag{33}$$

[7] Recall that (14) is already a simplification of the reality in which the extra cell interference should be wighted by the dynamic (evolving in time) factors capturing cells' activity.

$j=1,\ldots,J$.

We will also consider a (global) *mean cell* having, respectively, the traffic demand and cell load given by $\tilde{\rho} := \bar{\rho} = \frac{\rho}{\lambda}$ and $\tilde{\theta}$, and the remaining characteristics $\tilde{\rho}_c, \tilde{r}, \tilde{N}$ given, respectively by (31), (32) and (33) where the subscript j is dropped.

In the next section we shall evaluate the mean cell approximation (both globally and per type) by comparison to the characteristics of the typical cell obtained both from simulation and from real field measurements.

IV. Numerical results and model validation

In this section we present numerical results of the analysis of our model and compare them to the corresponding statistics obtained from some real field measurements. We show that the obtained results match the real field measurements. Our numerical assumptions, to be presented in Section IV-A2, correspond to an operational network in some big city in Europe in which two types of BS can be distinguished, conventionally called macro and micro base stations. [8]

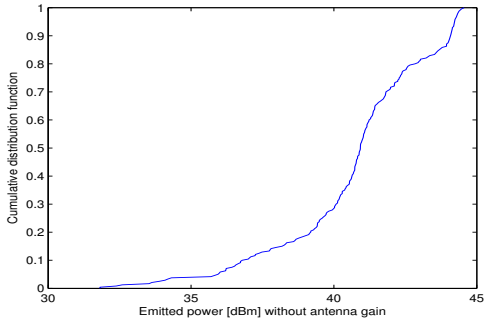

Fig. 1. CDF of the emitted antenna powers, without antenna gains, in the network consisting of micro (power $\leq$ 35dBm) and macro BS (power $\geq$ 35dBm). The average micro BS power is 33.42dBm and the average macro BS power is 41.26dBm. Adding antenna gains, which are equal respectively 14dBm and 17dBm for micro and macro BS, we obtain $P_2 = 47.42$dBm, $P_1 = 58.26$dBm.

The real field measurements are obtained using a methodology described in Section IV-A1. In Section IV-B the statistics obtained from these measuremetns will be compared to the performance of each category of BS calculated using the approach proposed in the present paper.

A. Model specification

1) Measurements: The raw data are collected using a specialized tool used by network maintenance engineers. This tool has an interface allowing to fetch values of several parameters for every base station 24 hours a day, at a time scale of one hour. For a given day, for every hour, we obtain information regarding the BS coordinates, type, power, traffic demand, number of users and cell load calculated as the percentage of the Transmission Time Intervals (TTI) scheduled for transmissions. Then we estimate the global cell performance metrics for the given hour averaging over time (this hour) and next over all considered network cells. The mean user throughput is calculated as the ratio of the mean traffic demand to the mean number of users. The mean traffic

[8] Let us explain what we mean here by macro and micro BS: Historically, the operator deployed first what we call here macro BS. Powers of these stations slightly vary around some mean value as a consequence of some local adaptations. We assume them constant. In order to cope with the increase of the traffic demand, new stations are added progressively. These new stations, which we call micro BS, emit with the power about 10 times smaller than the macro BS. Figure 1 shows the cumulative distribution function (CDF) of the antenna powers (without antenna gains).

demand ρ is also used as the input of our model. Knowing all cell coordinates, their types and the surface of the deployment region we deduce the network density and fraction of BS in the two tiers.

2) Numerical assumptions: The BS locations are generated as a realization of a Poisson point process of intensity $\lambda = \lambda_1 + \lambda_2 = 4.62\text{km}^{-2}$ (which corresponds to an average distance between two base stations of 0.5km) over a sufficiently large observation window which is taken to be the disc of radius 2.63km. The ratio of the micro to macro BS intensities equals $\lambda_2/\lambda_1 = 0.039$. The transmitted powers by macro and micro BS are $P_1 = 58.26$dBm, $P_2 = 47.42$dBm, respectively. The power (24) of the "equivalent" homogeneous network is $P = 58.03$dBm. The propagation loss due to distance is $l(x) = (K|x|)^\beta$ where $K = 7117\text{km}^{-1}$ and the path loss exponent $\beta = 3.8$. Shadowing is assumed log-normally distributed with standard deviation $\sigma = 10$dB and spatial correlation 0.05km. The technology is HSDPA (High-Speed Downlink Packet Access) with MMSE (Minimum Mean Square Error) receiver in the downlink. The peak bit-rate equals to 30% of the information theoretic capacity of the Rayleigh fading channel with AWGN; that is

$$R(\text{SINR}) = 0.3W\mathbb{E}\left[\log_2\left(1 + |H|^2\,\text{SINR}\right)\right] \qquad (34)$$

where the expectation $\mathbb{E}[\cdot]$ is with respect to the Rayleigh fading H of mean power $\mathbb{E}[|H|^2] = 1$, and $W = 5$MHz is the frequency bandwidth. A fraction $\epsilon = 10\%$ of the transmitted power is used by the pilot channel (which is always transmitted whether the BS serves users or not). [9] The antenna pattern is described in [1, Table A.2.1.1-2]. The noise power is -96dBm.

B. Results

We present now the results obtained form the analysis of our two-tier Poisson model conformal to a given region of the operational network, adopting both the typical cell approach described in Section III-B and the mean cell approach explained in Section III-C. The obtained results are compared to the respective quantities estimated in the given operational network. Error bars on all figures represent the standard deviation in the averaging over 10 realizations of the Poisson network in the Monte-Carlo estimation of the respective expectations.

