Adaptive Spatial Aloha, Fairness and Stochastic Geometry

Francois Baccelli
Depts. of Mathematics and ECE
University of Texas at Austin, USA
and
INRIA Rocquencourt Paris, France
Email: francois.baccelli@austin.utexas.edu

Chandramani Singh
INRIA-TREC
23 Avenue d'Italie
CS 81321
75214 Paris Cedex 13, France
Email: chandramani.singh@inria.fr

***Abstract*—This work aims at combining adaptive protocol design, utility maximization and stochastic geometry. We focus on a spatial adaptation of Aloha within the framework of ad hoc networks. We consider quasi-static networks in which mobiles learn the local topology and incorporate this information to adapt their medium access probability (MAP) selection to their local environment. We consider the cases where nodes cooperate in a distributed way to maximize the global throughput or to achieve either proportional fair or max-min fair medium access. In the proportional fair case, we show that nodes can compute their optimal MAPs as solutions to certain fixed point equations. In the maximum throughput case, the optimal MAPs are obtained through a Gibbs Sampling based algorithm. In the max min case, these are obtained as the solution of a convex optimization problem. In the proportional fair case we show that, when the nodes form a homogeneous Poisson point process (PPP) in the Euclidean plane, the distribution of the optimal MAP can be obtained from that of a certain shot noise process with respect to this Poisson point process and that the mean utility can also be derived from this distribution. Numerical results illustrate our findings and quantify the gains brought by spatial adaptation in such networks.**

I. Introduction

Stochastic geometry has recently been used for the analysis and performance evaluation of wireless (ad hoc as well as cellular) networks; in this approach, one models node locations as a spatial point process, e.g., homogeneous Poisson point processes (PPPs), and one computes various network statistics, e.g., interference, successful transmission probability, coverage (or, outage) probability etc. as spatial averages. This often leads to tractable performance metrics that are amenable to parametric optimization with respect to network parameters (node density, protocol parameters, etc.). More precisely, this approach yields spatial averages of the performance metrics for given network parameters; then the parameters can be chosen to optimize performance. This approach takes a macroscopic view of the network with the underlying assumption that all nodes in the network have identical statistical characteristics.

In practice, due to randomness and heterogeneity in networks, nodes need to adapt to local spatial and temporal conditions (e.g., channel conditions and topology) to reach optimum network wide performance. For example, nodes in wireless LANs adjust their window sizes based on acknowledgment feedback; in cellular networks nodes are scheduled based on channel conditions and adapt their transmit powers based on the measured SINRs, which in turn depend on the transmit powers set by other nodes. In all such scenarios, distributed adaptive algorithms are used to reach a desired network wide operating point e.g. that maximizing some utility. While the behavior of such distributed optimization protocols is often well understood on a given topology, there are usually no analytical characterizations of the statistical properties of the optimal state in large random and heterogeneous networks.

The main aim of this work is to use stochastic geometry to study spatial adaptations of medium access control in Aloha that aim at optimizing certain utilities.

Let us start with a review of the state of the art on Aloha. Wireless spectrum is well known to be a precious and scarce shared resource. Medium Access Control (MAC) algorithms are employed to coordinate access to the shared wireless medium. An efficient MAC protocol should ensure high system throughput, and should also distribute the available bandwidth fairly among the competing nodes. The simplest of the MAC protocols, Aloha and slotted Aloha, with a "random access" spirit, were introduced by Abramson [1] and Roberts [16] respectively. In these protocols, only one node could successfully transmit at a time. Reference [4] modeled node locations as spatial point processes, and also modeled channel fadings, interferences and SINR based reception. This allowed for spatial reuse and multiple simultaneous successful transmissions depending on SINR levels at the corresponding receivers. All the above protocols prescribe identical attempt probabilities for all the nodes. Reference [5] further proposed opportunistic Aloha in which nodes' transmission attempts are modulated by their channel conditions.

Among the initial attempts of MAP adaptation in Aloha, reference [10] analyzed protocol model and proposed stochastic approximation based strategies that were based on receiver feedback and were aimed at stabilizing the network. References [4], [5] also optimized nodes' attempt probabilities (or thresholds) in order to maximize the spatial density of successful transmissions. Reference [12] analyzed both plain and opportunistic Aloha in a network where all the nodes communicate to one access point. They assumed statistically identical Rayleigh faded channels with no dependence on geometry (i.e., no path loss components). They demonstrated a paradoxical behavior where plain Aloha yields better aggregate throughput than the opportunistic one. Reference [13] also studied optimal random access with SINR based reception. However, they considered constant channel gains. They developed a centralized algorithm that maximizes the network throughput, and also an algorithm that leads to max-min fair operation. Reference [18] modeled network as an undirected graph and studied Aloha under the protocol model. They designed distributed algorithms that are either proportional fair or max-min fair. Reference [11] built upon the model of [4], and formulated the channel access problem as a non-cooperative game among users. They considered throughput and delay as performance metrics and proposed pricing schemes that induce socially optimum behavior at equilibrium. However, they set time average quantities (e.g., throughput, delay) as utilities (or costs), and concentrated on symmetric Nash equilibria. Consequently, in their analysis, dependence on local conditions vanishes.

In none of the above Aloha models, nodes account for both wireless channel randomness and local topology for making their random access decisions, as we do in the present paper.[1]

There is a vast literature on the modeling of CSMA by stochastic geometry. The very nature of this MAC protocol is adaptive as each node senses the network and acts in order to ensure that certain exclusion rules are satisfied, namely that neighboring nodes do not access the channel simultaneously. However, CSMA as such is designed to guarantee a reasonable scheduling, not to optimize any utility of the throughput. The closest reference to our work is probably [6] where the authors study an adaptation of the exclusion range and of the transmit power of a CSMA node to the location of the closest interferer. This adaptation aims at maximizing the mean number of nodes transmitting per unit time and space (while respecting the above exclusion rules). This mean number is however only a surrogate of the rate. In addition, the adaptation is only with respect to the location of the nearest interferer.

We study spatial adaptation of Aloha in ad hoc networks. The network setting is described in Section II. We consider quasi-static networks in which mobiles learn the topology, and incorporate this information in their medium access probability (MAP) selection.

Section III is focused on the distributed algorithms that maximize the aggregate throughput or lead to either a proportional fair or max-min fair sharing of the network resources. In the proportional fair case, we show that nodes can compute the optimal MAPs as solutions to certain fixed point equations. In the maximum throughput case, the optimal MAPs are obtained through a Gibbs Sampling based algorithm. In the max min case, the optimal MAPs are obtained as the solution of a convex optimization problem.

Section IV contains the stochastic geometry results. The model features nodes forming a realization of a homogeneous PPP in the Euclidean plane. In such a network, we compute the MAP distribution for the proportional fair case using shot noise field theory. To the best of our knowledge, this distribution is the first example of successful combination of stochastic geometry and adaptive protocol design aimed at optimizing ceratin utility function within this Aloha setting. We show that the mean utility of a typical node can also be derived from this distribution.

The numerical results are gathered in Section V. The aim of this section is two-fold: 1) check the analytical results against simulation and 2) quantify the gains brought by adaption within this setting. More extensive simulation is provided in our technical report [17].

[1]In view of this distinction, we refer to the spatial Aloha protocol of [4] as plain Aloha.

II. Network Model

We model the ad-hoc wireless network as a set of transmitters and their corresponding receivers, all located in the Euclidean plane. This is often referred to as "bipole model" [3, Chapter 16]. There are N transmitter-receiver pairs communicating over a shared channel. The transmitters follow the slotted version of the Aloha MAC protocol. A transmitter, in each transmission attempt, sends one packet which occupies one slot. Each transmitter uses unit transmission power. We assume that each node has an infinite backlog of packets to transmit to its receiver. The Euclidean distance between transmitter j and receiver i is r_{ji}, and the path-loss exponent is α ($\alpha > 2$). We also assume Rayleigh faded channels with h_{ji} being the random fading between transmitter j and receiver i. Moreover, we assume that the random variables $h_{ij}, 1 \leq i \leq N, 1 \leq j \leq N$ are independent and identically distributed with mean $1/\mu$.[2] Thus all h_{ji}s have cumulative distribution function (CDF) $F(x) = 1 - e^{-\mu x}$ with $x \geq 0$. All the receivers are also subjected to white Gaussian thermal noise with variance w, which is also constant across slots. We assume that a receiver successfully receives the packet of the corresponding transmitter if the received SINR exceeds a threshold T.

Let e_i be the indicator variable indicating whether transmitter i transmits in a slot, and p_i be i's MAP. Thus $\mathbb{P}(e_i = 1) = p_i$. When node i transmits, the received SINR at the corresponding receiver is

$$\gamma_i = \frac{h_{ii} r_{ii}^{-\alpha}}{\sum_{j \neq i} e_j h_{ji} r_{ji}^{-\alpha} + w}.$$

Then the probability of successful reception q_i can be calculated as follows.

$$\begin{aligned}
&\mathbb{P}\left(\gamma_i \geq T | \{(h_{ji}, e_j) : j \neq i\}\right) \\
&= \mathbb{P}\left(h_{ii} \geq \sum_{j \neq i} e_j h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} T + \frac{wT}{r_{ii}^{-\alpha}} \middle| \{(h_{ji}, e_j) : j \neq i\} \right) \\
&= \exp\left(-\mu T \left(\sum_{j \neq i} e_j h_{ji} \left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} + \frac{w}{r_{ii}^{-\alpha}} \right) \right).
\end{aligned}$$

[2]The independence assumption is justified if the distance between two receivers is larger than the coherence distance of the wireless channel [3]. We assume this to be the case.

Thus

$$
\begin{aligned}
q_i &= \mathbb{E}_{\{(h_{ji},e_j):j\neq i\}} \exp\left(-\mu T\left(\sum_{j\neq i} e_j h_{ji}\left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} + \frac{w}{r_{ii}^{-\alpha}}\right)\right) \\
&= \mathrm{e}^{\frac{-\mu w T}{r_{ii}^{-\alpha}}} \prod_{j\neq i} \mathbb{E}_{(h_{ji},e_j)} \exp\left(-\mu e_j h_{ji}\left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} T\right) \\
&= \mathrm{e}^{\frac{-\mu w T}{r_{ii}^{-\alpha}}} \prod_{j\neq i} \mathbb{E}_{h_{ji}}\left((1-p_j) + p_j \exp\left(-\mu h_{ji}\left(\frac{r_{ji}}{r_{ii}}\right)^{-\alpha} T\right)\right) \\
&= \mathrm{e}^{-\mu w T r_{ii}^{\alpha}} \prod_{j\neq i}\left((1-p_j) + \frac{p_j}{1+1/b_{ji}}\right),
\end{aligned}
$$

where $b_{ji} = \frac{1}{T}\left(\frac{r_{ji}}{r_{ii}}\right)^{\alpha}$. Further simplifying,

$$
q_i = \mathrm{e}^{-\mu w T r_{ii}^{\alpha}} \prod_{j\neq i}\left(1 - \frac{p_j}{1+b_{ji}}\right). \tag{1}
$$

Then, the rate or throughput of transmitter i is given by $p_i q_i$.