Figure 2 shows the mean cell load together with the stable fraction of the network, both globally and separately for the two tiers, as functions of the mean traffic demand per cell $\bar\rho = \rho/\lambda$. The mean cell load is calculated using the two approaches: the typical cell and the mean cell one. The stable fraction of the network is available only in the typical cell approach. Figure 2 presents also 24 points presenting the mean cell load estimated from the real field measurements done during 24 different hours of some given day. Note a good fit of our results and the network measurements. Observe also that all real field measurements fall within the range of the traffic demand ($\bar\rho \leq 600$kbps) for which the stable fraction of the network for both network tiers is very close to 1. This is of course a consequence of a good dimensioning of the network. Interestingly these latter metrics allows us to reveal existing dimensioning margins. Specifically, we predict that there will be no unstable macro cells with the traffic demand slightly less than $\bar\rho \leq 700$kbps and the micro cells remain stable for much higher traffic demand of order $\bar\rho \approx 1000$kbps.

[9] It is taken into account by replacing $\min(\theta(Y), 1)$ in (14) by $\min(\theta(Y), 1)(1-\epsilon)+\epsilon$. Similar modification concerns $\tilde\theta$ in the right-hand-side of (30).

We move now to the mean number of users per cell presented on Figure 3, again, as function of the mean traffic demand per cell $\bar\rho = \rho/\lambda$. Both approaches (typical and mean cell) are adopted and the two network tiers are analyzed jointly and separately. As for the load, we present also 24 points corresponding the network measurements. Note a good fit of our model results and the network measurements. Note also that the prediction of the model performance for the traffic demand $\bar\rho \geq 700$kbps , where the fraction of unstable cell is non-negligible (cf Figure 2), is much more volatile. More precisely, the relatively large error-bars of the mean number of users for $\bar\rho \geq 700$kbps can be explained by a non-negligible probability of finding a cell whose load is just below 1. It still contributes to the calculation of the mean number of users (as we remove only strictly unstable cells) and makes the empirical mean very large for this simulation experiment.

Finally, Figure 4 shows the relation between the mean user throughput and the mean traffic demand per cell $\bar\rho$ obtained via the two modeling approaches and real field measurements. The global performance of the network and its macro-tier are quite well captured by our two modeling approaches. The micro-tier analysis via the typical cell and the real field measurements exhibit important volatility due to a relatively small number of such cells in the network. The mean cell model allows to predict however a macroscopic law in this regard.

V. Conclusions

A heterogeneous cellular network model allowing for different BS types (having different transmission powers) is proposed, aiming to help in performance evaluation and dimensioning of real (large, irregular) operation networks. It allows one to identify key laws relating the performance of the different base station types. In particular, we show how the mean load of different types of BS is related in a simple way to their transmission powers. The results of the model analysis are compared to real field measurement in an operational network showing its pertinence.

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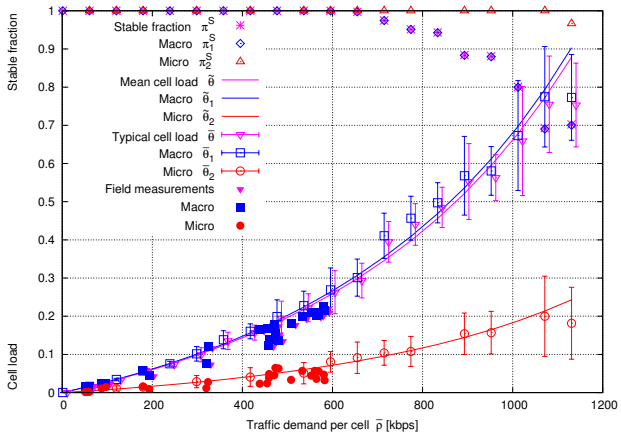

Fig. 2. Cell load versus traffic demand per cell.

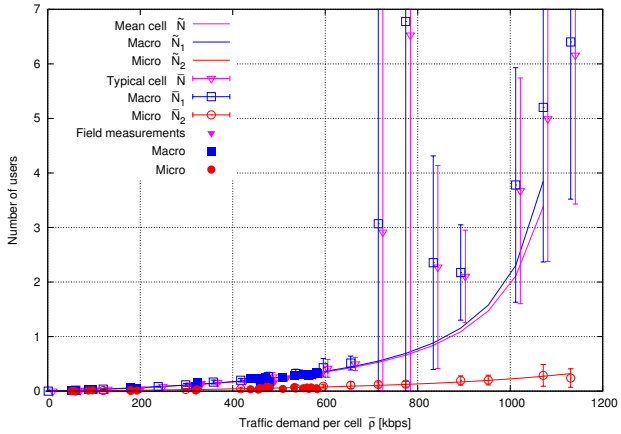

Fig. 3. Number of users per cell versus traffic demand per cell.

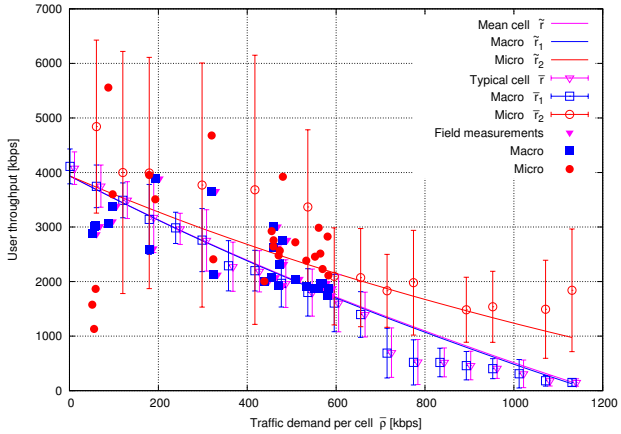

Fig. 4. Mean user throughput in the network versus traffic demand per cell.

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