In interference limited networks, the impact of thermal noise is negligible as compared to interference. We focus on such networks, and thus we ignore the thermal noise factor throughout.

III. Adaptive Spatial Aloha and Fairness

A. Maximum Throughput Medium Access

The throughput maximizing MAPs solve the following optimization problem.

$$
\begin{aligned}
\text{maximize} \quad & \Theta := \sum_i p_i \prod_{j\neq i}\left(1 - \frac{p_j}{1+b_{ji}}\right), \\
\text{subject to} \quad & 0 \le p_i \le 1,\ i \in \mathcal{N}.
\end{aligned}
$$

We first argue that the optimum in this optimization problem is attained at one of the vertices of the hypercube formed by the constraint set. To see this, suppose $\mathbf{p}^* \in [0,1]^{\mathcal{N}}$ is an optimal solution, and $p_i^* \in (0,1)$ for some $i \in \mathcal{N}$. Clearly,

$$
\begin{aligned}
&\frac{\partial \Theta}{\partial p_i}\Big|_{\mathbf{p}=\mathbf{p}^*} \\
&= \prod_{j\neq i}\left(1 - \frac{p_j^*}{1+b_{ji}}\right) - \sum_{j\neq i} \frac{p_j^*}{1+b_{ij}} \prod_{k\neq i,j}\left(1 - \frac{p_k^*}{1+b_{kj}}\right) \\
&= 0.
\end{aligned}
$$

Since the partial derivative is independent of p_i, p_i can be set to either 0 or 1 without reducing the value of the objective function. This proves our claim. In the following we focus only on such extreme solutions. Then the above problem is equivalent to finding an $\mathcal{M} \subset N$ such that $p_i = 1$ if and only if $i \in \mathcal{M}$ is an optimal solution. Thus we are interested in

$$
\underset{\mathcal{M}\subset\mathcal{N}}{\text{maximize}} \quad \sum_{i\in\mathcal{M}} \prod_{j\in\mathcal{M}\setminus\{i\}}\left(1 - \frac{1}{1+b_{ji}}\right).
$$

An iterative solution: We can pose this problem as a strategic form game with the users as players [15]. For each player its action a_i lies in $\{0,1\}$, and the utility function $u_i : \mathbf{a} \mapsto \mathbb{R}$ is given by

$$
\begin{aligned}
&u_i(0,\mathbf{a}_{-i}) = 0, \\
&u_i(1,\mathbf{a}_{-i}) = 1 - \prod_{j\in\mathcal{M}\setminus\{i\}}\left(1 - \frac{1}{1+b_{ji}}\right) \\
&\qquad - \sum_{j\in\mathcal{M}\setminus\{i\}} \frac{1}{1+b_{ij}} \prod_{k\in\mathcal{M}\setminus\{i,j\}}\left(1 - \frac{1}{1+b_{kj}}\right).
\end{aligned}
$$

This is a potential game with the above objective function as the potential function [14]. Thus the best response dynamics converges to a Nash equilibrium (NE). This algorithm can be implemented in a distributed fashion if each node i knows b_{ij}, b_{ji} for all j, and also $\mathcal{M}$ and $\prod_{k\in\mathcal{M}\setminus\{j\}}(1-(1+b_{kj})^{-1})$ for all $j \in \mathcal{M}$ after each iteration. However, a NE can be a suboptimal solution to the above optimization problem. To alleviate this problem, we propose a Gibbs sampler based distributed algorithm, wherein each node i chooses action 1 with probability

$$
p_i = \frac{\mathrm{e}^{u_i(1,\mathbf{a}_{-i})/\tau}}{1+\mathrm{e}^{u_i(1,\mathbf{a}_{-i})/\tau}}.
$$

The parameter τ is called the temperature. The Gibbs sampler dynamics converges to a steady state which is the Gibbs distribution associated with the aggregate throughput and the temperature τ [9]. In other words, we are led to the following distribution on the action profiles:

$$
\pi_\tau(\mathbf{a}) = u\mathrm{e}^{\sum_{i\in\mathcal{N}} u_i(\mathbf{a})},
$$

where u is a normalizing constant. When τ goes to 0 in an appropriate way (e.g., as $1/\log(1+t)$, where t is time), the distribution $\pi_\tau(\cdot)$ converges to a dirac mass at the action profile $\mathbf{a}^*$ with maximum aggregate utility if it is unique. Notice that the aggregate utility $\sum_{i\in\mathcal{N}} u_i(\mathbf{a})$ equals the aggregate throughput. Thus the action profile $\mathbf{a}^*$ is a solution to the original throughput optimization problem.

Remark 3.1: The first two terms in the utility of user i, $u_i(1,\mathbf{a}_{-i})$, can be seen as "selfish" part, whereas the last summation term is "altruistic" part. The user makes a decision based on whether the "selfish" part dominates or viceversa.

Remark 3.2: In a quasi-static network where topology continuously changes, although at a slower time scale, different sets of nodes are likely to be scheduled to transmit under different topologies. Thus, in terms of long term performance, maximum throughput medium access is not grossly unfair.

B. Proportional Fair Medium Access

The proportional fair medium access problem can be formulated as follows.

$$
\begin{aligned}
\text{maximize} \quad & \sum_i \log(p_i q_i), \\
\text{subject to} \quad & 0 \le p_i \le 1,\ i \in \mathcal{N}.
\end{aligned}
$$

The objective function can be rewritten as

$$
\sum_i \left(\log p_i + \sum_{j\neq i} \log\left(1 - \frac{p_j}{1+b_{ji}}\right)\right).
$$

We thus have a convex separable optimization problem. The derivative of the objective function with respect to p_i is

$$\frac{1}{p_i} - \sum_{j \neq i} \frac{1}{1 + b_{ij} - p_i}, \tag{2}$$

which is continuous and decreasing in p_i over $[0, 1]$. We conclude that at optimality, for each $1 \leq i \leq N$,

$$p_i = \begin{cases} f_i(p_i) := \left(\sum_{j \neq i} \frac{1}{1+b_{ij}-p_i} \right)^{-1} & \text{if } f_i(1) < 1, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that user i's optimal attempt probability is independent of others' attempt probabilities. In particular, if $f_i(1) < 1$, i can perform iterations $p_i^{k+1} = f_i(p_i^k)$ autonomously. Also,

$$f_i'(p_i) = - \sum_{j \neq i} \frac{1}{(1 + b_{ij} - p_i)^2} \left(\sum_{j \neq i} \frac{1}{1 + b_{ij} - p_i} \right)^{-2}.$$

Clearly, $|f_i'(p_i)| < 1$, i.e., $f_i(\cdot)$ is a contraction. Thus the fixed point iterations converge to the optimal p_i starting from any $p_i \in [0, 1]$.

Remark 3.3: The characterization of the optimal attempt probabilities reflects the altruistic behavior of users. More precisely, user i's attempt probability is a function of $\{b_{ij}, j \neq i\}$ which are measures of i's interference to all other users. In particular, if $\sum_{j \neq i} \frac{1}{b_{ij}} < 1$, i.e., if i's transmission does not cause significant interference to others, then i transmits in all the slots. Unlike the throughput maximization problem, there is no "selfish" component in the decision making rule.

Remark 3.4: As the target SINR $T \to \infty$, $b_{ij} \to 0$ for all i, j, and the proportional fair attempt probabilities satisfy

$$\frac{1}{p_i} = \sum_{j \neq i} \frac{1}{1 - p_i} = \frac{N - 1}{1 - p_i}$$

for all $i \in \mathcal{N}$. This yields $p_i = \frac{1}{N}$ for all $i \in \mathcal{N}$. This is expected, because in the limiting case a transmission can succeed if and only if there is no other concurrent transmission. This is hence Aloha without spatial reuse, and it is well known that in this case, the optimal MAP is $1/N$ asymptotically [7].

C. Max-min Fair Medium Access

Our analysis in this section follows [18], [19]. The max-min fair medium access problem can be formulated as

$$\begin{aligned} &\text{maximize} \quad \theta, \\ &\text{subject to} \quad \theta \leq p_i \prod_{j \neq i} \left(1 - \frac{p_j}{1 + b_{ji}} \right), \ i \in \mathcal{N}, \end{aligned}$$

where constraint functions are defined for all $\mathbf{p} \in [0, 1]^{\mathcal{N}}$. The following is an equivalent convex optimization problem (see [18] for details):

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\theta^2, \\ &\text{subject to} \quad \theta \leq \log p_i + \sum_{j \neq i} \log \left(1 - \frac{p_j}{1 + b_{ji}} \right), \ i \in \mathcal{N}. \end{aligned}$$

The Lagrange function of this problem is given by [8]

$$\frac{1}{2}\theta^2 + \sum_{i \in \mathcal{N}} \lambda_i \left(\theta - \log p_i - \sum_{j \neq i} \log \left(1 - \frac{p_j}{1 + b_{ji}} \right) \right),$$

with $\lambda_i \geq 0, i \in \mathcal{N}$ being the Lagrange multipliers.

Minimization of the Lagrange function (which is concave in $\mathbf{p}$ and θ) gives

$$p_i = \begin{cases} \lambda_i \left(\sum_{j \neq i} \frac{\lambda_j}{1+b_{ij}-p_i} \right)^{-1} & \text{if } \frac{1}{\lambda_i} \sum_{j \neq i} \frac{\lambda_j}{b_{ij}} > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

$$\theta = - \sum_{i \in N} \lambda_i. \tag{4}$$

Wang and Kar [18] suggest that the Lagrange multipliers be updated using the gradient projection method. More precisely, for all $i \in \mathcal{N}$,

$$\lambda_i(n+1) = \Bigg[\lambda_i(n) + \beta(n) \tag{5}$$

$$\left(\theta - \log p_i - \sum_{j \neq i} \log \left(1 - \frac{p_j}{1 + b_{ji}} \right) \right) \Bigg]^+, \tag{6}$$

where $\beta(n)$ is the step size at the nth iteration. Further more, [18, Theorem 2] implies that a solution arbitrary close to an optimal solution can be reached via appropriate choice of step sizes. However, all the users need to exchange variables in order to perform updates.

Finally, the *directed link graph* corresponding to our network is a directed graph in which each vertex stands for a user (i.e., a transmitter-receiver pair) in the network. There is an edge from vertex i to vertex j in the directed link graph if transmission of user i affects the success of transmission of user j. Two vertices i and j are said to be connected if either of the following two conditions hold:

1) there is an edge from i to j or viceversa,
2) there are vertices $v_0 = i, v_1, \ldots, v_{n-1}, v_n = j$ such that v_m and v_{m+1} are connected for $m = 0, 1, \ldots, n-1$.

Clearly, the directed link graph for our network model is a complete graph; for any pair of vertices i and j there is an edge from i to j and also from j to i. In particular, the directed link graph is a single *strongly connected component* [19]. Thus [19, Corollary 1] implies that the above optimization also obtains the lexicographic max-min fair MAPs that yield identical rates for all the users.

D. Closest Interferer Case

A user needs to know the entire topology, and in a few cases, also needs to communicate with all the nodes to implement the protocols developed in Sections III-A-III-C. In this section, we carry out analysis assuming that the aggregate interference at a receiver is dominated by transmission from the closest interferer. This is a reasonable approximation in moderately dense networks, specifically when the path loss attenuations are high. Throughout this section, we use the notation

$$c(i) := \underset{j\neq i}{\operatorname{argmin}}\, r_{ji},$$
$$C(i) := \{j : c(j) = i\}$$

for all $1 \leq i \leq N$; $c(i)$ is the closest interferer of node i, and $C(i)$ is the set of nodes to which node i is the closest interferer. We assume that there is a unique $c(i)$ for each i. Then, accounting only for the closest interferer, the approximate probability of successful transmission for node i is

$$\tilde{q}_i = 1 - \frac{p_{c(i)}}{1 + b_{c(i)i}}.$$

The analysis of Sections III-A-III-C can be adapted to this simplified scenario.

1) Maximum Throughput Medium Access: The throughput maximization problem can now be posed as follows.

$$\begin{aligned} \text{maximize} \quad & \tilde{\Theta} := \sum_i p_i \tilde{q}_i, \\ \text{subject to} \quad & 0 \leq p_i \leq N,\ i \in \mathcal{N}. \end{aligned}$$

As in Section III-A, we can argue that some $\mathbf{p}^* \in \{0,1\}^{\mathcal{N}}$ is optimum. Again, an equivalent optimization problem is

$$\underset{\mathcal{M}\subset\mathcal{N}}{\text{maximize}} \quad \sum_{i\in\mathcal{M}} \left(1 - \frac{\mathbb{1}\{c(i) \in \mathcal{M}\}}{1 + b_{c(i)i}}\right),$$

or alternatively,

$$\underset{\mathcal{M}\subset\mathcal{N}}{\text{maximize}} \quad \sum_{i\in\mathcal{M}} \left(1 - \sum_{j\in C(i)} \frac{\mathbb{1}\{j \in \mathcal{M}\}}{1 + b_{ij}}\right).$$

We now formulate a strategic form game among users, with action sets $\{0,1\}$ and utility functions given by

$$\begin{aligned} u_i(0, \mathbf{a}_{-i}) =& 0, \\ u_i(1, \mathbf{a}_{-i}) =& 1 - \frac{a_{c(i)}}{1 + b_{c(i)i}} - \sum_{j\in C(i)} \frac{a_j}{1 + b_{ij}}. \end{aligned}$$

Again a Gibbs sampler based algorithm yields the optimal set of transmitting users. Also, user i only needs to know the distances of user $c(i)$ and all the receivers in $C(i)$ and their actions to make its decision.

Remark 3.5: Notice that user i must choose $a_i = 1$ if

$$1 - \frac{1}{1 + b_{c(i)i}} - \sum_{j\in C(i)} \frac{1}{1 + b_{ij}} > 0.$$

Such users can set their actions to 1, and need not undergo Gibbs sampler updates.

Discussion: Consider a scenario where a node's closest interferer does not transmit, i.e., has zero attempt probability. Nonetheless, this node always has an active closest interferer (unless there are no other nodes in the network). A better approximation of the success probabilities, and hence of the throughput, is obtained by always accounting for the closest active interferer. Towards this, let us define

$$c(i, \mathcal{M}) := \underset{j\in\mathcal{M}, j\neq i}{\operatorname{argmin}}\, r_{ji},$$
$$C(i, \mathcal{M}) := \{j \in \mathcal{M} : c(j, \mathcal{M}) = i\},$$

for all $i \in \mathcal{M}$. We are now faced with the following optimization problem.

$$\underset{\mathcal{M}\subset\mathcal{N}}{\text{maximize}} \quad \sum_{i\in\mathcal{M}} \left(1 - \sum_{j\in C(i,\mathcal{M})} \frac{1}{1 + b_{ij}}\right).$$

We can now define users' utility functions as follows.

$$\begin{aligned} u_i(0, \mathbf{a}_{-i}) =& 0, \\ u_i(1, \mathbf{a}_{-i}) =& 1 - \frac{1}{1 + b_{c(i,\mathcal{M})i}} \\ & - \sum_{j\in C(i,\mathcal{M}\cup\{i\})} \left(\frac{1}{1 + b_{ij}} - \frac{1}{1 + b_{c(j,\mathcal{M})j}}\right), \end{aligned}$$

where $\mathcal{M} = \{j \in \mathcal{N} : j \neq i, a_j = 1\}$. The analogous distributed algorithm (Gibbs sampler based) can again be shown to lead to the optimal solution.

2) Other Cases: Proportional Fair and max-min fair MAPs can be obtained along the lines of Sections III-B and III-C respectively. The details are worked out in our technical report [17, Section III-D].

IV. Stochastic Geometry Analysis

A. Network and Communication Model

We now assume that the transmitting nodes are scattered on the Euclidian plane according to a homogeneous PPP of intensity λ. For each transmitter, its corresponding receiver is at distance r_0 in a random direction. The traffic and channel models are the same as in Section II. As before the transmitters use slotted Aloha to access the channel, and a receiver successfully receives the packet from its transmitter if the received SINR exceeds a threshold T. Finally, transmitters adapt their attempt probabilities as described in Section III.

Each transmitter is associated with a multi dimensional mark that carries information about the adaptive transmission probability and the transmission status. Let $\tilde{\Phi} = \{X_n, Z_n\}$ denote a marked point, where

- $\Phi = \{X_n\}$ denotes the PPP of intensity λ, representing the location of transmitters in the Euclidean plane.
- $\{Z_n = (\phi_n, p_n, e_n)\}$ denote the marks of the PPP $\tilde{\Phi}$, which consist of three components:
  - $\{\phi_n\}$ denote the angles from transmitters to receivers. These angles are i.i.d. and uniform on $[0, 2\pi]$ and independent of Φ. We will call them the primary marks.
  - $\{p_n\}$ denote the MAPs of the nodes; p_n is a secondary mark (i.e. a functionals of Φ and its primary marks, see below).
  - $\{e_n\}$ are indicator functions that take value one if a given node decides to transmit in a given time slot, and zero otherwise. Clearly, $\mathbb{P}(e_n = 1) = p_n = 1 - \mathbb{P}(e_n = 0)$. In particular, given p_n, e_n is independent of everything else including $\{e_m\}_{m\neq n}$.

The locations of the receivers will be denoted by $\Phi^r = \{Y_n = X_n + (r_0, \phi_n)\}$ with $(r_0, \phi) := (r_0 \cos\phi, r_0 \sin\phi)$. It follows from the displacement theorem [2] that Φ^r is also a homogeneous PPP of intensity λ.

The above assumptions will be referred to as the Poisson model. We will also consider below a more general case where the above marked point process is just stationary.

B. Proportional Fair Spatial Aloha

1) MAP distribution: Let us consider response functions $L : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ defined for each $0 \le p \le 1$ as follows

$$L_\rho(x, y) = \frac{\rho}{\frac{\|x-y\|^\alpha}{Tr_0^\alpha} + 1 - \rho}.$$

For all $0 \le \rho \le 1, x \in \mathbb{R}^2$, the shot noise field $J_{\Phi^r}(\rho, x)$ associated with the above response function and the marked point process $\tilde{\Phi}$ is

$$J_{\Phi^r}(\rho, x) = \int_{\mathbb{R}^2} L_\rho(x, y)\Phi^r(\mathrm{d}y) = \sum_{Y_n \in \Phi^r} L_\rho(x, Y_n).$$

Notice that this shot noise is not that representing the interference at x. It rather measures the effect of the presence of a transmitter at x on the whole set of receivers.

Consider a typical node at the origin, $X_0 = 0$, with marks p_0, ϕ_0. Let $\mathbb{P}^0$ denote the Palm distribution of the stationary marked point process $\tilde{\Phi}$ [2, Chapter 1]. The fixed point equation determining the MAP of node $X_0 = 0$ reads (see (2))

$$\frac{1}{p_0} = \sum_{n \neq 0} \frac{1}{\frac{\|Y_n\|^\alpha}{Tr_0^\alpha} + 1 - p_0}.$$

We have a similar equation for each node and the sequence $\{p_n\}$ is readily seen to be a sequence of marks of Φ and its primary marks.

It follows directly from monotonicity arguments that

$$\left\{\frac{1}{p_0} < \frac{1}{\rho}\right\} \quad \text{iff} \quad \left\{\sum_{n \neq 0} \frac{\rho}{\frac{\|Y_n\|^\alpha}{Tr_0^\alpha} + 1 - \rho} < 1\right\}.$$

Notice that we have not used the specific assumptions on the point process so far. Hence we have the following general connection between the optimal MAP distribution and the shot noise J_{Φ^r}:

Theorem 4.1: For all stationary marked point processes $\widetilde{\Phi}$ (not necessarily Poisson), for all $0 < \rho < 1$,

$$\mathbb{P}^0(p_0 > \rho) = \mathbb{P}^0\left(J_{\Phi^r \setminus \{Y_0\}}(\rho, 0) < 1\right),$$
$$\mathbb{P}^0(p_0 = 1) = \mathbb{P}^0\left(J_{\Phi^r \setminus \{Y_0\}}(1, 0) < 1\right),$$

with $\mathbb{P}^0$ the Palm distribution of $\tilde{\Phi}$.

We now use the fact that $\tilde{\Phi}$ is an independently marked point process [2, Definition 2.1]. From Slivnyak's theorem [2, Theorem 1.13],

$$\mathbb{P}^0\left(J_{\Phi^r \setminus \{Y_0\}}(\rho, 0) < 1\right) = \mathbb{P}\left(J_{\Phi^r}(\rho, 0) < 1\right)$$

for all $0 \le \rho \le 1$. Consequently,

$$\mathbb{P}^0(p_0 > \rho) = \mathbb{P}\left(J_{\Phi^r}(\rho, 0) < 1\right),$$
$$\mathbb{P}^0(p_0 = 1) = \mathbb{P}\left(J_{\Phi^r}(1, 0) < 1\right).$$

It follows from [2, Proposition 2.6] and from the fact that Φ^r is a homogeneous PPP that one can write the Laplace transform $\mathcal{L}_{J(\rho,0)}(s)$ of the shot noise $J_{\Phi^r}(\rho, 0)$ as

$$\mathcal{L}_{J(\rho,0)}(s) = \exp\left\{-2\pi\lambda \int_0^\infty \left(1 - \mathrm{e}^{-\frac{s\rho\bar{r}_0}{r^\alpha + (1-\rho)\bar{r}_0}}\right) r\mathrm{d}r\right\}, \quad (7)$$

where $\bar{r}_0 := Tr_0^\alpha$.

Theorem 4.2: Under the above Poisson assumptions, the attempt probability of the typical node has the distribution

$$\mathbb{P}^0(p_0 > \rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}_{J(\rho,0)}(iw) \frac{\mathrm{e}^{iw} - 1}{iw} \mathrm{d}w, \qquad (8)$$

with $\mathcal{L}_{J(\rho,0)}(\cdot)$ given by (7).

Proof: Let $g_\rho(\cdot)$ denote the density of the shot noise field $J_\Phi(\rho, 0)$. Then

$$\mathbb{P}^0(p_0 > \rho) = \int_0^1 g_\rho(t)\mathrm{d}t = \int_{-\infty}^{\infty} g_\rho(t)u(t)\mathrm{d}t,$$

where $u(t) = 1$ if $0 \le t \le 1$ and 0 otherwise. Now using Parseval's theorem

$$\mathbb{P}^0(p_0 > \rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_{J(\rho,0)}(w)\mathcal{F}_u^*(w)\mathrm{d}w,$$

with $\mathcal{F}_A(w) = \mathbb{E}\exp(-iwA)$ the Fourier transform of the real valued random variable A and B^* the complex conjugate of B. The claim follows after substituting $\mathcal{F}_u(w) = \frac{1-\mathrm{e}^{-iw}}{iw}$ and $\mathcal{F}_{J(\rho,0)}(w) = \mathcal{L}_{J(\rho,0)}(iw)$. ■

Remark 4.1: For $\alpha = 4$, the Laplace transform $\mathcal{L}_{J(p,0)}(s)$ can be simplified as

$$\mathcal{L}_{J(p,0)}(s) = \exp\left\{-2\pi\lambda\sqrt{(1-p)T}r_0^2 \int_0^1 \frac{1 - \mathrm{e}^{-spv^2/(1-p)}}{v^2\sqrt{1-v^2}}\mathrm{d}v\right\}.$$

2) Mean Utility: This subsection is devoted to the analysis of the mean value of the logarithm of the throughput of the typical node:

$$\theta = \mathbb{E}^0 \log((p_0)) + \mathbb{E}^0 \log((q_0)),$$

with q_0 as in (1). The first term can be evaluated using the cdf f of p_0. The second term can be rewritten as (see (1))

$$\mathbb{E}^0 \log((q_0)) = \mathbb{E}^0\left[\sum_{n \neq 0} \log\left(1 - \frac{p_n}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1}\right)\right].$$

Under the law $\mathbb{P}^0$, the points $\{X_n\}_{n\neq0}$ of Φ form a homogeneous PPP of intensity λ. However, the marks $\{p_n\}_{n\neq0}$ do not have the law identified in the last section. In fact, the mark p_n of a point X_n $(n \neq 0)$ satisfies the following modified fixed point equation:

$$\frac{1}{p_n} = \frac{1}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1 - p_n} + \sum_{m \neq 0, n} \frac{1}{\frac{\|X_n - Y_m\|^\alpha}{Tr_0^\alpha} + 1 - p_n},$$

with the convention that $p_n = 1$ if there is no solution in $[0,1]$. We can argue as above to conclude that $\frac{1}{p_n} < \frac{1}{\rho}$ iff

$$\frac{\rho}{\frac{\|X_n - Y_0\|^\alpha}{Tr_0^\alpha} + 1 - \rho} + \sum_{m \neq 0,n} \frac{\rho}{\frac{\|X_n - Y_m\|^\alpha}{Tr_0^\alpha} + 1 - \rho} < 1.$$

Conditioned on there being two nodes at 0 and x, the other points form a homogeneous PPP of intensity λ. This allows one to prove the following.

Theorem 4.3: Under the above Poisson assumptions, given that there is a node at 0 and a node at $x \in \mathbb{R}^2$, the attempt probability of the node at x has the distribution

$$\mathbb{P}^{0,x}(p_x > \rho) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}_{J_x(\rho,0)}(iw) \frac{\mathrm{e}^{iw} - 1}{iw} \mathrm{d}w,$$

with

$$\mathcal{L}_{J_x(\rho,0)}(s) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(-\frac{s\rho\bar{r}_0}{||x - (r_0,\phi)||^\alpha + (1-\rho)\bar{r}_0}\right) \mathrm{d}\phi \; \exp\left\{-2\pi\lambda \int_0^{\infty} \left(1 - \mathrm{e}^{-\frac{s\rho\bar{r}_0}{r^\alpha + (1-\rho)\bar{r}_0}}\right) r\mathrm{d}r\right\},$$

and $(r_0, \phi) := (r_0 \cos\phi, r_0 \sin\phi)$.
Due to the circular symmetry, the first integral in the above expression depends on x only through $\|x\|$. Thus the density of p_x also depends on $\|x\|$ only; it will be denoted by f_r when $\|x\| = r$. The density of p_0 identified in the last subsection will be denoted by f. The main result of this section is:

Theorem 4.4: Under the above Poisson assumptions, in the proportional fair case, the mean utility of a typical node is

$$\theta = \int_0^1 \log(u) f(\mathrm{d}u) + \frac{1}{2\pi} \int_{\phi\in(0,2\pi)} \int_{x\in\mathbb{R}^2} \int_v \log\left(1 - \frac{v\bar{r}_0}{||x - (r_0,\phi)||^\alpha + \bar{r}_0}\right) \mathrm{d}\phi f_{||x-(r_0,\phi)||}(\mathrm{d}v)\mathrm{d}x. \quad (9)$$

Proof: See [17, Appendix A]. ■

In the proportional fair case, each node computes its optimal MAP in one step as the solution of a fixed point equation that is almost surely well defined (in terms of a shot noise) even in the infinite Poisson population case. Unfortunately, this does not extend to the maximum throughput and max-min fairness cases. We discuss this in our technical report [17].

V. Numerical Results

A. Simulation Setting

We consider a two dimensional square plane with side length L, and N nodes placed independently over the plane according to the uniform distribution; this corresponds to $\lambda = N/L^2$ in the stochastic geometry model.[3] Each node has its receiver randomly located on the unit circle around it, again as per the uniform distribution. Thus $r_{ii} = 1$ for all i. We set $\alpha = 4$ and $T = 10$. To nullify the edge effect, we take into account only the nodes falling in the $L/2 \times L/2$ square around the center while computing various metrics. For each parameter set we calculate the average of the performance metric of interest over 1000 independent network realizations.

[3] A finite snapshot of a PPP would contain a Poisson distributed number of nodes. However, for large λL^2, the Poisson random variable with mean λL^2 is highly concentrated around its mean. Thus we can use λL^2 nodes for all the realizations in our simulation.

B. Joint Validation of the Analysis and the Simulation

We validate the analytical expression against the simulation for the case of proportional fair medium access. To illustrate, we plot the CDF of the MAP in Figure 1. Here we set $L = 40$ and consider two values of N, $N = 400$ and $N = 800$, which correspond to $\lambda = 0.5$ and $\lambda = 0.25$ respectively. The plots show that the stochastic geometry based formula (see Theorem 4.2) quite accurately predicts the nodes' behavior in simulation.

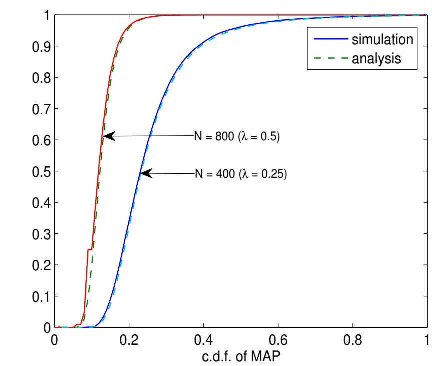

Fig. 1. CDF of the MAP for the proportional fair case.

C. Performance of the Adaptive Protocols

In this section we illustrate the performance of various adaptive schemes and their benefit over plain Aloha.

First we set $L = 20$ and $N = 50$. We consider the maximum throughput medium access, however, only accounting for the closest interferers. In figure 2, we show steady state behavior of our Gibbs sampling based algorithms; we have set the temperature $\tau(t) = 1/\log(1+t)$. As expected, the improved maximum throughput medium access (see the discussion at the end of Section III-D1) insures a better exclusion behavior. Under this scheme, a lesser number of nodes transmit, and neighboring nodes are unlikely to transmit simultaneously. So, this is expected to deliver better aggregate throughput.

Now we keep L fixed at 20, but vary N from 10 to 100; this corresponds to varying λ from 0.025 to 0.25 in the analytical expressions. We evaluate the aggregate throughputs of various Aloha schemes including plain Aloha. The average throughputs are plotted in Figure 3. *Although in some of the schemes we derive the attempt probabilities only considering the closest interferers, we always take into account the aggregate interference while calculating the throughput.* When the number of nodes is small, both the throughput maximizing medium access and plain aloha have identical performance; both prescribe attempt probabilities close to one for all the nodes. When the number of nodes increases beyond

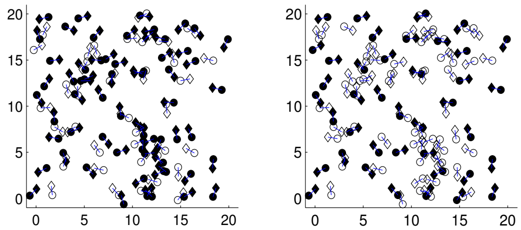

Fig. 2. Throughput maximizing medium access: There are 100 transmitter receiver pairs. The *diamonds* represent candidate transmitters, and the connected *circles* the corresponding receivers. The solid diamonds represent the nodes that transmit in all the slots; others never transmit. The left plot corresponds to the maximum throughput medium access and the right one to its improved version.

45, the throughput maximizing medium access significantly underestimates the interference, and thus its performance deteriorates. On the other hand, the aggregate interference based proportional fair scheme significantly outperforms plain Aloha in terms of aggregate throughput also. This benefit is sustained even as the number of nodes increases. We also notice that the improved version of maximum throughput medium access (see the discussion at the end of Section III-D1) yields best performance among all the schemes, and its performance does not deteriorate until a much higher number of nodes.

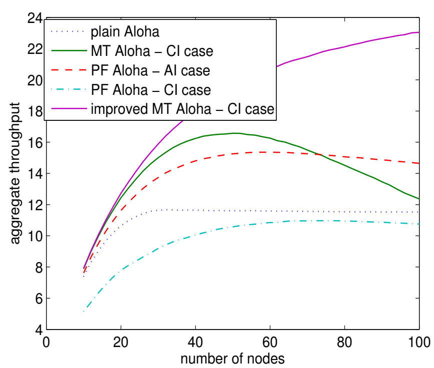

Fig. 3. Throughputs of various medium access schemes as a function of the number of nodes. MT, PF, CI and AI stand for *maximum throughput, proportional fair, closest interferer* and *aggregate interference* respectively.

VI. Conclusion

We have shown feasibility of performance analysis of distributed adaptive protocols that aim at maximizing some global utility in a large random network using stochastic geometry.

The proportionally fair adaptive Aloha is shown to have a tractable optimal MAP distribution. This distribution is obtained from the law of a certain shot noise field that describes the interference created by a typical node to all receivers but his. In the Poisson case, the distribution of the optimal MAP is obtained as a non-singular contour integral which is amenable to an efficient evaluation using classical numerical tools. The network performance at optimum can in turn be deduced from the latter using Campbell's formula, and thus the gains compared to plain Aloha can also be quantified.

This line of thought opens several research directions. The first is extension to other types of fairness, still in the framework of Aloha. The second and broader question is whether this approach can be extended to other MAC protocols. An example would be an adaptation of the exclusion radius of CSMA/CA to the environment of a node aiming at maximizing some utility of the throughput. Another general question concerns the evaluation of the "price of decentralization".

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